DOMAIN : $\boxed{\text{Characterising irreducibles}}$

IRREDUCIBLE : intuitively $\varepsilon = D;p_n$ s.t. p cannot be switched with previous steps

$\varepsilon = D;p$ irreducible iff $n = |\varepsilon| \geq 1 \ \wedge \ \forall\, \varepsilon \equiv^{sh}_{\sigma} \varepsilon' \quad \sigma(n-1) = n-1$

$(\Leftarrow)$ $\varepsilon = D;p$ $\qquad |pred(\varepsilon)| = |\{ [D'] \mid \exists \delta' \ \ D';\delta' = \varepsilon = D;\delta \}| = 1$

$(\Longrightarrow)$ let ε be irreducible $\qquad \varepsilon = D;p \equiv^{sh}_{\sigma} D';p' = \varepsilon'$

$n = |\varepsilon| \geq 1$

(otherwise $|\varepsilon| = 0 \Rightarrow \varepsilon = \bot$)

$\Rightarrow$ no predecessors

$\downarrow$

D, D' predecessors $\quad D \equiv_{\sigma'} D'$

since consuming, by Lemma 2.3.8 (restriction Lemma)

$p \equiv_{\sigma'} p'$ restricted

$\boxed{\text{INTERCHANGEABILITY}}$

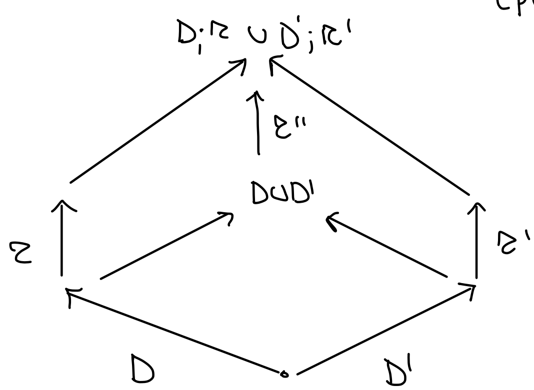
$\varepsilon = D;r$

$\varepsilon' = D';r'$ $\qquad \varepsilon \leftrightarrow \varepsilon'$ iff consistent and $\qquad i \leftrightarrow i' \Leftrightarrow \begin{cases} i \frown i' \ \& \\ p(i) \sqcup i' = i \cup p(i') \end{cases}$ $\overset{!}{=} i \sqcup i'$

$\Updownarrow$

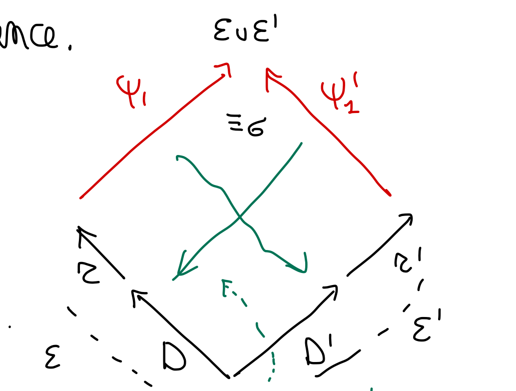
esiste $\varepsilon \cup \varepsilon' = \varepsilon;\varphi_1 \equiv_{\sigma} \varepsilon';\varphi_1'$

$\sigma(|\varepsilon|-1) = |\varepsilon'|-1$

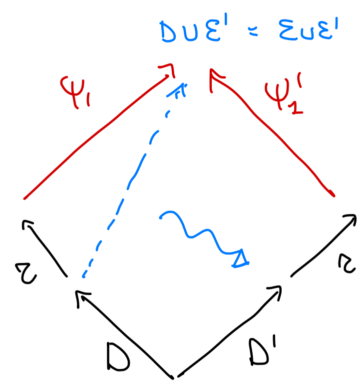


$(\Longrightarrow)$ Assume $\varepsilon \leftrightarrow \varepsilon'$. Then $\varepsilon, \varepsilon'$ consistent and $D = pred(\varepsilon)$ $\ D' = pred(\varepsilon')$
$\& \ D' \cup \varepsilon = D \cup \varepsilon' = \varepsilon \cup \varepsilon'$

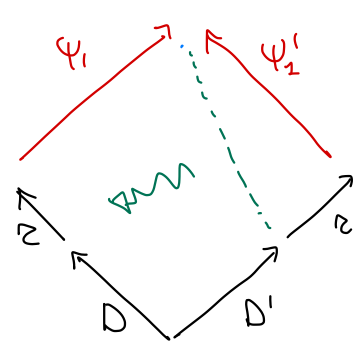
hence.



σ manda φ_1 in ε
σ^{-1} " φ_1' in ε'



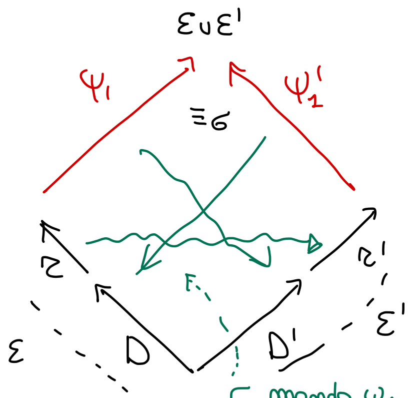
$\Rightarrow \ \sigma(|\varepsilon|-1) \leq |\varepsilon'|-1$



$\sigma^{-1}(|\varepsilon'|-1) \leq |\varepsilon|-1$

(⇐) Suppose $\mathcal{E}, \mathcal{E}'$ consistent & $\mathcal{E} \cup \mathcal{E}' = \mathcal{E};\varphi_2 \equiv_\sigma \mathcal{E}';\varphi_2'$ with

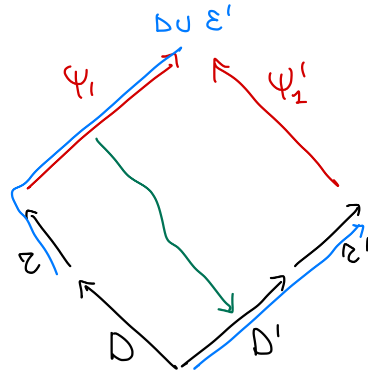
$\sigma(|\mathcal{E}| - 1) = |\mathcal{E}'| - 1$



σ manda φ_1 in $\mathcal{E}$

σ^{-1} '' φ_1' in $\mathcal{E}'$

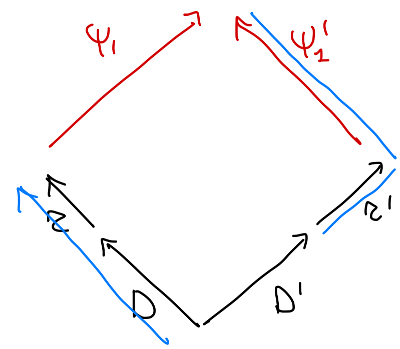
$\sigma(z) = z'$



hence $z;\varphi_1$ is a "minimal completion" for D with respect to $\mathcal{E}'$

"hence" $\mathcal{E} \cup \mathcal{E}' = D \cup \mathcal{E}'$

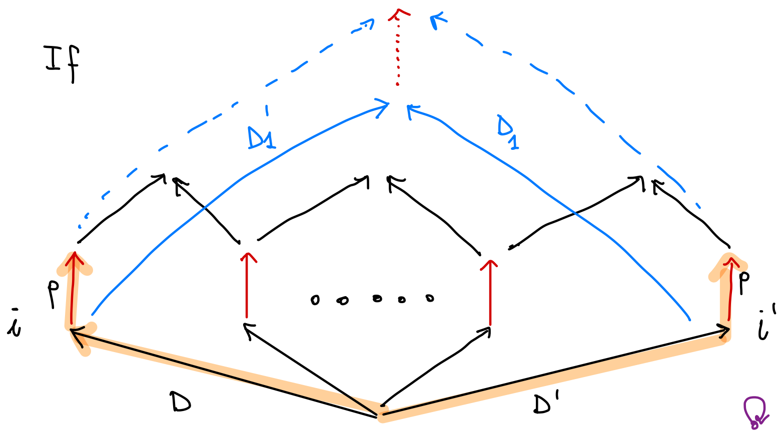
$\qquad\qquad\qquad\qquad \text{pred}(\mathcal{E})$



dually $\mathcal{E} \cup \mathcal{E}' = D' \cup \mathcal{E}$

SHOWING INTERCHANGEABILITY OF DOMAIN (axioms (I) & (II))

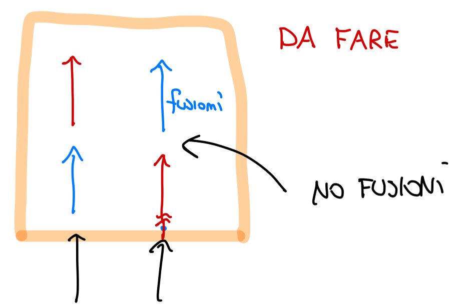
(I) If

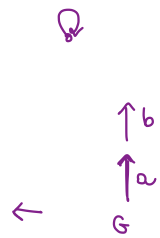


$i \leftrightarrow^* i'$

$\wedge$ $\quad \Rightarrow \quad i \leftrightarrow i'$

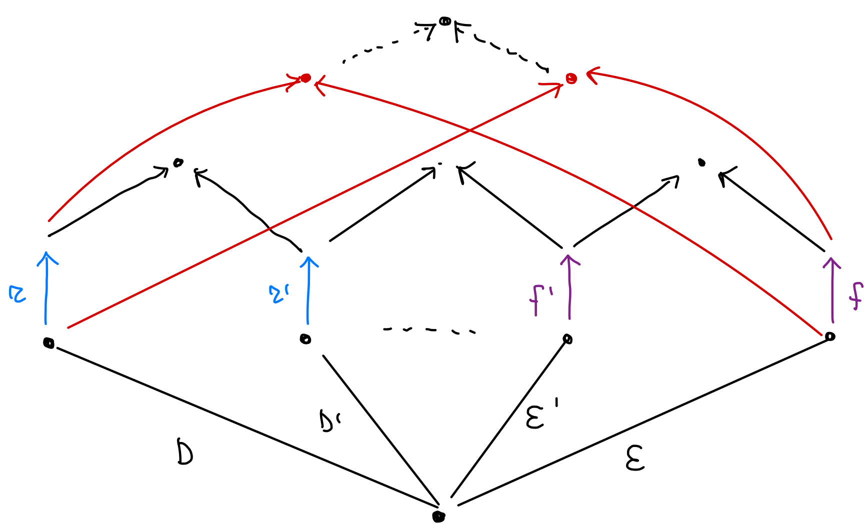
$p(i) \frown p(i')$





then the dotted lines exist

(II) Given



$i \leftrightarrow^* i'$ & $j \leftrightarrow^* j'$

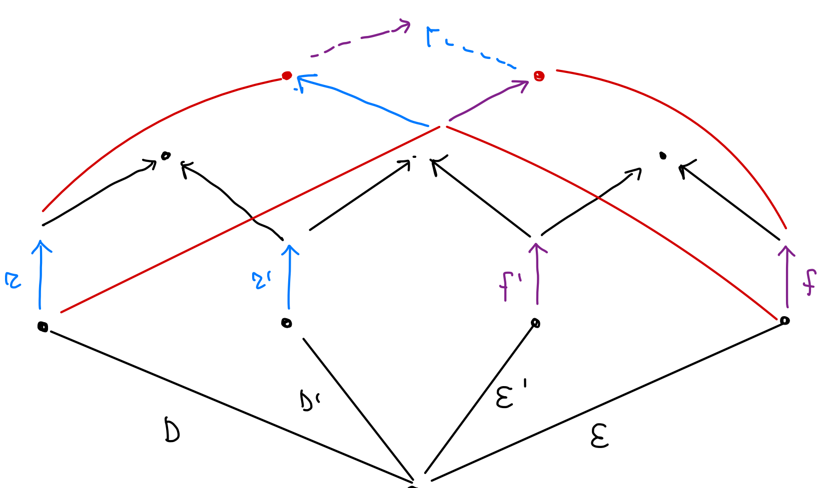
& $p(i) \frown j$ & $i \frown p(j)$

& $i' \frown j'$

$\Rightarrow \quad i \frown j$

then the dotted line exists ($D;r$ consistent with $\varepsilon;f$)

$\boxed{\text{Note}}$: we know that r and f can be translated



we should prove

that we can close

(they are indep.)

