The λ-calculus, from intuitionistic to classical logic

Webpage of the course

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ESSLLI Summer School, Bochum (Germany) *28/07/2025 – 01/08/2025*

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Lecture 2:

The λ-calculus

Read the notes: they are full of details, proofs, explanations, exercises, bibliography!

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Previously...

- What do we intuitively mean when we say that a function is computable
- That this relates to topology
- That $R = (\mathcal{P}(\mathbb{N}), \mathit{Scott})$ is a good topological space for modeling this
- That this topology only depends from the partial order $\subseteq$

  - That actually, Scott-topology and Scott-continuity themselves are a property about “approximations” in posets
  - That Scott-continuous functions are given by their restriction on the finite elements of R.
  - That continuous function embed into their base space. This is done via the retraction λ, fun, i.e. they satisfy (β)
  - That all Scott-continuous functions on R have fixed points!
  - That λ produces RE sets and fun preserves them

Outline

1. Extracting a formal language from R
2. Denotational Semantics of $\Lambda^{\vdash}$ in R
3. Full β Operational Semantics of Λ
4. Basic pen-and-paper fun(ctional) programming together!
5. Summary, exercises, bibliography

Extracting a formal language from R

Let's take inspiration by the fact that the set of RE sets is closed wrt the following rules:

$$\frac{}{\lambda(\lambda \circ \mathrm{curry}(\cdots(\lambda \circ \mathrm{curry}(\mathrm{proj}_i^n))\cdots))}$$

$$\frac{a \mapsto f(a) \quad \textit{computable}}{\lambda(f)} \qquad\qquad \frac{a \qquad b}{@(a, b)}$$

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$$\frac{a \mapsto f(a) \quad \textit{computable}}{\lambda(f)} \quad \begin{array}{c} \textit{function over } R \\ \downarrow \\ \textit{its encoding in } R \end{array} \qquad \frac{a \qquad b}{@(a,b)} \quad \textit{“application” in } R$$

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We would like a functional programming language to be closed wrt the following rules:

representation of proj *in the language*

abstract function
$\downarrow$
its reification in the language

“application” in the language

Extracting a formal language from R

Fix a countable set of variables.

Notation: $\underline{\mathtt{x}}$ is a finite set of variables.

The set $\Lambda^{\vdash}$ of λ-terms-with-context-variables is defined as:

$$(\mathtt{x} \in \underline{\mathtt{x}})\,\frac{}{\underline{\mathtt{x}} \vdash \mathtt{x}}$$

$$(\mathtt{y} \notin \underline{\mathtt{x}})\,\frac{\underline{\mathtt{x}}, \mathtt{y} \vdash \mathtt{M}}{\underline{\mathtt{x}} \vdash \lambda \mathtt{y}.\,\mathtt{M}} \qquad\qquad \frac{\underline{\mathtt{x}} \vdash \mathtt{M} \qquad \underline{\mathtt{x}} \vdash \mathtt{N}}{\underline{\mathtt{x}} \vdash \mathtt{M}\,\mathtt{N}}$$

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Remark: if $\underline{\mathtt{x}} \vdash \mathtt{M}$ then $\underline{\mathtt{x}}$ contains at least the free variables $FV(\mathtt{M})$ of $\mathtt{M}$.

The set Λ of λ-terms is defined as:

$$\mathtt{M} ::= \mathtt{x} \mid \lambda \mathtt{x}.\mathtt{M} \mid \mathtt{M}\,\mathtt{N} \qquad\qquad \textit{(for } \mathtt{x} \textit{ a variable)}$$

What's the actual formal definition of λ-terms?

That requires more carefulness than you think!

The "issue" of α-equivalence

$$\lambda \mathtt{x}.\mathtt{M} = \lambda \mathtt{y}.(\mathtt{M}\{\mathtt{x} := \mathtt{y}\}) \quad \textit{whenever } \mathtt{y} \notin FV(\mathtt{M})$$

In pen-and-paper research:

- Words quotiented by α-equivalence
- Trees quotiented by α-equivalence

In computer oriented research:

- De-Brujin indices
- Nominal sets
- Abstract syntax
- ...

