The Cayley–Dickson construction

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Abstract

This presentation, written for the third year of the Excellence Programme in the Bachelor's Degree in Mathematics, is aimed at giving the reader a simple yet deep understanding of the Cayley–Dickson construction of extensions of algebras over a field, mainly the real numbers $\mathbb{R}$, through the introduction of some important objects in nonassociative algebra.

1 Introduction

The quest for creating new extensions of the real numbers, recreating the process which had previously led to the discovery of complex numbers, was mainly addressed by the Irish mathematician sir William Rowan Hamilton. After he finished working on the formal definition of complex numbers as pairs of real numbers, he spent years trying to replicate a similar process in a three-dimensional space. However, later discoveries made clear that a three dimensional division algebra (whose definition we are goind to address) over $\mathbb{R}$ was impossible, as we will prove, and that what he needed in order to define a working multiplication was a four-dimensional algebra, later called quaternions ($\mathbb{H}$).

First, let's recall some basic definitions of objects we are going to deal with. In this context we will only consider $\mathbb{R}$-algebras, but almost everything can be applied to algebras on an arbitrary field $\mathbb{K}$.

Definition 1.1 (Algebra)**.** A $\mathbb{R}$-algebra is a $\mathbb{R}$-vector space $\mathfrak{A}$ equipped with a bilinear map $m : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, called multiplication, and a unit $1 \in \mathfrak{A}$ such that $m(1, a) = m(a, 1) = a$. As usual we will write $m(a, b) = ab$.

1. A norm on $\mathfrak{A}$ is a function $\|\cdot\| : \mathfrak{A} \to [0, +\infty[$ such that $(\mathfrak{A}, \|\cdot\|)$ is a normed vector space and is compatible with the multiplication (i.e. for all $a, b \in \mathfrak{A}$, $\|ab\| = \|a\|\|b\|$); the pair $(\mathfrak{A}, \|\cdot\|)$ is called a normed algebra.

2. An element $a \in \mathfrak{A}^* := \mathfrak{A} \setminus \{0\}$ is said to have a multiplicative inverse if there exists a $b \in \mathfrak{A}^*$ such that $ab = ba = 1$. If for each $a \in \mathfrak{A}^*$, a has a multiplicative inverse, $\mathfrak{A}$ is said to be a division algebra.

Remark 1.2. Every $\mathbb{R}$-algebra $\mathfrak{A}$ contains a subalgebra isomorphic to $\mathbb{R}$ given by

$$\mathbb{R}1 = \{x1 \mid x \in \mathbb{R}\}$$

where 1 is the unit of m and $x1$ is meant as the scalar multiplication of $\mathfrak{A}$ as a vector space over $\mathbb{R}$. From now on we will identify $\mathbb{R}$ with $\mathbb{R}1$ and write $\mathbb{R} \subseteq \mathfrak{A}$.

Definition 1.3 ($*$-Algebra)**.** A $*$-algebra is an algebra $\mathfrak{A}$ over $\mathbb{R}$ equipped with a conjugation, i.e. a $\mathbb{R}$-linear map $* : \mathfrak{A} \to \mathfrak{A}$, satisfying

$$a^{**} = a, \qquad (ab)^* = b^* a^* \qquad \text{for all } a, b \in \mathfrak{A}$$

A $*$-algebra over $\mathbb{R}$ is called real if $a = a^*$ for each $a \in \mathfrak{A}$ (i.e. the conjugation is trivial), and nicely normed if $a + a^* \in \mathbb{R}$ and $aa^* = a^* a \in \mathbb{R}_{>0}$ for each $a \in \mathfrak{A}^*$.

Remark 1.4. If a $*$-algebra is nicely normed, we can introduce the concept of real and imaginary part by defining for $a \in \mathfrak{A}$

$$\mathrm{Re}(a) := \frac{1}{2}(a + a^*) \in \mathbb{R}, \qquad \mathrm{Im}(a) := \frac{1}{2}(a - a^*) = a - \mathrm{Re}\, a.$$

Now we know all we need to prove why Hamilton's desire to build a three dimensional division algebra was impossible to achieve.

Lemma 1.5 ([She24])**.** *Let $\mathfrak{A}$ be an associative normed division $\mathbb{R}$-algebra with $i \in \mathfrak{A}$ such that $i^2 = -1$, then* $\dim \mathfrak{A} \neq 3$.

Remark 1.6. In this lemma we also require that $\mathfrak{A} \ni i$ with $i^2 = -1$ because we want $\mathfrak{A}$ to extend the complex numbers.

