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RATIONAL HOMOGENEOUS VARIETIES AND NESTING MAPS

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As for everything else, so for a mathematical theory:
beauty can be perceived but not explained.

Arthur Cayley

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Abstract

Rational homogeneous varieties are some of the most important building blocks in Algebraic Geometry, as they include the most studied amongst algebraic varieties, such as the projective space and the Grassmanians. This paper is a brief but possibly complete introduction to the theory of algebraic groups and rational homogeneous varieties, presented through some notable examples such as the flag manifolds, with a final focus on some recent developments and open problems revolving around them.

Contents

Introduction

In the vast field of Algebraic Geometry, an important problem which started emerging in the 1980s is that of contractions, namely maps from a normal algebraic variety X to another normal algebraic variety Y with connected fibres. These maps play a huge role in the study of algebraic varieties as they can be used to simplify the structure of a variety, in order to reduce it to a simpler — and thus more accessible — one. For example, if we consider the blow-up $\operatorname{Bl} X$ of an algebraic variety X, we can define a contraction $\operatorname{Bl} X \to X$ with respect to some divisor of X.

Contractions don't always have the same form, as they can appear as birational maps or as *fibre type* contractions, where the dimension of the codomain is smaller than that of the domain. Usually, given an algebraic variety, one can build a sequence of birational contractions in order to reduce it to a much simpler one — this is the main focus of the so-called *birational geometry* — but it isn't always possible (see, for instance, [KM98]): an important example of varieties that cannot be contracted birationally are the rational homogeneous varieties. These structures, characterised by having an algebraic group acting on them, are crucial building blocks of algebraic geometry and are amongst the most studied and well-known algebraic varieties, as they include the projective space and the Grassmanians.

The aim of this paper is therefore to give an introduction to the rational homogeneous varieties and algebraic groups, as presented in [SC19] and with the help of [Hum75; Hum72; Har92], and to address some of their main properties through some notable representatives, like quadrics and Grassmanians. In particular, we will describe how contractions will interact with some of them, appearing as projections between algebraic varieties of the same family.

Studying these maps, an obvious question arises: is it possible to build right inverses that are also morphisms of algebraic varieties? De Concini and Riechstein addressed this problem in [DCR03] and proved that, despite what one can imagine, such sections are not common amongst Grassmanians. The theory around this matter was later expanded in [MOSC20], where the authors found the conditions for a so-called *nesting map* to exist for a rational homogeneous variety which is a quotient of a semisimple algebraic group of classical type: the last chapter revolves around presenting their work from a *lower* and more *geometric* point of view, with the aim of being a first look into an open problem in Algebraic Geometry.

In our dissertation, we will describe three families of *nestings*, each of them associated to a certain type of Dynkin diagram, and we will find some similar patterns in how the nesting is

defined. However, in the case of the diagram B_3 something different happens: the section is built seeing $\mathbb{P}^7$ as the projective space on the (complexified) octonions $\mathbb{OP}^1$ and using the octonionic product to describe the map we want. This raises another interesting question that has not yet been answered: is there any link between the existence of a nesting map and the $\mathbb{C}$-$*$-algebras (as defined in [Bae02] and [Bor24]) obtained through the Cayley–Dickson construction? Can the nestings associated to other Dynkin diagrams be described through the (complexified) complex numbers or the (complexified) quaternions?

Chapter 1

Affine Algebraic Groups

1.1 Affine Varieties

In the classical definition, an *affine variety* is the set $\mathrm{Z}(S)$ of common zeros in $\mathbb{A}^n$ of a set of polynomials $S \subset \mathbb{K}[x_1, \ldots, x_n]$. By observing that $\mathrm{Z}(S) = \mathrm{Z}(\langle S \rangle)$, and thanks to the Hilbert Basis Theorem, S can be considered finite without any loss of generality. However, in order to work with algebraic groups, we will need a more intrinsic definition of affine variety, which will not be dependent on the environment $\mathbb{A}^n$. In order to do so, we must recall the concept of *coordinate ring* of an affine variety X, which is the $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]/\mathrm{I}(X)$, where $\mathrm{I}(X) \subset \mathbb{C}[x_1, \ldots, x_n]$ is the ideal of polynomials vanishing on X. Since (by the Nullstellensatz) $\mathrm{I}(X)$ is radical, it follows that $\mathbb{C}[X]$ is reduced.

First, consider an affine variety $X \subset \mathbb{A}^n$. It is a Noetherian topological space with the Zariski topology, which is defined through the basis $\{X_f \mid f \in \mathbb{C}[X]\}$, where $X_f = \{x \in X \mid f(x) \neq 0\}$. As a corollary of the Nullstellensatz, one can prove that there is a 1-1 correspondence between the closed subsets of X and the radical ideals of $\mathbb{C}[X]$, while the points of X correspond 1-1 with the maximal ideals of $\mathbb{C}[X]$: in this sense we can recover X from its coordinate ring.

Indeed, if we consider a reduced finitely generated $\mathbb{C}$-algebra $R = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$, then R is isomorphic to a quotient $\mathbb{K}[x_1, \ldots, x_n]/J$, where J is a radical ideal and the kernel of the map $x_i \mapsto \alpha_i$. Note that R is therefore isomorphic to the coordinate ring of an affine variety $X = \mathrm{Z}(J)$.

An advantage of this approach (we will be needing shortly) is that it can be used to give any principal open subset X_f of an irreducible variety X its own structure of affine variety. In fact, by defining R as the subalgebra of $\mathbb{C}(X)$ (the field of fractions of $\mathbb{C}[X]$) generated by $\mathbb{C}[X]$ and $1/f$, we obtain that R is a reduced finitely generated $\mathbb{C}$-algebra. Its maximal ideals correspond 1-1 with their intersections with $\mathbb{C}[X]$, which are the maximal ideals not containing f, and therefore to the points of X_f. By this construction, we have identified points of $(x_1, \ldots, x_n) \in X_f \subset X \subset \mathbb{A}^n$ with points $(x_1, \ldots, x_n, 1/f(x)) \in \mathbb{A}^{n+1}$ so that $X_f \cong (X \times \mathbb{A}^n) \cap \mathrm{Z}(f(x_1, \ldots, x_n)x_{n+1} - 1)$.

We also recall that a morphism of affine varieties $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ is a mapping $\varphi : X \to Y$

of the form $\varphi = (\varphi_1, \ldots, \varphi_m)$ where $\varphi_i \in \mathbb{K}[X]$. With a morphism is associated its *pullback* $\varphi^* : \mathbb{K}[Y] \to \mathbb{K}[X]$, defined by $\varphi^*([g]) = [g \circ \varphi]$: every morphism is completely determined by its pullback and vice versa. This establishes a contravariant functor between the category of affine $\mathbb{C}$-algebras (*i.e.* finitely generated reduced $\mathbb{C}$-algebras), with the $\mathbb{C}$-algebra homomorphisms as morphisms, and the category of affine varieties, with morphisms as previously defined.

1.2 Algebraic Groups

Definition 1.1 (Algebraic group)**.** An *algebraic group* is a variety G endowed with the structure of a group $(G, \cdot)$ such that the maps

$$\mu : \begin{array}{ccc} G \times G & \longrightarrow & G \\ (x, y) & \longmapsto & xy \end{array} \quad \text{and} \quad \iota : \begin{array}{ccc} G & \longrightarrow & G \\ x & \longmapsto & x^{-1} \end{array}$$

are morphism of varieties.

Remark 1.2. Here $G \times G$ is not equipped with the product topology, but the Zariski topology, as for a product of varieties.

As expected, a morphism of algebraic groups is a mapping $G \to H$ such that it is both a homomorphism of groups and a morphism of varieties. Since translation by an element is an isomorphism of varieties $G \to G$, every geometric property at one point can be transferred to any other point of G.

1.2.1 Classical examples

We now introduce some classical examples which will be crucial later. Note that for all these groups a projective version can be constructed considering matrices up to scalar multiples. Consider a fixed $n \in \mathbb{N}$.

General linear group. The general linear group $\mathsf{GL}(n + 1, \mathbb{C})$ consists of all $(n+1) \times (n+1)$ matrices with nonzero determinant. It is a principal open subset of the irreducible variety $\mathbb{A}^n$, and therefore it is an affine variety itself. Since matrix product and inversion are polynomial operations, it is clearly an algebraic group. The next groups will be subgroups of $\mathsf{GL}(k, \mathbb{C})$ for some k, so there will not be a need to prove the compatibility between the two structures.

Special linear group. The special linear group $\mathsf{SL}(n + 1, \mathbb{C})$ consists of the matrices with determinant 1 in $\mathsf{GL}(n+1, \mathbb{C})$: being the set of zeroes of $\det M - 1$, it is closed in $\mathsf{M}((n+1) \times (n+1), \mathbb{C}) \cong \mathbb{A}^{(n+1)^2}$. By computing its tangent space at the unity e, $\mathrm{T}_e \mathsf{SL}(n + 1, \mathbb{C}) = \mathfrak{sl}(n + 1, \mathbb{C})$, we obtain a Lie algebra of type A_n.

Symplectic group. The symplectic group $\mathsf{Sp}(2n, \mathbb{C})$ consists of all matrices $M \in \mathsf{GL}(2n, \mathbb{C})$ such that

$$M^\top \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \qquad \text{where } J_n = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

Its Lie algebra $\mathrm{T}_e\,\mathsf{Sp}(2n, \mathbb{C}) = \mathfrak{sp}(2n, \mathbb{C})$ is of type C_n.

Special orthogonal group (odd). The special orthogonal group $\mathsf{SO}(2n+1, \mathbb{C})$ consists of all matrices $M \in \mathsf{SL}(2n+1, \mathbb{C})$ such that

$$M^\top \Omega M = \Omega, \qquad \text{where } \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

Its Lie algebra $\mathrm{T}_e\,\mathsf{SO}(2n+1, \mathbb{C}) = \mathfrak{so}(2n+1, \mathbb{C})$ is of type B_n.