Ideas for rules which are well switching (18/05/2024)

[completare con caratterizzazione di Davide]

$$L \xleftarrow{\ell} K \xrightarrow{r} R$$

with L taken from $\boxed{\mathcal{L} \subseteq \mathcal{C}}$

$\ell \in \mathcal{M}$ (such that $\mathcal{C}$ $\mathcal{M}$-Adhesive)

$\mathcal{M}$?

$r \in \mathcal{Y}$ such that (1) mono $\subseteq \mathcal{Y}$

(2) $\mathcal{Y}$ closed by composition

(3) $\mathcal{Y}$ stable under $\boxed{\text{(3a) PO}}$ and (3b) PBs

(4) $\mathcal{Y}$ mono. for arrows with source in $\mathcal{L}$

$$L \overset{f}{\underset{g}{\rightrightarrows}} G \xrightarrow{h} G' \qquad f;h = g;h \Rightarrow f = g$$

Idea: Conditions above "can be expressed" by asking

probably

use this as a basic definition

$\mathcal{L}$ coreflective subcategory of $\mathcal{C}$

$$\mathcal{L} \underset{I}{\overset{F}{\underset{\top}{\rightleftarrows}}} \mathcal{C} \quad I \text{ inclusion}$$

e.g. for graphs $F(G)$ removes isolated nodes

and capture the properties (1), (2), (3b), (4) by $\mathcal{Y}$ = arrows whose image is mono in $\mathcal{L}$

while (3a) asked explicitly

Actually, the above is equivalent to the first formulation

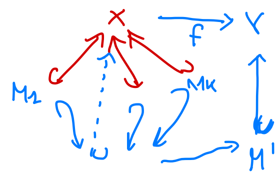
(see next page, DAVIDE KNOWS more)

ALCUNE NOTE UN PO' DISORDINATE SUL PUNTO DI CUI SOPRA

Given $\mathcal{C}$ with $\mathcal{E}$-$\mathcal{M}$ factorisation, $\mathcal{M}$-adhesive

(factorisation proper? $\mathcal{E} \subseteq$ epi, $\mathcal{M} \subseteq$ mono)

and $\mathcal{M}$ has unions



and a class $\mathcal{Y} \subseteq A(\mathcal{C})$

(1) $\mathcal{M} \subseteq \mathcal{Y}$

(2) $\mathcal{Y}$ closed by composition

(3) $\mathcal{Y}$ stable under $\boxed{\text{(3a) PO}}$ and (3b) PBs

Then if we denote by $\mathcal{L}$ the sub cat st. $\forall$ arrow ......

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "F"] \end{tikzcd}$$

[ok?]

$$FG = \bigcup \{ L \subseteq G \mid L \in \mathcal{L} \}$$

$$\begin{tikzcd} FG \arrow[d] & & \\ G & L' \in \mathcal{L} \arrow[l] \arrow[dl] & \\ L'' \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & H \arrow[r, "\circ"] & H' \end{tikzcd}$$

$$FG \rightrightarrows H \xrightarrow{\circ} H'$$

$\mathcal{C}$ $\quad \underline{L} \hookleftarrow K \xrightarrow{\circ} R$ $\quad \boxed{\begin{matrix} \mathcal{Y} \\ (1) - (3) \end{matrix}}$

Working in the above setting

(a) the rewriting system is well switching

$$\begin{tikzcd}
L_1 \arrow[d] & K_1 \arrow[l, hook'] \arrow[r, "\circ"] \arrow[d] & R_1 \arrow[d] & L_2 \arrow[dl] \arrow[dll, bend right, red, "\text{unique when it exists}"'] & K_2 \arrow[l] \arrow[r, "\circ"] \arrow[d] & R_2 \arrow[d] \\
G_0 & D_0 \arrow[l, hook'] \arrow[r, "\circ"] & G_1 & & D_1 \arrow[ll, hook'] \arrow[r, "\circ"] & G_2
\end{tikzcd}$$

(b) morphisms from graphs in the derivation to the colimit are in $\mathcal{Y}$
(hence the match in the derivation is uniquely determined by the image in the colimit) ← can we use this fact more? e.g. shift equivalent iff some colimit + something

(some starting graph) ?

$$\begin{tikzcd}
L_1 \arrow[d] & K_1 \arrow[l, hook'] \arrow[r, "\circ"] \arrow[d] & R_1 \arrow[d] & L_2 \arrow[dl] & K_2 \arrow[l] \arrow[r, "\circ"] \arrow[d] & R_2 \arrow[d] \\
G_0 \arrow[dr, "\circ"] & D_0 \arrow[l, hook'] \arrow[r, "\circ"] & G_1 \arrow[dl] \arrow[dr, "\circ"] & & D_1 \arrow[ll, hook'] \arrow[r, "\circ"] & G_2 \arrow[dll] \\
& G_{01} \arrow[dr, "\circ"] & & G_{12} \arrow[dl] & & \\
& & G_{012} & & &
\end{tikzcd}$$

All morphisms are compositions of arrows in $\mathcal{F}$
⇒ hence in $\mathcal{Y}$

(c) <u>Existence of a minimal quotient to which a rule applies</u>

If L has a match in a quotient G' of G

$$\begin{tikzcd}
& L \arrow[d, "m"] \\
G \arrow[r, two heads, "\circ"', "q"] & G'
\end{tikzcd}
\qquad
\begin{tikzcd}
& G'' \arrow[dr] & & L \arrow[ll] \arrow[dl, "m"] \\
G \arrow[ur, "\circ"] \arrow[rr, two heads, "\circ"] & & G' &
\end{tikzcd}$$

then there is a minimal quotient G'' for which a match exists

(all matches to quotients $m_1 \longrightarrow G_1 \overset{\circ}{\twoheadleftarrow} G$

factorises through G''

$$\begin{tikzcd}
& G'' \arrow[dr] & L \arrow[l, "m''"'] \arrow[d, "m_1"] \\
G \arrow[ur, "\circ"] \arrow[rr, two heads, "\circ"] & & G_1
\end{tikzcd}$$

In fact take the PB of m and q, and it image through F which is again a PB (since F is right adjoint

Additionaly consider the counit ε

$$\begin{tikzcd}
FL' \arrow[r, two heads, "\circ"] \arrow[d] \arrow[r, bend left] \arrow[dd, bend right] & FL \arrow[d, "Fm"] \\
FG \arrow[r, two heads, "Fq"] \arrow[d, "\varepsilon_G"'] & FG' \arrow[d, "\varepsilon_{G'}"] \\
G \arrow[r, two heads, "q"] & G'
\end{tikzcd}
\qquad \xleftarrow{F} \qquad
\begin{tikzcd}
L' \arrow[r] \arrow[d] & L \arrow[d, "m"] \\
G \arrow[r, two heads, "q"] & G'
\end{tikzcd}$$

take the pushout of the red arrows in $\mathcal{C}$

$$\begin{tikzcd}
FL' \arrow[r, two heads, "\circ"] \arrow[d] & FL \arrow[d, "Fm"] \arrow[ddr, bend left] & \\
FG \arrow[r, two heads, "Fq"] \arrow[d, "\varepsilon_G"'] & FG' \arrow[d, "\varepsilon_{G'}"] & \\
G \arrow[r, two heads, "q"] \arrow[rr, two heads, bend right, "\circ"] & G' & G'' \arrow[l, dotted]
\end{tikzcd}$$

$\boxed{\text{unclear if it is in } \mathcal{Y}, \text{ needed?}}$

since $Fm ; \varepsilon_{G'} = GF\,m = m$ the above is what we want.

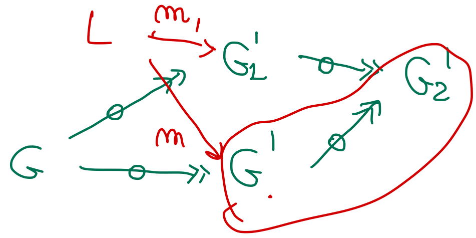
Si osserva che partendo da $G_2' \twoheadrightarrow G'$ ottengo lo stesso G''

$$\begin{tikzcd}
FL_2 \arrow[dr, "\sim"'] \arrow[rr, bend left] \arrow[ddr, bend right] & & \\
& FL' \arrow[r, two heads, "\circ"] \arrow[d] & FL \arrow[d, "Fm"] \arrow[dl, "F\bar{m}_2"'] \arrow[ddr, bend left] & \\
& FG \arrow[r, two heads, "Fq"] \arrow[d, "\varepsilon_G"'] \arrow[r, "Fq_2", bend left] & FG_2' \arrow[d, two heads, "F(\ )\ \text{iso}"] \\
& FG' \arrow[d, "\varepsilon_{G'}"] & \\
& G \arrow[r, two heads, "q"] \arrow[rr, two heads, bend right, "\circ"] & G' & G'' \arrow[l, dotted]
\end{tikzcd}$$

e quindi se ho due quozienti

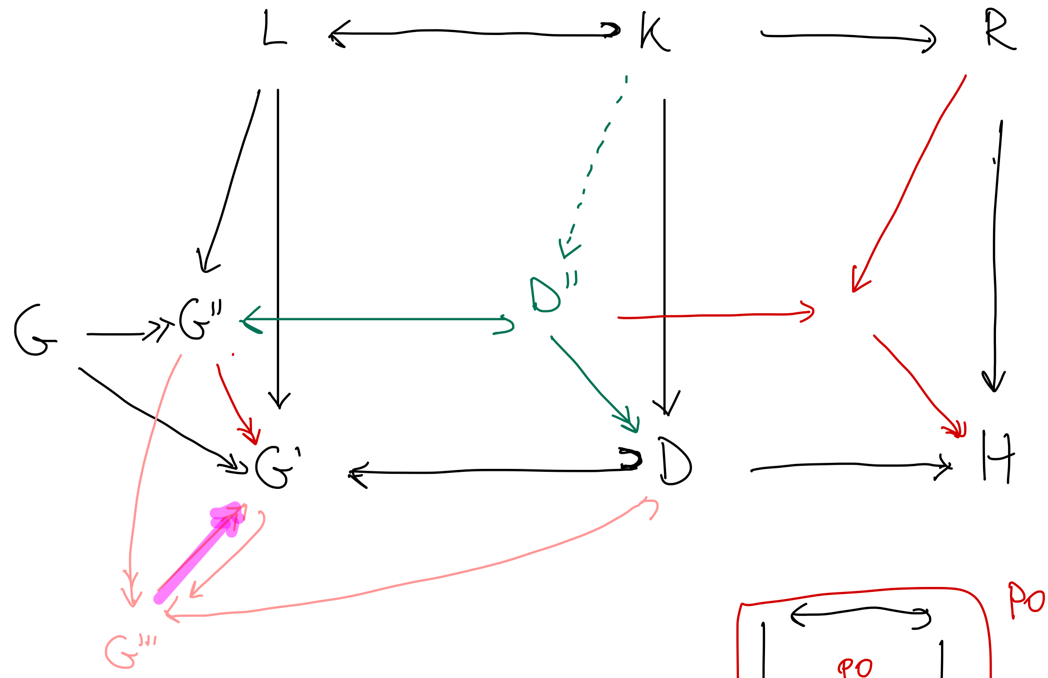
$G \twoheadrightarrow G_1'$ e $G \twoheadrightarrow G'$

considero il pushout e faccio lo stesso ragionamento.

