Lambda-Calculus, Taylor Expansion and (Tropical) Quantitative Semantics: an overview

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Outline

1. (Quantitative) Semantics of Programs
2. Quantitative Semantics: Linearity
3. Tropical Polynomials and Effectful Computation
4. Tropically Weighted Relational Semantics of PPCF
5. Overview of our recent results (CSL24 – Barbarossa, Pistone)
6. Future Work
7. Bonus: Finitness, Taylor, Generalised Metric Spaces

Outline

Program semantics $\Rightarrow$ properties of programs

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Qualitative:

- termination
- correctness
- program equivalence
- ...

Program semantics $\Rightarrow$ properties of programs

Qualitative:

- termination
- correctness
- program equivalence
- ...

Quantitative:

- probability of convergence
- probability of correctness
- errors, program similarity
- ...

Sensitivity Analysis

program metrics, quantitative equational theories

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Resource Analysis

linear logic, program differentiation, intersection types

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type $\mapsto$ metric space

program $\mapsto$ Lipschitz function

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$$M \overset{\epsilon}{\simeq} N \Rightarrow FM \overset{L\cdot\epsilon}{\simeq} FN$$

Metric Preservation

Sensitivity Analysis

program metrics, quantitative equational theories

Resource Analysis

linear logic, program differentiation, intersection types

type	$\mapsto$	metric space
program	$\mapsto$	Lipschitz function

type	$\mapsto$	vector space /module
program	$\mapsto$	smooth/analytic function

$$M \overset{\epsilon}{\simeq} N \Rightarrow FM \overset{L\cdot\epsilon}{\simeq} FN$$

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$$FM = \sum_{n=0}^{\infty} \frac{1}{!n} (\mathsf{D}^{(n)} F \cdot M^n) 0$$

Taylor Formula

λ-calculus (Purely functional, Turing–complete)

$$M ::= x \mid \lambda x.N \mid MN \qquad\qquad (\lambda x.M)N \to M\{x := N\}$$

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Simple types (Lose Turing-completeness, can be recovered easily)

$$\frac{}{x : A \vdash x : A} \qquad \frac{x : A \vdash M : B}{\vdash \lambda x.M : A \to B} \qquad \frac{\vdash M : A \to B \quad \vdash N : A}{\vdash MN : B}$$

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Categorical semantics in a Cartesian Closed Category $\mathcal{C}$

type A	$\mapsto$	object $[\![A]\!]$ in $\mathcal{C}$
program $x : A \vdash M : B$	$\mapsto$	morphism $[\![x : A \vdash M : B]\!] : [\![A]\!] \to [\![B]\!]$ in $\mathcal{C}$

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Resource Analysis

	Sensitivity Analysis	Resource Analysis
F uses input once		

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F uses input once	f non-expansive	
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F uses input once	f non-expansive	f linear $(\mathsf{D}^{(2)}F = 0)$
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F uses input k times	f k-Lipschitz	f polynomial $(\mathsf{D}^{(k+1)}F = 0)$
	$M \overset{\epsilon}{\simeq} N \Rightarrow FM \overset{k\cdot\epsilon}{\simeq} FN$	$FM = \sum_{n=0}^{k} \frac{1}{!n}(\mathsf{D}^{(n)}F \cdot M^n)0$

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	graded types $!_k A \multimap B$	intersection types $[A_1, \ldots, A_k] \multimap B$

Graded typed calculus

$$\frac{}{x :!_1 A \vdash x : A}$$

$$\frac{x :!_s C \vdash M :!_n A \multimap B \quad x :!_m C \vdash N : A}{x :!_{s+nm} C \vdash MN : B}$$

$$\frac{x :!_n A \vdash M : B}{\vdash \lambda x.M :!_n A \multimap B}$$

Resource calculus

$$\frac{}{x : [A] \vdash x : A}$$

$$\frac{x : [A_1, \cdots A_n] \vdash M : B}{\vdash \lambda x.M : [A_1, \cdots A_n] \multimap B}$$

$$\frac{\vdash M : [A_1, \cdots A_n] \multimap B \quad (\vdash N_i : A_i)_{i=1}^{n}}{\vdash M[N_1, \cdots N_n] : B}$$

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$\xrightarrow{\text{sensitivity analysis}}$

f k-Lipschitz

e.g. $f(x) = kx$

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resource analysis

f k-degree polynomial

e.g. $f(x) = x^k$

Quantitative Semantics: a Logarithmic Gap

$$\begin{tikzcd}[column sep=huge]
 & f\ k\text{-Lipschitz} \\
f \text{ uses input } k \text{ times} \arrow[ur, "\text{sensitivity analysis}"] \arrow[dr, "\text{resource analysis}"'] & \\
 & f\ k\text{-degree polynomial}
\end{tikzcd}$$

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Tropical Mathematics

a k-degree polynomial is a k-Lipschitz function!