Special orthogonal group (even). In this case, the special orthogonal group $\mathsf{SO}(2n, \mathbb{C})$ consists of all matrices $M \in \mathsf{SL}(2n, \mathbb{C})$ such that

$$M^\top \Omega M = \Omega, \qquad \text{where } \Omega = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Its Lie algebra $\mathrm{T}_e\,\mathsf{SO}(2n, \mathbb{C}) = \mathfrak{so}(2n, \mathbb{C})$ is of type D_n.

Note that in both cases, the matrix Ω can be any symmetric matrix with nonzero determinant.

1.3 The identity component

Let G be an algebraic group. Then, like every algebraic variety, it can be considered as a union of a finite number of irreducible components.

Proposition 1.3. *Let G be an algebraic group, then only one of its irreducible components contains the identity e.*

Proof. Let $X_1, \ldots, X_m$ be the irreducible components of G containing e. Then their product $X_1 \times \cdots \times X_m$ is an irreducible variety as well, and its image under the morphism $(x_1, \ldots, x_m) \mapsto x_1 \cdot \ldots \cdot x_m$ is an irreducible subset $X_1 \cdot \ldots \cdot X_m$ of G containing e, and therefore lies in some X_i. However, since $X_1 \cdot \ldots \cdot X_m$ contains e, it also contains each X_j, hence m can only be 1. $\square$

Definition 1.4 (Identity component)**.** Let G be an algebraic group. Then its unique irreducible component containing the unity e is called *identity component* and denoted G°.

Proposition 1.5. *Let G be an algebraic group. Then*

1. *G° is a normal subgroup of finite index in G, whose cosets are the connected as well as the irreducible components of G;*

2. *each closed subgroup of finite index in G contains G°.*

Proof.   1. For each $x \in G^\circ$, $x^{-1}G^\circ$ is an irreducible component of G containing e, therefore $x^{-1}G^\circ = G^\circ$ and $x^{-1} \in G^\circ$. Moreover, for each $g \in G^\circ$, gG° is an irreducible component of G containing $gg^{-1} = e$, hence $gG^\circ = G^\circ$ and, for each $h \in G^\circ$, $gh \in G^\circ$, so G° is a group.

Similarly, for any $x \in G$, $xG^\circ x^{-1}$ is an irreducible component of G containing $e = xex^{-1}$, and hence coincides with G°, which is therefore normal. Its cosets are translations of G°, hence they are irreducible components as well and must be of finite number as G is Noetherian. Since they are disjoint, they are also the irreducible components of G.

2. If H is a closed subgroup of finite index in G, then each of its cosets $g_1 H, \ldots, g_k H$ must be closed subvarieties of G, and therefore union of finitely many G°-cosets (distinct from H). Now, for every $i = 0, \ldots, k$, where $g_0 := e$

$$g_i H = G \smallsetminus \bigsqcup_{j \neq i} g_i H$$

and therefore is open. Since those cosets form a partition of G in both open and closed sets, they must be unions of connected components: in particular $H \supseteq G^\circ$. $\square$

Definition 1.6 (Connected algebraic group)**.** An algebraic group G is *connected* if $G = G^\circ$.

Remark 1.7. In the context of linear groups, the term *irreducible* has a complete different meaning.

1.4   Algebraic actions

Definition 1.8 (Algebraic action)**.** Let G be an algebraic group and X a variety. An *algebraic action* is a morphism $\alpha : G \times X \to X$, denoted $g \cdot x := \alpha(g, x)$, such that, for each $g, h \in G$, $x \in X$,

$$g \cdot (h \cdot x) = (gh) \cdot x \tag{A1}$$

$$1 \cdot x = x \tag{A2}$$

We also say that G acts *morphically* on X.

Remark 1.9. The set of fixed points $X^G := \{x \in X \mid gx = x \ \forall g \in G\}$ is a closed subset of X. Note that $X^G = \bigcap_{g \in G} X^g$ where $X^g := \{x \in X \mid gx = g\}$, and therefore it is sufficient to prove that for each g the set X^g is closed in X. Since α is a morphism (and therefore expressed by polynomials) and $X^g = \mathrm{Z}(\alpha(g, x) - x)$, it is closed indeed.

An important object associated to each (projective) variety X is its *automorphism group* $\mathrm{Aut}\, X$, which is an algebraic group as well and acts morphically on X.

Definition 1.10 (Homogeneous variety)**.** A projective variety X is called *homogeneous* if its group of automorphisms acts transitively on X, that is, if the *evaluation morphism*

$$\mathrm{ev} : \mathrm{Aut}\, X \times X \to X \qquad (g, x) \mapsto \mathrm{ev}(g, x) := g(x)$$

satisfies that given $x, y \in X$, there exists $g \in \mathrm{Aut}\, X$ such that $g(x) = y$.

1.5 Borel subgroups and semisimple groups

An important concept in the study of algebraic groups is that of *semisimple groups*. In order to define it, we recall the concept of *solvable group*.

Definition 1.11 (Solvable group)**.** Given a group G, its *derived series* is the sequence of subgroups
$$G \supset G^{(1)} \supset G^{(2)} \cdots \supset \cdots G^{(k-1)} \supset G^{(k)} \supset \cdots$$
where $G^{(k)} = [G^{(k-1)}, G^{(k-1)}] = \langle [g, h] \mid g, h \in G^{(k-1)} \rangle$ is the commutator subgroup of $G^{(k-1)}$. G is called *solvable* (or *soluble*) if $G^{(k)} = \{e\}$ for some k.

Remark 1.12. $G' := [G, G]$ is a characteristic subgroup of G, therefore every $G^{(i)}$ is normal in G.

Proposition 1.13. *The product of two solvable normal subgroups of a group G is solvable and normal in G.*

Proof. The product of two normal subgroups is clearly normal. Consider now the quotient G/H and the projection map $\pi : G \twoheadrightarrow G/H$: since $\pi(HN) = \pi(N)$ is solvable (as N is solvable), for some $k \in \mathbb{N}$, $\pi(HN)^{(k)} = eH$, therefore $(HN)^{(k)} = H$. Since H is solvable, so is HN. $\qquad\square$

Corollary 1.14. *Given an algebraic group G, it contains a unique maximal connected solvable normal closed subgroup, called* radical of G *and denoted by* $\mathrm{R}(G)$.

Definition 1.15 (Semisimple group)**.** A group G is called *semisimple* if and only if $\mathrm{R}(G) = \{e\}$.

Example 1.16.
The radical of $\mathsf{GL}(V)$ is equal to the subgroup of homotheties $\mathbb{C}^*e$, therefore the quotient $\mathsf{PGL}(V) := \mathsf{GL}(V)/\mathbb{C}^*e$ is semisimple.

Definition 1.17 (Borel subgroup)**.** Given an algebraic group G, a *Borel subgroup* of G is a connected, solvable, closed subgroup that is not contained properly in another subgroup of this kind.

Remark 1.18. One can prove that the radical of an algebraic group G is the identity component in the intersection of all Borel subgroups [Hum75, §20, Exercise 6].

Definition 1.19 (Parabolic subgroup)**.** A subgroup of an affine algebraic group G is called *parabolic* if it contains a Borel subgroup of G.

Example 1.20.
The main example of a connected solvable group is that of the upper triangular matrices $\mathsf{T}(V) \subset \mathsf{GL}(V)$ (where the concept of *triangular* needs to be expressed with respect to a certain choice of a basis). Furthermore, one can prove that $\mathsf{T}(V)$ is also a Borel subgroup, and that the parabolic groups are those of block triangular matrices.

The fact that the group $\mathsf{T}(V)$ of triangular matrices is a Borel subgroup of $\mathsf{GL}(V)$ is a crucial point in this theory thanks to the following theorem, which will allow us to consider every connected solvable subgroup $G \subset \mathsf{GL}(V)$ as a subgroup of $\mathsf{T}(V)$.

Theorem 1.21 (Lie–Kolchin)**.** *Let V be a finite dimensional complex vector space, and $G \subset \mathsf{GL}(V)$ be a connected solvable subgroup. Then there exists a basis of V such that the corresponding group $\mathsf{T}(V)$ contains G.*

1.6 The Lie algebra of an algebraic group

Now, we are interested in introducing some basic concepts about the Lie theory of algebraic groups that will be useful later. First, let's recall the definition of a Lie algebra.

Definition 1.22 (Lie algebra). A $\mathbb{C}$ vector space A, endowed with an operation

$$
\begin{aligned}
[\cdot,\cdot]: \quad A \times A \quad &\longrightarrow \quad A \\
(x,y) \quad &\longmapsto \quad [x,y]
\end{aligned}
$$

called the *bracket* or *commutator* of x and y, is called a Lie algebra if

(L1) the bracket operation is bilinear;

(L2) $[x,x] = 0$ for all $x \in A$;

(L3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \ \forall x,y,z, \in A$ (*Jacobi identity*).

Remark 1.23. (L1) and (L2) applied to $[x+y, x+y]$ imply anticommutativity

$$[x,y] = -[y,x]. \tag{L2'}$$

Conversely, since char $\mathbb{C} \neq 2$, it is clear that (L2') implies (L2).

Example 1.24.
The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ is the matrix algebra $\mathsf{M}(n \times n, \mathbb{C})$ viewed as a Lie algebra. This will turn out to be the Lie algebra associated to $\mathsf{GL}(n, \mathbb{C})$.

Let now G be an algebraic group and $A = \mathbb{C}[X]$ its coordinate ring: then G acts on A by left multiplication as follows

$$
\begin{aligned}
\lambda: \quad G \times A \quad &\longrightarrow \quad A \\
(x,f) \quad &\longmapsto \quad \lambda_x(f)
\end{aligned}
\qquad \text{where } \lambda_x(f)(y) = f(x^{-1}y).
$$

Hence, the Lie bracket of two derivations is still a derivation (where $\delta \in \mathrm{End}\, A$ is called a derivation if $\delta(fg) = f\delta(g) + \delta(f)g$ for all $f, g \in A$). Since the set $\mathrm{Der}\, A$ of the derivations of A is clearly a vector subspace of $\mathrm{End}\, A$, it is a Lie algebra as well. So is also the set of *left invariant derivations*

$$\mathscr{L}(G) := \{\delta \in \mathrm{Der}\, A \mid \delta \circ \lambda_x = \lambda_x \circ \delta \ \forall x \in G\},$$

which we call the *Lie algebra* of G.