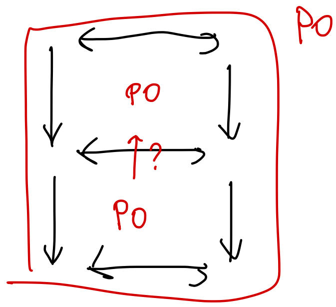


REWRITING VS QUOTIENTS

(1) Rewriting on a less quotiented object as long as there is a "coherent" match (commuting with the original one) is always possible



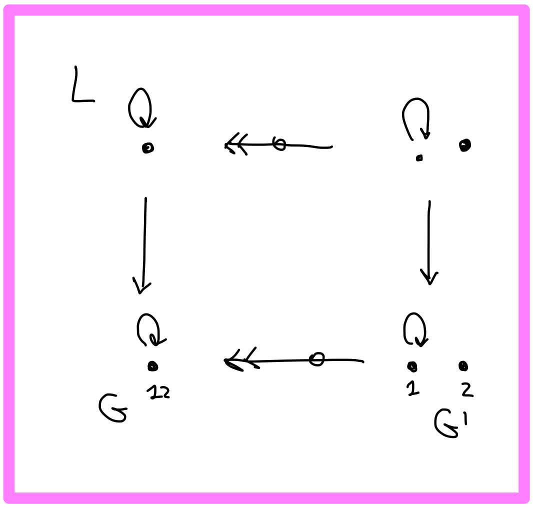
OK, me in B+



We assume $L \to G'$ allows for constructing DPO diagram.

We show that the same can be done for G'' "less quotiented".

NOTE: The previous version was using the fact that PB of L-objects along $\mathcal{Y}$-arrows is an L-object, which is false.



② Rewriting on a more quotiented object (as long as the quotient factorises through the colimit) is always possible [and does not change neither the target nor the colimit]

- **OBSERVE**: Given $D = p_1 \dots p_n$

$$
D \quad
\begin{tikzcd}
L_1 \arrow[d] & K_1 \arrow[l] \arrow[r] & R_1 \arrow[dr] & L_i \arrow[d] & K_i \arrow[l] \arrow[r] & R_i \arrow[dr] & L_{i+1} \arrow[dl] & K_{i+1} \arrow[l] \arrow[r] & R_{i+1} & \dots & L_n & K_n \arrow[l] \arrow[r] & R_n \\
G_1 \arrow[rrr, dash] & & & G_2 \arrow[rrr, dash] & & & G_{i+1} & & D_{n+1} & & & & G_{n+1}
\end{tikzcd}
$$

if $G_{i+1} \twoheadrightarrow G'_{i+1}$ factors through the colimit then I can continue the derivation from G'_{i+1}

$$
D' \quad
\begin{tikzcd}
L_1 \arrow[d] & K_1 \arrow[l] \arrow[r] & R_1 \arrow[dr] & L_i \arrow[d] & K_i \arrow[l] \arrow[r] & R_i \arrow[dr] & & L_{i+1} \arrow[dl, "m_{i+1}"'] \arrow[d] & K_{i+1} \arrow[l] \arrow[r] & R_{i+1} & \dots & L_n & K_n \arrow[l] \arrow[r] & R_n \arrow[d] \\
G_1 \arrow[rrr, dash] & & & G_2 \arrow[rrr, dash] & & & G_{i+1} \arrow[r, two heads] & G'_{i+1} & D'_{i+1} \arrow[l] & & & & & G'_{n+1}
\end{tikzcd}
$$

with D'_{i+1}, D'_{i+2} .... G'_{i+2} .. G'_{n+1} are quotients of the original

and $G'_{n+1} = G_{n+1}$

IDEAS FOR PROOF

$$
\begin{tikzcd}
\cdots & L_i \arrow[d] & K_i \arrow[l, hook'] \arrow[r] \arrow[d] & R_i \arrow[d] \\
& G_i \arrow[d, two heads] & D_i \arrow[l, hook'] \arrow[r] \arrow[d, two heads] & G_{i+1} \\
& G_i' \arrow[d, hook] & D_i' \arrow[l, hook'] & \\
& C & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& G_i \arrow[dl, hook'] \arrow[d, two heads] & D_i \arrow[l, hook'] \arrow[r] \arrow[d] & G_{i+1} \arrow[d] \arrow[dr] & ? \\
C_{0,i} \arrow[drr] & G_i' \arrow[dr, dashed] & D_i' \arrow[l, hook'] \arrow[r] \arrow[d, hook] & G_{i+1}' \arrow[dl, dashed] & C_{i+1,m} \arrow[dll, hook'] \\
& & C & &
\end{tikzcd}
$$

1 Starting from epi mono of $D_i \to C$ = epi mono of $D_i \to C_{i+1,m}$

$$
\begin{tikzcd}
& & D_i \arrow[dl, hook'] \arrow[dr, two heads] \arrow[dd, phantom, "1"] & & \\
& G_i \arrow[dl, hook'] \arrow[dr, two heads] \arrow[dd, phantom, "2"] & & D_i' \arrow[dl, hook'] \arrow[dr] & \\
C_{0i} \arrow[dr] & & G_i' \arrow[dl, hook'] \arrow[r, phantom, "3"] & & C_{i+1,m} \arrow[ddll, hook'] \\
& {} \arrow[dr, hook] & & & \\
& & C & &
\end{tikzcd}
$$

take p.o. (1), (2) and (3)

then $G_i \twoheadrightarrow G_i' \hookrightarrow C$ is

epi mono factorisation

This should imply that if I take epi-mono of G_i in C

$$
\begin{tikzcd}
& D_i \arrow[dl, hook'] \arrow[dr, two heads] \arrow[dd, phantom, "1"] & \\
G_i \arrow[dr, two heads] & & D_i \arrow[dl, hook'] \\
& G_i' \arrow[dr, hook] & \\
& & C
\end{tikzcd}
$$

and then epi-mono of $D_i \to G_i'$ then (1) is a PO (by uniqueness of the factorisation

② We can also perform a step along any quotient $G_i \twoheadrightarrow G_i''$ which factors through colimit

$$\begin{tikzcd}
G_i \arrow[d, two heads] & D_i \arrow[l, hook'] \arrow[dd, two heads] \\
G_i'' \arrow[d, two heads] & \\
G_i' \arrow[d, hook] & D_i' \arrow[l, hook'] \\
C &
\end{tikzcd}$$

epi mono of $D_i \to G_i'$ hence by (1) is a PO

Now take epi-mono of $D_i \to G_i''$ and then of $D_i'' \to G_i'$

$$\begin{tikzcd}
G_i \arrow[d, two heads] & & D_i \arrow[l, hook'] \arrow[dl] \arrow[ddd, two heads] \\
G_i'' \arrow[dd, two heads] & D_i'' \arrow[l, hook'] \arrow[d] & \\
& E_i' \arrow[dl, hook'] & \\
G_i' \arrow[d, hook] & & D_i' \arrow[ll, hook'] \\
C & &
\end{tikzcd}$$

By uniqueness of factorisation $E_i' \cong D_i'$

From the fact that the outer square is a PO we deduce that also (2) is a PO

$$\begin{tikzcd}
G_i \arrow[d, two heads, "m'"'] \arrow[dr, phantom, "(1)"] & D_i \arrow[l, hook'] \arrow[d, two heads] \\
G_i'' \arrow[d, "f'"'] \arrow[dd, bend right=40] \arrow[dr, phantom, "(2)"] & D_i'' \arrow[l, hook'] \arrow[d] \\
G_i' \arrow[d, dashed] & D_i' \arrow[l, hook'] \arrow[dl, bend left] \\
A &
\end{tikzcd}$$

- existence of mediating $G_i' \dashrightarrow A$ by the fact that the outer square is PO and m' epi
- uniqueness by f' epi

hence (1) is pushout by PO decomposition.

③ One need to complete the DPO

$$\begin{tikzcd}
K_i \arrow[r] \arrow[d] & R_i \arrow[d] & \\
D_i \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & G_{i+1} \arrow[d, dotted] \arrow[dr, bend left] & \\
D_i' \arrow[r, dashed] & G_{i+2} \arrow[r, two heads] & G_{i+1}''
\end{tikzcd}$$

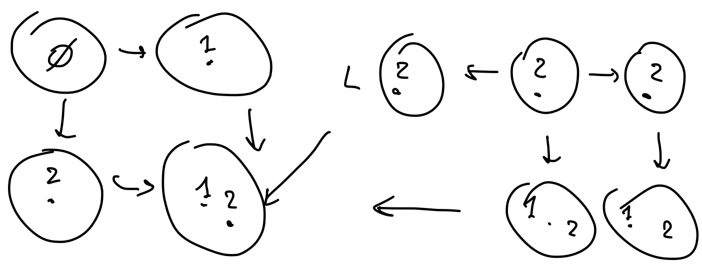
? { - it should work if $D_i \twoheadrightarrow D_i'$ is the image in the colimit (by (1)) }