Tropical semiring: $\mathbb{L} = ([0, +\infty], \min, +)$

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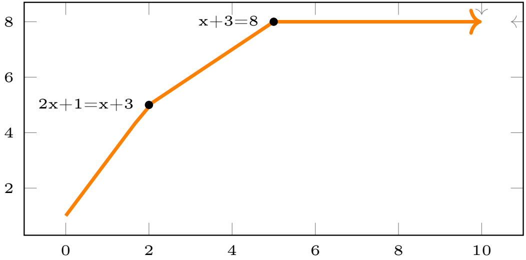

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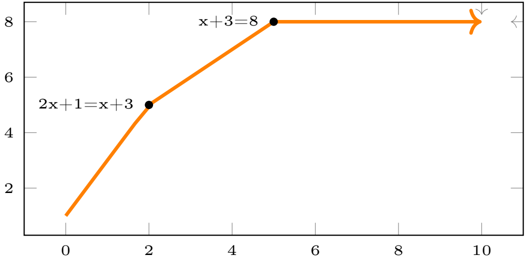
x_0 is a **tropical root** of $p(x)$ iff $p(x_0)$ is not differentiable



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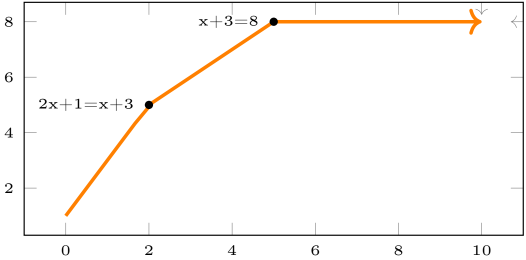
x_0 is a **tropical root** of $p(x)$ iff $p(x_0)$ is not differentiable

equivalently, iff the minimum $p(x_0)$ is attained twice



Intractable problems (e.g. root finding, optimization)

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$$\Big\downarrow \quad \begin{array}{l}\text{tropicalization:}\\ \quad + \mapsto \min \\ \quad \times \mapsto + \end{array}$$

Combinatorial (and sometimes tractable!) ones

- tropical roots are found in linear time
- likelihood estimation in statistical models
- machine learning (ReLU networks)
- optimal routing paths

f uses input k times

sensitivity analysis

f k-Lipschitz

e.g. $f(x) = kx$

resource analysis

f k-degree polynomial

e.g. $f(x) = x^k$

Tropical Mathematics

a k-degree polynomial is a k-Lipschitz function!

f uses input k times $\xrightarrow{\text{sensitivity analysis}}$ f k-Lipschitz

e.g. $f(x) = kx$

f uses input k times $\xrightarrow{\text{resource analysis}}$ f k-degree polynomial

tropical

e.g. $\boxed{f(x) = kx}$

Tropical Mathematics

a k-degree polynomial is a k-Lipschitz function!

$$FM \quad = \quad \sum_{n=0}^{\infty} \frac{1}{!n} (\mathsf{D}^{(n)} F \cdot M^n) 0$$

Tropical Taylor = Lipschitz Approximation

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$$FM \quad = \quad \inf \left\{ (\mathsf{D}^{(n)} F \cdot M^n) 0 \;\middle|\; n \in \mathbb{N} \right\}$$

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n-Lipschitz function

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F is the limit of more and more sensitive approximations

Outline

$$M ::= \mathrm{True} \mid \mathrm{False} \mid M \oplus_p M \quad (p \in [0,1] \cap \mathbb{Q})$$

$$M \oplus_p N \twoheadrightarrow_p M$$
$$M \oplus_p N \twoheadrightarrow_{1-p} N$$

$$(\text{True} \oplus_p \text{False}) \oplus_p \Big((\text{True} \oplus_p \text{False}) \oplus_p (\text{False} \oplus_p \text{True})\Big)$$

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$$q := 1 - p$$

$$\begin{aligned} P_{ll}(p,q) &= p^2 \\ P_{rll}(p,q) &= p^2 q \\ P_{rrr}(p,q) &= q^3 \end{aligned}$$

$$(\mathrm{True} \oplus_p \mathrm{False}) \oplus_p \Big((\mathrm{True} \oplus_p \mathrm{False}) \oplus_p (\mathrm{False} \oplus_p \mathrm{True})\Big)$$

$$\boxed{P_{\mathrm{True}}(p, q) = p^2 + p^2 q + q^3}$$

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Hidden Markov Model

$$P_{ll}(p, q) = p^2$$
$$P_{rll}(p, q) = p^2 q$$
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Maximum Likelihood problem:
supposing $M \twoheadrightarrow \text{True}$,
what is the most likely path?

