Switch equivalence and weak prime domains for fusions

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Abstract. A VERY NICE ABSTRACT

1 Introduction

A VERY NICE INTRODUCTION

2 $\mathcal{M}$-adhesive categories

This first section is devoted to recall the definition and the basic theory of $\mathcal{M}$-*adhesive categories* [2, 9, 14].

Notation. We stipulate here some notational conventions which will be used throughout this paper.

Given a category $\mathbf{X}$ we will not distinguish notationally between $\mathbf{X}$ and its class of objects: so that "$X \in \mathbf{X}$" means that X belongs to the class of objects of $\mathbf{X}$.

If 1 is a terminal object in a category $\mathbf{X}$, the unique arrow $X \to 1$ from another object X will be denoted by $!_X$. Similarly, if 0 is initial in $\mathbf{X}$ then $?_X$ will denote the unique arrow $0 \to X$. When $\mathbf{X}$ is **Set** and 1 is a singleton, δ_x will denote the arrow $1 \to X$ with value $x \in X$.

Finally, we will use the following notation for some special classes of arrows of a category $\mathbf{X}$:

– $\mathcal{A}(\mathbf{X})$ will denote the class of all arrows of $\mathbf{X}$;
– $\mathcal{M}(\mathbf{X})$ will denote the class of all monos of $\mathbf{X}$;
– $\mathcal{R}(\mathbf{X})$ will denote the class of all regular monos of $\mathbf{X}$.

2.1 The Van Kampen condition

The key property that $\mathcal{M}$-adhesive categories enjoy is given by the so-called *Van Kampen condition* [5, 13, 14]. We will recall it and examine some of its consequences.

Definition 1. *Let* $\mathbf{X}$ *be a category and consider the two diagrams below*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a" near end] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'" near start, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, "d" near start, crossing over] &
\end{tikzcd}
$$

We say that the left square is a Van Kampen square *if:*

1. *it is a pushout square;*
2. *whenever the right cube has pullbacks as back and left faces, then its top face is a pushout if and only if the front and right faces are pullbacks.*

Pushout squares which enjoy the "if" half of this condition are called stable*.*

Example 1. In any category $\mathbf{X}$, a square as the one below on the left, in which m is an isomorphism, is Van Kampen. Notice that, in this situation, n must be an isomorphism too.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a" near end] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'" near start, crossing over] \arrow[dd, "c"'] & & D' & \\
& A \arrow[rr, "f" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D \arrow[from=uu, "d" near start, crossing over] &
\end{tikzcd}
$$

Take a cube as the one above and suppose that its back faces are pullbacks. In particular we know that also m' is an isomorphism.

($\Rightarrow$) Suppose that the top face is a pushout, thus n' is an isomorphism and the right square is a pullback. The thesis for the front face follows at once since m', n', m and m are isomorphisms and the back face a pullback.

($\Leftarrow$) If all the vertical faces are pullbacks, then m' and n' are isomorphisms and this entails at once that the top face is a pushout.

Before proceeding further, we recall this classical result about pullbacks.

Lemma 1. *Let* $\mathbf{X}$ *be a category, and consider the following diagram in which the right square is a pullback.*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}
$$

Then the whole rectangle is a pullback if and only if the left square is one.

The previous result can be dualised to get an analogous lemma for pushouts.

Lemma 2. *Let $\mathbf{X}$ be a category, and consider the following diagram in which the left square is a pushout.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}$$

Then the whole rectangle is a pushout if and only if the right square is one.

The following proposition establishes a key property of Van Kampen squares with a mono as a side: they are not only pushouts, but also pullbacks.

Proposition 1. *Let $m\colon A \to C$ be a monomorphism in a category $\mathbf{X}$. Then every Van Kampen square*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

is also a pullback square and n is a monomorphism.

Proof. Take the following cube:

$$\begin{tikzcd}
& A \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_A"'] \arrow[dd, "\mathsf{id}_A"' near end] & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[dd, "\mathsf{id}_B"] \\
A \arrow[rr, "g" near end] \arrow[dd, "m"'] & & B \arrow[dd, "n"' near start] & \\
& A \arrow[rr, "g" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "f"'] & & D &
\end{tikzcd}$$

By construction the top face of the cube is a pushout and the back one a pullback. The left face is a pullback because m is mono. Thus the Van Kampen property yields that the front and the right faces are pullbacks and the thesis follows. $\square$

The previous proposition, in turn, allows us to establish the following results.

Lemma 3. *Let $m : X \to Y$ be a monomorphism in a category $\mathbf{X}$ and suppose that the left square below is Van Kampen, while all the vertical faces in the right* *cube are pullbacks.*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dl, "m'"'] \arrow[dd, "a"' near end] & & B' \arrow[dl, "n'"'] \arrow[dd, "b"] \\
C' \arrow[rr, "g'" near start] \arrow[dd, "c"'] & & D' \arrow[dd, "d" near start] & \\
& A \arrow[rr, "f" near end] \arrow[dl, "m"] & & B \arrow[dl, "n"] \\
C \arrow[rr, "g"'] & & D &
\end{tikzcd}
$$

Supppose, moreover that $d : D' \to D$ is mono, then $d \leq n$ if and only if $c \leq m$.

Remark 1. Recall that, given two monos $m : M \to X$ and $n : N \to X$ with the same codomain, $m \leq n$ means that there exists a, necessarily unique, $k : M \to N$ fitting in the triangle below:

$$
\begin{tikzcd}
M \arrow[rr, dotted, "k"] \arrow[dr, "m"'] & & N \arrow[dl, "n"] \\
& X &
\end{tikzcd}
$$

Notice that, if $m \leq n$ and $n \leq m$, then the arrow $k : M \to N$ is an isomorphism.

Remark 2. Notice that, since d is a mono and the front face a pullback, then c is a monomorphism too.

Proof. ($\Rightarrow$) By hypothesis there exists $k : D' \to B$ such that $n \circ k = d$. By Proposition 1, the bottom face of the cube is a pullback, thus there exists a unique $h : C' \to A$ as in the diagram below. In particular, this implies the thesis.

$$
\begin{tikzcd}
C' \arrow[r, "g'"] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, "c"'] & D' \arrow[dr, bend left, "k"] \arrow[ddr, "d" near end] & \\
& A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\
& C \arrow[r, "g"'] & D
\end{tikzcd}
$$

($\Leftarrow$) Let $h : C \to A$ be such that $c = m \circ h$. By the Van Kampen property the top face of the given cube is a pushout. Thus the dotted $k : D' \to B$ in the following diagram exists.

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "m'"'] \arrow[dr, bend left, "a"] & B' \arrow[d, "n'"] \arrow[ddr, bend left, "b"] & \\
C' \arrow[r, "g'"'] \arrow[dr, bend right, "h"'] & D' \arrow[dr, dotted, "k"] & \\
 & A \arrow[r, "f"'] & B
\end{tikzcd}$$

The thesis now follows at once. $\square$

Finally, we can show that stable pushouts enjoy a kind of *pullback-pushout decomposition* property.

Proposition 2. *Let* $\mathbf{X}$ *be a category and suppose that, in the diagram below, the whole rectangle is a stable pushout and the right square a pullback.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}$$

If the arrow k is a monomorphism, then both squares are pushouts.

Proof. We can begin noticinng that g, being the pullback of the monomorphism k, is monic too. Thus we can build the cube below, in which all the vertical faces are pullbacks.

$$\begin{tikzcd}
 & & X \arrow[rrr, "a"] \arrow[dl, "f"'] \arrow[ddd, "\mathsf{id}_X"] & & & A \arrow[dl, "h"] \arrow[ddd, "\mathsf{id}_A"] \\
 & Y \arrow[dl, "\mathsf{id}_Y"'] \arrow[ddd, "\mathsf{id}_Y"'] & & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[ddd, "\mathsf{id}_B"] & \\
Y \arrow[rrr, "b"] \arrow[ddd, "g"'] & & & B \arrow[ddd, "k"] & & \\
 & & X \arrow[rrr, "a"] \arrow[dl, "f"'] & & & A \arrow[dl, "h"] \\
 & Y \arrow[dl, "g"'] & & & B \arrow[dl, "k"] & \\
Z \arrow[rrr, "c"'] & & & C & &
\end{tikzcd}$$

By hypothesis the face is a stable pushout and so its top one is a pushout. Lemma 2 now entails that the right half of the rectangle with which we have started is a pushout too. $\square$

2.2 $\mathcal{M}$-adhesivity

In this section we will define the notion of $\mathcal{M},\mathcal{N}$-adhesivity and explore some of the consequence of such a property. Let us start fixing some terminology.

Definition 2. *Let $\mathcal{M}$ be a class of arrows of a category $\mathbf{X}$. We say that $\mathcal{M}$ is*

– stable under pushouts (pullbacks) *if for every pushout (pullbacks) square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "n"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

if $m \in \mathcal{M}$ ($n \in \mathcal{M}$) then $n \in \mathcal{M}$ ($m \in \mathcal{M}$);

– closed under composition *if $g, f \in \mathcal{M}$ implies $g \circ f \in \mathcal{M}$ whenever g and f are composable;*

– closed under decomposition *if $g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ implies $f \in \mathcal{M}$.*

Remark 3. Clearly, "decomposition" corresponds to "left cancellation", but we prefer to stick to the name commonly used in literature.

We are now ready to give the definition of $\mathcal{M}$-adhesive category

Definition 3 ([2, 12]). *Let $\mathbf{X}$ be a category and consider a subclass $\mathcal{M}$ of the class $\mathcal{M}(\mathbf{X})$ of monomorphisms such that:*

1. *$\mathcal{M}$ contains all isomorphisms and is closed under composition;*
2. *$\mathcal{M}$ is stable under pullbacks and pushouts.*

We say that $\mathbf{X}$ is $\mathcal{M}$-adhesive *if*

1. *for every $m\colon X \to Y$ in $\mathcal{M}$ and $g\colon Z \to Y$, a pullback square*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "n"'] & X \arrow[d, "m"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

exists, such pullbacks will be called $\mathcal{M}$-pullbacks*;*

2. *for every $m\colon X \to Y$ in $\mathcal{M}$ and $n\colon X \to Z$, a pushout square*

$$\begin{tikzcd} X \arrow[r, "n"] \arrow[d, "m"'] & Z \arrow[d, "q"] \\ Y \arrow[r, "p"'] & Q \end{tikzcd}$$

exists, such pushouts will be called $\mathcal{M}$-pushouts*;*

3. $\mathcal{M}$-pushouts are Van Kampen squares.

Remark 4. Our notion of $\mathcal{M}$-adhesivity is slightly different from the one of [2]: in that paper, $\mathcal{M}$ is assumed to be only stable under pullbacks. Notice, however, that if $\mathcal{M}$ contains all split monos, then stability under pushouts can be deduced from the other axioms [6, Prop. 5.1.21].

Remark 5. Adhesivity and *quasiadhesivity* as defined in [10, 14] coincide with $\mathcal{A}(\mathbf{X})$-adhesivity and $\mathcal{R}(\mathbf{X})$-adhesivity, respectively.

A first result we can prove regards closure under decomposition of $\mathcal{M}$.

Proposition 3. *Let $\mathcal{M}$ be a class of monomorphisms containing all isomorphisms and stable under pullbacks. For every arrow $f : X \to Y$ and monomorphism $m : Y \to Z$, if $m \circ f \in \mathcal{M}$ then $f \in \mathcal{M}$.*

Proof. Take the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathsf{id}_X"'] & Y \arrow[r, "\mathsf{id}_Y"] \arrow[d, "\mathsf{id}_Y"'] & Y \arrow[d, "m"] \\ X \arrow[r, "f"'] & Y \arrow[r, "m"'] & Z \end{tikzcd}$$

Since m is mono the right square is a pullback, the thesis now follows from Lemma 1.

Corollary 1. *In every $\mathcal{M}$-adhesive category $\mathbf{X}$, the class $\mathcal{M}$ is closed under decomposition.*

Another result which can be immediately established, with the aid of Proposition 1, is the following one.

Proposition 4. *Let $\mathbf{X}$ be an $\mathcal{M}$-adhesive category. Then $\mathcal{M}$-pushouts are also pullback squares.*

From Proposition 4, in turn, we can derive the following corollaries.

Corollary 2. *In a $\mathcal{M}$-adhesive category $\mathbf{X}$, every $m \in \mathcal{M}$ is a regular mono.*

Proof. Let m be an element of $\mathcal{M}$ and consider its pushout along itself.

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "m"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

By Proposition 4 this square is a pullback, proving that m is the equalizer of the arrows $f, g \colon Y \rightrightarrows Z$. $\square$

The following result now follows at once noticing that a regular monomorphism which is also epic is automatically an isomorphism.

Corollary 3. *If $\mathbf{X}$ is an $\mathcal{M}$-adhesive categories, then every epimorphisms in $\mathcal{M}$ is an isomorphisms. In particular, every adhesive category $\mathbf{X}$ is* balanced*: if a morphism is monic and epic, then it is an isomorphism.*

$\mathcal{M}$-adhesivity is well-behaved with respect to the comma construction [16], as shown by the following theorem.

Theorem 1 ([7, 8, 14]). *Let $\mathbf{A}$ and $\mathbf{B}$ be respectively an $\mathcal{M}$-adhesive and an $\mathcal{M}'$-adhesive category. Let also $L : \mathbf{A} \to \mathbf{C}$ be a functor that preserves $\mathcal{M}$-pushouts, and $R : \mathbf{B} \to \mathbf{C}$ be a functor which preserves pullbacks.Then $L{\downarrow}R$ is $\mathcal{M}{\downarrow}\mathcal{M}'$-adhesive, where*

$$\mathcal{M}{\downarrow}\mathcal{M}' := \{(h, k) \in \mathcal{A}(L{\downarrow}R) \mid h \in \mathcal{M}, k \in \mathcal{M}'\}$$

In particular, we can apply this result to slices over and under a given object.

Corollary 4. *Let X be an object of an $\mathcal{M}$-adhesive category $\mathbf{X}$. Then $\mathbf{X}/X$ and $X/\mathbf{X}$ are, respectively, is $\mathcal{M}/X$- and $X/\mathcal{M}$-adhesive, where*

$$\mathcal{M}/X := \{m \in \mathcal{A}(\mathbf{X}/X) \mid m \in \mathcal{M}\} \qquad X/\mathcal{M} := \{m \in \mathcal{A}(X/\mathbf{X}) \mid m \in \mathcal{M}\}$$

Another categorical construction which preserves $\mathcal{M}$-adhesivity property is the formation of the category of functors.

Theorem 2 ([7,8,14]). *If $\mathbf{X}$ is an $\mathcal{M}$-adhesive category, then for every small category $\mathbf{Y}$, the category $\mathbf{X}^{\mathbf{Y}}$ of functors $\mathbf{Y} \to \mathbf{X}$ is $\mathcal{M}^{\mathbf{Y}}$-adhesive, where*

$$\mathcal{M}^{\mathbf{Y}} := \{\eta \in \mathcal{A}(\mathbf{X}^{\mathbf{Y}}) \mid \eta_Y \in \mathcal{M} \text{ for every } Y \in \mathbf{Y}\}$$

We can list various examples of $\mathcal{M}$-adhesive categories (see [6, 7, 15]).

Topos, ipergrafi e grafi

Example 2.

GRAFI GERARCHICI

Example 3.

term graph

Example 4.

We end this section proving two properties of $\mathcal{M}$-adhesive categories: $\mathcal{M}$-pushout-pullback decomposition and uniqueness of pushouts complements.

Lemma 4 ($\mathcal{M}$-pushout-pullback decomposition). *Let $\mathbf{X}$ be an $\mathcal{M}$-adhesive category and suppose that, in the diagram below, the whole rectangle is a pushout and the right square a pullback.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\
A \arrow[r, "h"'] & B \arrow[r, "k"'] & C
\end{tikzcd}$$

Then the following statements hold true:

1. *if a belongs to $\mathcal{M}$ and k is a monomorphism, then both squares are pushouts and pullbacks;*
2. *if f and k are in $\mathcal{M}$, then both squares are pushouts and pullbacks.*

Proof. 1. By Proposition 2, it follows that both squares are pushouts. On the other hand, Proposition 4 entails that the whole rectangle is a pullback, thus the thesis follows from Lemma 1.

2. By hypothesis, g is the pullback of an arrow in $\mathcal{M}$, thus belongs to it. But then $g \circ f \in \mathcal{M}$ too and the whole rectangle is a $\mathcal{M}$-pushout. Therefore, by Proposition 4 a pullback, so that its left half is a pullback too, by Proposition 1. Moreover $k \circ h$ is in $\mathcal{M}$ as the pushout of $g \circ f$ and, by Corollary 1, we also know that $h \in \mathcal{M}$.
Using Lemma 2, it is enough to show that the left half of the original rectangle is a pushout. We can build the same cube of the previous point:

$$\begin{tikzcd}
& & X \arrow[rrr, "a"] \arrow[dl, "f"'] \arrow[ddd, "\mathsf{id}_X"] & & & A \arrow[dl, "h"] \arrow[ddd, "\mathsf{id}_A"] \\
& Y \arrow[dl, "\mathsf{id}_Y"'] \arrow[ddd, "\mathsf{id}_Y"'] & & & B \arrow[dl, "\mathsf{id}_B"'] \arrow[ddd, "\mathsf{id}_B"] & \\
Y \arrow[rrr, "b"] \arrow[ddd, "g"'] & & & B \arrow[ddd, "k"] & & \\
& & X \arrow[rrr, "a"] \arrow[dl, "f"'] & & & A \arrow[dl, "h"] \\
& Y \arrow[rrr, "b"] \arrow[dl, "g"'] & & & B \arrow[dl, "k"] & \\
Z \arrow[rrr, "c"'] & & & C & &
\end{tikzcd}$$

Its vertical faces are all pullbacks, those the top one is a pushout and we can conclude. $\square$

Let us turn our attention to pushout complements.

