The λ-calculus, from minimal to classical logic

Webpage of the course

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ESSLLI Summer School, Bochum (Germany) *28/07/2025 – 01/08/2025*

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Lecture 5:

Krivine's approach to classical logic

Read the notes: they are full of details, proofs, explanations, exercises, bibliography!

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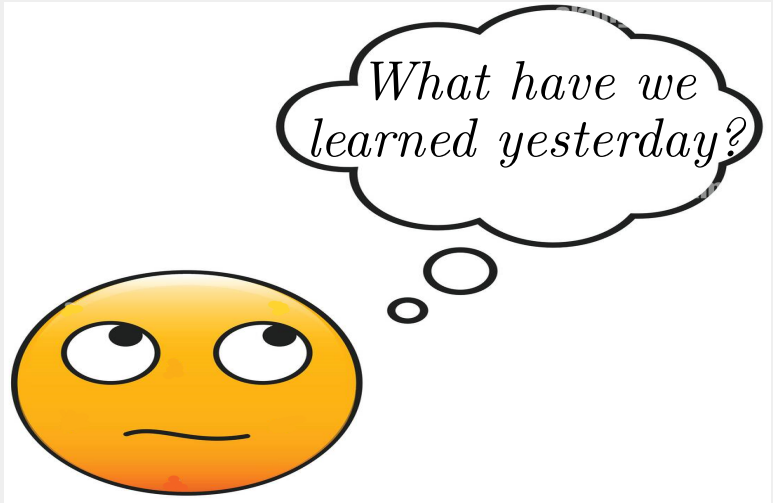
Previously...

- You have seen minimal logic
- You have seen that it corresponds to the simply-typed λ-calculus
- In the sense that formulas = types *and* cut-elimination = β-reduction
- Computational understanding of logic: proof $\rightarrow$ program



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a proof $\mathtt{x} : A \vdash \mathtt{M} : B$ *is* a (typed) program that computes the function

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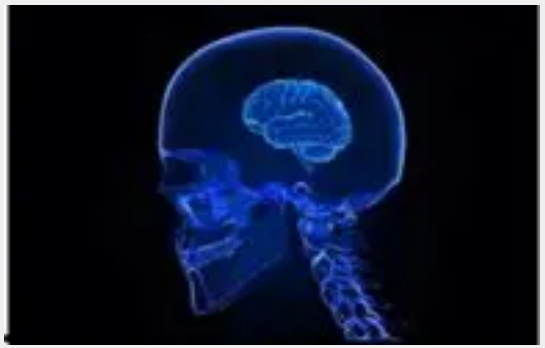
Yesterday: *minimal* logic



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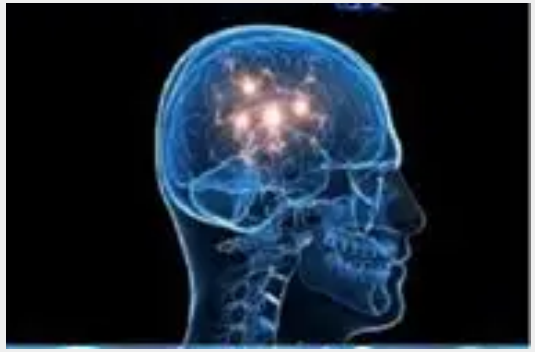

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Taking the above *literally* fails ("Joyal's lemma" in category theory, "Lafont's pairs" in sequent calculus,...) ... which we are not going to see

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What about *classical* logic?

Classical logic still has a computational content indeed!

type	$\rightsquigarrow$	realiser
purely functional	$\rightsquigarrow$	impure functional

Example

Classical realisability, $\neg\neg$+Dialectica, $\lambda\mu$-calculus, $\overline{\lambda}\mu\widetilde{\mu}$-calculus,...

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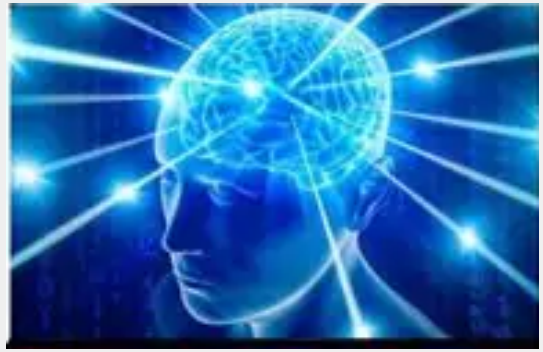

