Switch equivalence and weak prime domains for fusions

Paolo Baldan   Davide Castelnovo   Andrea Corradini
Fabio Gadducci

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Abstract

A VERY NICE ABSTRACT

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This first section is devoted to recall the definition and the basic theory of *$\mathcal{M}$-adhesive categories* [1, 7, 8, 13].

Notation. We stipulate here some notational conventions which will be used throughout this paper.

- Given a category $\mathbf{X}$ we will not distinguish notationally between $\mathbf{X}$ and its class of objects: so that "$X \in \mathbf{X}$" means that X belongs to the class of objects of $\mathbf{X}$.

- If 1 is a terminal object in a category $\mathbf{X}$, the unique arrow $X \to 1$ from another object X will be denoted by $!_X$. Similarly, if 0 is initial in $\mathbf{X}$ then $?_X$ will denote the unique arrow $0 \to X$.

- $\mathsf{Mor}(\mathbf{X})$, $\mathsf{Mono}(\mathbf{X})$ and $\mathsf{Reg}(\mathbf{X})$ will denote the class of all arrows, monos and regular monos of $\mathbf{X}$, respectively.

- Given an integer $n \in \mathbb{Z}$, $[0, n]$ will denote the set

  $$[0, n] := \{x \in \mathbb{N} \mid x \leq n\}$$

  Indeed, if $n < 0$, then $[0, n] = \emptyset$.

2.1 The Van Kampen property

The key property that $\mathcal{M}$-adhesive categories enjoy is given by the so-called *Van Kampen condition* [3, 12, 13]. We will recall it and examine some of its consequences. First of all we need to recall some terminology and facts regarding subclasses of $\mathsf{Mor}(\mathbf{X})$.

Definition 2.1. Let $\mathbf{X}$ be a category and $\mathcal{A}$ a subclass of $\mathsf{Mor}(\mathbf{X})$. We say that $\mathcal{A}$ is

- *stable under pushouts (pullbacks)* if for every pushout (pullbacks) square

  $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

  if $m \in \mathcal{A}$ ($n \in \mathcal{A}$) then $n \in \mathcal{A}$ ($m \in \mathcal{A}$);

- *closed under composition* if $g, f \in \mathcal{A}$ implies $g \circ f \in \mathcal{A}$ whenever g and f are composable;

- *closed under decomposition* if $g \circ f, g \in \mathcal{A}$ implies $f \in \mathcal{A}$.

Remark 2.2. In the previous definition, "decomposition" corresponds to "left cancellation", but we prefer to stick to the name commonly used in literature (see e.g. [10]).

So equipped, we can introduce the notion of *$\mathcal{A}$-Van Kampen square*.

Definition 2.3 (Van Kampen property)**.** Let $\mathbf{X}$ be a category and consider the two diagrams below

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a", near start] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'", near start, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f", near start] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, "d", near start, crossing over] &
\end{tikzcd}
$$

Given a class of arrows $\mathcal{A} \subseteq \mathsf{Mor}(\mathbf{X})$, we say that the left square *has the Van Kampen property relatively to* $\mathcal{A}$, or that it is a $\mathcal{A}$*-Van Kampen square* if:

1. is a pushout square;
2. whenever the right cube has pullbacks as back and left faces and the vertical arrows belong to $\mathcal{A}$, then its top face is a pushout if and only if the front and right faces are pullbacks.

Pushout squares which enjoy the "if" half of the second point of the condition above are called $\mathcal{A}$*-stable*.

We will call a $\mathsf{Mor}(\mathbf{X})$-Van Kampen ($\mathsf{Mor}(\mathbf{X})$-stable) square simply a *Van Kampen* (*stable*) square.

Before proceeding further, we recall this classical result about pullbacks.

Lemma 2.4. *Let* $\mathbf{X}$ *be a category, and consider the following diagram in which the right square is a pullback.*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}
$$

Then the whole rectangle is a pullback if and only if the left square is one.

The previous result can be dualised to get an analogous lemma for pushouts.

Lemma 2.5. *Let* $\mathbf{X}$ *be a category, and consider the following diagram in which the left square is a pushout.*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}
$$

Then the whole rectangle is a pushout if and only if the right square is one.

The following proposition establishes a key property of $\mathcal{A}$-Van Kampen squares with a mono as a side: they are not only pushouts, but also pullbacks.

Proposition 2.6. *Let $\mathcal{A}$ be a class of arrows stable under pushouts and containing all the isomorphisms. If $m\colon A \to C$ is mono and belongs to $\mathcal{A}$, then every $\mathcal{A}$-Van Kampen square*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

is also a pullback square and n is a monomorphism.

Proof. We can start considering the cube below and noticing that n, being the pushout of $m \in \mathcal{M}$, belongs to $\mathcal{A}$ too.

$$\begin{tikzcd}
& A \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_A"'] \arrow[dd, "\mathsf{id}_A"' near end] & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[dd, "\mathsf{id}_B"] \\
A \arrow[rr, "g" near start] \arrow[dd, "m"'] & & B \arrow[dd, "n" near start] & \\
& A \arrow[rr, "g" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "f"'] & & D &
\end{tikzcd}$$

By construction the top face of the cube is a pushout and the back one a pullback. The left face is a pullback because m is mono. Thus the $\mathcal{A}$-Van Kampen property yields that the front and the right faces are pullbacks. $\square$

The previous proposition, in turn, allows us to establish the following results.

Lemma 2.7. *Let $\mathcal{A}$ be a class of arrows stable under pullbacks, pushouts and containing all isomorphisms. Suppose that, the left square below is $\mathcal{A}$-Van Kampen, while the vertical faces in the right cube are pullbacks.*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \arrow[rr, "g'"] \arrow[dl, "m'"'] \arrow[dd, "a"' near end] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "f'" near start] \arrow[dd, "c"'] & & D' \arrow[dd, "d" near start] & \\
& A \arrow[rr, "g" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "f"'] & & D &
\end{tikzcd}$$

Supppose, moreover that $m : A \to C$ and $d : D' \to D$ are mono and that d belongs to $\mathcal{A}$. Then $d \leq n$ if and only if $c \leq m$.

Remark 2.8. Recall that, given two monos $m : M \to X$ and $n : N \to X$ with the same codomain, $m \leq n$ means that there exists a, necessarily unique and necessarily mono, $k : M \to N$ fitting in the triangle below:

$$\begin{tikzcd}
M \arrow[rr, "k"] \arrow[dr, "m"'] & & N \arrow[dl, "n"] \\
& X &
\end{tikzcd}$$

Notice that, if $m \leq n$ and $n \leq m$, then the arrow $k : M \to N$ is an isomorphism.

Remark 2.9. Notice that, since d is a mono and the vertical faces are pullbacks, then a, b and c are monomorphisms too. Moreover, n is mono by Proposition 2.6, so that even m' and n' are monos.

Proof. ($\Rightarrow$) By hypothesis there exists $k : D' \to B$ such that $n \circ k = d$. By Proposition 2.6, the bottom face of the cube is a pullback. Thus there exists a unique $h : C' \to A$ as in the diagram below, implying the thesis.

$$\begin{tikzcd}
C' \arrow[r, "f'"] \arrow[d, "h"] \arrow[dd, bend right, "c"'] & D' \arrow[d, "k"'] \arrow[dd, bend left, "d"] \\
A \arrow[r, "g"] \arrow[d, "m"] & B \arrow[d, "n"'] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

($\Leftarrow$) Let $h : C \to A$ be such that $c = m \circ h$. By the $\mathcal{A}$-Van Kampen property the top face of the given cube is a pushout. Thus the dotted $k : D' \to B$ in the following diagram exists.

$$\begin{tikzcd}
A' \arrow[r, "g'"] \arrow[d, "m'"] \arrow[dd, bend right, "a"'] & B' \arrow[d, "n'"'] \arrow[dd, bend left, "b"] \\
C' \arrow[r, "f'"'] \arrow[d, "h"] & D' \arrow[d, dotted, "k"'] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

Moreover, by construction we have

$$\begin{aligned} n \circ k \circ n' &= n \circ b \\ &= d \circ n' \end{aligned} \qquad \begin{aligned} n \circ k \circ f' &= n \circ g \circ h \\ &= f \circ m \circ h \\ &= f \circ c \\ &= d \circ f' \end{aligned}$$

We can therefore conclude that $n \circ k = d$. $\square$

Finally, we can show that $\mathcal{A}$-stable pushouts enjoy a kind of *pullback-pushout decomposition* property.

Proposition 2.10. *Let* $\mathbf{X}$ *be a category and* $\mathcal{A}$ *a class of arrows stable under pullbacks. Suppose that, in the diagram below, the whole rectangle is an* $\mathcal{A}$*-stable pushout and the right square a pullback.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}$$

If the arrow k *is in* $\mathcal{A}$ *and it is a monomorphism, then both squares are pushouts.*

Proof. We can begin noticing that g, being the pullback of k, is mono and in $\mathcal{A}$ too. Thus we can build the cube below, in which all the vertical faces are pullbacks, entailing that all the vertical arrows are in $\mathcal{A}$.

$$
\begin{tikzcd}
 & & X \arrow[rrr, "a"] \arrow[dl, "f"'] \arrow[ddd, "\mathsf{id}_X"] & & & A \arrow[dl, "h"] \arrow[ddd, "\mathsf{id}_A"] \\
 & Y \arrow[dl, "\mathsf{id}_Y"'] \arrow[ddd, "\mathsf{id}_Y"'] & & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[ddd, "\mathsf{id}_B"'] & \\
Y \arrow[rrr, "b"] \arrow[ddd, "g"'] & & & B \arrow[ddd, "k"] & & \\
 & & X \arrow[rrr, "a"] \arrow[dl, "f"'] & & & A \arrow[dl, "h"] \\
 & Y \arrow[dl, "g"'] & & & B \arrow[dl, "k"] & \\
Z \arrow[rrr, "c"'] & & & C & &
\end{tikzcd}
$$

By hypothesis the face is an $\mathcal{A}$-stable pushout and so its top one is a pushout. Using Lemma 2.5 we can conclude that the right half of the rectangle with which we have started is a pushout too. $\square$

2.2 $\mathcal{M}$-adhesivity

In this section we will define the notion of $\mathcal{M}$-adhesivity [1, 7, 8, 11, 13] and explore some of the consequence of such a property.

Definition 2.11 ($\mathcal{M}$-adhesive category)**.** Let $\mathbf{X}$ be a category and consider a subclass $\mathcal{M}$ of the class $\mathsf{Mono}(\mathbf{X})$ of monomorphisms such that:

1. $\mathcal{M}$ contains all isomorphisms and is closed under composition;
2. $\mathcal{M}$ is stable under pullbacks and pushouts.

We will use $m\colon X \rightarrowtail Y$ to denote that an arrow $m : X \to Y$ belongs to $\mathcal{M}$. $\mathbf{X}$ is said to be $\mathcal{M}$-*adhesive* if

1. for every $m\colon X \rightarrowtail Y$ in $\mathcal{M}$ and $g\colon Z \to Y$, a pullback square

$$
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, tail, "n"'] & X \arrow[d, tail, "m"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}
$$

   exists, such pullbacks will be called $\mathcal{M}$-*pullbacks*;

2. for every $m\colon X \rightarrowtail Y$ in $\mathcal{M}$ and $f\colon X \to Z$, a pushout square

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "m"'] & Z \arrow[d, "q"] \\
Y \arrow[r, "p"'] & Q
\end{tikzcd}
$$

   exists, such pushouts will be called $\mathcal{M}$-*pushouts*;

3. $\mathcal{M}$-pushouts are $\mathcal{M}$-Van Kampen squares.

A category $\mathbf{X}$ is said to be *strictly $\mathcal{M}$-adhesive* if $\mathcal{M}$-pushouts are Van Kampen squares.

Remark 2.12. Our notion of $\mathcal{M}$-adhesivity follows [7, 8] and is different from the one of [1]. What is called $\mathcal{M}$-adhesivity in that paper corresponds to our strict $\mathcal{M}$-adhesivity. Moreover, in [1] the class $\mathcal{M}$ is assumed to be only stable under pullbacks. However, if $\mathcal{M}$ contains all split monos, then stability under pushouts can be deduced from the other axioms [4, Prop. 5.1.21].

Remark 2.13. *Adhesivity* and *quasiadhesivity* as defined in [9,13] coincide with strict $\mathsf{Mono}(\mathbf{X})$-adhesivity and strict $\mathsf{Reg}(\mathbf{X})$-adhesivity, respectively.

A first result we can prove regards closure under decomposition of $\mathcal{M}$.

Proposition 2.14. *Let $\mathcal{A}$ be a class of arrows stable under pullbacks. For every arrow $f\colon X \to Y$ and monomorphism $m\colon Y \to Z$, if $m \circ f \in \mathcal{A}$ then $f \in \mathcal{A}$.*

Proof. Take the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathsf{id}_X"'] & Y \arrow[r, "\mathsf{id}_Y"] \arrow[d, "\mathsf{id}_Y"'] & Y \arrow[d, "m"] \\
X \arrow[r, "f"'] & Y \arrow[r, "m"'] & Z
\end{tikzcd}$$

Since m is mono the right square is a pullback, while the left square is a pullback by construction. By Lemma 2.4 the whole rectangle is a pullback and the thesis follows. □

Corollary 2.15. *In every $\mathcal{M}$-adhesive category $\mathbf{X}$, the class $\mathcal{M}$ is closed under decomposition.*

Another result which can be immediately established, with the aid of Proposition 2.6, is the following one.

Proposition 2.16. *Let $\mathbf{X}$ be an $\mathcal{M}$-adhesive category. Then $\mathcal{M}$-pushouts are also pullback squares.*

From Proposition 2.16, in turn, we can derive the following corollaries.

Corollary 2.17. *In a $\mathcal{M}$-adhesive category $\mathbf{X}$, every $m \in \mathcal{M}$ is a regular mono.*

Proof. Let m be an element of $\mathcal{M}$ and consider its pushout along itself.

$$\begin{tikzcd}
X \arrow[r, tail, "m"] \arrow[d, tail, "m"'] & Y \arrow[d, tail, "f"] \\
Y \arrow[r, tail, "g"'] & Z
\end{tikzcd}$$

By Proposition 2.16 this square is a pullback, proving that m is the equalizer of the arrows $f, g\colon Y \rightrightarrows Z$. □

The following result now follows at once noticing that a regular monomorphism which is also epic is automatically an isomorphism.

Corollary 2.18. *If* $\mathbf{X}$ *is an* $\mathcal{M}$*-adhesive categories, then every epimorphisms in* $\mathcal{M}$ *is an isomorphisms. In particular, every adhesive category* $\mathbf{X}$ *is* balanced*: if a morphism is monic and epic, then it is an isomorphism.*

$\mathcal{M}$-adhesivity is well-behaved with respect to the comma construction [15], as shown by the following theorem.

Theorem 2.19 ([6,13])**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be respectively an* $\mathcal{M}$*-adhesive and an* $\mathcal{M}'$*-adhesive category. Let also* $L : \mathbf{A} \to \mathbf{C}$ *be a functor that preserves* $\mathcal{M}$*-pushouts, and* $R : \mathbf{B} \to \mathbf{C}$ *be a functor which preserves pullbacks.Then* $L{\downarrow}R$ *is* $\mathcal{M}{\downarrow}\mathcal{M}'$*-adhesive, where*

$$\mathcal{M}{\downarrow}\mathcal{M}' := \{(h, k) \in \mathcal{A}(L{\downarrow}R) \mid h \in \mathcal{M}, k \in \mathcal{M}'\}$$

In particular, we can apply this result to slices over and under a given object.

Corollary 2.20. *Let* X *be an object of an* $\mathcal{M}$*-adhesive category* $\mathbf{X}$*. Then* $\mathbf{X}/X$ *and* $X/\mathbf{X}$ *are, respectively, is* $\mathcal{M}/X$*- and* $X/\mathcal{M}$*-adhesive, where*

$$\mathcal{M}/X := \{m \in \mathcal{A}(\mathbf{X}/X) \mid m \in \mathcal{M}\} \qquad X/\mathcal{M} := \{m \in \mathcal{A}(X/\mathbf{X}) \mid m \in \mathcal{M}\}$$

Another categorical construction which preserves $\mathcal{M}$-adhesivity property is the formation of the category of functors.

Theorem 2.21 ([6,13])**.** *If* $\mathbf{X}$ *is an* $\mathcal{M}$*-adhesive category, then for every small category* $\mathbf{Y}$*, the category* $\mathbf{X}^{\mathbf{Y}}$ *of functors* $\mathbf{Y} \to \mathbf{X}$ *is* $\mathcal{M}^{\mathbf{Y}}$*-adhesive, where*

$$\mathcal{M}^{\mathbf{Y}} := \{\eta \in \mathcal{A}(\mathbf{X}^{\mathbf{Y}}) \mid \eta_Y \in \mathcal{M} \textit{ for every } Y \in \mathbf{Y}\}$$

We can list various examples of $\mathcal{M}$-adhesive categories (see [4,5,14]).

Example 2.22. Topos, ipergrafi e grafi

Example 2.23. grafi semplici

Example 2.24. GRAFI GERARCHICI

Example 2.25. term graph

We end this section proving two properties of $\mathcal{M}$-adhesive categories: $\mathcal{M}$-pushout-pullback decomposition and uniqueness of pushouts complements.

Lemma 2.26 ($\mathcal{M}$-pushout-pullback decomposition)**.** *Let* $\mathbf{X}$ *be an* $\mathcal{M}$*-adhesive category and suppose that, in the diagram below, the whole rectangle is a pushout and the right square a pullback.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}$$

Then the following statements hold true:

1. *if* a *belongs to* $\mathcal{M}$ *and* k *is a monomorphism, then both squares are pushouts and pullbacks;*

2. if f and k are in $\mathcal{M}$, then both squares are pushouts and pullbacks.

Proof. 1. By Proposition 2.10, it follows that both squares are pushouts, thus the thesis follows from Proposition 2.16.

2. By hypothesis, g is the pullback of an arrow in $\mathcal{M}$, thus it belongs to it. But then $g \circ f \in \mathcal{M}$ too and the whole rectangle is a $\mathcal{M}$-pushout. Therefore, by Proposition 2.16 a pullback, so that its left half is a pullback too, by Proposition 2.6. Moreover $k \circ h$ is in $\mathcal{M}$ as the pushout of $g \circ f$ and, by Corollary 2.15, we also know that $h \in \mathcal{M}$.

   Using Lemma 2.5, it is enough to show that the left half of the original rectangle is a pushout. We can build the following cube:

$$\begin{tikzcd}
 & & X \arrow[rrr, "a"] \arrow[dl, tail, "f"'] \arrow[ddd, "\mathsf{id}_X"] & & & A \arrow[dl, tail, "h"] \arrow[ddd, "\mathsf{id}_A"] \\
 & Y \arrow[dl, "\mathsf{id}_Y"'] \arrow[ddd, "\mathsf{id}_Y"] & & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[ddd, "\mathsf{id}_B"] & \\
Y \arrow[rrr, "b"] \arrow[ddd, tail, "g"'] & & & B \arrow[ddd, tail, "k"] & & \\
 & & X \arrow[rrr, "a"] \arrow[dl, tail, "f"'] & & & A \arrow[dl, tail, "h"] \\
 & Y \arrow[rrr, "b"] \arrow[dl, tail, "g"'] & & & B \arrow[dl, tail, "k"] & \\
Z \arrow[rrr, "c"'] & & & C & &
\end{tikzcd}$$

   Its vertical faces are all pullbacks and all the vertical arrows are in $\mathcal{M}$, hence the top face is a pushout and we can conclude. ☐

Let us turn our attention to pushout complements.

Definition 2.27 (Pushout complement)**.** Let $f\colon X \to Y$ and $g\colon Y \to Z$ ba two composable arrows in a category $\mathbf{X}$. A *pushout complement* for the pair (f, g) is a pair (h, k) with $h\colon X \to W$ and $k\colon W \to Z$ such that the square below commutes and it is a pushout.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\
W \arrow[r, "k"'] & Z
\end{tikzcd}$$

Example 2.28. In a generic category $\mathbf{X}$, pushout complements may not exist: in $\mathbf{Set}$ the arrows $?_2\colon \emptyset \to 2$ and $!_2\colon 2 \to 1$ do not have a pushout complement.