Definition 4. *Let $f : X \to Y$ and $g : Y \to Z$ ba two composable arrows in a category* $\mathbf{X}$. *A* pushout complement *for the pair (f, g) is a pair (h, k) with $h : X \to W$ and $k : W \to Z$ such that the square below commutes and it is a pushout.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\
W \arrow[r, "k"'] & Z
\end{tikzcd}$$

Example 5. In a generic category $\mathbf{X}$, pushout complements may not exist: in the the category of **Set** the arrows $?_2 : \emptyset \to 2$ and $!_2 : 2 \to 1$ do not have a pushout complement.

Moreover, composable arrows $f : X \to Y$ and $g : Y \to Z$ may have pushout complements which are non-isomorphic: for instance, in **Set** the two squares below are both pushouts.

$$\begin{tikzcd} 2 \arrow[r, "!_2"] \arrow[d, "\mathsf{id}_2"'] & 1 \arrow[d, "\mathsf{id}_1"] \\ 2 \arrow[r, "!_2"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r, "!_2"] \arrow[d, "!_2"'] & 1 \arrow[d, "\mathsf{id}_1"] \\ 1 \arrow[r, "\mathsf{id}_1"'] & 1 \end{tikzcd}$$

Working in an $\mathcal{M}$-adhesive category we can amend the second defect.

Lemma 5 (Uniqueness of pushouts complements). *Let* $\mathbf{X}$ *be a* $\mathcal{M}$*-adhesive category. Given* $m : X \to Y$ *in* $\mathcal{M}$ *and* $n : Y \to Z$*, let* (h_1, k_1) *and* (h_2, k_2) *be pushout complements of* m *and* n *and* $W_1 = \mathsf{cod}(h_1)$*,* $W_2 = \mathsf{cod}(h_2)$*. Then there exists a unique isomorphism* $f : W_1 \to W_2$ *making the following diagram commutative.*

$$\begin{tikzcd} & X \arrow[r, "m"] \arrow[d, "h_1"'] \arrow[ddl, bend right, "h_2"'] & Y \arrow[d, "n"] \\ & W_1 \arrow[r, "k_1"'] \arrow[dl, dotted, "f"] & Z \\ W_2 \arrow[urr, bend right, "k_2"'] & & \end{tikzcd}$$

Proof. k_1 and k_2, being the pushout of m, are elements of $\mathcal{M}$ and thus are monomorphisms. In particular, k_2 is a monomorphism and this entails at once the uniqueness of f. Moreover, notice that the squares

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "h_1"'] & Y \arrow[d, "n"] \\ W_1 \arrow[r, "k_1"'] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "m"] \arrow[d, "h_2"'] & Y \arrow[d, "n"] \\ W_2 \arrow[r, "k_2"'] & Z \end{tikzcd}$$

are $\mathcal{M}$-pushouts and thus Van Kampen.

To prove the existence of the wanted f, we can start observing that k_1 and k_2, being the pushout of m, are elements of $\mathcal{M}$, so that we can consider the pullback square

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & W_1 \arrow[d, "k_1"] \\ W_2 \arrow[r, "k_2"'] & Z \end{tikzcd}$$

Since $k_1 \circ h_1 = k_2 \circ h_2$, there exists a unique $g : X \to P$ such that

$$p_1 \circ g = h_1 \qquad p_2 \circ g = h_2$$

We can then build the cubes

$$
\begin{tikzcd}
 & X \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_X"'] \arrow[dd, "\mathsf{id}_X" near end] & & P \arrow[dl, "p_2"'] \arrow[dd, "p_1"] \\
X \arrow[rr, "h_2" near start] \arrow[dd, "m"'] & & W_2 \arrow[dd, "k_2" near start] & \\
 & X \arrow[rr, "h_1" near end] \arrow[dl, "m"] & & W_1 \arrow[dl, "k_1"] \\
Y \arrow[rr, "n"'] & & Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[rr, "g"] \arrow[dl, "\mathsf{id}_X"'] \arrow[dd, "\mathsf{id}_X" near end] & & P \arrow[dl, "p_1"'] \arrow[dd, "p_2"] \\
X \arrow[rr, "h_1" near start] \arrow[dd, "m"'] & & W_1 \arrow[dd, "k_1" near start] & \\
 & X \arrow[rr, "h_2" near end] \arrow[dl, "m"] & & W_2 \arrow[dl, "k_2"] \\
Y \arrow[rr, "n"'] & & Z &
\end{tikzcd}
$$

Now, in both cubes the front and left faces are pullbacks, thus, by Lemma 1, their back face is a pullback too. Since $m \leq m$, Lemma 3 now entails that $k_1 \leq k_2$ and $k_2 \leq k_1$. Thus there exists an isomorphism $f : W_1 \circ W_2$ such that $k_1 = k_2 \circ f$. To see that $h_2 = f \circ h_1$, we can compute:

$$
\begin{aligned}
k_2 \circ f \circ h_1 &= k_1 \circ h_1 \\
&= n \circ m \\
&= k_2 \circ h_2
\end{aligned}
$$

The claim now follows since k_2 is a monomoprhism. □

3 Double pushout rewriting and derivations

A VERY NICE INTRODUCTION

3.1 Left-linear DPO-rewriting systems

We are now going to study rewriting systems in $\mathcal{M}, \mathcal{N}$-adhesive categories.

Definition 5 ([11,12]). *Let* $\mathbf{X}$ *be a* $\mathcal{M}$*-adhesive category, a* left $\mathcal{M}$-linear *rule* ρ *is a pair* (l, r) *of arrows with the same domain, such that* l *belongs to* $\mathcal{M}$*. The rule* ρ *is said to be* $\mathcal{M}$-linear *if* $r \in \mathcal{M}$ *too. A rule* ρ *is said to be* consuming *if* l *is not an isomorphism. We will represent a rule* ρ *as a span*

$$
\begin{tikzcd}
L & K \arrow[l, "l"'] \arrow[r, "r"] & R
\end{tikzcd}
$$

L *is the* left-hand side*,* R *is the* right-hand side *and* K *the* glueing object*.*

A left-linear DPO-rewriting system *is a pair* $(\mathbf{X}, \mathsf{R})$ *where* $\mathbf{X}$ *is a* $\mathcal{M}$*-adhesive category and* R *is a set of left* $\mathcal{M}$*-linear rules.* $(\mathbf{X}, \mathsf{R})$ *will be called* linear *if every rule in* R *is so.*

Given two objects G *and* H *and a rule* $\rho = (l, r)$ *in* R*, a* direct derivation $\mathscr{D}$ from G to H via ρ*, is a diagram as the one below, in which both squares are pushouts.*

$$
\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\
G & D \arrow[l, "f"] \arrow[r, "g"'] & H
\end{tikzcd}
$$

The arrow n is called the match *of the derivation, while h is its* back-match*. We will denote a direct derivation $\mathscr{D}$ between G and H as $\mathscr{D} : G \Mapsto H$.*

esempi di derivazione

Example 6.

Example 7. Let $\mathscr{D} : G \Mapsto H$ be the direct derivation

$$\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\
G & D \arrow[l, "f"] \arrow[r, "g"'] & H
\end{tikzcd}$$

If $\phi : G \to G'$ and $\psi : H \to H'$ are two isomorphisms, we can consider the direct derivation $\phi * \mathscr{D} * \psi : G' \Mapsto H'$ given by the following diagram.

$$\begin{tikzcd}
L \arrow[d, "\phi \circ n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "\psi \circ h"] \\
G' & D \arrow[l, "\phi \circ f"] \arrow[r, "\psi \circ g"'] & H
\end{tikzcd}$$

$\mathcal{M}$-adhesivity of $\mathbf{X}$ guarantess the essential uniqueness of the result obtained rewriting an object, as shown by the next proposition.

Proposition 5. *Let $\mathbf{X}$ be a $\mathcal{M}$-adhesive category. Suppose that the two direct derivations $\mathscr{D}$ and $\mathscr{D}'$ below, with the same match and applying the same left $\mathcal{M}$-linear rule ρ are given.*

$$\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\
G & D \arrow[l, "f"] \arrow[r, "g"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k'"] & R \arrow[d, "h'"] \\
G & D' \arrow[l, "f'"] \arrow[r, "g'"'] & H'
\end{tikzcd}$$

Then there exist isomorphisms $t : D \to D'$ and $s : H \to H'$ as in the following diagram.

$$\begin{tikzcd}
 & & & & R \arrow[dddl, "h"'] \arrow[dddr, "h'"] & \\
 & & & K \arrow[ur, "r"] \arrow[dll, "l"'] \arrow[dddl, "k"'] \arrow[dddr, "k'"] & & \\
 & L \arrow[dddl, "n"'] & & & & \\
 & & & H \arrow[rr, dotted, "s"] & & H' \\
 & & D \arrow[dll, "f"'] \arrow[ur, "g"] \arrow[rr, dotted, "t"] & & D' \arrow[ur, "g'"'] \arrow[dllll, bend left, "f'"] & \\
G & & & & &
\end{tikzcd}$$

Proof. By construction, the pairs (k, f) and (k', f') are pushout complements of l and n. Thus, the existence of the isomorphism $t : D \to D'$ follows from Lemma 5. Now, computing we have

$$\begin{aligned} g' \circ t \circ k &= g' \circ k' \\ &= h' \circ r \end{aligned}$$

Hence, we have the dotted $s : H \to H'$. To see that s is an isomorphism, consider the diagram

$$\begin{tikzcd} K \arrow[r, "k"'] \arrow[rr, bend left, "k'"] \arrow[d, "r"'] & D \arrow[r, "t"'] \arrow[d, "g"] & D' \arrow[d, "g'"] \\ R \arrow[r, "h"] \arrow[rr, bend right, "h'"'] & H \arrow[r, "s"] & H' \end{tikzcd}$$

By hypothesis the whole rectangle and its left half are pushouts, therefore, by Lemma 2 its right square is a pushout too. The claim now follows from the fact that the pushout of an isomorphism is an isomorphism. □

If we look to direct derivations as transitions, it is natural to consider them as edges in a direct graph. Taking objects as vertices objects led us to the following definition [12].

Definition 6. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewrityng system, with* $\mathbf{X}$ $\mathcal{M}$-*adhesive. The* DPO-derivation graph *of* $(\mathbf{X}, \mathsf{R})$ *is the (large) directed graph* $\mathsf{G}_R^{\mathbf{X}}$ *having as vertices the objects of* $\mathbf{X}$ *and in which an edge between* G *and* H *is a direct derivation* $\mathscr{D} : G \Mapsto H$. *A* derivation $\underline{\mathscr{D}}$ *between two objects* G *and* H *is a path between them in* $\mathsf{G}_R^{\mathbf{X}}$.

Remark 6. If $\mathbf{X}$ is not a small category , then $\mathsf{G}_R^{\mathbf{X}}$ has a proper class of vertices.

We can spell out more explicitly what a derivation $\mathscr{D}$ between G and H is. $\mathscr{D}$ is a, possibly empty, sequence $\{\mathscr{D}_i\}_{i=0}^{n}$ of direct derivations such that:

1. for every index i, $\mathscr{D}_i$ is a direct derivation $G_i \Mapsto G_{i+1}$;
2. $G_0 = G$ and $G_{n+1} = H$.

We will call the number $n + 1$ the *length* of the derivation, denoted by $\mathsf{length}(\underline{\mathscr{D}})$. We will also say that an empty derivation has length 0.

Moreover, if $\mathcal{D}_i$ is an application of the rule $\rho_i \in R$, then we can define an associated sequence of rules as $r(\underline{\mathscr{D}})$ as $\{\rho_i\}_{i=0}^{n}$.

Notation. Let $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{n}$ be a derivation. We will depict the i^{th} element $\mathscr{D}_i$ of $\underline{\mathscr{D}}$ as in the following diagram.

$$\begin{tikzcd} L_i \arrow[d, "m_i"'] & K_i \arrow[l, "l_i"'] \arrow[r, "r_i"] \arrow[d, "k_i"] & R_i \arrow[d, "h_i"] \\ G_i & D_i \arrow[l, "f_i"] \arrow[r, "g_i"'] & G_{i+1} \end{tikzcd}$$

Notice that, in particular, if $\underline{\mathscr{D}} : G \to H$, then $G_0 = G$ and $G_{n+1} = H$. When $\underline{\mathscr{D}}$ has length 1 we will suppress the indexes. In such case, we will also identify $\underline{\mathscr{D}}$ with its only element.

esempi di derivazione

Example 8.

Definition 7. *The* DPO-derivation category $\mathsf{C}_R^{\mathbf{X}}$ *of a left-linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$ *is the free category on* $\mathsf{G}_R^{\mathbf{X}}$.

Remark 7. Derivations, being paths in a graph, can be concatenate: given $\underline{\mathscr{D}} = \{\mathscr{D}\}_{i=0}^{n}$ between G and H and $\underline{\mathscr{D}'} = \{\mathscr{D}'_i\}_{i=0}^{m}$, their *concatenation* $\underline{\mathscr{D}} \cdot \underline{\mathscr{D}'}$ is the derivation $\{\mathscr{E}_i\}_{i=0}^{m+n+1}$ in which

$$\mathscr{E}_i := \begin{cases} \mathscr{D}_i & i \leq n \\ \mathscr{D}'_{i-(n+1)} & n < i \end{cases}$$

It is immediate to see that the concatenation of two non-empty derivations coincides with their composition in $\mathsf{C}_R^{\mathbf{X}}$.

?? allows us to compose derivations with isomorphisms.

Definition 8. *Given a left-linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$ *and aderivation* $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{n}$ *between* G *and* H*, the derivation* $\phi * \underline{\mathscr{D}} * \psi$ *between* G' *and* H' *is defined as*

$$\phi * \underline{\mathscr{D}} * \psi := \{\phi * \mathscr{D}_0 * \mathsf{id}_{G_1}\} \cdot \{\mathscr{D}_i\}_{i=1}^{n-1} \cdot \{\mathsf{id}_{G_n} * \mathscr{D}_n * \psi\}$$

Notation. In the following, we will use the notation $\phi * \mathscr{D}$, $\phi * \underline{\mathscr{D}}$ and $\mathscr{D} * \psi$, $\underline{\mathscr{D}} * \psi$ for the cases in which, respectively, $\psi = \mathsf{id}_H$ and $\phi = \mathsf{id}_G$. In particular

$$\phi * \underline{\mathscr{D}} * \psi = \{\phi * \mathscr{D}_0\} \cdot \{\mathscr{D}_i\}_{i=1}^{n-1} \cdot \{\mathscr{D}_n * \psi\}$$

We are often interested in an object of $\mathbf{X}$ only up to isomorphism. It is then useful to consider a version of $\mathsf{G}_R^{\mathbf{X}}$ in which vertices are classes of isomorphism of object of $\mathbf{X}$. In order to do so, some preliminary work is needed.

Definition 9. *[16] Let* $\mathbf{X}$ *be a category, we say that* $\mathbf{X}$ *is* skeletal *if, for every two objects* X *and* Y*, the existence of an isomorphism* $\phi : X \to Y$ *entails* $X = Y$*. A* skeleton *for a category* $\mathbf{X}$ *is a full subcategory* $\mathsf{sk}(\mathbf{X})$ *which is skeletal and such that the inclusion functor* $\mathsf{sk}(\mathbf{X}) \to \mathbf{X}$ *is an equivalence.*

Remark 8. By definition the inclusion $\mathsf{sk}(\mathbf{X}) \to \mathbf{X}$ is an equivalence. In particular, this mean that, for every objects X of $\mathbf{X}$ there exists $\pi(X)$ in $\mathsf{sk}(\mathbf{X})$ and an isomorphism $\phi_X : \pi(X) \to X$.

Proposition 6. *Every category* $\mathbf{X}$ *has a skeleton.*

Proof. For every object $X \in \mathbf{X}$, pick a single representative $\pi(X)$ of its isomorphism class. Let $\mathsf{sk}(\mathbf{X})$ be the full subcategory given by these objects. By definition $\mathsf{sk}(\mathbf{X})$ is skeletal and the inclusion functor is full, faithful and essentially surjective. □

Remark 9. The proof of **??** relies on the axiom of choice for classes.