Proof of Lemma 1.5. By way of contradiction, assume $\dim \mathfrak{A} = 3$. Since $1, i$ are linearly indipendent, thanks to Steinitz's theorem we can extend them to a basis $\beta = \{1, i, j\}$ of $\mathfrak{A}$. Now we are interested in computing the value of the product ij, which we should be able to write in terms of $ij = a + bi + cj$, with $a, b, c \in \mathbb{R}$. Consider

$$\begin{aligned} -j = i(ij) &= i(a + bi + cj) = ai + bi^2 + cij = \\ &= ai - b + c(a + bi + cj) = (ca - b) + (cb + a)i + c^2 j \end{aligned}$$

which, since the expression of a vector in terms of linear combination over a base is unique, this implies

$$\begin{cases} ca - b = 0 \\ cb - a = 0 \\ \boxed{c^2 = -1} \end{cases}$$

which leads to a contradiction with the fact that we picked $c \in \mathbb{R}$. $\square$

Remark 1.7. In this Lemma we have assumed $\mathfrak{A}$ to be associative, but it is also true under a weaker hypotesis, namely alternativity, which we will discuss in more detail later. Note that we are not using commutativity.

2 Constructing division algebras

After Hamilton's discovery of the quaternions, a young Arthur Cayley started expressing his interest in the subject, mainly working on their applications. So, in March 1845, he published a paper in which he also briefly described the octonions, which quickly became known as Cayley numbers. The construction of the octonions shown by Cayley was generalised by Leonard Dickson in 1919, by the process we are going to describe.

First, let's recall the usual construction of the complex numbers over $\mathbb{R}$ as pairs of real numbers: we consider $\mathbb{R}^2$ with the usual sum provided by its vector space structure and introduce a new multiplication

$$\begin{array}{rcl} \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ ((a,b),(c,d)) & \longmapsto & (a,b)\cdot(c,d) := (ac-db, ad+cb) \end{array}$$

and define $(\mathbb{C},+,\cdot)$ as $\mathbb{R}^2$ with the operations defined above. Note that this is compatible with the classical notation for elements of $\mathbb{C}$ as sums of a real and an imaginary part by considering $a+ib := (a,b)$. Since we want $\mathbb{C}$ to be a $*$-algebra, we also need to define a conjugation:

$$(a,b)^* := (a,-b).$$

We can repeat the same process, slightly modifying the definition of the product by introducing the conjugation and considering pairs of complex numbers, in order to construct the *quaternions*

$$\begin{array}{rcl} \mathbb{C}^2 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ ((a,b),(c,d)) & \longmapsto & (a,b)\cdot(c,d) := (ac-db^*, a^*d+cb). \end{array}$$

The same applies for the definition of the conjugation in $\mathbb{C}^2 := \mathbb{H}$:

$$(a,b)^* := (a^*,-b).$$

As for the complex numbers, this definition compatible with the usual construction of quaternions as linear combinations of the basis $\{1,i,j,k\}$ by considering $\mathbb{C}^2 \ni ((a,b),(c,d)) = a+ib+cj+dk \in \mathbb{H}$. By direct computation, one can prove the usual product laws for quaternions, that is $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$ and $ik = -j$.

Now, it is easy to observe that, since the conjugation over $\mathbb{R}$ is trivial (that is $a^* = a$ for each $a \in \mathbb{R}$), we can use the same definition of product and conjugation we described for $\mathbb{H}$ in constructing $\mathbb{C}$: this is the idea behind the Cayley–Dickson construction of algebras over $\mathbb{R}$, having a process we repeat for creating $2n$-dimensional extensions of $\mathbb{R}$.

2.1 The Cayley–Dickson construction

Consider a $*$-algebra $\mathfrak{A}$ over $\mathbb{R}$ with $\dim_{\mathbb{R}} \mathfrak{A} = n$ and such that

$$x + x^* \in \mathbb{R} \qquad \text{and} \qquad xx^* = x^*x \in \mathbb{R} \qquad \forall x \in \mathfrak{A} \tag{2.1}$$

(note that this is implied by $\mathfrak{A}$ being nicely normed). We are interested in constructing a new $*$-algebra $\mathfrak{A}'$ of dimension $2n$ over $\mathbb{R}$ with the same properties and having $\mathfrak{A}$ as a subalgebra.