Moreover, composable arrows $f\colon X \to Y$ and $g\colon Y \to Z$ may have pushout complements which are non-isomorphic: for instance, in $\mathbf{Set}$ the two squares below are both pushouts.

$$\begin{tikzcd}
2 \arrow[r, "!_2"] \arrow[d, "\mathsf{id}_2"'] & 1 \arrow[d, "\mathsf{id}_1"] \\
2 \arrow[r, "!_2"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[r, "!_2"] \arrow[d, "!_2"'] & 1 \arrow[d, "\mathsf{id}_1"] \\
1 \arrow[r, "\mathsf{id}_1"'] & 1
\end{tikzcd}$$

Working in an $\mathcal{M}$-adhesive category we can amend the second defect.

Lemma 2.29 (Uniqueness of pushouts complements)**.** *Let* $\mathbf{X}$ *be a* $\mathcal{M}$*-adhesive category. Given* $m\colon X \rightarrowtail Y$ *in* $\mathcal{M}$ *and* $n\colon Y \to Z$*, let* (h_1, k_1) *and* (h_2, k_2) *be pushout complements of* m *and* n *and* $W_1 = \mathsf{cod}(h_1)$*,* $W_2 = \mathsf{cod}(h_2)$*. Then there exists a unique isomorphism* $f\colon W_1 \to W_2$ *making the following diagram commutative.*

$$\begin{tikzcd}
 & X \arrow[r, tail, "m"] \arrow[d, "h_1"'] \arrow[ddl, bend right, "h_2"'] & Y \arrow[d, "n"] \\
 & W_1 \arrow[r, tail, "k_1"'] \arrow[dl, dotted, "f"] & Z \\
W_2 \arrow[urr, tail, bend right, "k_2"'] & &
\end{tikzcd}$$

Proof. By hypothesis k_1 and k_2, being the pushout of m, are elements of $\mathcal{M}$ and therefore are monomorphisms. In particular, k_2 is a monomorphism and this entails at once the uniqueness of f. Moreover, notice that the squares

$$\begin{tikzcd}
X \arrow[r, tail, "m"] \arrow[d, "h_1"'] & Y \arrow[d, "n"] \\
W_1 \arrow[r, tail, "k_1"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, tail, "m"] \arrow[d, "h_2"'] & Y \arrow[d, "n"] \\
W_2 \arrow[r, tail, "k_2"'] & Z
\end{tikzcd}$$

are $\mathcal{M}$-pushouts and thus $\mathcal{M}$-Van Kampen.

Now, take the $\mathcal{M}$-pullback square

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] & W_1 \arrow[d, tail, "k_1"] \\
W_2 \arrow[r, tail, "k_2"'] & Z
\end{tikzcd}$$

Since $k_1 \circ h_1 = k_2 \circ h_2$, there exists a unique $g\colon X \to P$ fitting in

$$\begin{tikzcd}
X \arrow[dr, dotted, "g"] \arrow[drr, bend left, "h_1"] \arrow[ddr, bend right, "h_2"'] & & \\
 & P \arrow[r, "p_1"] \arrow[d, "p_2"'] & W_1 \arrow[d, tail, "k_1"] \\
 & W_2 \arrow[r, tail, "k_2"'] & Z
\end{tikzcd}$$

We can then build the cubes

$$\begin{tikzcd}
 & X \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_X"'] \arrow[dd, "\mathsf{id}_X"] & & P \arrow[dl, "p_2"'] \arrow[dd, "p_1"] \\
X \arrow[rr, "h_2"] \arrow[dd, tail, "m"'] & & W_2 \arrow[dd, tail, "k_2"] & \\
 & X \arrow[rr, "h_1"] \arrow[dl, tail, "m"] & & W_1 \arrow[dl, tail, "k_1"] \\
Y \arrow[rr, "n"'] & & Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_X"'] \arrow[dd, "\mathsf{id}_X"] & & P \arrow[dl, "p_1"'] \arrow[dd, "p_2"] \\
X \arrow[rr, "h_1"] \arrow[dd, tail, "m"'] & & W_1 \arrow[dd, tail, "k_1"] & \\
 & X \arrow[rr, "h_2"] \arrow[dl, tail, "m"] & & W_2 \arrow[dl, tail, "k_2"] \\
Y \arrow[rr, "n"'] & & Z &
\end{tikzcd}$$

Now, in both cubes the front and left faces are pullbacks, thus, by Lemma 2.4, their back face is a pullback too. Since $m \leq m$, Lemma 2.7 now entails that $k_1 \leq k_2$ and $k_2 \leq k_1$. Thus there exists an isomorphism $f \colon W_1 \to W_2$ such that $k_1 = k_2 \circ f$. To see that $h_2 = f \circ h_1$, we can compute:

$$\begin{aligned} k_2 \circ f \circ h_1 &= k_1 \circ h_1 \\ &= n \circ m \\ &= k_2 \circ h_2 \end{aligned}$$

The claim now follows since k_2 is a monomorphism. □

3 DPO rewriting and derivations

$\mathcal{M}$-adhesive categories are the right context in which to perform abstract rewriting using the so-called "douple pushout approach" (DPO). We will recall the basic definitions and properties of this approach to abstract rewriting.

3.1 Left-linear DPO-rewriting systems

We are now going to study rewriting systems in $\mathcal{M}, \mathcal{N}$-adhesive categories.

Definition 3.1 ([10, 11])**.** Let $\mathbf{X}$ be a $\mathcal{M}$-adhesive category, a *left $\mathcal{M}$-linear* rule ρ is a pair (l, r) of arrows with the same domain, such that l belongs to $\mathcal{M}$. The rule ρ is said to be *$\mathcal{M}$-linear* if $r \in \mathcal{M}$ too. A rule ρ is said to be *consuming* if l is not an isomorphism. We will represent a rule ρ as a span

$$L \xleftarrow{\;l\;} K \xrightarrow{\;r\;} R$$

L is the *left-hand side*, R is the *right-hand side* and K the *glueing object*.

A *left-linear DPO-rewriting system* is a pair $(\mathbf{X}, \mathsf{R})$ where $\mathbf{X}$ is a $\mathcal{M}$-adhesive category and R is a set of left $\mathcal{M}$-linear rules. $(\mathbf{X}, \mathsf{R})$ will be called *linear* if every rule in R is $\mathcal{M}$-linear.

Given two objects G and H and a rule $\rho = (l, r)$ in R, a *direct derivation $\mathscr{D}$ from G to H applying the rule ρ*, is a diagram as the one below, in which both squares are pushouts.

$$\begin{tikzcd} L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\ G & D \arrow[l, "f"] \arrow[r, "g"'] & H \end{tikzcd}$$

The arrow n is called the *match* of the derivation, while h is its *back-match*. We will denote a direct derivation $\mathscr{D}$ between G and H as $\mathscr{D} \colon G \Mapsto H$.

Example 3.2.

esempi di derivazione

Remark 3.3. Let $\mathscr{D} \colon G \Mapsto H$ be the direct derivation

$$\begin{tikzcd} L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\ G & D \arrow[l, "f"] \arrow[r, "g"'] & H \end{tikzcd}$$

If $\phi\colon G' \to G$ and $\psi\colon H \to H'$ are two isomorphisms, we can consider the direct derivation $\phi * \mathscr{D} * \psi\colon G' \Mapsto H'$ given by the following diagram.

$$\begin{tikzcd}
L \arrow[d, "\phi^{-1}\circ n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "\psi\circ h"] \\
G' & D \arrow[l, "\phi^{-1}\circ f"] \arrow[r, "\psi\circ g"'] & H'
\end{tikzcd}$$

In particular, we will use $\phi*\mathscr{D}$ and $\mathscr{D}*\psi$ to denote $\phi*\mathscr{D}*\mathsf{id}_H$ and $\mathsf{id}_G*\mathscr{D}*\psi$.

$\mathcal{M}$-adhesivity of $\mathbf{X}$ guarantes the essential uniqueness of the result obtained rewriting an object, as shown by the next proposition.

Proposition 3.4. *Let $\mathbf{X}$ be a $\mathcal{M}$-adhesive category. Suppose that the two direct derivations $\mathscr{D}$ and $\mathscr{D}'$ below, with the same match and applying the same left $\mathcal{M}$-linear rule ρ are given.*

$$\begin{tikzcd}
L \arrow[d, "m"'] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\
G & D \arrow[l, tail, "f"] \arrow[r, "g"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[d, "m"'] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[d, "k'"] & R \arrow[d, "h'"] \\
G & D' \arrow[l, tail, "f'"] \arrow[r, "g'"'] & H'
\end{tikzcd}$$

Then there exist isomorphisms $t\colon D \to D'$ and $s\colon H \to H'$ as in the following diagram.

$$\begin{tikzcd}
 & & D' \arrow[dll, bend right, "f'"'] \arrow[r, "g'"] & H' \\
G & L \arrow[l, "m"'] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[u, "k'"] \arrow[d, "k"] & R \arrow[u, "h'"'] \arrow[d, "h"'] \\
 & & D \arrow[ull, tail, bend left, "f"] \arrow[r, "g"'] \arrow[uu, dotted, bend left=40, "t"] & H \arrow[uu, dotted, bend right=40, "s"']
\end{tikzcd}$$

Proof. By construction, the pairs (k, f) and (k', f') are pushout complements of l and n. Thus, the existence of the isomorphism $t\colon D \to D'$ follows from Lemma 2.29. Now, computing we have

$$\begin{aligned}
g' \circ t \circ k &= g' \circ k' \\
&= h' \circ r
\end{aligned}$$

Hence, we have the wanted $s\colon H \to H'$. To see that s is an isomorphism, consider the diagram

$$\begin{tikzcd}
K \arrow[r, "k"'] \arrow[rr, bend left, "k'"] \arrow[d, "r"'] & D \arrow[r, "t"'] \arrow[d, "g"] & D' \arrow[d, "g'"] \\
R \arrow[r, "h"] \arrow[rr, bend right, "h'"'] & H \arrow[r, "s"] & H'
\end{tikzcd}$$

By hypothesis the whole rectangle and its left half are pushouts, therefore, by Lemma 2.5 its right square is a pushout too. The claim now follows from the fact that the pushout of an isomorphism is an isomorphism. $\square$

If we look to direct derivations as transitions, it is natural to consider them as edges in a direct graph. Taking objects as vertices objects led us to the following definition [11].

Definition 3.5. Let $(\mathbf{X}, \mathsf{R})$ be a DPO-rewrityng system, with $\mathbf{X}$ $\mathcal{M}$-adhesive. The *DPO-derivation graph* of $(\mathbf{X}, \mathsf{R})$ is the (large) directed graph $\mathsf{G}_R^{\mathbf{X}}$ having as vertices the objects of $\mathbf{X}$ and in which an edge between G and H is a direct derivation $\mathscr{D}\colon G \Mapsto H$. A *derivation* $\underline{\mathscr{D}}$ between two objects G and H is a path between them in $\mathsf{G}_R^{\mathbf{X}}$. The *source* and *target* of $\underline{\mathscr{D}}$ are, respectively, G and H.

Remark 3.6. We can spell out more explicitly what a derivation $\mathscr{D}$ is. An *empty derivation* starting and ending in G is just G itself. A *non-empty derivation* $\mathscr{D}$ is a sequence $\{\mathscr{D}_i\}_{i=0}^n$ of direct derivations such that:

1. for every index i, $\mathscr{D}_i$ is a direct derivation $G_i \Mapsto G_{i+1}$;
2. $G_0 = G$ and $G_{n+1} = H$.

We will call the number $n+1$ the *length* of the derivation, denoted by $\mathsf{lg}(\underline{\mathscr{D}})$. We will also say that an empty derivation has length 0.

Moreover, if every $\mathscr{D}_i$ applies the rule $\rho_i \in R$, then we can define an associated sequence of rules as $r(\underline{\mathscr{D}})$ as $\{\rho_i\}_{i=0}^n$.

Remark 3.7. Consider a derivation $\underline{\mathscr{D}}$ in a DPO-rewriting system $(\mathbf{X}, \mathsf{R})$. We can take the subcategory $\Delta(\underline{\mathscr{D}})$ of $\mathbf{X}$ given by the arrows appearing in $\underline{\mathscr{D}}$. This subcategory comes equipped with an inclusion functor $I(\underline{\mathscr{D}})\colon \Delta(\underline{\mathscr{D}}) \to \mathbf{X}$. Moreover, we can further define $\Delta^{\downarrow}(\underline{\mathscr{D}})$ as the subcategory of $\Delta(\underline{\mathscr{D}})$ containing only the bottom row of the derivation.

Notation. Let $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n$ be a derivation. We will depict the i^{th} element $\mathscr{D}_i$ of $\underline{\mathscr{D}}$ as in the following diagram.

$$\begin{tikzcd}
L_i \arrow[d, "m_i"'] & K_i \arrow[l, "l_i"'] \arrow[r, "r_i"] \arrow[d, "k_i"] & R_i \arrow[d, "h_i"] \\
G_i & D_i \arrow[l, "f_i"] \arrow[r, "g_i"'] & G_{i+1}
\end{tikzcd}$$

Notice that, in particular, if $\underline{\mathscr{D}}\colon G \to H$, then $G_0 = G$ and $G_{n+1} = H$. When $\underline{\mathscr{D}}$ has length 1 we will suppress the indexes. In such case, we will also identify $\underline{\mathscr{D}}$ with its only element.

Example 3.8. esempi di derivazione

Definition 3.9. The *DPO-derivation category* $\mathsf{C}_R^{\mathbf{X}}$ of a DPO-rewriting system $(\mathbf{X}, \mathsf{R})$ is the category in which arrows between G and H are given by, possibly empty, derivations. Composition is concatenation of paths in $\mathsf{G}_R^{\mathbf{X}}$ and identities are given by empty derivations.

Remark 3.10. More explicitly, given $\underline{\mathscr{D}} = \{\mathscr{D}\}_{i=0}^n$ between G and H and $\underline{\mathscr{D}}' = \{\mathscr{D}_i'\}_{i=0}^m$, their concatenation $\underline{\mathscr{D}} \cdot \underline{\mathscr{D}}'$ is the derivation $\{\mathscr{E}_i\}_{i=0}^{m+n+1}$ in which

$$\mathscr{E}_i := \begin{cases} \mathscr{D}_i & i \leq n \\ \mathscr{D}'_{i-(n+1)} & n < i \end{cases}$$

Notice, moreover that, $\underline{\mathscr{D}} \cdot \underline{\mathscr{D}}'$ is equal to $\underline{\mathscr{D}}'$ if $\underline{\mathscr{D}}$ is empty, while it coincides with $\underline{\mathscr{D}}$ if $\underline{\mathscr{D}}'$ has length zero.

Remark 3.3 allows us to compose derivations with isomorphisms.

Definition 3.11. Let $(\mathbf{X}, \mathsf{R})$ be a a left-linear DPO-rewriting system. Given a derivation $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{n}$ between G and H and isomorphisms $\phi\colon G \to G'$, $\psi\colon H \to H'$, the derivations $\phi * \underline{\mathscr{D}}$ and $\underline{\mathscr{D}} * \psi$ are defined as

$$\phi * \underline{\mathscr{D}} := \begin{cases} G' & \mathsf{lg}(\underline{\mathscr{D}}) = 0 \\ \{\phi * \mathscr{D}_0\} \cdot \{\mathscr{D}_i\}_{i=1}^{n} & \mathsf{lg}(\underline{\mathscr{D}}) \neq 0 \end{cases}$$

$$\underline{\mathscr{D}} * \psi := \begin{cases} H' & \mathsf{lg}(\underline{\mathscr{D}}) = 0 \\ \{\mathscr{D}_i\}_{i=0}^{n-1} \cdot \{\mathscr{D}_n * \psi\} & \mathsf{lg}(\underline{\mathscr{D}}) \neq 0 \end{cases}$$

Moreover, if $\mathsf{lg}(\underline{\mathscr{D}}) > 0$, we define the derivation $\phi * \underline{\mathscr{D}} * \psi$ as

$$\phi * \underline{\mathscr{D}} * \psi = \{\phi * \mathscr{D}_0\} \cdot \{\mathscr{D}_i\}_{i=1}^{n-1} \cdot \{\mathscr{D}_n * \psi\}$$

Remark 3.12. When $\underline{\mathscr{D}}$ consists only in the direct derivation $\mathscr{D}$, then $\phi * \underline{\mathscr{D}} * \psi$ is the derivation of length one whose unique element is $\phi * \mathscr{D} * \psi$.

We are often interested in an object of $\mathbf{X}$ only up to isomorphism. It is therefore useful to consider a version of $\mathsf{G}_R^{\mathbf{X}}$ in which vertices are classes of isomorphism of object of $\mathbf{X}$. In order to do so, some preliminary work is needed.

Definition 3.13. [15] Let $\mathbf{X}$ be a category, we say that $\mathbf{X}$ is *skeletal* if, for every two objects X and Y, the existence of an isomorphism $\phi\colon X \to Y$ entails $X = Y$. A *skeleton* for a category $\mathbf{X}$ is a full subcategory $\mathsf{sk}(\mathbf{X})$ which is skeletal and such that the inclusion functor $\mathsf{sk}(\mathbf{X}) \to \mathbf{X}$ is an equivalence.

Remark 3.14. By definition the inclusion $\mathsf{sk}(\mathbf{X}) \to \mathbf{X}$ is an equivalence. In particular, this mean that, for every objects X of $\mathbf{X}$ there exists $\pi(X)$ in $\mathsf{sk}(\mathbf{X})$ and an isomorphism $\phi_X : \pi(X) \to X$.

Proposition 3.15. *Every category* $\mathbf{X}$ *has a skeleton.*

Proof. For every object $X \in \mathbf{X}$, pick a single representative $\pi(X)$ of its isomorphism class. Let $\mathsf{sk}(\mathbf{X})$ be the full subcategory given by these objects. By definition $\mathsf{sk}(\mathbf{X})$ is skeletal and the inclusion functor is full, faithful and essentially surjective. □

Remark 3.16. The proof of Proposition 3.15 relies on the axiom of choice for classes.

Remark 3.17. It is possible to proof that every two skeleta of a given category $\mathbf{X}$ are isomorphic (not only equivalent). For the remaining of this paper we assume that a skeleton $\mathsf{sk}(\mathbf{X})$ of $\mathbf{X}$ and a functor $\pi\colon \mathbf{X} \to \mathsf{sk}(\mathbf{X})$ are chosen once and for all.

Definition 3.18. Let $(\mathbf{X}, \mathsf{R})$ be a DPO-rewriting system, a *decorated derivation* between two objects G and H is a triple $(\underline{\mathscr{D}}, \alpha, \omega)$, where $\underline{\mathscr{D}}$ is a derivation between G and H, and $\alpha\colon \pi(G) \to G$ and $\omega\colon \pi(H) \to H$ are isomorphisms.

Notation. We will extend the use of the words length, source and target to decorated derivations in the obvious way, forgetting the decorations α and ω.

Example 3.19. A decorated derivation $(\underline{\mathscr{D}}, \alpha, \omega)$ with $\underline{\mathscr{D}}$ empty is just a span

$$G \xleftarrow{\alpha} \pi(G) \xrightarrow{\omega} G$$

in which both ω and α are isomorphisms.

As we are interested in objects only up to isomorphism, so we are interested in (decorated) derivations only up to some notion of coherent isomorphism between them. This is done with the help of Remark 3.7.