- more difficult with partial quotients unless we assume to have all PO's

INDEPENDENCE IN THE COLIMIT

For the linear case

$$\begin{tikzcd}
 & & & P_2 \cong P_1 \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
K \arrow[rr, hook] \arrow[d] & & R \arrow[dr] \arrow[drrrr, red] & & L' \arrow[dl] \arrow[dllll, red] & & K' \arrow[ll, hook'] \arrow[d] \\
D \arrow[rrr, hook] & & & G & & & D' \arrow[lll, hook']
\end{tikzcd}$$



independence pair

iff

PBs P_1 and P_2 coincide

since embeddings in the colimit are mono this transfers to colimit

$\Rightarrow$ $\boxed{\text{independence is global}}$

$\boxed{\text{What about left-linear derivations?}}$

<u>OBS 1</u>

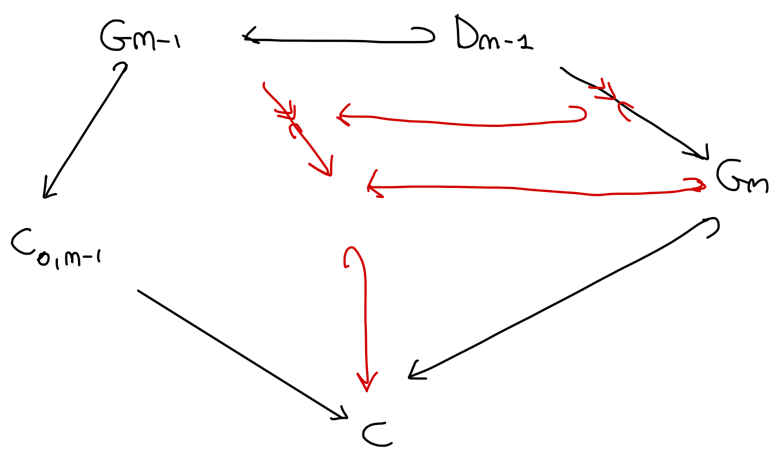
Given a derivation

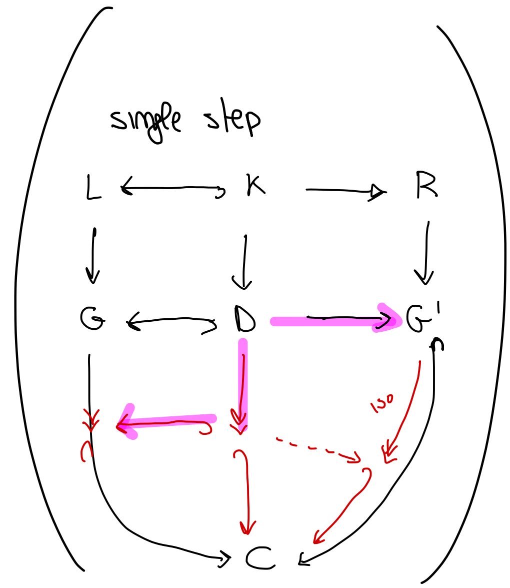
$$\begin{tikzcd}
L_1 \arrow[d] & K_1 \arrow[l, hook'] \arrow[r] \arrow[d] & R_1 \arrow[d] \; \cdots & L_{m-1} \arrow[d] & K_{m-1} \arrow[l, hook'] \arrow[r] \arrow[d] & R_{m-1} \arrow[d] & L_m \arrow[dl] & K_m \arrow[l, hook'] \arrow[r] \arrow[d] & R_m \arrow[d] \\
G_0 \arrow[drrrr] & D_0 \arrow[l, hook'] \arrow[r] & G_1 & G_{m-1} & D_{m-1} \arrow[l, hook'] \arrow[r] & G_{m-1} & & D_m \arrow[ll, hook'] \arrow[r] & G_m \arrow[dllll] \\
 & & & & C & & & &
\end{tikzcd}$$

then we can take the "image" in the colimit of the <u>last</u> step in

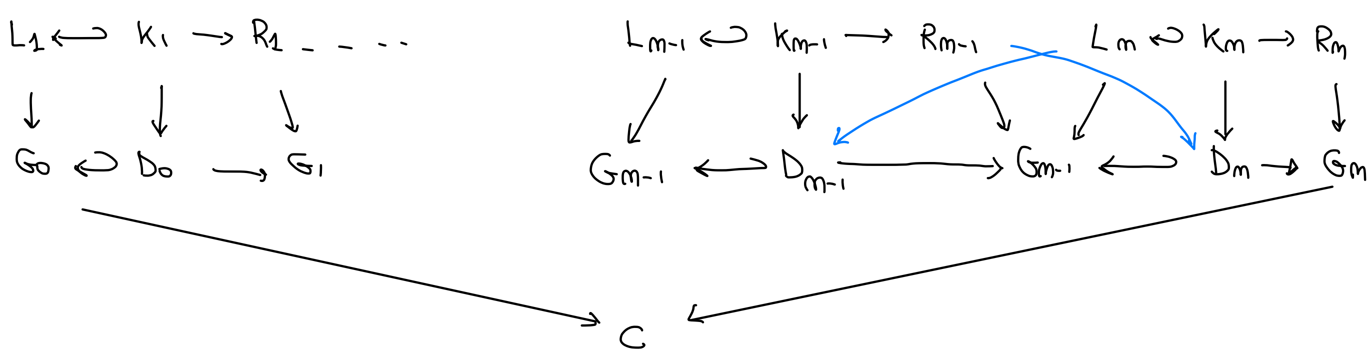
$$\begin{tikzcd}
G_{m-1} \arrow[d, two heads] & D_{m-1} \arrow[l, hook'] \arrow[r] \arrow[d, two heads] & G_m \arrow[d, two heads] \\
G'_{m-1} \arrow[dr] & D'_{m-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d] & G'_m \arrow[dl] \\
 & C &
\end{tikzcd}$$

in fact C is a pushout as follows

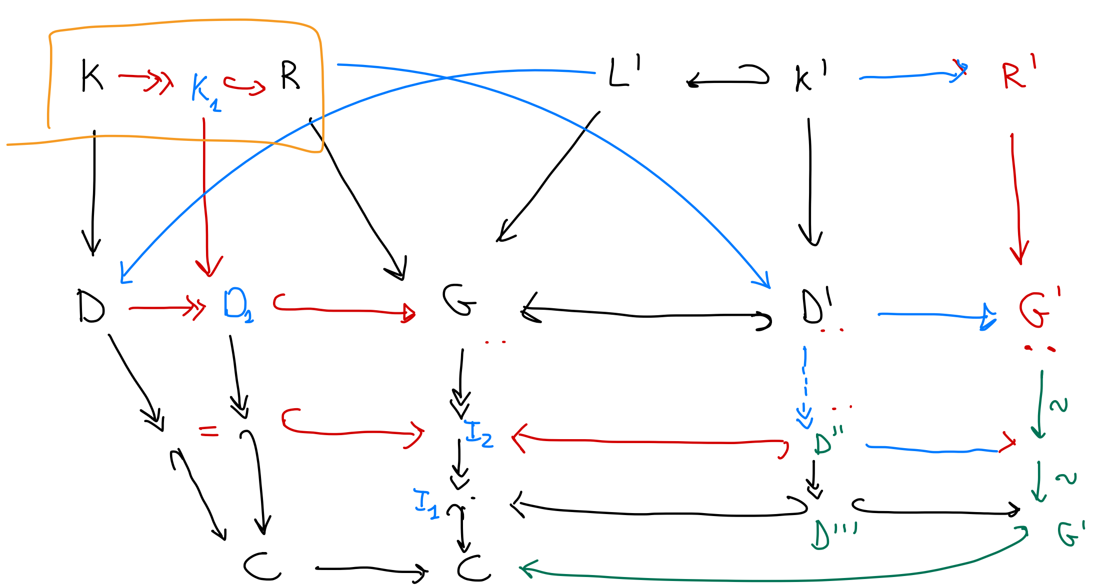




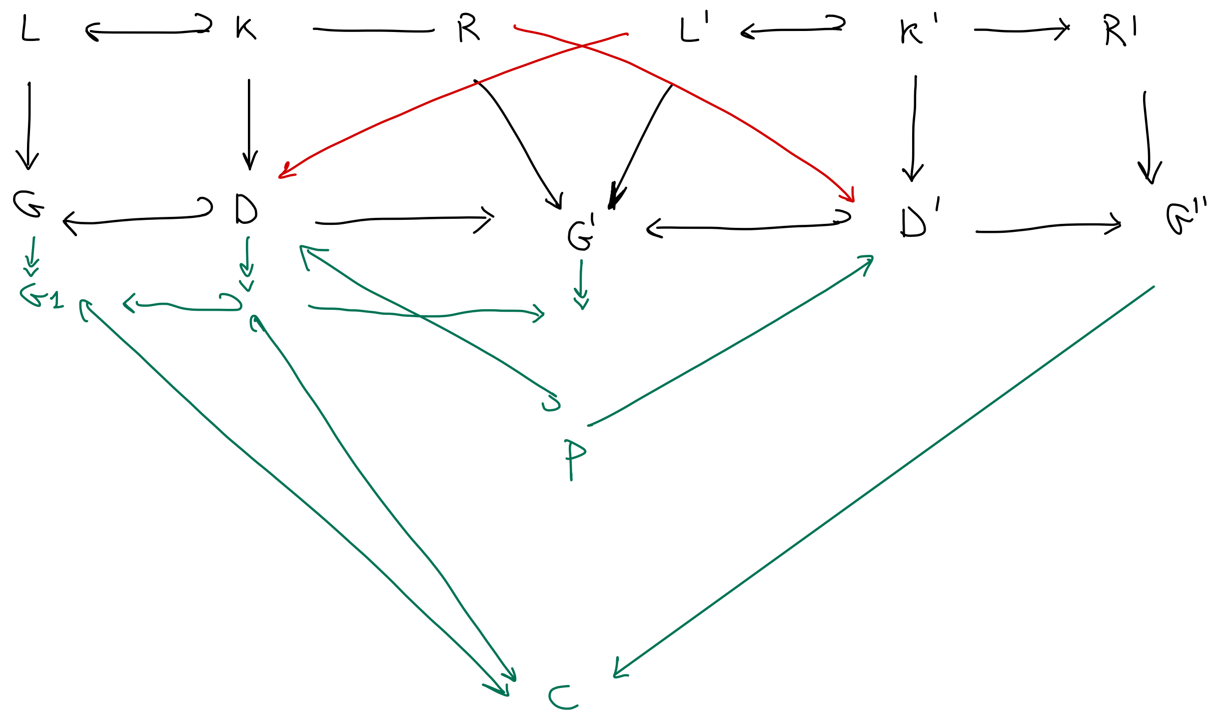
OBS2 Given a derivation, with the two last steps independent