In the notes:

- Graphs with built-in α-equivalence

For the lectures:

- Informal treatment and hoping all goes well...

$R^{\underline{\mathtt{x}}}$:= families of elements of R indexed by $\underline{\mathtt{x}}$. I.e. $\underline{a} \in R^{\underline{\mathtt{x}}}$ iff $\underline{a} = \{a_{\mathtt{x}} \mid \mathtt{x} \in \underline{\mathtt{x}},\ a_{\mathtt{x}} \in R\}$.

Definition (Semantics of $\Lambda^{\vdash}$ in R)

We define, by induction on $\underline{\mathtt{x}} \vdash \mathtt{M}$, its *$R$-interpretation* $[\![\underline{\mathtt{x}} \vdash \mathtt{M}]\!] : R^{\underline{\mathtt{x}}} \to R$ as:

$$
\begin{array}{lll}
[\![\underline{\mathtt{x}} \vdash \mathtt{x}]\!] := \mathrm{proj}^{\underline{\mathtt{x}}}_{\mathtt{x}}, \quad [\![\underline{\mathtt{x}} := \underline{a} \vdash \mathtt{x}]\!] & := & a_{\mathtt{x}} \\
[\![\underline{\mathtt{x}} \vdash \lambda \mathtt{y}.\mathtt{P}]\!], \quad [\![\underline{\mathtt{x}} := \underline{a} \vdash \lambda \mathtt{y}.\mathtt{P}]\!] & := & \lambda\,([\![\underline{\mathtt{x}} := \underline{a}, \mathtt{y} := (\_) \vdash \mathtt{P}]\!]) \quad \textit{if } \mathtt{y} \notin \underline{\mathtt{x}} \\
[\![\underline{\mathtt{x}} \vdash \mathtt{P}\,\mathtt{Q}]\!], \quad [\![\underline{\mathtt{x}} := \underline{a} \vdash \mathtt{P}\,\mathtt{Q}]\!] & := & @([\![\underline{\mathtt{x}} := \underline{a} \vdash \mathtt{P}]\!],\ [\![\underline{\mathtt{x}} := \underline{a} \vdash \mathtt{Q}]\!])
\end{array}
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Denotational Semantics of $\Lambda^{\vdash}$ in R

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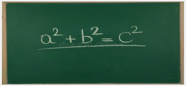
\end{array}
$$

Remember that in R we have:

$$\mathrm{fun} \circ \lambda = \mathrm{id}_{R \Rightarrow R} \tag{β}$$

Fix $\underline{\mathtt{y}}, \mathtt{x} \vdash \mathtt{M}$ with $\mathtt{x} \notin \underline{\mathtt{y}}$, and $\underline{\mathtt{y}} \vdash \mathtt{N}$. Then $\left[\!\!\left[\underline{\mathtt{y}} \vdash (\lambda \mathtt{x}.\mathtt{M})\,\mathtt{N}\right]\!\!\right]$ is the function:

$$\underline{a} \ \in \ R^{\underline{\mathtt{y}}} \ \mapsto \ \left[\!\!\left[\underline{\mathtt{y}} := \underline{a} \vdash (\lambda \mathtt{x}.\mathtt{M})\,\mathtt{N}\right]\!\!\right] = \left[\!\!\left[\underline{\mathtt{y}} := \underline{a}, \mathtt{x} := \left[\!\!\left[\underline{\mathtt{y}} := \underline{a} \vdash \mathtt{N}\right]\!\!\right] \vdash \mathtt{M}\right]\!\!\right] \in R$$



Denotational Semantics of $\Lambda^{\vdash}$ in R

$[\![\underline{\mathtt{y}} \vdash (\lambda \mathtt{x}.\mathtt{M})\, \mathtt{N}]\!]$ passes $[\![\underline{\mathtt{y}} := \underline{a} \vdash \mathtt{N}]\!]$ to $[\![\underline{\mathtt{y}} := \underline{a}, \mathtt{x} \vdash \mathtt{M}]\!]$ via $\mathtt{x}$.