Definition 3.20. Let $(\mathbf{X}, \mathsf{R})$ be a DPO-rewriting system, an *abstraction equivalence* between two derivations $\underline{\mathscr{D}}$ and $\underline{\mathscr{D}}'$ with the same length and such that $r(\underline{\mathscr{D}}) = r(\underline{\mathscr{D}}')$, is a family of isomorphisms $\{\phi_X\}_{X \in \Delta^{\downarrow}(\underline{\mathscr{D}})}$ such that, for every $i \in [0, \mathsf{lg}(\underline{\mathscr{D}})]$ the following diagram commutes

$$\begin{tikzcd}
G'_i & D'_i \arrow[l, "f'_i"'] \arrow[r, "g'_i"] & G'_{i+1} \\
L_i \arrow[u, "n'_i"] \arrow[d, "n_i"'] & K_i \arrow[l, "l_i"'] \arrow[r, "r_i"] \arrow[u, "k'_i"] \arrow[d, "k_i"'] & R \arrow[u, "h'_i"] \arrow[d, "h_i"'] \\
G_i \arrow[uu, bend right, "\phi_{G_i}"'] & D_i \arrow[l, "f_i"] \arrow[r, "g_i"'] \arrow[uu, bend right, "\phi_{D_i}"'] & G_{i+1} \arrow[uu, bend right, "\phi_{H_i}"']
\end{tikzcd}$$

Given two decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$, we say that they are *abstraction equivalent*, if $\mathsf{lg}(\underline{\mathscr{D}}) = \mathsf{lg}(\underline{\mathscr{D}}')$, $r(\underline{\mathscr{D}}) = r(\underline{\mathscr{D}}')$, and there exists an of abstraction equivalence between $\underline{\mathscr{D}}$ and $\underline{\mathscr{D}}$ such that the triangles below commute.

$$\begin{tikzcd}
& \pi(G_0) \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\
G_0 \arrow[rr, "\phi_{G_0}"'] & & G'_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& \pi(G_{n+1}) \arrow[dl, "\omega"'] \arrow[dr, "\omega'"] & \\
G_{n+1} \arrow[rr, "\phi_{G_{n+1}}"'] & & G'_{n+1}
\end{tikzcd}$$

We will use $\equiv^a$ to denote the resulting relation.

Remark 3.21. It is immediate to see that $\equiv^a$ is an equivalence relation. Indeed $(\underline{\mathscr{D}}, \alpha, \omega)$ is abstract equivalent to itself via the abstract equivalence with the identities as components. Furthermore, if $\{\phi_X\}_{X \in \Delta^{\downarrow}(\underline{\mathscr{D}})}$ witnesses $(\underline{\mathscr{D}}, \alpha, \omega) \equiv_a (\underline{\mathscr{D}}', \alpha', \omega')$, then considering $\{\phi_X^{-1}\}_{X \in \Delta^{\downarrow}(\underline{\mathscr{D}})}$ shows $(\underline{\mathscr{D}}', \alpha', \omega) \equiv_a (\underline{\mathscr{D}}, \alpha, \omega)$. Finally, transitivity is assured composing abstract equivalences.

We will denote by $[\underline{\mathscr{D}}, \alpha, \omega]_a$ is just the equivalence class of $(\underline{\mathscr{D}}, \alpha, \omega)$. Such equivalence classes will be called *abstract decorated derivation.*

Remark 3.22. Let $(\underline{\mathscr{D}}, \alpha, \omega)$ be an empty derivation from an object G and $(\underline{\mathscr{D}}', \alpha', \omega')$ a another empty one from G'. If $(\underline{\mathscr{D}}, \alpha, \omega) \equiv_a (\underline{\mathscr{D}}', \alpha', \omega')$ then an abstraction equivalence between $\underline{(D)}$ and $\underline{\mathscr{D}}'$ is just an isomorphism $\phi\colon G \to G'$, so that $\pi(G) = \pi(G')$. Moreover, such isomorphism must fit in the diagrams below.

$$\begin{tikzcd}
& \pi(G) \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\
G \arrow[rr, "\phi"'] & & G'
\end{tikzcd}
\qquad
\begin{tikzcd}
& \pi(G) \arrow[dl, "\omega"'] \arrow[dr, "\omega'"] & \\
G \arrow[rr, "\phi"'] & & G'
\end{tikzcd}$$

In particular, these two triangles imply that

$$\begin{aligned}\alpha' \circ \alpha^{-1} &= \phi \\ &= \omega' \circ \omega^{-1}\end{aligned}$$

Remark 3.23. Proposition 3.4 can be restated as saying that, given two direct derivations $\mathscr{D}$ and $\mathscr{D}'$ with the same match, there exists an abstract equivalence between them whose first component is an identity.

Example 3.24. Let $\underline{\mathscr{D}}$ be a derivation with source G and target H. Let also $\phi : G' \to G$ and $\psi : H \to H'$ be two isomorphisms. Then for every $X \in \Delta^{\downarrow}(\underline{\mathscr{D}})$ we can define

$$\varphi_X := \begin{cases} \phi^{-1} & X = G \\ \psi & X = H \\ \mathsf{id}_X & \text{otherwise} \end{cases}$$

It is immediate to see that the family $\{\phi_X\}_{X\in\Delta^{\downarrow}(\underline{\mathscr{D}})}$ is an abstraction equivalence between $\underline{\mathscr{D}}$ and $\phi * \underline{\mathscr{D}} * \psi$.

Definition 3.25. Let $(\underline{\mathscr{D}}, \alpha, \omega)$ be a decorated derivation between G and H and $(\underline{\mathscr{D}}', \alpha', \omega')$ one between H' and K. If H and H' are isomorphic, so that $\pi(H) = \pi(H')$, we define the *composite decorated derivation* putting

$$(\underline{\mathscr{D}}, \alpha, \omega) \cdot (\underline{\mathscr{D}}', \alpha', \omega') := \begin{cases} (\underline{\mathscr{D}}', \alpha' \circ \omega^{-1} \circ \alpha, \omega') & \mathsf{lg}(\underline{\mathscr{D}}) = 0 \\ (\underline{\mathscr{D}}, \alpha, \omega \circ (\alpha')^{-1} \circ \omega') & \mathsf{lg}(\underline{\mathscr{D}}') = 0 \text{ and } \mathsf{lg}(\underline{\mathscr{D}}) \neq 0 \\ (\underline{\mathscr{D}} * \omega^{-1} \cdot \alpha' * \underline{\mathscr{D}}', \alpha, \omega') & \text{otherwise} \end{cases}$$

Remark 3.26. Let $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ two composable decorated derivations such that $\mathsf{lg}(\underline{\mathscr{D}}) = n$ and $\mathsf{lg}(\underline{\mathscr{D}}') = m$. Then $(\underline{\mathscr{D}} * \omega^{-1} \cdot \alpha' * \underline{\mathscr{D}}', \alpha, \omega')$ has length $n + m$.

The next proposition justifies the use of decorations, guaranteeing that concatenation of abstract decorated derivations is well-defined.

Lemma 3.27. *Given a decorated derivation $(\underline{\mathscr{D}}, \alpha, \omega)$ between G and H and another one $(\underline{\mathscr{E}}, \beta, \xi)$ between E and K with $\pi(H) = \pi(E)$. If $(\underline{\mathscr{D}}', \alpha', \omega')$ and $(\underline{\mathscr{E}}', \beta', \xi')$ are two other decorated derivations such that*

$$[\underline{\mathscr{D}}, \alpha, \omega]_a = [\underline{\mathscr{D}}', \alpha', \omega']_a \qquad [\underline{\mathscr{D}}, \beta, \xi]_a = [\underline{\mathscr{E}}', \beta', \xi']_a$$

Then

$$[(\underline{\mathscr{D}}, \alpha, \omega) \cdot (\underline{\mathscr{E}}, \beta, \xi)]_a = [(\underline{\mathscr{D}}', \alpha', \omega') \cdot (\underline{\mathscr{E}}', \beta', \xi')]_a$$

Proof. Take two abstraction equivalences $\{\phi_X\}_{X\in\Delta^{\downarrow}(\underline{\mathscr{D}})}$ and $\{\varphi_X\}_{X\in\Delta^{\downarrow}(\underline{\mathscr{D}})}$ between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ and between $(\underline{\mathscr{E}}, \beta, \xi)$ and $(\underline{\mathscr{E}}', \beta', \xi')$, respectively. To fix the notation, suppose that goes from G' to H' and $(\underline{\mathscr{E}}', \beta', \xi')$ from E' to K'. We have three cases.

- $\mathsf{lg}(\underline{\mathscr{D}}) = 0$. Then, $\mathsf{lg}(\underline{\mathscr{D}}')$ is 0 too. By Definition 3.25 we have

$$\begin{aligned}(\underline{\mathscr{D}}, \alpha, \omega) \cdot (\underline{\mathscr{E}}, \beta, \xi) &= (\underline{\mathscr{E}}, \beta \circ \omega^{-1} \circ \alpha, \xi) \\ (\underline{\mathscr{D}}', \alpha', \omega') \cdot (\underline{\mathscr{E}}', \beta', \xi') &= (\underline{\mathscr{E}}', \beta' \circ (\omega')^{-1} \circ \alpha', \xi')\end{aligned}$$

Now, notice that G and H must coincide. Moreover, by Remark 3.22 we also know that $\pi(G) = \pi(G')$ too. The same Remark 3.22 entails that the inner squares of the following diagram are commutative, so that the whole rectangle commutes too.

$$\begin{tikzcd}
\pi(G) \arrow[r, "\alpha"] \arrow[d, "\mathsf{id}_{\pi(G)}"'] & G \arrow[r, "\omega^{-1}"] \arrow[d, "\phi_G"'] & \pi(G) \arrow[r, "\beta"] \arrow[d, "\mathsf{id}_{\pi(G)}"'] & E \arrow[d, "\varphi_E"] \\
\pi(G') \arrow[r, "\alpha'"'] & G' \arrow[r, "(\omega')^{-1}"'] & \pi(G') \arrow[r, "\beta'"'] & E'
\end{tikzcd}$$

We can then conclude that $\{\varphi_X\}_{X\in\Delta^{\downarrow}(\underline{\mathscr{D}})}$ witnesses the fact that $(\underline{\mathscr{E}}, \beta \circ \omega^{-1} \circ \alpha, \xi)$ is abstraction equivalent to $(\underline{\mathscr{E}}', \beta' \circ (\omega')^{-1} \circ \alpha', \xi')$.

- $\mathsf{lg}(\underline{\mathscr{D}}) \neq 0$ and $\mathsf{lg}(\underline{\mathscr{E}}) = 0$. As in the point above, we get that also $\underline{\mathscr{E}}'$ is an empty derivation, thus we have

$$(\underline{\mathscr{D}}, \alpha, \omega) \cdot (\underline{\mathscr{E}}, \beta, \xi) = (\underline{\mathscr{D}}, \alpha, \omega \circ \beta^{-1} \circ \xi)$$
$$(\underline{\mathscr{D}}', \alpha', \omega') \cdot (\underline{\mathscr{E}}', \beta', \xi') = (\underline{\mathscr{D}}', \alpha', \omega' \circ (\beta')^{-1} \circ \xi')$$

In this case we have that $E = K$ and that $\pi(E) = \pi(E')$. From Remark 3.22 we deduce that the diagram below commutes.

$$\begin{tikzcd}
\pi(E) \arrow[r, "\xi"] \arrow[d, "\mathsf{id}_{\pi(E)}"'] & E \arrow[r, "\beta^{-1}"] \arrow[d, "\varphi_E"'] & \pi(E) \arrow[r, "\omega"] \arrow[d, "\mathsf{id}_{\pi(E)}"'] & H \arrow[d, "\phi_H"] \\
\pi(E') \arrow[r, "\xi'"'] & E' \arrow[r, "(\beta')^{-1}"'] & \pi(E') \arrow[r, "\omega'"'] & H'
\end{tikzcd}$$

The thesis now follows at once.

- $\mathsf{lg}(\underline{\mathscr{D}}) \neq 0$ and $\mathsf{lg}(\underline{\mathscr{E}}) \neq 0$. In this case we have

$$(\underline{\mathscr{D}}, \alpha, \omega) \cdot (\underline{\mathscr{E}}, \beta, \xi) = (\underline{\mathscr{D}} * \omega^{-1} \cdot \beta * \underline{\mathscr{E}}, \alpha, \xi)$$
$$(\underline{\mathscr{D}}', \alpha', \omega') \cdot (\underline{\mathscr{E}}', \beta', \xi') = (\underline{\mathscr{D}}' * (\omega')^{-1} \cdot \beta' * \underline{\mathscr{E}'}, \alpha', \xi')$$

To fix the notation, suppose that $\underline{\mathscr{D}}$, $\underline{\mathscr{D}}'$, $\underline{\mathscr{E}}$ and $\underline{\mathscr{E}}'$ are given by

$$\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n \quad \underline{\mathscr{D}}' = \{\mathscr{D}_i'\}_{i=0}^n \quad \underline{\mathscr{E}} = \{\mathscr{E}_i\}_{i=0}^t \quad \underline{\mathscr{E}}' = \{\mathscr{E}_i'\}_{i=0}^t$$

Moreover, noticing that the rule applied by $\mathscr{D}_i$ and the one applied in $\mathcal{E}_i$ must coincide with, respectively, the one applied in $\mathscr{D}_i'$ and the one applied $\mathscr{E}_i'$. We will also assume that $\mathscr{D}_i$, $\mathscr{D}_i'$, $\mathscr{E}_i$ and $\mathscr{E}_i'$ are given, respectively, by the following four diagrams.

$$\begin{tikzcd}
L_{\underline{\mathscr{D}},i} \arrow[d, "m_{\underline{\mathscr{D}},i}"'] & K_{\underline{\mathscr{D}},i} \arrow[l, "l_{\underline{\mathscr{D}},i}"'] \arrow[r, "r_{\underline{\mathscr{D}},i}"] \arrow[d, "k_{\underline{\mathscr{D}},i}"'] & R_{\underline{\mathscr{D}},i} \arrow[d, "h_{\underline{\mathscr{D}},i}"] \\
G_i & D_i \arrow[l, "f_{\underline{\mathscr{D}},i}"] \arrow[r, "g_{\underline{\mathscr{D}},i}"'] & G_{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
L_{\underline{\mathscr{E}},i} \arrow[d, "m_{\underline{\mathscr{E}},i}"'] & K_{\underline{\mathscr{E}},i} \arrow[l, "l_{\underline{\mathscr{E}},i}"'] \arrow[r, "r_{\underline{\mathscr{E}},i}"] \arrow[d, "k_{\underline{\mathscr{E}},i}"'] & R_{\underline{\mathscr{E}},i} \arrow[d, "h_{\underline{\mathscr{E}},i}"] \\
E_i & F_i \arrow[l, "f_{\underline{\mathscr{E}},i}"] \arrow[r, "g_{\underline{\mathscr{E}},i}"'] & E_{i+1}
\end{tikzcd}$$

$$
\begin{tikzcd}
L_{\underline{\mathscr{D}},i} \arrow[d, "m_{\underline{\mathscr{D}}',i}"'] & K_{\underline{\mathscr{D}},i} \arrow[l, "l_{\underline{\mathscr{D}},i}"'] \arrow[r, "r_{\underline{\mathscr{D}},i}"] \arrow[d, "k_{\underline{\mathscr{D}}',i}"'] & R_{\underline{\mathscr{D}},i} \arrow[d, "h_{\underline{\mathscr{D}}',i}"] \\
G'_i & D'_i \arrow[l, "f_{\underline{\mathscr{D}}',i}"] \arrow[r, "g_{\underline{\mathscr{D}}',i}"'] & G'_{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
L_{\underline{\mathscr{E}},i} \arrow[d, "m_{\underline{\mathscr{E}}',i}"'] & K_{\underline{\mathscr{E}},i} \arrow[l, "l_{\underline{\mathscr{E}},i}"'] \arrow[r, "r_{\underline{\mathscr{E}},i}"] \arrow[d, "k_{\underline{\mathscr{E}}',i}"'] & R_{\underline{\mathscr{E}},i} \arrow[d, "h_{\underline{\mathscr{E}}',i}"] \\
E'_i & F'_i \arrow[l, "f_{\underline{\mathscr{E}}',i}"] \arrow[r, "g_{\underline{\mathscr{E}}',i}"'] & E'_{i+1}
\end{tikzcd}
$$

Now, for every $X \in \Delta^{\downarrow}(\underline{\mathscr{D}} * \omega^{-1} \cdot \beta * \underline{\mathscr{E}})$ we can define

$$
\psi_X := \begin{cases} \phi_X & X \in \Delta^{\downarrow}(\underline{\mathscr{D}}) \text{ and } X \neq H \\ \varphi_X & X \in \Delta^{\downarrow}(\underline{\mathscr{E}}) \text{ and } X \neq E \\ \mathsf{id}_{\pi(H)} & X = \pi(H) \end{cases}
$$

Notice that, since $\psi_G = \phi_G$ and $\psi_K = \varphi_K$ we have at once the commutativity of the triangles

$$
\begin{tikzcd}
& \pi(G) \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\
G \arrow[rr, "\psi_G"'] & & G'
\end{tikzcd}
\qquad
\begin{tikzcd}
& \pi(K) \arrow[dl, "\xi"'] \arrow[dr, "\xi'"] & \\
K \arrow[rr, "\psi_K"'] & & K'
\end{tikzcd}
$$

To show that $\{\psi_X\}_{\Delta^{\downarrow}(\underline{\mathscr{D}} * \omega^{-1} \cdot \beta * \underline{\mathscr{E}})}$ is an abstraction equivalence, it is now enough to prove the commutativity of the diagrams

$$
\begin{tikzcd}
G'_n & D'_n \arrow[l, "f_{\underline{\mathscr{D}}',n}"'] \arrow[r, "(\omega')^{-1} \circ g_{\underline{\mathscr{D}}',n}"] \arrow[dr, bend right, "g_{\underline{\mathscr{D}}',n}"'] & \pi(H') \\
& & H' \arrow[u, "(\omega')^{-1}"] \\
L_{\underline{\mathscr{D}},n} \arrow[uu, "m_{\underline{\mathscr{D}}',n}"] \arrow[dd, "m_{\underline{\mathscr{D}},n}"'] & K_{\underline{\mathscr{D}},n} \arrow[l, "l_{\underline{\mathscr{D}},n}"'] \arrow[r, "r_{\underline{\mathscr{D}},n}"] \arrow[uu, "k_{\underline{\mathscr{D}}',n}"'] \arrow[dd, "k_{\underline{\mathscr{D}},n}"'] & R_{\underline{\mathscr{D}},n} \arrow[u, "h_{\underline{\mathscr{D}}',n}"'] \arrow[d, "h_{\underline{\mathscr{D}},n}"'] \\
& & H \arrow[d, "\omega^{-1}"'] \\
G_n \arrow[uuuu, bend right, "\phi_{G_n}"'] & D_n \arrow[l, "f_{\underline{\mathscr{D}},n}"] \arrow[r, "\omega^{-1} \circ g_{\underline{\mathscr{D}},n}"'] \arrow[ur, bend left, "g_{\underline{\mathscr{D}},n}"] \arrow[uuuu, bend right, "\phi_{D_n}"'] & \pi(H) \arrow[uuuu, bend right, "\mathsf{id}_{\pi(H)}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\pi(E') & F'_0 \arrow[l, "(\beta')^{-1} \circ f_{\underline{\mathscr{E}}',0}"'] \arrow[r, "g_{\underline{\mathscr{E}}',0}"] \arrow[dl, bend left, "f_{\underline{\mathscr{E}}',0}"] & E'_1 \\
E' \arrow[u, "(\beta')^{-1}"'] & & \\
L_{\underline{\mathscr{E}},0} \arrow[u, "m_{\underline{\mathscr{E}}',0}"'] \arrow[d, "m_{\underline{\mathscr{E}},0}"] & K_{\underline{\mathscr{E}},0} \arrow[l, "l_{\underline{\mathscr{E}},0}"'] \arrow[r, "r_{\underline{\mathscr{E}},0}"] \arrow[uu, "k_{\underline{\mathscr{E}}',0}"'] \arrow[dd, "k_{\underline{\mathscr{E}},i}"] & R_{\underline{\mathscr{E}},0} \arrow[uu, "h_{\underline{\mathscr{E}}',0}"'] \arrow[dd, "h_{\underline{\mathscr{E}},0}"] \\
E \arrow[d, "\beta^{-1}"] & & \\
\pi(E) \arrow[uuuu, bend left, "\mathsf{id}_{\pi(E)}"] & F_0 \arrow[l, "\beta^{-1} \circ f_{\underline{\mathscr{E}},0}"] \arrow[r, "g_{\underline{\mathscr{E}},0}"'] \arrow[ul, bend right, "f_{\underline{\mathscr{E}},0}"'] \arrow[uuuu, bend left, "\varphi_{F_0}"] & E_1 \arrow[uuuu, bend left, "\varphi_{E_1}"]
\end{tikzcd}
$$