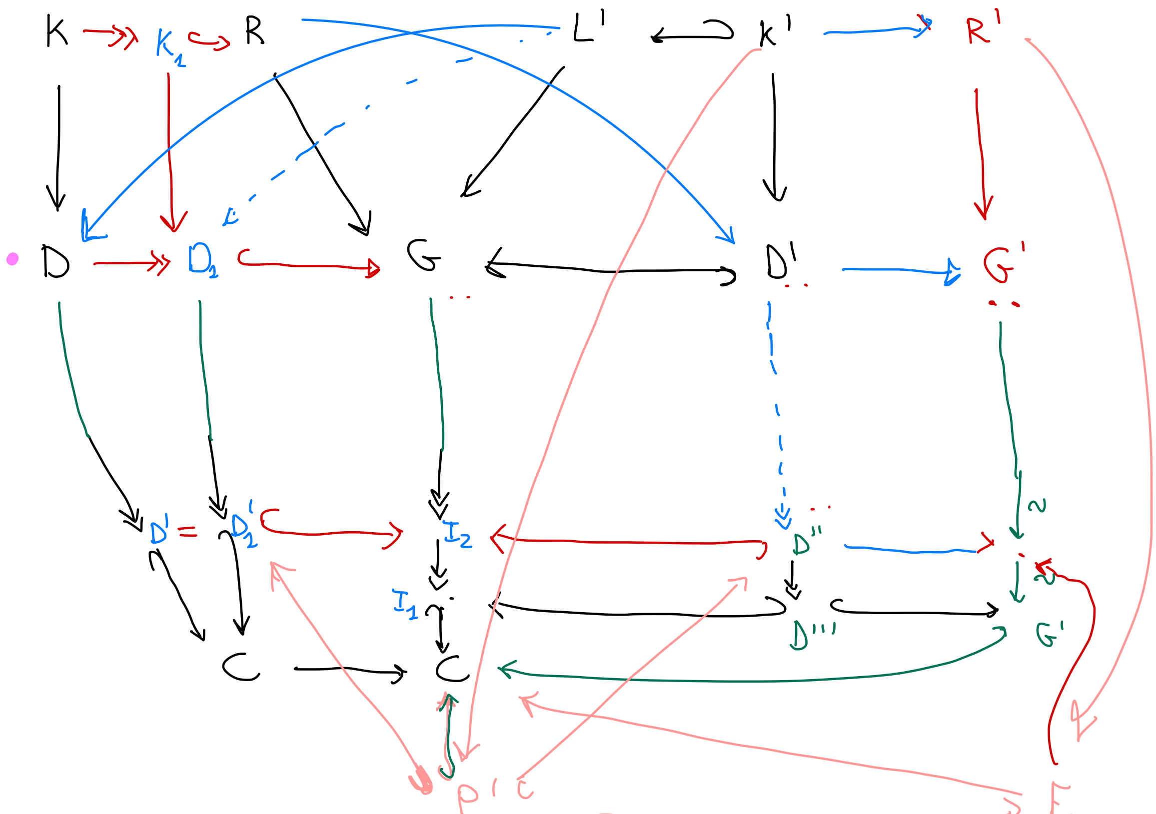


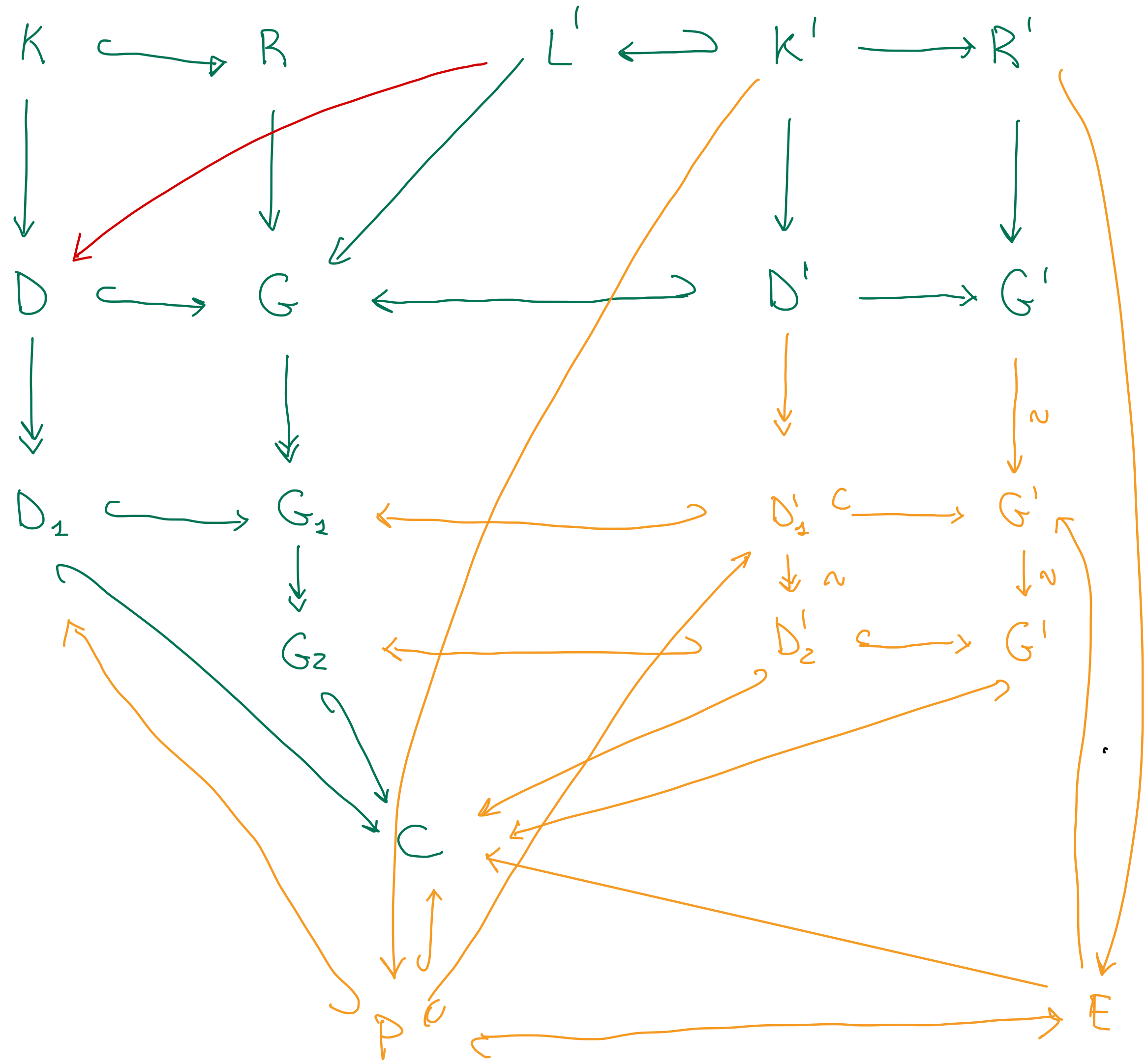
then we can take the "image" in the colimit of step m

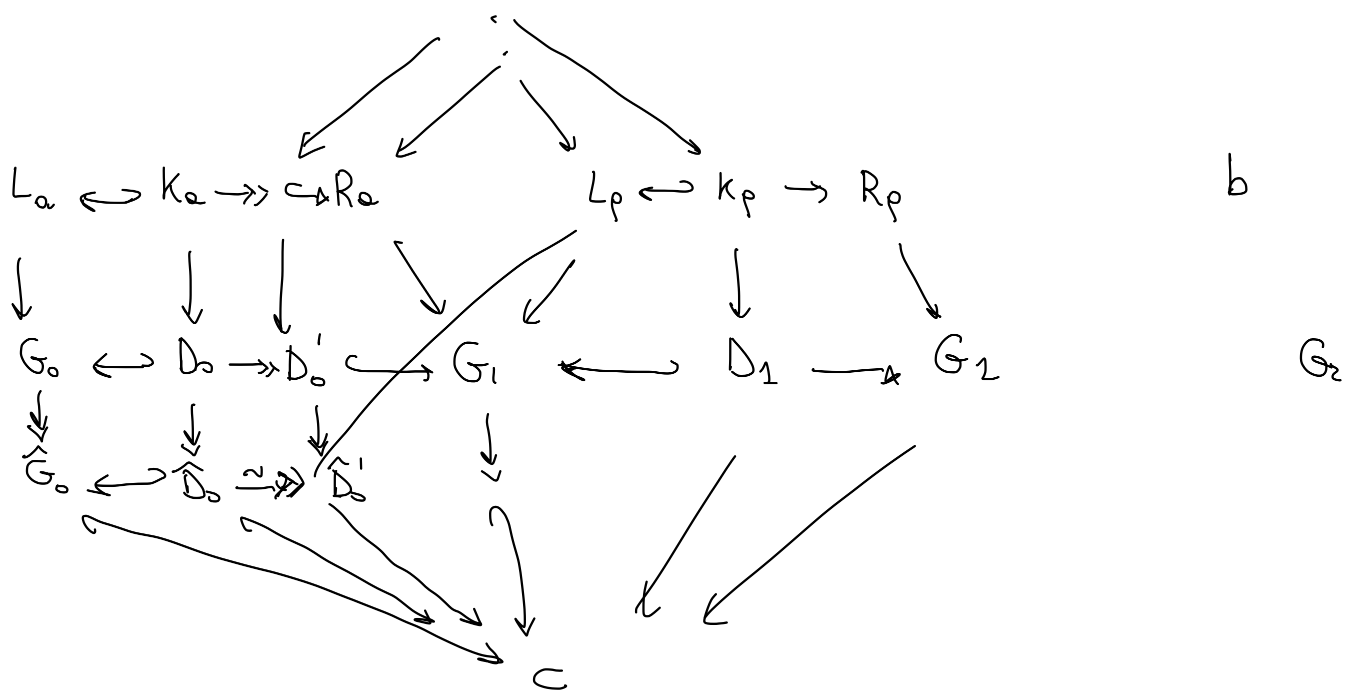


independence $\iff$ $PB(K_1 \to C, L' \to C)$ e $PB(R \to C, L' \to C)$ same


L
K
R
L'
K'
R'
G
D
G'
D'
G''
G1
P
C


K
K1
R
L'
K'
R'
D
D1
G
D'
G'
D' = D1'
I2
D''
I1
C
C
D'''
G'
P'C
E


K
R
L'
K'
R'
D
G
D'
G'
D₁
G₁
D'₁
G'
G₂
D'₂
G'
C
P
E


Lₐ
Kₐ
Rₐ
Lₚ
Kₚ
Rₚ
b
G₀
D₀
D'₀
G₁
D₁
G₂
G₂
Ĝ₀
D̂₀
D̂'₀
C

DERIVATIONS UP TO FUSION

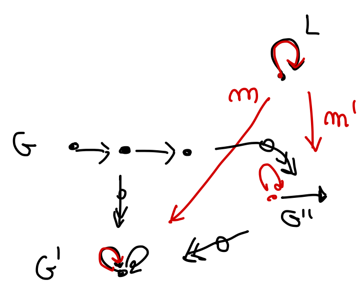
$$G \rightleftarrows A$$

$\uparrow$ grafi $\qquad$ $\uparrow$ grafi senza nodi isolati

Recall: we are working with rules

$$L \hookleftarrow K \rightarrowtail R$$

where, in particular, we know that given a match into a $\twoheadrightarrow$ quotient there is a least quotient to which the rule can be applied. i.e.



$$\begin{tikzcd}
 & & L \arrow[dll, "m"'] \arrow[d, dashed, "\mu(e,m)"] \\
G \arrow[d, "e"'] \arrow[rr, dashed] & & G'' \arrow[dll, dashed] \\
G' & &
\end{tikzcd}$$

The idea which is discussed in the subsequent pages is to have

- <u>DPO step up to fusion</u>

$G \Rightarrow H$

$$\begin{tikzcd}
 & L \arrow[d, "m"] & K \arrow[l] \arrow[r] \arrow[d] & R \arrow[d] \\
G \arrow[r, two heads, "e"'] & G' & D \arrow[l] \arrow[r] & H
\end{tikzcd}$$

<u>standard</u> if $m = \mu(e,m)$

> NOTE: Ordinary DPO is a special case

- <u>Sequential independence for steps up to fusion</u>

$$\begin{tikzcd}
 & L_0 \arrow[d, "m_0"'] & K_0 \arrow[l] \arrow[r] \arrow[d] & R_0 \arrow[d] \arrow[rrd, bend left, red] & L_1 \arrow[d, "m_1"'] \arrow[lld, bend right, red] & K_1 \arrow[l] \arrow[r] \arrow[d] & R_1 \arrow[d] \\
G_0 \arrow[r, two heads] & G_0' \arrow[d] & D_0 \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, "PO"', near start, shift right=3] & G_1 \arrow[r, two heads] & G_1' \arrow[d] & D_1 \arrow[l] \arrow[r] & G_2 \\
 & G_0'' & D_0' \arrow[l] \arrow[rr] & & G_1'' & &
\end{tikzcd}$$

independence in the colimit. ?