Remark 10. It is possible to proof that every two skeleta of a given category $\mathbf{X}$ are isomorphic (not only equivalent). For the remaining of this paper we assume that a skeleton $\mathsf{sk}(\mathbf{X})$ of $\mathbf{X}$ and a functor $\pi : \mathbf{X} \to \mathsf{sk}(\mathbf{X})$ are chosen once and for all.

Definition 10. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system, a* decorated derivation *between two objects* G *and* H *is a triple* $(\underline{\mathscr{D}}, \alpha, \omega)$*, where* $\underline{\mathscr{D}}$ *is a derivation between* G *and* H*, and* $\alpha : \pi(G) \to G$ *and* $\omega : \pi(H) \to H$ *are isomorphisms.*

As we are interested in objects only up to isomorphism, so we are interested in (decorated) derivations only up to some notion of coherent isomorphism between them.

Definition 11. *Let* $\mathscr{D} : G \Mapsto H$ *and* $\mathscr{D}' : G' \Mapsto H'$ *be two direct derivations given by the following two diagrams*

$$\begin{tikzcd} L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\ G & D \arrow[l, "f"] \arrow[r, "g"'] & H \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, "n'"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k'"] & R \arrow[d, "h'"] \\ G' & D' \arrow[l, "f'"] \arrow[r, "g'"'] & H' \end{tikzcd}$$

An abstraction equivalence $\phi : \mathscr{D} \to \mathscr{D}'$ *is a triple* (ϕ_1, ϕ_2, ϕ_3) *of isomorphism fitting in the diagram below.*

$$\begin{tikzcd} & & & R \arrow[dddl, "h"'] \arrow[dddrr, "h'"] & & \\ & & K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dddl, "k"'] \arrow[dddrr, "k'"] & & & \\ & L \arrow[dddl, "n"'] \arrow[dddr, "n'"] & & & & \\ & & H \arrow[rrr, "\phi_3"] & & & H' \\ & D \arrow[ur, "g"'] \arrow[dl, "f"'] \arrow[rrr, "\phi_2"] & & & D' \arrow[ur, "g'"'] \arrow[dll, "f'"] & \\ G \arrow[rr, "\phi_1"'] & & G' & & & \end{tikzcd}$$

Given two decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$, $(\underline{\mathscr{D}'}, \alpha', \omega')$ *with* $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{n}$, $\underline{\mathscr{D}'} = \{\mathscr{D}'_i\}_{i=0}^{n}$ *such that* $\mathscr{D}_i : G_i \Mapsto G_{i+1}$ *and* $\mathscr{D}'_i : G'_i \Mapsto G'_{i+1}$*, we will say that* $(\underline{\mathscr{D}}, \alpha, \omega)$ *and* $(\underline{\mathscr{D}'}, \alpha', \omega')$ *are* abstraction equivalent*, if* $r(\underline{\mathscr{D}}) = r(\underline{\mathscr{D}'})$ *and there exists a family* $\{\phi^i\}_{i=0}^{n}$ *of abstraction equivalences* ϕ^i *between* $\mathscr{D}_i$ *and* $\mathscr{D}'_i$ *such that, for every index* i, $\phi_3^i = \phi_1^{i+1}$ *and the following diagrams commute.*

$$\begin{tikzcd} & \pi(G_0) \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ G_0 \arrow[rr, "\phi_1^0"'] & & G'_0 \end{tikzcd} \qquad \begin{tikzcd} & \pi(G_{n+1}) \arrow[dl, "\omega"'] \arrow[dr, "\omega'"] & \\ G_{n+1} \arrow[rr, "\phi_3^n"'] & & G'_{n+1} \end{tikzcd}$$

In such situation we will say that $(\underline{\mathscr{D}}, \alpha, \omega)$ *and* $(\underline{\mathscr{D}}', \alpha', \omega')$ *are* abstraction equivalent. *We will use* $\equiv^a$ *to denote the resulting equivalence relation, while* $[\underline{\mathscr{D}}, \alpha, \omega]_a$ *will denote the equivalence class of* $(\underline{\mathscr{D}}, \alpha, \omega)$*, called an* abstract decorated derivation.

Remark 11. Proposition 5 can be restated as saying that, given two direct derivations $\mathscr{D}$ and $\mathscr{D}'$ with the same match, there exists an abstract equivalence between them whose first component is an identity.

Definition 12. *Let* $(\underline{\mathscr{D}}, \alpha, \omega)$ *be a decorated derivation between* G *and* H *and* $(\underline{\mathscr{D}}', \alpha', \omega')$ *one between* H' *and* K*. If* H *and* H' *are isomorphic, so that* $\pi(H) = \pi(H')$*, we define the* composite decorated derivation $(\underline{\mathscr{D}}, \alpha, \omega) \cdot (\underline{\mathscr{D}}', \alpha', \omega')$ *as* $((\underline{\mathscr{D}} * \omega^{-1}) \cdot ((\alpha')^{-1} * \underline{\mathscr{D}}'), \alpha, \omega')$*.*

Remark 12. Given two composable decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ such that $\underline{\mathscr{D}}$ and $\underline{\mathscr{D}}'$ have length n and m, respectively, then $(\underline{\mathscr{D}} * \omega^{-1}) \cdot ((\alpha')^{-1} * \underline{\mathscr{D}}')$ has length $n + m$.

The next proposition justifies the use of decorations, guaranteeing that concatenation of abstract decorated derivations is weel-defined.

Proposition 7. *Given a decorated derivation* $(\underline{\mathscr{D}}, \alpha, \omega)$ *between* G *and* H *and another one* $(\underline{\mathscr{E}}, \beta, \xi)$ *between* E *and* K *with* $\pi(H) = \pi(E)$*. If* $(\underline{\mathscr{D}}', \alpha', \omega')$ *and* $(\underline{\mathscr{E}}', \beta', \xi')$ *are two other decorated derivations such that* $[\underline{\mathscr{D}}, \alpha, \omega]_a = [\underline{\mathscr{D}}', \alpha', \omega']_a$ *and* $[\underline{\mathscr{D}}, \beta, \xi]_a = [\underline{\mathscr{E}}', \beta', \xi']_a$*, then*

$$[(\underline{\mathscr{D}}, \alpha, \omega) \cdot (\underline{\mathscr{E}}, \beta, \xi)]_a = [(\underline{\mathscr{D}}', \alpha', \omega') \cdot (\underline{\mathscr{E}}', \beta', \xi')]_a$$

Proof. To fix the notation, let $\underline{\mathscr{D}}$, $\underline{\mathscr{D}}'$, $\underline{\mathscr{E}}$ and $\underline{\mathscr{E}}'$ be given by

$$\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n \quad \underline{\mathscr{D}}' = \{\mathscr{D}'_i\}_{i=0}^n \quad \underline{\mathscr{E}} = \{\mathscr{E}_i\}_{i=0}^m \quad \underline{\mathscr{E}}' = \{\mathscr{E}'_i\}_{i=0}^m$$

Let also $\{\phi^i\}_{i=0}^n$ and $\{\varphi^i\}_{i=0}^m$ be families of abstract equivalences between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ and between $(\underline{\mathscr{E}}, \beta, \xi)$ and $(\underline{\mathscr{E}}', \beta', \xi')$, respectively. For every $i \in [0, n + m - 1]$ define

$$\psi^i := \begin{cases} \phi^i & i < n \\ (\phi_1^n, \phi_2^n, \mathsf{id}_{\pi(H)}) & i = n \\ (\mathsf{id}_{\pi(E)}, \varphi_2^0, \varphi_3^0) & i = n+1 \\ \varphi^{i-n+1} & i > n+1 \end{cases}$$

Notice suppose that $h_n : R_n \to H, h_n : R_n \to H'$ are the back-matches of $\underline{\mathscr{D}}_n$ and $\underline{\mathscr{D}}'_n$ and that the following diagram is given

$$\begin{tikzcd}
 & & & R_n \arrow[dddl, "h_n"] \arrow[dddr, "h'_n"] & \\
 & & K_n \arrow[ur, "r_n"] \arrow[dl, "l_n"'] \arrow[dddl, "k_n"'] \arrow[dddr, "k'_n"] & & \\
 & L_n \arrow[dddl, "m_n"'] \arrow[dddr, "m'_n"] & & & \\
 & & H \arrow[rr, "\phi_3^n"] & & H' \\
 & D_n \arrow[ur, "g_n"] \arrow[rr, "\phi_2^n"] \arrow[dl, "f_n"'] & & D'_n \arrow[ur, "g'_n"'] \arrow[dl, "f'_n"] & \\
G_n \arrow[rr, "\phi_1^n"'] & & G'_n & &
\end{tikzcd}$$

Then we have

$$\begin{aligned}
\mathsf{id}_{\pi(H)} \circ \omega^{-1} \circ h_n &= (\omega')^{-1} \circ \phi_3^n \circ \omega \circ \omega^{-1} \circ h_n \\
&= (\omega')^{-1} \circ \phi_3^n \circ h_n \\
&= (\omega')^{-1} \circ h'_n
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{id}_{\pi(H)} \circ \omega^{-1} \circ g_n &= (\omega')^{-1} \circ \phi_3^n \circ \omega \circ \omega^{-1} \circ g_n \\
&= (\omega')^{-1} \circ \phi_3^n \circ g'_n \\
&= (\omega')^{-1} \circ g'_n \circ \phi_2^n
\end{aligned}$$

Therefore ψ^n is an abstract equivalence between $\mathscr{D}_n * \omega^{-1}$ and $\mathscr{D}'_n * (\omega')^{-1}$.

Similarly, if $\hat{m}_0 : \hat{L}_0 \to E, \hat{m}'_0 : L'_0 \to E'$ the matches of $\underline{\mathscr{E}}_0$, $\underline{\mathscr{E}}'_0$, we can consider the diagram

$$\begin{tikzcd}
 & & & \hat{R}_0 \arrow[dddl, "\hat{h}_0"] \arrow[dddr, "\hat{h}'_0"] & \\
 & & \hat{K}_0 \arrow[ur, "\hat{r}_0"] \arrow[dl, "\hat{l}_0"'] \arrow[dddl, "\hat{k}_0"'] \arrow[dddr, "k'_0"] & & \\
 & \hat{L}_n \arrow[dddl, "\hat{m}_0"'] \arrow[dddr, "\hat{m}'_0"] & & & \\
 & & \hat{G}_1 \arrow[rr, "\varphi_3^0"] & & \hat{G}'_1 \\
 & \hat{D}_0 \arrow[ur, "\hat{g}_0"] \arrow[rr, "\varphi_2^0"] \arrow[dl, "\hat{f}_0"'] & & \hat{D}'_0 \arrow[ur, "\hat{g}'_0"'] \arrow[dl, "\hat{f}'_0"] & \\
E \arrow[rr, "\varphi_1^0"'] & & E' & &
\end{tikzcd}$$

and computing we get

$$\begin{aligned}\mathsf{id}_{\pi(E)} \circ \alpha^{-1} \circ \hat{m}_0 &= (\alpha')^{-1} \circ \varphi_1^0 \circ \alpha \circ \alpha^{-1} \circ \hat{m}_0 \\ &= (\alpha')^{-1} \circ \varphi_1^0 \circ \hat{m}_0 \\ &= (\alpha')^{-1} \circ \hat{m}'_n\end{aligned}$$

together with

$$\begin{aligned}\mathsf{id}_{\pi(E)} \circ \alpha^{-1} \circ \hat{f}_0 &= (\alpha')^{-1} \circ \varphi_1^0 \circ \alpha \circ \alpha^{-1} \circ \hat{f}_0 \\ &= (\alpha')^{-1} \circ \varphi_1^0 \circ \hat{f}_0 \\ &= (\alpha')^{-1} \circ \hat{f}'_0 \circ \varphi_2^0\end{aligned}$$

The previous two identities show that ψ^{n+1} is an abstract equivalence between $\mathscr{E}_0$ and $\mathscr{E}'_0$. By hypothesis $\pi(H) = \pi(E)$ and, moreover, we have

$$\begin{aligned}\psi_1^0 \circ \alpha &= \phi_1^0 \circ \alpha \qquad & \psi_3^{n+m-1} \circ \xi &= \varphi_3^m \circ \xi \\ &= \alpha' & &= \xi'\end{aligned}$$

Therefore the thesis now follows. □

Definition 13. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewrityng system, with* $\mathbf{X}$ *an* $\mathcal{M}$*-adhesive category. The category* $[\mathsf{C}]_R^{\mathbf{X}}$ *is defined as follows:*

- *objects are isomorphism classes of objects of* $\mathbf{X}$*;*
- *a non-identity arrow* $[G] \to [H]$ *is an equivalence class* $[\underline{\mathscr{D}}, \alpha, \omega]_a$ *of a decorated derivation between* G' *and* H' *for some* G' *and* H' *such that* $\pi(G') = G$ *and* $\pi(H') = H$*;*
- *composition is concatenation of abstract decorated derivations;*
- *identities are added formally.*

3.2 Consistent permutations

Given a left-linear DPO-rewriting system $(\mathbf{X}, \mathsf{R})$, a derivation $\underline{\mathscr{D}}$ determines a diagram $\Delta(\underline{\mathscr{D}})$ in $\mathbf{X}$. We can then wonder if such a diagram has a colimit. Clealry if $\underline{\mathscr{D}}$ is the empty derivation then a colimit for $\Delta(\underline{\mathscr{D}})$ amount to an initial object in $\mathbf{X}$. If $\underline{\mathscr{D}}$ is non-empty we can use the following result.

Lemma 6. *Let* $\mathbf{X}$ *be an* $\mathcal{M}$*-adhesive category and* $(\mathbf{X}, \mathsf{R})$ *a left-linear DPO-rewriting system over it. The following hold true:*

1. *if* $\underline{\mathscr{D}}$ *is a derivation of length one, whose unique element is given by the following diagram*

$$\begin{tikzcd} L \arrow[d, "m"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\ G & D \arrow[l, "f"] \arrow[r, "g"'] & H \end{tikzcd}$$

then the colimiting cocone $(\llparenthesis \underline{\mathscr{D}} \rrparenthesis, \{\iota_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ *exists and fits in the diagram below, where the bottom face is a pushout;*

$$\begin{tikzcd}
 & K \arrow[dl, "l"'] \arrow[rr, "r"] \arrow[d, "k"] & & R \arrow[d, "h"] \\
L \arrow[d, "m"'] & D \arrow[dl, "f"] \arrow[rr, "g"] & & H \arrow[dl, "\iota_H"] \\
G \arrow[rr, "\iota_G"'] & & \llparenthesis \underline{\mathscr{D}} \rrparenthesis &
\end{tikzcd}$$

2. *for every non-empty derivation* $\underline{\mathscr{D}}$ *from* G *to* H*, the diagram* $\Delta(\underline{\mathscr{D}})$ *has a colimit* $(\llparenthesis \underline{\mathscr{D}} \rrparenthesis, \{\iota_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ *such that* ι_H *belong to* $\mathcal{M}$*;*
3. *let* $\underline{\mathscr{D}}$ *be the concatenation* $\underline{\mathscr{D}}_1 \cdot \underline{\mathscr{D}}_2$ *of two derivations* $\underline{\mathscr{D}}_1 = \{\mathscr{D}_{1,i}\}_{i=0}^{n_1}$ *between* G *and* H *and* $\underline{\mathscr{D}}_2 = \{\mathscr{D}_{2,j}\}_{j=0}^{n_2}$ *between* H *and* T*, then the colimiting cocone* $(\llparenthesis \underline{\mathscr{D}} \rrparenthesis, \{\iota_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ *exists too and there is a pushout square*

$$\begin{tikzcd}
H \arrow[r, "\iota_{2,H}"] \arrow[d, "\iota_{1,H}"'] & \llparenthesis \underline{\mathscr{D}}_2 \rrparenthesis \arrow[d, "p_2"] \\
\llparenthesis \underline{\mathscr{D}}_1 \rrparenthesis \arrow[r, "p_1"'] & \llparenthesis \underline{\mathscr{D}} \rrparenthesis
\end{tikzcd}$$

where $(\llparenthesis \underline{\mathscr{D}}_1 \rrparenthesis, \{\iota_{1,X}\}_{X \in \Delta(\underline{\mathscr{D}}_1)})$ *and* $(\llparenthesis \underline{\mathscr{D}}_2 \rrparenthesis, \{\iota_{2,X}\}_{X \in \Delta(\underline{\mathscr{D}}_2)})$ *are the colimiting cocone for* $\Delta(\underline{\mathscr{D}}_1)$ *and* $\Delta(\underline{\mathscr{D}}_2)$*, respectively;*
4. *given a non-empty derivation* $\underline{\mathscr{D}} = \{\mathscr{D}\}_{i=0}^{n}$ *between* G *and* H*, for every* $j \in n+1$ *different from* 0 *the following diagram is a pushout*

$$\begin{tikzcd}
D_j \arrow[r, "g_j"] \arrow[d, "f_j"'] & G_{j+1} \arrow[r, "\hat{\iota}_{G_{j+1}}"] \arrow[ddl, bend left, "\iota'_{G_j}"] & \llparenthesis \hat{\underline{\mathscr{D}}} \rrparenthesis \arrow[dd, "p_2"] \\
G_j \arrow[d, "\iota'_{G_j}"'] & & \\
\llparenthesis \underline{\mathscr{D}}' \rrparenthesis \arrow[rr, "p_1"'] & & \llparenthesis \underline{\mathscr{D}} \rrparenthesis
\end{tikzcd}$$

where $\hat{\underline{\mathscr{D}}} = \{\mathscr{D}_i\}_{i=j+1}^{n}$*,* $\underline{\mathscr{D}}' = \{\mathscr{D}_i\}_{i=0}^{j}$ *and* $p_1 : \llparenthesis \underline{\mathscr{D}}' \rrparenthesis \to \llparenthesis \underline{\mathscr{D}} \rrparenthesis$ *and* $p_2 : \llparenthesis \hat{\underline{\mathscr{D}}} \rrparenthesis \to \llparenthesis \underline{\mathscr{D}} \rrparenthesis$ *are the arrows induced by the cocones,* $(\llparenthesis \underline{\mathscr{D}} \rrparenthesis, \{\iota_X\}_{X \in \Delta(\underline{\mathscr{D}}')})$ *and* $(\llparenthesis \underline{\mathscr{D}} \rrparenthesis, \{\iota_X\}_{X \in \Delta(\underline{\mathscr{D}}')})$*, respectively.*