To see this, in turn, it is enough to show that the squares

$$
\begin{tikzcd}
R_{\underline{\mathscr{D}},n} \arrow[r, "h_{\underline{\mathscr{D}'},n}"] \arrow[d, "h_{\underline{\mathscr{D}},n}"'] & H' \arrow[d, "(\omega')^{-1}"] \\
H \arrow[r, "\omega^{-1}"'] & \pi(H)
\end{tikzcd}
\qquad
\begin{tikzcd}
L_{\underline{\mathscr{E}},0} \arrow[r, "m_{\underline{\mathscr{E}'},0}"] \arrow[d, "m_{\underline{\mathscr{E}},0}"'] & E' \arrow[d, "(\beta')^{-1}"] \\
E \arrow[r, "\beta^{-1}"'] & \pi(E)
\end{tikzcd}
$$

$$
\begin{tikzcd}
D_n \arrow[rr, "\phi_{D_n}"] \arrow[d, "g_{\underline{\mathscr{D}},n}"'] & & D'_n \arrow[d, "g_{\underline{\mathscr{D}'},n}"] \\
H \arrow[r, "\omega^{-1}"'] & \pi(H) & H' \arrow[l, "(\omega')^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F_0 \arrow[rr, "\varphi_{F_0}"] \arrow[d, "f_{\underline{\mathscr{E}},0}"'] & & F'_0 \arrow[d, "f_{\underline{\mathscr{E}'},0}"] \\
E \arrow[r, "\beta^{-1}"'] & \pi(E) & E' \arrow[l, "(\beta')^{-1}"]
\end{tikzcd}
$$

are commutative. For the first ones, we have

$$
\begin{aligned}
\omega^{-1} \circ h_{\underline{\mathscr{D}},n} &= \mathsf{id}_{\pi(H)} \circ \omega^{-1} \circ h_{\underline{\mathscr{D}},n} \\
&= (\omega')^{-1} \circ \phi_H \circ \omega \circ \omega^{-1} \circ h_{\underline{\mathscr{D}},n} \\
&= (\omega')^{-1} \circ \phi_H \circ h_{\underline{\mathscr{D}},n} \\
&= (\omega')^{-1} \circ h_{\underline{\mathscr{D}'},n}
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{-1} \circ m_{\underline{\mathscr{E}},0} &= \mathsf{id}_{\pi(E)} \circ \beta^{-1} \circ m_{\underline{\mathscr{E}},0} \\
&= (\beta')^{-1} \circ \varphi_E \circ \beta \circ \beta^{-1} \circ m_{\underline{\mathscr{E}},0} \\
&= (\beta')^{-1} \circ \varphi_E \circ m_{\underline{\mathscr{E}},0} \\
&= (\beta')^{-1} \circ m_{\underline{\mathscr{E}'},0}
\end{aligned}
$$

The commutativity of the second row of diagrams follows from

$$
\begin{aligned}
\omega^{-1} \circ g_{\underline{\mathscr{D}},n} &= \mathsf{id}_{\pi(H)} \circ \omega^{-1} \circ g_{\underline{\mathscr{D}},n} \\
&= (\omega')^{-1} \circ \phi_H \circ \omega \circ \omega^{-1} \circ g_{\underline{\mathscr{D}},n} \\
&= (\omega')^{-1} \circ \phi_H \circ g_{\underline{\mathscr{D}},n} \\
&= (\omega')^{-1} \circ g_{\underline{\mathscr{D}'},n} \circ \phi_{D_n} r
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{-1} \circ f_{\underline{\mathscr{E}},0} &= \mathsf{id}_{\pi(E)} \circ \beta^{-1} \circ f_{\underline{\mathscr{E}},0} \\
&= (\beta')^{-1} \circ \varphi_E \circ \beta \circ \beta^{-1} \circ f_{\underline{\mathscr{E}},0} \\
&= (\beta')^{-1} \circ \varphi_E \circ f_{\underline{\mathscr{E}},0} \\
&= (\beta')^{-1} \circ f_{\underline{\mathscr{E}'},n} \circ \varphi_{F_0}
\end{aligned}
$$

The thesis now follows. □

Definition 3.28. Let $(\mathbf{X}, \mathsf{R})$ be a DPO-rewrityng system, with $\mathbf{X}$ an $\mathcal{M}$-adhesive category. The category $[\mathsf{C}]_R^{\mathbf{X}}$ is defined as follows:

- objects are isomorphism classes of objects of $\mathbf{X}$;
- an arrow $[G] \to [H]$ is an equivalence class $[\underline{\mathscr{D}}, \alpha, \omega]_a$ of a decorated derivation between G' and H' for some G' and H' such that $\pi(G') = G$ and $\pi(H') = H$;
- composition is concatenation of abstract decorated derivations;
- the identity on $[G]$ is $[G, \alpha, \alpha]_a$, where α is any isomorphism $\pi(G) \to G$.

3.2 Consistent permutations

Given DPO-rewriting system $(\mathbf{X}, \mathsf{R})$, we have already noted in Remark 3.7 that a derivation $\underline{\mathscr{D}}$ determines a diagram $\Delta(\underline{\mathscr{D}})$ in $\mathbf{X}$. We can then wonder if such a diagram has a colimit. Clearly if $\underline{\mathscr{D}}$ is the empty derivation G then a colimit for $\Delta(\underline{\mathscr{D}})$ is simply the object G. More generally, we have the following result.

Lemma 3.29. *Let $\mathbf{X}$ be an $\mathcal{M}$-adhesive category and $(\mathbf{X}, \mathsf{R})$ a left-linear DPO-rewriting system over it. The following properties hold true.*

1. *If $\underline{\mathscr{D}}$ is a derivation from G to H, then the diagram $\Delta(\underline{\mathscr{D}})$ has a colimit $(\langle\!|\underline{\mathscr{D}}|\!\rangle, \{\iota_X\}_{X\in\Delta(\underline{\mathscr{D}})})$ such that ι_H belongs to $\mathcal{M}$.*
2. *Let $\underline{\mathscr{D}}$ be the concatenation $\underline{\mathscr{D}}_1 \cdot \underline{\mathscr{D}}_2$ of two derivations $\underline{\mathscr{D}}_1 = \{\mathscr{D}_{1,i}\}_{i=0}^{n_1}$ between G and H and $\underline{\mathscr{D}}_2 = \{\mathscr{D}_{2,j}\}_{j=0}^{n_2}$ between H and T, then the colimiting cocone $(\langle\!|\underline{\mathscr{D}}|\!\rangle, \{\iota_X\}_{X\in\Delta(\underline{\mathscr{D}})})$ exists too and there is a pushout square*

$$\begin{tikzcd} H \arrow[r, "\iota_{2,H}"] \arrow[d, "\iota_{1,H}"'] & \langle\!|\underline{\mathscr{D}}_2|\!\rangle \arrow[d, "p_2"] \\ \langle\!|\underline{\mathscr{D}}_1|\!\rangle \arrow[r, "p_1"'] & \langle\!|\underline{\mathscr{D}}|\!\rangle \end{tikzcd}$$

where $(\langle\!|\underline{\mathscr{D}}_1|\!\rangle, \{\iota_{1,X}\}_{X\in\Delta(\underline{\mathscr{D}}_1)})$ and $(\langle\!|\underline{\mathscr{D}}_2|\!\rangle, \{\iota_{2,X}\}_{X\in\Delta(\underline{\mathscr{D}}_2)})$ are the colimiting cocone for $\Delta(\underline{\mathscr{D}}_1)$ and $\Delta(\underline{\mathscr{D}}_2)$, respectively.

Remark 3.30. Let $I : \Delta^{\downarrow}(\mathscr{D}) \to \Delta(\mathscr{D})$ be the inclusion functor. It is immediate to see that such functor is *final* [15]. This means that for every functor $F \colon \Delta(\mathscr{D}) \to \mathbf{Y}$ we have:

1. if $(C, \{c_X\}_{X\in\Delta^{\downarrow}(\mathscr{D})})$ is colimiting for $F \circ I$, then there exists a colimiting cocone $(D, \{d_X\}_{X\in\Delta(\mathscr{D})})$ for F;
2. $(C, \{c_X\}_{X\in\Delta^{\downarrow}(\mathscr{D})})$ and $(D, \{d_X\}_{X\in\Delta(\mathscr{D})})$ are colimiting for, repsectively, $F\circ I$ and F, then the canonical arrow $\phi\colon C \to D$ induced by $(D, \{d_X\}_{X\in\Delta^{\downarrow}(\mathscr{D})})$ is an isomorphism.

Proof. 1. Let us proceed by induction on the length of $\underline{\mathscr{D}}$.

- $\mathsf{lg}(\mathscr{D}) = 0$. then the $\langle\!|\underline{\mathscr{D}}|\!\rangle$ is simply $(G, \{\mathsf{id}_G\})$ and $\mathsf{id}_G \in \mathcal{M}$.

- $\mathsf{lg}(\mathscr{D}) = 0$. Suppose that $\mathscr{D}$ has as its single component the derivation

$$\begin{tikzcd}
L \arrow[d, "m"'] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\
G & D \arrow[l, tail, "f"] \arrow[r, "g"'] & H
\end{tikzcd}$$

  The arrow f is the pushout of l and so it is in in $\mathcal{M}$. We can thus consider the $\mathcal{M}$-pushout square

$$\begin{tikzcd}
D \arrow[r, "g"] \arrow[d, tail, "f"'] & H \arrow[d, tail, "p"] \\
G \arrow[r, "q"'] & P
\end{tikzcd}$$

  Since $p \in \mathcal{M}$, the thesis follows immediately from Remark 3.30.

- $\mathsf{lg}(\mathscr{D}) \geq 2$. Let $\underline{\mathscr{D}}$ be $\{\mathscr{D}_i\}_{i=0}^{n}$ with $n \geq 1$. Let also $\underline{\mathscr{D}}'$ be $\{\mathscr{D}_i\}_{i=0}^{n-1}$ and $\rho_n = (l_n, r_n)$ be the rule applied in $\mathscr{D}_n$. The pushout of l_n is the arrow $f_n \colon D_n \to G_n$ is in $\mathcal{M}$ and, by inductive hypothesis, $\iota_{G_n} \colon G_n \to \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle$ is in $\mathcal{M}$ too. Thus, we can consider the diagram below, having a pushout as its lower half.

$$\begin{tikzcd}
L_n \arrow[d, "m_n"'] & K_n \arrow[l, tail, "l_n"'] \arrow[r, "r_n"] \arrow[d, "k_n"] & R_n \arrow[d, "h_n"] \\
G_n \arrow[d, tail, "\iota'_{G_n}"'] & D_n \arrow[l, tail, "f_n"] \arrow[r, "g_n"'] & H \arrow[d, tail, "q"] \\
\langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle \arrow[rr, "p"'] & & P
\end{tikzcd}$$

  Notice that, as in the point above, the arrow $q \colon H \to P$ is the pushout of an element in $\mathcal{M}$, therefore it is enough to show that the diagram so constructed provides a colimiting cocone for $\Delta(\underline{\mathscr{D}})$.

  Let $(C, \{c_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ be a cocone, since $\Delta(\underline{\mathscr{D}}')$ is a subdiagram of $\Delta(\underline{\mathscr{D}})$, we get another cocone $(c, \{c_X\}_{X \in \Delta(\underline{\mathscr{D}}')})$ which induces an arrow $c' \colon \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle \to C$ such that

$$\begin{aligned}
c' \circ \iota_{G_n} \circ f_n &= c_{G_n} \circ f_n \\
&= c_{D_n} \\
&= c_H \circ g_n
\end{aligned}$$

  Therefore the arrows c' and c_H induce a morphism $c \colon P \to C$ and the thesis now follows at once.

2. As a first step, notice that $(\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle, \{\iota_X\}_{X \in \Delta(\underline{\mathscr{D}}_1)})$ and $(\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle, \{\iota_X\}_{X \in \Delta(\underline{\mathscr{D}}_2)})$ are cocone on, respectively, $\Delta(\underline{\mathscr{D}}_1)$ and $\Delta(\underline{\mathscr{D}}_2)$. Hence, there exist two arrows $p_1 \colon \langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle$, $p_2 \colon \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle$ such that, for every $X \in \Delta(\underline{\mathscr{D}}_1)$ and $Y \in \Delta(\underline{\mathscr{D}}_2)$

$$p_1 \circ \iota_{1,X} = \iota_X \qquad p_2 \circ \iota_{2,Y} = \iota_{2,Y}$$

In particular, this entails the commutativity of the square

$$\begin{tikzcd}
H \arrow[r, "\iota_{2,H}"] \arrow[d, tail, "\iota_{1,H}"'] \arrow[dr, "\iota_H"] & \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \arrow[d, "p_2"] \\
\langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \arrow[r, "p_1"'] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}$$

Let us now show that the square above is a pushout. Take two arrows $a\colon \langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \to C$, $b\colon \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \to C$ such that

$$a \circ \iota_{1,H} = b \circ \iota_{2,H}$$

We can use the previous equality to define a cocone $(C, \{c_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ putting:

$$c_X := \begin{cases} a \circ \iota_{1,X} & X \in \Delta(\underline{\mathscr{D}}_1) \\ b \circ \iota_{2,X} & X \in \Delta(\underline{\mathscr{D}}_2) \end{cases}$$

From this, we can deduce at once the existence of a unique $c\colon \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to C$ such that

$$c \circ \iota_X = c_X$$

By construction, for every $X \in \Delta(\underline{\mathscr{D}}_1)$ and $Y \in \Delta(\underline{\mathscr{D}}_2)$ we have

$$\begin{aligned} c \circ p_1 \circ \iota_{1,X} &= c \circ \iota_X \\ &= c_X \\ &= a \circ \iota_{1,X} \\ &= a \circ p_1 \circ \iota_{1,X} \end{aligned} \qquad\qquad \begin{aligned} c \circ p_2 \circ \iota_{2,Y} &= c \circ \iota_Y \\ &= c_Y \\ &= b \circ \iota_{2,Y} \\ &= b \circ p_2 \circ \iota_{2,Y} \end{aligned}$$

Therefore

$$c \circ p_1 = a \qquad c \circ p_2 = b$$

For uniqueness, suppose that $c'\colon \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to C$ is such that

$$c' \circ p_1 = a \qquad c' \circ p_2 = b$$

Then, for every $X \in \Delta(\underline{\mathscr{D}})$ we have

$$\begin{aligned} c' \circ \iota_X &= \begin{cases} c' \circ p_1 \circ \iota_{1,X} & X \in \Delta(\underline{\mathscr{D}}_1) \\ c' \circ p_2 \circ \iota_{2,X} & X \in \Delta(\underline{\mathscr{D}}_2) \end{cases} \\ &= \begin{cases} a \circ \iota_{1,X} & X \in \Delta(\underline{\mathscr{D}}_1) \\ b \circ \iota_{2,X} & X \in \Delta(\underline{\mathscr{D}}_2) \end{cases} \\ &= c_X \\ &= c \circ \iota_X \end{aligned}$$

showing that $c' = c$ as wanted. □

Corollary 3.31. *Let $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n$ a derivation of length $n+1$ and fix an index $j \in [0, n]$. Define*

$$\underline{\mathscr{D}}_1^j := \{\mathscr{D}_i\}_{i=0}^{j-1} \qquad \underline{\mathscr{D}}_2^j = \{\mathscr{D}_j\} \qquad \underline{\mathscr{D}}_3^j := \{\mathscr{D}_i\}_{i=j+1}^n$$

with the convention that $\underline{\mathscr{D}}_1^0$ and $\underline{\mathscr{D}}_3^n$ are the empty derivation on, respectively, G_0 and G_n. Then the square below is a pushout and a pullback

$$\begin{tikzcd}
D_j \arrow[r, "g_j"] \arrow[d, tail, "f_j"'] \arrow[ddrr, "\iota_{D_j}"] & G_{j+1} \arrow[r, "\iota_{3,G_{j+1}}"] \arrow[ddr, "\iota_{G_{j+1}}"] & \langle\!\langle \underline{\mathscr{D}}_3^j \rangle\!\rangle \arrow[dd, "p_2"] \\
G_j \arrow[d, tail, "\iota_{1,G_j}"'] \arrow[drr, "\iota_{G_j}"'] & & \\
\langle\!\langle \underline{\mathscr{D}}_1^j \rangle\!\rangle \arrow[rr, "p_1"'] & & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}$$

Where the two arrows $p_1\colon \langle\!\langle \underline{\mathscr{D}}_1^j \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle$, $p_2\colon \langle\!\langle \underline{\mathscr{D}}_3^j \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle$ are induced by the cocones $(\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle, \{\iota_X\}_{X\in\Delta(\underline{\mathscr{D}}_1^j)})$ and $(\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle, \{\iota_X\}_{X\in\Delta(\underline{\mathscr{D}}_3^j)})$, respectively.

Remark 3.32. If $\underline{\mathscr{D}}$ is empty then $\underline{\mathscr{D}}_1^j, \underline{\mathscr{D}}_2^j$ and $\underline{\mathscr{D}}_3^j$ are empty too.

Proof. We can notice that $\underline{\mathscr{D}} = \underline{\mathscr{D}}_1^j \cdot \underline{\mathscr{D}}_2^j \cdot \underline{\mathscr{D}}_3^j$. By the first and the second point of Lemma 3.29 then we get the following diagram, in which all squares are $\mathcal{M}$-pushouts.

$$\begin{tikzcd}
D_j \arrow[r, "g_j"] \arrow[d, tail, "f_j"'] & G_{j+1} \arrow[r, "\iota_{3,G_{j+1}}"] \arrow[d, "\iota_{2,G_{j+1}}"'] \arrow[dd, bend left, "\iota_{1,2,G_{j+1}}"] & \langle\!\langle \underline{\mathscr{D}}_3^j \rangle\!\rangle \arrow[dd, "p_2"] \\
G_j \arrow[r, "\iota_{2,G_j}"] \arrow[d, tail, "\iota_{1,G_j}"'] & \langle\!\langle \underline{\mathscr{D}}_2^j \rangle\!\rangle \arrow[d, "a"'] & \\
\langle\!\langle \underline{\mathscr{D}}_1^j \rangle\!\rangle \arrow[r, "b"] \arrow[rr, bend right, "p_1"'] & \langle\!\langle \underline{\mathscr{D}}_1^j \cdot \underline{\mathscr{D}}_2^j \rangle\!\rangle \arrow[r, "c"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}$$

Applying Lemma 2.5 twice we get that the whole square is an $\mathcal{M}$-pushout. Then the thesis follows from Proposition 2.16. $\square$

Remark 3.33. In particular, considering $j = 0$ or $j = n$, we get that the following two squares are $\mathcal{M}$-pushouts and, thus, pullbacks.

$$\begin{tikzcd}
D_0 \arrow[r, "\iota_{3,D_0}"] \arrow[d, tail, "f_0"'] & \langle\!\langle \underline{\mathscr{D}}_3^0 \rangle\!\rangle \arrow[d, tail, "p_2"] \\
G \arrow[r, "\iota_G"'] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
D_n \arrow[r, "g_n"] \arrow[d, "\iota_{1,D_n}"'] & G_n \arrow[d, tail, "\iota_{G_n}"] \\
\langle\!\langle \underline{\mathscr{D}}_1^n \rangle\!\rangle \arrow[r, "p_1"'] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}$$

Corollary 3.34. *Let $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n$ be a derivation between G and H. Let j and k be two indexes less or equal than $n+1$ and suppose that $j < k$. Consider two arrows $a\colon T \to G_j$, $b\colon T \to G_k$. If $\iota_{G_j} \circ a = \iota_{G_k} \circ b$, then there exist a unique arrow $c\colon T \to D_j$ such that*

$$f_j \circ c = a \qquad \iota_{D_j} \circ c = \iota_{G_k} \circ b$$

Proof. Consider the diagram

$$
\begin{tikzcd}
T \arrow[rr, "b"] \arrow[d, dotted, "c"] \arrow[dd, bend right, "a"'] & & G_k \arrow[d, "\iota_{3,G_k}"'] \arrow[ddd, bend left, "\iota_{G_k}"] \\
D_j \arrow[r, "g_j"] \arrow[d, tail, "f_j"] \arrow[ddrr, "\iota_{D_j}"] & G_{j+1} \arrow[r, "\iota_{3,G_{j+1}}"] \arrow[ddr, "\iota_{G_{j+1}}"] & \langle\!\langle \underline{\mathscr{D}}^j_3 \rangle\!\rangle \arrow[dd, "p_2"'] \\
G_j \arrow[d, tail, "\iota_{1,G_j}"'] \arrow[drr, "\iota_{G_j}"'] & & \\
\langle\!\langle \underline{\mathscr{D}}^j_1 \rangle\!\rangle \arrow[rr, "p_1"'] & & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}
$$

Thanks to Corollary 3.31 we know that the bottom right rectangle in the diagram above is a pullback and the thesis follows at once. $\square$

So equipped, we can introduce the notion of *consistent permutation*.

