

# A taxonomy of categories for relations\*

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## Abstract

The study of categories abstracting the structural properties of relations has been extensively developed over the years, resulting in a rich and diverse body of work. This paper strives to provide a modern and comprehensive presentation of these “categories for relations”, including their enriched version, further showing how they arise as Kleisli categories of suitable symmetric monoidal monads. The resulting taxonomy aims at bringing clarity and organisation to the numerous related concepts and frameworks occurring in the literature.

## 1 Introduction

Category theory, from its very beginnings, was conceived as an abstraction of the notions of set and function. This intuition is clearly expressed in the first sentence of the introduction of the well-known book by Mac Lane [?] ]

Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows. Each arrow  $f : X \rightarrow Y$  represents a function; that is, a set  $X$ , a set  $Y$ , and a rule  $x \mapsto f(x)$  which assigns to each element  $x \in X$  an element  $f(x) \in Y$ .

Almost the whole of category theory focussed for years on the paradigm that the maps are the counterpart in the category at hand of total functions, and essential use of this was made in forming the definitions.

Late 1980s witnessed the emergence of a new perspective, aimed at exploring the relational aspects of algebra and logic. During this period, three fundamental works were presented, shaping the development of categories aimed at abstracting the properties of relations: the paper on *cartesian bicategories*

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by Carboni and Walters [?] and the one on *p-categories* by Robinson and Rosolini [?], and the book on *allegories* by Freyd and Scedrov [?].

The relevance of cartesian bicategories and p-categories in mathematics and computer science has increased in the last years, see for example [? ? ?]. Indeed, the crucial intuitions behind such notions and their pervasiveness in many applications has led several authors to further develop this kind of structures.

Such a process has therefore led to the development and introduction of new categories, which are very similar in nature but originated in very different contexts and thus have different notations.

Relevant examples include the *categories of information transformers* [?] and the *CD categories* [?] and its *affine* variant, which is the basis for a recent approach to categorical probability, where they are dubbed *Markov categories* [?] based on the interpretation of arrows as generalised Markov kernels.

Independently, the notion of *gs-monoidal category* was introduced to present a categorical characterisation of *term graphs* [?], as well as its 2-categorical counterpart suitable to describe term graph *rewriting* [?]. Their study was pursued in a series of papers (see e.g. [? ?] among others), including their application to the functorial semantics of relational and partial algebras [? ?].

Another fundamental concept in category theory, which has deep connections with those mentioned above, is the notion of a monad. In particular, the Kleisli category associated with a monad provides a framework for understanding generalised morphisms between objects, capturing both the structure of mappings and the effects described by the monad. In computer science, Kleisli categories have been widely used to model *computations* and *effects*. This connection was first formalised in [?], where it was shown how monads provide a rigorous mathematical foundation for these notions.

The purposes of this work is twofold. The first is to arrange and revisit these categorical structures, and possibly their 2-categorical versions in the form of preorder-enriched categories, in a modern and comprehensive way, appropriately comparing them to provide a single reference where they can be analysed. In fact, the large number of similar notions, presented under different names and in different contexts, makes it increasingly challenging to navigate the literature and to have a clear framework connecting all these kinds of categories. The second purpose is to conduct a study on the Kleisli categories of suitable monads on such categorical structures, with the aim of further investigating, for all the possible ramifications, the well-known fact that the Kleisli category of a symmetric monoidal monad on a cartesian category is symmetric monoidal, see [?]. This analysis is also extended to the enriched context.

As a result, we provide an overview of the main categories that abstract the properties of relations, and we show how they are related to each other. To present the various notions, we will use the language of string diagrams [?], which is widely used today in both mathematics and computer science.

The key notions in this presentation are that of *garbage* and *share* categories, as well as their dual and enriched versions, which we identify as the crucial cores of all the other notions. Our presentation aims to be as modular as we can, in order to highlight the key differences between the various notions analysed, and

so that the reader can easily navigate the paper and focus on the specific aspects of interest. This modular approach will bring us to alternative characterisations of some known categories, such as restriction and Markov categories.

