The λ-calculus, from intuitionistic to classical logic

Webpage of the course

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The λ-calculus, from intuitionistic to classical logic

What will you (hopefully!) learn: You will have an idea of some basic but fundamental aspects of the relationship between (*λ-calculus based*) programming language theory and (*Curry-Howard based*) mathematical logic.

Lecture 1: Topology/domain-theory of the "graph model" over $\mathcal{P}(\mathbb{N})$
Davide Barbarossa
Lecture 2: The λ-calculus
Davide Barbarossa
Lecture 3: Category theory for denotational semantics
Giulio Guerrieri
Lecture 4: Curry-Howard and intuitionistic logic
Giulio Guerrieri
Lecture 5: Krivine's approach to classical logic
Davide Barbarossa

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Lecture 1:

Topology/domain-theory of the "graph model" over $\mathcal{P}(\mathbb{N})$

Read the notes: they are full of details, proofs, explanations, exercises, bibliography!

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What do we mean by *computable*?

Problem: Compute $\sqrt{S}$, for some given $S \in \mathbb{R}$

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Problem: Compute $\sqrt{54899321104}$

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\begin{aligned}
x_0 &:= \textit{any positive number} \\
x_{n+1} &:= \frac{1}{2}\left(x_n + \frac{54899321104}{x_n}\right)
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$\overset{\text{Heron}}{\cancel{\text{We}}}$ defined a function

$$H : \mathbb{N} \to \mathbb{R}, \qquad H(n) := x_n$$

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$$\lim_{n \to +\infty} H(n) = \sqrt{54899321104} = H(+\infty) = H\left(\sup_{n < +\infty} n\right)$$

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$\overset{\text{Eratosthenes}}{\cancel{\text{We}}}$ defined a function

$$E : \mathbb{N} \to \mathcal{P}(\mathbb{N}), \qquad E(n) := \textit{set of the first } n \textit{ prime numbers}$$

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What property do the share from the fact that they are given by an algorithm?

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(potentially) infinitary entity	$a \in \mathbb{N} \cup \{+\infty\}$	$r \in \mathbb{R}$
finitary entity	$n \in \mathbb{N}$	$q \in \mathbb{Q}$
approximation of infinitary by finitary	$n < +\infty$	$q \in \mathbb{Q}$ is "close" to $r \in \mathbb{R}$
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A generic f

For all approximation d of $f(x)$,
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For all finite portion of the output,
there is a finite portion of the input such that
the latter suffices to compute the former

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Let's make this rigorous: a topology over $\mathcal{P}(\mathbb{N})$

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For today and tomorrow, R means $\mathcal{P}(\mathbb{N})$. For $a \in R$, let $\uparrow a := \{b \in R \mid b \supseteq a\}$.

Definition

Let us take as open sets $O \subseteq R$ the ones of shape $O = \bigcup_{e \in I} \uparrow e$, for some $I \subseteq R_{\mathit{fin}}$.

This is possible as $\{\uparrow e \subseteq R \mid e \subseteq_{\mathit{fin}} \mathbb{N}\} \subseteq \mathcal{P}(R)$ is a *base* for a topology on $\mathcal{P}(\mathbb{N})$.

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Theorem

The open sets of our topology are exactly the $O \subseteq R$ which satisfy the following condition for all $a \in R$:

$$a \in O \Leftrightarrow O \ni e \subseteq_{fin} a, \textit{ for some } e \in R.$$

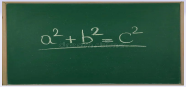
Let's make this rigorous: a topology over $\mathcal{P}(\mathbb{N})$

Theorem

Let $f : R \to R$. The following are equivalent:

1. *f is continuous.*
2. *f is monotone for $\subseteq$ and, for all $a, d \in R$ we have*
$$d \subseteq_{\text{fin}} f(a) \quad \Longrightarrow \quad \text{there is } \; e \subseteq_{\text{fin}} a \; \text{ such that } \; d \subseteq f(e).$$
3. *for all $a \in R$ we have*
$$f(a) = \bigcup_{e \subseteq_{fin} a} f(e).$$

Theorem



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The Scott-topology of a poset

A poset is the data of a set X and a partial order relation $\leq$ (i.e. reflexive, antisymmetric and transitive).

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Example

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A poset is the data of a set X and a partial order relation $\leq$ (i.e. reflexive, antisymmetric and transitive).
A subset D of a poset X is called *directed* whenever $D \neq \emptyset$ and all $d, d' \in D$ admit a common upper bound in D.

Example

In R, the set $D = \{\{5, 2\}, \emptyset, \{8\}\}$ is not directed, because D contains $\{5, 2\}, \{8\}$ but no element of D is bigger than both $\{5, 2\}$ and $\{8\}$.

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In R, the set $D = \{e \in R \mid e \subseteq_{\text{fin}} \{5,2\}\} = \{\emptyset, \{5,2\}, \{5\}, \{2\}\}$ is directed (in this case it even has a maximum element).

