The λ-calculus, from intuitionistic to classical logic

Webpage of the course

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ESSLLI Summer School, Bochum (Germany) *28/07/2025 – 01/08/2025*

The λ-calculus, from intuitionistic to classical logic

What will you (hopefully!) learn: You will have an idea of some basic but fundamental aspects of the relationship between (λ-*calculus based*) programming language theory and (*Curry-Howard based*) mathematical logic.

Lecture 1: Topology/domain-theory of the "graph model" over $\mathcal{P}(\mathbb{N})$
Davide Barbarossa

Lecture 2: The λ-calculus
Davide Barbarossa

Lecture 3: Category theory for denotational semantics
Giulio Guerrieri

Lecture 4: Curry-Howard and intuitionistic logic
Giulio Guerrieri

Lecture 5: Krivine's approach to classical logic
Davide Barbarossa

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Lecture 1:

Topology/domain-theory of the "graph model" over $\mathcal{P}(\mathbb{N})$

Read the notes: they are full of details, proofs, explanations, exercises, bibliography!

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Outline

What do we mean by *computable*?

Problem: Compute $\sqrt{S}$, for some given $S \in \mathbb{R}$

What do we mean by *computable*?

Problem: Compute $\sqrt{54899321104}$

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x_0 &:= \textit{any positive number} \\
x_{n+1} &:= \frac{1}{2}\left(x_n + \frac{54899321104}{x_n}\right)
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$\overset{\text{Heron}}{\cancel{\text{We}}}$ defined a function

$$H : \mathbb{N} \to \mathbb{R}, \qquad H(n) := x_n$$

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(potentially) infinitary entity	$a \in \mathbb{N} \cup \{+\infty\}$	$r \in \mathbb{R}$
finitary entity	$n \in \mathbb{N}$	$q \in \mathbb{Q}$
approximation of infinitary by finitary	$n < +\infty$	$q \in \mathbb{Q}$ is "close" to $r \in \mathbb{R}$
it works because	$\sup_{n \in \mathbb{N}} n = +\infty$	$\mathbb{Q}$ is dense in $\mathbb{R}$

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For all finite portion of the output,
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We set $R \coloneqq \mathcal{P}(\mathbb{N})$ and $R_{\mathit{fin}} \coloneqq \mathcal{P}_{\mathit{fin}}(\mathbb{N})$. For $a \in R$, let $\uparrow a := \{b \in R \mid b \supseteq a\}$.

Definition

The open sets $O \subseteq R$ are the ones of shape $O = \bigcup_{e \in I} \uparrow e$, for some $I \subseteq R_{\mathit{fin}}$.

This is possible as $\{\uparrow e \subseteq R \mid e \subseteq_{\mathit{fin}} \mathbb{N}\} \subseteq \mathcal{P}(R)$ is a *base* for a topology on $\mathcal{P}(\mathbb{N})$.

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Theorem

The open sets of our topology are exactly the $O \subseteq R$ which satisfy the following condition for all $a \in R$:

$$a \in O \Leftrightarrow O \ni e \subseteq_{\mathit{fin}} a, \ \textit{for some } e \in R.$$

Theorem

Let $f : R \to R$. The following are equivalent:

1. *f is continuous.*
2. *f is monotone for $\subseteq$ and, for all $a, d \in R$ we have*
$$d \subseteq_{\mathrm{fin}} f(a) \quad \Longrightarrow \quad \text{there is } \ e \subseteq_{\mathrm{fin}} a \ \text{ such that } \ d \subseteq f(e).$$
3. *for all $a \in R$ we have*
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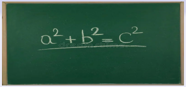

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A poset is the data of a set X and a partial order relation $\leq$ (i.e. reflexive, antisymmetric and transitive).

The Scott-topology of a poset

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Example

R with $\subseteq$ is a poset.

The Scott-topology of a poset

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A subset D of a poset X is called *directed* whenever $D \neq \emptyset$ and all $d, d' \in D$ admit an upper bound in D (i.e. there is $d'' \in D$ such that $d \leq d''$ and $d' \leq d''$).

Example

In R, the set $D = \{\{5,2\}, \emptyset, \{8\}\}$ is not directed, because D contains $\{5,2\}, \{8\}$ but no element of D is bigger than both $\{5,2\}$ and $\{8\}$.