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$\rightarrow$ find $\omega_0 \in \{ll, rll, rrr\}$ maximizing $P(M \twoheadrightarrow_{\omega_0} \mathrm{True} \mid M \twoheadrightarrow \mathrm{True})$:

$$P_{\omega_0}(p, q) = \max_\omega P_\omega(p, q)$$

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Maximum Likelihood problem:
supposing $M \twoheadrightarrow \text{True}$,
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$$-\log\, P_{\omega_0}(p,q) = \min_\omega\{-\log\, P_\omega(p,q)\}$$

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$x := -\log\ p, \quad y := -\log\ q$

Hidden Markov Model

$$P_{ll}(p,q) = p^2$$
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Maximum Likelihood problem:
supposing $M \twoheadrightarrow$ True,
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Hidden Markov Model

$$\mathtt{t}P_{ll}(x,y) = 2x$$
$$P_{rll}(p,q) = p^2 q$$
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$$\begin{aligned} \mathtt{t}P_{ll}(x, y) &= 2x \\ \mathtt{t}P_{rll}(x, y) &= 2x + y \\ \mathtt{t}P_{rrr}(x, y) &= 3y \end{aligned}$$

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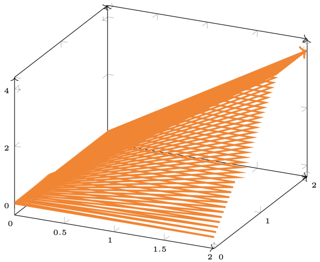

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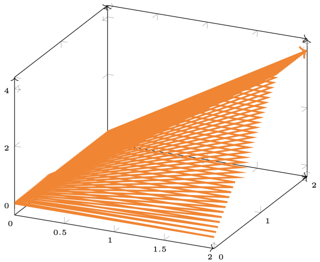
<u>tropical roots</u> $\mapsto$ line $y = \frac{2}{3}x$



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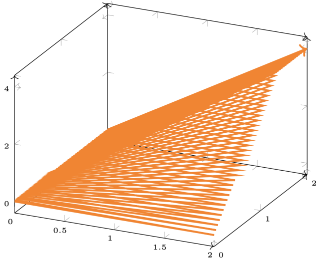
- rrr most likely as soon $y \leq \frac{2}{3}x$
- ll most likely otherwise



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<u>tropical roots</u> $\mapsto$ line $y = \frac{2}{3}x$

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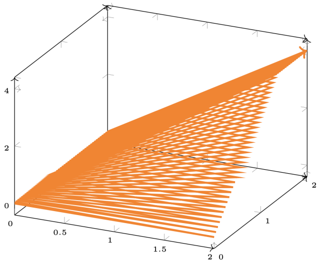
- ll most likely otherwise



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$$M := \mathbf{fix}.(\lambda x.\mathrm{True} \oplus_p x) \to (\lambda x.\mathrm{True} \oplus_p x)M \to \mathrm{True} \oplus_p M$$

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Theorem. For any term M : Nat of PPCF and $n \in \mathbb{N}$, $[\![M]\!] \in \mathbb{L}^{\mathbb{N}}$ and

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Theorem. [$\mathbb{L}$Mod $\simeq$ $\mathbb{L}$CCat is a model of ST$\partial\lambda$C] The equivalent categories of $\mathbb{L}$-modules and complete generalized metric spaces form a model of ST$\partial\lambda$C which extends the $\mathbb{L}$-weighted relational model.

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Thank you!

Outline

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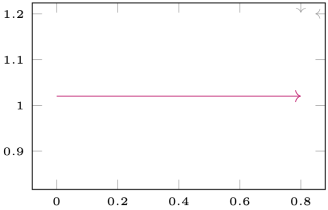

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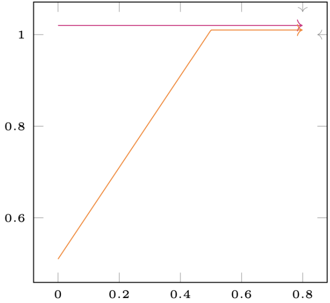

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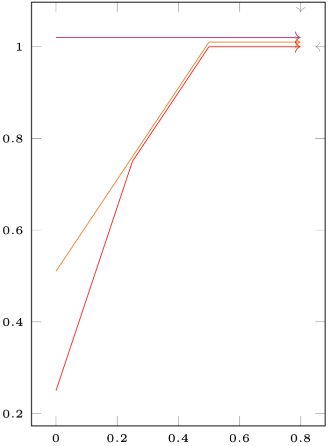