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In R, given $a \in R$, the set $D = \{e \in R \mid e \subseteq_{\text{fin}} a\} =: \downarrow_{\text{fin}} a = \mathcal{P}_{\text{fin}}(a)$ is directed: the upper bound in D of $e, e' \in D$ is $e \cup e' \in D$ (in this case it is even their sup).

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Definition

Let $(X, \leq)$ be a poset. Its *Scott-topology* is defined by declaring open the subsets U of X such that:

1. U is upward closed (i.e. $U \ni x \leq y \Rightarrow U \ni y$)
2. for all directed $D \subseteq X$ which admits $\bigvee D \in U$, we have $D \cap U \neq \emptyset$.

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Example (Proposition)

Our topology on R coincides with the Scott-topology of the poset $(R, \subseteq)$.

The Scott-topology of a poset

In a poset $(X, \leq)$, we say that an element $e \in X$ is *compact* iff for all directed $D \subseteq X$ admitting $\bigvee D$, we have: if $e \leq \bigvee D$ then $e \leq d$ for some $d \in D$.

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In $(R, \subseteq)$, the set of compact elements is R_{fin}.

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Example (Proposition)

In $(R, \subseteq)$, the set of compact elements is R_{fin}.

Theorem

Let X, Y be posets and $f : X \to Y$. The following are equivalent:

1. *f is continuous wrt the Scott-topologies on X, Y.*
2. *For all directed $D \subseteq X$ admitting $\bigvee D$ in X, there is $\bigvee(fD)$ in Y and:*

$$\bigvee(fD) = f(\bigvee D). \qquad \text{(Scott-Continuity)}$$

If moreover X, Y are "algebraic" (like R), then the above are equivalent to:

3. *f is monotone for $\leq$ and, for all $a \in X$ and $d \in Y$, we have*

$$d \text{ compact } \leq f(a) \quad \Longrightarrow \quad \text{there is } e \text{ compact } \leq a \text{ such that } d \leq f(e).$$

4. *For all $a \in X$ we have $f(a) = \bigvee_{e \text{ compact} \leq a} f(e)$*

Encoding (continuous) functions as points

"*We can encode everything into* $\mathbb{N}$*!* "

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$$\mathrm{pair} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad \mathrm{pair}(n, m) := 2^n(2m+1) - 1$$

$$\mathrm{list} : \mathbb{N}^* \to \mathbb{N} \qquad \mathrm{list}([]) := 0 \quad \textit{and} \quad \mathrm{list}(n :: l) := 1 + \mathrm{pair}(n, \mathrm{list}(l))$$

$$\langle \_, \_ \rangle : \mathbb{N}^* \times \mathbb{N} \to \mathbb{N} \qquad \langle l, n \rangle := \mathrm{pair}(\mathrm{list}(l), n)$$

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Corollary

We have a bijective encoding $\langle \_ , \_ \rangle$ *of* $\mathbb{N}^* \times \mathbb{N}$ *into* $\mathbb{N}$

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Consider the following two auxiliary functions:

$$\begin{array}{ll} \mathrm{set} : \mathbb{N}^* \to R_{\mathit{fin}} & \mathrm{set}(n_1, \ldots, n_k) := \{n_1, \ldots, n_k\} \\ \mathrm{kl} : R \to \mathcal{P}(\mathbb{N}^*) & \mathrm{kl}(a) := \{l \in \mathbb{N}^* \mid \mathrm{set}(l) \subseteq a\} \end{array}$$

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Definition

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\begin{array}{ll}
\mathrm{fun} : R \to (R \Rightarrow R), & \mathrm{fun}(a)(b) := \{n \in \mathbb{N} \mid \textit{ there is } l \in \mathrm{kl}(b) \textit{ s.t. } \langle l, n\rangle \in a\} \\
@ : R \times R \to R, & @ := \mathrm{uncurry}(\mathrm{fun}) \\
\lambda : (R \Rightarrow R) \to R, & \lambda(f) := \{\langle l, m\rangle \in \mathbb{N} \mid l \in \mathbb{N}^*,\ m \in f(\mathrm{set}(l))\}
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$$@ : R \times R \to R, \quad @ := \mathrm{uncurry}(\mathrm{fun})$$

$$\lambda : (R \Rightarrow R) \to R, \quad \lambda(f) := \{\langle l, m \rangle \in \mathbb{N} \mid l \in \mathbb{N}^*, \, m \in f(\mathrm{set}(l))\}$$

Theorem

We indeed have $\mathrm{Im}(\mathrm{fun}) \subseteq (R \Rightarrow R)$.

"We can encode everything into $\mathbb{N}$!"