Consider $\mathfrak{A}' = \mathfrak{A}^2$ equipped with sum and multiplication by scalars defined componentwise and the multiplication built as follows:

$$(a_1,a_2)(a_1',a_2') = (a_1a_1' + \mu a_2'a_2^*, a_1^*a_2' + a_1'a_2) \qquad \forall\,(a_1,a_2),(a_1',a_2') \in \mathfrak{A}'$$

for some fixed $\mu \in \mathbb{R}^*$.

Remark 2.1. By direct computation, the it follows that

1. $1_{\mathfrak{A}'} = (1_{\mathfrak{A}}, 0)$ is the unity element for $\mathfrak{A}'$;
2. $\mathfrak{A} = \{(a,0) \,|\, a \in \mathfrak{A}\} \cong \mathfrak{A}$ is a subalgebra of $\mathfrak{A}'$;

3. $v := (0,1)$ is such that $v^2 = \mu 1$ and $\mathfrak{A}' = \mathfrak{A} \oplus v\mathfrak{A}$ (where $\oplus$ is the direct sum of vector spaces). With this identification, we can write each $(a_1, a_2) \in \mathfrak{A}'$ as $a_1 + va_2$ and the product is given by

$$(a_1 + va_2)(a_1' + va_2') = (a_1 a_1' + \mu a_2' a_2^*) + v(a_1^* a_2' + a_1' a_2).$$

As we did in Remark 1.2, we will write $\mathfrak{A} \subset \mathfrak{A}'$ for $\mathfrak{A}$.

We also wanted $\mathfrak{A}'$ to be a $*$-algebra, so we need to define an conjugation:

$$(a_1, a_2)^* = (a_1^*, -a_2).$$

It is quite obvious that $*$ is linear and that $(x^*)^* = x$ for all $x \in \mathfrak{A}'$, while the other axiom follows by a simple computation

$$\begin{aligned}
[(a_1, a_2)(a_1', a_2')]^* &= (a_1 a_1' + \mu a_2' a_2^*, a_1^* a_2' + a_1' a_2)^* = \\
&= ((a_1 a_1')^* + \mu(a_2' a_2^*)^*, -a_1^* a_2' - a_1' a_2) \\
&= (a_1'^* a_1^* + \mu a_2 a_2'^*, -a_1' a_2 - a_1^* a_2') = \\
&= (a_1'^*, -a_2')(a_1^*, -a_2) = (a_1', a_2')^*(a_1, a_2)^*.
\end{aligned}$$

and, thanks to (2.1), we have that the same is true for $\mathfrak{A}'$:

$$(a, b) + (a, b)^* = (a, b) + (a^*, -b) = (a + a^*, 0) \in \mathbb{R}$$

$$(a, b)(a, b)^* = (a, b)(a^*, -b) = (aa^* - \mu bb^*, -a^* b + a^* b) = (aa^* - \mu bb^*, 0) \in \mathbb{R}$$

$$(a, b)^*(a, b) = (a^*, -b)(a, b) = (aa^* - \mu bb^*, (a^*)^* b - ab) = (aa^* - \mu bb^*, 0) \in \mathbb{R}.$$

From now on our interest will be limited to the classical construction, where $\mu = -1$. We have already defined $\mathbb{C} = \mathbb{R}'$ and $\mathbb{H} = \mathbb{C}'$.

Definition 2.2 (Octonions and sedenions)**.** The 8-dimensional algebra $\mathbb{R}$-algebra obtained by applying the Cayley–Dickson construction with $\mu = -1$ to the quaternions is called octonions, and written $\mathbb{O} = \mathbb{H}'$. The 16-dimensional $\mathbb{R}$-algebra obtained by applying the Cayley–Dickson construction with $\mu = -1$ to the octonions is called sedenions, and written $\mathbb{S} = \mathbb{O}'$.

3 Properties of Cayley–Dickson algebras

We have shown some basic structure properties which are preserved by the Cayley–Dickson construction, but it is common for $\mathfrak{A}'$ to lose some of the properties that are true for $\mathfrak{A}$. In this section we will explain some of the reasons behind this loss of structure, but first we need to give some definitions.