Since algebraic groups are smooth algebraic varieties, another interesting topic to study is their tangent spaces. Moreover, since left multiplication is a transitive action of G onto itself and, for each $g \in G$, the homomorphism α_g given by $\alpha_g(h) = gh$ is an isomorphism, it is sufficient to study the tangent space at a fixed point. Since groups have a privileged element, the unity e, it is natural to study the tangent space at e $\mathfrak{g} = \mathrm{T}_e\, G$.

Despite the tangent space and the Lie algebra may seem unrelated, one can prove that they are indeed essentially the same.

Theorem 1.25. *Let G be an algebraic group, $\mathfrak{g} = \mathrm{T}_e G$ and $\mathscr{L}(G)$ as above. Define θ :*
$\mathscr{L}(G) \to \mathfrak{g}$ *as*
$$\theta(\delta) : A \to \mathbb{C}, \qquad \theta(\delta)(f) = \delta(f)(e).$$

Then θ is a vector space isomorphism.

Thanks to this result, one can consider $\mathfrak{g}$ to be the Lie algebra of G by defining a Lie bracket

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad [\mathsf{x}, \mathsf{y}] = \theta([\eta(\mathsf{x}), \eta(\mathsf{y})]) \qquad \forall \mathsf{x}, \mathsf{y} \in \mathfrak{g},$$

where on the left-hand side is the newly defined bracket and on the right-hand side is that of $\mathscr{L}(G)$ and $\eta = \theta^{-1}$ is defined as follows:

$$\eta(\mathsf{x}) : A \to A \qquad \eta(\mathsf{x})(f)(x) = \mathsf{x}(\lambda_{x^{-1}} f).$$

Example 1.26 (The Lie algebras of $\mathsf{GL}(n, \mathbb{C})$ and $\mathsf{SL}(n, \mathbb{C})$)**.**

In order work $\mathsf{GL}(n, \mathbb{C})$ as and algebraic variety, we need to consider its embedding in $\mathbb{A}^{n^2+1}$ as

$$\mathsf{GL}(n, \mathbb{C}) = \mathrm{Z}(y \det(x_{ij})_{ij} - 1)$$

where we put coordinates x_{ij} and y onto $\mathbb{A}^{n^2+1}$. To compute its tangent space, we first need to compute the derivative of the determinant:

$$\frac{\partial}{\partial x_{ij}} \det(a_{k\ell})_{k\ell} = \frac{\partial}{\partial x_{ij}} \left(\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{k=1}^n x_{k\sigma(k)} \right) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \frac{\partial}{\partial x_{ij}} \left(\prod_{k=1}^n x_{k\sigma(k)} \right) =$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \sum_{\ell=1}^n \frac{\partial x_{\ell\sigma(\ell)}}{\partial x_{ij}} \prod_{k \neq \ell} x_{k\sigma(k)} = \sum_{\substack{\sigma \in S_n \\ \sigma(i)=j}} \mathrm{sgn}(\sigma) \prod_{k \neq i} x_{k\sigma(k)}$$

which, evaluated at $e = (\delta_{\ell m})_{\ell m}$, gives

$$\frac{\partial}{\partial x_{ij}}\bigg|_e \det(a_{k\ell})_{k\ell} = \sum_{\substack{\sigma \in S_n \\ \sigma(i)=j}} \mathrm{sgn}(\sigma) \prod_{k \neq i} \delta_{k\sigma(k)} = \begin{cases} 1 & () \in \{\sigma \in S_n \mid \sigma(i) = j\} \\ 0 & \text{otherwise} \end{cases} = \delta_{ij},$$

Now, we can use it to compute the tangent space. Let $f := y \det(x_{ij})_{ij} - 1$, then

$$\frac{\partial f}{\partial x_{ij}}\bigg|_e = \left(y \frac{\partial}{\partial x_{ij}} \det(x_{ij})_{ij} \right)\bigg|_e = \delta_{ij}$$

$$\frac{\partial f}{\partial y}\bigg|_e = \det(x_{ij})_{ij}\big|_e = \det I = 1,$$

hence
$$\mathrm{T}_e\, \mathsf{GL}(n,\mathbb{C}) = \mathrm{Z}((x_{ij},y) \cdot (\delta_{ij},1)) = \mathrm{Z}(\mathrm{tr}(x_{ij})_{ij} - y) \cong \mathsf{M}(n \times n, \mathbb{C})$$

via the isomorphism $\mathsf{M}(n \times n, \mathbb{C}) \ni (x_{ij})_{ij} \mapsto (x_{ij}, -\mathrm{tr}(x_{ij})_{ij}) \in \mathrm{Z}(\mathrm{tr}(x_{ij})_{ij} - y)$.

A similar computation leads to the tangent space of $\mathsf{SL}(n,\mathbb{C}) = \mathrm{Z}(\det(x_{ij})_{ij} - 1) \subset \mathsf{M}(n \times n, \mathbb{C})$.
Let $g = \det(x_{ij})_{ij} - 1$, then

$$\left.\frac{\partial g}{\partial x_{ij}}\right|_e = \left.\frac{\partial}{\partial x_{ij}}\right|_e \det(x_{ij})_{ij} = \delta_{ij}$$

and therefore

$$\mathrm{T}_e\, \mathsf{SL}(n,\mathbb{C}) = \mathrm{Z}((x_{ij})_{ij} \cdot (\delta_{ij})_{ij}) = \mathrm{Z}(\mathrm{tr}(x_{ij})_{ij}).$$

Now, we have to compute the Lie bracket of $\mathfrak{gl}(n,\mathbb{C}) = \mathrm{T}_e\, \mathsf{GL}(n,\mathbb{C}) \cong \mathsf{M}(n \times n, \mathbb{C})$. Consider $\mathsf{x}, \mathsf{y} \in \mathfrak{gl}(n,\mathbb{C})$, then by definition

$$[\mathsf{x},\mathsf{y}] = \theta([\eta(\mathsf{x}),\eta(\mathsf{y})]).$$

For all $f \in \mathbb{C}[\mathsf{GL}(n,\mathbb{C})]$, by definition of θ and η,

$$[\mathsf{x},\mathsf{y}](f) = \theta([\eta(\mathsf{x}),\eta(\mathsf{y})])(f)(e) = [\eta(\mathsf{x}),\eta(\mathsf{y})](f)(e) = (\eta(\mathsf{x}) \circ \eta(\mathsf{y}) - \eta(\mathsf{y}) \circ \eta(\mathsf{x}))\,(f)(e) =$$
$$= \eta(\mathsf{x})\big(\mathsf{y}(\lambda_{e^{-1}}f)\big) - \eta(\mathsf{y})\big(\mathsf{x}(\lambda_{e^{-1}}f)\big) = \mathsf{x}(\lambda_{e^{-1}}\mathsf{y}(f)) - \mathsf{y}(\lambda_{e^{-1}}\mathsf{x}(f)) =$$
$$= \mathsf{x}(\mathsf{y}(f)) - \mathsf{y}(\mathsf{x}(f)) = (\mathsf{x}\mathsf{y} - \mathsf{y}\mathsf{x})(f),$$

where the use of "$\circ$" is meant to stress the fact that the product of derivations is the composition. This proves that the Lie bracket of $\mathfrak{gl}(n,\mathbb{C})$ (and therefore of $\mathfrak{sl}(n,\mathbb{C})$) is the matrix commutator.

In the next example, we will introduce a group which will become part of some other interesting examples, the special orthogonal group of 8×8 matrices.

Example 1.27 (The Lie Algebra of $\mathsf{SO}(8)$)**.**
Here we will be considering the group $\mathsf{SO}(8) = \mathsf{SO}(8,Q,\mathbb{C})$, where Q is the matrix

$$Q = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \in \mathsf{M}(8 \times 8, \mathbb{C}).$$

In Example 1.26, we computed the tangent space of $\mathsf{SL}(n,\mathbb{C})$ using the fact that it is the zero-locus of the partial derivatives of the equations describing it as a variety. In this case, we will describe a much simpler process involving the fact that $\mathrm{T}_e\, \mathsf{SO}(8)$ is the set of tangent vectors to all curves contained in $\mathsf{SO}(8)$ and passing through e.

Given a matrix $A \in \mathsf{SO}(8)$, a curve through A is (at least locally) of the form $s \mapsto A + sX$,

for some $X \in \mathsf{M}(8 \times 8, \mathbb{C})$ such that for any s must satisfy

$$(A + sX)^\top Q(A + sX) = Q,$$

and can be rewritten as

$$\underbrace{A^\top QA}_{Q} + s(A^\top QX + X^\top QA) + s^2(X^\top QX) = Q$$

or

$$f := s(A^\top QX + X^\top QA) + s^2(X^\top QX) = 0.$$

Since we are interested in computing the tangent bundle at $e = I_8$, then we shall consider $A = I_8$ and $s = 0$: then the tangent space is the set of all matrices X such that

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f = \left.\left(QX + X^\top Q + s^2(X^\top QX)\right)\right|_{s=0} = QX + X^\top Q.$$

In particular, by considering

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

with $X_{ij} \in \mathsf{M}(4 \times 4, \mathbb{C})$, the equation describing the tangent space can be rewritten as

$$\begin{pmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{pmatrix} + \begin{pmatrix} X_{21}^\top & X_{11}^\top \\ X_{22}^\top & X_{12}^\top \end{pmatrix} = 0 \qquad \Longleftrightarrow \qquad \begin{cases} X_{21} = -X_{21}^\top \\ X_{12} = -X_{12}^\top \\ X_{11} = -X_{22}^\top \end{cases},$$

hence

$$\mathfrak{so}(8) = \left\{ X \in \mathfrak{sl}(8) \mid QX + X^\top Q = 0 \right\} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \middle| \begin{smallmatrix} X_{21}=-X_{21}^\top, \\ X_{12}=-X_{12}^\top, \\ X_{11}=-X_{22}^\top \end{smallmatrix} \right\}.$$

In particular, both X_{21} and X_{12} have dimension 6, while X_{11} has dimension 16 and X_{22} is completely determined by X_{11}, so in total the tangent space must have dimension $16 + 6 + 6 = 28$.

$$\text{these can be chosen} \ \rightarrow \begin{pmatrix} 0 & -* & -* & -* \\ \boxed{*} & 0 & -* & -* \\ \boxed{*} & \boxed{*} & 0 & -* \\ \boxed{*} & \boxed{*} & \boxed{*} & 0 \end{pmatrix} \ \begin{matrix} \leftarrow \ \text{these are completely determined} \\ \text{by the others.} \end{matrix}$$

In conclusion, since $\mathsf{SO}(8)$ is a subspace of $\mathsf{SL}(8)$, the Lie Bracket of $\mathsf{SO}(8)$ is the same of $\mathsf{SL}(8)$.