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Outline

2nd order classical logic

Formulas:

$$A \quad ::= \quad X \;\mid\; A \to A$$

Proofs:

$$\frac{}{\underline{A},\quad B \vdash \quad B}$$

$$\frac{\underline{A} \vdash \quad B \to C \qquad \underline{A} \vdash \quad B}{\underline{A} \vdash \quad C}$$

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$$\frac{\underline{\mathtt{x}} : \underline{A}, \mathtt{y} : B \vdash \mathtt{M} : C}{\underline{\mathtt{x}} : \underline{A} \vdash \lambda \mathtt{y}.\,\mathtt{M} : B \to C}$$

Proof-like terms:

$$\mathtt{M} \quad ::= \quad \mathtt{x} \;\mid\; \mathtt{M}\,\mathtt{N} \;\mid\; \lambda \mathtt{x}.\mathtt{M}$$

2nd order classical logic

Arithmetic expressions:

$$e \quad ::= \quad n \mid a \mid f(e, \ldots, e) \qquad (for\ n \in \mathbb{N}, a \in \mathcal{V}_1, f \in \mathcal{S}_k)$$

Formulas:

$$A \quad ::= \quad X(e, \ldots, e) \mid A \to A \mid \forall c.A \mid \forall^k X.A \qquad (for\ X \in \mathcal{V}_2^k)$$

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$$\frac{\underline{A} \vdash \quad \forall c.B}{\underline{A} \vdash \quad B\{c := e\}}$$

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Operational semantics of λ-calculus + `callcc`

OPS via:	*programs run...*	*E.g.*	*Logic*
β-reduction	by themselves	$\mathtt{M} \twoheadrightarrow_\beta \mathtt{N}$	intuitionistic
Abstract Machine	by interaction with execution stacks	$\mathtt{M} \star \pi \rightarrow \mathtt{N} \star \rho$	classical

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$$\textit{Proof-Like terms} \quad \mathtt{P} ::= \mathtt{x} \mid \lambda\mathtt{x}.\mathtt{P} \mid \mathtt{P}\,\mathtt{P} \mid \mathtt{callcc}$$

$$\textit{Terms} \quad \mathtt{M} ::= \mathtt{x} \mid \lambda\mathtt{x}.\mathtt{M} \mid \mathtt{M}\,\mathtt{M} \mid \mathtt{callcc} \mid \mathtt{k}_\pi$$

$$\textit{Stacks} \quad \pi ::= [\,] \mid \mathtt{M} :: \pi$$

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Operational Semantics via a (simplified) KAM:

$$\textit{Weak Head } \beta\textit{-reduction} \begin{cases} \textbf{(push)} & \mathtt{M}\,\mathtt{N} \star \pi \to \mathtt{M} \star \mathtt{N} :: \pi \\ \textbf{(pop)} & \lambda\mathtt{x}.\mathtt{M} \star \mathtt{N} :: \pi \to \mathtt{M}\{\mathtt{x} := \mathtt{N}\} \star \pi \end{cases}$$

$$\textit{Continuation passing style} \begin{cases} \textbf{(save)} & \mathtt{callcc} \star \mathtt{M} :: \pi \to \mathtt{M} \star \mathtt{k}_\pi :: \pi \\ \textbf{(restore)} & \mathtt{k}_\pi \star \mathtt{M} :: \rho \to \mathtt{M} \star \pi \end{cases}$$

Operational semantics of λ-calculus + `callcc`

Arena

Player		Opponent
$\mathtt{k}_\pi$	$\star$	$\mathtt{M} :: \rho$
	$\downarrow$	
$\mathtt{M}$	$\star$	π

Operational semantics of λ-calculus + `callcc`

Arena			Weapons	
Player		Opponent	Witnesses	Counterwitnesses
$\mathtt{k}_\pi$	$\star$	$\mathtt{M} :: \rho$		
	$\downarrow$		$\mathtt{M} \in \mathcal{W}(A)$	$\pi \in \mathcal{C}(A)$
$\mathtt{M}$	$\star$	π		

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	$\cap$			
	$\perp\!\!\!\perp$			

Operational semantics of λ-calculus + callcc

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$\mathtt{k}_\pi$	$\star$	$\mathtt{M} :: \rho$		
	$\downarrow$		$\mathtt{M} \in \mathcal{W}(A)$	$\pi \in \mathcal{C}(A)$
			$\mathtt{k}_\pi \in \mathcal{W}(A \to \bot)$	
$\mathtt{M}$	$\star$	π		

$$\Longrightarrow \qquad \mathtt{k}_\pi \in \mathcal{W}(\neg A) \quad \textit{for all } \pi \in \mathcal{C}(A)$$