3.1 Alternativity and power-associativity

Consider an algebra $\mathfrak{A}$ as in Definition 1.1 and a subset $X \subset \mathfrak{A}$. Then the subalgebra generated by X is defined as usual as the smallest subalgebra $\langle X \rangle$ of $\mathfrak{A}$ containing X, that is

$$\langle X \rangle = \bigcap \{A \subset \mathfrak{A} \mid A \text{ subalgebra of } \mathfrak{A},\ X \subset A\}.$$

Definition 3.1 (Weak associativity)**.** An algebra $\mathfrak{A}$ is said to be

1. Power-associative if any subalgebra $\langle\{x\}\rangle$ generated by any single element $x \in \mathfrak{A}$ is associative;

2. Flexible if for each $x, y \in \mathfrak{A}$,
$$x(yx) = (xy)x;$$

3. Alternative if for each $x, y \in \mathfrak{A}$,
$$(xx)y = x(xy) \qquad \text{and} \qquad (xy)y = x(yy).$$

Definition 3.2 (Associator)**.** Let $\mathfrak{A}$ be an algebra, the associator is the trilinear form

$$\begin{array}{rccc} [\cdot,\cdot,\cdot]: & \mathfrak{A}\times\mathfrak{A}\times\mathfrak{A} & \longrightarrow & \mathfrak{A} \\ & (a,b,c) & \longmapsto & (ab)c - a(bc) \end{array}$$

Remark 3.3. Just as the commutator $[a, b] = ab - ba$ gives information about the commutativity of an algebraic structure, in fact $\mathfrak{A}$ is commutative iff it vanishes for each $a, b \in \mathfrak{A}$, the associator gives information about its associativity.

Now let us prove a proposition which will justify the names we gave to these weakly associative algebras we just defined. In order to state it we need to introduce the concept of powers of a single element, which are defined recursively as $x^1 = x$ and $x^{n+1} = xx^n$.

Proposition 3.4. *Let $\mathfrak{A}$ be an algebra, than*

1. *$\mathfrak{A}$ is power associative iff $x^i x^j = x^{i+j}$ for each $x \in \mathfrak{A}$, $i, j \in \mathbb{N}^*$.*

2. *$\mathfrak{A}$ is alternative iff the associator is a trilinear alternating form.*

Proof. 1. First assume that $x^i x^j = x^{i+j}$ holds for each $x \in \mathfrak{A}$ and let us prove that $\langle x\rangle$ is associative: it follows by direct computation by showing that

$$\left[\left(\sum_{i=1}^{l}\lambda_i x^i\right)\left(\sum_{j=1}^{m}\mu_j x^j\right)\right]\left(\sum_{k=1}^{n}\eta_k x^k\right) = \left(\sum_{i=1}^{l}\lambda_i x^i\right)\left[\left(\sum_{j=1}^{m}\mu_j x^j\right)\left(\sum_{k=1}^{n}\eta_k x^k\right)\right].$$

The other implication is obvious by observing that $x^i, x^j \in \langle x\rangle$ for all $x \in \mathfrak{A}$ and for all $i, j \in \mathbb{N}^*$.

2. It follows directly from the definition that

$$\begin{array}{ccc} (xx)y = x(xy) & \iff & [x,x,y] = 0 \quad \text{and} \\ (xy)y = x(yy) & \iff & [x,y,y] = 0. \end{array}$$

Assuming that these equations hold, let us prove that $[x, y, x] = 0$.

$$\begin{aligned} [x,y,x] &= (xy)x - x(yx) + \underline{(x(x+y))(x+y) - x((x+y)(x+y))} = \\ &= (xy)x - x(yx) + (xx)x + (xy)x + (xx)y + (xy)y + \\ &\quad - x(xx) - x(xy) - x(yx) - x(yy) = 0 \end{aligned}$$ □

Remark 3.5. Note that what we proved in 2. is that any alternative algebra satisfies the flexible identity, that is every alternative algebra is also flexible. Note that in a flexible algebra, we can write unambiguously xax for $x(ax) = (xa)x$.

Remark 3.6. If an algebra $\mathfrak{A}$ is power associative, it is common to write $\mathbb{R}[x] = \langle\{x\}\rangle$ for the subalgebra generated by a single element $x \in \mathfrak{A}$. Note that $\mathbb{R}[x]$ is isomorphic to a quotient of the polinomial ring in one indeterminate.

Lemma 3.7 (Moufang identities)**.** *For x, y, a in an alternative algebra $\mathfrak{A}$, the following hold:*

$$(xax)y = x[a(xy)], \tag{3.1}$$

$$y(xax) = [(yx)a]x, \tag{3.2}$$

$$(xy)(ax) = x(ya)x. \tag{3.3}$$

Proof. Start by observing that (3.1) and (3.2) are the same property by swapping left and right multiplication. Thanks to this idea we can only prove (3.1), since the proof of (3.2) would then be similar. Note that we will make large use of the fact that every alternating multilinear form (in this case the associator) is also antisymmetric.

