2.2.4 Beam Elements model

Solid element were found to be too computationally expensive while meta-material approaches where considered oversimplified.
A modelling strategy needs then to be adopted in order to simplify the solid FE model.
The most logical way to do that, is to neglect from the analysis certain components of stress that contribute only secondarily to the fatigue life of the structure, in order to reduce the size of the problem.
When a metallic structure is loaded in tension, fatigue crack will more easily nucleate in areas where a tensile stress concentration is present. As previously described, several experimental works show that fatigue failure can be originated in bending dominated lattices in case of compression-compression cyclic loading. This singular phenomenon was modelled and explained (de Krijger *et al.* 2016) with the effect of tensile stresses generated from bending in the micro-structure (Figure 2.3). This suggest that the modelling approach adopted should be at least able to evaluate pure axial forces and bending moments to describe this phenomenon.
Simple rod elements can't be used to this purpose since they only evaluate axial stress generated by axial forces.
Simple beam elements due to their low computational costs and their capability to evaluate bending moments, seems to be the perfect choice.

2.2.5 Saint Venant principle

Since Lattice micro-structure is generally characterized by slender ligaments with one dimension predominant with respect to the others, Saint Venant simplified problem for slender bodies is introduced. The starting assumption is the famous Saint-Venant principle:

While statically equivalent systems of forces acting on a body produce substantially different local effects, the stresses at sections distant from the surface of loading are essentially the same.
Adhémar Jean Claude Barré de Saint-Venant (1797-1886).

Taking as a reference a beam loaded in pure tension in two specific points of its cross section (Figure 2.7), the axial stress distribution near the application point is the one represented in section A-A. According to Saint Venant Principle, in section B-B the stress becomes constant in the cross section. We may then apply the following theory in points far away from load application.

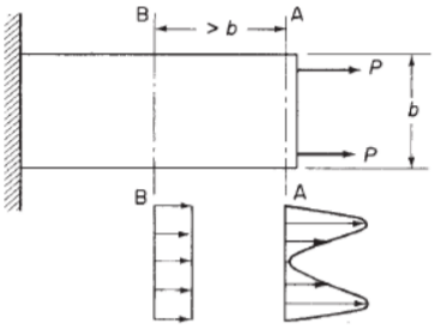

Figure 2.7: Saint Venant's Principle

In order to correctly apply Saint Venant's principle, the elastic problem domain needs to satisfy certain requirements:

- *Cylindrical solids with simply connected sections*
- *Length large when compared to the cross section*
- *Body forces negligible*
- *Surface forces applied only on the bases*
- *No applied displacements*
- *Isotropic homogeneous elastic material*

Lattice structure ligaments not always fulfill the previous assumptions.

Low relative density lattice's ligaments can be considered slender bodies but increasing relative density the ligaments become thicker and the model loses its applicability.

The unidirectional assumption can be enforced with the following equation:

$$\sigma_{yy} = \sigma_{zz} = \tau_{yz} = 0 \tag{2.1}$$

This represents the biggest simplification since the out of plane shear and out of axis normal stresses are neglected from the analysis.

This reduces the degrees of freedom of every element and, as a consequence, the computational costs.

The only stress components acting on an infinitesimal cross sectional area are then:

$$\sigma_{xx}, \tau_{xz}, \tau_{xy} \tag{2.2}$$

.

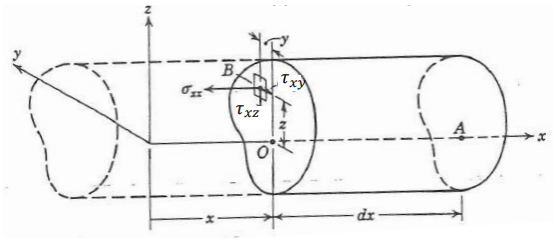

Figure 2.8: Beam Element notation

The equilibrium problem can then be imposed considering conventions in Figure 2.8.

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} &= 0 \end{aligned} \tag{2.3}$$

Thanks to the linear theory adopted and the superposition principle, effects of resultants forces and moments can be separated and studied individually. Normal stress generated by axial forces is simply evaluated as:

$$\sigma_{xx} = \frac{P}{A} \tag{2.4}$$

Concerning the bending moment, Euler Bernoulli theory is used. The basic assumption is that cross-sectional planes of the beam remain plane and normal to the axis as the beam deforms (Figure 2.9). Thanks to this assumption is possible to determine the deflection of any point in terms of the deflection of the beam axis.