Details follow (not always consistent...)

DERIVATIONS UP TO FUSION

<u>prelude</u>: the aim is to characterise the fact that the second step uses only fusions from the first

$$\begin{tikzcd}
L_0 \arrow[d] & K_0 \arrow[l] \arrow[r] \arrow[d] & R_0 \arrow[dr] & & L_1 \arrow[dl] & K_1 \arrow[l] \arrow[r] \arrow[d] & R_1 \arrow[d] \\
G_0 & D_0 \arrow[l] \arrow[rr] & & G_1 & & D_1 \arrow[ll] \arrow[r] & G_2
\end{tikzcd}$$

<u>sequential independent up to fusion</u> if $\exists$ $G_0 \twoheadrightarrow G_0'$ s.t.

$$\begin{tikzcd}
L_0 \arrow[d] \arrow[dd, bend left] & K_0 \arrow[l] \arrow[r] \arrow[dd] & R_0 \arrow[ddr] \arrow[ddrrr] & & L_1 \arrow[ddl] \arrow[ddlll] & K_1 \arrow[l] \arrow[r] \arrow[dd] & R_1 \arrow[dd] \\
G_0 \arrow[d, two heads] & & & & & & \\
G_0' & D_0' \arrow[l] \arrow[rr] & & G_1 & & D_1 \arrow[ll] \arrow[r] & G_2
\end{tikzcd}$$

and we can switch them to (should be the same as asking the above in colimit)

$$\begin{tikzcd}
& L_1 \arrow[d] \arrow[dl, dashed] & K_1 \arrow[l] \arrow[r] \arrow[d] & R_1 \arrow[dr] \arrow[drrr] & & L_0 \arrow[dl] \arrow[dlll] & K_0 \arrow[l] \arrow[r] \arrow[d] & R_0 \arrow[d] \\
G_0 \arrow[r, two heads] & G_0^m & E_1 \arrow[l] \arrow[rr] & & F_1 & & E_0 \arrow[ll] \arrow[r] & G_2
\end{tikzcd}$$

where $G_0^m \twoheadrightarrow G_0$ is the smallest quotient for

$$\begin{tikzcd}
& L \arrow[d] \\
G_0 \arrow[r, two heads] & G_0'
\end{tikzcd}$$

$$\begin{tikzcd}
& L_0 \arrow[dl] \arrow[ddr] & \\
G_0^m \arrow[drr, two heads] & & \\
& & G_0'
\end{tikzcd}$$

observe that it does not depend on the choice of G_0'

given

$$\begin{tikzcd}
G_0 \arrow[r, two heads] \arrow[dr, two heads] & G_0' \arrow[r, two heads] & G_0^c \\
& G_0'' \arrow[ur, two heads] &
\end{tikzcd}$$

we have

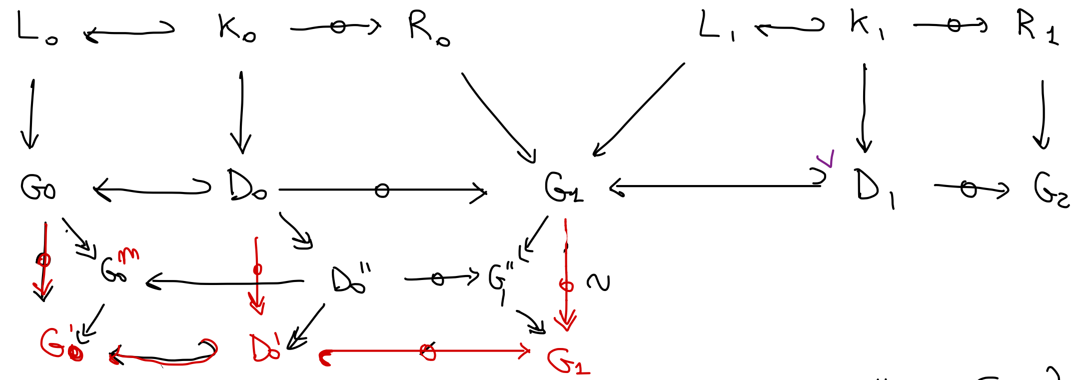
$\simeq$ colimit

hence both are the same as for G_0^c

<u>IDEA</u> first I switch to G_0' and then take the smallest quotient G_0^m

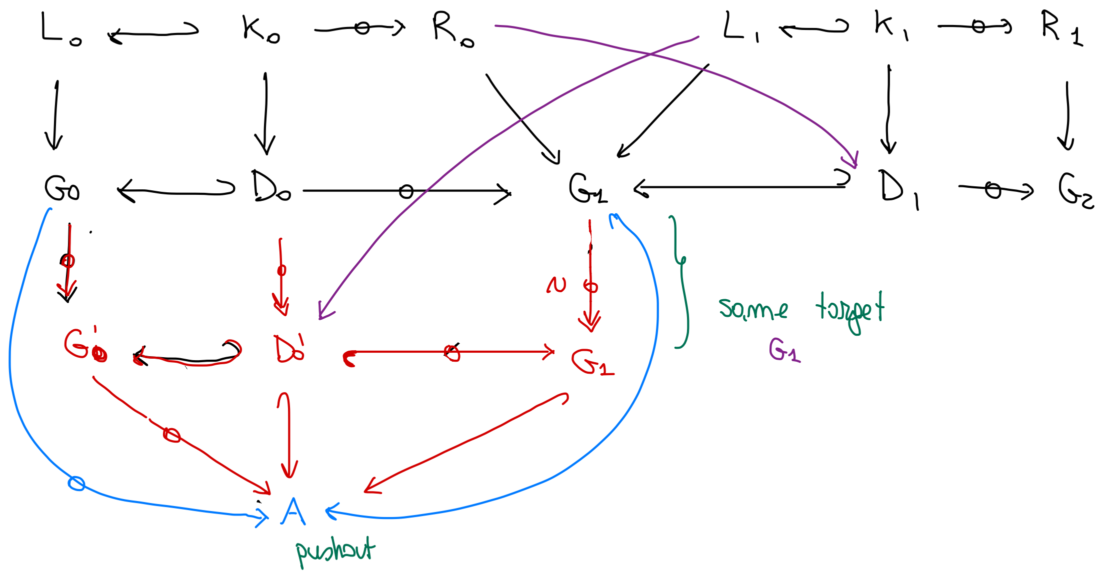
$$\begin{tikzcd}
& L_1 \arrow[dl] \arrow[dd] & K_1 \arrow[l] \arrow[r] \arrow[dl] \arrow[dd] & R_1 \arrow[d] \arrow[ddr] & & L_0 \arrow[ddl] & K_0 \arrow[l] \arrow[r] \arrow[dl] \arrow[dd] & R_0 \arrow[d] \arrow[dd, bend left] \\
G_0^m \arrow[d] & E_1' \arrow[l] \arrow[rr] \arrow[d] & & F_1' \arrow[dr] & \cdots & E_0' \arrow[rr] \arrow[d] & & G_2' \arrow[d] \\
G_0 \arrow[u] \arrow[r] & G_0' & E_1 \arrow[l] \arrow[rr] & & F_1 & E_0 \arrow[l] \arrow[rr] & & G_2
\end{tikzcd}$$

<u>Note</u> : If in the original derivation I start from G_0^m I still get to G_1

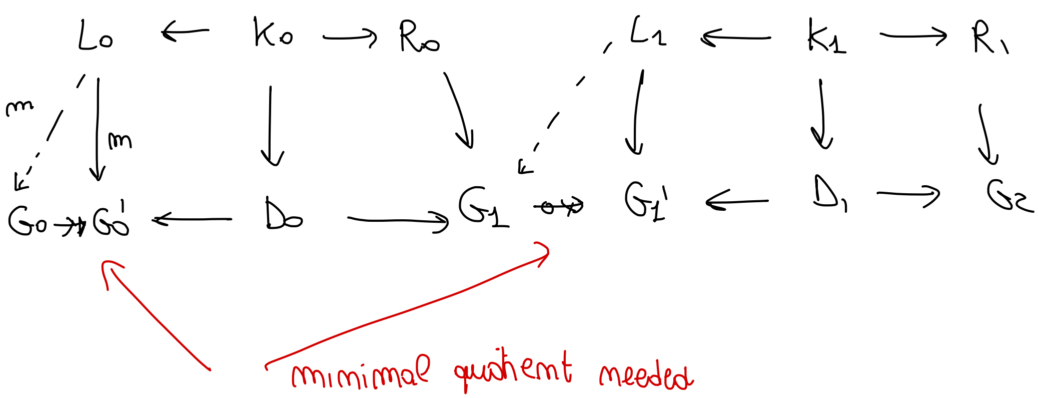


$\Rightarrow \quad G'' \sim G_1$? ok

If useful I think that one can show that whenever independence up to fusion holds it holds in the quotient obtained by taking PO of 1st step as below



Can we generalise the above to derivations in which formally one starts from an object but actually the rules are applied to (the smallest) quotient to which the rule is applicable



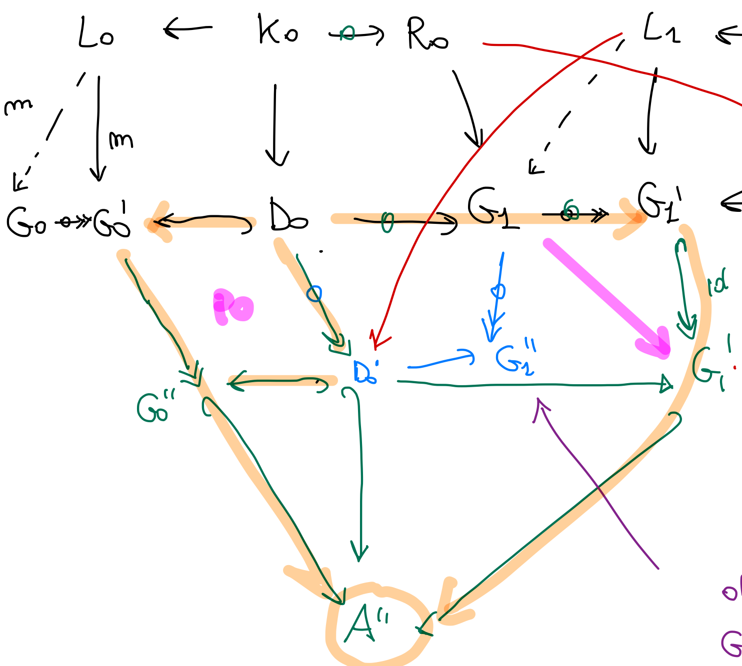
* Derivations up to fusion

$D = \rho_1 \rho_2 \cdots \rho_m$ with $\rho_i : G_i' \to G_{i+1}$

$G_i \twoheadrightarrow G_i'$

↑ minimal quotient need for applying the rule

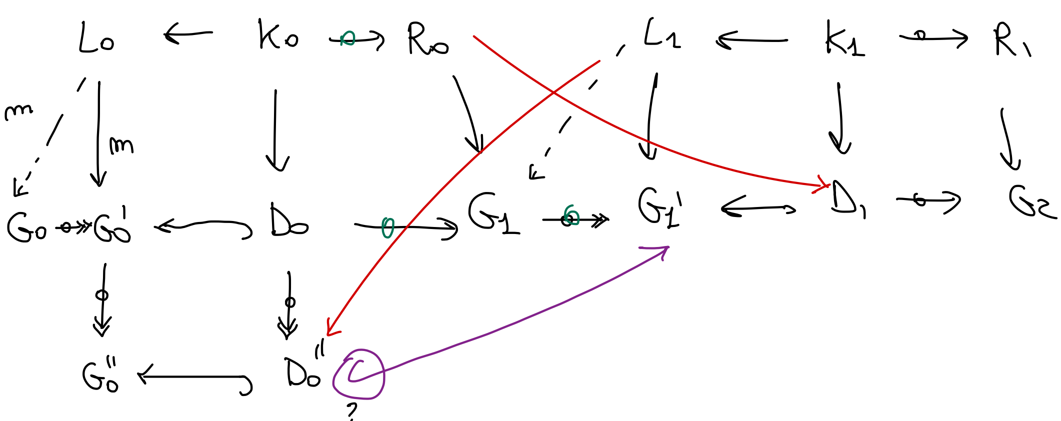
sequential independence (up to fusion) for derivations up to fusion