$$\begin{aligned}
(xax)y - x[a(xy)] &= (xax)y \underline{- (xa)(xy) + (xa)(xy)} - x[a(xy)] && \text{(add and subtract } (xa)(xy)\text{)}\\
&= [xa, x, y] + [x, a, xy]\\
&= -[x, xa, y] - [x, xy, a]\\
&= -[x(xa)]y + x[(xa)y] - [x(xy)]a + x[(xy)a]\\
&= -(x^2a)y - (x^2y)a + x[(xa)y + (xy)a] && \text{(complete the associator)}\\
&= \cancel{-[x^2, a, y] - [x^2, y, a]} - x^2(ay) - x^2(ya) +\\
&\quad + x[(xa)y + (xy)a]\\
&= x[-x(ay) - x(ya) + (xa)y + (xy)a]\\
&= x([x, a, y] + [x, y, a]) = 0
\end{aligned}$$

Now it remains only to prove (3.3)

$$\begin{aligned}
(xy)(ax) - x(ya)x &= [x, y, ax] + x[y(ax) - (ya)x] && \text{(complete the associator)}\\
&= -[x, ax, y] - x[y, a, x]\\
&= -(xax)y + x[(ax)y] - x[y, a, x]\\
&= -(xax)y + x[(ax)y - [y, a, x]] && \text{(Equation (3.1))}\\
&= -x[a(xy)] + x[(ax)y - [y, a, x]]\\
&= -x[a(xy) - (ax)y + [y, a, x]]\\
&= -x[-[a, x, y] + [y, a, x]] = 0 \qquad \square
\end{aligned}$$

Remark 3.8. In the proof of the next Theorem we will also need an equivalent form of the Moufang identity (3.2):

$$[y, xa, x] = -[y, x, a]x, \tag{3.4}$$

for all $x, y, a \in \mathfrak{A}$ since

$$[y, xa, x] = [y(xa)]x - y(xax) \underset{(3.2)}{=} [y(xa)]x - [(yx)a]x = -[y, x, a]x.$$

Moreover, by expanding

$$[y, (x + z)a, (x + z)] = -[y, (x + z), a](x + z),$$

it follows that

$$[y, xa, z] + [y, za, x] = -[y, x, a]z - [y, z, a]x. \tag{3.5}$$

Theorem 3.9 (Artin - [Sch66, Thm. 3.1]). *The subalgebra generated by any two elements of an alternative algebra is associative.*

Proof. Consider any two elements $x, y \in \mathfrak{A}$, we will write $p = p(x, y)$ for a nonassociative product $z_1 z_2 \cdots z_t$ (with some distribution of parenthesis) of t factors z_i chosen between x or y. We also denote the degree of this product as $\partial p := t$ and say that p begins with z_1. In order to prove the associativity of $\langle\{x, y\}\rangle$ it is sufficient to prove that $[p, q, r] = 0$ for all nonassociative products $p = p(x, y), q = q(x, y), r = r(x, y)$ as defined above.

We shall prove this by strong induction on $n = \partial p + \partial q + \partial r$ (if $n < 3$ the result is vacuous, so we can consider $n \geq 3$). Let us assume that $[p, q, r] = 0$ for each p, q, r with $\partial p + \partial q + \partial r < n$. Since $\partial p < n$, we note that we can apply the inductive assumption and conclude that the parenthesis are not necessary in writing it. Now, p, q and r are products of only x and y, so at least two of them must start with the same factor: without loss of generality we can assume q and r both start with x. Now we can consider three cases:

- if $\partial q = \partial r = 1$, i.e. $q = r = x$, then $[p, q, r] = [p, x, x] = 0$ by definition of alternativity;
- if only one between q and r has degree > 1 (say $q = xq'$ and $r = x$), by use of (3.4) and assumption of induction the we conclude that

$$[p, q, r] = [p, xzq', x] = -[p, q', x]q' = 0;$$

- if $\partial q > 1$ and $\partial r > 1$, then $q = xq'$ and $r = xr'$, where $\partial q' = \partial q - 1$ and $\partial r' = \partial r - 1$. By putting $y = xr'$, $a = q'$ and $z = p$ in (3.5), we have

$$\begin{aligned}
[p, q, r] &= [p, xq', xr'] = -[xr', xq', p] && \text{(Equation (3.5))} \\
&= [xr', pq', x] + \cancel{[xr', x, q']}p + \cancel{[xr', p, q']}x && \text{(assumption of induction)} \\
&= [xr', pq', x] = -[pq', xr', x] = && \text{(Equation (3.4))} \\
&= [pq', x, r']x = 0 && \text{(assumption of induction).}
\end{aligned}$$

$\square$

Corollary 3.10. *Any alternative algebra is power-associative.*

Remark 3.11. It is clear that an alternative division algebra cannot have zero-divisors, in fact for each $a, b \in \mathfrak{A}$ such that $ab = 0$ we have $b = a^{-1}ab = 0$ if $a \neq 0$ or $a = abb^{-1} = 0$ if $b \neq 0$. If its dimension over $\mathbb{R}$ is finite, the opposite is true, as well.