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Example

In R, the set $D = \{e \in R \mid e \subseteq_{\text{fin}} \{5,2\}\} = \{\emptyset, \{5,2\}, \{5\}, \{2\}\}$ is directed (in this case it even has a maximum element).

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In R, given $a \in R$, the set $D = \{e \in R \mid e \subseteq_{\text{fin}} a\} =: \downarrow_{\text{fin}} a = \mathcal{P}_{\text{fin}}(a)$ is directed: the upper bound in D of $e, e' \in D$ is $e \cup e' \in D$ (in this case it is even their sup).

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Definition

Let $(X, \leq)$ be a poset. Its *Scott-topology* is defined by declaring open the subsets U of X such that:

1. U is upward closed (i.e. $U \ni x \leq y \Rightarrow U \ni y$)
2. for all directed $D \subseteq X$ which admits a lub $\bigvee D \in U$, we have $D \cap U \neq \emptyset$.

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Example (Proposition)

Our topology on R coincides with the Scott-topology of the poset $(R, \subseteq)$.

The Scott-topology of a poset

In a poset $(X, \leq)$, we say that an element $e \in X$ is *compact* iff for all directed $D \subseteq X$ admitting $\bigvee D$, we have: if $e \leq \bigvee D$ then $e \leq d$ for some $d \in D$.

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In $(R, \subseteq)$, the set of compact elements is R_{fin}.

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Example (Proposition)

In $(R, \subseteq)$, the set of compact elements is R_{fin}.

Theorem

Let X, Y be posets and $f : X \to Y$. The following are equivalent:

1. *f is continuous wrt the Scott-topologies on X, Y.*
2. *For all directed $D \subseteq X$ admitting $\bigvee D$ in X, there is $\bigvee(fD)$ in Y and:*

$$\bigvee(fD) = f(\bigvee D). \qquad \text{(Scott-Continuity)}$$

If moreover X, Y are "algebraic" (like R), then the above are equivalent to:

3. *f is monotone (i.e. $a \leq b$ implies $f(a) \leq f(b)$) and, for all $a \in X$, $d \in Y$,*

$$d \text{ compact } \leq f(a) \quad \Longrightarrow \quad \text{there is } e \text{ compact } \leq a \text{ such that } d \leq f(e).$$

4. *For all $a \in X$ we have $f(a) = \bigvee_{e \text{ compact } \leq a} f(e)$*

Encoding (continuous) functions as points

"*We can encode everything into* $\mathbb{N}$*!*"

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$$\mathrm{pair} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad \mathrm{pair}(n, m) := 2^n(2m+1) - 1$$

$$\mathrm{list} : \mathbb{N}^* \to \mathbb{N} \qquad \mathrm{list}([]) := 0 \quad \textit{and} \quad \mathrm{list}(n :: l) := 1 + \mathrm{pair}(n, \mathrm{list}(l))$$

$$\langle \_, \_ \rangle : \mathbb{N}^* \times \mathbb{N} \to \mathbb{N} \qquad \langle l, n \rangle := \mathrm{pair}(\mathrm{list}(l), n)$$

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Corollary

We have a bijective encoding $\langle \_ , \_ \rangle$ *of* $\mathbb{N}^* \times \mathbb{N}$ *into* $\mathbb{N}$

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Consider the following two auxiliary functions:

$$\begin{array}{ll} \mathrm{set} : \mathbb{N}^* \to R_{\mathit{fin}} & \mathrm{set}(n_1, \dots, n_k) \coloneqq \{n_1, \dots, n_k\} \\ \mathrm{kl} : R \to \mathcal{P}(\mathbb{N}^*) & \mathrm{kl}(a) \coloneqq \{l \in \mathbb{N}^* \mid \mathrm{set}(l) \subseteq a\} \end{array}$$

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Encoding (continuous) functions as points

"*We can encode everything into* $\mathbb{N}$*!*"

Corollary

We have a bijective encoding $\langle \_\, , \_ \rangle$ *of* $\mathbb{N}^* \times \mathbb{N}$ *into* $\mathbb{N}$

Consider the following two auxiliary functions:

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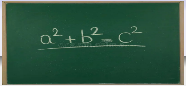

Encoding (continuous) functions as points

Theorem

- $@$, fun *and* λ *are continuous.*
- *The pair* (λ, fun) *defines a topological retraction of* R *onto* $\mathrm{Im}(\lambda) \approx (R \Rightarrow R)$, *i.e.*
$$\mathrm{fun} \circ \lambda = \mathrm{id}_{R \Rightarrow R}. \tag{β}$$
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$$\lambda(\mathrm{fun}(f)) \supseteq f \quad \textit{and} \quad \lambda(f) = \bigcup_{\mathrm{fun}(a)=f} a.$$

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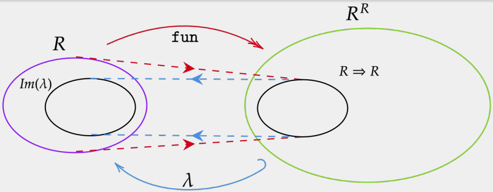

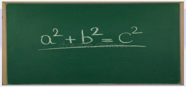

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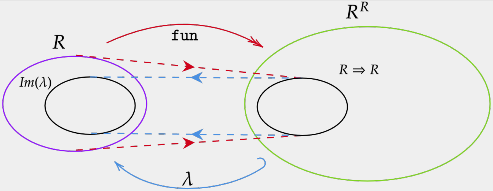

Interesting properties of continuous functions!

All continuous $f : R \to R$ admit fixed points.

The function

$$Y : (R \Rightarrow R) \to R \qquad Y(f) := @(\Delta_f, \Delta_f)$$

where $\Delta_f := \lambda(f \circ @ \circ \delta) \in R$ and $\delta : R \to R \times R$ is the diagonal, is a *fixed point combinator*, i.e. for all $f : R \Rightarrow R$, we have

$$f(Y(f)) = Y(f).$$

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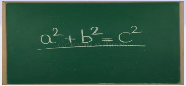

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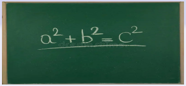

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Theorem

The set of RE sets is closed wrt the following rules:

$$\frac{}{\lambda(\lambda \circ \mathrm{curry}(\cdots(\lambda \circ \mathrm{curry}(\mathrm{proj}_i^n))\cdots))}$$

$$\frac{f : R \Rightarrow R \ \textit{computable}}{\lambda(f)} \qquad\qquad \frac{a \qquad b}{@(a, b)}$$

Summary, exercises, bibliography

- What do we intuitively mean when we say that a function is computable
- That this relates to topology
- That $R = (\mathcal{P}(\mathbb{N}), \mathit{Scott})$ is a good topological space for modeling this
- That its topology only depends on the partial order $\subseteq$
  - That actually, Scott-topology and Scott-continuity themselves are a property about "approximations" in posets
  - That Scott-continuous functions are given by their restriction on the finite elements of R. Also, they embed into their base space. This is done via the retraction (λ, fun), i.e. equation (β)
  - That all Scott-continuous functions on R have fixed points
  - That λ produces RE sets and fun preserves them

- Do the proofs of the statements in the slides
- Look at **our notes** on the webpage of the course, there are plenty of **exercises**
- The exercises have **solutions** (but try to do them by yourself before looking at them!)

- Chapter 1 and Section 1 of Chapter 18 of:
  Henk P. Barendregt, The lambda-calculus, its syntax and semantics, 1984,
  https://www.sciencedirect.com/bookseries/studies-in-logic-and-the-foundations-of-mathematics/vol/103
- Domain theory chapter of:
  Handbook of logic in computer science: semantic structures. 1995,
  https://dl.acm.org/doi/book/10.5555/218742 (an updated version is available here)
- Chapter 1 of:
  Amadio R., Curien P-L, Domains and lambda-calculi, 1996,
  https://www.cambridge.org/core/books/domains-and-lambdacalculi/4C6AB6938E436CFA8D5A8533B76A7F23
- Chapter 1, 5 of:
  PhD thesis of Giulio Manzonetto, Models and theories of lambda calculus, 2008,
  https://www.irif.fr/~gmanzone/ManzonettoPhdThesis.pdf
- This conference of **Dana Scott**