Chapter 2

Rational Homogeneous Varieties

2.1 Quotients of algebraic groups

The work we did in the last section of the previous chapter leads to the following result, which enables us to give a structure of algebraic variety to the left coset space G/H of an affine algebraic group G with respect to a closed subgroup H.

Theorem 2.1 (Chevalley's Linearization Theorem). *Given an affine algebraic group G and a closed subgroup $H \subset G$, there exists a rational faithful representation V of G such that, considering the associated action of $\mathbb{P}(V^\vee)$, H is the stabiliser G_x of a point $x \in \mathbb{P}(V^\vee)$. In particular, the left coset space G/H, identified with the orbit by G of x in $\mathbb{P}(V^\vee)$, is a smooth open set in a projective variety.*

Remark 2.2. H need not be normal, therefore the coset space G/H may not be a group. Indeed, in the following sections we will be considering mainly cases in which H is not a normal subgroup of G.

Chevalley's Linearization Theorem can be used to prove the following proposition:

Proposition 2.3. *Given an algebraic group G and a Borel subgroup of maximal dimension, the quotient G/B is a projective variety.*

Note that the conjugate of a Borel subgroup is obviously a Borel subgroup. Moreover, by the previous proposition, one can prove that all the Borel subgroups of an affine algebraic group G are conjugated and that the quotients of G/B of G by a Borel subgroup B are isomorphic as projective varieties.

Definition 2.4 (RH variety). A smooth projective variety obtained as a quotient of an

affine connected algebraic group (that is, admitting a transitive algebraic action of an affine connected algebraic group) is called a *rational homogeneous variety* (a *RH variety* for short).

We are now able to state the following corollary, which is a crucial step in the classification of RH varieties.

Corollary 2.5. *The quotient of an affine algebraic group G by a subgroup P is projective if and only if P is parabolic.*

Moreover, since every Borel subgroup B of an affine algebraic group G contains $\mathrm{R}(G)$, while studying the classification of RH varieties we may reduce to the case of quotients of semisimple groups:

Corollary 2.6. *Every rational homogeneous variety X admits a transitive algebraic action of a semisimple group G, and a surjective morphism $G/B \twoheadrightarrow X$, where $B \subset G$ is a Borel subgroup.*

2.2   Flag Manifolds

Let V be a $\mathbb{C}$-vector space of finite dimension $n+1$, then for each increasing sequence of positive integers $d_1 < \cdots < d_k < \dim V - 1$ we can form the variety of *flags*

$$\mathbb{F}(d_1, \ldots, d_k; \mathbb{P}(V)) = \left\{ (\Lambda_1, \ldots, \Lambda_k) \mid \dim \Lambda_i = d_i \text{ and } \Lambda_i \subset \Lambda_{i+1} \ \forall i = 1, \ldots, k-1 \right\}$$
$$\subset \mathbb{G}(d_1, \mathbb{P}(V)) \times \cdots \times \mathbb{G}(d_k, \mathbb{P}(V)).$$

In particular, if $d_1, \ldots, d_k = 0, 1, \ldots, \dim V - 2$, the corresponding flag manifold is known as *complete flag*. This is clearly a subvariety of the product of Grassmanians: in fact if we write $\Lambda_i = [u_1 \wedge \cdots \wedge u_{d_i}]$ and $\Lambda_{i+1} = [v_1 \wedge \cdots \wedge v_{d_{i+1}}]$ then the condition $\Lambda_i \subset \Lambda_{i+1}$ translates to

$$(v_1 \wedge \cdots \wedge v_{d_{i+1}}) \wedge u_j = 0 \qquad \forall j = 1, \ldots, d_i,$$

which is clearly a closed condition.

Note that a similar definition can be stated regarding the vector spaces associated to each linear subspace, as the map

$$(V_1, \ldots, V_k) \mapsto (\mathbb{P}(V_1), \ldots, \mathbb{P}(V_k))$$

defines a bijection between the *vector* and the *projective* version of the flag:

$$\{(V_1, \ldots, V_k) \mid \dim V_i = d_i + 1, V_i \subset V_i + 1\} \leftrightarrow \{(\Lambda_1, \ldots, \Lambda_k) \mid \dim V_i = d_i, \Lambda_i \subset \Lambda_i + 1\}.$$

Given this bijection we might use both definitions, choosing the one that better fits our needs in each situation.

Remark 2.7. It is possible to define a natural morphism among flag varieties, called the *natural projection*: by considering a subset

$$\{d'_1, \ldots, d'_h\} \subsetneq \{d_1, \ldots, d_k\}$$

such that $d'_1 < d'_2 < \cdots < d'_h$ and $d_1 < d_2 < \cdots < d_k$, then there exists a surjective morphism of varieties

$$\begin{aligned} p: \quad \mathbb{F}(d_1, \ldots, d_k, \mathbb{P}(V)) &\longrightarrow \quad \mathbb{F}(d'_1, \ldots, d'_k, \mathbb{P}(V)) \\ (\Lambda_1, \ldots, \Lambda_k) &\longmapsto \quad (\Lambda_{j_1}, \ldots, \Lambda_{j_k}) \end{aligned}$$

where j_i is such that $d_{j_i} = d'_i$. One can easily prove that this morphism is *equivariant* with respect to the action of $\mathsf{GL}(V)$ (i.e. it commutes with the automorphisms given by the elements of $\mathsf{GL}(V)$).

We have introduced the flag manifolds because they are, along with the Grassmanians, a notable example of RH varieties, as the next proposition states.

Proposition 2.8. *The action of* $\mathsf{PGL}(V)$ *on* $\mathbb{F}(d_1, \ldots, d_k; \mathbb{P}(V))$ *defined by*

$$\omega \cdot (\Lambda_1, \ldots, \Lambda_k) := (\omega(\Lambda_1), \ldots, \omega(\Lambda_k))$$

is algebraic and transitive.

Proof. The action is clearly algebraic. Consider $(\Lambda_1, \ldots, \Lambda_k), (\Sigma_1, \ldots, \Sigma_k) \in \mathbb{F}(d_1, \ldots, d_k)$, and, for each i, fix W_i such that $\Lambda_i = \mathbb{P}(W_i)$ and consider a basis $u_0, \ldots, u_{d_1}$ of W_i. Then, complete it to a basis of W_i for each $i = 2, \ldots, k$ and, finally, to a basis of the whole vector space V. By repeating the same process for each Σ_i, we obtain another basis v_i of V. Those two basis induce an automorphism $T \in \mathsf{GL}(V)$, which induces a map $\omega = [T] \in \mathsf{PGL}(V)$ which, by definition, maps Λ_i to Σ_i. □

Remark 2.9. One can observe that the restriction to SL of the quotient map $\mathsf{GL}(n+1, \mathbb{C}) \twoheadrightarrow \mathsf{PGL}(n+1, \mathbb{C})$ is surjective. This means that the same action described in Proposition 2.8 can be interpreted as an action of $\mathsf{SL}(n+1, \mathbb{C})$ on $\mathbb{F}(d_1, \ldots, d_k, \mathbb{P}(V))$: this means $\mathbb{F}(d_1, \ldots, d_k, \mathbb{P}(V))$ can be interpreted as a quotient of the semisimple group $\mathsf{SL}(n+1, \mathbb{C})$.

In Example 1.20 we noted that (with respect to a certain basis) the subgroup of upper triangular matrices $\mathsf{T}(V) \subset \mathsf{GL}(V)$ is a Borel subgroup and that the groups of block triangular matrices are parabolic. The same holds true by considering their corresponding subgroups of $\mathsf{PGL}(V)$. Now, equipped with the notion of flags and with the theory we have introduced, we

can describe their quotients. Consider the following upper block triangular matrix

$$
M = \begin{bmatrix}
\boxed{M_{11}} & M_{12} & \cdots & M_{1k} \\
0 & \boxed{M_{22}} & \cdots & M_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \boxed{M_{k+1,k+1}}
\end{bmatrix}
$$

where $M_{ij} \in \mathsf{M}_{a_i \times a_j}(\mathbb{C})$, where the a_i are recursively defined such that $\sum_{i=1}^{j} a_i = d_j + 1$ (where $d_{k+1} = n+1$) and call $P = P(d_1, \ldots, d_k) \subset \mathsf{PGL}(n+1, \mathbb{C})$ the group of those matrices. Then the action of P on $\mathbb{P}(V)$ has the property of fixing every linear subspace generated by $\{[e_1], \ldots, [e_{d_j}]\}$ for each $j = 1, \ldots, k$. Hence, we are able to describe the quotient

$$
\mathsf{PGL}(n+1, \mathbb{C})/P(d_1, \ldots, d_k) = \mathbb{F}(d_1, \ldots, d_k, \mathbb{P}(V)).
$$

In particular, if $\{d_1, \ldots, d_k\} = \{1, \ldots, n-1\}$ then $P(d_1, \ldots, d_k) = \mathsf{T}(V)$ and the quotient is the complete flag $\mathsf{PGL}(n+1, \mathbb{C})/\mathsf{T}(V) = \mathbb{F}(1, \ldots, n-1, \mathbb{P}(V))$.