Corollary 3.12. *A nicely normed alternative $*$-algebra over $\mathbb{R}$ has a natural definition of a norm given by*

$$\|a\|^2 := aa^*$$

and it has multiplicative inverses by defining

$$a^{-1} = \frac{a^*}{\|a\|^2} \qquad \forall a \in \mathfrak{A}^*.$$

Proof. Consider the map $\langle \cdot, \cdot \rangle : \mathfrak{A}^2 \to \mathbb{R}$ given by $\langle a, b \rangle = \mathrm{Re}(ab^*) = \frac{1}{2}(ab^* + ba^*)$. We observe that

- it is symmetric: $\langle a, b \rangle = \frac{1}{2}(ab^* + ba^*) = \frac{1}{2}(ba^* + ab^*) = \langle b, a \rangle$;

- it is bilinear: for each $a_1, a_2, b \in \mathfrak{A}$, $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\begin{aligned}\langle \lambda_1 a_1 + \lambda_2 a_2, b\rangle &= \mathrm{Re}((\lambda_1 a_1 + a_2)b^*) = \mathrm{Re}(\lambda_1 a_1 b^* + \lambda_2 a_2 b^*)\\ &= \frac{1}{2}(\lambda_1 a_1 b^* + \lambda_2 a_2 b^* + \lambda_1 b a_1^* + \lambda_2 b a_2^*)\\ &= \frac{\lambda_1}{2}(a_1 b^* + b a_1^*) + \frac{\lambda_2}{2}(a_2 b^* + b a_2^*) = \lambda_1 \langle a_1, b\rangle + \lambda_2 \langle a_2, b\rangle.\end{aligned}$$

- it is positive definite: since $\mathfrak{A}$ is nicely normed, for each $a \in \mathfrak{A}$, $\langle a, a\rangle = \mathrm{Re}(aa^*) = aa^* \in [0, +\infty[$ and $aa^* \in \mathbb{R}_{>0}$ $\forall a \in \mathfrak{A}^*$.

So we obtain that $\langle \cdot, \cdot \rangle$ is a scalar product, so, by defining $\|a\|^2 := \langle a, a\rangle = \mathrm{Re}(aa^*) = aa^*$ we obtain that $\|\cdot\|$ is a norm of vector spaces. In order to prove that it is a norm in the sense of algebras, we need to verify that $\|ab\| = \|a\|\|b\|$ for each $a, b \in \mathfrak{A}$. First we observe that, since $a^* = (a + a^*) - a = 2\,\mathrm{Re}(a) - a \in \mathbb{R}[a]$, a, a^*, b and b^* are elements of the subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ generated by $\{a, b\}$, which is associative by Artin's Theorem. So, by direct computation, it follows that

$$\|ab\|^2 = (ab)(ab)^* = (ab)(b^* a^*) = (a(bb^*))a^* = (aa^*)(bb^*) = \|a\|^2\|b\|^2.$$

where we used the fact that, since $bb^* \in \mathbb{R}$, it commutes with elements of $\mathfrak{A}$.

To conclude, it is sufficient to observe that $aa^{-1} = (aa^*)/\|a\|^2 = (aa^*)/(aa^*) = 1$. □

3.2 Associativity and the Cayley–Dickson construction

Consider a $*$-algebra $\mathfrak{A}$ and let $\mathfrak{A}'$ be the product of the Cayley–Dickson construction applied to $\mathfrak{A}$. Then the following results hold [Bae02, Prop. 1-5].

Proposition 3.13. *If $\mathfrak{A}$ is an algebra over $\mathbb{R}$, $\mathfrak{A}'$ is never real.*

Proof. We only need to observe that $(a, b)^* = (a^*, -b)$ and that there is at least one $b \in \mathfrak{A}$ such that $b \neq -b$ ($\mathrm{char}\,\mathbb{R} \neq 2$). □

Proposition 3.14. *$\mathfrak{A}$ real (and thus commutative) $\iff$ $\mathfrak{A}'$ is commutative.*

Proof.

