On shift equivalence for non-linear rules*

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Abstract. Sequential independence is a key ingredient in the concurrency theory of DPO graph transformation, since it states when consecutive rewriting steps can be switched, thus identifying them as causally unrelated. The associated theory of shift equivalence on rewriting sequences is well-understood for linear rules, including its connection with standard models for concurrent systems such as event structures. This is not so for rules that are only left-linear, i.e., such that their right-hand side may merge graph items. More precisely, the correspondence with event structures fails, since merging may hinder the causal dependency between rewriting steps. We argue that a stricter notion of sequential independence is needed for such rules, and we prove that our proposal is well-behaved with respect to the concatenation of rewriting steps.

Keywords: Shift equivalence, non-linear rules, DPO rewriting.

1 Introduction

Shortly, shift equivalence can be defined as the smallest equivalence relation on derivations, i.e. sequences of rewriting steps, that can be obtained by the iterated switch of sequentially independent steps. A crucial observation, which represented our starting point, is that the notion of sequential independence traditionally adopted for linear graph transfomation systems is not adequate when we allow rules to be non-right linear, i.e., in the presence of fusions. Consider the rules in Fig. 1, typed over the type graph T, where numbers are used to represent the mappings from the interface to the left- and right-hand sides.

Now, let us consider an initial graph with a single node. We can then obtain the derivation $\rho_1 = \delta_1; \delta_2; \delta_3$ in Fig. 2, where δ_2 and δ_3 are sequential independent according to the standard notion.

We can exchange the applications of p_2 and p_3, thus getting the derivation $\rho_2 = \delta_1; \delta_3'; \delta_2'$ in Figure 3. Now, one can exchange again p_3 and p_2, reaching the derivation $\rho_3 = \delta_1; \delta_2'; \delta_3$ in Figure 4, where in δ_2' production p_2 uses the node of the start graph and thus "does not depend" on p_1.

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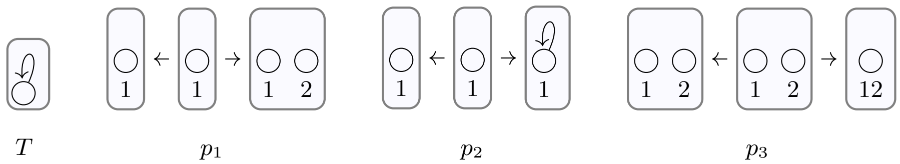


Fig. 1: The type graph and the rules of the grammar of the example.

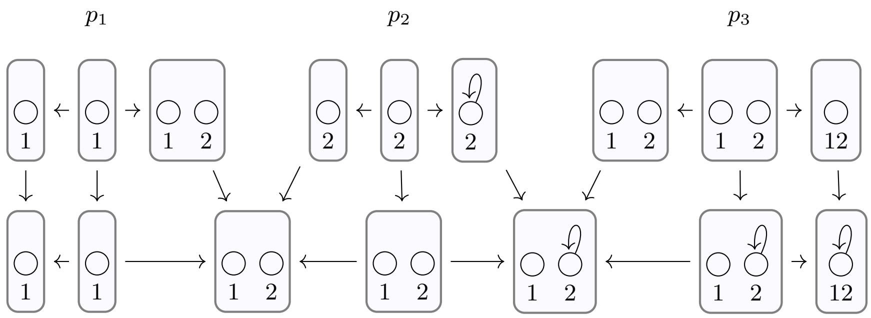


Fig. 2: The derivation ρ_1.

The fact that if we switch two steps and then switch them back, we may obtain a derivation that is not coincident with the original one has unpleasant consequences. For instance, if we restrict ρ_1' and ρ_3' to the first two steps, we get non-equivalent derivations $\delta_1; \delta_2 \not\equiv^{sh} \delta_1; \delta_2'$, but $\rho_1 \equiv^{sh}_{\sigma} \rho_3$, with σ mapping δ_1 to δ_1 and δ_2 to δ_2'. Such a behaviour with respect to composition does not allow to obtain a causal semantics for graph transformation systems. More precisely, it does not allow to associate an event structure to a derivation. Thus, the notion of sequential independence has to be refined. The point is that in the associated shift equivalence we want to forget who made the fusion, but we should not forget (as it happens in the example) that some fusions occurred.

Shift equivalence is made stricter by asking that two steps are sequentially independent if the result of their switch is uniquely determined. Intuitively, when there are several way of performing the exchange, it means that the first step has executed some fusions that the second step is using in an essential way, hence they should not considered independent. Formally, this boils down to require that in the definition of sequential independence the interchange pair is unique.

The paper builds on this intuition: it introduces this stricter notion of sequential independence (Section xxx) and prove that the associated theory of shift equivalence on derivations is well-behaved with respect to the concatenation of rewriting steps (Section xxx). Finally, it outlines how the relationship with event structure works, wrapping up the paper with comparison with related woks and hinting at further direction of research (Section xxx).

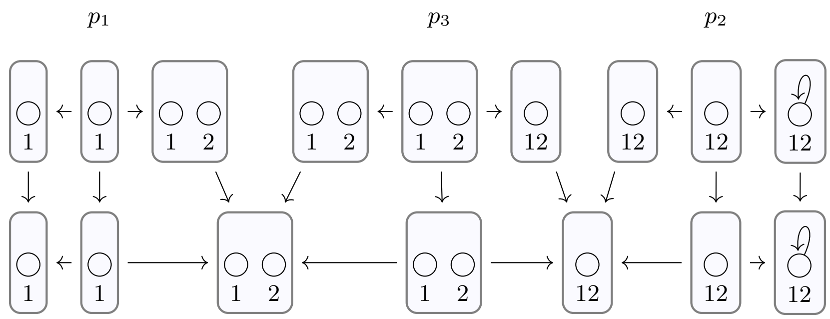


Fig. 3: The derivation ρ_2.

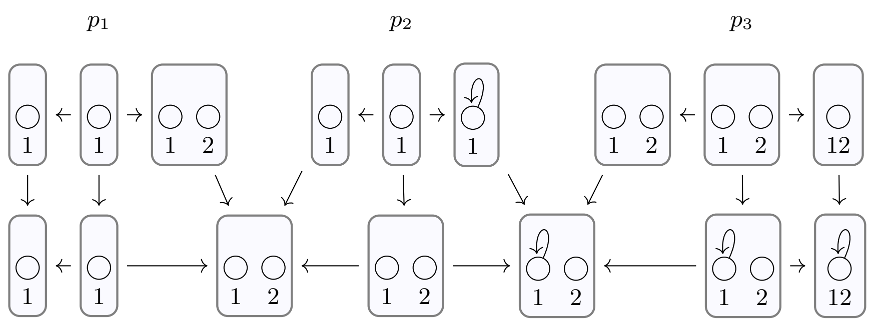


Fig. 4: The derivation ρ_3.