This same idea can be generalised to every semisimple group, as the following definition states.

Definition 2.10 (Generalised complete flag). Given a semisimple group G and a Borel subgroup $B \subset G$, the variety G/B is called the (*generalised*) *complete flag variety associated to G*.

Despite $\mathsf{PGL}(n+1, \mathbb{C})$ not being a semisimple group, this definition is compatible with the previous theory by Remark 2.9.

The same process described above for flags can be repeated with quotients of PSO and PSp groups, obtaining the so-called orthogonal flags and Lagrangian (or isotropic) flags.

Example 2.11.

To clarify what we just described, let's discuss an example of a flag in $\mathbb{P}^3$. The parabolic subgroups of $\mathsf{PGL}(4, \mathbb{C})$ p'resented above are

$$
P(0) = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \right\} \quad
P(1) = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\} \quad
P(0,1) = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\}
$$

$$
P(2) = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\} \quad
P(0,2) = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\} \quad
P(1,2) = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}.
$$
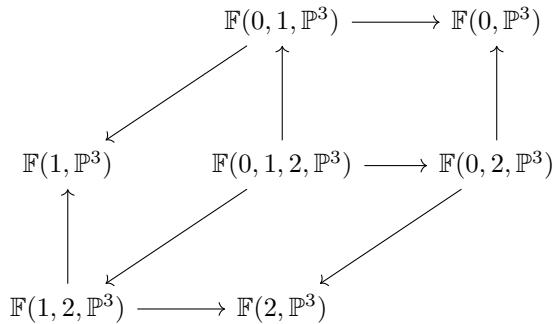
while the Borel subgroup is

$$B = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}$$

and their associated flags are

- $\mathbb{F}(0, \mathbb{P}^4)$: the set of points of $\mathbb{P}^4$, which is isomorphic to $\mathbb{P}^4$;

- $\mathbb{F}(1, \mathbb{P}^4)$: the set of all linear subspaces of dimension 1 in $\mathbb{P}^4$, which coincides with $\mathbb{G}(1, \mathbb{P}^4)$;

- $\mathbb{F}(2, \mathbb{P}^4)$: the set of all linear subspaces of dimension 2 in $\mathbb{P}^4$, which coincides with $\mathbb{G}(2, \mathbb{P}^4)$ and is isomorphic to $\mathbb{P}^{4\vee}$;

- $\mathbb{F}(0, 1, \mathbb{P}^4)$: the set of pairs (p, Λ) such that $p \in \Lambda \subset \mathbb{P}^4$, where p is a point and Λ is a linear subspace of dimension 1;

- $\mathbb{F}(0, 2, \mathbb{P}^4)$: the set of pairs (p, Σ) such that $p \in \Sigma \subset \mathbb{P}^4$, where p is a point and Σ is a linear subspace of dimension 2;

- $\mathbb{F}(1, 2, \mathbb{P}^4)$: the set of pairs (Λ, Σ) such that $\Lambda \subset \Sigma \subset \mathbb{P}^4$, where Λ is a linear subspace of dimension 1 and Σ is a linear subspace of dimension 2;

- $\mathbb{F}(0, 1, 2, \mathbb{P}^4)$, the complete flag (associated to B): the set of triples (p, Λ, Σ), such that $p \in \Lambda \subset \Sigma \subset \mathbb{P}^4$, where p is a point, Λ a linear subspace of dimension 1 and Σ a linear subspace of dimension 2.

Considering the projection maps described in Remark 2.7, these flags can be arranged as follows:

A crucial fact one can prove is that every RH variety is isomorphic to a generalised flag, where G can be seen as a matrix group.

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2.3 Isotropic flags

Consider the group $\mathsf{SO}(2n) = \mathsf{SO}(2n, \Omega, \mathbb{C})$, where Ω is the matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in \mathsf{M}(2n \times 2n, \mathbb{C})$$

as in Example 1.27. Then the matrix Ω defines a $(2n-2)$-dimensional smooth quadric $\mathcal{Q}^{2n-2} = Z(\sigma) \subset \mathbb{P}^{2n}$, where $\sigma(\mathbf{x}) = \mathbf{x}^\top \Omega \mathbf{x}$, described by the equation

$$x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n} = 0.$$

In order to study its properties, we need to introduce some concepts.

Definition 2.12 (Isotropic subspace)**.** A vector subspace $W \subset \mathbb{C}^{2n}$ is *isotropic with respect to* Ω if

$$v_1^\top \Omega v_2 = 0 \qquad \forall v_1, v_2 \in W.$$

Note that it is sufficient to prove so for a basis of W.

Definition 2.13 (Polar subspace)**.** Given a vector subspace $W \subset \mathbb{C}^{2n}$, its *polar subspace with respect to* Ω is

$$W_\Omega^\perp := \left\{ w \in \mathbb{C}^{2n} \mid v^\perp \Omega w = 0 \quad \forall v \in W \right\}.$$

Remark 2.14. If W is isotropic, it is clear by definition that $W \subset W_\Omega^\perp$. Moreover, since $W_\Omega^\perp = (\tilde{W})^\perp$, where $\tilde{W} = \{\Omega v \mid v \in W\}$, and $\det \Omega \neq 0$, it follows that

$$\dim W_\Omega^\perp = \dim(\tilde{W})^\perp = 2n - \dim \tilde{W} = 2n - \dim W.$$

Lemma 2.15. *Let $W \subset \mathbb{C}^{2n}$ be a vector subspace. Then the following statements are equivalent:*

(a) W is isotropic with respect to Ω;

(b) $W \subset W_\Omega^\perp$

(c) $\mathbb{P}(W) \subset \mathcal{Q}^{2n-2}$

Proof. The equivalence between (a) and (b) is obvious, we will prove that between (a) and (c). Consider $[v_1], [v_2] \in \mathbb{P}(W)$: then by definition of vector subspace also $[v_1 + v_2] \in \mathbb{P}(W) \subset \mathcal{Q}^{2n-2}$, hence

$$0 = (v_1 + v_2)^\top \Omega (v_1 + v_2) = v_1^\top \Omega v_1 + v_1^\top \Omega v_2 + v_2^\top \Omega v_1 + v_2^\top \Omega v_2 = 2 v_1^\top \Omega v_2$$

by symmetry of Ω and by the fact that $[v_1], [v_2] \in \mathbb{P}(W)$. Now, since $\operatorname{char} \mathbb{C} \neq 2$, this leads to the fact that $v_1^\top \Omega v_2 = 0$, *i.e.* W is isotropic: this proves that (c) implies (a). The converse is trivial. $\qquad \square$

Remark 2.16. An isotropic subspace $W \subset \mathbb{C}^{2n}$ can be at most 4-dimensional: this is because it must be a subset of its polar $W_\Omega^\perp$, which has dimension $2n - \dim W$:

$$\dim W \leq 2n - \dim W \quad \Longleftrightarrow \quad 2 \dim W \leq 2n \quad \Longleftrightarrow \quad \dim W \leq n.$$

Definition 2.17 (Isotropic Flag)**.** Given a symmetric matrix $\Omega \in \mathsf{M}_{2n \times 2n}(\mathbb{C})$, an *orthogonal flag* is a variety

$$\mathbb{F}(d_1, \ldots, d_r, \mathcal{Q}^{2n-2}) := \{(\Lambda_1, \ldots, \Lambda_r) \in \mathbb{F}(d_1, \ldots, d_r, \mathbb{P}^7) \mid \Lambda_r \subset \mathcal{Q}^{2n-2}\}$$
$$\subset \mathbb{F}(d_1, \ldots, d_r, \mathbb{P}(\mathbb{C}^m)).$$

If $\{d_1, \ldots, d_r\} = \{1, \ldots, n\}$, then the flag is called *complete*, and we will write

$$\mathbb{F}(1, \ldots, n, \mathcal{Q}^{2n-2}) =: \mathbb{F}(\Omega).$$

Note that we don't need to express other conditions about $\Lambda_1, \ldots, \Lambda_{r-1}$ being subsets of $\mathcal{Q}^6$: this follows directly by the definition of flags and the fact that $\Lambda_r \subset \mathcal{Q}^6$. The same applies if we consider the analogue definition for vector spaces: for each $i = 1, \ldots, r-1$, since the polar subspaces flip inclusions,

$$W_i \subset W_r \subset (W_r)_\Omega^\perp \subset (W_1)_\Omega^\perp.$$

This means that, as we did before, we can work with both the *vector* and the *projective* version of these flags: in this context, we will prefer the former. To distinguish them while working, we will write $\mathbb{F}$ for the projective flag and F for the vector flag. In particular, if we consider the latter, we can extend the chain of subspaces of the complete flag to include some of higher dimension: in particular we will have

$$V_1 \subsetneq \cdots \subsetneq V_n \subsetneq V_{n+1} \subsetneq \cdots \subsetneq V_{2n-1} \subsetneq \mathbb{C}^{2n}$$

where $V_i^\perp = V_{2n-i}^\perp$. Note that in this "extension" we make no choice, so the flag is completely determined by the first n elements $V_1, \ldots, V_n$.