$\boxed{\Rightarrow}$ By direct computation,

$$(a, b)(c, d) = (ac - db^*, a^* d + cb) = (ac - db, ad + cb) = (ca - bd^*, d^* a + bc) = (c, d)(a, b)$$

$\boxed{\Leftarrow}$ Consider a fixed $a \in \mathfrak{A}$, then

$$(a, 0) = (0, -1)(0, a) = (0, a)(0, -1) = (a^*, 0) \qquad \Longrightarrow \qquad a = a^*.$$

□

Proposition 3.15. *$\mathfrak{A}$ is commutative and associative $\iff$ $\mathfrak{A}'$ is associative.*

Proof.

$\boxed{\Rightarrow}$ By direct computation,

$$\begin{aligned}
(a,b)[(c,d)(e,f)] &= (a,b)(ce - fd^*, c^*f - ed) = \\
&= \big(a(ce - fd^*) - (c^*f - ed)b^*, a^*(c^*f - ed) + (ce - fd^*)b\big) = \\
&= \big(\underline{ace} - \underline{\underline{afd^*}} - \dashuline{c^*fb^*} - \uwave{edb^*}, \underline{a^*c^*f} - \uwave{a^*ed} + \underline{\underline{ceb}} - \dashuline{fd^*b}\big)
\end{aligned}$$

$$\begin{aligned}
[(a,b)(c,d)](e,f) &= (ac - db^*, a^*d + cb)(e,f) = \\
&= \big((ac - db^*)e - f(a^*d + cb)^*, (ac - db^*)^*f - (a^*d + cb)e\big) = \\
&= \big(\underline{ace} - \uwave{db^*e} - \underline{\underline{fad^*}} - \dashuline{fc^*b^*}, \underline{a^*c^*f} - \dashuline{d^*bf} - \uwave{a^*de} + \underline{\underline{cbe}}\big)
\end{aligned}$$

By matching the similarly underlined terms, we conclude that

$$(a,b)[(c,d)(e,f)] = [(a,b)(c,d)](e,f).$$

$\boxed{\Leftarrow}$ Since $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{A}'$, it follows that $\mathfrak{A}$ is associative. Now, for each $a, b \in \mathfrak{A}$, since $\mathfrak{A}'$ is associative,

$$(0, ba) = (0,a)(b,0) = (0,a)[(0,-b^*)(0,1)] = [(0,a)(0,-b^*)](0,1) = (a^*b^*, 0)(0,1) = (0, ab)$$

thus $\mathfrak{A}$ is commutative. □

Proposition 3.16. *$\mathfrak{A}$ is nicely normed $\iff$ $\mathfrak{A}'$ is nicely normed.*

Proof. Since $\mathfrak{A} \subset \mathfrak{A}'$, it is clear that if $\mathfrak{A}'$ is nicely normed, so is $\mathfrak{A}$. Let's prove the opposite. Consider $(a,b) \in \mathfrak{A}'$, then by direct computation

$$(a,b) + (a,b)^* = (a,b) + (a^*, -b) = (a + a^*, 0) \in \mathfrak{A} \subset \mathfrak{A}'$$

and, since $a + a^* \in \mathbb{R} \subset \mathfrak{A}$, it follows that $(a + a^*, 0) \in \mathbb{R} \subset \mathfrak{A} \subset \mathfrak{A}'$. A similar approach leads to the proof of the other axiom:

$$(a,b)(a,b)^* = (a,b)(a^*, -b) = (aa^* + bb^*, -a^*b + a^*b) = (aa^* + bb^*, 0) \in \mathfrak{A} \subset \mathfrak{A}'$$

$$(a,b)^*(a,b) = (a^*, -b)(a,b) = (a^*a + b^*b, (a^*)^*b - ab) = (aa^* + bb^*, 0) = (a,b)(a,b)^*$$

and, since $aa^* + bb^* \in \mathbb{R}_{\geq 0}1 \subset \mathfrak{A}$, $(aa^* + bb^*, 0) \in \mathbb{R}_{\geq 0}1 \subset \mathfrak{A} \subset \mathfrak{A}'$ □

Proposition 3.17. *$\mathfrak{A}$ is associative and nicely normed $\iff$ $\mathfrak{A}'$ is alternative and nicely normed.*

Proof. Thanks to Proposition 3.16, we can assume both algebras to be nicely normed.