Remark 13. There is a version of point 4 of Lemma 6 for $j = 0$ which follows directly from points 1 and 3. Indeed, $\underline{\mathscr{D}} = \underline{\mathscr{D}}_1 \cdot \underline{\mathscr{D}}_2$ where $\underline{\mathscr{D}}_1 = \{\mathscr{D}_0\}$ and $\underline{\mathscr{D}}_2 = \{\mathscr{D}_i\}_{i=1}^{n}$. By point 1 and 3 of Lemma 6, the two halves of the rectangle below are pushouts. By Lemma 2 the whole diagram is then a pushout too.

$$\begin{tikzcd}
D_0 \arrow[r, "g_0"] \arrow[d, "f_0"'] & G_1 \arrow[r, "\iota_{2,G_1}"] \arrow[d, "\iota_{1,G_1}"] & \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \arrow[d, "p_2"] \\
G \arrow[r, "\iota_{1,G}"'] & \langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \arrow[r, "p_1"'] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}$$

Notice that this entails that the square below is also a pushout.

$$\begin{tikzcd}
D_0 \arrow[r, "\iota_{2,D_0}"] \arrow[d, "\iota_{1,D_0}"'] & \langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \arrow[d, "p_2"] \\
\langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \arrow[r, "p_1"'] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}$$

Proof. 1. We can start noticing that f, being the pushout of l is in $\mathcal{M}$ and thus it admits a pushout along g, giving us the diagram

$$\begin{tikzcd}
L \arrow[d, "m"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\
G \arrow[dr, bend right, "q"'] & D \arrow[l, "f"] \arrow[r, "g"'] & H \arrow[dl, bend left, "p"] \\
 & P &
\end{tikzcd}$$

Clearly, precomposing p and q with the arrows appearing in $\mathscr{D}$, we can extend the pushout (P, p, q), to a cocone $(P, \{j_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ on $\Delta(\underline{\mathscr{D}})$. Let $(C, \{c_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ be another cocone, for $\Delta(\underline{\mathscr{D}})$, in particular we have

$$\begin{aligned} c_H \circ g &= c_D \\ &= c_G \circ f \end{aligned}$$

Hence, there exists a unique $c : P \to C$ such that $c \circ q = c_G$ and $c \circ p = c_H$. But these identities are enough to deduce that $(P, \{j_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ is colimiting.

2. Let us proceed by induction.
   - If $n = 0$, then the claim follows immediately from point 1, noticing that ι_H is the pushout of f and thus an element of $\mathcal{M}$.
   - Let $\underline{\mathscr{D}}$ be $\{\mathscr{D}_i\}_{i=0}^{n}$ and suppose that $n \geq 1$. Let also $\underline{\mathscr{D}}'$ be $\{\mathscr{D}_i\}_{i=0}^{n-1}$ and $\rho_n = (l_n, r_n)$ be the rule applied in $\mathscr{D}_n$. The pushout of l_n is the arrow $f_n : D_n \to G_n$ is in $\mathcal{M}$ and, by inductive hypothesis, $\iota_{G_n} : G_n \to \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle$ is in $\mathcal{M}$ too. Thus, we can consider the diagram below, having a pushout

as its lower half.

$$
\begin{tikzcd}
L_n \arrow[d, "m_n"'] & K_n \arrow[l, "l_n"'] \arrow[r, "r_n"] \arrow[d, "k_n"] & R_n \arrow[d, "h_n"] \\
G_n \arrow[d, "\iota'_{G_n}"'] & D_n \arrow[l, "f_n"] \arrow[r, "g_n"'] & H \arrow[d, "q"] \\
\langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle \arrow[rr, "p"'] & & P
\end{tikzcd}
$$

Notice that, as in the point above, the arrow $q : H \to P$ is the pushout of an element in $\mathcal{M}$, therefore it is enough to show that the diagram so constructed provides a colimiting cocone for $\Delta(\underline{\mathscr{D}})$.
Let $(C, \{c_X\}_{X\in\Delta(\underline{\mathscr{D}})})$ be a cocone, since $\Delta(\underline{\mathscr{D}}')$ is a subdiagram of $\Delta(\underline{\mathscr{D}})$, we get another cocone $(c, \{c_X\}_{X\in\Delta(\underline{\mathscr{D}}')})$ which induces an arrow $c' : \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle \to C$ such that

$$
\begin{aligned}
c' \circ \iota_{G_n} \circ f_n &= c_{G_n} \circ f_n \\
&= c_{D_n} \\
&= c_H \circ g_n
\end{aligned}
$$

Therefore the arrows c' and c_H induce a morphism $c : P \to C$ and the thesis now follows at once.

3. As a first step, notice that $(\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle, \{\iota_X\}_{X\in\Delta(\underline{\mathscr{D}}_1)})$ and $(\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle, \{\iota_X\}_{X\in\Delta(\underline{\mathscr{D}}_2)})$ are cocone on, respectively, $\Delta(\underline{\mathscr{D}}_1)$ and $\Delta(\underline{\mathscr{D}}_2)$. Hence, there exist two arrows $p_1 : \langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle$, $p_2 : \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle$ such that, for every $X \in \Delta(\underline{\mathscr{D}}_1)$ and $Y \in \Delta(\underline{\mathscr{D}}_2)$

$$
p_1 \circ \iota_{1,X} = \iota_X \qquad p_2 \circ \iota_{2,Y} = \iota_{2,Y}
$$

In particular, this entails the commutativity of the square

$$
\begin{tikzcd}
H \arrow[r, "\iota_{2,H}"] \arrow[d, "\iota_{1,H}"'] \arrow[dr, "\iota_H"] & \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \arrow[d, "p_2"] \\
\langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \arrow[r, "p_1"'] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}
$$

Let us now show that the square above is a pushout. Take two arrows $a : \langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \to C$, $b : \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \to C$ such that

$$
a \circ \iota_{1,H} = b \circ \iota_{2,H}
$$

We can use the previous equality to define a cocone $(C, \{c_X\}_{X\in\Delta(\underline{\mathscr{D}})})$ putting:

$$
c_X := \begin{cases} a \circ \iota_{1,X} & X \in \Delta(\underline{\mathscr{D}}_1) \\ b \circ \iota_{2,X} & X \in \Delta(\underline{\mathscr{D}}_2) \end{cases}
$$

From this, we can deduce at once the existence of a unique $c : \llangle \underline{\mathscr{D}} \rrangle \to C$ such that

$$c \circ \iota_X = c_X$$

By construction, for every $X \in \Delta(\underline{\mathscr{D}}_1)$ and $Y \in \Delta(\underline{\mathscr{D}}_2)$ we have

$$\begin{aligned} c \circ p_1 \circ \iota_{1,X} &= c \circ \iota_X & \qquad c \circ p_2 \circ \iota_{2,Y} &= c \circ \iota_Y \\ &= c_X & &= c_Y \\ &= a \circ \iota_{1,X} & &= b \circ \iota_{2,Y} \\ &= a \circ p_1 \circ \iota_{1,X} & &= b \circ p_2 \circ \iota_{2,Y} \end{aligned}$$

Therefore

$$c \circ p_1 = a \qquad c \circ p_2 = b$$

For uniqueness, suppose that $c' : \llangle \underline{\mathscr{D}} \rrangle \to C$ is such that

$$c' \circ p_1 = a \qquad c' \circ p_2 = b$$

Then, for every $X \in \Delta(\underline{\mathscr{D}})$ we have

$$\begin{aligned} c' \circ \iota_X &= \begin{cases} c' \circ p_1 \circ \iota_{1,X} & X \in \Delta(\underline{\mathscr{D}}_1) \\ c' \circ p_2 \circ \iota_{2,X} & X \in \Delta(\underline{\mathscr{D}}_2) \end{cases} \\ &= \begin{cases} a \circ \iota_{1,X} & X \in \Delta(\underline{\mathscr{D}}_1) \\ b \circ \iota_{2,X} & X \in \Delta(\underline{\mathscr{D}}_2) \end{cases} \\ &= c_X \\ &= c \circ \iota_X \end{aligned}$$

showing that $c' = c$ as wanted.

4. Let $a : \llangle \hat{\underline{\mathscr{D}}} \rrangle \to C$, $b : \llangle \underline{\mathscr{D}}' \rrangle \to C$ be two arrows fitting in the diagram

$$\begin{tikzcd}
D_j \arrow[r, "g_j"] \arrow[d, "f_j"'] \arrow[ddrr, "\iota_{D_j}"] & G_{j+1} \arrow[r, "\hat{\iota}_{G_{j+1}}"] & \llangle \hat{\underline{\mathscr{D}}} \rrangle \arrow[dd, "p_2"] \arrow[dddr, bend left, "a"] & \\
G_j \arrow[d, "\iota'_{G_j}"'] & & & \\
\llangle \underline{\mathscr{D}}' \rrangle \arrow[rr, "p_1"'] \arrow[drrr, bend right, "b"'] & & \llangle \underline{\mathscr{D}} \rrangle \arrow[dr, dotted, "c"] & \\
 & & & C
\end{tikzcd}$$

We can define a cocone $(C, \{c_X\}_{X \in \Delta(\underline{\mathscr{D}})})$ putting

$$c_X := \begin{cases} a \circ \hat{\iota}_X & X \in \Delta(\hat{\underline{\mathscr{D}}}) \\ b \circ \iota'_X & X \in \Delta(\underline{\mathscr{D}}') \\ b \circ \iota'_{G_j} \circ f_j & X = D_j \end{cases}$$

Thus we get a unique arrow $c : \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to C$ such that

$$c \circ \iota_X = c_X$$

Now, for every $X \in \Delta(\underline{\mathscr{D}}')$ and $Y \in \Delta(\underline{\hat{\mathscr{D}}})$ we have

$$\begin{aligned} c \circ p_1 \circ \iota'_X &= c \circ \iota_X & \qquad c \circ p_2 \circ \hat{\iota}_Y &= c \circ \iota_Y \\ &= c_X & &= c_Y \\ &= b \circ \iota'_X & &= a \circ \hat{\iota}_Y \\ &= b \circ p_1 \circ \iota'_X & &= a \circ p_2 \circ \hat{\iota}_Y \end{aligned}$$

so that we can deduce

$$c \circ p_1 = a \qquad c \circ p_2 = b$$

To see that such c is unique, suppose that $c' : \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to C$ is given such that

$$c' \circ p_1 = b \qquad c' \circ p_2 = a$$

Then, for every $X \in \Delta(\underline{\mathscr{D}})$ we have

$$\begin{aligned} c' \circ \iota_X &= \begin{cases} c' \circ p_1 \circ \iota'_X & X \in \Delta(\underline{\mathscr{D}}') \\ c' \circ p_2 \circ \hat{\iota}_X & X \in \Delta(\underline{\hat{\mathscr{D}}}) \\ c' \circ \iota_{D_j} & X = D_j \end{cases} \\ &= \begin{cases} b \circ \iota'_X & X \in \Delta(\underline{\mathscr{D}}') \\ a \circ \hat{\iota}_X & X \in \Delta(\underline{\hat{\mathscr{D}}}) \\ c' \circ p_1 \circ \iota'_{G_j} \circ f_j & X = D_j \end{cases} \\ &= \begin{cases} b \circ \iota'_X & X \in \Delta(\underline{\mathscr{D}}') \\ a \circ \hat{\iota}_X & X \in \Delta(\underline{\hat{\mathscr{D}}}) \\ b \circ \iota'_{G_j} \circ f_j & X = D_j \end{cases} \\ &= c_X \\ &= c \circ \iota_X \end{aligned}$$

allowing us to deduce that $c' = c$. $\square$

Corollary 5. *Let $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{n}$ be a derivation between G and H. Let j and k be two indexes less or equal than $n+1$ and suppose that $j < k$. Consider two arrows $a : T \to G_j$, $b : T \to G_k$. If $\iota_{G_j} \circ a = \iota_{G_k} \circ b$, then there exist arrow $c : T \to D_j$ such that*

$$f_j \circ c = a \qquad \iota_{D_j} \circ c = \iota_{G_k} \circ b$$

Proof. We split the cases.

– $j = 0$. Consider $\underline{\mathscr{D}}_1 := \{\mathscr{D}_0\}$ and $\underline{\mathscr{D}}_2 := \{\mathscr{D}_i\}_{i=1}^{n}$. Consider the diagram below in which, by **??**, the rectangle on the bottom right is a pushout.

$$
\begin{tikzcd}
T \arrow[rr, "b"] \arrow[d, dotted, "t"] \arrow[ddd, bend right, "a"'] & & G_k \arrow[d, "\iota_{2,G_k}"'] \arrow[ddd, bend left, "\iota_{G_k}"] \\
D_0 \arrow[r, "g_0"] \arrow[d, "f_0"] & G_1 \arrow[r, "\iota_{2,G_1}"] & \langle\!\langle \underline{\mathscr{D}}_2 \rangle\!\rangle \arrow[dd, "p_2"'] \\
G \arrow[d, "\iota_{1,G}"] \arrow[drr, "\iota_G"] & & \\
\langle\!\langle \underline{\mathscr{D}}_1 \rangle\!\rangle \arrow[rr, "p_1"'] & & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}
$$

Since f_0 and $\iota_{1,G}$ are in $\mathcal{M}$ the same rectangle is an $\mathcal{M}$-pushout and so, by Proposition 4 is a pullback and the thesis follows.

– $j \neq 0$. We proceed as in the point above. Using point 4 of Lemma 6 we know that the bottom right rectangle in the diagram below is a pushout.

$$
\begin{tikzcd}
T \arrow[rr, "b"] \arrow[d, dotted, "t"] \arrow[ddd, bend right, "a"'] & & G_k \arrow[d, "\iota_{2,G_k}"'] \arrow[ddd, bend left, "\iota_{G_k}"] \\
D_j \arrow[r, "g_j"] \arrow[d, "f_j"] & G_{j+1} \arrow[r, "\hat{\iota}_{G_{j+1}}"] & \langle\!\langle \underline{\hat{\mathscr{D}}} \rangle\!\rangle \arrow[dd, "p_2"'] \\
G_j \arrow[d, "\iota'_{G_j}"] \arrow[drr, "\iota_{G_j}"] & & \\
\langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle \arrow[rr, "p_1"'] & & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle
\end{tikzcd}
$$

Point 2 of Lemma 6 entails that ι'_{G_j} is in $\mathcal{M}$, and so is f_j as it is the pushout of l_j. Thus the rectangle we are considering is an $\mathcal{M}$-pushout and the thesis follows again from Proposition 4. □

So equipped, we can introduce the notion of *consistent permutation.*

Definition 14. *Let* $\mathbf{X}$ *be an* $\mathcal{M}$*-adhesive category and consider a left-linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$ *on it. Take two non-empty decorated derivations* $(\underline{\mathscr{D}}, \alpha, \omega)$ *and* $(\underline{\mathscr{D}}', \alpha', \omega')$ *such that*

– $\underline{\mathscr{D}}$ *and* $\underline{\mathscr{D}}'$ *have the same length;*
– *if* $\underline{\mathscr{D}}$ *is a derivation between* G *and* H *amd* $\underline{\mathscr{D}}'$ *one between* G' *and* H', *then* $G \simeq G'$ *and* $H \simeq H'$.