- A'' pushout of orange arrows
- take D_0', G_0'' images in A, image of G_1' is G_1'

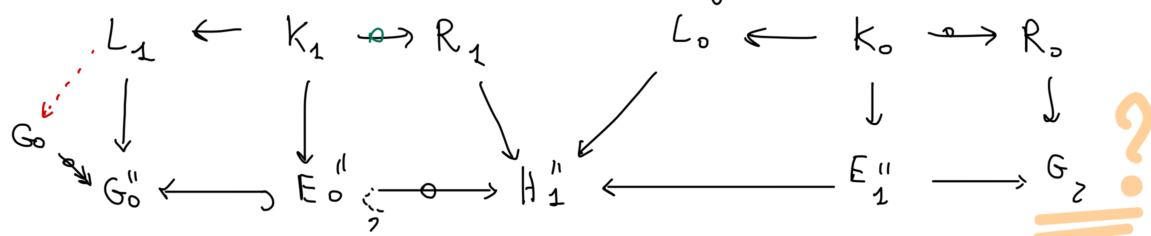
observe that G_1'' must coincide with G_1' since it is a quotient of G_1 and L_1 maps to it? ?

If the above holds then whenever the derivations are independent we are in the following situation



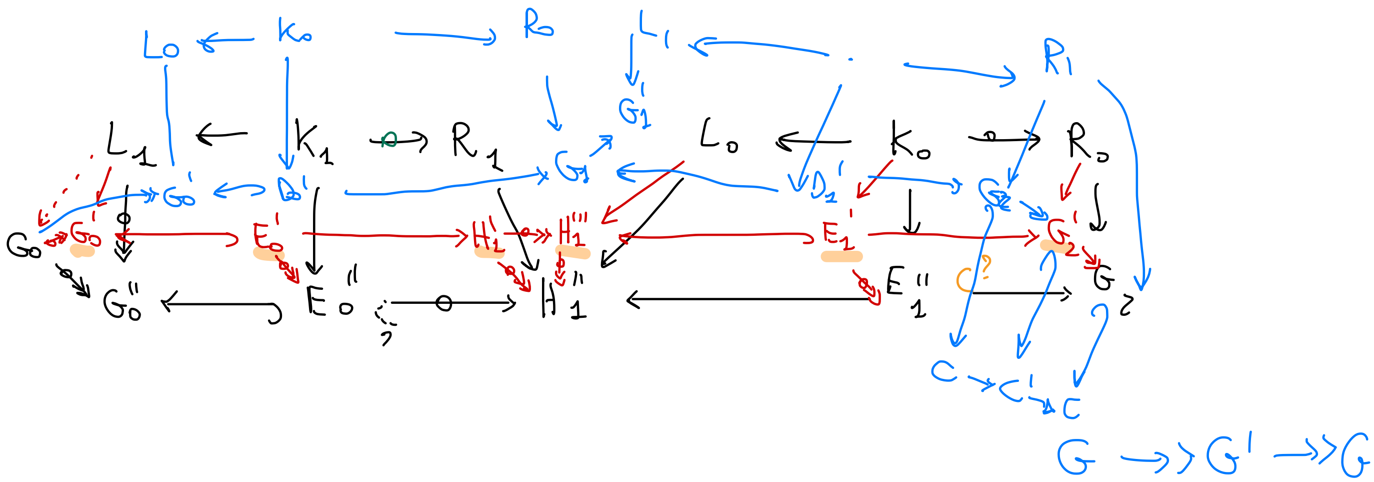
?
(but not relevant, I think)

hence we can switch the applications starting from G_0''



da mostrare che arrivo sempre a G_2!

and then take the applications from the least quotient ⇝



idea: scambiando e riscambiando ottengo

e poi coerente e concludo.

$$G_2 \twoheadrightarrow G_2' \twoheadrightarrow G_2$$

[wild conjecture]!

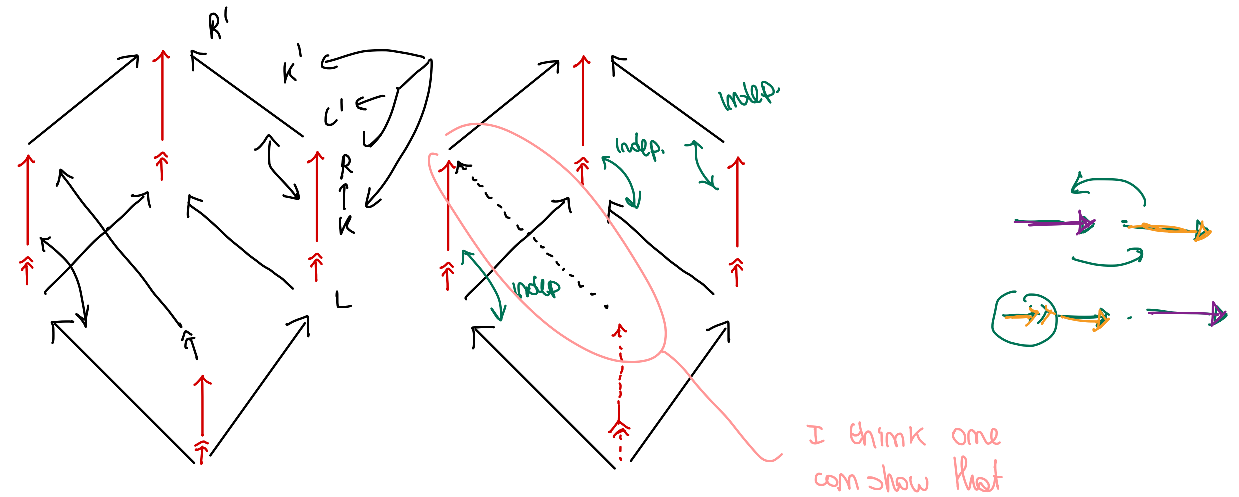
$\boxed{\text{equivalence up to fusion}}$: Given $D, \mathcal{E} : G \to G'$

$D \equiv_f \mathcal{E}$ iff $\mathrm{colimit}(D) = \mathrm{colimit}(\mathcal{E})$

∗ $\boxed{\text{Can independence be characterised in the colimit ?}}$ (up to fusion)

since independent steps up to fusion are independent steps in a quotient (which factorises through the colimit) the same considerations done for "standard" steps apply

→ including independence preserved in the situation below



I think one can show that if one of the steps is not up to fusion it remains so.

NOT SURE HOW TO PROCEED

$$\begin{tikzcd}
L_0 \arrow[d, "m"] & K_0 \arrow[l] \arrow[r] \arrow[d] & R_0 \arrow[d] \arrow[drr, bend left] \arrow[rr, bend right] & L_1 \arrow[dl, dashed] \arrow[d] \arrow[dll, bend right] & K_1 \arrow[l] \arrow[r] \arrow[d] & R_1 \arrow[d] \\
G_0 \arrow[d] & D_0 \arrow[l] \arrow[r] \arrow[d] & G_1 \arrow[r] & G_1' & D_1 \arrow[l] \arrow[r] & G_2 \\
G_0'' & D_0'' \arrow[l] & C \arrow[ur, bend right] & & &
\end{tikzcd}$$

?
(but not relevant, I think)

$\boxed{\underline{\text{idea}} : \text{characterise } \longleftrightarrow^*}$!

if $\rho ; \varepsilon \leftrightarrow^* \rho' ; \varepsilon'$ then

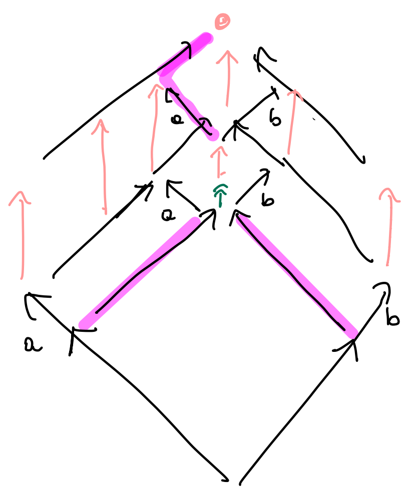
$$\rho ; \varepsilon \equiv_\sigma \rho_1 ; \varepsilon_1 ; \rho_2$$

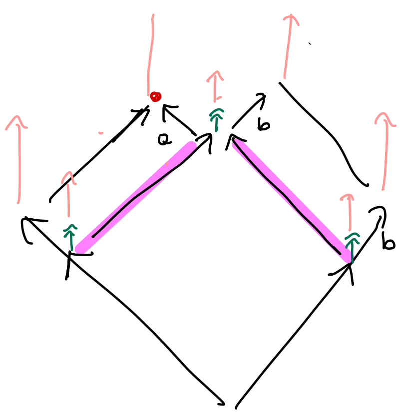
$$\rho' ; \varepsilon' \equiv_{\sigma'} \underbrace{\rho_1 ; \varepsilon_1}_{\text{up to fusion}} ; \rho_2'$$

with $\rho_1 ; \varepsilon_1$ derivation up to fusion

ρ_2, ρ_2' ordinary derivations

STRUCTURE OF INDUCTIVE PROOF FOR CHARACTERISING INTERCHANGEABLE IRREDUCIBLES





DOMAIN : RECALL

IRREDUCIBLE: intuitively $\varepsilon = D;p$ s.t. p cannot be switched with previous steps

$\varepsilon = D;p$ irreducible iff $n = |\varepsilon| \geq 1 \;\wedge\; \forall\, \varepsilon \equiv^{sh}_{\sigma} \varepsilon' \quad \sigma(n-1) = n-1$