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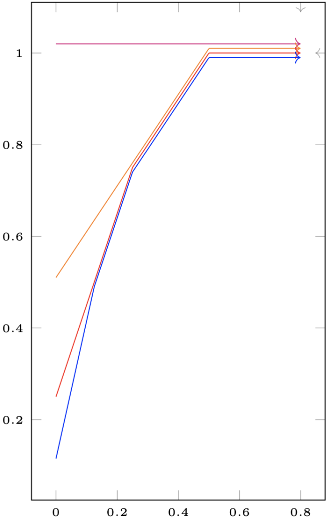

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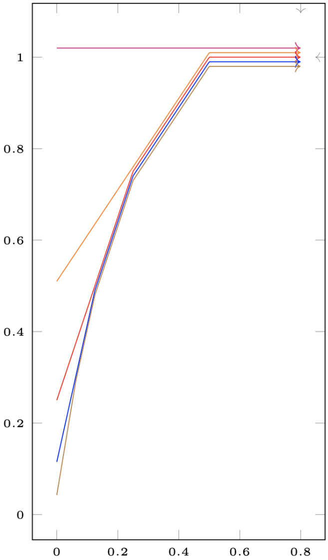

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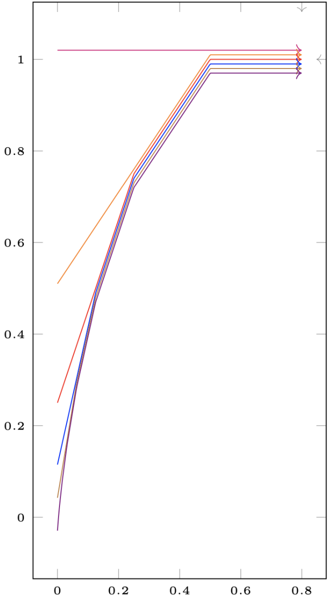

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φ is "locally" a polynomial:

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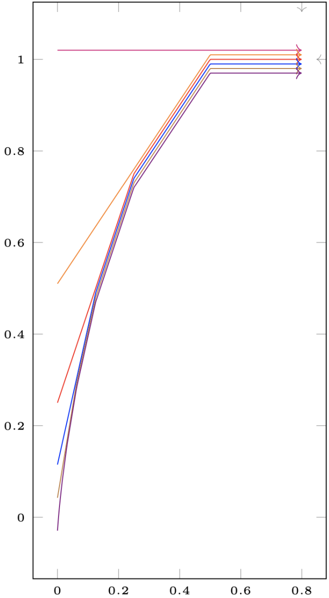

Theorem. Let $f : [0, +\infty]^k \to [0, +\infty]$ be a tropical power series given by

$$f(x) = \inf_{i \in I}\{n_i x + c_i\}.$$

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$\rightarrow$ if we can compute the polynomial $[\![M]\!]_n|_{[\epsilon,+\infty]}$ for ϵ small enough, then we can compute maximum likelihood values for M.

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- otherwise, f is locally Lipschitz:

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$$\begin{aligned} \mathcal{T}(x) &= \{x\} \\ \mathcal{T}(\lambda x.M) &= \{\lambda x.t \mid t \in \mathcal{T}(M)\} \\ \mathcal{T}(MN) &= \{t\langle u_1, \dots, u_n\rangle \mid n \in \mathbb{N}, t \in \mathcal{T}(M), u_1, \dots, u_n \in \mathcal{T}(N)\} \end{aligned}$$

Theorem. For any simply typed λ-term M,

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Recall that, for $M : A \to B$, $[\![M]\!]$ is only locally Lipschitz: for any $x \in [\![A]\!]$, there is some Lipschitz constant L_x that holds "around" x.
Can we approximate L_x?

Corollary. Let $M : A \to B$ and $N : A$. For all $t \in \mathcal{T}(M)$ and $\delta > 0$, unless $[\![t]\!]([\![N]\!]) \neq \infty$, the map $[\![M]\!](x)$ is $\frac{[\![t]\!]([\![N]\!]+3\delta)}{\delta}$-Lipschitz over the open ball $B_\delta([\![N]\!])$.

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Theorem. $\mathbb{L}\mathsf{Mod}_! \simeq \mathbb{L}\mathsf{CCat}_!$ extends $\mathbb{L}\mathsf{Rel}_!$ as a model of the ST$\partial\lambda$C