If $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n$ *and* $\underline{\mathscr{D}}' = \{\mathscr{D}'_i\}_{i=0}^n$ *with assciated squence of rules* $r(\underline{\mathscr{D}}) = \{\rho_i\}_{i=0}^n$ *and* $r(\underline{\mathscr{D}}') = \{\rho'_i\}_{i=0}^n$*, a* consistent permutation *between* $(\underline{\mathscr{D}}, \alpha, \omega)$ *and* $(\underline{\mathscr{D}}', \alpha', \omega')$ *is a permutation* $\sigma : [0,n] \to [0,n]$ *such that, for every* $i \in [0,n]$, $\rho_i = \rho'_{\sigma(i)}$ *and, moreover, there exists a* mediating isomorphism $\xi_\sigma : \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle$ *fitting in the following diagrams, where* m_i, m'_i, h_i *and* h'_i *are, respectively, the matches and back-matches of* $\mathscr{D}_i$ *and* $\mathscr{D}'_i$*.*

$$\begin{tikzcd}
\pi(G) \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & G \arrow[r, "\iota_G"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi"] \\
G' \arrow[rr, "\iota'_{G'}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi(H) \arrow[r, "\omega"] \arrow[d, "\omega'"'] & H \arrow[r, "\iota_H"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi"] \\
H' \arrow[rr, "\iota'_{H'}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}$$

$$\begin{tikzcd}
L_i \arrow[r, "m_i"] \arrow[d, "m'_{\sigma(i)}"'] & G_i \arrow[r, "\iota_{G_i}"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi_\sigma"] \\
G'_{\sigma(i)} \arrow[rr, "\iota'_{G'_{\sigma(i)}}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
R_i \arrow[r, "h_i"] \arrow[d, "h'_{\sigma(i)}"'] & G_{i+1} \arrow[r, "\iota_{G_{i+1}}"] & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[d, "\xi_\sigma"] \\
G'_{\sigma(i)+1} \arrow[rr, "\iota'_{G'_{\sigma(i)+1}}"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle
\end{tikzcd}$$

Remark 14. The commutativity of the last two rectangles in **??**, is equivalent to the commutativity of the following bigger diagram.

$$\begin{tikzcd}
 & & & R_i \arrow[dddl, "h_i"] \arrow[dddr, "h'_{\sigma(i)}"] & \\
 & & K_i \arrow[ur, "r"] \arrow[dl, "l_i"'] \arrow[dddl, "k_i"] \arrow[dddr, "k'_{\sigma(i)}"] & & \\
 & L_i \arrow[dddl, "m_i"'] \arrow[dddr, "m'_{\sigma(i)}"] & & & \\
 & & G_{i+1} \arrow[dddl, bend left, "\iota_{G_{i+1}}"] & & G'_{\sigma(i)+1} \arrow[dddl, bend left, "\iota'_{G'_{\sigma(i)+1}}"] \\
 & D_i \arrow[ur, "g_i"] \arrow[dl, "f_i"'] \arrow[dd, "\iota_{D_i}"'] & & D'_\sigma(i) \arrow[ur, "g'_{\sigma(i)}"] \arrow[dl, "f'_{\sigma(i)}"'] \arrow[dd, "\iota'_{D'_\sigma(i)}"] & \\
G_i \arrow[dr, bend right, "\iota_{G_i}"'] & & G'_{\sigma(i)} \arrow[dr, bend right, "\iota'_{G'_{\sigma(i)}}"] & & \\
 & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[rr, "\xi_\sigma"'] & & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle &
\end{tikzcd}$$

Indeed, to prove the commutativity of the central pentagon we can compute to get:

$$\begin{aligned}
\xi_\sigma \circ \iota_{D_i} \circ k_i &= \xi_\sigma \circ \iota_{G_i} \circ f_i \\
&= \xi_\sigma \circ \iota_{G_i} \circ m_i \circ l_i \\
&= \iota_{G'_{\sigma(i)}} \circ m'_i \circ l_i \\
&= \iota_{G'_{\sigma(i)}} \circ f'_{\sigma(i)} \circ k'_i \\
&= \iota_{D'_\sigma(i)} \circ k'_i
\end{aligned}$$

Remark 15. Notice that, in particular, the previous diagram entails

$$\xi_\sigma \circ \iota_{L_i} = \iota'_{L_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{K_i} = \iota'_{K_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{R_i} = \iota'_{R_{\sigma(i)}}$$

Remark 16. Let $\sigma : [0,n] \to [0,n]$ be a consistent permutation between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$, then σ^{-1} is a consistent permutation between $(\underline{\mathscr{D}}', \alpha', \omega')$ and $(\underline{\mathscr{D}}, \alpha, \omega)$: indeed, it is enough to consider, as mediating isomorphism, the inverse ξ_σ^{-1} of ξ_σ.

Remark 17. Consistent permutations can be composed. Indeed, given decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$, $(\underline{\mathscr{D}}', \alpha', \omega')$ and $(\hat{\underline{\mathscr{D}}}, \hat{\alpha}, \hat{\omega})$, if σ is a consistent permutation between the first two and σ' one between the second and the third, then we have diagrams

$$\begin{tikzcd}
 & G \arrow[r, "\iota_G"] & \Lbag \underline{\mathscr{D}} \Rbag \arrow[d, "\xi_\sigma"] \\
\pi(G) \arrow[ur, bend left, "\alpha"] \arrow[r, "\alpha'"] \arrow[dr, bend right, "\hat{\alpha}"'] & G' \arrow[r, "\iota'_{G'}"] & \Lbag \underline{\mathscr{D}}' \Rbag \arrow[d, "\xi_{\sigma'}"] \\
 & \hat{G} \arrow[r, "\hat{\iota}_{\hat{G}}"'] & \Lbag \hat{\underline{\mathscr{D}}} \Rbag
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H \arrow[r, "\iota_H"] & \Lbag \underline{\mathscr{D}} \Rbag \arrow[d, "\xi_\sigma"] \\
\pi(H) \arrow[ur, bend left, "\omega"] \arrow[r, "\omega'"] \arrow[dr, bend right, "\hat{\omega}"'] & H' \arrow[r, "\iota'_{H'}"] & \Lbag \underline{\mathscr{D}}' \Rbag \arrow[d, "\xi_{\sigma'}"] \\
 & \hat{H} \arrow[r, "\hat{\iota}_{\hat{H}}"'] & \Lbag \hat{\underline{\mathscr{D}}} \Rbag
\end{tikzcd}$$

$$\begin{tikzcd}
 & G_i \arrow[r, "\iota_{G_i}"] & \Lbag \underline{\mathscr{D}} \Rbag \arrow[d, "\xi_\sigma"] \\
L_i \arrow[ur, bend left, "m_i"] \arrow[r, "m'_{\sigma(i)}"] \arrow[dr, bend right, "\hat{m}_{\sigma'(\sigma(i))}"'] & G'_{\sigma(i)} \arrow[r, "\iota'_{G'_{\sigma(i)}}"] & \Lbag \underline{\mathscr{D}}' \Rbag \arrow[d, "\xi_{\sigma'}"] \\
 & \hat{G}_{\sigma'(\sigma(i))} \arrow[r, "\hat{\iota}_{\hat{G}_{\sigma'(\sigma(i))}}"'] & \Lbag \hat{\underline{\mathscr{D}}} \Rbag
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G_i \arrow[r, "\iota_{G_i}"] & \Lbag \underline{\mathscr{D}} \Rbag \arrow[d, "\xi_\sigma"] \\
R_i \arrow[ur, bend left, "h_i"] \arrow[r, "h'_{\sigma(i)}"] \arrow[dr, bend right, "\hat{h}_{\sigma'(\sigma(i))}"'] & G'_{\sigma(i)} \arrow[r, "\iota'_{G'_{\sigma(i)}}"] & \Lbag \underline{\mathscr{D}}' \Rbag \arrow[d, "\xi_{\sigma'}"] \\
 & \hat{G}_{\sigma'(\sigma(i))} \arrow[r, "\hat{\iota}_{\hat{G}_{\sigma'(\sigma(i))}}"'] & \Lbag \hat{\underline{\mathscr{D}}} \Rbag
\end{tikzcd}$$

We deduce at once that $\sigma' \circ \sigma$ is a consistent permutation with mediating isomorphism given by $\xi_{\sigma'} \circ \xi_\sigma$.

Example 9. Let $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$ be two abstraction equivalent non-empty decorated derivations of length $n+1$. Then the identity permutation $\mathsf{id}_{[0,n]}$ is a consistent permutation between them. In such a case $\xi_{\mathsf{id}_{[0,n]}}$ is simply the isomorphism induced by any family $\{\phi^i\}_{i=0}^n$ witnessing $(\underline{\mathscr{D}}, \alpha, \omega) \equiv^a (\underline{\mathscr{D}}', \alpha', \omega')$.

esempi di cose consistenti utile per example 17

Example 10.

Proposition 8. *Let $\sigma, \sigma' : [0,n] \rightrightarrows [0,n]$ be two consistent permutations between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$. Then the following hold true:*

1. $\xi_\sigma \circ \iota_{G_0} = \xi_{\sigma'} \circ \iota_{G_0}$;

2. $\xi_\sigma \circ \iota_{G_i} = \xi_{\sigma'} \circ \iota_{G_i}$ *for every index* $i \in [0, n]$ *such that* $\sigma_{|[0,i]} = \sigma'_{|[0,i]}$.

Proof. 1. This follows at once noticing that both $\xi_\sigma \circ \iota_{G_0}$ and $\xi_{\sigma'} \circ \iota_{G_0}$ are equal to $\iota'_{G'_0} \circ \alpha' \circ \alpha^{-1}$.

2. If $\sigma(0) \neq \sigma'(0)$ there is nothing to show. Otherwise, let j be the maximum of the set

$$\{i \in [0, n] \mid \sigma_{|[0,i]} = \sigma'_{|[0,i]}\}$$

We proceed by induction on $i \in [0, j]$.

- If $i = 0$ the thesis follows from point 1.
- If $i > 0$, we know that there is a pushout square

$$\begin{tikzcd} K_{i-1} \arrow[r, "r_{i-1}"] \arrow[d, "k_{i-1}"'] & R_{i-1} \arrow[d, "h_{i-1}"] \\ D_{i-1} \arrow[r, "g_{i-1}"'] & G_i \end{tikzcd}$$

By **??** we know that

$$\begin{aligned} \xi_\sigma \circ \iota_{G_i} \circ h_{i-1} &= \xi_\sigma \circ \iota_{R_{i-1}} \\ &= \iota'_{R_{\sigma(i-1)}} \\ &= \iota'_{R_{\sigma'(i-1)}} \\ &= \xi_{\sigma'} \circ \iota_{R_{i-1}} \\ &= \xi_{\sigma'} \circ \iota_{G_i} \circ h_{i-1} \end{aligned}$$

But, by the induction hypothesis we also have

$$\begin{aligned} \xi_\sigma \circ \iota_{G_i} \circ g_{i-1} &= \xi_\sigma \circ \iota_{D_{i-1}} \\ &= \xi_\sigma \circ \iota_{G_{i-1}} \circ f_{i-1} \\ &= \xi_{\sigma'} \circ \iota_{G_{i-1}} \circ f_{i-1} \\ &= \xi_{\sigma'} \circ \iota_{D_{i-1}} \\ &= \xi_{\sigma'} \circ \iota_{G_i} \circ g_{i-1} \end{aligned}$$

The thesis now follows. □

Now, notice that, given a consistent permutation σ, $\xi_\sigma : \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle$, we already know, by **??** that

$$\xi_\sigma \circ \iota_{L_i} = \iota'_{L_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{K_i} = \iota'_{K_{\sigma(i)}} \quad \xi_\sigma \circ \iota_{R_i} = \iota'_{R_{\sigma(i)}}$$

Moreover, for $\xi_\sigma \circ \iota_{D_i}$ must be $\xi_\sigma \circ \iota_{G_i} \circ f_i$. Thus it is enough to define $\xi_\sigma \circ \iota_{G_i}$ for every $i \in [0, n]$. But then **??** entails the following.

Corollary 6. *Given a consistent permutation* σ *between* $(\underline{\mathscr{D}}, \alpha, \omega)$ *and* $(\underline{\mathscr{D}}', \alpha', \omega')$, *the mediating isomorphism* $\xi_\sigma : \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \to \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle$ *is unique.*

Proposition 9. il caso in cui sigma si de-componga come due permu

Proof. contenuto... ☐

We are now ready to prove the central result of this section.

Lemma 7. *Let $(\mathbf{X}, R)$ be a left-linear DPO-rewriting system. Consider two consistent permutation $\sigma, \sigma' : [0, n] \rightrightarrows [0, n]$ between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$. Suppose that $\sigma \neq \sigma'$ and let j be the minimum index such that $\sigma(j) \neq \sigma'(j)$. Let also $r(\underline{\mathscr{D}})$ be $\{\rho_i\}_{i=0}^{n}$. Then the following hold true:*

1. *if $j = 0$, then the rule ρ_0 is not consuming;*
2. *if $j \neq 0$ then the rule ρ_{j-1} is not consuming.*

Proof. 1. Let k be $\sigma^{-1}(\sigma'(0))$ and notice that, since $\sigma(0) \neq \sigma'(0)$, then $0 < k$. By the first point of **??** we can consider the diagram

$$
\begin{tikzcd}
 & L_0 \arrow[dl, bend right, "m_0"'] \arrow[d, "m'_{\sigma'(0)}"] \arrow[dr, bend left, "m_k"] & \\
G_0 \arrow[d, "\iota_{G_0}"'] \arrow[drr, "\iota_{G_0}"] & G'_{\sigma'(0)} \arrow[d, "\iota_{G'_{\sigma'(0)}}"] & G_k \arrow[d, "\iota_{G_k}"] \\
\llparenthesis \underline{\mathscr{D}} \rrparenthesis \arrow[r, "\xi_{\sigma'}"'] & \llparenthesis \underline{\mathscr{D}}' \rrparenthesis & \llparenthesis \underline{\mathscr{D}} \rrparenthesis \arrow[l, "\xi_{\sigma}"]
\end{tikzcd}
$$

From **??**, we can conclude that there exists $c : L_0 \to D_0$ such that $f_0 \circ c = m_0$. We thus have the solid part of the commutative diagram below.

$$
\begin{tikzcd}
L_0 \arrow[dr, dotted, "t"] \arrow[drr, bend left, "\mathsf{id}_{L_0}"] \arrow[ddr, bend right, "c"'] & & \\
 & K_0 \arrow[r, "l_0"] \arrow[d, "k_0"'] & L_0 \arrow[d, "m_0"] \\
 & D_0 \arrow[r, "f_0"'] & G_0
\end{tikzcd}
$$

The internal square is an $\mathcal{M}$-pushout and thus a pullback, by Proposition 4, so that we have the existence of the dotted $t : L_0 \to K_0$. Therefore $\mathsf{id}_{L_0} = l_0 \circ t$, proving that l_0 is an epimorphism. The thesis now follows from Corollary 2.

2. Let k be $\sigma^{-1}(\sigma'(j-1))$ and notice that $\rho_{j-1} = \rho_k$. By definition of j, $\sigma_{|[0,j-1]} = \sigma'_{|[0,j-1]}$, thus the second point of Proposition 5 yields the diagram

$$\begin{tikzcd}
 & L_{j-1} \arrow[dl, bend right, "m_{j-1}"'] \arrow[d, "m'_{\sigma'(j-1)}"'] \arrow[dr, bend left, "m_k"] & \\
G_{j-1} \arrow[d, "\iota_{G_{j-1}}"'] \arrow[drr, "\iota_{G_{j-1}}"'] & G'_{\sigma'(j-1)} \arrow[d, "\iota_{G'_{\sigma'(j-1)}}"] & G_k \arrow[d, "\iota_{G_k}"] \\
\langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[r, "\xi_{\sigma'}"'] & \langle\!\langle \underline{\mathscr{D}}' \rangle\!\rangle & \langle\!\langle \underline{\mathscr{D}} \rangle\!\rangle \arrow[l, "\xi_{\sigma}"]
\end{tikzcd}$$

Let a be $\min(j-1,k)$, by **??**, there exists $c : L_{j-1} \to D_a$ such that $f_a \circ c = m_a$. As before this yields the solid part of the following diagram

$$\begin{tikzcd}
L_{j-1} \arrow[dr, dotted, "t"] \arrow[drr, bend left, "\mathsf{id}_{L_{j-1}}"] \arrow[ddr, bend right, "c"'] & & \\
 & K_{j-1} \arrow[r, "l_{j-1}"] \arrow[d, "k_a"'] & L_{j-1} \arrow[d, "m_a"] \\
 & D_a \arrow[r, "f_a"'] & G_a
\end{tikzcd}$$

The existence of the dotted $t : L_{j-1} \to K_{j-1}$ follows from Proposition 4 and we can conclude. □

Corollary 7. *Let $(\mathbf{X}, R)$ ba left-linear DPO-rewriting system and suppose that every rule in R is consuming. For every two non-empty decorated derivations $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{D}}', \alpha', \omega')$, there exists at most one consistent permutation between them.*

Corollary 8.

prefisso e suffisso

Proof. contenuto... □

4 Shift equivalence and traces

A VERY NICE INTRO

4.1 Sequentially independendent and switchable derivations

A VERY NICE INTRO

Definition 15. *Let $(\mathbf{X}, \mathsf{R})$ be a left-linear DPO-rewriting system with $\mathbf{X}$ an $\mathcal{M}$-adhesive category. Let also $\mathscr{D} : G \Mapsto H$ and $\mathscr{D}' : H \Mapsto T$ be the two direct derivations depicted below.*

$$
\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\
G & D \arrow[l, "f"] \arrow[r, "g"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
L' \arrow[d, "n'"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\
H & D' \arrow[l, "f'"] \arrow[r, "g'"'] & T
\end{tikzcd}
$$

An independence pair *between $\mathscr{D}$ and $\mathscr{D}'$, is a pair of arrows $i_1 : R \to D'$ and $i_2 : L' \to D$ such that the following diagram commutes.*

$$
\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"'] & R \arrow[dr, "h"'] \arrow[drrr, dotted, bend left, "i_1"] & & L' \arrow[dl, "n'"] \arrow[dlll, dotted, bend right, "i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\
G & D \arrow[l, "f"] \arrow[rr, "g"'] & & H & & D' \arrow[ll, "f'"] \arrow[r, "g'"'] & T
\end{tikzcd}
$$

We will say that $\mathscr{D}$ and $\mathscr{D}'$ are weakly sequentially independent *if an independence pair exists. If such independence pair is unique we will say that $\mathscr{D}$ and $\mathscr{D}'$ are* sequentially independent*.*

sequential independence

Example 11.

sequential independence che serva anche per es successivo

Example 12.