Definition 3.35 (Consistent permutation)**.** Let $\mathbf{X}$ be an $\mathcal{M}$-adhesive category and consider a left-linear DPO-rewriting system $(\mathbf{X}, \mathsf{R})$ on it. Take two non-empty decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ with the same length and with isomorphic sources and targets.

If $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n$ and $\underline{\mathscr{D}}' = \{\mathscr{D}'_i\}_{i=0}^n$ with assciated sequence of rules $r(\underline{\mathscr{D}}) = \{\rho_i\}_{i=0}^n$ and $r(\underline{\mathscr{D}}') = \{\rho'_i\}_{i=0}^n$, a *consistent permutation* between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ is a permutation $\sigma\colon [0,n] \to [0,n]$ such that, for every $i \in [0,n]$, $\rho_i = \rho'_{\sigma(i)}$ and, moreover, there exists a *mediating isomorphism* $\xi_\sigma\colon \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle$ fitting in the following diagrams, where m_i, m'_i, h_i and h'_i are, respectively, the matches and back-matches of $\mathscr{D}_i$ and $\mathscr{D}'_i$.

$$
\begin{tikzcd}
\pi(G_0) \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & G_0 \arrow[r, "\iota_{G_0}"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi_\sigma"] \\
G'_0 \arrow[rr, "\iota'_{G'_0}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi(G_{n+1}) \arrow[r, "\omega"] \arrow[d, "\omega'"'] & G_{n+1} \arrow[r, tail, "\iota_{G_{n+1}}"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi_\sigma"] \\
G'_{n+1} \arrow[rr, tail, "\iota'_{G'_{n+1}}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}
$$

$$
\begin{tikzcd}
L_i \arrow[r, "m_i"] \arrow[d, "m'_{\sigma(i)}"'] & G_i \arrow[r, "\iota_{G_i}"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi_\sigma"] \\
G'_{\sigma(i)} \arrow[rr, "\iota'_{G'_{\sigma(i)}}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
R_i \arrow[r, "h_i"] \arrow[d, "h'_{\sigma(i)}"'] & G_{i+1} \arrow[r, "\iota_{G_{i+1}}"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi_\sigma"] \\
G'_{\sigma(i)+1} \arrow[rr, "\iota'_{G'_{\sigma(i)+1}}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}
$$

Remark 3.36. The commutativity of the last two rectangles in Definition 3.35, is equivalent to the commutativity of the following bigger diagram.

$$\begin{tikzcd}
G_i \arrow[rrr, bend left=30, "\iota_{G_i}"] & D_i \arrow[l, tail, "f_i"'] \arrow[r, "g_i"] \arrow[rr, bend left=20, "\iota_{D_i}"] & G_{i+1} \arrow[r, "\iota_{G_{i+1}}"'] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[dd, "\xi_\sigma"] \\
L_i \arrow[u, "n_i"] \arrow[d, "n'_{\sigma(i)}"'] & K_i \arrow[l, tail, "l_i"'] \arrow[r, "r_i"] \arrow[u, "k_i"'] \arrow[d, "k'_{\sigma(i)}"] & R \arrow[u, "h_i"] \arrow[d, "h'_{\sigma(i)}"'] & \\
G'_{\sigma(i)} \arrow[rrr, bend right=30, "\iota'_{G'_{\sigma(i)}}"'] & D'_{\sigma(i)} \arrow[l, tail, "f'_{\sigma(i)}"'] \arrow[r, "g'_i"] \arrow[rr, bend right=20, "\iota'_{D'_{\sigma(i)}}"'] & G'_{\sigma(i)+1} \arrow[r, "\iota'_{G'_{\sigma i+1}}"] & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}$$

This follows at once from the following chain of identities::

$$\begin{aligned}
\xi_\sigma \circ \iota_{D_i} \circ k_i &= \xi_\sigma \circ \iota_{G_i} \circ f_i \\
&= \xi_\sigma \circ \iota_{G_i} \circ m_i \circ l_i \\
&= \iota_{G'_{\sigma(i)}} \circ m'_i \circ l_i \\
&= \iota_{G'_{\sigma(i)}} \circ f'_{\sigma(i)} \circ k'_i \\
&= \iota_{D'_\sigma(i)} \circ k'_i
\end{aligned}$$

Remark 3.37. Notice that, in particular, the previous diagram entails

$$\xi_\sigma \circ \iota_{L_i} = \iota'_{L_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{K_i} = \iota'_{K_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{R_i} = \iota'_{R_{\sigma(i)}}$$

Remark 3.38. Let $\sigma\colon [0,n] \to [0,n]$ be a consistent permutation between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$, then its inverse σ^{-1} is a consistent permutation between $(\underline{\mathscr{D}}', \alpha', \omega')$ and $(\underline{\mathscr{D}}, \alpha, \omega)$. Indeed, it is enough to consider, as mediating isomorphism, the inverse ξ_σ^{-1} of ξ_σ.

Remark 3.39. Consistent permutations can be composed. Indeed, given decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$, $(\underline{\mathscr{D}}', \alpha', \omega')$ and $(\hat{\underline{\mathscr{D}}}, \hat{\alpha}, \hat{\omega})$ all of length n, if σ is a consistent permutation between the first two and σ' one between the second and the third, then we have diagrams

$$
\begin{tikzcd}
& G_0 \arrow[r, "\iota_{G_0}"] & \llparenthesis \underline{\mathscr{D}} \rrparenthesis \arrow[d, "\xi_\sigma"] \\
\pi(G_0) \arrow[ur, bend left, "\alpha"] \arrow[r, "\alpha'"] \arrow[dr, bend right, "\hat{\alpha}"'] & G'_0 \arrow[r, "\iota'_{G'_0}"] & \llparenthesis \underline{\mathscr{D}}' \rrparenthesis \arrow[d, "\xi_{\sigma'}"] \\
& \hat{G}_0 \arrow[r, "\hat{\iota}_{\hat{G}_0}"'] & \llparenthesis \underline{\hat{\mathscr{D}}} \rrparenthesis
\end{tikzcd}
\qquad
\begin{tikzcd}
& G_n \arrow[r, tail, "\iota_{G_n}"] & \llparenthesis \underline{\mathscr{D}} \rrparenthesis \arrow[d, "\xi_\sigma"] \\
\pi(G_n) \arrow[ur, bend left, "\omega"] \arrow[r, "\omega'"] \arrow[dr, bend right, "\hat{\omega}"'] & G'_n \arrow[r, tail, "\iota'_{G'_n}"] & \llparenthesis \underline{\mathscr{D}}' \rrparenthesis \arrow[d, "\xi_{\sigma'}"] \\
& \hat{G}_n \arrow[r, tail, "\hat{\iota}_{\hat{G}_n}"'] & \llparenthesis \underline{\hat{\mathscr{D}}} \rrparenthesis
\end{tikzcd}
$$

$$
\begin{tikzcd}
& G_i \arrow[r, "\iota_{G_i}"] & \llparenthesis \underline{\mathscr{D}} \rrparenthesis \arrow[d, "\xi_\sigma"] \\
L_i \arrow[ur, bend left, "m_i"] \arrow[r, "m'_{\sigma(i)}"] \arrow[dr, bend right, "\hat{m}_{\sigma'(\sigma(i))}"'] & G'_{\sigma(i)} \arrow[r, "\iota'_{G'_{\sigma(i)}}"] & \llparenthesis \underline{\mathscr{D}}' \rrparenthesis \arrow[d, "\xi_{\sigma'}"] \\
& \hat{G}_{\sigma'(\sigma(i))} \arrow[r, "\hat{\iota}_{\hat{G}_{\sigma'(\sigma(i))}}"'] & \llparenthesis \underline{\hat{\mathscr{D}}} \rrparenthesis
\end{tikzcd}
\qquad
\begin{tikzcd}
& G_i \arrow[r, "\iota_{G_i}"] & \llparenthesis \underline{\mathscr{D}} \rrparenthesis \arrow[d, "\xi_\sigma"] \\
R_i \arrow[ur, bend left, "h_i"] \arrow[r, "h'_{\sigma(i)}"] \arrow[dr, bend right, "\hat{h}_{\sigma'(\sigma(i))}"'] & G'_{\sigma(i)} \arrow[r, "\iota'_{G'_{\sigma(i)}}"] & \llparenthesis \underline{\mathscr{D}}' \rrparenthesis \arrow[d, "\xi_{\sigma'}"] \\
& \hat{G}_{\sigma'(\sigma(i))} \arrow[r, "\hat{\iota}_{\hat{G}_{\sigma'(\sigma(i))}}"'] & \llparenthesis \underline{\hat{\mathscr{D}}} \rrparenthesis
\end{tikzcd}
$$

We deduce at once that $\sigma' \circ \sigma$ is a consistent permutation with mediating isomorphism given by $\xi_{\sigma'} \circ \xi_\sigma$.

Example 3.40. RISCRIVERE Let $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ be two abstraction equivalent non-empty decorated derivations of length $n+1$. Then the identity permutation $\mathsf{id}_{[0,n]}$ is a consistent permutation between them. In such a case $\xi_{\mathsf{id}_{[0,n]}}$ is simply the isomorphism induced by any abstraction equivalence $\{\phi_X\}_{X \in \Delta(X)}$.

If $\underline{\mathscr{D}}$ and $\underline{\mathscr{D}}'$ are empty, then the converse also holds: indeed in such a case a mediating isomorphism provides the wanted abstraction equivalence.

Example 3.41.

identità consistente non implica astrazione

Proposition 3.42. *Let $\sigma, \sigma' \colon [0,n] \rightrightarrows [0,n]$ be two consistent permutations between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$. Then the following hold true:*

1. $\xi_\sigma \circ \iota_{G_0} = \xi_{\sigma'} \circ \iota_{G_0}$;
2. $\xi_\sigma \circ \iota_{G_i} = \xi_{\sigma'} \circ \iota_{G_i}$ *for every index $i \in [0,n]$ such that* $\sigma_{|[0,i]} = \sigma'_{|[0,i]}$.

Proof.

1. This follows at once noticing that both $\xi_\sigma \circ \iota_{G_0}$ and $\xi_{\sigma'} \circ \iota_{G_0}$ are equal to $\iota'_{G'_0} \circ \alpha' \circ \alpha^{-1}$.
2. If $\sigma(0) \neq \sigma'(0)$ there is nothing to show. Otherwise, let j be the maximum of the set
$$\{i \in [0,n] \mid \sigma_{|[0,i]} = \sigma'_{|[0,i]}\}$$
We proceed by induction on $i \in [0,j]$.
   - If $i = 0$ the thesis follows from point 1.

- If $i > 0$, we know that there is a pushout square

$$\begin{tikzcd} K_{i-1} \arrow[r, "r_{i-1}"] \arrow[d, "k_{i-1}"'] & R_{i-1} \arrow[d, "h_{i-1}"] \\ D_{i-1} \arrow[r, "g_{i-1}"'] & G_i \end{tikzcd}$$

By Remark 3.37 we know that

$$\begin{aligned} \xi_\sigma \circ \iota_{G_i} \circ h_{i-1} &= \xi_\sigma \circ \iota_{R_{i-1}} \\ &= \iota'_{R_{\sigma(i-1)}} \\ &= \iota'_{R_{\sigma'(i-1)}} \\ &= \xi_{\sigma'} \circ \iota_{R_{i-1}} \\ &= \xi_{\sigma'} \circ \iota_{G_i} \circ h_{i-1} \end{aligned}$$

But, by the induction hypothesis we also have

$$\begin{aligned} \xi_\sigma \circ \iota_{G_i} \circ g_{i-1} &= \xi_\sigma \circ \iota_{D_{i-1}} \\ &= \xi_\sigma \circ \iota_{G_{i-1}} \circ f_{i-1} \\ &= \xi_{\sigma'} \circ \iota_{G_{i-1}} \circ f_{i-1} \\ &= \xi_{\sigma'} \circ \iota_{D_{i-1}} \\ &= \xi_{\sigma'} \circ \iota_{G_i} \circ g_{i-1} \end{aligned}$$

The thesis now follows. □

Now, notice that, given a consistent permutation $\sigma\colon [0,n] \to [0,n]$, we already know, by Remark 3.37 that

$$\xi_\sigma \circ \iota_{L_i} = \iota'_{L_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{K_i} = \iota'_{K_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{R_i} = \iota'_{R_{\sigma(i)}}$$

Moreover, $\xi_\sigma \circ \iota_{D_i}$ must be $\xi_\sigma \circ \iota_{G_i} \circ f_i$. Thus ξ_σ is uniquely determined by $\xi_\sigma \circ \iota_{G_i}$ for every $i \in [0,n]$. In particular, Proposition 3.42 entails the following.

Corollary 3.43. *For every consistent permutation σ between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$, the mediating isomorphism $\xi_\sigma\colon \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle$ is unique.*

We are now ready to prove the central result of this section.

Lemma 3.44. *Let $(\mathbf{X}, R)$ be a left-linear DPO-rewriting system. Consider two consistent permutation $\sigma, \sigma'\colon [0,n] \rightrightarrows [0,n]$ between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$. Suppose that $\sigma \neq \sigma'$ and let j be the minimum index such that $\sigma(j) \neq \sigma'(j)$. Let also $r(\underline{\mathscr{D}})$ be $\{\rho_i\}_{i=0}^n$. Then the following hold true:*

1. *if $j = 0$, then the rule ρ_0 is not consuming;*
2. *if $j \neq 0$ then the rule ρ_{j-1} is not consuming.*

Proof. 1. Let k be $\sigma^{-1}(\sigma'(0))$ and notice that, since $\sigma(0) \neq \sigma'(0)$, then $0 < k$. By the first point of Proposition 3.42 we can consider the diagram

$$\begin{tikzcd}
 & L_0 \arrow[dl, bend right, "m_0"'] \arrow[d, "m'_{\sigma'(0)}"'] \arrow[dr, bend left, "m_k"] & \\
G_0 \arrow[d, "\iota_{G_0}"'] \arrow[drr, "\iota_{G_0}" near start] & G'_{\sigma'(0)} \arrow[d, "\iota_{G'_{\sigma'(0)}}"] & G_k \arrow[d, "\iota_{G_k}"] \\
\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[r, "\xi_{\sigma'}"'] & \langle\!\langle \underline{\mathscr{D}'} \rangle\!\rangle & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[l, "\xi_{\sigma}"]
\end{tikzcd}$$

From Corollary 3.34, we can conclude that there exists $c\colon L_0 \to D_0$ such that $f_0 \circ c = m_0$. We thus have the solid part of the commutative diagram below.

$$\begin{tikzcd}
L_0 \arrow[dr, dotted, "t"] \arrow[drr, bend left, "\mathsf{id}_{L_0}"] \arrow[ddr, bend right, "c"'] & & \\
 & K_0 \arrow[r, tail, "l_0"] \arrow[d, "k_0"'] & L_0 \arrow[d, "m_0"] \\
 & D_0 \arrow[r, tail, "f_0"'] & G_0
\end{tikzcd}$$

The internal square is an $\mathcal{M}$-pushout and thus a pullback, by Proposition 2.16, so that we have the existence of the dotted $t\colon L_0 \to K_0$. Therefore $\mathsf{id}_{L_0} = l_0 \circ t$, proving that l_0 is an epimorphism. The thesis now follows from Corollary 2.17.

2. Let k be $\sigma^{-1}(\sigma'(j-1))$ and notice that $\rho_{j-1} = \rho_k$. By definition of j, $\sigma_{|[0,j-1]} = \sigma'_{|[0,j-1]}$, thus the second point of Proposition 3.4 yields the diagram

$$\begin{tikzcd}
 & L_{j-1} \arrow[dl, bend right, "m_{j-1}"'] \arrow[d, "m'_{\sigma'(j-1)}"'] \arrow[dr, bend left, "m_k"] & \\
G_{j-1} \arrow[d, "\iota_{G_{j-1}}"'] \arrow[drr, "\iota_{G_{j-1}}" near start] & G'_{\sigma'(j-1)} \arrow[d, "\iota_{G'_{\sigma'(j-1)}}"] & G_k \arrow[d, "\iota_{G_k}"] \\
\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[r, "\xi_{\sigma'}"'] & \langle\!\langle \underline{\mathscr{D}'} \rangle\!\rangle & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[l, "\xi_{\sigma}"]
\end{tikzcd}$$

Let a be $\min(j-1, k)$, by Corollary 3.34, there exists $c\colon L_{j-1} \to D_a$ such that $f_a \circ c = m_a$. As before this yields the solid part of the following diagram

$$\begin{tikzcd}
L_{j-1} \arrow[dr, dotted, "t"] \arrow[drr, bend left, "\mathsf{id}_{L_{j-1}}"] \arrow[ddr, bend right, "c"'] & & \\
 & K_{j-1} \arrow[r, tail, "l_{j-1}"] \arrow[d, "k_a"'] & L_{j-1} \arrow[d, "m_a"] \\
 & D_a \arrow[r, tail, "f_a"'] & G_a
\end{tikzcd}$$

The existence of the dotted $t\colon L_{j-1} \to K_{j-1}$ follows from Proposition 2.16 and we can conclude. $\square$

Corollary 3.45 (Uniqueness of consistent permutation)**.** *Let $(\mathbf{X}, R)$ ba left-linear DPO-rewriting system and suppose that every rule in R is consuming. For every two non-empty decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$, there exists at most one consistent permutation between them.*

We end this section proving two rather technical result about consistent permutations between composite decorated derivations.

Proposition 3.46. *Let $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ be two abstract decorated derivations such that*

$$(\underline{\mathscr{D}}, \alpha, \omega) = (\underline{\mathscr{D}}_1, \alpha_1, \omega_1) \cdot (\underline{\mathscr{D}}_2, \alpha_2, \omega_2) \qquad (\underline{\mathscr{D}}', \alpha', \omega') = (\underline{\mathscr{D}}'_1, \alpha'_1, \omega'_1) \cdot (\underline{\mathscr{D}}'_2, \alpha'_2, \omega'_2)$$

If $\sigma\colon [0, \mathsf{lg}(\underline{\mathscr{D}}_1) - 1] \to [0, \mathsf{lg}(\underline{\mathscr{D}}_1) - 1]$ is a consistent permutation between $(\underline{\mathscr{D}}_1, \alpha_1, \omega_1)$ and $(\underline{\mathscr{D}}'_1, \alpha'_1, \omega'_1)$ and $\tau : [0, \mathsf{lg}(\underline{\mathscr{D}}_2) - 1] \to [0, \mathsf{lg}(\underline{\mathscr{D}}_2) - 1]$ one between $(\underline{\mathscr{D}}_2, \alpha_2, \omega_2)$ and $(\underline{\mathscr{D}}'_2, \alpha'_2, \omega'_2)$, then

$$\sigma + \tau\colon [0, \mathsf{lg}(\underline{\mathscr{D}}) - 1] \to [0, \mathsf{lg}(\underline{\mathscr{D}}) - 1] \qquad n \mapsto \begin{cases} \sigma(n) & n < \mathsf{lg}(\underline{\mathscr{D}}_1) \\ \tau(n - \mathsf{lg}(\underline{\mathscr{D}}_1)) & \mathsf{lg}(\underline{\mathscr{D}}_1) \leq n \end{cases}$$

is a consistent permutation between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$.