Remark 2.18. Consider an isotropic vector subspace $W \subset \mathbb{C}^{2n}$. Then $W \subset W_\Omega^\perp$ and we can consider a basis $\{v_1, \ldots, v_k\}$ of W and complete it to a basis $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_{2n-k}\}$

of $W^\perp$. Then we have that $\sigma|_{V_\Omega^\perp}$ can be written with respect to β as

$$\Omega_\beta := \mathrm{M}_\beta\big(\sigma|_{V_\Omega^\perp}\big) = \left(\begin{array}{c|c} \mathbf{0}_{k \times k} & \mathbf{0}_{k \times (2n-2k)} \\ \hline \mathbf{0}_{(2n-2k) \times k} & \Omega'_\beta \end{array}\right),$$

where $\Omega \in \mathrm{M}_{(2n-2k) \times (2n-2k)}(\mathbb{C})$ as we must have $(V_\Omega^\perp)_\Omega^\perp = V$, and we know that $v_i \Omega v_j = 0$ for each $i = 1, \ldots, k$ and $j = 1, \ldots, 2n - k$. Moreover, the quadric associated to Ω_β is a quadric cone of vertex $\mathbb{P}(W)$ on the quadric associated to Ω'_β.

Lemma 2.19. *Every isotropic vector $v_1 \neq 0$ is contained in an isotropic complete flag*

Proof. We proceed by induction on n: if $n = 1$, the proof is trivial. Consider now the vector subspace $\langle v_i \rangle_\Omega^\perp$: then the restriction of σ to $\langle v_i \rangle_\Omega^\perp$ has rank $2n - 1$. If we write $\langle v_1 \rangle_\Omega^\perp = \langle v_1 \rangle \oplus W$, then we can repeat the same process and obtain that $\mathrm{rk}\, \sigma|_W = 2(n-1)$: therefore, by induction there exists an isotropic complete flag in W

$$\langle v_2 \rangle \subsetneq \langle v_2, v_3 \rangle \subsetneq \cdots \subsetneq \langle v_2, v_3 \rangle_\Omega^\perp \subsetneq \langle v_2 \rangle_\Omega^\perp \subsetneq W.$$

We can use it to build an isotropic flag of $\mathbb{C}^{2n}$ containing v_1 by taking the direct sum with $\langle v_1 \rangle$:

$$\langle v_1 \rangle \subsetneq \langle v_1, v_2 \rangle \subsetneq \langle v_1, v_2, v_3 \rangle \subsetneq \cdots \subsetneq \langle v_1, v_2, v_3 \rangle_\Omega^\perp \subsetneq \langle v_1, v_2 \rangle_\Omega^\perp \subsetneq W \oplus \langle v_1 \rangle = \langle v_1 \rangle_\Omega^\perp \subsetneq \mathbb{C}^{2n} \quad \square$$

Lemma 2.20. *Let $V_1 \subset \cdots \subset V_{2n-1}$ be an isotropic flag with respect to Ω. Then there exists a basis $\beta = \{v_1, \ldots, v_{2n}\}$ of $\mathbb{C}^{2n}$ such that*

(1) $V_i = \langle v_1, \ldots, v_i \rangle$ for each $i = 1, \ldots, k$;

(2) the matrix representing σ with respect to β is

$$\mathrm{M}_\beta(\sigma) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Proof. Consider $\beta' = \{v'_1, \ldots, v'_k, v'_{k+1}, \ldots, v'_{2k}\}$ such that $V_i = \langle v'_1, \ldots, v'_i \rangle$ for each $i = 1, \ldots, k$. Then we know that for each $1 \leq i < j \leq 2n$, $v_i \in V_i \subset V_j \subset (V_j)_\Omega^\perp$, so in particular we must have that $v_i \Omega v_j = 0$: in the matrix representing σ with respect to the basis β', this translates in the fact that the entry in position (i, j) must be zero for each $i < j$:

$$\mathrm{M}_{\beta'}(\sigma) = \left(\begin{array}{c|c} \begin{smallmatrix} 0 \\ \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ * \\ * \end{smallmatrix} \\ \hline & \begin{smallmatrix} * \end{smallmatrix} \end{array}\right) = \left(\begin{array}{c|c} 0 & A \\ \hline A^\top & B \end{array}\right),$$

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where A is an anti-triangular nondegenerate matrix and B is symmetric. Then we want to find a matrix $P \in \mathsf{GL}(2n, \mathbb{C})$ such that $P v_i = v_i$ for each $i = 1, \ldots, k$ and $P^\top \mathrm{M}_{\beta'}(\sigma) P = \Omega$. The first condition can be achieved by imposing

$$P = \left(\begin{array}{c|c} I & C \\ \hline 0 & D \end{array} \right),$$

with $C, D \in \mathsf{M}_{n \times n}(\mathbb{C})$. The latter follows by the next computation:

$$
\begin{aligned}
P^\top \mathrm{M}_{\beta'}(\sigma) P &= \left(\begin{array}{c|c} I & 0 \\ \hline C^\top & D^\top \end{array} \right) \left(\begin{array}{c|c} 0 & A \\ \hline A^\top & B \end{array} \right) \left(\begin{array}{c|c} I & C \\ \hline 0 & D \end{array} \right) \\
&= \left(\begin{array}{c|c} I & 0 \\ \hline C^\top & D^\top \end{array} \right) \left(\begin{array}{c|c} 0 & AD \\ \hline A^\top & A^\top C + BD \end{array} \right) \\
&= \left(\begin{array}{c|c} 0 & AD \\ \hline D^\top A^\top & C^\top AD + D^\top A^\top C + D^\top BD \end{array} \right)
\end{aligned}
$$

which, if we choose $D = A^{-1}$ and $C = C^\top = -\frac{1}{2}((A^{-1})^\top B A^{-1})$, reduces to

$$P^\top \mathrm{M}_{\beta'}(\sigma) P = \left(\begin{array}{c|c} 0 & I \\ \hline I & C^\top + C + (A^{-1})^\top B (A^{-1}) \end{array} \right) = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array} \right) = \Omega.$$

This choice of P defines a basis β' meeting the requirements of the Lemma, hence the proof is concluded. $\square$

Corollary 2.21. *Let* $\mathrm{F}(\Omega)$ *be the algebraic variety parametrising the* Ω*-isotropic flags on* $\mathbb{C}^{2n}$. *Then the group* $\mathsf{O}(2n, \Omega, \mathbb{C}) = \{P \in \mathsf{GL}(2n, \mathbb{C}) \mid P^\top \Omega P = \Omega\}$ *acts transitively on* $\mathrm{F}(\Omega)$.

Proof. Consider any isotropic flag $V_1 \subset \ldots, V_{2n-1}$, then by the previous Lemma there exists a basis $\beta = \{v_1, \ldots, v_{2n}\}$ such that $V_i = \langle v_1, \ldots, v_i \rangle$ for each $i = 1, \ldots, k$. If we define $\varphi \in \mathrm{Aut}\, \mathbb{C}^{2n}$ as $\varphi(e_i) = v_i$, then we have that $\varphi(V_1) \subset \cdots \subset \varphi(V_{2n-1})$ is the canonical flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{2n-1} \rangle$. Since by (2) of the same Lemma the matrix P representing φ satisfies $P^\top \Omega P = \Omega$, the proof is concluded. $\square$

2.3.1 Special isotropic flags

Until now, we have solely considered orthogonal matrices. However, our goal is to use only those with positive determinant: the next part will be devoted to the description of what changes if we restrict to this particular case. The determinant is indeed a polynomial map $\mathsf{O}(2n, \Omega, \mathbb{C}) \to \mathbb{C}^*$ and a group morphism, which has image $\{\pm 1\}$, hence its preimages define two connected components: the one with positive determinant is also a subgroup and is known as $\mathsf{SO}(2n, \Omega, \mathbb{C})$. This structure of $\mathsf{O}(2n, \Omega, \mathbb{C})$ translates on that of $\mathrm{F}(\Omega)$ as an algebraic action with two orbits: the one containing the canonical flag is called $\mathrm{SF}(\Omega) := \mathsf{SO}(2n, \Omega, \mathbb{C}) \cdot (\text{canonical flag})$.

> **Definition 2.22** (Special isotropic flag). The *special isotropic flag* is the orbit of the canonical flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{n-1} \rangle$ of the action of $\mathsf{SO}(2n, \Omega, \mathbb{C})$ on $\mathrm{F}(\Omega)$.

Since $\mathsf{SO}(2n, \Omega, \mathbb{C})$ defines two orbits in $\mathrm{F}(\Omega)$, we can write

$$\mathrm{F}(\Omega) = \mathrm{SF}(\Omega) \sqcup E \cdot \mathrm{SF}(\Omega),$$

where $E \in \mathsf{O}(2n, \Omega, \mathbb{C})$ has negative determinant.

Consider now a basis $\beta = \{v_1, \ldots, v_{n+1}\}$ of V_{n+1}, where $\{v_1, \ldots, v_{n-1}\}$ is a basis of V_{n-1} and $\{v_1, \ldots, v_n\}$ is a basis of V_n. Then, as we noted before,

$$\mathrm{M}_\beta\big(\sigma|_{V_{k+1}}\big) = \left(\begin{array}{c|c} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times 2} \\ \hline \mathbf{0}_{2 \times n} & A \end{array} \right),$$
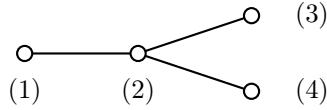
where $A \in \mathsf{M}_{2 \times 2}(\mathbb{C})$ is a lower triangular matrix. If we consider the algebraic variety associated to A, we get two points in $\mathbb{P}^1$, hence $\mathcal{Q} \cap \mathbb{P}\big((V_{n-1})^{\perp}_{\mathcal{Q}}\big)$ is a cone on those same two points of vertex $\mathbb{P}(V_{n-1})$. Now, a cone on two points is clearly the union of two linear spaces of dimension $n-1$, one of which is $\mathbb{P}(V_n)$. Hence, we conclude that each isotropic vector space of dimension $n-1$ is contained in exactly two isotropic vector spaces of dimension n: in particular, each partial flag

$$V_1 \subset \cdots \subset V_{n-1}$$
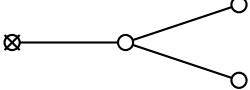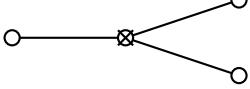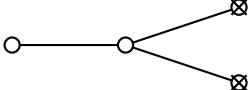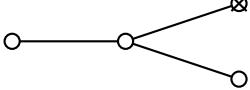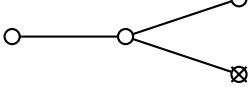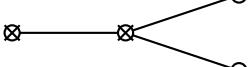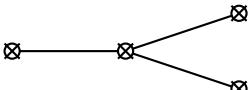
can be completed in two ways

$$V_1 \subset \cdots \subset V_{n-1} \overset{\subset \, V_n \, \subset}{\underset{\subset \, V_n' \, \subset}{}} V_{n-1}^{\perp} \subset \cdots \subset V_1^{\perp}. \tag{2.1}$$