Remark 18. Let (i_1, i_2) and (j_1, j_2) be independence pairs for the direct derivations $\mathscr{D}$ and $\mathscr{D}'$. Notice that, by definition, we have

$$
\begin{aligned}
f' \circ i_1 &= h \\
&= f' \circ j_1
\end{aligned}
$$

On the other hand, $f' : D' \to H$ is the pushout of $l' : K' \to L'$ and so it is in $\mathcal{M}$, implying $j_1 = i_1$. If, moreover, we suppose that the rule ρ applied in $\mathscr{D}$ is linear, then $g : D \to H$ is in $\mathcal{M}$ too, hence, from the equation

$$
\begin{aligned}
g \circ i_2 &= h \\
&= g \circ j_2
\end{aligned}
$$

we can deduce that $i_2 = j_2$, too.

Summing up, if $(\mathbf{X}, \mathsf{R})$ is a linear DPO-rewriting system, then sequential independence and weak sequential independence coincide.

When working with linear rewriting systems, (weakly) sequential independent direct derivations can be switched, producing two new (weakly) sequential independent direct derivations between the same objects [14, Thm. 7.7] . This is no more the case if the rules are only left-linear, as shown by the next example.

pensare ad un esempio in cui l'indipendenza non basta

Example 13.

To fix this problem, we adapt the notion of *canonical filler* from [12].

Definition 16. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system with* $\mathbf{X}$ $\mathcal{M}$*-adhesive. Let also* $\mathscr{D} : G \Rrightarrow H$ *and* $\mathscr{D}' : H \Rrightarrow T$ *be the two derivations depicted below.*

$$\begin{tikzcd} L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "h"] \\ G & D \arrow[l, "f"] \arrow[r, "g"'] & H \end{tikzcd} \qquad \begin{tikzcd} L' \arrow[d, "n'"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\ H & D' \arrow[l, "f'"] \arrow[r, "g'"'] & T \end{tikzcd}$$

Since f' *is in* $\mathcal{M}$*, we can moreover consider a pullback square*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "p'"'] & D \arrow[d, "g"] \\ D' \arrow[r, "f'"'] & H \end{tikzcd}$$

A filler *between* $\mathscr{D}$ *and* $\mathscr{D}'$ *is given by a pair of arrows* $u : K \to P$ *and* $u' : K' \to P$ *satisfying the following conditions*

1. $p \circ u = k$*,* $p' \circ u' = k'$ *and there exists a pushout square*

$$\begin{tikzcd} K' \arrow[r, "r'"] \arrow[d, "u'"'] & R' \arrow[d, dotted, "j'"] \\ P \arrow[r, dotted, "q'"'] & Q' \end{tikzcd}$$

2. *there exist arrows* $i_1 : R \to D'$*,* $i_2 : L' \to D$ *satisfying* $f' \circ i_1 = h$*,* $g \circ i_2 = n'$ *and such that the following squares are pushouts*

$$\begin{tikzcd} K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, dotted, "i_1"] \\ P \arrow[r, "p'"'] & D' \end{tikzcd} \qquad \begin{tikzcd} K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, dotted, "i_2"] \\ P \arrow[r, "p"'] & D \end{tikzcd}$$

Remark 19. Let $\mathscr{D}$ and $\mathscr{D}'$ be two switchable direct derivations. Then the existence of a filler allows us to build the solid part of the diagram below.

$$
\begin{tikzcd}
 & & R \arrow[dr, "h"] \arrow[ddrr, bend right, "i_1"] & & L' \arrow[dl, "n'"'] \arrow[ddll, bend left, "i_2"'] & & \\
 & K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dr, "k"] \arrow[ddrr, bend right, "u"] & & H & & K' \arrow[ul, "l'"'] \arrow[dl, "k'"'] \arrow[dr, "r'"] \arrow[ddll, bend left, "u'"'] & \\
L \arrow[dr, "n"'] \arrow[ddrr, dotted, bend right, "j"'] & & D \arrow[ur, "g"] \arrow[dl, "f"'] & & D' \arrow[ul, "f'"'] \arrow[dr, "g'"] & & R' \arrow[dl, "h'"'] \arrow[ddll, bend left, "j'"] \\
 & G & & P \arrow[ul, "p"'] \arrow[ur, "p'"] \arrow[dl, dotted, "q"'] \arrow[dr, "q'"] & & T & \\
 & & Q \arrow[ul, dotted, "s"] & & Q' \arrow[ur, dotted, "t"'] & &
\end{tikzcd}
$$

Let us complete this diagram defining the dotted arrows. We can start noticing that, since $l \in \mathcal{M}$, there exists a pushout square

$$
\begin{tikzcd}
K \arrow[r, "l"] \arrow[d, "u"'] & L \arrow[d, "j"] \\
P \arrow[r, "q"'] & Q
\end{tikzcd}
$$

Moreover, the existence of the wanted $s : Q \to G$ and $t : Q' \to T$ follows from the following equalities

$$
\begin{aligned}
f \circ p \circ u &= f \circ k \qquad & g' \circ p' \circ u' &= g' \circ k' \\
&= n \circ l & &= h' \circ r'
\end{aligned}
$$

We can prove some other properties of the arrows appearing in the diagram above. The three rectangles below are pushouts and their left halves are pushouts too. Therefore, by Lemma 2, also their right halves are pushouts.

$$
\begin{tikzcd}
K \arrow[r, "u"'] \arrow[rr, bend left, "u"] \arrow[d, "r"'] & P \arrow[r, "p"'] \arrow[d, "p'"] & D \arrow[d, "g"] \\
R \arrow[r, "i_1"] \arrow[rr, bend right, "h"'] & D' \arrow[r, "f'"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "u'"'] \arrow[rr, bend left, "k'"] \arrow[d, "l'"'] & P \arrow[r, "p'"'] \arrow[d, "p"] & D' \arrow[d, "f'"] \\
L' \arrow[r, "i_2"] \arrow[rr, bend right, "n'"'] & D \arrow[r, "g"] & H
\end{tikzcd}
$$

$$
\begin{tikzcd}
K \arrow[r, "u"'] \arrow[rr, bend left, "k"] \arrow[d, "l"'] & P \arrow[r, "p"'] \arrow[d, "q"] & D \arrow[d, "f"] \\
L \arrow[r, "j"] \arrow[rr, bend right, "n"'] & Q \arrow[r, "s"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "u'"'] \arrow[rr, bend left, "k'"] \arrow[d, "r'"'] & P \arrow[r, "p'"'] \arrow[d, "q'"] & D' \arrow[d, "g'"] \\
R' \arrow[r, "j'"] \arrow[rr, bend right, "h'"'] & Q' \arrow[r, "t"] & T
\end{tikzcd}
$$

Notice, moreover, that p, q are the pushouts of l' and l, respectively, thus they are elements of $\mathcal{M}$. By Proposition 4 the squares

$$
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "p'"'] & D \arrow[d, "g"] \\
D' \arrow[r, "f'"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "q"'] & D \arrow[d, "f"] \\
Q \arrow[r, "s"'] & G
\end{tikzcd}
$$

are pullbacks too.

Notice that if there is a filler between $\mathscr{D}$ and $\mathscr{D}'$ are switchable, then they are weakly sequentially independent: indeed, if a filler between them exists, then (i_1, i_2) is an independence pair. On the other hand, **??** shows that not every independence pair arises in this way.

Definition 17. *Let $\mathscr{D} : H \Rrightarrow H$, $\mathscr{D}' : H \Rrightarrow T$ be two direct derivations in a left-linear DPO-rewriting system $(\mathbf{X}, \mathsf{R})$. An independence pair (i_1, i_2) between $\mathscr{D}$ and $\mathscr{D}'$ is* good *if there exists a filler (u, u') between them such that the squares below are pushouts.*

$$
\begin{tikzcd}
K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \\
P \arrow[r, "p'"'] & D'
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "i_2"] \\
P \arrow[r, "p"'] & D
\end{tikzcd}
$$

We will say that $\mathscr{D}$ and $\mathscr{D}'$ are switchable *if a good independence pair between them exists. If such a pair is unique, we will say that $\mathscr{D}$ and $\mathscr{D}'$ are* uniquely switchable*. We will use the notation $\mathscr{D} \updownarrow \mathscr{D}'$ to mean that $\mathscr{D}$ and $\mathscr{D}'$ are switchable, while $\mathscr{D} \updownarrow_! \mathscr{D}'$ will denote that they are uniquely so.*

If every independence pair is good we will say that $(\mathbf{X}, \mathsf{R})$ is tame*.*

Remark 20. Given a good independence pair (i_1, i_2) between $\mathscr{D}$ and $\mathscr{D}'$, there is a unique filler such that (u, u') such that

$$
\begin{tikzcd}
K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \\
P \arrow[r, "p'"'] & D'
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "i_2"] \\
P \arrow[r, "p"'] & D
\end{tikzcd}
$$

are pushouts. Indeed, for every other filler (v, v'), it must be that

$$\begin{aligned} p \circ u &= k & \qquad p \circ u' &= i_2 \circ l' \\ &= p \circ v & &= p \circ v' \end{aligned}$$

Since the arrow $p : P \to D$, which is the pullback of f', is in $\mathcal{M}$ we conclude that $(u, u') = (v, v')$.

Remark 21. Clearly in a tame left-linear DPO-rewriting system two direct derivations are sequentially independent if and only if they are unquely switchable.

A source of tame left-linear DPO-rewriting systems is given by the linear ones, as shown by the following proposition.

Proposition 10. *Every linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$ *is tame.*

Proof. Suppose that $\mathbf{X}$ is $\mathcal{M}$-adhesive and let (i_1, i_2) be an independence pair between $\mathscr{D} : G \Mapsto H$ and $\mathscr{D}' : H \Mapsto T$. We have a diagram

$$\begin{tikzcd}
L \arrow[d, "n"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[dr, "h"'] \arrow[drrr, bend left, "i_1"] & & L' \arrow[dl, "n'"] \arrow[dlll, bend right, "i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "k'"] & R' \arrow[d, "h'"] \\
G & D \arrow[l, "f"] \arrow[rr, "g"'] & & H & & D' \arrow[ll, "f'"] \arrow[r, "g'"'] & T
\end{tikzcd}$$

Pulling back g along f', we get another diagram

$$\begin{tikzcd}
& & R \arrow[dr, "h"] \arrow[ddrr, bend right, "i_1"'] & & L' \arrow[dl, "n'"'] \arrow[ddll, bend left, "i_2"] & & \\
& K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dr, "k"] \arrow[ddrr, dotted, bend right, "u"'] & & H & & K' \arrow[ul, "l'"'] \arrow[dl, "k'"'] \arrow[dr, "r'"] \arrow[ddll, dotted, bend left, "u'"] & \\
L \arrow[dr, "n"'] & & D \arrow[ur, "g"] \arrow[dl, "f"'] & & D' \arrow[ul, "f'"'] \arrow[dr, "g'"] & & R' \arrow[dl, "h'"] \\
& G & & P \arrow[ul, "p"'] \arrow[ur, "p'"] & & T &
\end{tikzcd}$$

Now, if we compute we get

$$\begin{aligned} f' \circ i_1 \circ r &= h \circ r & \qquad g \circ i_2 \circ l' &= n' \circ l' \\ &= g \circ k & &= f' \circ k' \end{aligned}$$

Therefore the two dotted arrows $u : K \to P$ and $u' : K' \to P$ exist. We have to show that they satisfy the two conditions in the definition of a filler.

1. By construction $p \circ u = k$ and $p' \circ u' = k'$. Since $(\mathbf{X}, \mathsf{R})$ is linear, then $r' : K' \to R'$ belongs to $\mathcal{M}$, thus it admits a pushout along $u' : K' \to P$, as wanted.

2. Take the following two rectangles

$$\begin{tikzcd}
K \arrow[r, "u"'] \arrow[rr, bend left, "u"] \arrow[d, "r"'] & P \arrow[r, "p"'] \arrow[d, "p'"] & D \arrow[d, "f"] \\
R \arrow[r, "i_1"] \arrow[rr, bend right, "h"'] & D' \arrow[r, "f'"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
K' \arrow[r, "u'"'] \arrow[rr, bend left, "k'"] \arrow[d, "l'"'] & P \arrow[r, "p'"'] \arrow[d, "p"] & D' \arrow[d, "f'"] \\
L' \arrow[r, "i_2"] \arrow[rr, bend right, "n'"'] & D \arrow[r, "g"] & H
\end{tikzcd}$$

By hypothesis r and l' are in $\mathcal{M}$, thus f' and g belong to it too. The first point of Lemma 4 yields the thesis. $\square$

Remark 22. [3] identifies a large class of (quasi)adhesive categories with the property that every left linear DPO-rewriting system on them is tame. We adapt these results to our context in Appendix A.

We are now going to justify the choice of the name for the relation $\updownarrow$, showing that two switchable direct derivations $\mathscr{D}$ and $\mathscr{D}'$ can be actually switched.

Let (i_1, i_2) be a good independence pair and consider the following diagram: the solid part exists by the definition of a filler, while the two new dotted arrows $v : Q \to J$ and $v' : Q' \to J$ are obtained as the pushout of $q : P \to Q$, which is in $\mathcal{M}$ by Remark 14, along $q' : P \to Q'$.

$$\begin{tikzcd}
 & & R \arrow[dr, "h"] \arrow[drr, bend right, "i_1"] & & L' \arrow[dl, "n'"'] \arrow[dll, bend left, "i_2"'] & & \\
 & K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dr, "k"] \arrow[drr, bend right, "u"'] & & H & & K' \arrow[ul, "l'"'] \arrow[dl, "k'"'] \arrow[dr, "r'"] \arrow[dll, bend left, "u'"] & \\
L \arrow[dr, "n"] \arrow[drr, bend right, "j"'] & & D \arrow[ur, "g"] \arrow[dl, "f"'] & & D' \arrow[ul, "f'"'] \arrow[dr, "g'"] & & R' \arrow[dl, "h'"'] \arrow[dll, bend left, "j'"] \\
 & G & & P \arrow[ul, "p"'] \arrow[ur, "p'"] \arrow[dl, "q"'] \arrow[dr, "q'"] & & T & \\
 & & Q \arrow[ul, "s"] \arrow[dr, dotted, "v"'] & & Q' \arrow[ur, "t"] \arrow[dl, dotted, "v'"] & & \\
 & & & J & & &
\end{tikzcd}$$

Since, by Remark 14, all the curved rectangles are pushouts, as well as the bottom square, we can state the following definition.