Proof. Let us start fixing some notation. Given $X_1 \in \Delta(\underline{\mathscr{D}}_1)$, $X_2 \in \Delta(\underline{\mathscr{D}}_2)$, $X'_1 \in \Delta(\underline{\mathscr{D}}'_1)$, $X'_2 \in \Delta(\underline{\mathscr{D}}'_2)$, we will denote their coprojections into, respectively, $\llparenthesis \underline{\mathscr{D}}_1 \rrparenthesis$, $\llparenthesis \underline{\mathscr{D}}'_2 \rrparenthesis$, $\llparenthesis \underline{\mathscr{D}}'_1 \rrparenthesis$, $\llparenthesis \underline{\mathscr{D}}'_2 \rrparenthesis$ by $\iota_{1,X_1} : X_1 \to \llparenthesis \underline{\mathscr{D}}_1 \rrparenthesis$, $\iota_{1,X_2} : X_2 \to \llparenthesis \underline{\mathscr{D}}_1 \rrparenthesis$, $\iota'_{1,X'_1} : X'_1 \to \llparenthesis \underline{\mathscr{D}}'_1 \rrparenthesis$ and $\iota'_{2,X'_2} : X'_2 \to \llparenthesis \underline{\mathscr{D}}_1 \rrparenthesis$.

Now that these preliminaries matters have been addressed, we will proceed with the rest of the proof.. We split the proof in three cases.

- $\mathsf{lg}(\underline{\mathscr{D}}_1) = 0$. Thus $\mathsf{lg}(\underline{\mathscr{D}}'_1) = 0$ too and we have

$$(\underline{\mathscr{D}}, \alpha, \omega) = (\underline{\mathscr{D}}_2, \alpha_2 \circ \omega_1^{-1} \circ \alpha_1, \omega_2)$$
$$(\underline{\mathscr{D}}', \alpha', \omega') = (\underline{\mathscr{D}}'_2, \alpha'_2 \circ (\omega'_1)^{-1} \circ \alpha'_1, \omega'_2)$$

Moreover, in this case σ must be $\mathsf{id}_\emptyset$ and $\sigma + \tau$ must be equal to τ. The thesis now follows from the commutativity of the following diagram.

$$\begin{tikzcd}
 & G_{1,0} \arrow[dd, "\xi_{\mathsf{id}_\emptyset}"] \arrow[dr, "\omega_1^{-1}"] & & G_{2,0} \arrow[r, "\iota_{2,G_{2,0}}"] & \llparenthesis \underline{\mathscr{D}}_2 \rrparenthesis \arrow[dd, "\xi_\tau"] \\
\pi(G_{1,0}) \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha'_1"'] & & \pi(G_{2,0}) \arrow[ur, "\alpha_2"] \arrow[dr, "\alpha'_2"'] & & \\
 & G'_{1,0} \arrow[ur, "(\omega'_1)^{-1}"'] & & G'_{2,0} \arrow[r, "\iota'_{2,G_{2,0}}"'] & \llparenthesis \underline{\mathscr{D}}'_2 \rrparenthesis
\end{tikzcd}$$

- $\mathsf{lg}(\underline{\mathscr{D}}_2) = 0$. As before this implies that also $\mathsf{lg}(\underline{\mathscr{D}}'_2)$ is 0. Applying Definition 3.25 we get

$$
\begin{aligned}
(\underline{\mathscr{D}}, \alpha, \omega) &= (\underline{\mathscr{D}}_1, \alpha_1, \omega_1 \circ (\alpha_2)^{-1} \circ \omega_2) \\
(\underline{\mathscr{D}}', \alpha', \omega') &= (\underline{\mathscr{D}}'_1, \alpha'_1, \omega'_1 \circ (\alpha'_2)^{-1} \circ \omega'_2)
\end{aligned}
$$

Since $\underline{\mathscr{D}}_2$ is empty, then $\tau = \mathsf{id}_\emptyset$ and $\sigma + \tau = \sigma$. Let n be $\lg(\underline{\mathscr{D}}_1)$, as before the thesis follows from the diagram below.

$$
\begin{tikzcd}
 & G_{2,0} \arrow[dd, "\xi_{\mathsf{id}_\emptyset}"] \arrow[dr, "\alpha_2^{-1}"] & & G_{1,n} \arrow[r, tail, "\iota_{1,G_{1,n}}"] & \llbracket \underline{\mathscr{D}}_2 \rrbracket \arrow[dd, "\xi_{\sigma_1}"] \\
\pi(G_{2,0}) \arrow[ur, "\omega_2"] \arrow[dr, "\omega'_2"'] & & \pi(G_{1,n}) \arrow[ur, "\omega_1"] \arrow[dr, "\omega'_1"'] & & \\
 & G'_{2,0} \arrow[ur, "(\alpha'_1)^{-1}"'] & & G'_{1,n} \arrow[r, tail, "\iota'_{1,G_{1,n}}"'] & \llbracket \underline{\mathscr{D}}'_2 \rrbracket
\end{tikzcd}
$$

- $\mathsf{lg}(\underline{\mathscr{D}}_1) \neq 0$ and $\mathsf{lg}(\underline{\mathscr{D}}_2) \neq 0$. Thus $\underline{\mathscr{D}}'_1$ and $\underline{\mathscr{D}}'_2$ are non-empty too. In this case we have:

$$
\begin{aligned}
(\underline{\mathscr{D}}, \alpha, \omega) &= (\underline{\mathscr{D}}_1 * \omega_1^{-1} \cdot \alpha_2 * \underline{\mathscr{D}}_2, \alpha_1, \omega_2) \\
(\underline{\mathscr{D}}', \alpha', \omega') &= (\underline{\mathscr{D}}'_1 * (\omega'_1)^{-1} \cdot \alpha'_2 * \underline{\mathscr{D}}'_2, \alpha'_1, \omega'_2)
\end{aligned}
$$

Let us assume that $\underline{\mathscr{D}}_1$, $\underline{\mathscr{D}}'_1$, $\underline{\mathscr{D}}_2$ and $\underline{\mathscr{D}}'_2$ are given by

$$
\underline{\mathscr{D}}_1 = \{\mathscr{D}_{1,i}\}_{i=0}^{n} \quad \underline{\mathscr{D}}'_1 = \{\mathscr{D}'_{1,i}\}_{i=0}^{n} \quad \underline{\mathscr{D}}_2 = \{\mathscr{D}_{2,i}\}_{i=0}^{t} \quad \underline{\mathscr{D}}'_2 = \{\mathscr{D}'_{2,i}\}_{i=0}^{t}
$$

By definition of consistent permutation, the rule applied by $\mathscr{D}_{1,i}$ and the one applied in $\mathscr{D}_{2,i}$ must coincide with, respectively, the one applied in $\mathscr{D}'_{1,i}$ and the one applied $\mathscr{D}'_{2,1}$. Let $\mathscr{D}_{1,i}$, $\mathscr{D}'_{1,i}$, $\mathscr{D}_{2,i}$ and $\mathscr{D}'_{2,i}$ be given, by the following four diagrams.

$$
\begin{tikzcd}
L_{1,i} \arrow[d, "m_{1,i}"'] & K_{1,i} \arrow[l, tail, "l_{1,i}"'] \arrow[r, "r_{1,i}"] \arrow[d, "k_{1,i}"'] & R_{1,i} \arrow[d, "h_{1,i}"] \\
G_{1,i} & D_{1,i} \arrow[l, tail, "f_{1,i}"] \arrow[r, "g_{1,i}"'] & G_{1,i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
L_{1,i} \arrow[d, "m'_{1,i}"'] & K_{1,i} \arrow[l, tail, "l_{1,i}"'] \arrow[r, "r_{1,i}"] \arrow[d, "k'_{1,i}"'] & R_{1,i} \arrow[d, "h'_{1,i}"] \\
G'_{1,i} & D'_{1,i} \arrow[l, tail, "f'_{1,i}"] \arrow[r, "g'_{1,i}"'] & G'_{1,i+1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
L_{2,i} \arrow[d, "m_{2,i}"'] & K_{2,i} \arrow[l, tail, "l_{2,i}"'] \arrow[r, "r_{2,i}"] \arrow[d, "k_{2,i}"'] & R_{2,i} \arrow[d, "h_{2,i}"] \\
G_{2,i} & D_{2,i} \arrow[l, tail, "f_{2,i}"] \arrow[r, "g_{2,i}"'] & G_{2,i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
L_{2,i} \arrow[d, "m'_{2,i}"'] & K_{2,i} \arrow[l, tail, "l_{2,i}"'] \arrow[r, "r_{2,i}"] \arrow[d, "k'_{2,i}"'] & R_{2,i} \arrow[d, "h'_{2,i}"] \\
G'_{2,i} & D'_{2,i} \arrow[l, tail, "f'_{2,i}"] \arrow[r, "g'_{2,i}"'] & G'_{2,i+1}
\end{tikzcd}
$$

Let $(\llbracket \underline{\mathscr{D}}_1 * (\omega_1)^{-1} \rrbracket, \{j_X\}_{X \in \Delta(\underline{\mathscr{D}}_1 * (\omega_1)^{-1})})$, $(\llbracket \underline{\mathscr{D}}'_1 * (\omega'_1)^{-1} \rrbracket, \{j'_X\}_{X \in \Delta(\underline{\mathscr{D}}'_1 * (\omega'_1)^{-1})})$, $(\llbracket \underline{\mathscr{D}}_1 * (\omega_1)^{-1} \rrbracket, \{j_X\}_{X \in \Delta(\underline{\mathscr{D}}_1 * (\omega_1)^{-1})})$ and $(\llbracket \underline{\mathscr{D}}'_1 * (\omega'_1)^{-1} \rrbracket, \{j'_X\}_{X \in \Delta(\underline{\mathscr{D}}'_1 * (\omega'_1)^{-1})})$

be colimiting cocones. Define two other cocones and $\underline{\mathscr{D}}'_1 * (\omega'_1)^{-1} \cdot \alpha'_2 * \underline{\mathscr{D}}'_2$ Using point 2 of Lemma 3.29 we get two pushout squares

$$\pi(H)$$

Now, notice that () and *bb* are cocones on ... so that there exists... such that. □

Corollary 3.47.

prefisso e suffisso

Proof. contenuto... □

4 Shift equivalence and traces

A VERY NICE INTRO

4.1 Sequentially independent and switchable derivations

A VERY NICE INTRO

Definition 4.1. Let $(\mathbf{X}, \mathsf{R})$ be a left-linear DPO-rewriting system with $\mathbf{X}$ an $\mathcal{M}$-adhesive category. Let also $\mathscr{D}\colon G \Mapsto H$ and $\mathscr{D}'\colon H \Mapsto T$ be the two direct derivations depicted below.

$$\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] & & L' \arrow[d, "n'"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\
G & D \arrow[l, "f"] \arrow[r, "g"'] & H & & H & D' \arrow[l, "f'"] \arrow[r, "g'"'] & T
\end{tikzcd}$$

An *independence pair* between $\mathscr{D}$ and $\mathscr{D}'$, is a pair of arrows $i_1\colon R \to D'$ and $i_2\colon L' \to D$ such that the following diagram commutes.

$$\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"'] & R \arrow[dr, "h"'] \arrow[rrr, dotted, "i_1"] & & L' \arrow[dl, "n'"] \arrow[lll, dotted, "i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\
G & D \arrow[l, "f"] \arrow[rr, "g"'] & & H & & D' \arrow[ll, "f'"] \arrow[r, "g'"'] & T
\end{tikzcd}$$

We will say that $\mathscr{D}$ and $\mathscr{D}'$ are *weakly sequentially independent* if an independence pair exists. If such independence pair is unique we will say that $\mathscr{D}$ and $\mathscr{D}'$ are *sequentially independent*.

Example 4.2.

sequential independence

Example 4.3.

sequential independence che serva anche per es successivo

Remark 4.4. Let (i_1, i_2) and (j_1, j_2) be independence pairs for the direct derivations $\mathscr{D}$ and $\mathscr{D}'$. Notice that, by definition, we have

$$\begin{aligned} f' \circ i_1 &= h \\ &= f' \circ j_1 \end{aligned}$$

On the other hand, $f' \colon D' \to H$ is the pushout of $l' \colon K' \to L'$ and so it is in $\mathcal{M}$, implying $j_1 = i_1$. If, moreover, we suppose that the rule ρ applied in $\mathscr{D}$ is linear, then $g \colon D \to H$ is in $\mathcal{M}$ too, hence, from the equation

$$\begin{aligned} g \circ i_2 &= h \\ &= g \circ j_2 \end{aligned}$$

we can deduce that $i_2 = j_2$, too.

Summing up, if $(\mathbf{X}, \mathsf{R})$ is a linear DPO-rewriting system, then sequential independence and weak sequential independence coincide.

When working with linear rewriting systems, (weakly) sequential independent direct derivations can be switched, producing two new (weakly) sequential independent direct derivations between the same objects [13, Thm. 7.7] . This is no more the case if the rules are only left-linear, as shown by the next example.

Example 4.5.

pensare ad un esempio in cui l'indipendenza non basta

To fix this problem, we adapt the notion of *canonical filler* from [11].

Definition 4.6. Let $(\mathbf{X}, \mathsf{R})$ be a left-linear DPO-rewriting system with $\mathbf{X}$ $\mathcal{M}$-adhesive. Let also $\mathscr{D} \colon G \Mapsto H$ and $\mathscr{D}' \colon H \Mapsto T$ be the two derivations depicted below.

$$\begin{tikzcd} L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\ G & D \arrow[l, "f"] \arrow[r, "g"'] & H \end{tikzcd} \qquad \begin{tikzcd} L' \arrow[d, "n'"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\ H & D' \arrow[l, "f'"] \arrow[r, "g'"'] & T \end{tikzcd}$$

Since f' is in $\mathcal{M}$, we can moreover consider a pullback square

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "p'"'] & D \arrow[d, "g"] \\ D' \arrow[r, "f'"'] & H \end{tikzcd}$$

A *filler* between $\mathscr{D}$ and $\mathscr{D}'$ is given by a pair of arrows $u \colon K \to P$ and $u' \colon K' \to P$ satisfying the following conditions

1. $p \circ u = k$, $p' \circ u' = k'$ and there exists a pushout square

$$\begin{tikzcd} K' \arrow[r, "r'"] \arrow[d, "u'"'] & R' \arrow[d, dotted, "j'"] \\ P \arrow[r, dotted, "q'"'] & Q' \end{tikzcd}$$

2. there exist arrows $i_1 \colon R \to D'$, $i_2 \colon L' \to D$ satisfying $f' \circ i_1 = h$, $g \circ i_2 = n'$ and such that the following squares are pushouts

$$\begin{tikzcd} K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, dotted, "i_1"] \\ P \arrow[r, "p'"'] & D' \end{tikzcd} \qquad \begin{tikzcd} K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, dotted, "i_2"] \\ P \arrow[r, "p"'] & D \end{tikzcd}$$

Remark 4.7. Let $\mathscr{D}$ and $\mathscr{D}'$ be two switchable direct derivations. Then the existence of a filler allows us to build the solid part of the diagram below.

$$
\begin{tikzcd}
 & & R \arrow[dr, "h"] \arrow[ddrr, bend right, "i_1"'] & & L' \arrow[dl, "n'"'] \arrow[ddll, bend left, "i_2"] & & \\
 & K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dr, "k"] \arrow[ddrr, bend right, "u"'] & & H & & K' \arrow[ul, "l'"'] \arrow[dl, "k'"'] \arrow[dr, "r'"] \arrow[ddll, bend left, "u'"] & \\
L \arrow[dr, "n"'] \arrow[ddrr, dotted, bend right, "j"'] & & D \arrow[ur, "g"] \arrow[dl, "f"'] & & D' \arrow[ul, "f'"'] \arrow[dr, "g'"] & & R' \arrow[dl, "h'"'] \arrow[ddll, bend left, "j'"] \\
 & G & & P \arrow[ul, "p"'] \arrow[ur, "p'"] \arrow[dl, dotted, "q"'] \arrow[dr, "q'"] & & T & \\
 & & Q \arrow[ul, dotted, "s"] & & Q' \arrow[ur, dotted, "t"'] & &
\end{tikzcd}
$$

Let us complete this diagram defining the dotted arrows. We can start noticing that, since $l \in \mathcal{M}$, there exists a pushout square

$$
\begin{tikzcd}
K \arrow[r, "l"] \arrow[d, "u"'] & L \arrow[d, "j"] \\
P \arrow[r, "q"'] & Q
\end{tikzcd}
$$

Moreover, the existence of the wanted $s \colon Q \to G$ and $t \colon Q' \to T$ follows from the following equalities

$$
\begin{aligned}
f \circ p \circ u &= f \circ k & \qquad g' \circ p' \circ u' &= g' \circ k' \\
&= n \circ l & &= h' \circ r'
\end{aligned}
$$

We can prove some other properties of the arrows appearing in the diagram above. The three rectangles below are pushouts and their left halves are pushouts too. Therefore, by Lemma 2.5, also their right halves are pushouts.

$$
\begin{tikzcd}
K \arrow[r, "u"'] \arrow[rr, bend left, "u"] \arrow[d, "r"'] & P \arrow[r, "p"'] \arrow[d, "p'"] & D \arrow[d, "g"] \\
R \arrow[r, "i_1"] \arrow[rr, bend right, "h"'] & D' \arrow[r, "f'"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "u'"'] \arrow[rr, bend left, "k'"] \arrow[d, "l'"'] & P \arrow[r, "p'"'] \arrow[d, "p"] & D' \arrow[d, "f'"] \\
L' \arrow[r, "i_2"] \arrow[rr, bend right, "n'"'] & D \arrow[r, "g"] & H
\end{tikzcd}
$$

$$
\begin{tikzcd}
K \arrow[r, "u"'] \arrow[rr, bend left, "k"] \arrow[d, "l"'] & P \arrow[r, "p"'] \arrow[d, "q"] & D \arrow[d, "f"] \\
L \arrow[r, "j"] \arrow[rr, bend right, "n"'] & Q \arrow[r, "s"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "u'"'] \arrow[rr, bend left, "k'"] \arrow[d, "r'"'] & P \arrow[r, "p'"'] \arrow[d, "q'"] & D' \arrow[d, "g'"] \\
R' \arrow[r, "j'"] \arrow[rr, bend right, "h'"'] & Q' \arrow[r, "t"] & T
\end{tikzcd}
$$

Notice, moreover, that p, q are the pushouts of l' and l, respectively, thus they are elements of $\mathcal{M}$. By Proposition 2.16 the squares

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "p'"'] & D \arrow[d, "g"] \\ D' \arrow[r, "f'"'] & H \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] & D \arrow[d, "f"] \\ Q \arrow[r, "s"'] & G \end{tikzcd}$$

are pullbacks too.

Notice that if there is a filler between $\mathscr{D}$ and $\mathscr{D}'$ are switchable, then they are weakly sequentially independent: indeed, if a filler between them exists, then (i_1, i_2) is an independence pair. On the other hand, Example 4.5 shows that not every independence pair arises in this way.

Definition 4.8. Let $\mathscr{D} \colon H \Rrightarrow H$, $\mathscr{D}' \colon H \Rrightarrow T$ be two direct derivations in a left-linear DPO-rewriting system $(\mathbf{X}, \mathsf{R})$. An independence pair (i_1, i_2) between $\mathscr{D}$ and $\mathscr{D}'$ is *good* if there exists a filler (u, u') between them such that the squares below are pushouts.

$$\begin{tikzcd} K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \\ P \arrow[r, "p'"'] & D' \end{tikzcd} \qquad \begin{tikzcd} K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "i_2"] \\ P \arrow[r, "p"'] & D \end{tikzcd}$$

We will say that $\mathscr{D}$ and $\mathscr{D}'$ are *switchable* if a good independence pair between them exists. If such a pair is unique, we will say that $\mathscr{D}$ and $\mathscr{D}'$ are *uniquely switchable*. We will use the notation $\mathscr{D} \updownarrow \mathscr{D}'$ to mean that $\mathscr{D}$ and $\mathscr{D}'$ are switchable, while $\mathscr{D} \updownarrow_! \mathscr{D}'$ will denote that they are uniquely so.

If every independence pair is good we will say that $(\mathbf{X}, \mathsf{R})$ is *tame*.