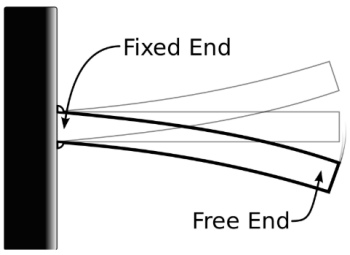

Figure 2.9: Euler Bernoulli bending

Summing up the effects of bending and axial force, a 3d formulation for axial stress can be derived:

$$\sigma_{xx} = \frac{P}{A} - y\frac{M_{zz}}{I_{zz}} - z\frac{M_{yy}}{I_{yy}} \tag{2.5}$$

This description is chosen since it provides an estimation of axial stresses due to bending moments, useful for the fatigue analysis, while keeping the computational costs as low as possible. The numerical formulation of the Euler Bernoulli beam problem is described by the linear system of force displacement. The local element stiffness matrix for a 2d beam element is the following:

$$K_e = EI \begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & -\frac{12}{l^3} & \frac{6}{l^2} \\ & \frac{4}{l} & -\frac{6}{l^2} & \frac{2}{l} \\ & & \frac{12}{l^3} & -\frac{6}{l^2} \\ sym. & & & \frac{4}{l} \end{bmatrix} \tag{2.6}$$

The local stiffness matrix is assembled in the stiffness matrix which relates global displacements to global forces. Finally the linear system is solved

$$Ku = F \tag{2.7}$$

2.2.6 Limitations

The main limitation of the previously described approach, is that it fails to correctly predict the stress near constrains and load application points.
This means that the biggest error will be in the ligament connections, which is a crucial area for fatigue propagation as previously described.
A second limitation is that increasing the Lattice relative density, the average ligament thickness is increased and so the slenderness assumption loses its applicability.
Increasing the relative density, the stress distribution becomes more tri-dimensional and so beam element simplified model is no more fitting the reality.
As a conclusion it can be said that applicability of a beam element approach is limited to low relative density lattices.

2.3 Modelling Fatigue

2.3.1 Progressive failure approach

The correct way to calculate lattices fatigue life, would be to solve crack propagation equations in every site of nucleation until failure of ligaments.
This way the progressive loss of stiffness could be evaluated.
The big complication is that, after every ligament rupture, stress redistributes and the loading condition changes. This means that every time a rupture happens, a new FE Analysis is required and crack propagation needs to be evaluated according to the new loading conditions.
In addition to that, crack opening analysis is extremely complex to conduct. The described procedure, although being correct, is extremely expensive from a computational point of view. A simplified modelling approach should then be adopted also in the fatigue analysis. The first simplification that can be made, is assuming that lattice properties remain unchanged between two ligaments failures.
This is equivalent to assuming that the propagation of cracks inside a ligament structure contributes to the weakening of the lattice less than the ligament rupture itself.
The second simplification to be made is assuming, at least in the first place, that the crack path direction and shape is determined by the stress redistribution after every ligament failure.
To summarize:

- *Material properties remain constant between two different ligament failures*
- *The position of the successive failure is mainly determined by the stress redistribution*

Those two assumptions are the basis of the progressive failure approach (Zargarian *et al.* 2016). Progressive failure approach consists in simulating fatigue crack propagation by predicting every ligament failure.

Each time a rupture in the micro scale occurs, stress is redistributed and the surrounding ligaments experience higher stress amplitudes during the following cycles. In order to take into account of this, an iterative analysis is required. Each step of the simulation is a failure event and corresponds to the failure of one single micro structure.

After a first FE simulation of the intact structure, fluctuating and mean stress is determined for every ligament using beam element theory.

After that, a mean stress correction is applied using Goodman equation:

$$\frac{\sigma_a}{S_{Nf}} + \frac{\sigma_m}{S_u} = 1 \tag{2.8}$$

At this point Basquin relation can be used to evaluate every ligament's life using the equivalent stress amplitude.

$$N = C(\Delta S)^b \tag{2.9}$$

It's extremely important to know the average ligament properties to have a correct estimation of fatigue life.

The fatigue life material properties C and b can be defined as intercept and slope of the S-N logarithmic fatigue graph.

In order to have a correct evaluation of material parameters, experimental tests should be conducted on single ligaments but this procedure is extremely expensive and time consuming.

