Quantum Information

M2 Cybersecurity, documents allowed, 50mn

Exercise 1:

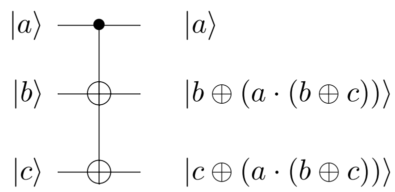
1. Write the Identity on two qubits as sum of projectors on the standard basis states (projectors on the vectors $|ab\rangle$ where $a, b \in \{0, 1\}$).

The Fredkin gate is a 3-qubit gate that acts on the basis states as follows : $\forall a, b, c \in \{0, 1\}$,

$$|a, b, c\rangle \mapsto |a, b \oplus (a \cdot (b \oplus c)), c \oplus (a \cdot (b \oplus c))\rangle$$

where $\oplus$ denotes the XOR gate (sum mod 2) and $\cdot$ denotes the AND gate (product).

The Fredkin gate is depicted as follows:



2. For any $b, c \in \{0, 1\}$, what is the image by the Fredkin gate of the states $|0, b, c\rangle$ and $|1, b, c\rangle$?

3. For any arbitrary one-qubit states $|\varphi\rangle = \alpha |0\rangle + \beta |1\rangle$ and $|\varphi'\rangle = \alpha' |0\rangle + \beta' |1\rangle$, what is the image by the Fredkin gate of $|a\rangle \otimes |\varphi\rangle \otimes |\varphi'\rangle$ when $a = 0$ and when $a = 1$?

In the following we want to analyse the circuit represented in Figure 1.

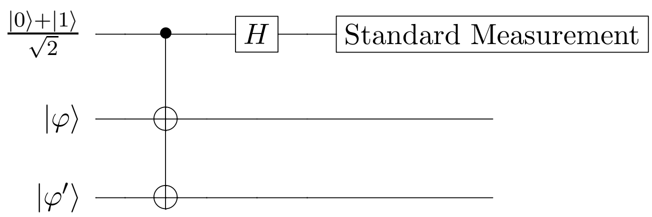


Figure 1: Circuit using the Fredkin gate

4. What is the state of the system immediately after the application of the Fredkin gate?

Let s be the inner product $s = \langle \varphi' | \varphi \rangle$ and $|s|^2 = ss^*$.

5. What is the state (density matrix) of the first qubit (the qubit initially in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$) after the application of the Fredkin gate?
6. What is the state of the system immediately after the application of the Hadamard gate? (recall that the Hadamard gate is defined on the basis states by : $|a\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^a |1\rangle)$ for $a \in \{0, 1\}$).

The first qubit is measured in the standard basis. Let p_0 be the probability that the classical outcome is 0.

7. What is the state of the system after the measurement if the classical result of the measurement is 0?
8. Using $|s|^2$, give the probability p_0 that the classical outcome is 0.
9. What can be checked using the circuit described in Figure 1? (Hint: consider the case $|s| = 1$)