2 Preliminaries [to be redone]

Definition 1 (sequential independence). *Consider a derivation* $G \overset{p_1/m_1}{\Longrightarrow} H \overset{p_2/m_2}{\Longrightarrow} M$ *as in Fig. 5. Then, its components are* sequentially independent *if there exists a unique* independence pair *among them, i.e., two graph morphisms* $i_1 : R_1 \to D_2$ *and* $i_2 : L_2 \to D_1$ *such that* $l_2^* \circ i_1 = m_{R_1}$, $r_1^* \circ i_2 = m_{L_2}$.

Observe that this notion reduces to the usual one for linear rules. In fact when r_1^* and l_2^* are mono the uniqueness requirement is trivially satisfied. Instead, for left-injective rules i_1 is still unique, but i_2 might be not (as it happens, e.g., in the example before).

Say more about the example

Observe that the notion is not symmetric: if $\rho = G \overset{p_1/m_1}{\Longrightarrow} H \overset{p_2/m_2}{\Longrightarrow} M$ is sequential independent and we switch it, we could obtain a pair which is not because it fails to satisfy uniqueness of the independence pair (this happens, e.g., in our example, where $\delta_2; \delta_3$ is sequential independent, but $\delta_3'; \delta_2'$ is not). The idea is to define shift equivalence as $\rho \equiv_{sh} \rho'$ when *each of the two* can be obtained from the other by exchanging sequential independent derivations. More

$$
\begin{tikzcd}
L_1 \arrow[d, "m_{L_1}"'] & K_1 \arrow[l, "l_1"'] \arrow[r, "r_1"] \arrow[d, dashed, "m_{K_1}"] & R_1 \arrow[dr, "m_{R_1}"] \arrow[drrr, dotted, "i_1" near end] & & L_2 \arrow[dl, "m_{L_2}"'] \arrow[dlll, dotted, "i_2"' near end] & K_2 \arrow[l, "l_2"'] \arrow[r, "r_2"] \arrow[d, dashed, "m_{K_2}"] & R_2 \arrow[d, "m_{R_2}"] \\
G & D_1 \arrow[l, "l_1^*"] \arrow[rr, "r_1^*"'] & & H & & D_2 \arrow[ll, "l_2^*"] \arrow[r, "r_2^*"'] & M
\end{tikzcd}
$$

Fig. 5: Sequential independence for $\rho = G \overset{p_1/m_1}{\Longrightarrow} H \overset{p_2/m_2}{\Longrightarrow} M$.

$$
\begin{tikzcd}
\delta_1 & & & & \rho & & & & \delta_n \\
L_1 \arrow[d] & K_1 \arrow[l, tail] \arrow[r] \arrow[d] & R_1 \arrow[d] \arrow[drrrrr, dashed, "i_{1,n}" near start] & L_2 \arrow[dl] & \cdots \arrow[l, dotted, tail] \arrow[r, dotted] & R_{n-1} \arrow[dr] & L_n \arrow[d] \arrow[dlllll, dashed, "i_{n,1}"' near start] & K_n \arrow[l, tail] \arrow[r] \arrow[d] & R_n \arrow[d] \\
G_1 & D_1 \arrow[l, tail] \arrow[r, tail] \arrow[drrr, dotted] & G_2 \arrow[drr, dotted] & D_2 \arrow[l, tail] \arrow[rr, dotted, no head] \arrow[dr, dotted] & & D_{n-1} \arrow[dl, dotted] & G_n \arrow[l] \arrow[r, tail] \arrow[dll, dotted] & D_n \arrow[r] \arrow[dlll, dotted] & G_{n+1} \\
& & & & C & & & &
\end{tikzcd}
$$

Fig. 6: Independence.

precisely, we have a shift relation, which is used for defining the equivalence as in Fig. 7. It could be better to have a slightly less elegant but more direct definition. Write $\rho \rightsquigarrow^{sh}_{(i,i+1)} \rho'$ when $\rho = \rho_1; \rho_2; \rho_3$ and $\rho' = \rho_1; \rho_2', \rho_3$ with $|\rho_1| = i - 1$, $|\rho_2| = 2$ and $\rho_2' \in IC(\rho_2)$. Then we write $\rho \sqsubseteq^{sh}_{\sigma} \rho'$ if $\rho = \rho_0 \rightsquigarrow^{sh}_{\sigma_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\sigma_n} \rho'$ with $\sigma = \sigma_1; \ldots; \sigma_n$ for some $n \geq 0$ (with the convention that for $n = 0$ we get the identity).

The restricted notion of sequential independence ensures that the interchange operator produces a result that is unique up to abstraction equivalence.

Lemma 1 (IC is unique). *Let ρ be a derivation such that $|\rho| = 2$. If ρ is sequential independent, let us write $IC(\rho) = \{\rho' \mid \rho \rightsquigarrow^{sh} \rho'\}$. Then for all $\rho', \rho'' \in IC(\rho)$ we have $\rho' \equiv \rho''$. Moreover, if ρ is sequential independent then for all $\rho' \in IC(\rho)$, if ρ' is sequential independent and $\rho'' \in IC(\rho')$, we have $\rho \equiv \rho''$.*

It is convenient (at the beginning, at least) to work with shift on derivations where the source and target graph is concrete and the middle can be taken up to iso. I think that everything can be then easily transferred to ctc-equivalence. We say that ρ and ρ' are abstract shift-equivalent and write $\rho \sim \rho'$.

A crucial observation is that independence can be defined also for non-contiguous steps.

Definition 2 (independence). *Let $\delta_1; \rho; \delta_n$ be a derivation as in Fig. 6, where δ_1 and δ_n are direct derivations and ρ is a generic, possibly empty derivation. We say that δ_1 and δ_n are independent if there is a unique independence pair*

$i_{n,1}$ and $i_{1,n}$ satisfying the expected commutation, where C is the colimit of the bottom line (excluding G_1, G_{n+1}).

The fact that the notion of independence makes sense is proved by the following lemma, showing that the notion is preserved by the shift relation.

