On Dialectica and Differentiation

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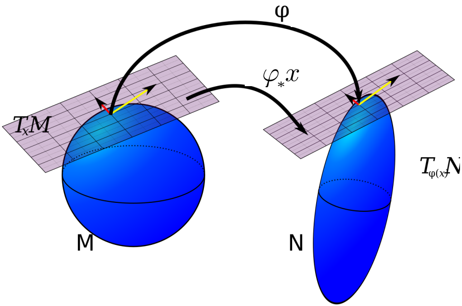
Seminar at University of Sussex, Brighton (UK)

13/11/2024

Differentiation



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Cartesian differential categories (∼'09)

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Cartesian tangent categories ('14)

$$\frac{f : A \to B}{Tf : TA \to TB}$$

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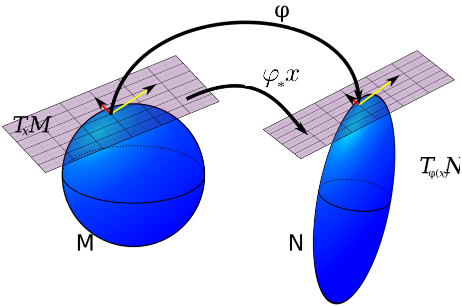

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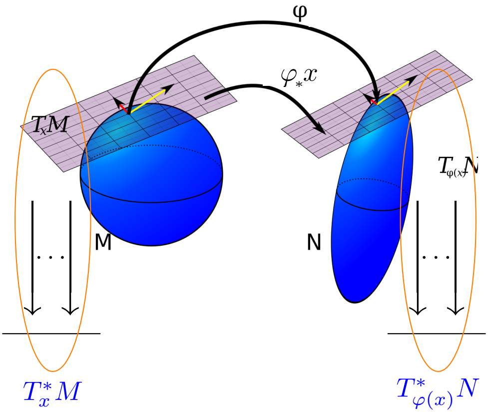

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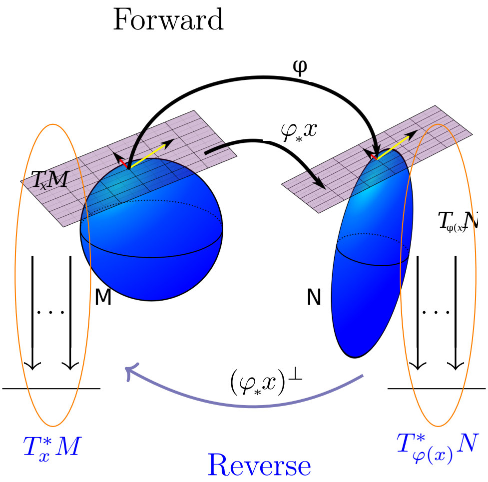

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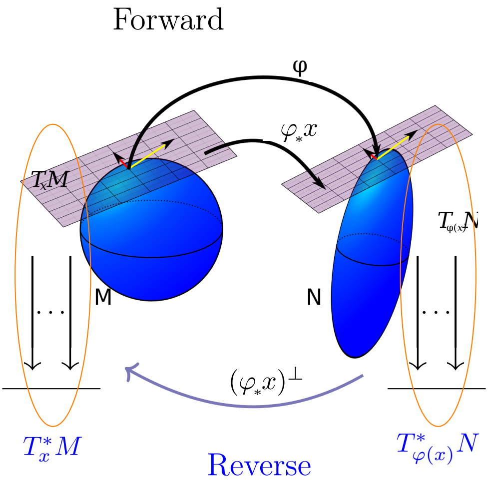
Cartesian reverse diff. categories ('20)

$$\frac{f : A \to B}{Rf : A \times B \to A}$$

Cart. reverse tangent categories ('24)

$$\frac{f : A \to B}{T^*f : f^*T^*B \to T^*A}$$



Dialectica (overview)

	Source $\to$ Target		
Gödel ('58)	$A \in \mathrm{HA}$	$\longmapsto$	$A_D\{w, c\} \in \mathrm{T}$
		such that	
	$\vdash_{\mathrm{HA}} A$	$\Longrightarrow$	$\vdash_{\mathrm{T}} A_D\{\mathtt{M}, c\}$ *for some* $\mathtt{M} \in \mathrm{T}$