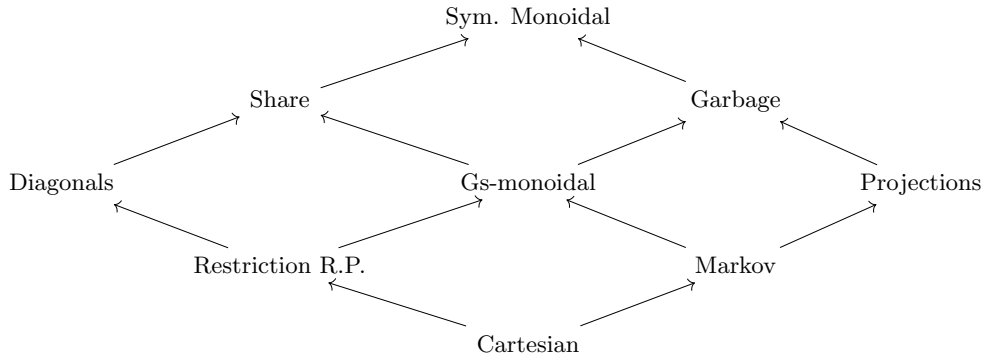
Finally, in this work we provide a taxonomy for Kleisli categories in the context of gs-monoidal categories and to present new examples. Following the philosophy and motivations behind this work, the aim here as well is to showcase, recall, and present the notions of affine and relevant monads [? ? ] (and their enriched version), as well as the characterisations of their Kleisli categories, in the simplest, most general, and modular way possible.

Examples of this kind of monads arise in a quite natural way when we consider action monads. In this context, the affine or relevant monoidal structure of the monad is determined by the condition that the base category is connected or special respectively. Similar examples can be obtained by taking instances of the semiring monad, proving also examples in the enriched case.

The paper is structured as follows. In Section 2 we provide a background on the notions of interest, most importantly gs-monoidal categories, focussing on their relationships with Markov and restriction categories. In Section 3 we characterise the structure of Kleisli categories for symmetric monoidal monads. In Section 4 we study oplax cartesian categories, the order-enriched version of gs-monoidal categories, and how they are related to cartesian bicategories. Finally, in Section 5 we draw some conclusions and outline future research directions.

## 2 A taxonomy of gs-monoidal categories

The overall, one-dimensional taxonomy of the various “categories for relations” we are going to analyse here is represented by the diagram below, where the presence of an arrow  $X \rightarrow Y$  means that a category falling in the class  $X$  (e.g. restriction categories with restriction products) also belong to the class  $Y$  (categories that are either gs-monoidal or with diagonals)



Monoidal categories were introduced by Bénabou [? ] and later finitely axiomatised by Mac Lane in [? ], with the term “monoidal category” first appearing in a 1966 paper by Eilenberg and Kelly [? ]. Nowadays, the literature on monoidal

categories is very extensive. Hence, for a standard presentation we refer to [? ], while for a complete list of references we refer to the *nlab* page<sup>1</sup>.

A monoidal category is a category equipped with a tensor product operation and a unit object, satisfying certain coherence conditions. It provides a framework for studying structures where objects can be combined and interactions are modelled algebraically. Cartesian categories are one of the leading examples of monoidal categories, where the tensor product is the categorical product.

As shown by Fox in [? ], cartesian categories are precisely symmetric monoidal categories in which every object is equipped with a comonoidal structure given by  $\nabla_X : X \rightarrow X \times X$  and  $!_X : X \rightarrow I$ , both natural in  $X$ .

By relaxing the requirement about the existence and the naturality of these two families of arrows one obtains a series of well-known categories.

Requiring the existence of both families but not their naturality, one obtains the notion of gs-monoidal category, introduced by Corradini and Gadducci in [? ? ] in the context of algebraic presentations of graphical formalisms. These categories abstract the properties of cartesian product of sets in the category of relations on one hand, and on the other they have the right structure to distinguish between relations, partial functions, total relations and functions.

Requiring the existence of both families but naturality only for comultiplication  $\nabla_X$ , one obtains categories apt to abstract the notion of *partial* function. These categories have been presented in equivalent forms with the names *restriction categories with restriction products* by Cockett and Lack in [? ? ? ], *p-categories* by Rosolini and Robinson in [? ] and *partial categories* by Curien and Obtulowicz in [? ]. Requiring only the existence of comultiplication  $\nabla_X$  and its naturality, one obtains the notion of *categories with diagonals*, introduced by Jacobs in [? ] in the context of linear logic. The existence of comultiplication  $\nabla_X$  without naturality gives rise to the notion of *share categories*.

