

# On Dialectica and Differentiation

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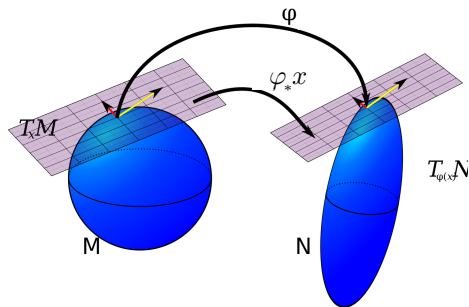
Department of Computer Science



Seminar at University of Sussex, Brighton (UK)

*06/11/2024*

# Differentiation



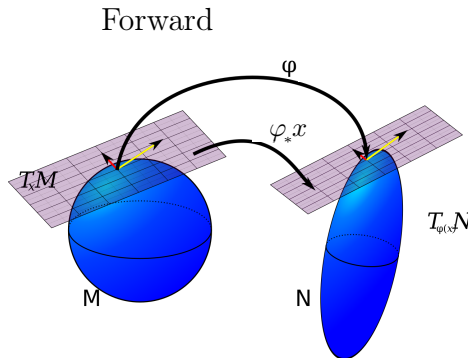
# Differentiation

Cartesian differential categories

$$\frac{f : A \rightarrow B}{Df : A \times A \rightarrow B}$$

Cartesian tangent categories

$$\frac{f : A \rightarrow B}{Tf : TA \rightarrow TB}$$



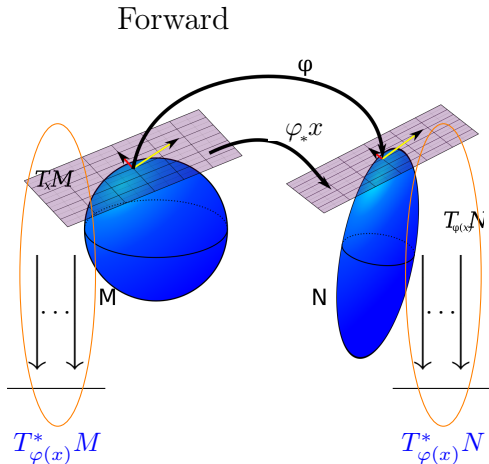
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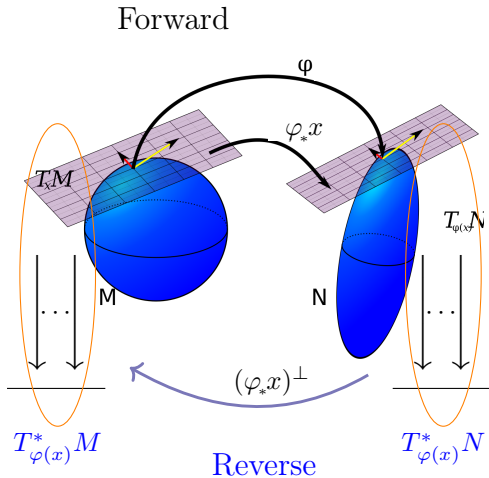
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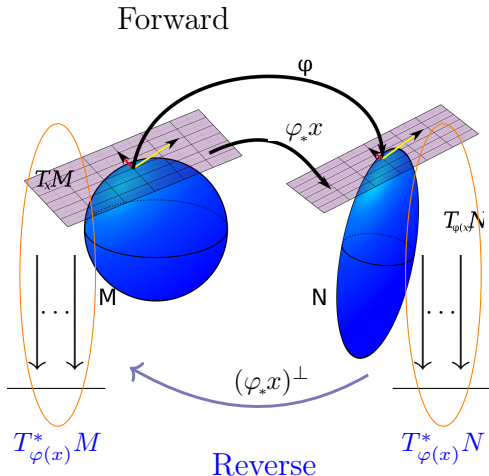
$$\frac{f : A \rightarrow B}{Tf : TA \rightarrow TB}$$