Dynkin diagrams. The situation described above can be represented, in the case $n = 4$, with a diagram, where every node represents a family of subspaces.
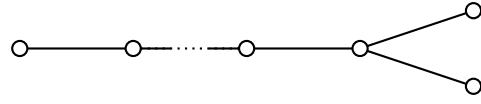
Here, node (1) represents the points ($\mathbb{P}^0$) of the quadric, node (2) the lines ($\mathbb{P}^1$), and the nodes (3) and (4) represent the two families of $\mathbb{P}^3$s. The $\mathbb{P}^2$s are given by the intersection of two $\mathbb{P}^3$s, one for each family. This diagram can be used for describing a Grassmanian (or a flag) by marking the dots: a marked node represents the fact that the associated flag contains elements of that family. For example:
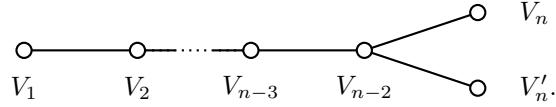
	$\mathbb{G}(0, \mathcal{Q}^6)$
	$\mathbb{G}(1, \mathcal{Q}^6)$
	$\mathbb{G}(2, \mathcal{Q}^6)$
	$\mathsf{SO}(8) \cdot \mathbb{P}(\langle e_0, e_1, e_2, e_3 \rangle)$
	$\mathsf{SO}(8) \cdot \mathbb{P}(\langle e_0, e_1, e_2, e_7 \rangle)$
	$\mathbb{F}(0, 1, \mathcal{Q}^6)$
	Complete flag of $\mathbb{Q}^6$

This type of diagram, known as *Dynkin diagram*, plays a crucial role not only in describing flag manifolds, but also it is a key element in the description and classification of semisimple Lie

algebras. The one listed above is that of a Lie algebra of type D_n (in this case $n = 4$), which in general has the following shape.

If we compare this diagram with (2.1), we can easily understand what each node represents:

Note that no node is associated to V_{k-1}, as it can be retrieved by computing the intersection of V_k and V'_k.

Chapter 3

Nestings

3.1 Introduction to nesting maps

In [DCR03], De Concini and Reichstein introduced the concept of *nesting maps* as morphisms between two Grassmanians $n : \mathbb{G}(k, \mathbb{P}^n) \to \mathbb{G}(r, \mathbb{P}^n)$, where $k < r$, and such that $\Lambda \subset n([\Lambda])$ for each $\Lambda \in \mathbb{G}(k, \mathbb{P}^n)$. In the same paper, they showed that maps of this kind can exist only if n is odd and $\{k, r\} = \{0, n-1\}$, where a point $[v]$ is mapped to $\mathbb{P}(v_\Omega^\perp)$, where $v_\Omega^\perp$ is the orthogonal complement to v with respect to an alternating form.

A more general approach to nesting maps is presented in [MOSC20], where the definition is generalised to the case of RH varieties. Consider the flag $\mathbb{F}(k, r, \mathbb{P}^n)$, then two natural projections to the Grassmanians of dimension k and r are defined, thus the existence of the nesting map n is equivalent to that of a section of the projection $\pi_k : \mathbb{F}(k, r, \mathbb{P}^n) \to \mathbb{G}(k, \mathbb{P}^n)$:

$$
\begin{array}{ccc}
 & \mathbb{F}(k, r, \mathbb{P}^n) & \\
 \overset{s}{\nearrow} \quad \swarrow \scriptstyle{\pi_k} & & \searrow \scriptstyle{\pi_r} \\
\mathbb{G}(k, \mathbb{P}^n) & \overset{n}{\dashrightarrow} & \mathbb{G}(r, \mathbb{P}^n)
\end{array}
$$

then we can use the other projection to define $n = \pi_r \circ s$. This same idea can be generalised to any kind of RH variety. For example, given $\{d'_1, \ldots, d'_h\} \subsetneq \{d_1, \ldots, d_k\} \subset \{0, \ldots, n-1\}$, we can study the existence of a section of the natural projection

$$
\mathbb{F}(d_1, \ldots, d_k, \mathbb{P}^n) \longrightarrow\!\!\!\rightarrow \mathbb{F}(d'_1, \ldots, d'_h, \mathbb{P}^n).
$$

In this section, we will use the term *nesting* to refer to this type of section, rather than the first definition, since it is more practical to address the problem from this point of view. Since RH varieties can be classified through marked Dynkin diagrams, the problem can be rewritten making use of this notation. Given a semisimple group G and a parabolic subgroup P, the quotient G/P can be described by marking a subset I of the set D of nodes of the Dynkin diagram $\mathcal{D}$ of G, and will be denoted by $\mathcal{D}(I)$. Moreover, the parabolic subgroups $P' \subset G$ contained in P correspond
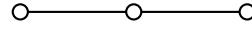
to the subsets I' of D containing I (as seen in §2.3), so a nesting map can be represented as a section of the projection

$$\mathcal{D}(I') \longrightarrow\!\!\!\!\!\rightarrow \mathcal{D}(I).$$

By writing $I' = I \sqcup J$, we say that the section is a *nesting of type* $(\mathcal{D}, I, J)$.

Example 3.1.

- In Example 2.11, we considered the flags of $\mathbb{P}^3$, which are described by the Dynkin diagram A_3.
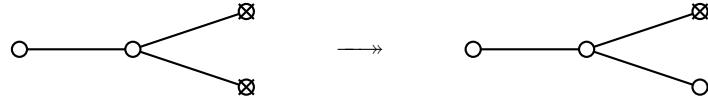

  An example of projection map is $\mathbb{F}(0, 2, \mathbb{P}^3) \to \mathbb{F}(0, \mathbb{P}^3)$, which is described by the Dynkin diagrams

  and a section of this map is a nesting of type $(\mathsf{A}_3, 1, 3)$.

- In §2.3, we described the case of a Dynkin diagram of type D_4, where the projection map $\mathbb{F}(3, 4, \mathcal{Q}^6) \twoheadrightarrow \mathbb{F}(3, \mathcal{Q}^6)$ is described by

  and a section of this map is a nesting of type $(\mathsf{D}_4, 3, 4)$.

Despite not seeming such a strong condition, the existence of nesting maps is not a common property in RH varieties: in [MOSC20], the authors proved the following theorem about existence of nesting maps of type $(\mathcal{D}, I, J)$ for Dynkin diagrams of classical type.

Theorem 3.2. *Let G be a simple algebraic group whose associated Dynkin diagram $\mathcal{D}$ is of classical type, and let I, J be two disjoint non-empty sets of nodes of $\mathcal{D}$ such that $(\mathcal{D}, I, J)$ admits a nesting. Then $(\mathcal{D}, I, J)$ is isomorphic to one of the following:*

$$(\mathsf{A}_{2m-1}, 1, 2m-1) \quad m \geq 2, \qquad (\mathsf{B}_3, 1, 3), \qquad (\mathsf{D}_n, n-1, n) \quad n \geq 4.$$

The following sections are devoted to presenting some examples of these three families of nesting maps.

3.2 Nestings of type $(\mathsf{A}_{2m-1}, 1, 2m-1)$

In this section we will describe the simplest case of nestings, the one described in [DCR03], where classical flags are involved. We will be considering the projection

$$\pi : \mathbb{F}(I \sqcup J, \mathbb{P}^{2m-1}) \to \mathbb{F}(I, \mathbb{P}^n)$$

and constructing a section in the case of n odd and $I = \{1\}$, $J = \{2m-1\}$. Fix a skew-symmetric matrix $\Omega \in \mathsf{GL}(2m-1, \mathbb{C})$ and consider the map

$$\sigma_\Omega : \quad \mathbb{F}(0, \mathbb{P}^{2m-1}) \quad \longrightarrow \quad \mathbb{F}(0, 2, \mathbb{P}^{2m-1})$$
$$\mathbf{p} \quad \longmapsto \quad (\mathbf{p}, \mathbf{p}_\Omega^\perp)$$
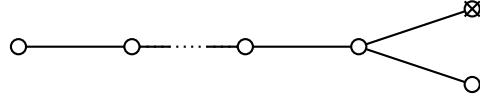
where, if $\mathbf{p} = [v]$, we define

$$\mathbf{p}_\Omega^\perp := \{[w] \mid v^\top \Omega w = 0\} \in \mathbb{G}(2, \mathbb{P}^{2m-1}),$$

then clearly σ_Ω is a morphism and $\pi \circ \sigma_\Omega = \mathrm{Id}_{\mathbb{F}(0, \mathbb{P}^{2m-1})}$, hence it is a nesting map of type $(\mathsf{A}_{2m-1}, 1, 2m-1)$. One can prove (*cf.* [MOSC20, Proposition 4.2]) that every nesting of this type must be structured in this fashion.

3.3 Nestings of type $(\mathsf{D}_n, n-1, n)$

Consider the Dynkin diagram $\mathsf{D}_n(n-1)$,

which describes one of the two families of linear subspaces Λ^{n-1} of dimension $n-1$ inside the quadric $\mathcal{Q}^{2n-2}$. Then we want to describe a way to associate to each Λ^{n-1} a subspace Σ^{n-2} of dimension $n-2$ such that

$$\Sigma^{n-2} \subset \Lambda^{n-1}.$$

Since we know that the intersection of Λ^{n-1} with a general hyperplane has dimension $n-2$, this idea might lead to a solution to our problem: indeed, if we consider a non-isotropic vector $v \in \mathbb{C}^{2n}$ (*i.e.* a point $\mathbf{p} = [v] \notin \mathcal{Q}^{2n-2}$), then it corresponds to an hyperplane $H_v := \{[w] \in \mathbb{P}^{2n} \mid v^\top \Omega w = 0\}$ non-tangent to the quadric, which has the properties we need to construct the map

$$\sigma_v : \quad \mathsf{D}_n(n-1) \quad \longrightarrow \quad \mathsf{D}_n(n-1, n)$$
$$\Lambda^{n-1} \quad \longmapsto \quad \Lambda^{n-1} \cap H_v,$$

which can be proved to be a morphism of algebraic varieties.