Lemma 2 (independence is global). *The notion of independence for contiguous derivation steps coincides with sequential independence.*

Moreover, independence is invariant under the shift-relation, i.e.,

- *if $\delta_1;\rho;\delta_2$ and $\rho \sqsubseteq^{sh} \delta'$ then in $\delta_1;\rho;\delta_2$ the steps δ_1, δ_2 are independent iff they are in $\delta_1;\delta';\delta_2$.*
- *if $\delta_1;\rho;\delta';\delta_2$ and $\delta_2';\delta'' \in IC(\delta';\delta_2)$ then in $\delta_1;\rho;\delta';\delta_2$ the steps δ_1 and δ_2 are independent iff they are in $\delta_1;\rho;\delta_2'$.*
- *same as before, but with shift on the left.*

Proof. To be done. The fact that sequential independence coincides with independence for contiguous derivations is easy.

A *permutation* on $[1,n]$ is a bijection $\sigma : [1,n] \to [1,n]$. We call it a *transposition* when there are $i,j \in [1,n]$, $i \neq j$ such that $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(k) = k$ otherwise. Such a transposition is normally denoted as (i,j). Whenever $j = i+1$ we call it an *adjacent transposition*.

Given a permutation σ on $[1,n]$, the set of inversions of σ is $inv(\sigma) = \{(i,j) \mid i,j \in [1,n-1] \ \wedge \ i < j \ \wedge \ \sigma(i) > \sigma(j)\}$. Clearly, σ is the identity iff $inv(\sigma) = 0$.

Now, if $\sigma \neq id$ there is at least an inversion of the kind $(i,i+1)$. Then we have that $\sigma = (i,i+1);\sigma'$ with $|inv(\sigma')| = |inv(\sigma)|-1$. Hence we can inductively express σ as a (minimal length) composition of adjacent transpositions.

Lemma 3 (on permutations). *Let σ, σ' be permutations on $[1,n]$ and $\sigma = \upsilon_1;\sigma';\upsilon_2$ where υ_1 and υ_2 are transpositions with $\upsilon_1 = (i,j)$ and $\upsilon_2 = (\sigma'(i),\sigma'(j))$. Then $\sigma = \sigma'$.*

Moreover, let υ_1, υ_2 be adjacent transpositions and $\sigma' = \upsilon_1;\ldots,\upsilon_m$ a minimal length decomposition of σ' in terms of adjacent transpositions (that thus operate on inversions). Then, for all $h \in [1,m]$ with $\upsilon_h = (l,m)$ there exists $k \in [1,m]$ such that $\upsilon_k = (\upsilon_{1,h-1}^{-1};\upsilon_1;\upsilon_{1,k-1}(l), \upsilon_{1,h-1}^{-1};\upsilon_1;\upsilon_{1,k-1}(m))$.

Proof. The first part is easy. The second is a consequence of the fact that the minimal decomposition in terms of adjacent transpositions is essentially unique, in the sense that it consists of exchanges of all the inversions. To be formalised.

In words, the lemma above says that if a permutation is decomposed as $\sigma = \upsilon_1;\sigma';\upsilon_2$ with $\sigma' = \upsilon_1';\ldots,\upsilon_n'$, where υ_1 and υ_2 performs a switch and its reverse, and the inner decomposition is minimal, then we can strip the first and last switch and get a decomposition of the same permutation that performs the same switches but possibly in a different order.

Lemma 4 (no need of useless shifts). *Let ρ, ρ' be direct derivations and assume $\rho \sqsubseteq^{sh}_{\sigma} \rho'$. Then $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_n} \rho'$ where $\sigma = \upsilon_1;\ldots;\upsilon_n$ and $\upsilon_j \in inv(\upsilon_{j,n})$ for $j \in \{1,\ldots,n\}$.*

Proof. Let $\rho \sqsubseteq^{sh}_{\sigma} \rho'$. This means that $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_n} \rho'$ with $\sigma = \upsilon_1; \ldots; \upsilon_n$ for some $n \geq 0$.

We show that if some υ_j does not act on an inversion, i.e., $\upsilon_j \notin inv(\sigma_{j,n})$ then we can shorten the proof of $\rho \sqsubseteq^{sh}_{\sigma} \rho'$ of two steps. This will allow us to conclude that we can reduce it to a proof only acting on inversions, as desired.

The key observation is that, if $\upsilon_j \notin inv(\sigma_{j,n})$ then, intuitively, the "same exchange" must be performed in the reverse direction by some υ_h $(h > j)$. Formally, if $\upsilon_j = (i, i+1)$ and $(i, i+1) \notin inv(\upsilon_{i,n})$ then there exists $h > j$ such that $\upsilon_{j,h-1}(i) = \upsilon_{j,h-1}(i+1) + 1$ and $\upsilon_h = (\upsilon_{j,h-1}(i+1), \upsilon_{j,h-1}(i+1) + 1)$. Among all the pairs j, h of this kind, take one with minimal $h - i$. This means that in the middle all transpositions are inversions of $\upsilon_{i+1,h-1}$.

By Lemma 3, υ_i and υ_h can be stripped, i.e., $\upsilon_{i,h} = \upsilon_{i+1,h-1}$. Moreover, again by Lemma 3, $\upsilon_{i+1}, \ldots, \upsilon_{h-1}$ perform exchanges that were already done in the original proof, and thus, since independence is invariant under shift equivalence, as expressed by Lemma 2 they are allowed. Moreover, since by Lemma 1 the result of switching is uniquely determined (up to iso), the resulting derivation is the same as before, hence we can strip υ_i and υ_h, and conclude.

Then we can prove that whenever $\rho \sqsubseteq^{sh}_{\sigma} \rho'$, each decomposition of σ in terms of inversions leads to a valid derivation of ρ' from ρ.

Lemma 5 (independence is global/2). *Let ρ, ρ' be direct derivations. Let $\rho \sqsubseteq^{sh}_{\sigma} \rho'$, with $\sigma \neq id$ and let $\sigma = (i, i+1); \sigma'$ where $(i, i+1) \in inv(\sigma)$. Then there exists ρ'' such that $\rho \sqsubseteq^{sh}_{(i,i+1)} \rho'' \sqsubseteq^{sh}_{\sigma'} \rho'$.*

Proof. Let $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_n} \rho'$ where $\sigma = \upsilon_1; \ldots; \upsilon_n$ and $\upsilon_j \in inv(\upsilon_{j,n})$ for $j \in \{1, \ldots, n\}$, be the decomposition provided by Lemma 4.