Remark 4.9. Given a good independence pair (i_1, i_2) between $\mathscr{D}$ and $\mathscr{D}'$, there is a unique filler such that (u, u') such that

$$\begin{tikzcd} K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \\ P \arrow[r, "p'"'] & D' \end{tikzcd} \qquad \begin{tikzcd} K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "i_2"] \\ P \arrow[r, "p"'] & D \end{tikzcd}$$

are pushouts. Indeed, for every other filler (v, v'), it must be that

$$\begin{aligned} p \circ u &= k & \qquad p \circ u' &= i_2 \circ l' \\ &= p \circ v & &= p \circ v' \end{aligned}$$

Since the arrow $p \colon P \to D$, which is the pullback of f', is in $\mathcal{M}$ we conclude that $(u, u') = (v, v')$.

Remark 4.10. Clearly in a tame left-linear DPO-rewriting system two direct derivations are sequentially independent if and only if they are unquely switchable.

A source of tame left-linear DPO-rewriting systems is given by the linear ones, as shown by the following proposition.

Proposition 4.11. *Every linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$ *is tame.*

Proof. Suppose that $\mathbf{X}$ is $\mathcal{M}$-adhesive and let (i_1, i_2) be an independence pair between $\mathscr{D}\colon G \Mapsto H$ and $\mathscr{D}'\colon H \Mapsto T$. We have a diagram

$$
\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[dr, "h"'] \arrow[drrr, bend left, "i_1"] & & L' \arrow[dl, "n'"] \arrow[dlll, bend right, "i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\
G & D \arrow[l, "f"] \arrow[rr, "g"'] & & H & & D' \arrow[ll, "f'"] \arrow[r, "g'"'] & T
\end{tikzcd}
$$

Pulling back g along f', we get another diagram

$$
\begin{tikzcd}
& & R \arrow[dr, "h"] \arrow[ddrr, bend right, "i_1"'] & & L' \arrow[dl, "n'"'] \arrow[ddll, bend left, "i_2"] & & \\
& K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dr, "k"] \arrow[ddrr, dotted, bend right, "u"'] & & H & & K' \arrow[ul, "l'"'] \arrow[dl, "k'"'] \arrow[dr, "r'"] \arrow[ddll, dotted, bend left, "u'"] & \\
L \arrow[dr, "n"'] & & D \arrow[ur, "g"] \arrow[dl, "f"] & & D' \arrow[ul, "f'"'] \arrow[dr, "g'"] & & R' \arrow[dl, "h'"] \\
& G & & P \arrow[ul, "p"] \arrow[ur, "p'"'] & & T &
\end{tikzcd}
$$

Now, if we compute we get

$$
\begin{aligned}
f' \circ i_1 \circ r &= h \circ r & \qquad g \circ i_2 \circ l' &= n' \circ l' \\
&= g \circ k & &= f' \circ k'
\end{aligned}
$$

Therefore the two dotted arrows $u\colon K \to P$ and $u'\colon K' \to P$ exist. We have to show that they satisfy the two conditions in the definition of a filler.

1. By construction $p \circ u = k$ and $p' \circ u' = k'$. Since $(\mathbf{X}, \mathsf{R})$ is linear, then $r'\colon K' \to R'$ belongs to $\mathcal{M}$, thus it admits a pushout along $u'\colon K' \to P$, as wanted.

2. Take the following two rectangles

$$
\begin{tikzcd}
K \arrow[r, "u"'] \arrow[rr, bend left, "u"] \arrow[d, "r"'] & P \arrow[r, "p"'] \arrow[d, "p'"] & D \arrow[d, "f"] \\
R \arrow[r, "i_1"] \arrow[rr, bend right, "h"'] & D' \arrow[r, "f'"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "u'"'] \arrow[rr, bend left, "k'"] \arrow[d, "l'"'] & P \arrow[r, "p'"'] \arrow[d, "p"] & D' \arrow[d, "f'"] \\
L' \arrow[r, "i_2"] \arrow[rr, bend right, "n'"'] & D \arrow[r, "g"] & H
\end{tikzcd}
$$

By hypothesis r and l' are in $\mathcal{M}$, thus f' and g belong to it too. The first point of Lemma 2.26 yields the thesis. $\square$

Remark 4.12. [2] identifies a large class of (quasi)adhesive categories with the property that every left linear DPO-rewriting system on them is tame. We adapt these results to our context in Appendix A.

Definition 4.13 (Very tameness). very tame

Remark 4.14. tutto coincide con tutto

We are now going to justify the choice of the name for the relation $\updownarrow$, showing that two switchable direct derivations $\mathscr{D}$ and $\mathscr{D}'$ can be actually switched.

Let (i_1, i_2) be a good independence pair and consider the following diagram: the solid part exists by the definition of a filler, while the two new dotted arrows $v\colon Q \to J$ and $v'\colon Q' \to J$ are obtained as the pushout of $q\colon P \to Q$, which is in $\mathcal{M}$ by Remark 4.7, along $q'\colon P \to Q'$.

$$\begin{tikzcd}
& & R \arrow[dr, "h"] \arrow[ddrr, bend right, "i_1"] & & L' \arrow[dl, "n'"'] \arrow[ddll, bend left, "i_2"'] & & \\
& K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dr, "k"] \arrow[ddrr, bend right, "u"'] & & H & & K' \arrow[ul, "l'"'] \arrow[dl, "k'"'] \arrow[dr, "r'"] \arrow[ddll, bend left, "u'"] & \\
L \arrow[dr, "n"] \arrow[ddrr, bend right, "j"'] & & D \arrow[dl, "f"'] \arrow[ur, "g"] & & D' \arrow[ul, "f'"'] \arrow[dr, "g'"] & & R' \arrow[dl, "h'"] \arrow[ddll, bend left, "j'"] \\
& G & & P \arrow[ul, "p"'] \arrow[ur, "p'"] \arrow[dl, "q"] \arrow[dr, "q'"] & & T & \\
& & Q \arrow[ul, "s"] \arrow[dr, dotted, "v"'] & & Q' \arrow[ur, "t"] \arrow[dl, dotted, "v'"] & & \\
& & & J & & &
\end{tikzcd}$$

Since, by Remark 4.7, all the curved rectangles are pushouts, as well as the bottom square, we can state the following definition.

Definition 4.15. Let $(\mathbf{X}, R)$ be a left-linear DPO-rewriting system and suppose that $\mathbf{X}$ is $\mathcal{M}$-adhesive. Given a good independence pair (i_1, i_2) between $\mathscr{D}\colon G \Mapsto H$ and $\mathscr{D}'\colon H \Mapsto T$, if (u, u') is the associate filler, we define other two direct derivations $S_{i_1,i_2}(\mathscr{D}')\colon G \Mapsto J$ and $S_{i_1,i_2}(\mathscr{D})\colon J \Mapsto T$ as follows:

$$\begin{tikzcd}
L' \arrow[d, "f \circ i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "q \circ u'"'] & R' \arrow[d, "v' \circ j'"'] \\
G & Q \arrow[l, "s"] \arrow[r, "v"'] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[d, "v \circ j"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "q' \circ u"] & R \arrow[d, "g' \circ i_1"] \\
J & Q' \arrow[l, "v'"] \arrow[r, "t"'] & T
\end{tikzcd}$$

The *switching* $S_{i_1,i_2}(\mathscr{D}, \mathscr{D}')$ of $\mathscr{D}$ and $\mathscr{D}'$ is the derivation $S_{i_1,i_2}(\mathscr{D}') \cdot S_{i_1,i_2}(\mathscr{D})$.

Remark 4.16. Notice that (j', j) is an independence pair for $S_{i_1,i_2}(\mathscr{D}')$ and $S_{i_1,i_2}(\mathscr{D})$. This is witnessed by the following diagram, commutative by construction.

$$
\begin{tikzcd}
L' \arrow[d, "f \circ i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "q \circ u'"'] & R' \arrow[dr, "v' \circ j'"'] \arrow[drrr, bend left, "j'"] & & L \arrow[dl, "v \circ j"] \arrow[dlll, bend right, "j"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "q' \circ u"] & R \arrow[d, "g' \circ i_1"] \\
G & Q \arrow[l, "s"] \arrow[rr, "v"'] & & J & & Q' \arrow[ll, "v'"] \arrow[r, "t"'] & T
\end{tikzcd}
$$

Theorem 4.17 (Local Church-Rosser Theorem)**.** *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system with* $\mathbf{X}$ *an* $\mathcal{M}$*-adhesive category. Then every filler between two direct derivations* $\mathscr{D} \colon G \Mapsto H$ *and* $\mathscr{D}' \colon H \Mapsto T$ *is a filler also for* $S_{u,u'}(\mathscr{D}') \colon G \Mapsto J$ *and* $S_{u,u'}(\mathscr{D}) \colon J \Mapsto H$*. In particular, if* $\mathscr{D} \updownarrow \mathscr{D}'$*, then* $S_{u,u'}(\mathscr{D}') \updownarrow S_{u,u'}(\mathscr{D})$*.*

Proof. By definition of filler, we have two pushout square

$$
\begin{tikzcd}
K \arrow[r, "l"] \arrow[d, "u"'] & L \arrow[d, "j"] \\
P \arrow[r, "q"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "r'"] \arrow[d, "u'"'] & R' \arrow[d, "j'"] \\
P \arrow[r, "q'"'] & Q'
\end{tikzcd}
$$

In particular, q is an arrow of $\mathcal{M}$, therefore, by Proposition 2.16, the square below is a pullback.

$$
\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "q'"'] & Q \arrow[d, "v"] \\
Q' \arrow[r, "v'"'] & J
\end{tikzcd}
$$

To prove our claim, it is now enough to show that (u', u) is a filler between $S_{u,u'}(\mathscr{D}')$ and $S_{u,u'}(\mathscr{D})$.

1. As for the first point of Definition 4.6, the only non obvious part is the existence of a pushout of u along r. But, since (u, u') is a filler between $\mathscr{D}$ and $\mathscr{D}'$, we know that such a pushout exists: it is enough to take the square

$$
\begin{tikzcd}
K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \\
P \arrow[r, "p'"'] & D'
\end{tikzcd}
$$

2. For the second point, notice that the arrows j and j' fit in the squares

$$
\begin{tikzcd}
K' \arrow[r, "r'"] \arrow[d, "u'"'] & R' \arrow[d, "j'"] \\
P \arrow[r, "q"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "l"] \arrow[d, "u"'] & L \arrow[d, "j"] \\
P \arrow[r, "q'"'] & Q'
\end{tikzcd}
$$

By Remark 4.16 we know that (j, j') is an independence pair. The results above now implies that (j, j') is good. □

The previous remark allow us to further switch the direct derivation $S_{i_1,i_2}(\mathscr{D})$ and $S_{i_2,i_2}(\mathscr{D}')$. The following lemma guarantees us that, in this way, we get back a derivation which is abstraction equivalent to $\mathscr{D} \cdot \mathscr{D}'$. introdurre decorazione

Lemma 4.18. *Let $\mathscr{D} \colon G \Mapsto H$ and $\mathscr{D}' \colon H \Mapsto T$ be two direct derivations in a left-linear DPO-rewriting system $(\mathbf{X}, \mathsf{R})$ and let (i_1, i_2) be a good independence pair between them. Then $S_{j',j}(S_{i_1,i_2}(\mathscr{D}, \mathscr{D}'))$ is abstraction equivalent to $\mathscr{D} \cdot \mathscr{D}'$.*

Proof. Let (u, u') be the filler associated to (i_1, i_2). By Theorem 4.17, (u', u) is a filler between $S_{i_1,i_2}(\mathscr{D}')$ and $S_{i_1,i_2}(\mathscr{D})$. Thus we have a diagram as the one below.

$$\begin{tikzcd}
 & & R' \arrow[dr, "v'\circ j'"] \arrow[ddrr, bend right, "j'"] & & L \arrow[dl, "v\circ j"'] \arrow[ddll, bend left, "j"'] & & \\
 & K' \arrow[ur, "r'"] \arrow[dl, "l'"'] \arrow[dr, "q\circ u'"] \arrow[ddrr, bend right, "u'"'] & & J & & K \arrow[ul, "l"'] \arrow[dr, "r"] \arrow[dl, "q'\circ u"'] \arrow[ddll, bend left, "u"] & \\
L' \arrow[dr, "f\circ i_2"] \arrow[ddrr, bend right, "c_2"'] & & Q \arrow[ur, "v"] \arrow[dl, "s"'] & & Q' \arrow[ul, "v'"'] \arrow[dr, "t"] & & R \arrow[dl, "g'\circ i_1"'] \arrow[ddll, bend left, "c_1"] \\
 & G & & P \arrow[ul, "q"'] \arrow[ur, "q'"] \arrow[dl, "e'"'] \arrow[dr, "e"] & & T & \\
 & & E' \arrow[ul, "a"] \arrow[dr, "w'"'] & & E \arrow[ur, "b"'] \arrow[dl, "w"] & & \\
 & & & F & & &
\end{tikzcd}$$

Now, to ease the notation, let $S_{j',j}(S_{i_1,i_2}(\mathscr{D}, \mathscr{D}'))$ be $\mathscr{E}_0 \cdot \mathscr{E}_1$, then $\mathscr{E}_0$ and $\mathscr{E}_1$ are the direct derivations given by the diagrams

$$\begin{tikzcd}
L \arrow[d, "s\circ j"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "e'\circ u"'] & R \arrow[d, "w\circ c_1"'] \\
G & E' \arrow[l, "a"] \arrow[r, "w'"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
L' \arrow[d, "w'\circ c_2"] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "e\circ u'"] & R' \arrow[d, "t\circ j'"] \\
F & E \arrow[l, "w"] \arrow[r, "b"'] & R
\end{tikzcd}$$

Notice, moreover, that, since the squares

$$\begin{tikzcd}
K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "c_2"] \\
P \arrow[r, "e'"'] & E'
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "c_1"] \\
P \arrow[r, "e"'] & E
\end{tikzcd}$$

are pushouts, we have isomorphisms $\phi' \colon D \to E'$, $\phi \colon D' \to E$ making the following diagrams commutative.

$$
\begin{tikzcd}
K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "i_2"] \arrow[dr, bend left, "c_2"] & \\
P \arrow[r, "p"] \arrow[rr, bend right, "e'"'] & D \arrow[r, dotted, "\phi'"] & E'
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \arrow[dr, bend left, "c_1"] & \\
P \arrow[r, "p'"] \arrow[rr, bend right, "e"'] & D' \arrow[r, dotted, "\phi"] & E
\end{tikzcd}
$$

In particular, we have

$$
\begin{aligned}
a \circ \phi' \circ i_2 &= a \circ c_2 & a \circ \phi' \circ p' &= a \circ e' & b \circ \phi \circ i_1 &= b \circ c_1 & b \circ \phi \circ p &= b \circ e \\
&= f \circ i_2 & &= s \circ q & &= g' \circ i_1 & &= t \circ q' \\
& & &= f \circ p & & & &= g' \circ p'
\end{aligned}
$$

and this shows that

$$a \circ \phi' = f \qquad b \circ \phi = g'$$

Now, since ϕ' is an isomorphism and by Proposition 2.16, the two halves of the rectangle

$$
\begin{tikzcd}
K \arrow[r, "\mathsf{id}_K"] \arrow[d, "(\phi')^{-1} \circ e' \circ u"'] & K \arrow[r, "l"] \arrow[d, "e' \circ u"'] & L \arrow[d, "s \circ j"] \\
D \arrow[r, "\phi'"'] & E' \arrow[r, "a"'] & G
\end{tikzcd}
$$

are pullbacks. Thus the whole diagram is a pullback. But, by construction $s \circ j = n$ and we have already proved that $a \circ \phi' = f$. We then conclude that ther exists an isomorphism $\zeta \colon K \to K$ which makes the diagram below commutative

$$
\begin{tikzcd}
K \arrow[r, dotted, "\zeta"'] \arrow[rr, bend left, "l"] \arrow[dr, bend right, "(\phi')^{-1} \circ e' \circ u"'] & K \arrow[r, "l"'] \arrow[d, "n"'] & L \arrow[d, "n"] \\
& D \arrow[r, "f"'] & G
\end{tikzcd}
$$

The commutativity of the upper triangle entails $l \circ \zeta = l$. Since l is an element of $\mathcal{M}$ we can deduce that. $\zeta = \mathsf{id}_K$. From this, we conclude that

$$e' \circ u = \phi' \circ k$$

As a next step, notice the existence of ϕ and ϕ', together with Remark 4.7, entails the existence of a third isomorphism $\psi \colon H \to F$ fitting in the diagram below.

$$
\begin{tikzcd}
P \arrow[r, "p"'] \arrow[rr, bend left, "e'"] \arrow[d, "p'"] \arrow[dd, bend right, "e"'] & D \arrow[r, "\phi'"'] \arrow[d, "g"] & E' \arrow[d, "w'"] \\
D' \arrow[r, "f'"'] \arrow[d, "\phi"] & H \arrow[r, dotted, "\psi"] & F \\
E \arrow[urr, bend right, "w"'] & &
\end{tikzcd}
$$

Now, if we compute, we get

$$
\begin{aligned}
\psi^{-1} \circ w \circ c_1 &= f' \circ \phi^{-1} \circ c_1 \\
&= f' \circ i_1 \\
&= h
\end{aligned}
$$

Summing up, we have just build the diagram below.

$$
\begin{tikzcd}
 & & & R \arrow[dddl, "h"'] \arrow[dddr, "w \circ c_1"] & \\
 & & K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dddl, "k"'] \arrow[dddr, "e' \circ u"] & & \\
 & L \arrow[dddl, "n"'] \arrow[dddr, "s \circ j"] & & & \\
 & & H \arrow[rr, "\psi"] & & F \\
 & D \arrow[dl, "f"] \arrow[ur, "g"'] \arrow[rr, "\phi'"] & & E' \arrow[ur, "w'"'] \arrow[dl, "a"] & \\
G \arrow[rr, "\mathsf{id}_G"'] & & G & &
\end{tikzcd}
$$

Next, we already know that

$$
t \circ j' = h' \quad w \circ \phi = \psi \circ f' \quad b \circ \phi = g'
$$

If we compute further, we also get

$$
\begin{aligned}
\psi^{-1} \circ w' \circ c_2 &= g \circ (\phi')^{-1} \circ c_2 \qquad & \phi^{-1} \circ e \circ u' &= p' \circ u' \\
&= g \circ i_2 & &= k' \\
&= n'
\end{aligned}
$$

These equations allow us to conclude that the following diagram commutes.

$$
\begin{tikzcd}
 & & & R' \arrow[dddl, "h'"'] \arrow[dddr, "t \circ j'"] & \\
 & & K' \arrow[ur, "r"] \arrow[dl, "l'"'] \arrow[dddl, "k"'] \arrow[dddr, "e \circ u'"] & & \\
 & L' \arrow[dddl, "n'"'] \arrow[dddr, "w' \circ c_2"] & & & \\
 & & T \arrow[rr, "\mathsf{id}_T"] & & T \\
 & D' \arrow[dl, "f'"] \arrow[ur, "g'"'] \arrow[rr, "\phi"] & & E \arrow[ur, "b"'] \arrow[dl, "w"] & \\
H \arrow[rr, "\psi"'] & & F & &
\end{tikzcd}
$$

Putting together the two diagrams above we get the thesis. $\square$

Our next step is to relate derivations which are equal "up to switching".

Definition 4.19. Let $(\mathbf{X}, \mathsf{R})$ be a left-linear DPO-rewriting system. Given two direct derivations $\mathscr{D}\colon G \Mapsto H$ and $\mathscr{D}'\colon H \Mapsto T$, we say that $\mathscr{D}$ and $\mathscr{D}'$ are *properly switchable* if $\mathscr{D} \updownarrow_! \mathscr{D}'$ and $S_{i_1,i_2}(\mathscr{D}') \updownarrow_! S_{i_1,i_2}(\mathscr{D})$, where (i_2, i_2) is a good independence pair betwwen $\mathscr{D}$ and $\mathscr{D}'$. In such a case, we will write $\mathscr{D} \Updownarrow \mathscr{D}'$.