Cartesian reverse differential categories

$$\frac{f : A \rightarrow B}{Rf : A \times B \rightarrow A}$$

Cartesian reverse tangent categories

$$\frac{f : A \rightarrow B}{T^*f : f^*T^*B \rightarrow T^*A}$$



# Dialectica (overview)

Source  $\rightarrow$  Target

Gödel  
(’58)

$$A \in \text{HA} \quad \mapsto \quad A_D\{w, c\} \in \text{T}$$

*such that*

$$\vdash_{\text{HA}} A \quad \implies \quad \vdash_{\text{T}} A_D\{M, c\} \text{ for some } M \in \text{T}$$

# Dialectica (overview)

	Source $\rightarrow$ Target		
Gödel (’58)	$A \in \mathbf{HA}$ $\vdash_{\mathbf{HA}} A$	$\longmapsto$ <i>such that</i> $\implies$	$A_D\{w, c\} \in \mathbf{T}$ $\vdash_{\mathbf{T}} A_D\{\mathbf{M}, c\}$ for some $\mathbf{M} \in \mathbf{T}$
Pédrot (Diller- Nahm) (’15)	$A \in \Lambda$ $\mathbf{M} \in \Lambda$ $\mathbf{x} : A \vdash_{\Lambda} \mathbf{M} : B$	$\longmapsto$ $\longmapsto$ <i>such that</i> $\implies$	$W(A), C(A) \in \mathbf{P}$ $\mathbf{M}^\bullet, \mathbf{M}_x \in \mathbf{P}$ (for $x$ variable) $\begin{cases} \mathbf{x} : W(A) \vdash_{\mathbf{P}} \mathbf{M}^\bullet : W(B) \\ \mathbf{x} : W(A) \vdash_{\mathbf{P}} \mathbf{M}_x : C(B) \rightarrow \mathcal{M}[C(A)] \end{cases}$



# Dialectica (Transformation)

	$\alpha$	$E \rightarrow F$
W	$\alpha_W$	$W(E) \rightarrow W(F)$ $\times$ $W(E) \times C(F) \rightarrow \mathcal{M}[C(E)]$
C	$\alpha_C$	$W(E) \times C(F)$

	$x$	$\lambda x.M$	PQ
$(\_)\bullet$	$x$	$\left\langle \begin{array}{c} \lambda x.M^\bullet \\ \lambda \pi.(\lambda x.M_x)\pi^1\pi^2 \end{array} \right\rangle$	$P^{\bullet 1}Q^\bullet$
$(\_)_y$	$\begin{cases} \lambda \pi.[\pi], & x = y \\ \lambda \pi.0, & y \neq y \end{cases}$	$\lambda \pi.(\lambda x.M_y)\pi^1\pi^2$	$\lambda \pi. \left( \begin{array}{c} P_y\langle Q^\bullet, \pi \rangle \\ + \\ P^{\bullet 2}\langle Q^\bullet, \pi \rangle \gg Q_y \end{array} \right)$

# A model $(\mathcal{C}, !)$ of Classical Differential Linear Logic

Arrows in  $\mathcal{C}$ :  $A \xrightarrow{f} B$  (linear)    Arrows in  $\mathcal{C}_!$ :  $A \xrightarrow{f} B := !A \xrightarrow{f} B$  (non-linear)

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Cartesian  
+SMC  
+Seely

$$\frac{A \quad B}{A \& B}$$

$$\frac{A \quad B}{A \otimes B}$$

$$\overline{! \top}$$

$$\overline{\text{ev}_{A,B} : [A \multimap B] \otimes A \multimap B}$$

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$\star$ -autonomous  
comm.  
monoids  
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$$\overline{0_{A,B} : A \multimap B}$$

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*DiLL magic*

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$\mathcal{C}_!$  is a model of differential  $\lambda$ -calculus where we can transpose linear arrows:

$$\frac{f : A \rightarrow B}{Df : A \times A \rightarrow B} \quad (\text{in } \mathcal{C}_!, E \times F := E \& F)$$

# Dialectica and (Categorical) Differentiation

$$\sim_B \subseteq \{\vdash_{\mathbf{P}} \mathbf{M} : W(B)\} \times \mathcal{C}_!(\top, B)$$

$$\bowtie_B^A \subseteq \{\vdash_{\mathbf{P}} \mathbf{M} : C(B) \rightarrow \mathcal{M}[C(A)]\} \times \mathcal{C}_!(A, B) \times \mathcal{C}(B^\perp, A^\perp)$$