We know that steps corresponding to i and $i+1$ are exchanged at some point, namely there is j such that $\upsilon_j = (\upsilon_{1,j-1}(i), \upsilon_{1,j-1}(i+1))$. By Lemma 2 the steps i and $i+1$ are independent already at the beginning and thus we can exchange them first. More precisely,

$$\rho \rightsquigarrow^{sh}{}_{(i,i+1)} \rho'' \rightsquigarrow^{sh}{}_{\upsilon_{1,j-1}} \rho_j \rightsquigarrow^{sh}{}_{\upsilon_{j+1,n}} \rho'.$$

This can be proved by induction on j.

Lemma 6 (more on shift). *if $\rho_1; \rho'_1 \sqsubseteq^{sh}_{\sigma|\sigma'} \rho_2; \rho'_2$, with $|\rho_1| = |\rho_2|$ and $\sigma : [1, |\rho_1| - 1]] \to [1, |\rho_2| - 1]]$ then $\rho_1 \sqsubseteq^{sh}_{\sigma} \rho_2$ and $\rho'_1 \sqsubseteq^{sh}_{\sigma'} \rho'_2$.*

Proof. The results follows by the fact that, from Lemma 4, we derive that

$$\rho_1; \rho'_1 \rightsquigarrow^{sh}_{\upsilon_1|id_{[1..|\rho'_1|]}} \ldots \rightsquigarrow^{sh}_{\upsilon_h|id_{[1..|\rho'_1|]}} \rho_2; \rho'_1 \rightsquigarrow^{sh}_{id_{[1..|\rho_1|]}|\upsilon'_1} \ldots \rightsquigarrow^{sh}_{id_{[1..|\rho_1|]}|\upsilon'_k} \rho_2; \rho'_2$$

where $\upsilon_1; \ldots; \upsilon_h$ is a decomposition of σ in terms of inversions and, similarly, $\upsilon'_1; \ldots; \upsilon'_k$ is a decomposition of σ'. Then $\rho_1 \rightsquigarrow^{sh}_{\upsilon_1} \ldots \rightsquigarrow^{sh}_{\upsilon_h} \rho_2$ and analogously $\rho'_1 \rightsquigarrow^{sh}_{\upsilon'_1} \ldots \rightsquigarrow^{sh}_{\upsilon'_k} \rho'_2$, which leads to the desired conclusion.

Note that above we assume that ρ_1 and ρ'_1 ends in the same graph. To be formalised.

Lemma 7 (uniqueness of permutation). *if $\rho \sqsubseteq^{sh}_{\sigma_1} \rho'$ and $\rho \sqsubseteq^{sh}_{\sigma_2} \rho'$ then $\sigma_1 = \sigma_2$. In particular, if $\rho_1 \sqsubseteq^{sh}_{\sigma_1} \rho_1$ then $\sigma_1 = id$.*

Proof. We proceed by induction on the length of the derivations $\ell = |\rho| = |\rho'|$ and by an inner induction on the number of inversions in σ_1. When $\ell = 0$, the result is trivial. Let us assume $\ell > 0$ and let $i_1 = \sigma_1^{-1}(1)$, $i_2 = \sigma_2^{-1}(1)$. Let $L'_1 \xleftarrow{l'_1} K'_1 \xrightarrow{r'_1} R'_1)$ be the rule used by the first production in ρ'. By Lemma 17(1) in the paper [the implication that holds] σ_1 and σ_2 are consistent permutations of ρ into ρ'. Since they are consuming, it is easy to see that $i_1 = \sigma_1^{-1}(1)$ is uniquely determined by the (equivalence class) of an item in the initial graph consumed by L'_1 and the same applies to $i_2 = \sigma_2^{-1}(1)$, hence we conclude that $i_1 = i_2$.

Now, if $i_1 = i_2 = 1$, then $\sigma_j = (1,1) \mid \sigma'_j$ for $j \in \{1,2\}$ and $\rho = \delta; \rho_1$, $\rho' = \delta; \rho'_1$ where δ is the first direct derivation, and $\rho_1 \sqsubseteq^{sh}_{\sigma'_j} \rho'_1$ for $j \in \{1,2\}$. Since $|\rho_1| = \ell - 1$ we conclude by inductive hypothesis that $\sigma'_1 = \sigma'_2$ and thus $\sigma_1 = \sigma_2$, as desired.

If instead $i_1 = i_2 = i > 1$, since $(i_1 - 1, i_1)$ is an inversion for both σ_1 and σ_2, by Lemma 5 we can perform such a shift first, obtaining

$$\rho \rightsquigarrow^{sh}_{(i-1,i)} \rho_1 \sqsubseteq^{sh}_{(i-1,i);\sigma_1} \rho' \quad \text{and} \quad \rho \rightsquigarrow^{sh}_{(i-1,i)} \rho_1 \sqsubseteq^{sh}_{(i-1,i);\sigma_2} \rho'$$

Since $(i-1,i);\sigma_1$ has an inversion less than σ_1 we conclude by inductive hypothesis that $(i-1,i);\sigma_1 = (i-1,i);\sigma_2$, whence $\sigma_1 = \sigma_2$.

Note that the above works also with the liberal definition of sequential independence.

Lemma 8 (Lemma 17(2) in the paper). *if $\rho; \rho_1 \sqsubseteq^{sh}_{\sigma} \rho; \rho_2$ then $\sigma = id_{[1..|\rho|]} \mid \sigma'$ and $\rho_1 \sqsubseteq^{sh}_{\sigma'} \rho_2$.*

Proof (skecth). Productions are 'uniquely identified by the items they consume, hence productions in ρ are necessarily mapped to themselves, i.e., $\sigma = id_{[1..|\rho|]} \mid \sigma'$. Then we conclude $\rho_1 \sqsubseteq^{sh}_{\sigma'} \rho_2$ by Lemma 6.