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Pédrot (Diller-Nahm) ('15)	$A \in \Lambda$	$\longmapsto$	$W(A), C(A) \in \mathbf{P}$
	$\mathtt{M} \in \Lambda$	$\longmapsto$	$\mathtt{M}^\bullet, \mathtt{M}_\mathtt{x} \in \mathbf{P}$ *(for* $\boldsymbol{x}$ *variable)*
		such that	
	$\mathtt{x} : A \vdash_\Lambda \mathtt{M} : B$	$\Longrightarrow$	$\begin{cases} \mathtt{x} : W(A) \vdash_{\mathbf{P}} \mathtt{M}^\bullet : W(B) \\ \mathtt{x} : W(A) \vdash_{\mathbf{P}} \mathtt{M}_\mathtt{x} : C(B) \to \mathcal{M}[C(A)] \end{cases}$

Dialectica (Transformation)

	α	$E \to F$
W	α_W	$W(E) \to W(F)$ $\times$ $W(E) \times C(F) \to \mathcal{M}[C(E)]$
C	α_C	$W(E) \times C(F)$

	$\mathtt{x}$	$\lambda\mathtt{x}.\mathtt{M}$	$\mathtt{PQ}$
$(\_)^\bullet$	$\mathtt{x}$	$\left\langle \begin{array}{c} \lambda\mathtt{x}.\mathtt{M}^\bullet \\ \lambda\pi.(\lambda\mathtt{x}.\mathtt{M}_\mathtt{x})\pi^1\pi^2 \end{array} \right\rangle$	$\mathtt{P}^{\bullet\,1}\mathtt{Q}^\bullet$
$(\_)_\mathtt{y}$	$\begin{cases} \lambda\pi.[\pi], & \mathtt{x} = \mathtt{y} \\ \lambda\pi.0, & \mathtt{y} \neq \mathtt{y} \end{cases}$	$\lambda\pi.(\lambda\mathtt{x}.\mathtt{M}_\mathtt{y})\pi^1\pi^2$	$\lambda\pi.\left(\begin{array}{c} \mathtt{P}_\mathtt{y}\langle \mathtt{Q}^\bullet, \pi\rangle \\ + \\ \mathtt{P}^{\bullet\,2}\langle \mathtt{Q}^\bullet, \pi\rangle \gg\!= \mathtt{Q}_y \end{array} \right)$

A model $(\mathcal{C}, !)$ of Classical Differential Linear Logic

Arrows in $\mathcal{C}$*:* $A \overset{f}{\multimap} B$ *(linear)* *Arrows in* $\mathcal{C}_!$*:* $A \xrightarrow{f} B := !A \overset{f}{\multimap} B$ *(non-linear)*

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Cartesian
+SMC
+Seely

$$\frac{A \quad B}{A \& B} \qquad \frac{A \quad B}{A \otimes B} \qquad \frac{}{!\top}$$

$$\frac{}{\mathrm{ev}_{A,B} : [A \multimap B] \otimes A \multimap B}$$

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$$\frac{f : A \multimap B}{f^{\perp} : B^{\perp} \multimap A^{\perp}}$$

$$\frac{}{f^{\perp\perp} = f}$$

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DiLL magic

$$\frac{f : A \to B}{\partial f : A \otimes !A \multimap B} \qquad \frac{\dfrac{f : A \to B}{f_* : A \to [A \multimap B]} \quad a : !\top \multimap A}{f_* a : A \multimap B}$$

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$\mathcal{C}_!$ is a model of differential λ-calculus where we can transpose linear arrows:

$$\frac{f : A \to B}{Df : A \times A \to B} \qquad (\textit{in } \mathcal{C}_!,\ E \times F := E \& F\ )$$

Dialectica and (Categorical) Differentiation

$$\sim_B \quad \subseteq \quad \{\vdash_{\mathbf{P}} \mathtt{M} : W(B)\} \quad \times \quad \mathcal{C}_!(\top, B)$$

$$\bowtie_B^A \quad \subseteq \quad \{\vdash_{\mathbf{P}} \mathtt{M} : C(B) \to \mathcal{M}[C(A)]\} \quad \times \quad \mathcal{C}_!(A, B) \times \mathcal{C}(B^\perp, A^\perp)$$