$\mathbf{M} \sim_{E \rightarrow F} f$	<p>for all <math>H \sim_E e</math>, we have <math>\mathbf{M}^1 H \sim_F f _e : F</math></p> <p>and <math>\lambda \pi. \mathbf{M}^2 \langle H, \pi \rangle \bowtie_F^E \left( \begin{array}{c} \lambda^{-1} f : E \rightarrow F \\ ((\lambda^{-1} f)_* e)^\perp : F^\perp \multimap E^\perp \end{array} \right)</math></p>
$\mathbf{M} \bowtie_{E \rightarrow F}^A \left( \begin{array}{c} f \\ g \end{array} \right)$	<p>for all <math>H \sim_E e</math>, we have</p> <p><math>\lambda \pi. \mathbf{M} \langle H, \pi \rangle \bowtie_F^A \left( \begin{array}{c} f _e : A \rightarrow F \\ g^\perp _e^\perp : F^\perp \multimap A^\perp \end{array} \right)</math></p>



# The theorem

Let  $\mathbf{x} : A \vdash_{\Lambda} \mathbf{M} : B$ . Then:

$$1) \quad (\lambda \mathbf{x} . \mathbf{M})^{\bullet} \quad \sim_{A \rightarrow B} \quad \llbracket \lambda \mathbf{x} . \mathbf{M} \rrbracket : [A \rightarrow B]$$

$$2) \quad (\lambda \mathbf{x} . \mathbf{M}_{\mathbf{x}}) \mathbf{N} \quad \bowtie_B^A \quad \left( \begin{array}{ll} \llbracket \mathbf{M} \rrbracket & : \quad A \rightarrow B \\ (\llbracket \mathbf{M} \rrbracket_* a)^{\perp} & : \quad B^{\perp} \multimap A^{\perp} \end{array} \right) \quad \text{for all } \mathbf{N} \sim_A a.$$

Moral:

$$((\lambda \mathbf{x} . \mathbf{M})^{\bullet}, \lambda \mathbf{x} . \mathbf{M}_{\mathbf{x}})$$

“represents” the pair  $(\llbracket M \rrbracket, R\llbracket M \rrbracket)$ , where

$$R\llbracket M \rrbracket : A \times B^{\perp} \rightarrow A^{\perp}$$

is the reverse differential of  $\llbracket M \rrbracket$ .

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Not really, we are just formulating it differently.
- Is this correspondence astonishing/magic? Can we find some "reason" clarifying it?  
Definitely yes at first sight... but then we can clearly understand its reason by looking at the categorical framework behind it.



# Lens Categories

The category  $\text{Lens}(\mathcal{L})$  of lenses over  $\mathcal{L}$  is defined as follows:

- objects: arrows in  $\mathcal{L}$ , which we think as fibre bundles and we write  $p : \binom{\alpha}{A}$
- arrows from  $p : \binom{\alpha}{A}$  to  $q : \binom{\beta}{B}$  are the data of both a  $f : A \rightarrow B$  in  $\mathcal{L}$  and a span  $\alpha \xleftarrow{F} f^*\beta \xrightarrow{\bar{f}} \beta$  in  $\mathcal{L}$ , taken from the following pullback diagram:

$$\begin{array}{ccccc}
 \alpha & \xleftarrow{F} & f^*\beta & \xrightarrow{\bar{f}} & \beta \\
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Let  $\mathcal{ELens}(\mathcal{L})$  be the full subcategory of  $\text{Lens}(\mathcal{L})$  of trivial bundles, i.e. first projections. Concretely:

- Objects are first projections  $\pi_1 : \binom{A \times X}{A}$
- An arrow from  $\pi_1 : \binom{A \times X}{A}$  to  $\pi_1 : \binom{B \times Y}{B}$  is given by an  $f : A \rightarrow B$  and a span  $A \times X \xleftarrow{F} A \times Y \xrightarrow{f \times 1} B \times Y$  such that  $F; \pi_1^{A,X} = \pi_1^{A,Y}$ .

# Differentiation through the lens of lenses

Let  $\mathcal{L}$  be a Cartesian (closed, if you want  $\lambda$ -calculus) differential category where from the differential  $Df$  of a function  $f$  (a primitive data in  $\mathcal{L}$ ) we can define the reverse differential  $Rf$  of  $f$ . (Think of  $\mathcal{L} := \mathcal{C}_!$  of the first part).