Lemma 9 (Lemma 20 in the paper). *Let ψ and ψ' be decorated derivations. Then the following hold:*

1. *Let ψ_1, ψ'_1 be such that $\psi; \psi_1 \equiv^c_\sigma \psi'; \psi'_1$ and let $n = |\{j \in [|\psi|, |\psi;\psi_1| - 1] \mid \sigma(j) < |\psi'|\}|$. Then for all ϕ_2, ϕ'_2 such that $\psi; \phi_2 \equiv^c \psi'; \phi'_2$ it holds $|\phi_2| \geq n$ and there are ψ_2, ψ'_2, ψ_3 such that*
   - $\psi; \psi_1 \equiv^c \psi; \psi_2; \psi_3$
   - $\psi; \psi_2 \equiv^c \psi'; \psi'_2$
   - $|\psi_2| = n$
2. *Let ψ_1, ψ'_1, ψ_2, ψ'_2 be such that $\psi; \psi_1 \equiv^c_{\sigma_1} \psi'; \psi'_1$ and $\psi; \psi_2 \equiv^c_{\sigma_2} \psi'; \psi'_2$ with ψ_1, ψ_2 of minimal length. Then $\psi_1 \equiv^c_\sigma \psi_2 \cdot \nu$, where $\nu : \mathsf{t}(\psi_2) \to \mathsf{t}(\psi_2)$ is some graph isomorphism and $\sigma(j) = \sigma_2^{-1}(\sigma_1(j + |\psi|)) - |\psi|$ for $j \in [0, |\psi_1| - 1]$.*

Proof (sketch). Point 1: The proof should be similar to the one we have already, basing the rearrangement on Lemma 5. Roughly we get $\psi; \psi_1 \equiv^c \psi_a; \psi_b; \psi_2; \psi_3$ and $\psi'; \psi'_1 \equiv^c \psi'_a; \psi'_b; \psi'_2; \psi'_3$ with

$$\psi_a; \psi_b; \psi_2; \psi_3 \equiv^c_{\sigma_a|\sigma_{b2}|\sigma_2} \psi'_a; \psi'_b; \psi'_2; \psi'_3$$

with σ_a over $[1, |\psi_a| - 1]$, σ_{b2} over $[1, |\psi_b; \psi_2| - 1]$ "mapping ψ_b to ψ'_2 and ψ'_b to ψ_2", σ_3 over $[1, |\psi_2| - 1]$. We deduce by Lemma 6 that $\psi_a \equiv^c_{\sigma_a} \psi'_a$, $\psi_b; \psi_2 \equiv^c_{\sigma_{b2}} \psi'_b; \psi'_2$ and $\psi_3 \equiv^c_{\sigma_3} \psi'_3$.

Now, for all ϕ_2, ϕ'_2 such that $\psi; \phi_2 \equiv^c \psi'; \phi'_2$ we can perform the same splitting and this leads to the conclusion that $|\phi| \geq n$ and also it can be expressed as $\psi_2; \phi'_2$, that should subsume also the second point. □

Another idea could be to try to use the inductive characterisation in Figure 7 (but I do not see how to mange the transitive rule).

$$\text{(SH-id)}\ \frac{}{\rho \sqsubseteq^{sh}_{id} \rho} \qquad \text{(SH-exch)}\ \frac{\rho = \upsilon_1; \upsilon_2 \quad \rho' \in IC(\rho)}{\rho \sqsubseteq^{sh}_{(1,2)} \rho'}$$

$$\text{(SH-comp)}\ \frac{\rho_1 \sqsubseteq^{sh}_{\upsilon_1} \rho'_1,\ \rho_2 \sqsubseteq^{sh}_{\upsilon_2} \rho'_2,\ \tau(\rho_1) = \sigma(\rho_2)}{\rho_1; \rho_2 \sqsubseteq^{sh}_{\upsilon_1|\upsilon_2} \rho'_1; \rho'_2}$$

$$\text{(SH-tr)}\ \frac{\rho \sqsubseteq^{sh}_{\sigma} \rho',\ \rho' \sqsubseteq^{sh}_{\sigma'}, \rho''}{\rho \sqsubseteq^{sh}_{\sigma' \circ \sigma} \rho''} \qquad \text{(SH-eq)}\ \frac{\rho \sqsubseteq^{sh}_{\sigma} \rho' \quad \rho' \sqsubseteq^{sh}_{\sigma^{-1}} \rho}{\rho' \equiv^{sh}_{\sigma} \rho}$$

$$\text{(C-abs)}\ \frac{\psi \equiv^a \psi'}{\psi \equiv^c_{id} \psi'} \quad \text{(C-sh)}\ \frac{\rho \equiv^{sh}_{\sigma} \rho'}{\langle \alpha, \rho, \omega \rangle \equiv^c_{\sigma} \langle \alpha, \rho', \omega \rangle} \quad \text{(C-tr)}\ \frac{\psi \equiv^c_{\sigma} \psi' \quad \psi' \equiv^c_{\sigma'} \psi''}{\psi \equiv^c_{\sigma' \circ \sigma} \psi''}$$

Fig. 7: Inductive definition of $\sqsubseteq^{sh}$ and $\equiv^c$, for $\sigma_1 \mid \sigma_2$ the pairing of two permutations.

3 Proving invariance of independence with respect to shift

This is the way I (AC) propose to organize the proof that sequential independence is invariant with respect to shift. I copy and update some results of the previous part. Not everything is formalized completely.

The structure of the proof should be applicable to any notion of *sequential independence* for which we can prove the following facts:

Definition 3 (conditions on sequential independence).

1. *Sequential independent steps can be switched. If two consecutive steps δ_i and δ_{i+1} of a derivation ρ are SI (sequential independent, written $\delta_i \langle \mathrm{SI} \rangle \delta_{i+1}$), then there is a constructive way to obtain a derivation ρ' differing from ρ only for steps δ'_i and δ'_{i+1}, such that $rule(\delta_i) = rule(\delta'_{i+1})$ and $rule(\delta_{i+1}) = rule(\delta'_i)$. In this case we write $\rho \rightsquigarrow^{sh}_i \rho'$.*
2. *Switching is deterministic up to iso. If $\delta_i \langle \mathrm{SI} \rangle \delta_{i+1}$ in ρ and $\rho \rightsquigarrow^{sh}_i \rho'$, $\rho \rightsquigarrow^{sh}_i \rho''$, then $\rho' \equiv \rho''$.*
3. *Sequential independence is preserved by immediate switch. If $\delta_i \langle \mathrm{SI} \rangle \delta_{i+1}$ in ρ and $\rho \rightsquigarrow^{sh}_i \rho'$, then $\delta'_i \langle \mathrm{SI} \rangle \delta'_{i+1}$ in ρ'.*
4. *Switch is reversible. If $\rho \rightsquigarrow^{sh}_i \rho'$ then $\rho' \rightsquigarrow^{sh}_i \rho$.*
5. *As a consequence of the previous points, if $\rho \rightsquigarrow^{sh}_i \rho'$ and $\rho' \rightsquigarrow^{sh}_i \rho''$ then $\rho \equiv \rho''$.*

I think that all conditions are satisfied by the standard definition of SI in DPO with linear rules and by our stronger notion of SI (uniqueness of the two mediating morphism **and** same property of the switched steps) in DPO with possibly non-right-linear rules.