For the current work standard material charts are used and size and manufacturing effect on ligament's material properties are neglected.

Once life of every element is determined, the minimum life element is removed and damage is updated in every structure.

In this way a memory effect of previous stress cycle can be introduced in the structure.

The damage accumulation during the previous cycles is updated through Miner's rule.

The remaining life at i^{th} failure event of the j^{th} ligament n_j^i can be evaluated from previous events.

$$n_j^i = Nf_j^i(1 - \sum_{s=1}^{i-1} \frac{n_{min}^s}{Nf_j^s}) \tag{2.10}$$

Where Nf_j^i is the expected fatigue life derived from Basquin relation. The summation involves all the previous failure events and every time needs to be updated.

At every iteration, the minimum remaining life element is removed.

By adding the remaining life of the removed element at every iteration, a total fatigue life of the specimen can be evaluated.

2.3.2 Limitations

The first and most important limitation of the adopted model is regarding the use of Basquin relation for fatigue life estimations.

It's impossible to know exactly the properties of every ligament since they can vary due to manufacturing related imperfections.

If an error is committed in the evaluation of the ligament properties, it will propagate after every failure producing high divergence in the resulting life.

A second limitation is given by the bad applicability of Basquin relation for low cycle fatigue.

Lattice's structure ligaments fail at different cycles, and so certain structures can experience a very short fatigue life.

The use of a mean stress correction, adds other errors to the final fatigue life calculated, since it makes use of an approximated model (Goodman).

Another limitation is the fact that failed elements are removed from the model

after every cycle. This could actually describe the reality for tension tension fatigue, but in case of compression-compression, removed elements will still be able to carry part of the load after failure. This is not taken into account in the adopted model.

2.4 Modelling approach overview

The model used for evaluate stress is Beam element model because of the low computational costs. As previously described an error is introduced in the surface stress evaluation near ligament's connections. This error is repeated at every ligament failure (since a new FE analysis is conducted).

The fatigue modelling introduces an error in the parent material property estimation. The Basquin parameters used are not describing imperfections introduced by manufacturing. In addition to that there's an error introduced by the mean stress correction factor.

Errors in fatigue life evaluation are repeated at every failure event as described in Figure 2.10

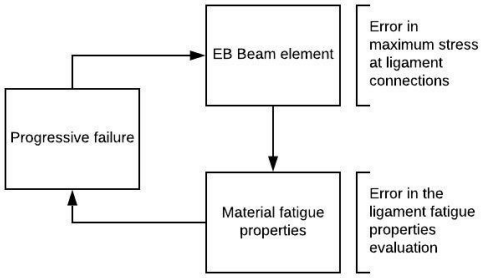

Figure 2.10: Errors introduced by modelling approach

3.1 Generation of Geometry

The first part of the program consists of the generation of the lattice geometry. The user is allowed to select between different Lattice unit cells both 2D and 3D.

Geometrical parameters as structure thickness, cell size and relative density can be imposed.

The lattice pattern is generated by symmetrical repetition of the unit cell in the three directions of space.

The Geometrical generation module allows also to create functional grading of relative density.

Two different codes are used for the 2D case and 3D case. This way the degrees of freedom of the bidimensional lattice analysis can be reduced.