Corollary

We have a bijective encoding $\langle \_, \_ \rangle$ of $\mathbb{N}^* \times \mathbb{N}$ into $\mathbb{N}$

Consider the following two auxiliary functions:

$$\begin{array}{ll} \mathrm{set} : \mathbb{N}^* \to R_{fin} & \mathrm{set}(n_1, \ldots, n_k) := \{n_1, \ldots, n_k\} \\ \mathrm{kl} : R \to \mathcal{P}(\mathbb{N}^*) & \mathrm{kl}(a) := \{l \in \mathbb{N}^* \mid \mathrm{set}(l) \subseteq a\} \end{array}$$

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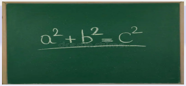

Encoding (continuous) functions as points

Theorem

- *@, fun and λ are continuous.*
- *The pair (λ, fun) defines a topological retraction of R onto $\mathrm{Im}(\lambda) \approx (R \Rightarrow R)$, i.e.*
$$\mathrm{fun} \circ \lambda = \mathrm{id}_{R \Rightarrow R}. \tag{β}$$
- *For all $f : R \Rightarrow R$ we have:*
$$\lambda(\mathrm{fun}(f)) \supseteq f \quad \textit{and} \quad \lambda(f) = \bigcup_{\mathrm{fun}(a)=f} a.$$

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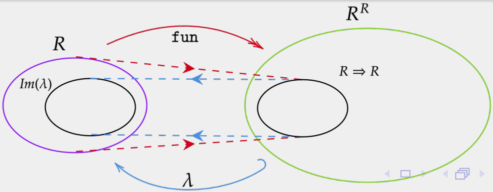
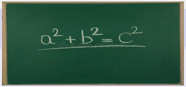
Definition

The structure $(R, \lambda, @)$ is called the *graph model.*





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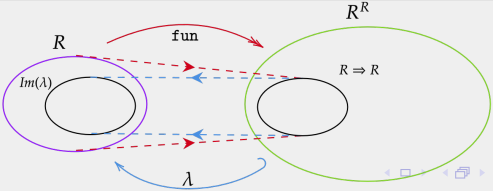

Interesting properties of continuous functions!

All continuous $f : R \to R$ admit fixed points.

The function

$$Y : (R \Rightarrow R) \to R \qquad Y(f) := @(\Delta_f, \Delta_f)$$

where $\Delta_f := \lambda(f \circ @ \circ \delta) \in R$ and $\delta : R \to R \times R$ is the diagonal, is a *fixed point combinator*, i.e. for all $f : R \Rightarrow R$, we have

$$f(Y(f)) = Y(f).$$

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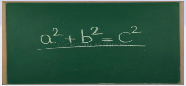

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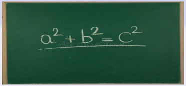

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Theorem

The set of RE sets is closed wrt the following rules:

$$\frac{}{\lambda(\lambda \circ \mathrm{curry}(\cdots(\lambda \circ \mathrm{curry}(\mathrm{proj}_i^n))\cdots))}$$

$$\frac{f : R \Rightarrow R \text{ and computable}}{\lambda(f)} \qquad \frac{a \qquad b}{@(a, b)}$$

Topology/domain-theory of the graph model over $\mathcal{P}(\mathbb{N})$

- What do we intuitively mean when we say that a function is computable
- That this relates to topology
- That $R = (\mathcal{P}(\mathbb{N}), Scott)$ is a good topological space for modeling this
- That its topology only depends from the partial order $\subseteq$
  - That actually, Scott-topology and Scott-continuity themselves are a property about "approximations" in posets
  - That Scott-continuous functions are given by their restriction on the finite elements of R. Also, they embed into their base space. This is done via the retraction (λ, fun), i.e. equation (β)
  - That all Scott-continuous functions on R have fixed points
  - That λ produces RE sets and fun preserves them

Some exercises and details

- Do the proofs of the statements in the slides
- Look at my **notes** on the webpage of the course: there are plenty more **exercises** and **details**
- The exercises have **solutions** (but try to do them by yourself before looking at them!)

Some (classic) bibliography to go further

- Chapter 1 and Section 1 of Chapter 18 of:
  Henk P. Barendregt, The lambda-calculus, its syntax and semantics, 1984,
  https://www.sciencedirect.com/bookseries/studies-in-logic-and-the-foundations-of-mathematics/vol/103
- Domain theory chapter of:
  Handbook of logic in computer science: semantic structures. 1995,
  https://dl.acm.org/doi/book/10.5555/218742 (an updated version is available here)
- Chapter 1 of:
  Amadio R., Curien P-L, Domains and lambda-calculi, 1996,
  https://www.cambridge.org/core/books/domains-and-lambdacalculi/4C6AB6938E436CFA8D5A8533B76A7F23
- Chapter 1, 5 of:
  PhD thesis of Giulio Manzonetto, Models and theories of lambda calculus, 2008,
  https://www.irif.fr/~gmanzone/ManzonettoPhdThesis.pdf
- This conference of **Dana Scott, 2019**. The first half is an interesting historical background, and the second half includes what we saw today