Proposition 1 ($\langle$si$\rangle$ satisfies all the conditions...). *Let's call the relation of Def. 1* weak sequential independence*. Then it satisfies conditions 1 and 2 of Definition 3 (by the classical Church-Rosser Theorem and uniqueness of the mediating morphisms), but not the remaining ones.*

Let sequential independence *be defined as follows: $\delta_i \langle \mathrm{SI} \rangle \delta_{i+1}$ in ρ iff they are weak sequential independent, and if ρ' is obtained by ρ switching δ_i and δ_{i+1} (which is possible by condition 1 and deterministic up to iso by condition 2) then δ'_i and δ'_{i+1} are weak sequential independent in ρ'.*

Then $\langle \mathrm{SI} \rangle$ satisfies all the conditions of Definition 3.

Proof. Conditions 1 and 2 already hold for weak sequential independence. Condition 3 holds by definition. **Condition 4 should be proved explicitly, I fear...**

Definition 4 (Invariance under shift). *A binary relation R on consecutive steps of derivations is* invariant under shift equivalence up to n *if whenever $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_k} \rho'$ with $\sigma = \upsilon_1; \ldots; \upsilon_k$ and $k \leq n$ (thus $\rho \equiv^{sh}_\sigma \rho'$ with a proof made of at most n switches of steps) and $\sigma(i+1) = \sigma(i) + 1$, then $\delta_i \; R \; \delta_{i+1}$ in ρ if and only if $\delta'_{\sigma(i)} \; R \; \delta'_{\sigma(i+1)}$ in ρ'. Relation R is* invariant under shift equivalence*, tout-court, if it is so up to any n.*

In words, two consecutive steps are related by R in a derivation iff they are related by R in any shift-equivalent derivation in which they are again consecutive.

The following is Lemma 4 with a bound n on the length of the proof that two derivations are shift equivalent. Also, it uses as assumption the fact that $\langle \mathrm{SI} \rangle$is invariant w.r.t. shift up to $n-1$: the fact that $n-1$ is sufficient is exploited later in the proof of the main result. I think that it works, because the assumption is used only at the end, when the two reverse transpositions were already stripped, thus the length is $n-2$.

Lemma 10 (no need of useless shifts). *Given an integer n, let us assume that $\langle \mathrm{SI} \rangle$ is invariant under shift up to $n-1$. Let ρ, ρ' be direct derivations* *such that* $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_k} \rho'$ *with* $k \leq n$*, and let* $\sigma = \upsilon_1; \ldots; \upsilon_k$*. Then* $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon'_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon'_h} \rho'$ *where* $\sigma = \upsilon'_1; \ldots; \upsilon'_h$*,* $h \leq k$ *and* $\upsilon'_j \in inv(\upsilon'_{j,h})$ *for* $j \in \{1, \ldots, h\}$*.*

Proof. Let $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_k} \rho'$. We show that if some υ_j does not act on an inversion, i.e., $\upsilon_j \notin inv(\upsilon_{j,k})$ then we can shorten the proof of $\rho \equiv^{sh}_{\sigma} \rho'$ of two steps. This allow us to conclude that we can reduce the original proof to one which is not longer and only acting on inversions, as desired.

The key observation is that, if $\upsilon_j \notin inv(\sigma_{j,k})$ then, intuitively, the "same exchange" must be performed in the reverse direction by some υ_s, with $s > j$. Formally, if $\upsilon_j = (i, i+1)$ and $(i, i+1) \notin inv(\upsilon_{i,k})$ then there exists $s > j$ such that $\upsilon_{j,s-1}(i) = \upsilon_{j,s-1}(i+1) + 1$ and $\upsilon_s = (\upsilon_{j,s-1}(i+1), \upsilon_{j,s-1}(i+1)+1)$. Among all the pairs j, s of this kind, take one with minimal $s - j$. This means that in the middle all transpositions are inversions of $\upsilon_{j+1,s-1}$.

By Lemma 3, υ_j and υ_s can be stripped, i.e., $\upsilon_{j,s} = \upsilon_{j+1,s-1}$. Moreover, again by Lemma 3, $\upsilon_{i+1}, \ldots, \upsilon_{s-1}$ perform exchanges that were already done in the original proof, and thus, since independence is invariant under shift equivalence up to $n-1$, they are allowed. Moreover, since by condition 2 of Definition 3 the result of switching is uniquely determined (up to iso), the resulting derivation is the same as before, hence we can strip υ_j and υ_s, and conclude.

The next one is Lemma 5, again bounded w.r.t. the length of the proof of shift equivalence. **Note:** It is not used in the following.

Then we can prove that whenever $\rho \equiv^{sh}_{\sigma} \rho'$, each decomposition of σ in terms of inversions leads to a valid valid proof of the shift equivalence of ρ and ρ.

Lemma 11 (independence is global/2). *Given an integer* n*, let us assume that* $\langle \textsc{si} \rangle$ *is invariant under shift up to* n*. Let* ρ*,* ρ' *be direct derivations such that* $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_k} \rho'$*, with* $0 < k \leq n$*,* $\sigma = \upsilon_1; \ldots; \upsilon_k$ *and* $\upsilon_j \in inv(\upsilon_{j,k})$ *for* $j \in \{1, \ldots, k\}$*. Let* $\sigma = (i, i+1); \sigma'$ *where* $(i, i+1) \in inv(\sigma)$*. Then there exists* ρ'' *such that* $\rho \equiv^{sh}_{(i,i+1)} \rho'' \equiv^{sh}_{\sigma'} \rho'$*.*

Proof. We know that steps corresponding to i and $i+1$ are exchanged at some point, namely there is j such that $\upsilon_j = (\upsilon_{1,j-1}(i), \upsilon_{1,j-1}(i+1))$. Since $\langle \textsc{si} \rangle$ is invariant up to n, the steps i and $i+1$ are independent already at the beginning and thus we can exchange them first. More precisely,

$$\rho \rightsquigarrow^{sh}{}_{(i,i+1)} \rho'' \rightsquigarrow^{sh}{}_{\upsilon_{1,j-1}} \rho_j \rightsquigarrow^{sh}{}_{\upsilon_{j+1,k}} \rho'.$$

This can be proved by induction on j.