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We have a functor  $\mathcal{L} \rightarrow \mathcal{ELens}(\mathcal{L})$  defined by:

$$A \quad \mapsto \quad \pi_1 : \left( A \times A^\perp \right)_A$$

$$A \xrightarrow{f} B \quad \mapsto \quad \left( f, \quad A \times A^\perp \xleftarrow{\langle \pi_1, Rf \rangle} A \times B^\perp \xrightarrow{f \times 1} B \times B^\perp \right).$$

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where  $Rf : A \times B \rightarrow A$  is the reverse differential of  $f$  (a primitive data in  $\mathcal{L}$ ).

# Differentiation through the lens of lenses

Let  $\mathcal{L}$  be a reverse tangent category. This means that  $\mathcal{L}$  has a tangent functor  $T$  giving tangent bundles  $p_A : (T^*_A A)$  of objects  $A$  and giving tangent arrows  $Tf : T^*_A A \rightarrow T^*_B B$  for arrows  $f : A \rightarrow B$ , and we can “reverse”  $T$  in order to get cotangent bundles  $p_A^* : (T^* A)$  and arrows in the pullback diagram below:

$$\begin{array}{ccccc}
 T^* A & \xleftarrow{T^* f} & f^* T^* B & \xrightarrow{\bar{f}} & T^* B \\
 & \searrow p_A^* & \downarrow f^* p_B & \lrcorner & \downarrow p_B^* \\
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where  $f^* T^* B$  is the diff. geometry formulation of the reverse differential of  $f$ .

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We have a functor  $\mathcal{L} \rightarrow \text{Lens}(\mathcal{L})$  defined by:

$$A \mapsto p_A^* : (T^{*A}_A)$$

$$A \xrightarrow{f} B \mapsto (f, T^*A \xleftarrow{T^*f} f^*T^*B \xrightarrow{\bar{f}} T^*B).$$



# Expressing Dialectica as a functor

$$\Lambda_{\text{cat}} \rightarrow \mathcal{E}\text{Lens}(\mathbf{P}_{\text{cat}})$$

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$$\Lambda_{\text{cat}} \rightarrow \mathcal{ELens}(\mathbf{P}_{\text{cat}})$$

- An object  $A$  is sent to  $(z : W(A) \times \mathcal{M}[C(A)] \vdash_{\mathbf{P}} z^1 : W(A))$ ;
- An arrow  $(z : A \vdash_{\Lambda} M : B)$  in  $\Lambda_{\text{cat}}$  from  $A$  to  $B$  is sent to the arrow in  $\mathcal{ELens}(\mathbf{P}_{\text{cat}})$  from  $(z : W(A) \times \mathcal{M}[C(A)] \vdash_{\mathbf{P}} z^1 : W(A))$  to  $(z : W(B) \times \mathcal{M}[C(B)] \vdash_{\mathbf{P}} z^1 : W(B))$  given by the following diagram:

$$\begin{array}{ccccc}
 W(A) \times \mathcal{M}[C(A)] & \xleftarrow{\langle z^1, z^2 \bowtie M_{(z^1)} \rangle} & W(A) \times \mathcal{M}[C(B)] & \xrightarrow{\langle M^\bullet, z^2 \rangle} & W(B) \times \mathcal{M}[C(B)] \\
 & \searrow z^1 & \downarrow z^1 & & \downarrow z^1 \\
 & & W(A) & \xrightarrow{M^\bullet} & W(B)
 \end{array}$$

# Expressing Dialectica as a functor

$$\Lambda_{\text{cat}} \rightarrow \mathcal{ELens}(\mathbf{P}_{\text{cat}})$$

Moral:

The Dialectica transformation of  $\lambda$ -calculus encodes (reverse) Differentiation *because* it is a transformation into a category of Lenses, the latter being the abstract setting for Reverse Differentiation.

## Final comments

- I didn't talk about Dialectica categories. I could have said something (ask me if you are interested)
- Explore categorical framework to reverse a Cartesian closed differential category in order to define Cartesian closed reverse differential/tangent categories
- Reverse differential  $\lambda$ -calculus? There is an interesting paper from Ong and Mak.

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THANK YOU!