Take two derivations $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{n}$ and $\underline{\mathscr{D}}' = \{\mathscr{D}'_i\}_{i=0}^{n}$ with the same length and between the same G_0 and G_n. We say that $\underline{\mathscr{D}}'$ is *obtained by a proper switch from* $\underline{\mathscr{D}}$ if there exists an index $j < n$ such that

1. for every $i \notin \{j, j+1\}$, $\mathscr{D}_i = \mathscr{D}'_i$;
2. $\mathscr{D}_j \Updownarrow \mathscr{D}_{j+1}$;
3. $\mathscr{D}'_j \cdot \mathscr{D}'_{j+1} = S_{i_1,i_2}(\mathscr{D}, \mathscr{D}')$.

In such a case, we will write $\underline{\mathscr{D}} \rightsquigarrow_j \underline{\mathscr{D}}'$ to denote that $\underline{\mathscr{D}}'$ is obtained by a proper switch between $\mathscr{D}_j$ and $\mathscr{D}_{j+1}$.

We will say that $\underline{\mathscr{D}}$ is *switch equivalent* to $\underline{\mathscr{D}}'$, if there exists a sequence, $\{\underline{\mathscr{D}}_i\}_{i=0}^{n}$ of derivations such that

1. $\underline{\mathscr{D}}_0 = \underline{\mathscr{D}}$ and $\underline{\mathscr{D}}_n = \underline{\mathscr{D}}'$;
2. for every $i < n$, $\underline{\mathscr{D}}_{i+1}$ is obtained by a proper switch from $\underline{\mathscr{D}}_i$.

We will write $\underline{\mathscr{D}} \equiv^s \underline{\mathscr{D}}'$ to denote that $\underline{\mathscr{D}}$ is switch equivalent to $\underline{\mathscr{D}}'$.

Example 4.20. il punto due sopra è necessario

We can now prove some properties of switch equivalence.

Lemma 4.21. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system. Then the following hold true:* Def. 3 della bozza di Andrea

1.

2.

3.

4.

Proof. 1.

2.

3.

4. ∎

Example 4.22. esempio sul perché weakly independence non è invertibile

Lemma 4.23. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$*. Then the following hold true*

1. *If* $\mathscr{D}$ *and* $\mathscr{D}'$ *are two direct derivations such that* $\mathscr{D} \updownarrow \mathscr{D}'$*, then for every filler* (u, u') *between them, the function*

$$\tau \colon 2 \to 2 \qquad x \mapsto \begin{cases} 1 & x = 0 \\ 0 & x = 1 \end{cases}$$

defines a consistent permutation between $\mathscr{D} \cdot \mathscr{D}'$ *and* $S_{i_1,i_2}(\mathscr{D}, \mathscr{D}')$*;*

2. *if $\underline{\mathscr{D}}$ and $\underline{\mathscr{D}}'$ are two switch equivalent derivation, then there exists a consistent permutation between them.*

Proof. 1.

2. ◻

This, together with ???

Corollary 4.24. unicità

Example 4.25. permutazione consistente non implica scambiabilità

4.1.1 Graphical rewriting systems

ci sono un sacco di sistemi very tame

4.2 On the globality of $\Updownarrow$

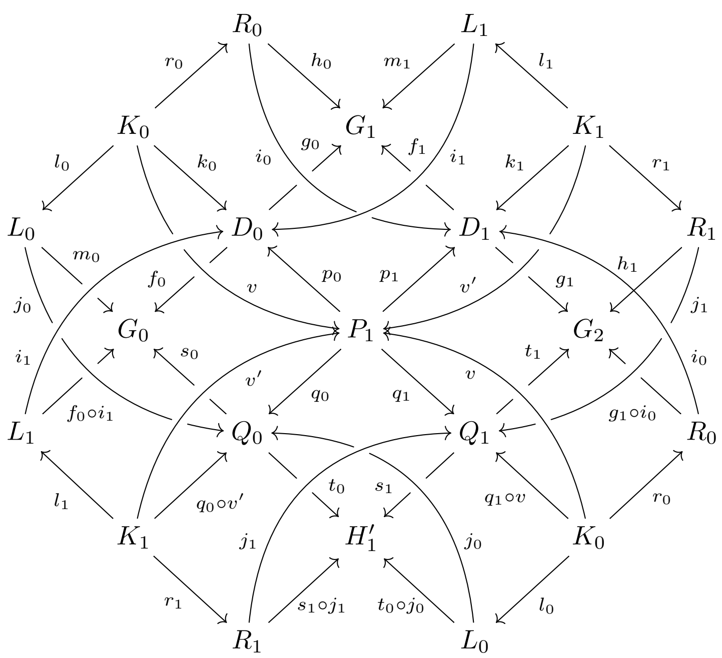
intro

Lemma 4.26 (Three steps Lemma)**.** *Let $(\mathbf{X}, \mathsf{R})$ be a left-linear DPO-rewriting system with $\mathbf{X}$ an $\mathcal{M}$-adhesive category. Consider a derivation $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{2}$ and suppose that (i_0, i_1) is a good independence pair between $\mathscr{D}_0$ and $\mathscr{D}_1$, (a_0, a_1) one between $\mathscr{D}_1$ and $\mathscr{D}_2$ and (e_0, e_1) one between $\mathscr{D}_0$ and $S_{a_0,a_1}(\mathscr{D}_2)$. Then the following properties hold true.*

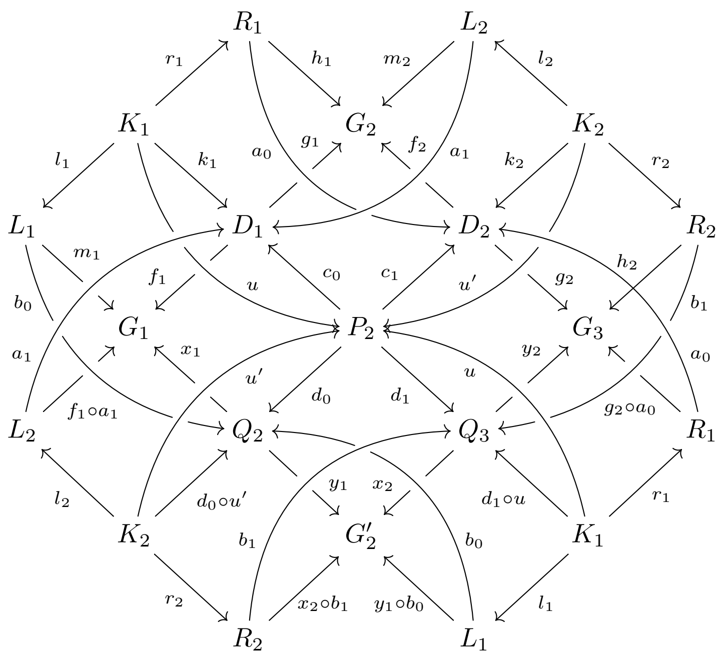
1. *$S_{e_0,e_1}(\mathscr{D}_0)$ and $S_{a_0,a_1}(\mathscr{D}_1)$ are weakly sequentially independent.*

2. *If $S_{i_0,i_1}(\mathscr{D}_0) \updownarrow_! \mathscr{D}_2$ with a good independence pair (α_0, α_1), then $S_{i_0,i_1}(\mathscr{D}_1)$ and $S_{\alpha_0,\alpha_1}(\mathscr{D}_2)$ are weakly sequentially independent.*

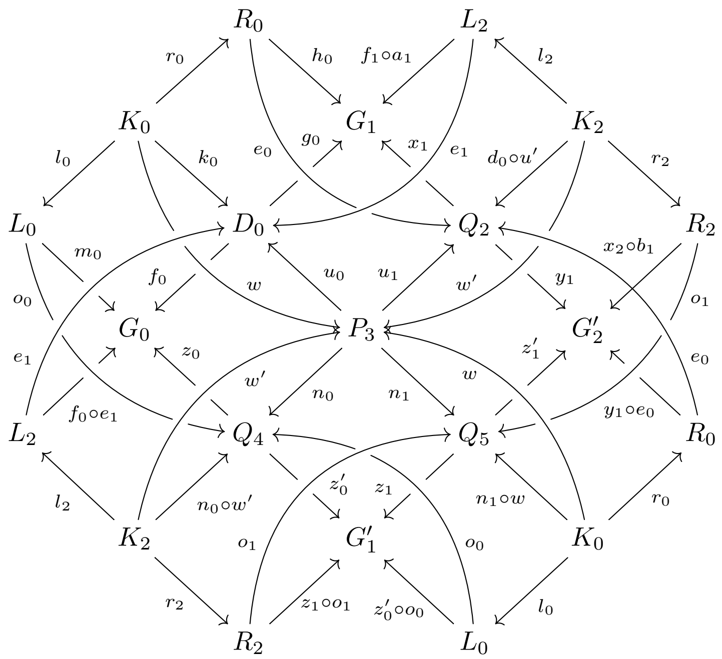
Proof. As a preliminary step, we are going to use Definitions 4.6 and 4.15 to get some diagrams. First of all, let (v, v') be the filler between $\mathscr{D}_0$ and $\mathscr{D}_1$ associated to (i_0, i_1), then we have



Secondly, the filler (u, u') induced by (a_0, a_1) between $\mathscr{D}_1$ and $\mathscr{D}_2$ yields:



Finally, the filler (w, w') between $\mathscr{D}_0$ and $S_{a_0,a_1}(\mathscr{D}_2)$ given by (e_0, e_1) provides us with:



So equipped we can turn to the prove of our claims.

1. We have to construct the two dotted arrows in the diagram below.

$$\begin{tikzcd}
L_0 & K_0 \arrow[l, "l_0"'] \arrow[r, "r_0"] \arrow[d, "n_1 \circ w"'] & R_0 \arrow[dr, "\beta_1"'] \arrow[drrr, dotted, "y_1 \circ e_0"] & & L_1 \arrow[dl, "\beta_0"] \arrow[dlll, dotted, "y_1 \circ b_0"'] & K_1 \arrow[l, "l_1"'] \arrow[r, "r_1"] \arrow[d, "d_1 \circ u"] & R_1 \arrow[d, "g_2 \circ a_1"] \\
G_1' \arrow[from=u, "z_0' \circ o_0"'] & Q_5 \arrow[l, "z_1"] \arrow[rr, "z_1'"'] & & G_2' & & Q_3 \arrow[ll, "x_2"] \arrow[r, "y_2"'] & G_3
\end{tikzcd}$$

Consider the arrows $i_0\colon R_0 \to D_1$ and $e_0\colon R_0 \to Q_2$. An easy computation shows that

$$\begin{aligned} f_1 \circ i_0 &= h_0 \\ &= x_1 \circ e_0 \end{aligned}$$

entailing the existence of the dotted $\beta_0'\colon R_0 \to P_2$ in the diagram

$$\begin{tikzcd}
R_0 \arrow[dr, dotted, "\beta_0'"] \arrow[drr, bend left, "i_0"] \arrow[ddr, bend right, "e_0"'] & & \\
& P_2 \arrow[r, "c_1"] \arrow[d, "d_0"] & D_1 \arrow[d, "f_1"] \\
& Q_2 \arrow[r, "x_1"'] & G_1
\end{tikzcd}$$

If we define $\beta_0\colon R_0 \to Q_3$ as $d_1 \circ \beta_1'$, then we easily get that

$$\begin{aligned} x_2 \circ \beta_0 &= x_2 \circ d_2 \circ \beta_0' \\ &= y_1 \circ d_0 \circ \beta_0' \\ &= y_1 \circ e_0 \end{aligned}$$

To define β_1, we proceed similarly. First consider $i_1\colon L_1 \to D_0$ and $b_0\colon L_1 \to Q_2$ and notice that

$$\begin{aligned} g_0 \circ i_1 &= m_1 \\ &= x_1 \circ b_0 \end{aligned}$$

implying the existence of $\beta_1'\colon L_1 \to P_3$ fitting in the diagram below.

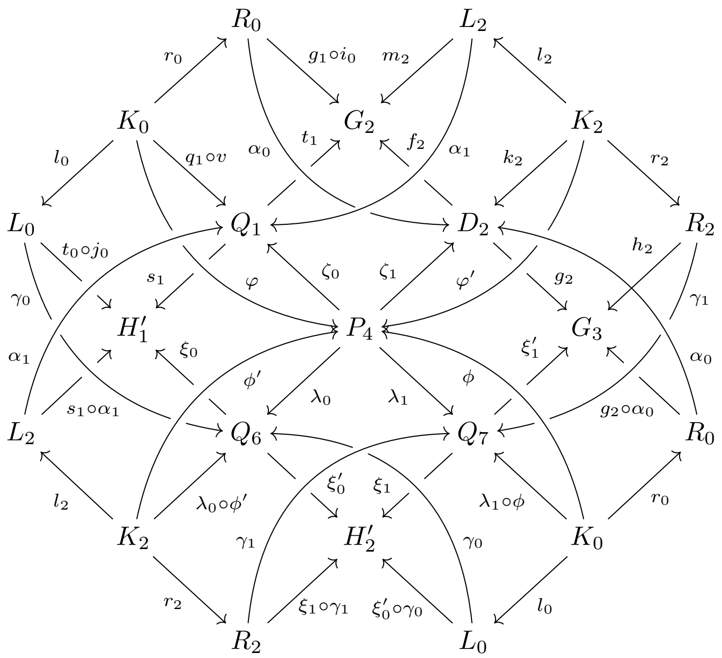
$$\begin{tikzcd}
L_1 \arrow[dr, dotted, "\beta_1'"] \arrow[drr, bend left, "i_1"] \arrow[ddr, bend right, "b_0"'] & & \\
& P_3 \arrow[r, "u_0"] \arrow[d, "u_1"] & D_0 \arrow[d, "g_0"] \\
& Q_2 \arrow[r, "x_1"'] & G_1
\end{tikzcd}$$

Let $\beta_1\colon L_1 \to Q_5$ be $n_1 \circ \beta_1'$, then

$$\begin{aligned} z_1' \circ \beta_1 &= z_1' \circ n_1 \circ \beta_1' \\ &= y_1 \circ u_1 \circ \beta_1' \\ &= y_1 \circ b_0 \end{aligned}$$

Therefore, (β_0, β_1) is the wanted independence pair.

2. In addition to the three diagram above, we have a fourth one given by the filler (φ_0, φ_1) associated to (α_0, α_1).



Our aim is to construct the dotted arrow in the following diagram.

$$
\begin{tikzcd}
L_1 \arrow[d, "f_0 \circ i_0"'] & K_1 \arrow[l, "l_1"'] \arrow[r, "r_1"] \arrow[d, "q_0 \circ v'"'] & R_1 \arrow[dr, "s_1 \circ j_1"'] \arrow[rrrd, dotted, "\epsilon_0"] & & L_2 \arrow[dl, "s_1 \circ \alpha_1"] \arrow[llld, dotted, "\epsilon_1"'] & K_2 \arrow[l, "l_2"'] \arrow[r, "r_2"] \arrow[d, "\lambda_0 \circ \phi'"] & R_2 \arrow[d, "\xi_1 \circ \gamma_1"] \\
G_0 & Q_0 \arrow[l, "s_0"] \arrow[rr, "t_0"'] & & H_1' & & Q_6 \arrow[ll, "\xi_0"] \arrow[r, "\xi_0'"'] & H_2'
\end{tikzcd}
$$

Let us start considering $j_1 \colon R_1 \to Q_1$ and $a_0 \colon R_1 \to D_2$. We have

$$
\begin{aligned}
f_2 \circ a_0 &= h_1 \\
&= t_1 \circ j_1
\end{aligned}
$$

and thus we get an arrow $\epsilon_0' \colon R_1 \to P_4$ which makes the diagram below commutative.

$$
\begin{tikzcd}
R_1 \arrow[dr, dotted, "\epsilon_0'"] \arrow[drr, bend left, "j_1"] \arrow[ddr, bend right, "a_0"'] & & \\
& P_4 \arrow[r, "\zeta_0"] \arrow[d, "\zeta_1"'] & Q_1 \arrow[d, "t_1"] \\
& D_2 \arrow[r, "f_2"'] & G_2
\end{tikzcd}
$$

We can then define $\epsilon_0 \colon R_1 \to Q_6$ to be $\lambda_0 \circ \epsilon_0'$. For such an arrow we have a chain of identities:

$$\begin{aligned}
\xi_0 \circ \epsilon_0 &= \xi_0 \circ \lambda_0 \circ \epsilon_0' \\
&= s_1 \circ \zeta_0 \circ \epsilon_0' \\
&= s_1 \circ j_1
\end{aligned}$$

Next, to define $\epsilon_1 \colon L_2 \to Q_0$ we take $e_1 \colon L_2 \to D_0$ and $a_1 \colon L_2 \to D_1$. By definition we have $g_0 \circ e_1 = f_1 \circ a_1$, giving us the dotted arrow below

$$\begin{tikzcd}
L_2 \arrow[dr, dotted, "\epsilon_1'"] \arrow[drr, bend left, "a_1"] \arrow[ddr, bend right, "e_0"'] & & \\
& P_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] & D_1 \arrow[d, "f_1"] \\
& D_0 \arrow[r, "g_0"'] & G_1
\end{tikzcd}$$

Now, notice that

$$\begin{aligned}
t_1 \circ q_1 \circ \epsilon_1' &= g_1 \circ p_1 \circ \epsilon_1' & &= g_1 \circ a_1 \\
&= m_2
\end{aligned}$$

Hence $(\alpha_0, q_1 \circ \epsilon_1')$ is an independence pair for $S_{i_0,i_1}(\mathscr{D}_0)$ and $\mathscr{D}_2$. By hypothesis $S_{i_0,i_1}(\mathscr{D}_0) \mathrel{\updownarrow!} \mathscr{D}_2$ and thus $q_1 \circ \epsilon_1'$ must coincide with α_1. Now, let $\epsilon_1 \colon L_2 \to Q_0$ be $q_0 \circ \epsilon_1'$. Computing we get

$$\begin{aligned}
t_0 \circ \epsilon_1 &= t_0 \circ q_0 \circ \epsilon_1' \\
&= s_1 \circ q_1 \circ \epsilon_1' \\
&= s_1 \circ \alpha_1
\end{aligned}$$

Allowing us to conclude that $S_{i_0,i_1}(\mathscr{D}_1) \updownarrow S_{\alpha_0,\alpha_1}(\mathscr{D}_2)$. $\square$

4.3 Concatenable traces

Lemma 4.27. abstract equivalence and switch

Proof. contenuto... $\square$

Definition 4.28. tracce

Before moving forward, we will prove some other useful properties of the switch equivalence relation.

Lemma 4.29. lemma 19

Proof. contenuto... $\square$

Theorem 4.30. preordine

Proof. contenuto... $\square$

5 Domains for DPO-rewriting

5.1 weak domains

5.2 From adhesive grammars to weak domains

6 Conclusions and further work

VERY NICE CONCLUSIONS

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A A note on fillers and sequential independence

In Proposition 4.11 we proved that, in the linear case, the existence of an indpendence pair between two derivation is equivalent to that of a filler between them. This result can be further refined: in a [2] a class $\mathbb{B}$ of (quasi)adhesive category is defined for which the local Church-Rosser Theorem holds even for left-linear DPO-rewriting system. In our language, and given Theorem 4.17 and **??**, this amount to prove that, for elements of $\mathbb{B}$, every independence pair induces a filler.

Definition A.1. Let $\mathbf{X}$ be a category, we say that $\mathbf{X}$ satisfies

- the *mixed decomposition* property if for every diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\ A \arrow[r, "h"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

  whose outer boundary is a pushout and in which k is a monomorphisms,

- the *pushout decomposition* property

Lemma A.2. *contenuto...*

Proof. contenuto... □

Corollary A.3. *contenuto...*

The following result shows that the mixed and pushout decomposition properties guarantee that every independence pair gives rise to a filler.

Theorem A.4.

filler e classe B+

Proof. □

Our next step is to identify sufficient conditions for a category **X** to satisfy the mixed and pushout decomposition properties.

Definition A.5.

classe B e class B+

Example A.6.

esempi

Example A.7.

esempi

Proposition A.8.

da B a B+

Proof. contenuto... □

Lemma A.9.

due proprietà classe B

Proof. □

Corollary A.10.

due proprietà classe B+

Corollary A.11.

filler e classe B+

B Rewriting systems on Set.

C On permutations

inserire risutati sulle permutazione che abbiamo usato