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3.4 Nestings of type $(\mathbf{B}_3, 1, 3)$

3.4.1 The octonionic structure of $\mathbb{P}^7$

In the previous cases, we used the information we had about the spaces we were working in to give them a richer structure, we then used to build a nesting. In this case, the means by which we will be giving a richer structure to $\mathbb{P}^7$ are slightly different, and make use of the *complexified octonions* $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$: this $\mathbb{C}$-$*$-algebra is built the same way as the classic octonions (by iterating the Cayley–Dickson construction) but starting with $\mathbb{C}$ instead of $\mathbb{R}$. Another (and maybe more useful) way of describing them is as the $\mathbb{C}$-$*$-algebra
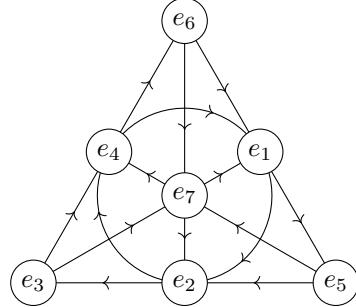
$$\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} := \bigoplus_{i=0}^{7} \mathbb{C}e_i,$$

endowed with the product described by the relations

$$e_i^2 = -1, \quad e_i \cdot e_j = -e_j \cdot e_i \qquad \forall i,j = 1,\ldots,7$$

$$e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_4 = e_5, \quad e_1 \cdot e_6 = e_7, \quad e_2 \cdot e_4 = -e_6, \quad e_2 \cdot e_5 = e_7, \quad e_3 \cdot e_4 = e_7, \quad e_3 \cdot e_5 = e_6,$$

which can be described with the following diagram, known as the Fano plane (*cf.* [Bae02, §2.1]).

Here, we will sloppily refer to $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ as simply $\mathbb{O}$.

We must also recall that the octonions have a natural definition of conjugation, given by $(x_1, x_2, \ldots, x_7)^* = (x_1, -x_2, \ldots, -x_7)$, and a norm $N : \mathbb{O} \to \mathbb{C}$ given by $N(x) = xx^* = x^*x = \sum_{i=0}^{7} x_i^2$. Moreover, an element $x \in \mathbb{O}$ is invertible if and only if $N(x) \neq 0$, with inverse $x^{-1} = x^*/N(x)$.

Using these facts, we can define two special quadrics in $\mathbb{P}^7$,

$$\mathcal{Q}^6 := \{[x] \mid x \in \mathbb{O}, x^*x = 0\} \qquad \text{and} \qquad \mathcal{Q}^5 = \mathcal{Q}^6 \cap \mathbb{P}(\mathrm{Im}_{\mathbb{O}}) = \{[x] \mid x \in \mathbb{O}, x^2 = 0\},$$

where $\mathrm{Im}_{\mathbb{O}} = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_7\}$ is the set of purely imaginary octonions and the characterisation

of $\mathcal{Q}^5$ is given by the fact that

$$0 = x^2 = \begin{pmatrix} x_0^2 - \sum_{i=1}^7 x_i^2 \\ 2x_0 x_1 \\ \vdots \\ 2x_0 x_7 \end{pmatrix} \qquad \Longleftrightarrow \qquad x_0 = 0 \quad \text{and} \quad \sum_{i=1}^7 x_i^2 = 0.$$

Now, given a vector $v \in \mathbb{O}$, we can define a 3-dimensional linear subspace

$$\Lambda_v := \{[w] \mid w \in \mathbb{O}, v \cdot w = 0\} \subset \mathbb{P}^7 :$$

indeed, if we set (without loss of generality) $v = e_1 + ie_2$, for each $w = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \in \mathbb{O}$, by direct computation

$$v \cdot w = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \left.\begin{array}{l} -x_1 - ix_2 = 0 \\ x_2 - ix_1 = 0 \end{array}\right] \\ \left.\begin{array}{l} x_0 + ix_4 = 0 \\ -x_4 + ix_0 = 0 \end{array}\right] \\ \left.\begin{array}{l} x_3 + ix_6 = 0 \\ -x_6 + ix_3 = 0 \end{array}\right] \\ \left.\begin{array}{l} -x_7 - ix_5 = 0 \\ x_5 - ix_7 = 0 \end{array}\right] \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} x_1 + ix_2 = 0 \\ x_0 + ix_4 = 0 \\ x_3 + ix_6 = 0 \\ x_7 + ix_5 = 0. \end{cases}$$

Therefore, one can observe that Λ_v is described by 4 independent equations, so it has codimension 4 and hence is 3-dimensional.

Despite being the most straightforward, this is not the only way we can associate to a vector $v \in \mathbb{O}$ a linear subspace: indeed, if we fix an invertible element $a \in \mathbb{O}$, it can be used to define a new product $\circledast_a$ on $\mathbb{O}$, given by

$$v \circledast_a w := (v \cdot (w \cdot a)) \cdot a^{-1},$$

which is clearly bilinear and such that $v \circledast_a 1 = 1 \circledast_a v = 1$. Then it follows that this product fixes $\mathcal{Q}^6$ and $\mathcal{Q}^5$ as

$$v \circledast_a v = (v \cdot (v \cdot a)) \cdot a^{-1} = (vv) \cdot (aa^{-1}) = v \cdot v,$$

$$v \circledast_a v^* = (v \cdot (v^* \cdot a)) \cdot a^{-1} = (v \cdot v^*) \cdot a \cdot a^{-1} = v \cdot v^*,$$

since by Artin's Theorem (*cf.* [Bor24, Theorem 3.9]) the subalgebra of $(\mathbb{O}, +, \cdot)$ generated by v and a is associative. Therefore, by means of this new product, we might define another 3-

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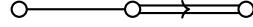
dimensional subspace as

$$\Lambda_v^a := \{[w] \mid w \in \mathbb{O}, v \circledast_a w = 0\} \subset \mathbb{P}^7,$$

which has exactly the same properties as Λ_v.

3.4.2 The nesting maps

Consider the Dynkin diagram B_3,

then the marking $\mathsf{B}_3(1)$ corresponds to the quadric $\mathcal{Q}^5$ described above and the marking $\mathsf{B}_3(3)$ corresponds to

$$\mathsf{B}_3(3) = \{\Sigma^2 \in \mathbb{G}(2, \mathbb{P}^7) \mid \Sigma^2 \subset \mathcal{Q}^5\} \simeq \mathcal{Q}^6,$$

hence a nesting is a map $\sigma : \mathsf{B}_2(1) \to \mathsf{B}_3(3)$ mapping a point $\mathbf{p}$ to a plane containing it.

By using the machinery involving the octonions described above, we can observe that $\mathcal{Q}^5$ can be seen as the space parametrising a family of planes in itself: indeed, if we define

$$\sigma_1([x]) := \Lambda_x \cap \mathcal{Q}^5 = \{[y] \in \mathcal{Q}^5 \mid x \cdot y = 0\},$$

then $\sigma_1([x])$ is clearly a plane in $\mathcal{Q}^5$ containing $[x]$, hence σ_1 is the nesting we are looking for. As in the previous cases, we can use the structure we gave to the space we are working in to start from this idea of a nesting and build some more: by using the new product $\circledast_a$ we defined on $\mathbb{O}$, we can construct a nesting for any invertible $a \in \mathbb{O}$ (*i.e.* $a \notin \mathcal{Q}^6$) as

$$\sigma_a([x]) := \Lambda_x^a \cap \mathcal{Q}^5 = \{[y] \in \mathcal{Q}^5 \mid x \circledast_a y = 0\}.$$

Bibliography

[Bae02]   John C. Baez. *The Octonions*. 2002. arXiv: math/0105155 [math.RA]. URL: https://arxiv.org/abs/math/0105155.

[Bor24]   Davide Borra. *The Cayley–Dickson Construction*. 2024. URL: https://davideborra.github.io/?r=pem.

[DCR03]   Corrado De Concini and Zinovy Reichstein. "Nesting maps of Grassmannians". In: (2003). arXiv: math/0312039 [math.AG]. URL: https://arxiv.org/abs/math/0312039.

[Har92]   Joe Harris. *Algebraic geometry: a first course*. eng. Graduate texts in mathematics, 133. Springer, 1992. ISBN: 0387977163.

[Hum72]   James E. Humphreys. *Introduction to Lie algebras and representation theory*. eng. Graduate texts in mathematics, 9. Springer, 1972. ISBN: 0387900535.

[Hum75]   James E. Humphreys. *Linear algebraic groups*. eng. Graduate texts in mathematics, 21. Springer, 1975. ISBN: 0387901086.

[KM98]   Janos Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Mathematics. Cambridge University Press, 1998.

[MOSC20]   Roberto Muñoz, Gianluca Occhetta, and Luis E. Solá Conde. "Nestings of Rational Homogeneous Varieties". In: *Transformation Groups* 27.1 (July 2020), 189–223. ISSN: 1531-586X. DOI: 10.1007/s00031-020-09597-x. URL: http://dx.doi.org/10.1007/s00031-020-09597-x.

[SC19]   Luis E. Solá Conde. *Geometry of rational homogeneous spaces. Notes for the 42nd Autumn School in Algebraic Geometry held in Lukecin (Poland)*. eng. Sept. 2019. URL: https://drive.google.com/file/d/1USdZzvDFn9lTjiIOd9x38bGpS-wT5WpL/view?usp=sharing.