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Such substitution *is itself the interpretation of a λ-term-with-context-variables*!

Definition (Substitution)

Given $\mathtt{M}, \mathtt{N} \in \Lambda$, $\mathtt{x} \in \mathrm{Var}$, define the term $\mathtt{M}\{\mathtt{x} := \mathtt{N}\} \in \Lambda$ by induction:

$$\mathtt{x}\{\mathtt{x} := \mathtt{N}\} := \mathtt{N} \qquad \mathtt{z}\{\mathtt{x} := \mathtt{N}\} := \mathtt{z} \qquad (\mathtt{P}\,\mathtt{Q})\{\mathtt{x} := \mathtt{N}\} := (\mathtt{P}\{\mathtt{x} := \mathtt{N}\})(\mathtt{Q}\{\mathtt{x} := \mathtt{N}\})$$

$$(\lambda\mathtt{z}.\mathtt{P})\{\mathtt{x} := \mathtt{N}\} := \lambda\mathtt{z}.(\mathtt{P}\{\mathtt{x} := \mathtt{N}\}), \quad \textit{if } \mathtt{z} \notin FV(\mathtt{N}) \cup \{\mathtt{x}\}.$$

Denotational Semantics of $\Lambda^{\vdash}$ in R

$\left[\!\left[\underline{\mathtt{y}} \vdash (\lambda \mathtt{x}.\mathtt{M})\,\mathtt{N} \right]\!\right]$ passes $\left[\!\left[\underline{\mathtt{y}} := \underline{a} \vdash \mathtt{N} \right]\!\right]$ to $\left[\!\left[\underline{\mathtt{y}} := \underline{a}, \mathtt{x} \vdash \mathtt{M} \right]\!\right]$ via $\mathtt{x}$.

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Example

$$\begin{array}{lcll} (\lambda \mathtt{y}.\,\mathtt{x})\{\mathtt{x} := \mathtt{y}\} & = & (\lambda \mathtt{v}.\,\mathtt{x})\{\mathtt{x} := \mathtt{y}\} & \textit{for } \mathtt{v} \neq \mathtt{x}, \ \textit{by } \alpha \\ & = & \lambda \mathtt{v}.\,\mathtt{y} & \textit{for } \mathtt{v} \neq \mathtt{x}, \mathtt{y}, \ \textit{by def of substitution} \\ & = & \lambda \mathtt{x}.\,\mathtt{y} & \textit{for } \mathtt{x} \neq \mathtt{y}, \ \textit{by } \alpha \end{array}$$

$\llbracket \underline{\mathtt{y}} \vdash (\lambda \mathtt{x}.\mathtt{M})\,\mathtt{N} \rrbracket$ passes $\llbracket \underline{\mathtt{y}} := \underline{a} \vdash \mathtt{N} \rrbracket$ to $\llbracket \underline{\mathtt{y}} := \underline{a}, \mathtt{x} \vdash \mathtt{M} \rrbracket$ via $\mathtt{x}$.

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Theorem ($(R, \lambda, @)$ is a model of λ-calculus)

For all $\underline{\mathtt{y}}, \mathtt{x} \vdash \mathtt{M}$ *with* $\mathtt{x} \notin \underline{\mathtt{y}}$, *and* $\underline{\mathtt{y}} \vdash \mathtt{N}$, *we have:*

$$\llbracket \underline{\mathtt{y}} \vdash (\lambda \mathtt{x}.\,\mathtt{M})\,\mathtt{N} \rrbracket = \llbracket \underline{\mathtt{y}} \vdash \mathtt{M}\{\mathtt{x} := \mathtt{N}\} \rrbracket : R^{\underline{\mathtt{y}}} \to R. \qquad (\llbracket \beta \rrbracket)$$

The proof is by induction on $\underline{\mathtt{y}}, \mathtt{x} \vdash \mathtt{M}$ (but careful with α-equivalence!).