Requiring the existence of both families but naturality only for counit  $!_X$ , one obtains categories apt to abstract the notion of *total* relation. These categories have been presented in equivalent forms with the names *affine CD-categories* by Cho and Jacobs in [? ] and Markov categories by Fritz in [? ] in the context of categorical probability theory. Requiring only the existence of discharger  $!_X$  and its naturality, one obtains the notion of *categories with projections*, introduced by Jacobs in [? ] in the context of linear logic. The existence of counit  $!_X$  without naturality gives rise to the notion of *garbage categories*.

The above categories are among the most commonly used in the literature of “categories for relations”. However, we will also discuss some minor variants such as their duals, special, connected, Frobenius, and bialgebraic categories.

## 2.1 Share categories

**Definition 2.1.** A *share category* is a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  together with a commutative cosemigroup structure for each object  $X$ , consisting of a comultiplication

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<sup>1</sup><https://ncatlab.org/nlab/show/monoidal+category>

$$\text{copy}_X = \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ X \end{array}$$

which is coassociative and cocommutative

$$\begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ X \end{array} = \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ X \end{array} \quad \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ X \end{array} = \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ X \end{array}$$

These cosemigroup structures must be multiplicative with respect to the monoidal structure, meaning that they satisfy the equations

$$\begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ X \otimes Y \end{array} = \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ X \end{array} \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ Y \end{array} \quad \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ I \end{array} = \begin{array}{c} \text{---} \text{---} \\ \text{---} \bullet \text{---} \\ | \\ I \end{array}$$

Symbolically, we also write  $\nabla_X : X \rightarrow X \otimes X$  for the share structure arrow and call it **duplicator**.

**Example 2.2.** The monoidal category  $(\mathcal{N}, +, 0)$ , where  $\mathcal{N}$  is the posetal category of natural numbers and the monoidal operation  $+$  is given by the usual sum of natural numbers is a share category, where the comultiplication is given by the arrow  $n \leq n + n$ .

**Example 2.3.** The leading example of share category is the category of sets and *partial functions*. We will see in Example 2.27 how the share structure of this category can be derived from the gs-monoidal structure of the category of sets and relations.

**Example 2.4.** The monoidal category  $(\mathbf{Set}, \times, \{\bullet\})$  of sets and functions with the direct product as the monoidal operator is a share category, where the comultiplication is given by the functions  $X \rightarrow X \times X$  such that  $a \mapsto \langle a, a \rangle$ .

The following lemma shows that a share structure can be equivalently given in terms of a monoidal transformation.

**Lemma 2.5.** *Given a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  and the two trivial strong symmetric monoidal functors given by the identity functor*

$$(\text{id}, \text{id}_I, \text{id}_{+\otimes-}) : \mathcal{C} \rightarrow \mathcal{C}$$

*and the functor  $\otimes(-, -) : \mathcal{C} \rightarrow \mathcal{C}$  which sends  $X$  to  $X \otimes X$*

$$(\otimes(-, -), \lambda_I, \text{id}_+ \otimes \gamma_{+,-} \otimes \text{id}_-) : \mathcal{C} \rightarrow \mathcal{C}$$

*a **share** structure is given by a monoidal transformation*

$$\text{copy}_- : (\text{id}, \text{id}_I, \text{id}_{+\otimes-}) \rightarrow (\otimes(-, -), \lambda_I, \text{id}_+ \otimes \gamma_{+,-} \otimes \text{id}_-).$$

**Remark 2.6.** It is now immediate to see that a share category is a *category with diagonals* [?, Def. 2.1] if  $\text{copy}_-$  is a natural transformation.