Definition 18. *Let $(\mathbf{X}, R)$ be a left-linear DPO-rewriting system and suppose that $\mathbf{X}$ is $\mathcal{M}$-adhesive. Given a good independence pair (i_1, i_2) between $\mathscr{D} : G \Mapsto$*

H and $\mathscr{D}'$: $H \Rrightarrow T$, if (u, u') is the associate filler, we define other two direct derivations $S_{i_1,i_2}(\mathscr{D}') : G \Rrightarrow J$ and $S_{i_1,i_2}(\mathscr{D}) : J \Rrightarrow T$ as follows:

$$\begin{tikzcd} L' \arrow[d, "f\circ i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "q\circ u'"'] & R' \arrow[d, "v'\circ j'"'] \\ G & Q \arrow[l, "s"] \arrow[r, "v"'] & J \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, "v\circ j"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "q'\circ u"] & R \arrow[d, "g'\circ i_1"] \\ J & Q' \arrow[l, "v'"] \arrow[r, "t"'] & T \end{tikzcd}$$

The switching $S_{i_1,i_2}(\mathscr{D}, \mathscr{D}')$ *of $\mathscr{D}$ and $\mathscr{D}'$ is the derivation $S_{i_1,i_2}(\mathscr{D}') \cdot S_{i_1,i_2}(\mathscr{D})$.*

Remark 23. Let (i_1, i_2) be a good independence pair between the derivation $\mathscr{D} : G \Rrightarrow H$ and $\mathscr{D}' : H \Rrightarrow T$. Let also $\hat{\underline{\mathscr{D}}}$ be the derivation consisting only in $S_{i_1,i_2}(\mathscr{D})$. We have a diagram

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & Q \arrow[r, "v"] \arrow[d, "s"] & J \arrow[r, "\hat{\iota}_J"] & \llbracket \hat{\underline{\mathscr{D}}} \rrbracket \arrow[d, "\hat{p}"] \\ D \arrow[r, "f"'] & G \arrow[rr, "\iota_G"'] & & \llbracket S_{i_1,i_2}(\mathscr{D}, \mathscr{D}') \rrbracket \end{tikzcd}$$

By **??** and remark 14, its two halves are pullbacks and so, by Lemma 1 the whole rectangle is a pullback too. Moreover, notice that

$$\begin{aligned} \hat{\iota}_J \circ v \circ q &= \hat{\iota}_J \circ v' \circ q' \\ &= \hat{\iota}_{Q'} \circ q' \end{aligned}$$

which entails that the rectangle below is a pullback too.

$$\begin{tikzcd} P \arrow[r, "q'"] \arrow[d, "p"'] & Q' \arrow[rr, "\hat{\iota}_{Q'}"] & & \llbracket \hat{\underline{\mathscr{D}}} \rrbracket \arrow[d, "\hat{p}"] \\ D \arrow[r, "f"'] & G \arrow[rr, "\iota_G"'] & & \llbracket S_{i_1,i_2}(\mathscr{D}, \mathscr{D}') \rrbracket \end{tikzcd}$$

Remark 24. Notice that (j', j) is an independence pair for $S_{i_1,i_2}(\mathscr{D}')$ and $S_{i_1,i_2}(\mathscr{D})$. This is witnessed by the following diagram, commutative by construction.

$$\begin{tikzcd} L' \arrow[d, "f\circ i_2"'] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "q\circ u'"'] & R' \arrow[dr, "v'\circ j'"'] \arrow[drrr, bend left, "j"] & & L \arrow[dl, "v\circ j"] \arrow[dlll, bend right, "j'"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "q'\circ u"] & R \arrow[d, "g'\circ i_1"] \\ G & Q \arrow[l, "s"] \arrow[rr, "v"'] & & J & & Q' \arrow[ll, "v'"] \arrow[r, "t"'] & T \end{tikzcd}$$

Proposition 11. *Let $(\mathbf{X}, \mathsf{R})$ be a left-linear DPO-rewriting system with $\mathbf{X}$ an $\mathcal{M}$-adhesive category. Then every filler between two direct derivations $\mathscr{D} : G \Rrightarrow H$ and $\mathscr{D}' : H \Rrightarrow T$ is a filler also for $S_{u,u'}(\mathscr{D}') : G \Rrightarrow J$ and $S_{u,u'}(\mathscr{D}) : J \Rrightarrow H$. In particular, if $\mathscr{D} \updownarrow \mathscr{D}'$, then $S_{u,u'}(\mathscr{D}') \updownarrow S_{u,u'}(\mathscr{D})$.*

questo è Church-Rosser

Remark 25.

Proof. By definition of filler, we have two pushout square

$$\begin{tikzcd} K \arrow[r, "l"] \arrow[d, "u"'] & L \arrow[d, "j"] \\ P \arrow[r, "q"'] & Q \end{tikzcd} \qquad \begin{tikzcd} K' \arrow[r, "r'"] \arrow[d, "u'"'] & R' \arrow[d, "j'"] \\ P \arrow[r, "q'"'] & Q' \end{tikzcd}$$

In particular, q is an arrow of $\mathcal{M}$, therefore, by Proposition 4, the square below is a pullback.

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "q'"'] & Q \arrow[d, "v"] \\ Q' \arrow[r, "v'"'] & J \end{tikzcd}$$

To prove our claim, it is now enough to show that (u', u) is a filler between $S_{u,u'}(\mathscr{D}')$ and $S_{u,u'}(\mathscr{D})$.

1. As for the first point of Definition 12, the only non obvious part is the existence of a pushout of u along r. But, since (u, u') is a filler between $\mathscr{D}$ and $\mathscr{D}'$, we know that such a pushout exists: it is enough to take the square

$$\begin{tikzcd} K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \\ P \arrow[r, "p'"'] & D' \end{tikzcd}$$

2. For the second point, notice that the arrows j and j' fit in the squares

$$\begin{tikzcd} K' \arrow[r, "r'"] \arrow[d, "u'"'] & R' \arrow[d, "j'"] \\ P \arrow[r, "q"'] & Q \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "l"] \arrow[d, "u"'] & L \arrow[d, "j"] \\ P \arrow[r, "q'"'] & Q' \end{tikzcd}$$

By Remark 16 we know that (j, j') is an independence pair. The results above now implies that (j, j') is good.

The previous remark allow us to further switch the direct derivation $S_{i_1,i_2}(\mathscr{D})$ and $S_{i_2,i_2}(\mathscr{D}')$. The following lemma guarantees us that, in this way, we get back a derivation which is abstraction equivalent to $\mathscr{D} \cdot \mathscr{D}'$.

introdurre decorazione

Lemma 8. *Let $\mathscr{D} : G \Rrightarrow H$ and $\mathscr{D}' : H \Rrightarrow T$ be two direct derivations in a left-linear DPO-rewriting system $(\mathbf{X}, \mathsf{R})$ and let (i_1, i_2) be a good independence pair between them. Then $S_{j',j}(S_{i_1,i_2}(\mathscr{D}, \mathscr{D}'))$ is abstraction equivalent to $\mathscr{D} \cdot \mathscr{D}'$.*

Proof. Let (u, u') be the filler associated to (i_1, i_2). By Proposition 8, (u', u) is a filler between $S_{i_1,i_2}(\mathscr{D}')$ and $S_{i_1,i_2}(\mathscr{D})$. Thus we have a diagram as the one below.

$$\begin{tikzcd}
 & & R' \arrow[dr, "v'\circ j'"] \arrow[ddrr, bend right, "j'"'] & & L \arrow[dl, "v\circ j"'] \arrow[ddll, bend left, "j"] & & \\
 & K' \arrow[ur, "r'"] \arrow[dl, "l'"'] \arrow[dr, "q\circ u'"] \arrow[ddrr, bend right, "u'"'] & & J & & K \arrow[ul, "l"'] \arrow[dr, "r"] \arrow[dl, "q'\circ u"'] \arrow[ddll, bend left, "u"] & \\
L' \arrow[dr, "f\circ i_2"] \arrow[ddrr, bend right, "c_2"'] & & Q \arrow[ur, "v"] \arrow[dl, "s"'] & & Q' \arrow[ul, "v'"'] \arrow[dr, "t"] & & R \arrow[dl, "g'\circ i_1"'] \arrow[ddll, bend left, "c_1"] \\
 & G & & P \arrow[ul, "q"'] \arrow[ur, "q'"] \arrow[dl, "e'"'] \arrow[dr, "e"] & & T & \\
 & & E' \arrow[ul, "a"] \arrow[dr, "w'"'] & & E \arrow[ur, "b"'] \arrow[dl, "w"] & & \\
 & & & F & & &
\end{tikzcd}$$

Now, to ease the notation, let $S_{j',j}(S_{i_1,i_2}(\mathscr{D}, \mathscr{D}'))$ be $\mathscr{E}_0 \cdot \mathscr{E}_1$, then $\mathscr{E}_0$ and $\mathscr{E}_1$ are the direct derivations given by the diagrams

$$\begin{tikzcd}
L \arrow[d, "s\circ j"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "e'\circ u"'] & R \arrow[d, "w\circ c_1"'] \\
G & E' \arrow[l, "a"] \arrow[r, "w'"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
L' \arrow[d, "w'\circ c_2"] & K' \arrow[l, "l'"'] \arrow[r, "r'"] \arrow[d, "e\circ u'"] & R' \arrow[d, "t\circ j'"] \\
F & E \arrow[l, "w"] \arrow[r, "b"'] & R
\end{tikzcd}$$

Notice, moreover, that, since the squares

$$\begin{tikzcd}
K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "c_2"] \\
P \arrow[r, "e'"'] & E'
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "c_1"] \\
P \arrow[r, "e"'] & E
\end{tikzcd}$$

are pushouts, we have isomorphisms $\phi' : D \to E'$, $\phi : D' \to E$ making the following diagrams commutative.

$$\begin{tikzcd}
K' \arrow[r, "l'"] \arrow[d, "u'"'] & L' \arrow[d, "i_2"] \arrow[dr, bend left, "c_2"] & \\
P \arrow[r, "p"] \arrow[rr, bend right, "e'"'] & D \arrow[r, dotted, "\phi'"] & E'
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "r"] \arrow[d, "u"'] & R \arrow[d, "i_1"] \arrow[dr, bend left, "c_1"] & \\
P \arrow[r, "p'"] \arrow[rr, bend right, "e"'] & D' \arrow[r, dotted, "\phi"] & E
\end{tikzcd}$$

In particular, we have

$$
\begin{aligned}
a \circ \phi' \circ i_2 &= a \circ c_2 \\
&= f \circ i_2
\end{aligned}
\qquad
\begin{aligned}
a \circ \phi' \circ p' &= a \circ e' \\
&= s \circ q \\
&= f \circ p
\end{aligned}
\qquad
\begin{aligned}
b \circ \phi \circ i_1 &= b \circ c_1 \\
&= g' \circ i_1
\end{aligned}
\qquad
\begin{aligned}
b \circ \phi \circ p &= b \circ e \\
&= t \circ q' \\
&= g' \circ p'
\end{aligned}
$$

and this shows that

$$a \circ \phi' = f \qquad b \circ \phi = g'$$

Now, since ϕ' is an isomorphism and by Proposition 4, the two halves of the rectangle

$$
\begin{tikzcd}
K \arrow[r, "\mathsf{id}_K"] \arrow[d, "(\phi')^{-1} \circ e' \circ u"'] & K \arrow[r, "l"] \arrow[d, "e' \circ u"'] & L \arrow[d, "s \circ j"] \\
D \arrow[r, "\phi'"'] & E' \arrow[r, "a"'] & G
\end{tikzcd}
$$

are pullbacks. Thus the whole diagram is a pullback. But, by construction $s \circ j = n$ and we have already proved that $a \circ \phi' = f$. We then conclude that ther exists an isomorphism $\zeta : K \to K$ which makes the diagram below commutative

$$
\begin{tikzcd}
K \arrow[r, dotted, "\zeta"'] \arrow[rr, bend left, "l"] \arrow[dr, bend right, "(\phi')^{-1} \circ e' \circ u"'] & K \arrow[r, "l"'] \arrow[d, "n"'] & L \arrow[d, "n"] \\
 & D \arrow[r, "f"'] & G
\end{tikzcd}
$$

The commutativity of the upper triangle entails $l \circ \zeta = l$. Since l is an element of $\mathcal{M}$ we can deduce that. $\zeta = \mathsf{id}_K$. From this, we conclude that

$$e' \circ u = \phi' \circ k$$

As a next step, notice the existence of ϕ and ϕ', together with Remark 14, entails the existence of a third isomorphism $\psi : H \to F$ fitting in the diagram below.

$$
\begin{tikzcd}
P \arrow[r, "p"'] \arrow[rr, bend left, "e'"] \arrow[d, "p'"] \arrow[dd, bend right, "e"'] & D \arrow[r, "\phi'"'] \arrow[d, "g"] & E' \arrow[d, "w'"] \\
D' \arrow[r, "f'"'] \arrow[d, "\phi"] & H \arrow[r, dotted, "\psi"] & F \\
E \arrow[urr, bend right, "w"'] & &
\end{tikzcd}
$$

Now, if we compute, we get

$$
\begin{aligned}
\psi^{-1} \circ w \circ c_1 &= f' \circ \phi^{-1} \circ c_1 \\
&= f' \circ i_1 \\
&= h
\end{aligned}
$$

Summing up, we have just build the diagram below.

$$
\begin{tikzcd}
 & & & R \arrow[dddl, "h"'] \arrow[dddr, "w\circ c_1"] & \\
 & & K \arrow[ur, "r"] \arrow[dl, "l"'] \arrow[dddl, "k"'] \arrow[dddr, "e'\circ u"] & & \\
 & L \arrow[dddl, "n"'] \arrow[dddr, "s\circ j"] & & & \\
 & & H \arrow[rr, "\psi"] & & F \\
 & D \arrow[ur, "g"'] \arrow[rr, "\phi'"] \arrow[dl, "f"'] & & E' \arrow[ur, "w'"'] \arrow[dl, "a"] & \\
G \arrow[rr, "\mathsf{id}_G"'] & & G & &
\end{tikzcd}
$$

Next, we already know that

$$t \circ j' = h' \quad w \circ \phi = \psi \circ f' \quad b \circ \phi = g'$$

If we compute further, we also get

$$
\begin{aligned}
\psi^{-1} \circ w' \circ c_2 &= g \circ (\phi')^{-1} \circ c_2 & \qquad \phi^{-1} \circ e \circ u' &= p' \circ u' \\
&= g \circ i_2 & &= k' \\
&= n' & &
\end{aligned}
$$

These equations allow us to conclude that the following diagram commutes.

$$
\begin{tikzcd}
 & & & R' \arrow[dddl, "h'"'] \arrow[dddr, "t\circ j'"] & \\
 & & K' \arrow[ur, "r"] \arrow[dl, "l'"'] \arrow[dddl, "k"'] \arrow[dddr, "e\circ u'"] & & \\
 & L' \arrow[dddl, "n'"'] \arrow[dddr, "w'\circ c_2"] & & & \\
 & & T \arrow[rr, "\mathsf{id}_T"] & & T \\
 & D' \arrow[ur, "g'"'] \arrow[rr, "\phi"] \arrow[dl, "f'"'] & & E \arrow[ur, "b"'] \arrow[dl, "w"] & \\
H \arrow[rr, "\psi"'] & & F & &
\end{tikzcd}
$$

Putting together the two diagrams above we get the thesis. $\square$

Our next step is to relate derivations which are equal "up to switching".

Definition 19. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system. Given two direct derivations* $\mathscr{D} : G \Mapsto H$ *and* $\mathscr{D}' : H \Mapsto T$*, we say that* $\mathscr{D}$ *and* $\mathscr{D}'$ *are* properly switchable *if* $\mathscr{D} \updownarrow_! \mathscr{D}'$ *and* $S_{i_1,i_2}(\mathscr{D}') \updownarrow_! S_{i_1,i_2}(\mathscr{D})$*, where* (i_2, i_2) *is a good independencepair betwwen* $\mathscr{D}$ *and* $\mathscr{D}'$*. In such a case, we will write* $\mathscr{D} \Updownarrow \mathscr{D}'$*.*

Take two derivations $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^n$ *and* $\underline{\mathscr{D}}' = \{\mathscr{D}'_i\}_{i=0}^n$ *with the same length and between the same* G_0 *and* G_n*. We say that* $\underline{\mathscr{D}}'$ *is* obtained by a proper switch from $\underline{\mathscr{D}}$ *if there exists an index* $j < n$ *such that*

1. *for every* $i \notin \{j, j+1\}$, $\mathscr{D}_i = \mathscr{D}'_i$;
2. $\mathscr{D}_j \Updownarrow \mathscr{D}_{j+1}$;
3. $\mathscr{D}'_j \cdot \mathscr{D}'_{j+1} = S_{i_1,i_2}(\mathscr{D}, \mathscr{D}')$.

In such a case, we will write $\underline{\mathscr{D}} \rightsquigarrow_j \underline{\mathscr{D}}'$ *to denote that* $\underline{\mathscr{D}}'$ *is obtained by a proper switch between* $\mathscr{D}_j$ *and* $\mathscr{D}_{j+1}$.

We will say that $\underline{\mathscr{D}}$ *is* switch equivalent *to* $\underline{\mathscr{D}}'$, *if there exists a sequence,* $\{\underline{\mathscr{D}}_i\}_{i=0}^{n}$ *of derivations such that*

1. $\underline{\mathscr{D}}_0 = \underline{\mathscr{D}}$ *and* $\underline{\mathscr{D}}_n = \underline{\mathscr{D}}'$;
2. *for every* $i < n$, $\underline{\mathscr{D}}_{i+1}$ *is obtained by a proper switch from* $\underline{\mathscr{D}}_i$.

We will write $\underline{\mathscr{D}} \equiv^s \underline{\mathscr{D}}'$ *to denote that* $\underline{\mathscr{D}}$ *is switch equivalent to* $\underline{\mathscr{D}}'$.

Example 14.

il punto due sopra è necessario

We can now prove some properties of switch equivalence.

Def. 3 della bozza di Andrea

Lemma 9. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system. Then the following hold true:*

1.
2.
3.
4.

Proof. 1.
2.
3.
4. $\square$

Example 15.

esempio sul perché weakly independence non è invertibile

Lemma 10. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$. *Then the following hold true*

1. *If* $\mathscr{D}$ *and* $\mathscr{D}'$ *are two direct derivations such that* $\mathscr{D} \updownarrow \mathscr{D}'$, *then for every filler* (u, u') *between them, the function*
$$\tau : 2 \to 2 \qquad x \mapsto \begin{cases} 1 & x = 0 \\ 0 & x = 1 \end{cases}$$
defines a consistent permutation between $\mathscr{D} \cdot \mathscr{D}'$ *and* $S_{i_1,i_2}(\mathscr{D}, \mathscr{D}')$;
2. *if* $\underline{\mathscr{D}}$ *and* $\underline{\mathscr{D}}'$ *are two switch equivalent derivation, then there exists a consistent permutation between them.*

Proof. 1.
2. $\square$

This, together with ???