Full β Operational Semantics of Λ

$$\left[\!\!\left[\underline{\mathtt{y}} \vdash (\lambda \mathtt{x}.\,\mathtt{M})\,\mathtt{N}\right]\!\!\right] = \left[\!\!\left[\underline{\mathtt{y}} \vdash \mathtt{M}\{\mathtt{x} := \mathtt{N}\}\right]\!\!\right] : R^{\underline{\mathtt{y}}} \to R. \qquad ([\![\beta]\!])$$

Full β Operational Semantics of Λ

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Full β Operational Semantics of Λ

$$(\lambda \mathtt{x}.\, \mathtt{M})\, \mathtt{N} \quad = \quad \mathtt{M}\{\mathtt{x} := \mathtt{N}\}$$

Full β Operational Semantics of Λ

$$(\lambda \mathtt{x}.\, \mathtt{M})\, \mathtt{N} \quad \rightarrow_\beta \quad \mathtt{M}\{\mathtt{x} := \mathtt{N}\}$$

Full β Operational Semantics of Λ

Definition

The *full β-reduction* $\to_\beta \subseteq \Lambda \times \Lambda$ is defined by the following inductive rules:

$$\frac{}{(\lambda \mathtt{x}.\,\mathtt{M})\,\mathtt{N} \to_\beta \mathtt{M}\{\mathtt{x} := \mathtt{N}\}} \qquad \frac{\mathtt{M} \to_\beta \mathtt{N}}{\lambda \mathtt{x}.\mathtt{M} \to_\beta \lambda \mathtt{x}.\mathtt{N}} \qquad \frac{\mathtt{M} \to_\beta \mathtt{M}'}{\mathtt{M}\,\mathtt{N} \to_\beta \mathtt{M}'\,\mathtt{N}} \qquad \frac{\mathtt{N} \to_\beta \mathtt{N}'}{\mathtt{M}\,\mathtt{N} \to_\beta \mathtt{M}\,\mathtt{N}'}$$

Let $\twoheadrightarrow_\beta$ be the reflexive and transitive closure of $\to_\beta$.
Let $=_\beta$ be the symmetric closure of $\twoheadrightarrow_\beta$.
A term $\mathtt{M}$ is *normal* if there is no possible $\to_\beta$-reduction from $\mathtt{M}$.
A normal term $\mathtt{N}$ is a *normal form of* $\mathtt{M}$ if $\mathtt{M} \twoheadrightarrow_\beta \mathtt{N}$.
A *redex* is any term of shape $(\lambda \mathtt{x}.\,\mathtt{M})\,\mathtt{N}$.

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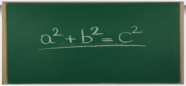
Let $\mathtt{I} := \lambda \mathtt{x}.\,x$, let $\delta := \lambda \mathtt{x}.\,\mathtt{x}\,\mathtt{x}$ and $\Omega := \delta\,\delta$. Show that:



$$\Omega \rightarrow_\beta \Omega \quad \text{and} \quad (\lambda \mathtt{x}.\mathtt{I})\,\Omega =_\beta \mathtt{I}\mathtt{I}$$

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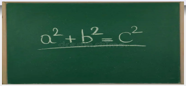

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For $\mathtt{f}$ a variable, does $\Delta_{\mathtt{f}}\Delta_{\mathtt{f}}$ have a normal form? Show that: $\Delta_{\mathtt{I}}\Delta_{\mathtt{I}} \twoheadrightarrow_\beta \Delta_{\mathtt{I}}\Delta_{\mathtt{I}}$

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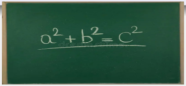

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For $\mathtt{f}$ a variable, does $\Delta_{\mathtt{f}}\Delta_{\mathtt{f}}$ have a normal form? Show that: $\Delta_{\mathtt{I}}\Delta_{\mathtt{I}} \twoheadrightarrow_\beta \Delta_{\mathtt{I}}\Delta_{\mathtt{I}}$
Let $\Psi := \lambda \mathtt{x}.\lambda \mathtt{y}.\, \mathtt{y}(\mathtt{x}\,\mathtt{x})$ and $\chi := \Psi\,\Psi$. Show that:

$$\chi\,\mathtt{M} \twoheadrightarrow_\beta \mathtt{M}\,\chi \quad \text{and} \quad \chi \twoheadrightarrow_\beta \lambda \mathtt{x}.\, \mathtt{x}\,\chi$$

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Definition

A *fixed point combinator* is a term $\mathtt{F}$ such that $\mathtt{F}\,\mathtt{M} =_\beta \mathtt{M}\,(\mathtt{F}\,\mathtt{M})$ for all term $\mathtt{M}$.
There are lots of them!