The following main lemma needs a proof.

Lemma 12 (Invariance of $\langle$si$\rangle$ under immediate shift).

1. *Let* $\rho = \delta_1; \delta_2; \delta_3$ *be a derivation such that* $\rho \rightsquigarrow^{sh}_{(2,3)} \rho' \rightsquigarrow^{sh}_{(1,2)} \rho''$*, with* $\delta_2 \langle \textsc{si} \rangle \delta_3$ *in* ρ*,* $\rho' = \delta_1; \delta'_3; \delta'_2$*,* $\delta_1 \langle \textsc{si} \rangle \delta'_3$ *in* ρ'*, and that* $\rho'' = \delta''_3; \delta'_1; \delta'_2$*. Then* $\delta_1 \langle \textsc{si} \rangle \delta_2$ *in* ρ *if and only if* $\delta'_1 \langle \textsc{si} \rangle \delta'_2$ *in* ρ''*.*

The situation can be represented as in the following diagram, where each transposition is performed, downwards, only if the two starting steps are sequential independent. The proposition states that the steps ① are sequential independent if and only if so are the steps ②. For brevity, we abbreviate this statement to $SI(①) \Leftrightarrow SI(②)$.

$$
\begin{array}{rcccccc}
 & & & ① & & & \\
\rho & = & \delta_1 & ; & \delta_2 & ; & \delta_3 \\
{\scriptstyle (2,3)}\wr_{sh} & & & & & & \\
\rho' & = & \delta_1 & ; & \delta'_3 & ; & \delta'_2 \\
{\scriptstyle (1,2)}\wr_{sh} & & & & & & \\
\rho'' & = & \delta''_3 & ; & \delta'_1 & ; & \delta'_2 \\
 & & & & & ② &
\end{array}
$$

2. *Also a symmetric statement holds, for which we show only the graphical representation. Again, we show that $SI(①) \Leftrightarrow SI(②)$.*

$$
\begin{array}{rcccccc}
 & & & & & ① & \\
\rho & = & \delta_1 & ; & \delta_2 & ; & \delta_3 \\
{\scriptstyle (1,2)}\wr_{sh} & & & & & & \\
\rho' & = & \delta'_2 & ; & \delta'_1 & ; & \delta_3 \\
{\scriptstyle (2,3)}\wr_{sh} & & & & & & \\
\rho'' & = & \delta'_2 & ; & \delta'_3 & ; & \delta''_1 \\
 & & & ② & & &
\end{array}
$$

Proof. **To be done...**

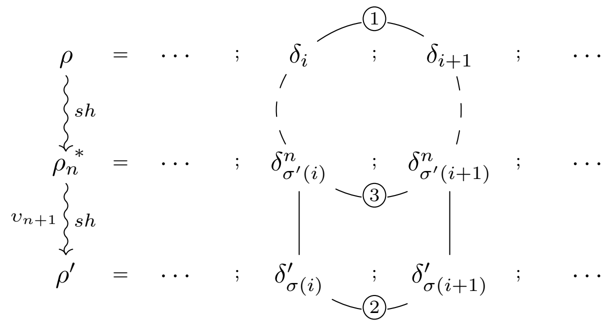
We are now able to state and prove the main result.

Theorem 1 (Invariance of $\langle$si$\rangle$). *Relation $\langle$SI$\rangle$ is invariant under shift equivalence, as for Definition 4.*

Proof. Let $\rho \equiv^{sh}_{\sigma} \rho'$ be two shift equivalent derivations and let $1 \leq i < |\rho|$ such that $\sigma(i+1) = \sigma(i)+1$. We have to show that $\delta_i \ \langle\text{SI}\rangle \ \delta_{i+1}$ in ρ iff $\delta_{\sigma(i)} \ \langle\text{SI}\rangle \ \delta_{\sigma(i+1)}$ in ρ'.

Since $\rho \equiv^{sh}_{\sigma} \rho'$, we have $\rho = \rho_0 \rightsquigarrow^{sh}_{\upsilon_1} \rho_1 \ldots \rightsquigarrow^{sh}_{\upsilon_n} \rho'$, with $\sigma = \upsilon_1; \ldots; \upsilon_n$. We proceed by induction on n. If $n = 0$, then $\sigma = id$ and the statement trivially holds. Now we assume that the statement holds for each $k \leq n$, and we show that it holds for $n+1$. Let σ' be defined as $\sigma' = \upsilon_1; \ldots; \upsilon_n$, thus $\sigma = \sigma'; \upsilon_{n+1}$. We proceed by case analysis on the last adjacent transposition $\upsilon_{n+1} = (z, z+1)$.

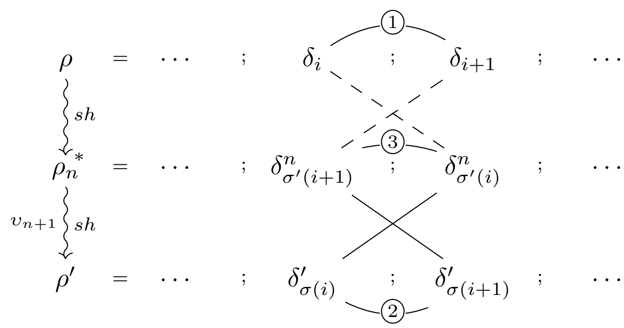
$[z+1 < \sigma'(i)$ **or** $\sigma'(i+1) < z]$ In this case the last transposition does not affect the steps $\delta^n_{\sigma'(i)}$ and $\delta^n_{\sigma'(i+1)}$ of ρ_n, as depicted here:



But is it obvious that if two steps keep their position the remain shift-equivalent? Maybe this is another condition...