Corollary 9.

unicità

Example 16.

permutazione consistente non implica scambiabilità

4.2 Proper switchability is global

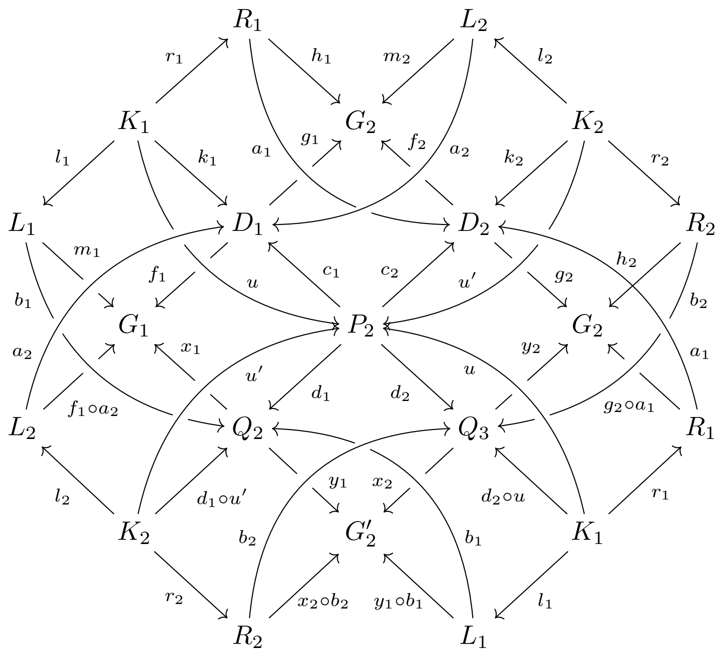
intro

Lemma 11. *Let* $(\mathbf{X}, \mathsf{R})$ *be a left-linear DPO-rewriting system with* $\mathbf{X}$ *an* $\mathcal{M}$*-adhesive category. Consider a derivation* $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^{2}$*. If* (i_0, i_1) *is a good independence pair between* $\mathscr{D}_0$ *and* $\mathscr{D}_1$*,* (a_1, a_2) *one between* $\mathscr{D}_1$ *and* $\mathscr{D}_2$ *and* $\mathscr{D}_0 \updownarrow S_{a_1,a_2}(\mathscr{D}_2)$ *with good independence pair* (e_0, e_1)*, then* $S_{e_0,e_1}(\mathscr{D}_0)$ *and* $S_{a_1,a_2}(\mathscr{D}_1)$ *are weakly sequentially independent.*

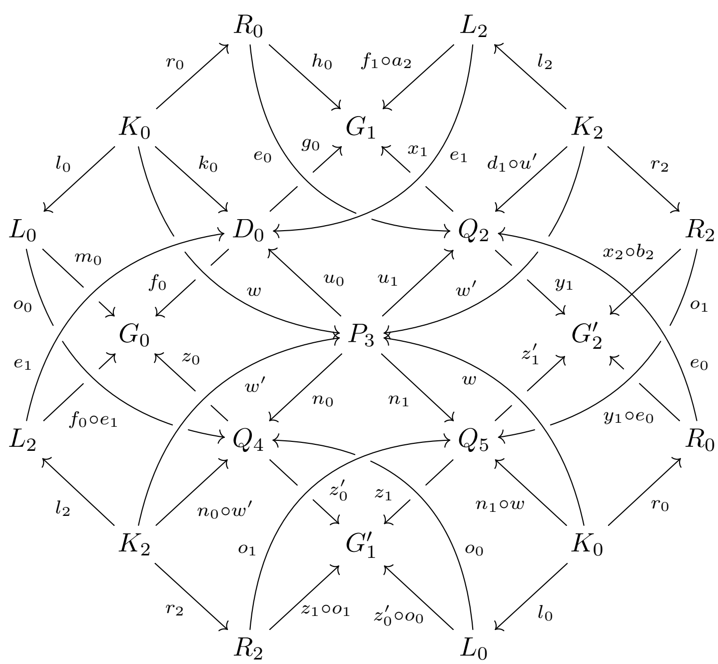
Proof. We can use Definition 12 and **??** to get some diagrams. First of all, let (v, v') be the filler between $\mathscr{D}_0$ and $\mathscr{D}_1$ associated to (i_1, i_2), then we have

$$\begin{tikzcd}
 & & R_0 \arrow[dr, "h_0"] \arrow[ddrr, bend right, "i_0"] & & L_1 \arrow[dl, "m_1"'] \arrow[ddll, bend left, "i_1"'] & & \\
 & K_0 \arrow[ur, "r_0"] \arrow[dl, "l_0"'] \arrow[dr, "k_0"] \arrow[ddrr, bend right, "v"] & & G_1 & & K_1 \arrow[ul, "l_1"'] \arrow[dr, "r_1"] \arrow[dl, "k_1"'] \arrow[ddll, bend left, "v'"] & \\
L_0 \arrow[dr, "m_0"] \arrow[ddrr, bend right, "j_0"] & & D_0 \arrow[ur, "g_0"] \arrow[dl, "f_0"'] & & D_1 \arrow[ul, "f_1"'] \arrow[dr, "g_1"] & & R_1 \arrow[dl, "h_1"'] \arrow[ddll, bend left, "j_1"'] \\
 & G_0 & & P_1 \arrow[ul, "p_0"'] \arrow[ur, "p_1"] \arrow[dl, "q_0"] \arrow[dr, "q_1"'] & & G_2 & \\
L_1 \arrow[ur, "f_0\circ i_1"'] \arrow[uurr, bend left, "i_1"] & & Q_0 \arrow[ul, "s_0"'] \arrow[dr, "t_0"] & & Q_1 \arrow[ur, "t_1"] \arrow[dl, "s_1"'] & & R_0 \arrow[ul, "g_1\circ i_0"] \arrow[uull, bend right, "i_0"'] \\
 & K_1 \arrow[ul, "l_1"] \arrow[ur, "q_0\circ v'"'] \arrow[dr, "r_1"'] \arrow[uurr, bend left, "v'"] & & G'_1 & & K_0 \arrow[ul, "q_1\circ v"] \arrow[ur, "r_0"'] \arrow[dl, "l_0"] \arrow[uull, bend right, "v"'] & \\
 & & R_1 \arrow[ur, "s_1\circ j_1"'] \arrow[uurr, bend left, "j_1"] & & L_0 \arrow[ul, "t_0\circ j_0"] \arrow[uull, bend right, "j_0"'] & &
\end{tikzcd}$$

Secondly, the filler (u, u') induced by (a_1, a_2) between $\mathscr{D}_1$ and $\mathscr{D}_2$ yields:



Finally, the filler (w, w') between $\mathscr{D}_0$ and $S_{u,u'}(\mathscr{D}_2)$ given by (e_0, e_1) provides us with:



We have to construct the two dotted arrows in the diagram below.

$$\begin{tikzcd}
L. \arrow[d, "z_0' \circ o_0"'] & K_0 \arrow[l, "l_0"'] \arrow[r, "r_0"] \arrow[d, "n_1 \circ w"'] & R_0 \arrow[dr, "y_1 \circ e_0"'] \arrow[drrr, dotted, "\alpha_1"] & & L_1 \arrow[dl, "y_1 \circ b_1"] \arrow[dlll, dotted, "\alpha_2"'] & K_1 \arrow[l, "l_1"'] \arrow[r, "r_1"] \arrow[d, "d_2 \circ u"] & R_1 \arrow[d, "g_2 \circ a_1"] \\
G_1' & Q_5 \arrow[l, "z_1"] \arrow[rr, "z_1'"'] & & G_2' & & Q_3 \arrow[ll, "x_2"] \arrow[r, "y_2"'] & G_2
\end{tikzcd}$$

Consider the arrows $i_0 : R_0 \to D_1$ and $e_0 : R_0 \to Q_2$. An easy computation shows that

$$\begin{aligned} f_1 \circ i_0 &= h_0 \\ &= x_1 \circ e_0 \end{aligned}$$

entailing the existence of the dotted $\alpha_1' : R_0 \to P_2$ in the diagram

$$\begin{tikzcd}
R_0 \arrow[dr, dotted, "\alpha_1'"] \arrow[drr, bend left, "i_0"] \arrow[ddr, bend right, "e_0"'] & & \\
& P_2 \arrow[r, "c_2"] \arrow[d, "d_1"'] & D_1 \arrow[d, "f_1"] \\
& Q_2 \arrow[r, "x_1"'] & G_1
\end{tikzcd}$$

If we define $\alpha_1 : R_0 \to Q_3$ as $d_2 \circ \alpha_1'$, then we easily get that

$$\begin{aligned} x_2 \circ \alpha_1 &= x_2 \circ d_2 \circ \alpha_1' \\ &= y_1 \circ d_1 \circ \alpha_1' \\ &= y_1 \circ e_0 \end{aligned}$$

For α_2, we proceed similarly. First consider $i_1 : L_1 \to D_0$ and $b_1 : L_1 \to Q_2$ and notice that

$$\begin{aligned} g_0 \circ i_1 &= m_1 \\ &= x_1 \circ b_1 \end{aligned}$$

implying the existence of $\alpha_2' : L_1 \to P_3$ fitting in the diagram below.

$$\begin{tikzcd}
L_1 \arrow[dr, dotted, "\alpha_2'"] \arrow[drr, bend left, "i_1"] \arrow[ddr, bend right, "b_1"'] & & \\
& P_3 \arrow[r, "u_0"] \arrow[d, "u_1"'] & D_0 \arrow[d, "g_0"] \\
& Q_2 \arrow[r, "x_1"'] & G_1
\end{tikzcd}$$

Let $\alpha_2 : L_1 \to Q_5$ be $n_1 \circ \alpha_2'$, then

$$\begin{aligned} z_1' \circ \alpha_2 &= z_1' \circ n_1 \circ \alpha_2' \\ &= y_1 \circ u_1 \circ \alpha_2' \\ &= y_1 \circ b_1 \end{aligned}$$

The thesis now follows. $\square$

Corollary 10. *Given a tame left-linear DPO-rewriting system* $(\mathbf{X}, \mathsf{R})$ *with* $\mathbf{X}$ *an* $\mathcal{M}$*-adhesive category and a decorated derivation* $(\underline{\mathscr{D}}, \alpha, \omega)$ *with* $\underline{\mathscr{D}} = \{\mathscr{D}_i\}_{i=0}^2$*. Then the following are true:*

1. *suppose that* (a_1, a_2) *and* (e_0, e_1) *are independence pairs witnessing* $\mathscr{D}_1 \updownarrow_! \mathscr{D}_2$ *and* $\mathscr{D}_0 \updownarrow_! S_{a_1,a_2}(\mathscr{D}_2)$ *respectively, if* $\mathscr{D}_0 \updownarrow_! \mathscr{D}_1$*, then* $S_{e_0,e_1}(\mathscr{D}_0) \updownarrow_! S_{a_2,a_2}(\mathscr{D}_1)$*;*
2.

ripulire le ipotesi: serve scambiabilità propria tra 1 e 2

Proof. 1. Let (i_1, i_2) be the unique independence pair between $\mathscr{D}_0$ and $\mathscr{D}_1$. By tameness and **??** we know that $S_{e_0,e_1}(\mathscr{D}_0) \updownarrow S_{a_1,a_2}(\mathscr{D}_1)$. Let (α_1, α_2) be the independence pair constructed in the proof of **??**. Take another independence pair (β_1, β_2), by Remark 11 we already know that $\beta_1 = \alpha_1$. Let also $\underline{\mathscr{E}}$ be the derivation $S_{e_0,e_1}(S_{a_1,a_2}(\underline{\mathscr{D}}_2)) \cdot S_{e_0,e_1}(\mathscr{D}_0) \cdot S_{a_1,a_2}(\underline{\mathscr{D}}_1)$. Let $\sigma : [0,2] \to [0,2]$ be the cycle $(0,1,2)$. By **??** we know that σ is a consistent permutation between $(\underline{\mathscr{D}}, \alpha, \omega)$ and $(\underline{\mathscr{E}}, \alpha, \omega)$. In particular, we have

sistemare il fatto che qua utilizziamo derivazioni di 3 passi

$$\begin{aligned} \iota_{\underline{\mathscr{E}},G_0} \circ f_0 \circ i_1 &= \xi_\sigma \circ \iota_{\underline{\mathscr{D}},G_0} \circ f_0 \circ i_1 \\ &= \xi_\sigma \circ \iota_{\underline{\mathscr{D}},D_0} \circ i_1 \\ &= \xi_\sigma \circ \iota_{\underline{\mathscr{D}},G_1} \circ g_0 \circ i_1 \\ &= \xi_\sigma \circ \iota_{\underline{\mathscr{D}},G_1} \circ m_1 \\ &= \iota_{\underline{\mathscr{E}},G_2'} \circ y_1 \circ b_1 \\ &= \iota_{\underline{\mathscr{E}},G_2'} \circ z_1' \circ \beta_2 \\ &= \iota_{\underline{\mathscr{E}},Q_5} \circ \beta_2 \end{aligned}$$

Using **??** we can deduce the existence of a unique arrow $\beta_2' : L_1 \to P_3$ fitting in the diagram below

$$\begin{tikzcd} L_1 \arrow[dr, dotted, "\beta_2'"] \arrow[drr, bend left, "\beta_2"] \arrow[ddr, bend right, "i_1"'] & & \\ & P_3 \arrow[r, "n_1"] \arrow[d, "u_0"'] & Q_5 \arrow[d, "\iota_{\mathscr{E},Q_5}"] \\ & D_0 \arrow[r, "\iota_{\mathscr{E},G_0}"'] & \langle\!\langle \underline{\mathscr{E}} \rangle\!\rangle \end{tikzcd}$$

If we further compute, we get

$$\begin{aligned} y_1 \circ u_1 \circ \beta_2' &= z_1' \circ n_1 \circ \beta_2' = \\ &= z_1' \circ \beta_2 \\ &= y_1 \circ b_1 \end{aligned}$$

Thus $(u_1 \circ \beta_2', b_2)$ is an independence pair between $S_{a_1,a_2}(\underline{\mathscr{D}}_2)$ and $S_{a_1,a_2}(\underline{\mathscr{D}}_1)$ therefore, by hypothesis $u_1 \circ \beta_2' = b_1$. But this implies that β_2' and α_2' are equal, giving us the thesis.

ripulire le ipotesi: serve scambiabilità propria tra 1 e 2

2. □

sistemare il fatto che qua utilizziamo derivazioni di 3 passi

We need another lemma, similar to **??**.

nell'altro senso

Lemma 12.

fare

Proof. □

passare a $\updownarrow_!$ e $\Updownarrow$

Corollary 11.

Corollary 12 (Invariance under immediate shift). *contenuto...*

ricopiare tutta l'induzione di Andrea

4.3 Concatenable traces

abstract equivalence and switch

Lemma 13.

Proof. contenuto...

tracce

Definition 20.

Before moving forward, we will prove some other useful properties of the switch equivalence relation.

lemma 19

Lemma 14.

Proof. contenuto...

preordine

Theorem 3.

Proof. contenuto...

5 Domains for DPO-rewriting

5.1 weak domains

5.2 From adhesive grammars to weak domains

6 Conclusions and further work

VERY NICE CONCLUSIONS

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A On permutations

inserire risutati sulle permutazione che abbiamo usato

B A note on fillers and sequential independence

In Proposition 7 we proved that, in the linear case, the existence of an indpendence pair between two derivation is equivalent to that of a filler between them. This result can be further refined: in a [3] a class $\mathbb{B}$ of (quasi)adhesive category is defined for which the local Church-Rosser Theorem holds even for left-linear DPO-rewriting system. In our language, and given Proposition 8 and Remark 17, this amount to prove that, for elements of $\mathbb{B}$, every independence pair induces a filler.

Definition 21. *Let* $\mathbf{X}$ *be a category, we say that* $\mathbf{X}$ *satisfies*

– *the* mixed decomposition *property if for every diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\ A \arrow[r, "h"'] & B \arrow[r, "k"'] & C \end{tikzcd}$$

whose outer boundary is a pushout and in which k *is a monomorphisms,*

– *the* pushout decomposition *property*

Lemma 15. *contenuto...*

Proof. contenuto... □

Corollary 13. *contenuto...*

The following result shows that the mixed and pushout decomposition properties guarantee that every independence pair gives rise to a filler.

filler e classe B+

Theorem 4.
Proof. □

Our next step is to identify sufficient conditions for a category $\mathbf{X}$ to satisfy the mixed and pushout decomposition properties.

classe B e class B+

Definition 22.

esempi

Example 17.

esempi

Example 18.

Proposition 12.

da B a B+

Proof. contenuto... □

Lemma 16.

due proprietà classe B

Proof. □

Corollary 14.

due proprietà classe B+

Corollary 15.

filler e classe B+