$\mathtt{M} \sim_{E \to F} f$	*for all* $\mathtt{H} \sim_E e$, *we have* $\quad \mathtt{M}^1\mathtt{H} \sim_F f\big\vert_e : F$ *and* $\quad \lambda\pi.\mathtt{M}^2\langle \mathtt{H}, \pi\rangle \bowtie_F^E \begin{pmatrix} \lambda^{-1} f : E \to F \\ ((\lambda^{-1} f)_* e)^\perp : F^\perp \multimap E^\perp \end{pmatrix}$
$\mathtt{M} \bowtie_{E \to F}^A \binom{f}{g}$	*for all* $\mathtt{H} \sim_E e$, *we have* $\lambda\pi.\mathtt{M}\langle \mathtt{H}, \pi\rangle \bowtie_F^A \begin{pmatrix} f\big\vert_e : A \to F \\ g^\perp\big\vert_e^\perp : F^\perp \multimap A^\perp \end{pmatrix}$

The theorem

Let $\mathtt{x} : A \vdash_\Lambda \mathtt{M} : B$. Then:

$$1) \quad (\lambda \mathtt{x}.\mathtt{M})^\bullet \quad \sim_{A\to B} \quad [\![\lambda\mathtt{x}.\mathtt{M}]\!] : [A \to B]$$

$$2) \quad (\lambda \mathtt{x}.\mathtt{M}_\mathtt{x})\mathtt{N} \quad \bowtie^A_B \quad \begin{pmatrix} [\![\mathtt{M}]\!] & : & A \to B \\ ([\![\mathtt{M}]\!]_* a)^\perp & : & B^\perp \multimap A^\perp \end{pmatrix} \quad \textit{for all } \mathtt{N} \sim_A a.$$

Moral:

$$(\lambda \mathtt{x}.\mathtt{M}^\bullet \,,\, \lambda \mathtt{x}.\mathtt{M}_\mathtt{x})$$

"represents" the pair $([\![\mathtt{M}]\!] \,,\, R[\![\mathtt{M}]\!])$, where

$$R[\![\mathtt{M}]\!] : A \times B^\perp \to A^\perp$$

is the reverse differential of $[\![\mathtt{M}]\!]$.

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- Is this correspondence astonishing/magic? Can we find some "reason" clarifying it?
  Definitely yes at first sight... but then we can clearly understand its reason by looking at the categorical framework behind it.

Lens Categories

The category $\mathrm{Lens}(\mathcal{L})$ of lenses over $\mathcal{L}$ is defined as follows:

- objects: arrows in $\mathcal{L}$, which we think as fibre bundles and we write $p : \binom{\alpha}{A}$
- arrows from $p : \binom{\alpha}{A}$ to $q : \binom{\beta}{B}$ are the data of both a $f : A \to B$ in $\mathcal{L}$ and a span $\alpha \xleftarrow{F} f^*\beta \xrightarrow{\overline{f}} \beta$ in $\mathcal{L}$, taken from the following pullback diagram:

$$\begin{tikzcd}
\alpha \arrow[drr, "p"'] & f^*\beta \arrow[l, "F"'] \arrow[r, "\overline{f}"] \arrow[d, "f^*q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \beta \arrow[d, "q"] \\
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Let $\mathcal{E}\mathrm{Lens}(\mathcal{L})$ be the full subcategory of $\mathrm{Lens}(\mathcal{L})$ of trivial bundles, i.e. first projections. Concretely:

- Objects are first projections $\pi_1 : \binom{A \times X}{A}$
- An arrow from $\pi_1 : \binom{A \times X}{A}$ to $\pi_1 : \binom{B \times Y}{B}$ is given by an $f : A \to B$ and a span $A \times X \xleftarrow{F} A \times Y \xrightarrow{f \times 1} B \times Y$ such that $F; \pi_1^{A,X} = \pi_1^{A,Y}$.

Differentiation through the lens of lenses

Let $\mathcal{L}$ be a Cartesian (closed, if you want λ-calculus) differential category where from the differential Df of a function f (a primitive data in $\mathcal{L}$) we can define the reverse differential Rf of f. (Think of $\mathcal{L} := \mathcal{C}_!$ of the first part).