**Definition 2.7.** For share categories  $\mathcal{C}$  and  $\mathcal{D}$ , a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  equipped with a lax symmetric monoidal structure

$$\psi: \otimes \circ (F \times F) \rightarrow F \circ \otimes, \quad \psi_0: I \rightarrow F(I)$$

is **relevant** if the following diagram commutes for all  $X$  in  $\mathcal{C}$

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\nabla_X)} & F(X \otimes X) \\ \searrow \nabla_{FX} & & \nearrow \psi_{X,X} \\ & F(X) \otimes F(X) & \end{array} \quad (1)$$

**Definition 2.8.** An arrow  $f: X \rightarrow Y$  in a share category is called **copyable** or **functional** if

$$\begin{array}{c} Y \quad Y \\ \swarrow \quad \searrow \\ \bullet \\ \downarrow \\ \boxed{f} \\ \downarrow \\ X \end{array} = \begin{array}{c} Y \quad Y \\ \downarrow \quad \downarrow \\ \boxed{f} \quad \boxed{f} \\ \swarrow \quad \searrow \\ \bullet \\ \downarrow \\ X \end{array}$$

Therefore, the notion category with diagonals [?, Def. 2.1] can be easily rephrased in terms of share category.

**Lemma 2.9.** A share category has diagonals if and only if every arrow is functional.

**Example 2.10.** The share categories  $(\mathcal{N}, +, 0)$  and  $(\mathbf{Set}, \times, \{\bullet\})$  presented in Example 2.2 and Example 2.4 respectively are both categories with diagonals.

**Corollary 2.11.** The sub-category  $\mathcal{C}\text{-Fun}$  of functional arrows is a category with diagonals.

We close recalling the notion of positivity from [?, Def. 11.22].

**Definition 2.12.** A share category is called **positive** if for every arrows  $f: X \rightarrow Y$  and  $g: Y \rightarrow W$  such that  $g \circ f$  is functional then

$$\begin{array}{c} Y \quad W \\ \downarrow \quad \downarrow \\ \boxed{f} \quad \boxed{g} \\ \downarrow \quad \downarrow \\ \quad \boxed{f} \\ \swarrow \quad \searrow \\ \bullet \\ \downarrow \\ X \end{array} = \begin{array}{c} Y \quad W \\ \downarrow \quad \downarrow \\ \quad \boxed{g} \\ \swarrow \quad \searrow \\ \bullet \\ \downarrow \\ \boxed{f} \\ \downarrow \\ X \end{array}$$

## 2.2 Categories with garbage

**Definition 2.13.** A *category with garbage* is a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  together with a distinguished arrow for each object  $X$

$$\text{dis}_X = \begin{array}{c} \bullet \\ | \\ X \end{array}$$

These arrows must be multiplicative with respect to the monoidal structure, meaning that they satisfy the equations

$$\begin{array}{c} \bullet \\ | \\ X \otimes Y \end{array} = \begin{array}{c} \bullet \\ | \\ X \end{array} \quad \begin{array}{c} \bullet \\ | \\ Y \end{array} = \begin{array}{c} \bullet \\ | \\ I \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

Symbolically, we also write  $!_X : X \rightarrow I$  for the garbage structure arrow and call it **discharger**.

**Example 2.14.** The monoidal category  $(\mathcal{N}^{\text{op}}, +, 0)$ , where  $(\mathcal{N}, +, 0)$  is the category defined in Example 2.2, is a category with garbage.

**Example 2.15.** The leading example of a category with garbage is the category of sets and *total relations*, while a leading counterexample is the share category of partial functions of Example 2.3 (and of course vice versa). Also for this case we will see in Example 2.27 how the garbage structure of this category can be derived from the gs-monoidal structure of the category of sets and relations.

**Example 2.16.** The monoidal category  $(\mathbf{Set}, \times, \{\bullet\})$  of sets and functions with the direct product of Example 2.4 as the monoidal operator is also a category with garbage, where the arrows are given by the function  $a \mapsto \bullet$ .

The following lemma shows that garbage structure can be equivalently given in terms of a monoidal transformation.

**Lemma 2.17.** Given a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  and the two trivial strong symmetric monoidal functors given by the identity

$$(\text{id}, \text{id}_I, \text{id}_{+\otimes-}) : \mathcal{C} \rightarrow \mathcal{C}$$

and the constant value  $I$  functor

$$(I, \text{id}_I, \lambda_I^{-1}) : \mathcal{C} \rightarrow \mathcal{C}$$

a **garbage** structure is given by a monoidal transformation

$$\text{dis}_- : (\text{id}, \text{id}_I, \text{id}_{+\otimes-}) \rightarrow (I, \text{id}_I, \lambda_I^{-1}).$$

**Remark 2.18.** Similarly to what occurs for share categories, a category with garbage is a *category with projections* [?, Def. 2.1] if  $\text{dis}_-$  is a natural transformation. Notice that categories with projections are also called *semicartesian categories* in [? ?].