Operational semantics of λ-calculus + callcc

Arena

Player		Opponent
callcc	$\star$	$\mathtt{M} :: \pi$
	$\downarrow$	
$\mathtt{M}$	$\star$	$\mathtt{k}_\pi :: \pi$

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	$\downarrow$		$\mathtt{M} \in \mathcal{W}(\neg A \to \bot)$	$\pi \in \mathcal{C}(A)$
$\mathtt{M}$	$\star$	$\mathtt{k}_\pi :: \pi$		
	$\curlywedge$			
	$\perp\!\!\!\perp$			

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			$\texttt{callcc} \in \mathcal{W}(\neg\neg A \to A)$	
$\texttt{M}$	$\star$	$\texttt{k}_\pi :: \pi$		

$$\Longrightarrow \qquad \texttt{callcc} \Vdash \neg\neg A \to A$$

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Player		Opponent	Witnesses	Counterwitnesses
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			$\texttt{callcc} \in \mathcal{W}(\neg\neg A \to A)$	
$\texttt{M}$	$\star$	$\texttt{k}_\pi :: \pi$		

$$\Longrightarrow \qquad \texttt{callcc} \Vdash \neg\neg A \to A$$

Similar argument for

$$\lambda\texttt{x}.\,\texttt{callcc}\,\texttt{x} \Vdash \neg\neg A \to A \quad \textit{and} \quad \lambda\texttt{x}.\,\texttt{callcc}(\lambda\texttt{y}.\,\texttt{x}\,\texttt{y}) \Vdash \neg\neg A \to A$$

Definition (Realisability semantics of formulas)

$$C^{\perp\!\!\!\perp} := \{t \in \mathbb{W} \mid t \perp\!\!\!\perp \pi \text{ for all } \pi \in C\}$$

$$\mathcal{W}(\_) := \mathcal{C}(\_)^{\perp\!\!\!\perp}$$

	$\mathbb{W}$	$\mathbb{C}$	$\perp\!\!\!\perp \subseteq \mathbb{W} \times \mathbb{C}$	$*$
Tarski	$\{\Box\}$	$\{\dagger\}$	$\emptyset$	$\Rightarrow$
Krivine	Λ	Λ^*	*pole*	*cons*

$$\begin{aligned}
\mathcal{C}(X(e_1, \ldots, e_k)) &= [\![X]\!]([\![e_1]\!], \ldots, [\![e_k]\!]) \\
\mathcal{C}(A \to B) &= \mathcal{W}(A) * \mathcal{C}(B) \\
\mathcal{C}(\forall c.A) &= \bigcup_{m \in \mathbb{N}} \mathcal{C}(A\{c := m\}) \\
\mathcal{C}(\forall^m Y.A) &= \bigcup_{Q : \mathbb{N}^m \to \mathcal{P}(\mathbb{C})} \mathcal{C}(A\{Y := Q\})
\end{aligned}$$

Realisability relation: $\mathtt{M} \Vdash A$ whenever $\mathtt{M}$ is a closed proof like term in $\mathcal{W}(A)$

Not only $\lambda \mathtt{x}.\, \mathtt{callcc}(\lambda \mathtt{y}.\, \mathtt{x}\, \mathtt{y}) \Vdash \neg\neg A \to A$ but even:

$$\vdash \lambda \mathtt{x}.\, \mathtt{callcc}(\lambda \mathtt{y}.\, \mathtt{x}\, \mathtt{y}) \ : \ \neg\neg A \to A$$

In fact, typing $\Rightarrow$ realising. The converse fails: that's precisely what we want!

Adequacy Theorem

Let

$$\mathtt{x}_1 : A_1, \ldots, \mathtt{x}_m : A_m \vdash \mathtt{M} : B$$

and fix an interpretation of the (free) 1st and 2nd order variables of $A_1, \ldots, A_m, B$.
For all closed terms $\mathtt{N}_1, \ldots, \mathtt{N}_m$, we have:

$$\mathtt{N}_1 \in \mathcal{W}(A_1)\,, \ \ldots\ , \mathtt{N}_m \in \mathcal{W}(A_m) \quad \Longrightarrow \quad \mathtt{M}\{\vec{\mathtt{x}} := \vec{\mathtt{N}}\} \in \mathcal{W}(B).$$

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Corollary

$$\vdash \mathtt{M} : A \implies \mathtt{M} \Vdash A$$

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In other words, a proof $\mathtt{x} : A \vdash \mathtt{M} : B$ defines a proof-term $\lambda\mathtt{x}.\,\mathtt{M}$ that computes (for each interpretation of variables and notion of winning process), the function

$$\mathtt{N} \in \mathcal{W}(A) \quad \mapsto \quad \mathtt{M}\{\mathtt{x} := \mathtt{N}\} \in \quad \mathcal{W}(B)$$

Realisability formalises BHK by extending the literal Curry-Howard one!