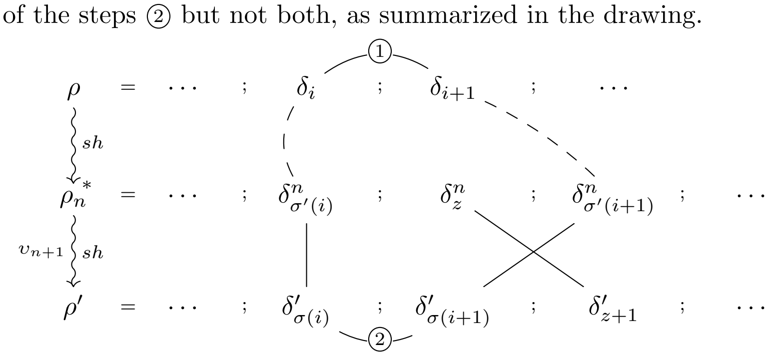
Therefore clearly $SI(2) \Leftrightarrow SI(3)$. Furthermore $SI(1) \Leftrightarrow SI(3)$ by induction hypothesis, because $\rho \overset{sh}{\rightsquigarrow}^{*} \rho_n$ has length n, and thus we conclude $SI(1) \Leftrightarrow SI(2)$.

[$z = \sigma'(i+1)$ **and** $\sigma'(i) = z+1$] In this case the last transposition switches the steps $\delta^n_{\sigma'(i+1)}$ and $\delta^n_{\sigma'(i)}$ of ρ_n, as depicted here:

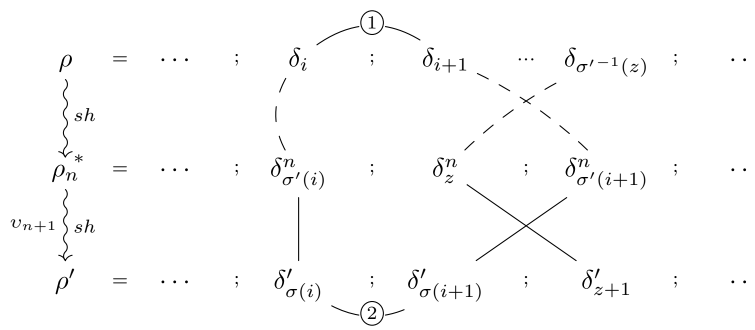


Clearly we have $SI(3)$ because the two steps are switched, and thus also $SI(2)$ by conditions 3 and 4 of Definition 3. Furthermore, as evident from the drawing the steps δ_i and δ_{i+1} of ρ must have been switched by a transposition $\upsilon_j = (v, v+1)$ with $1 \leq j \leq n$. Therefore we have that steps δ^{j-1}_v and δ^{j-1}_{v+1} are sequential independent in ρ^{j-1}, and by induction hypothesis (because $j \leq n$) we infer $SI(1)$, allowing us to conclude that $SI(1) \Leftrightarrow SI(2)$.

[$z+1 = \sigma'(i+1)$**, thus** $z = \sigma(i+1)$] In this case the transposition υ_{n+1} affects one of the steps 2 but not both, as summarized in the drawing.

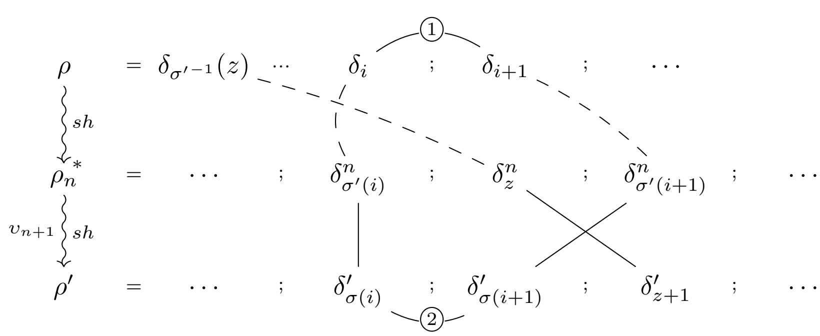


There are two subcases, depending on whether step δ^n_z originates from a step in ρ after δ_{i+1} or before δ_i. In the first case we have:

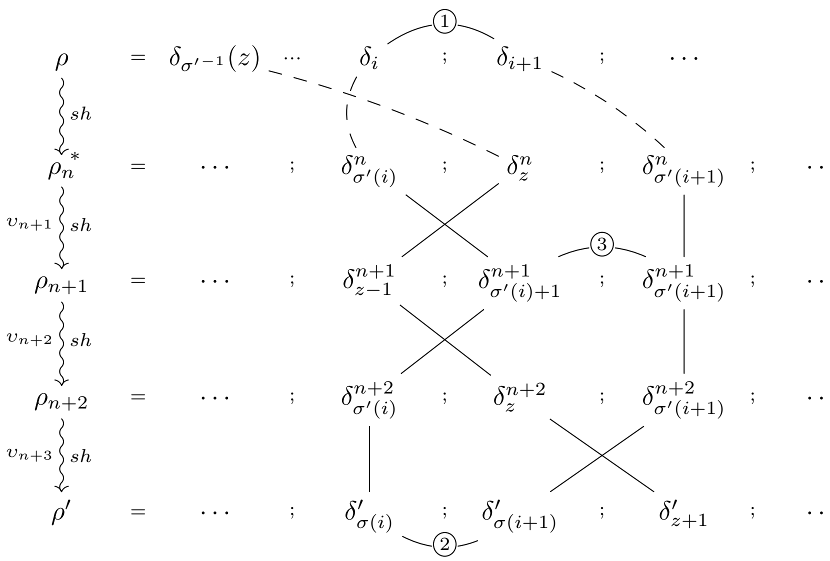


As evident form the drawing, in this case there are certainly some useless transpositions. Thus by Lemma 10 (used on proofs of length at most $n+1$, with the induction hypothesis up to length n) there exists an equivalent proof of $\rho \equiv^{sh}_{\sigma} \rho'$ of length at most $n-1$, thus we can conclude by inductive hypothesis.

The second case is depicted as follows:



We see that two steps from which $\delta^n_{\sigma'(i)}$ and δ^n_z originate were switched by a transposition υ_j with $1 \leq j \leq n$ therefore by induction hypothesis we infer that $\delta^n_{\sigma'(i)}$ and δ^n_z are sequentially independent. Thus we can modify the proof as follows, switching such steps twice:



Now we can conclude observing that $SI(②) \Leftrightarrow SI(③)$ by point 2 of Lemma 12, and that $SI(①) \Leftrightarrow SI(③)$ can be proved by applying the first subcase of this step of the proof.

Lemma 12 considera solo derivazioni di lunghezza 3. Anche qui serve un lemma che garantisce che vale anche in un contesto piu' ampio.

[$z = \sigma'(i)$, thus $z + 1 = \sigma(i)$] This case is completely symmetrical to the previous one, exploiting in the last subcase point 1 of Lemma 12.

References