INTERCHANGEABILITY

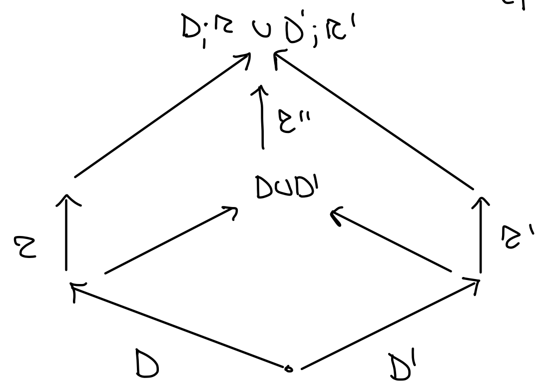
$\varepsilon = D;r$

$\varepsilon' = D';r'$

$i \leftrightarrow i' \Leftrightarrow$

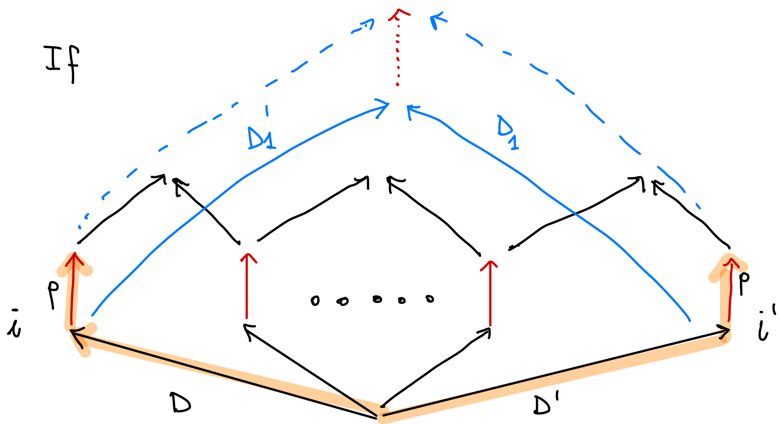
$\varepsilon \leftrightarrow \varepsilon'$ iff consistent and $\left\{ \begin{array}{l} i \frown i' \;\& \\ p(i) \sqcup i' = i \sqcup p(i') \end{array} \right\}$

$\overset{!}{=} i \sqcup i'$

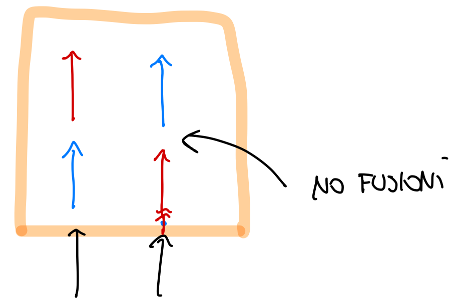


NEED to show that the domain is interchangeable (axioms (I) & (II))

(I) If

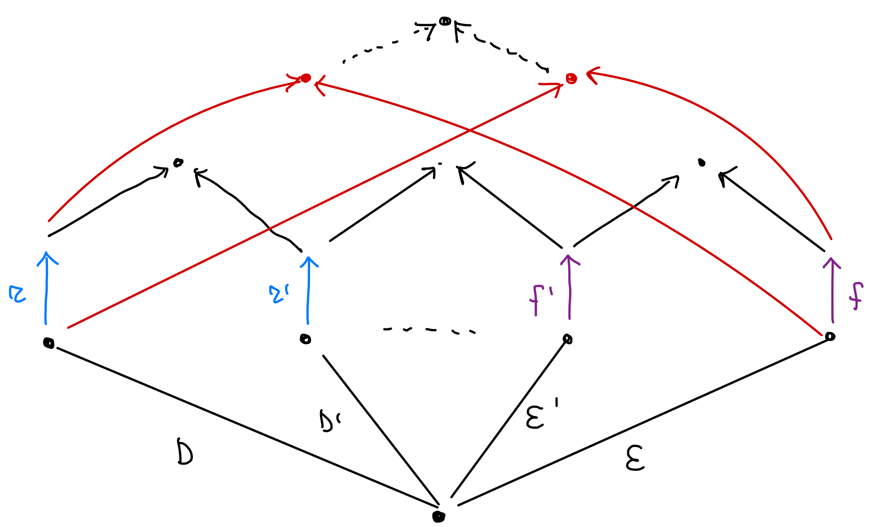


$i \leftrightarrow^* i' \;\wedge\; p(i) \frown p(i') \Rightarrow i \leftrightarrow i'$



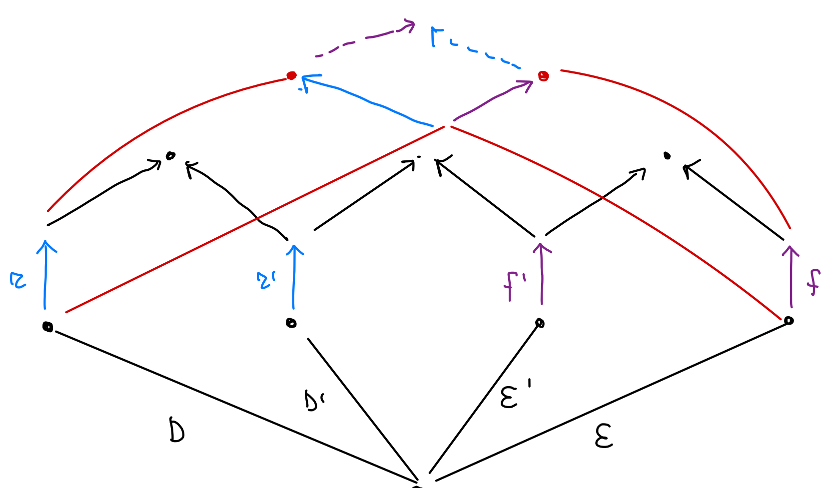
then the dotted lines exist

(II) Given



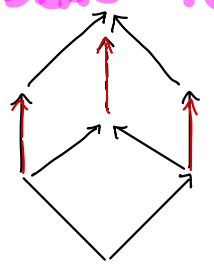
then the dotted line exists (D;τ consistent with ε;f)

$\boxed{\text{Note}}$: we know that τ and f can be translated

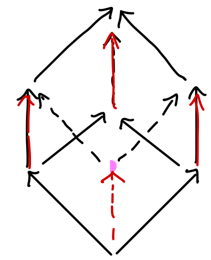


we should prove that we can close (they are indep.)

IDEAS TO CONCLUDE



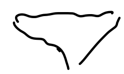
up to fusion



Idea: axioms (I) and (II) are a consequence of coherence in prime domains

maybe here they say that the "prime" completion of a weak domain arise by repeatedly applying axiom on the left (i.e. adding • does not add anything above

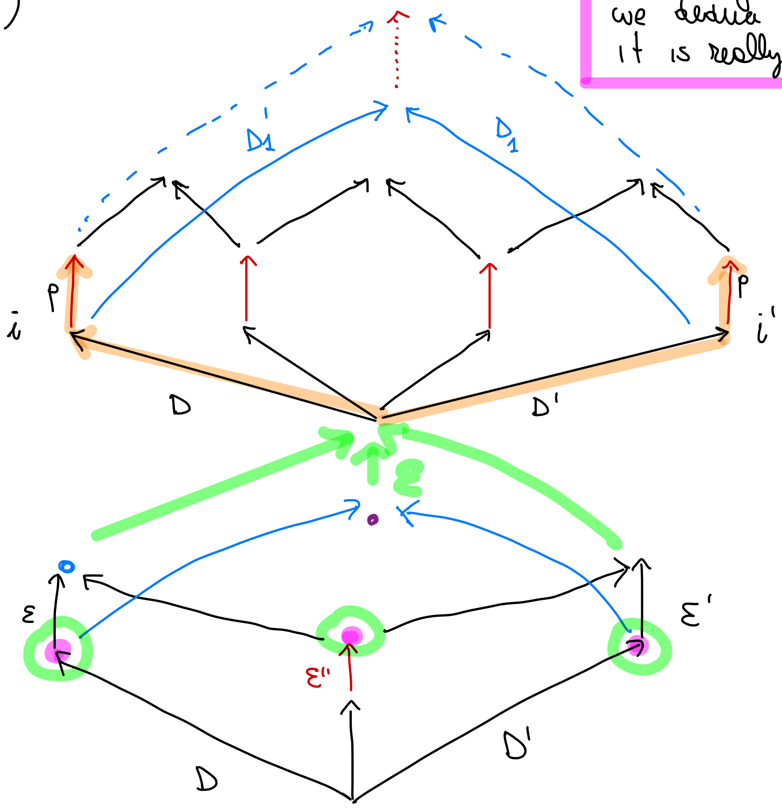
given a set of derivations

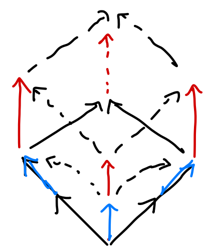


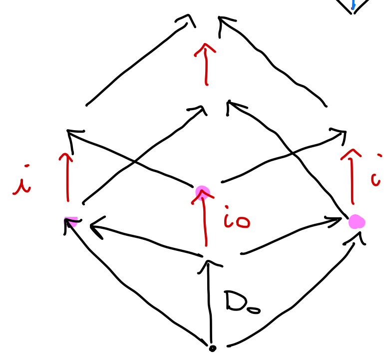
⇐

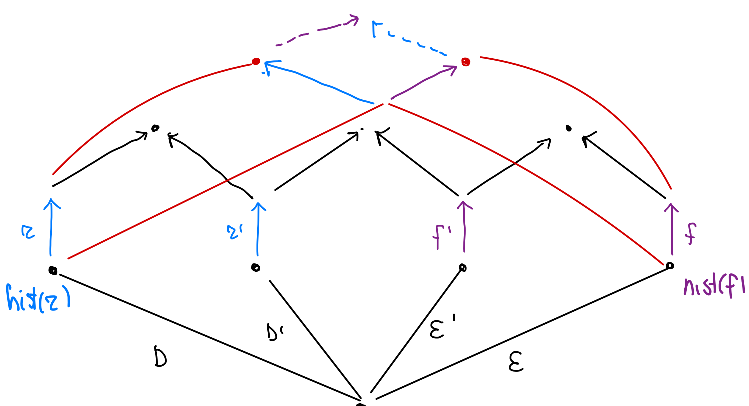
if working up to fusion we deduce something above, is it really there?

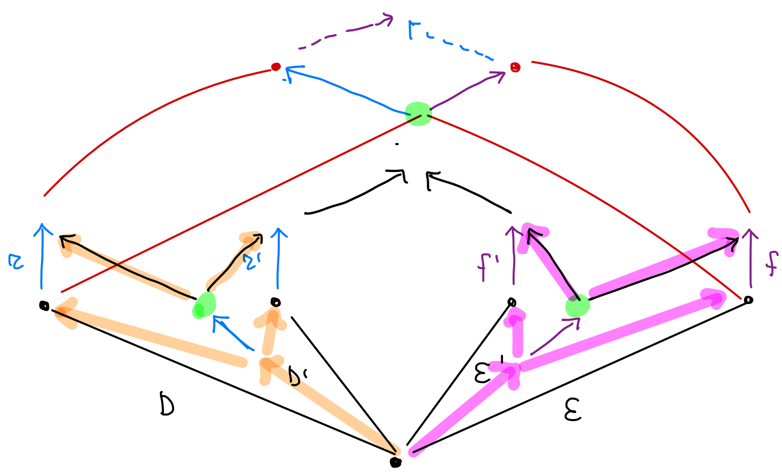
(I)











$r \leftrightarrow^* r'$ $\qquad f \leftrightarrow^* f'$ $\qquad \Rightarrow \qquad r \frown f$

$r' \frown f'$ $\qquad r \frown hist(f)$ $\qquad f \frown hist(r)$