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Corollary

Let $\mathcal{T}$ be a theory of PA2 (or ZF/+CH/+C+...). If all axioms A of $\mathcal{T}$ are realised by programs $\mathtt{N}_A \Vdash A$, then: $\underline{\mathtt{x}} : \underline{A} \vdash \mathtt{M} : B \Longrightarrow \mathtt{M}\{\underline{\mathtt{x}} := \underline{\mathtt{N}}\} \Vdash B$.

E.g., $A =$*countable/dependent axiom of choice, ultrafilter axiom on* $\mathbb{N}$, *Continuum Hypothesis,...*

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(The computational content of) Classical logic is about the interaction with some notion of environment

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Proof.

True for all (that I know) computational approaches of classical logic: classical realisability, Game Semantics, Linear Logic, $\lambda\mu$-calculus, $\neg\neg$+Dialectica $\square$

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Now that we disclosed the "hidden" computational content of classical logic, we want to realise axioms, not only theorems!

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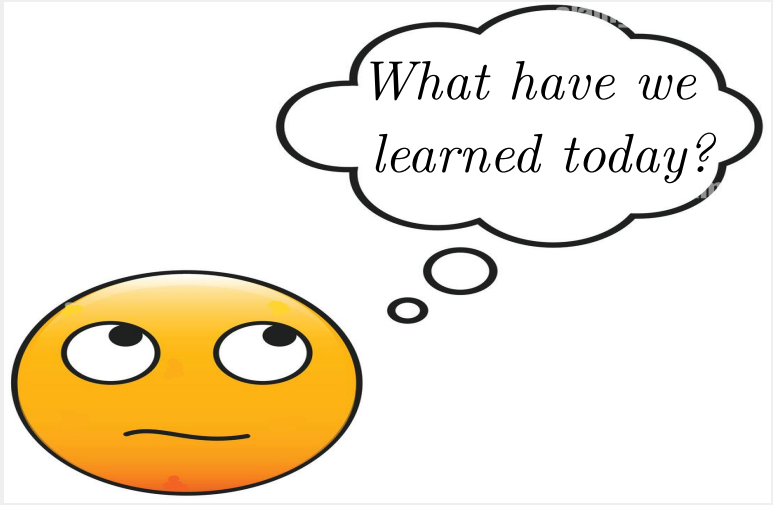
Summary, Exercises, Bibliography

- We have seen proof-terms for classical 2nd order logic
- We have given them an operational semantics in terms of a KAM which manipulates continuations
- We have defined the realisability semantics by refining Tarski
- We have seen that realisability is adequate wrt provability



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- Look at our **notes** on the webpage of the course, there are plenty of **details**, **proofs** and **exercises**.

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Exercise

The 2nd order encoding of $A \vee B$ is:
$A \vee B \;:=\; \forall^0 X.\, (A \to X) \to (B \to X) \to X$. With that, show (by hand) that:

$$\texttt{callcc}(\lambda \texttt{yvh.}\, \texttt{h}(\lambda \texttt{x.}\, \texttt{y}(\lambda \texttt{zw.}\, \texttt{zx}))) \;\Vdash\; A \vee \neg A.$$

This is actually the proof-term of a derivation of the excluded middle from *Consequentia Mirabilis* (itself an instance of Peirce's law). Do *not* use adequacy though.

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One million dollars exercise

Find a program `M` such that `M` $\Vdash$ full Axiom of Choice

[*Hint (?): Krivine proved that one exists.*]

Summary, Exercises, Bibliography

- Where the idea of callcc with Peirce law was introduced:
  A formulae-as-type notion of control, Timothy G. Griffin, 1990,
  https://dl.acm.org/doi/10.1145/96709.96714
- A standard introduction to the topic:
  Realizability in classical logic, Jean-Luis Krivine, 2004,
  https://www.irif.fr/~krivine/articles/Luminy04.pdf
- A very nice and clear PhD manuscript on the topic:
  On Forcing and Classical Realizability, Lionel Rieg, 2014,
  https://www-verimag.imag.fr/~riegl/assets/thesis-color.pdf
- To go further (one cool example among many possible):
  A program for the full Axiom of Choice, Jean-Luis Krivine, 2021,
  https://www.irif.fr/~krivine/articles/A_program_for_full_AC.pdf