Full β Operational Semantics of Λ

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A normal term $\mathtt{N}$ is a *normal form of* $\mathtt{M}$ if $\mathtt{M} \twoheadrightarrow_\beta \mathtt{N}$.

Definition

A *fixed point combinator* is a term $\mathtt{F}$ such that $\mathtt{F}\,\mathtt{M} =_\beta \mathtt{M}\,(\mathtt{F}\,\mathtt{M})$ for all term $\mathtt{M}$.
There are lots of them!

Example

Remember $\Delta_\mathtt{f} = \lambda \mathtt{x}.\mathtt{f}(\mathtt{x}\mathtt{x})$. Show that $\mathtt{Y} := \lambda \mathtt{f}.\,\Delta_\mathtt{f}\Delta_\mathtt{f}$ is a fixed-point combinator

Full β Operational Semantics of Λ

Definition

The *full β-reduction* $\rightarrow_\beta \subseteq \Lambda \times \Lambda$ is defined by the following inductive rules:

$$\frac{}{(\lambda \mathtt{x}.\,\mathtt{M})\,\mathtt{N} \rightarrow_\beta \mathtt{M}\{\mathtt{x} := \mathtt{N}\}} \qquad \frac{\mathtt{M} \rightarrow_\beta \mathtt{N}}{\lambda \mathtt{x}.\mathtt{M} \rightarrow_\beta \lambda \mathtt{x}.\mathtt{N}} \qquad \frac{\mathtt{M} \rightarrow_\beta \mathtt{M}'}{\mathtt{M}\,\mathtt{N} \rightarrow_\beta \mathtt{M}'\,\mathtt{N}} \qquad \frac{\mathtt{N} \rightarrow_\beta \mathtt{N}'}{\mathtt{M}\,\mathtt{N} \rightarrow_\beta \mathtt{M}\,\mathtt{N}'}$$

Let $\twoheadrightarrow_\beta$ be the reflexive and transitive closure of $\rightarrow_\beta$.
Let $=_\beta$ be the symmetric closure of $\twoheadrightarrow_\beta$.
A term $\mathtt{M}$ is *normal* if there is no possible $\rightarrow_\beta$-reduction from $\mathtt{M}$.
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There are lots of them!

Theorem (Church, Rosser)

The abstract rewriting system $(\Lambda, \rightarrow_\beta)$ is confluent.
$\Longrightarrow$ *If a term has a normal form, then it is unique.*

Full β Operational Semantics of Λ as a model of computation

Can we see $(\Lambda, \to_\beta)$ as a model of computation? Need to choose:

some data-type A
(that terms computes on)

some encoding $\ulcorner A \urcorner$ *of* A

for each term `M`
a function (if any) $f_{\mathtt{M}}$ *on* $\ulcorner A \urcorner$

Full β Operational Semantics of Λ as a model of computation

Can we see $(\Lambda, \to_\beta)$ as a model of computation? Yes!

data-type A *(that terms computes on)*	$\longrightarrow$	$\mathbb{N}$
encoding $\ulcorner a \urcorner$ *of* $a \in A$	$\longrightarrow$	*Church encoding* $\ulcorner n \urcorner$ *of* $n \in \mathbb{N}$
for each term $\mathtt{M}$ *a function (if any)* $f_{\mathtt{M}}$ *on* $\ulcorner A \urcorner$	$\longrightarrow$	$f_{\mathtt{M}}(\ulcorner n \urcorner) := \mathrm{nf}_\beta(\mathtt{M} \ulcorner n \urcorner)$ *if it is a Church numeral*

A function $f : A \to A$ is implemented by $\mathtt{M}$ when

$$f_{\mathtt{M}} \ulcorner a \urcorner = \ulcorner f(a) \urcorner$$

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A function $f : \mathbb{N} \to \mathbb{N}$ is implemented by $\mathtt{M}$ when

$$M\ulcorner n \urcorner =_\beta \ulcorner f(n) \urcorner$$

How expressive this is?