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We have a functor $\mathcal{L} \to \mathcal{E}\mathrm{Lens}(\mathcal{L})$ defined by:

$$
\begin{array}{lcc}
A & \mapsto & \pi_1 : \binom{A \times A^\perp}{A} \\[2ex]
A \xrightarrow{f} B & \mapsto & \left(\quad f \quad , \quad A \times A^\perp \xleftarrow{\langle \pi_1, Rf \rangle} A \times B^\perp \xrightarrow{f \times 1} B \times B^\perp \quad \right).
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\end{array}
$$

where $Rf : A \times B \to A$ is the reverse differential of f (a primitive data in $\mathcal{L}$).

Differentiation through the lens of lenses

Let $\mathcal{L}$ be a reverse tangent category. This means that $\mathcal{L}$ has a tangent functor T giving tangent bundles $p_A : \binom{TA}{A}$ of objects A and giving tangent arrows $Tf : TA \to TA$ for arrows $f : A \to B$, and we can "reverse" T in order to get cotangent bundles $p_A^* : \binom{T^*A}{A}$ and arrows in the pullback diagram below:

$$\begin{tikzcd}
T^*A \arrow[drr, "p_A^*"'] & f^*T^*B \arrow[l, "T^*f"'] \arrow[r, "\overline{f}"] \arrow[dr, "f^*p_B^*"', pos=0.3] \arrow[drr, phantom, "\lrcorner", very near start] & T^*B \arrow[dr, "p_B^*"] & \\
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where T^*f is the diff. geometry formulation of the reverse differential of f.

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We have a functor $\mathcal{L} \to \mathrm{Lens}(\mathcal{L})$ defined by:

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A & \mapsto & p^*_A : \binom{T^*A}{A} \\
A \xrightarrow{f} B & \mapsto & \left(\quad f \quad , \quad T^*A \xleftarrow{T^*f} f^*T^*B \xrightarrow{\overline{f}} T^*B \quad \right).
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Expressing Dialectica as a functor $\Lambda_{\mathrm{cat}} \to \mathcal{E}\mathrm{Lens}(\mathbf{P}_{\mathrm{cat}})$

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- An object A is sent to the typed term $\mathtt{z} : W(A) \times \mathcal{M}[C(A)] \vdash_{\mathbf{P}} \mathtt{z}^1 : W(A)$
- An arrow $\mathtt{z} : A \vdash_{\Lambda} M : B$ in Λ_{cat} from A to B is sent to the arrow in $\mathcal{E}\mathrm{Lens}(\mathbf{P}_{\mathrm{cat}})$ from $\mathtt{z} : W(A) \times \mathcal{M}[C(A)] \vdash_{\mathbf{P}} \mathtt{z}^1 : W(A)$ to $\mathtt{z} : W(B) \times \mathcal{M}[C(B)] \vdash_{\mathbf{P}} \mathtt{z}^1 : W(B)$ given by the following diagram:

$$
\begin{tikzcd}
W(A) \times \mathcal{M}[C(A)] \arrow[drr, "\mathtt{z}^1"'] & & W(A) \times \mathcal{M}[C(B)] \arrow[ll, "\langle \mathtt{z}^1, \mathtt{z}^2 \gg= \mathtt{M}_{(\mathtt{z}^1)} \rangle"'] \arrow[rr, "\langle \mathtt{M}^\bullet, \mathtt{z}^2 \rangle"] \arrow[d, "\mathtt{z}^1"'] \arrow[drr, phantom, "\lrcorner", very near start] & & W(B) \times \mathcal{M}[C(B)] \arrow[d, "\mathtt{z}^1"] \\
& & W(A) \arrow[rr, "\mathtt{M}^\bullet"'] & & W(B)
\end{tikzcd}
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Expressing Dialectica as a functor $\Lambda_{\mathrm{cat}} \to \mathcal{E}\mathrm{Lens}(\mathbf{P}_{\mathrm{cat}})$

Moral:

The Dialectica transformation of λ-calculus encodes (reverse) Differentiation *because* it is a transformation into a category of Lenses, the latter being the abstract setting for Reverse Differentiation.

Final comments

- I didn't talk about Dialectica categories. I could have said something (ask me if you are interested)
- Explore categorical framework to reverse a Cartesian closed differential category in order to define Cartesian closed reverse differential/tangent categories
- Reverse differential λ-calculus? There is an interesting paper from Ong and Mak.

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THANK YOU!