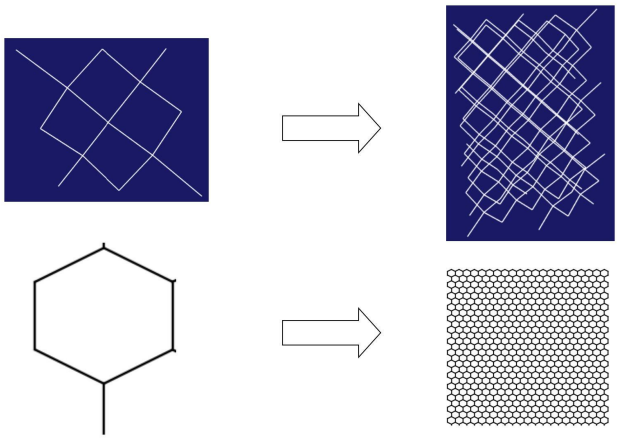

Figure 3.1: Generation of the geometry 2D and 3D

3.2 Fatigue Analysis

The Fatigue analysis module requires to perform two different FE calculations for each structure failure. The loading cycle data are inserted in the model through Maximum Force on the top plate of the specimen and R ratio.
Two FE analysis are performed corresponding to maximum and minimum macroscopic loading. This way a stress cycle can be determined for every beam element constituting the structure.
For the calculation of the beam element displacements Anastruct (R.Vink 2019) module was used for the 2D analysis, while for the three dimensional case Pynite (J.Wock 2019) module was used.
Both those Python open source classes calculate beam elements force and displacements, but they were adapted to calculate stresses and deal with lattice structures.
After FE analysis, maximum stress is examined in order to see if it's bigger than ultimate strength of the base material.
If one ore more elements surpass ultimate stress level, static failure is supposed to happen, and the element with the higher stress is removed and the analysis repeated.
Thanks to this procedure, if several elements are supposed to statically fail, they can be removed in a progressive way to take into account the stress redistribution after every static failure.
If stress levels are consistent with fatigue analysis, mean stress correction is applied and life of each structure is calculated using the Basquin relation.
Damage is updated using Miner's Rule of linear damage accumulation. Statically failed ligaments don't contribute to the damage accumulation and to the general fatigue life.
The process is iterated and repeated until the final collapse is reached.

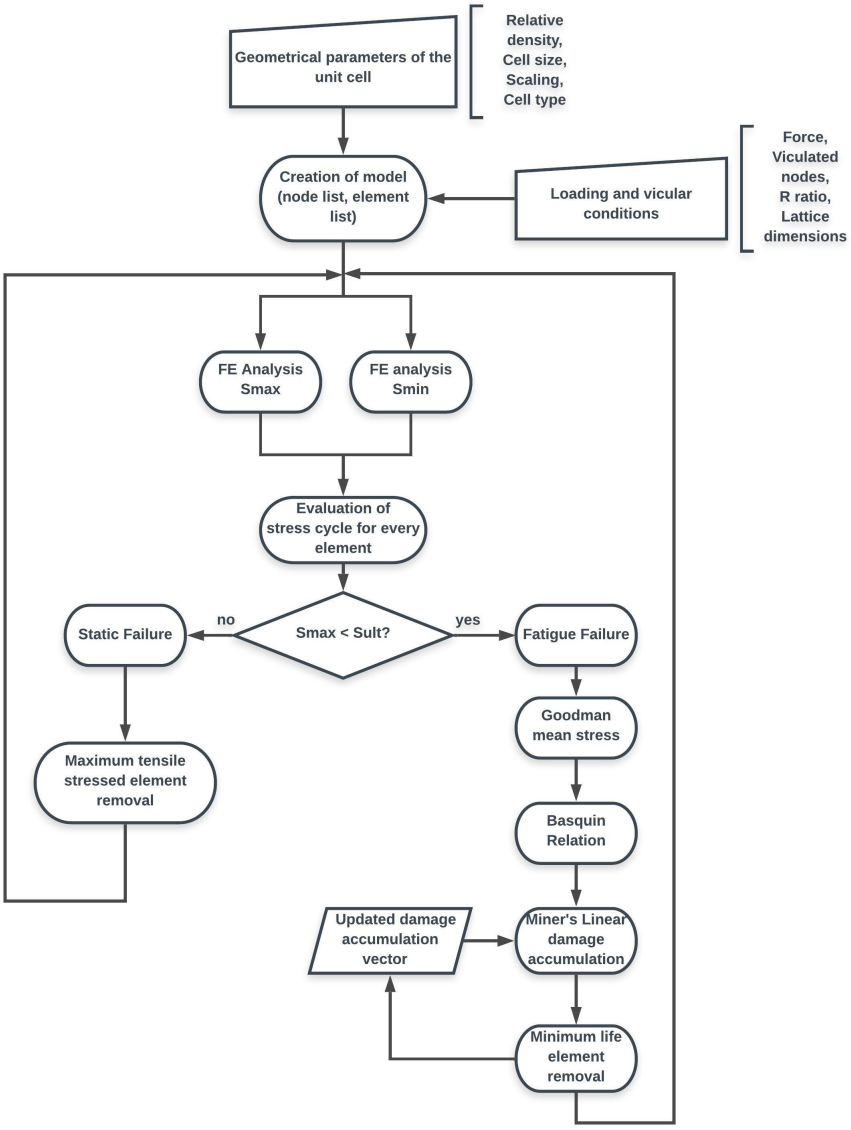

Figure 3.2: Algorithm flow chart