Full β Operational Semantics of Λ as a model of computation

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A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is implemented by $\mathtt{M}$ when

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How expressive this is?

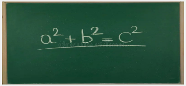
$(\Lambda, \rightarrow_\beta)$ under Church encoding is Turing-complete!

Under Church encoding, the *partial* functions $\mathbb{N} \rightarrow \mathbb{N}$ which are implementable in $(\Lambda, \rightarrow_\beta)$ are exactly the Turing-computable ones.

We did **not** mention how to handle *partiality* in all this discussion, it would require going deeper!

Basic pen-and-paper fun(ctional) programming together!

Church encoding of Booleans:



$$\textsc{true} := \lambda \mathtt{x}.\lambda \mathtt{y}.\, \mathtt{x} \qquad \textsc{false} := \lambda \mathtt{x}.\lambda \mathtt{y}.\, \mathtt{y}$$

✎ Implement the following functions:

$$\mathit{not} : \mathrm{Bool} \to \mathrm{Bool} \quad \mathit{and} : \mathrm{Bool} \times \mathrm{Bool} \to \mathrm{Bool}$$

Church encoding of natural numbers – *using notation:* $\mathtt{M}^{(n)}\,\mathtt{x} := \mathtt{M}\,(\,\mathtt{M}\,(\ldots\,(\,\mathtt{M}\,\mathtt{x}\,)))$

$$\ulcorner n \urcorner := \lambda \mathtt{f}.\lambda \mathtt{x}.\, \mathtt{f}^{(n)}\,\mathtt{x}$$

✎ Implement the following functions:

$$\mathit{successor} : \mathrm{nat} \to \mathrm{nat} \quad \mathit{addition} : \mathrm{nat} \times \mathrm{nat} \to \mathrm{nat} \quad \mathit{is\text{-}zero\text{-}test} : \mathrm{nat} \to \mathrm{Bool}$$

Suppose to have a term $\textsc{pred} \in \Lambda$ that implements the predecessor: $\mathrm{nat} \to \mathrm{nat}$

$$\textsc{pred}\,\ulcorner 0 \urcorner \;=_\beta\; \ulcorner 0 \urcorner\,, \qquad \textsc{pred}\,\ulcorner n+1 \urcorner \;=_\beta\; \ulcorner n \urcorner$$

✎ Implement the following functions:

$$\mathit{subtraction} : \mathrm{nat} \times \mathrm{nat} \to \mathrm{nat} \quad \mathit{less\text{-}than\text{-}or\text{-}equal\text{-}to\text{-}test} : \mathrm{nat} \times \mathrm{nat} \to \mathrm{Bool}$$

✎ Using a MULT $\in \Lambda$ implementing multiplication, implement the factorial:

$$(\_)! : \mathbb{N} \to \mathbb{N}, \qquad 0! := 1, \quad (n+1)! := n!\,(n+1)$$

- Inspired by the graph model, we have defined a very basic functional programming language
- We have given it a denotational semantics in the graph model
- We have given it a operational semantics
- Even if minimal, the OPS makes it a Turing-complete programming language
- We have programmed some basic functions on simple datatypes

- Do the proofs of the statements in the slides
- Look at my **notes** on the webpage of the course, there are plenty of **exercises**
- The exercises have **solutions** (but try to do them by yourself before looking at them!)

- Chapter 2 of:
  Amadio R., Curien P-L, Domains and lambda-calculi, 1996,
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  Henk P. Barendregt, The lambda-calculus, its syntax and semantics, 1984,
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  https://www.irif.fr/~gmanzone/ManzonettoPhdThesis.pdf
- To go (much) further:
  A Lambda Calculus Satellite, Henk Barendregt and Giulio Manzonetto, 2022,
  https://www.collegepublications.co.uk/logic/mlf/?00035