$\boxed{\Rightarrow}$ Here we will prove only the left alternativity, the other will follow from a similar computartion:

$$\begin{aligned}
(a,b)[(a,b)(c,d)] &= (a,b)(ac - db^*, a^*d + cb) = \\
&= \big(a(ac - db^*) - (a^*d + cb)b^*, a^*(a^*d + cb) + (ac - db^*)b\big) = \\
&= \big(aac - \underline{adb^* - a^*db^*} - \underline{cbb^*}, a^*a^*d + \underline{a^*cb + acb} - db^*b\big) = \\
&= \big(aac - db^*a - db^*a^* - bb^*c, a^*a^*d + ca^*b + cab - db^*b\big)
\end{aligned}$$

were we used the fact that, since $a + a^* \in \mathbb{R}$ and bb^* in $\mathbb{R}$, it commutes in the product:

$$\begin{cases} adb^* + a^*db^* = (a + a^*)db^* = db^*(a + a^*) = db^*a + db^*a^*, \\ cbb^* = c(bb^*) = (bb^*)c = bb^*c, \\ a^*cb + acb = (a^* + a)cb = c(a^* + a)b = ca^*b + cab. \end{cases}$$

Now, we expand the other form of the left alternative identity

$$\begin{aligned} [(a, b)(a, b)](c, d) &= (aa - bb^*, a^*b + ab)(c, d) = \\ &= \big((aa - bb^*)c - d(a^*b + ab)^*, (aa - bb^*)^*d + c(a^*b + ab)\big) = \\ &= \big(aac - bb^*c - db^*a - db^*a^*, a^*a^*d - bb^*d + ca^*b + cab\big) \end{aligned}$$

and, by matching as before, we conclude that $\mathfrak{A}'$ is left-alternative.

$\boxed{\Leftarrow}$ Since $\mathfrak{A}'$ is alternative, so $\mathfrak{A}$ is. Now, let us prove that $\mathfrak{A}$ is associative. Consider $a, b, c \in \mathfrak{A}$, then (considering the expansion above)

$$(a, b)[(a, b)(0, c)] = (a, b)(-cb^*, a^*c) = \big(-a(cb^*) - (a^*c)b^*, a^*(a^*c) - (cb^*)b\big)$$

$$\begin{aligned} [(a, b)(a, b)](0, c) &= (aa - bb^*, a^*b + ab)(c, d) = \big(-c(a^*b + ab)^*, (aa - bb^*)c\big) = \\ &= \big(-c(b^*a) - c(b^*a^*), (aa)c - (bb^*)c\big). \end{aligned}$$

Now, focus on the first component: since $\mathfrak{A}'$ is alternative, the two products must be the same, so it must be

$$-a(cb^*) - (a^*c)b^* = -c(b^*a) - c(b^*a^*) \tag{3.6}$$

now, in the right part of the equation, we can collect c and then b^* and, since $a + a^* \in \mathbb{R}$, thus it commutes and associate in the product, we can rewrite

$$c(b^*a) + c(b^*a^*) = c[b^*(a + a^*)] = (a + a^*)(cb^*) = a(cb^*) + a^*(cb^*).$$

Now, by substituting this into (3.6), we conclude that

$$\cancel{a(cb^*)} + (a^*c)b^* = \cancel{a(cb^*)} + a^*(cb^*) \qquad \Longrightarrow \qquad (a^*c)b^* = a^*(cb^*),$$

that is $\mathfrak{A}$ is associative. □

Now we recall without proof a result by Schafer about flexible algebras [Sch54, Thm. 1]

Theorem 3.18 (Schafer)**.** *If an algebra $\mathfrak{A}$ over $\mathbb{R}$ is flexible, then also $\mathfrak{A}'$ is flexible.*

Finally, we can use these general results to prove which of the properties we are interested in hold in each of the $\mathbb{R}$-algebras we discussed earlier:

Corollary 3.19. *Since $\mathbb{R}$ is a real, commutative, associative, nicely normed $*$-algebra over itself, it follows that*

- $\mathbb{C}$ *is a commutative, associative, nicely normed $*$-algebra;*
- $\mathbb{H}$ *is an associative nicely normed $*$-algebra, but it is not commutative.*
- $\mathbb{O}$ *is an alternative nicely normed $*$-algebra, but it is neither commutative nor associative.*

- *The* sedenions $\mathbb{S}$ *are a nicely normed flexible* $*$*-algebra, but they are neither commutative nor alternative.*

Remark 3.20. The main problem with sedenions (and everything that comes after them) is that they are no more a division algebra. In fact, one can prove that sedenions have zero divisors: considering e_i as the vector in $\mathbb{R}^{16}$ with 1 in the i-th component and 0 elsewhere, by direct computation it follows that

$$(e_3 + e_{10})(e_6 - e_{15}) = 0.$$

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