**Definition 2.19.** For categories with garbage  $\mathcal{C}$  and  $\mathcal{D}$ , a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  equipped with a lax symmetric monoidal structure

$$\psi: \otimes \circ (F \times F) \rightarrow F \circ \otimes, \quad \psi_0: I \rightarrow F(I)$$

is **affine** if the following diagram commutes for all  $X$  in  $\mathcal{C}$

$$\begin{array}{ccc} F(X) & \xrightarrow{F(!_X)} & F(I) \\ & \searrow & \nearrow \psi_0 \\ & I & \end{array} \quad (2)$$

**Definition 2.20.** An arrow  $f: X \rightarrow Y$  in a category with garbage is called **discardable** or **total** if

$$\begin{array}{c} \bullet \\ | \\ \boxed{f} \\ | \\ X \end{array} = \begin{array}{c} \bullet \\ | \\ X \end{array}$$

**Lemma 2.21.** A category with garbage has projections if and only if every arrow is total.

**Remark 2.22.** Notice that in a garbage category every arrow is total if and only if the object  $I$  is terminal.

**Corollary 2.23.** The sub-category  $\mathcal{C}\text{-Tot}$  of total arrows is a category with projections.

### 2.3 Gs-monoidal categories

The original notion of *gs-monoidal category* introduced in [? ?] can be presented combining the previous notions of share and garbage category.

**Definition 2.24.** A **gs-monoidal category** is a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  with share and garbage structures such that for each object  $X$

$$\begin{array}{c} \bullet \\ \curvearrowright \\ \bullet \\ | \\ X \end{array} = \begin{array}{c} | \\ X \end{array}$$



In other words, each object is equipped with a commutative comonoid structure. We can now close this section with a result that is basically stated in [? ].

**Lemma 2.25.** *A gs-monoidal category is cartesian monoidal if and only if  $\nabla_-$  and  $!_-$  are natural transformations.*

**Corollary 2.26.** *The sub-category  $\mathcal{C}\text{-TFun}$  of total and functional arrows is cartesian monoidal.*

**Example 2.27.** The category  $(\mathbf{Set}, \times, \{\bullet\})$  of sets and functions with the direct product is gs-monoidal, and in fact cartesian monoidal. The category  $(\mathbf{Rel}, \times, \{\bullet\})$  of sets and relations with the composition of  $a \subseteq X \times Y$  with  $b \subseteq Y \times Z$  given by

$$b \circ a := \{(x, z) \mid \exists y \in Y, (x, y) \in a \wedge (y, z) \in b\} \subseteq X \times Z$$

and the monoidal operation given by the direct product of sets is the leading example of gs-monoidal category [? ]. In this category, the copyable arrows are precisely the partial functions, and the discardable arrows are the total relations.

**Example 2.28.** Recently, an alternative category of relations  $(\mathbf{Rel}^\vee, \otimes, \{\bullet\})$  has been investigated [? ? ]. The category  $\mathbf{Rel}^\vee$  has the same objects and arrows as  $\mathbf{Rel}$ , but the composition of a relation  $a \subseteq X \times Y$  with  $b \subseteq Y \times Z$  is given by

$$b \circ a = \{(x, z) \mid \forall y \in Y, (x, y) \in a \vee (y, z) \in b\} \subseteq X \times Z$$

and the identity arrow is given by  $\text{id}_X = \{(x, y) \mid x \neq y\} \subseteq X \times X$ . The tensor product is defined on objects as the direct product of sets (as in the previous case), and on two arrows  $a \subseteq X \times Y$  and  $c \subseteq Z \times V$  is given by

$$a \otimes c = \{((x, z), (y, v)) \mid (x, y) \in a \vee (z, v) \in c\}$$

This category is gs-monoidal with the following structure arrows

$$\nabla_X = \{(x, (y, z)) \mid x \neq y \vee x \neq z\} \subseteq X \times (X \times X) \quad !_X = \emptyset \subseteq X \times I$$

Note that  $\mathbf{Rel}^\vee$  is isomorphic to  $\mathbf{Rel}$  via a “complement functor”, which is strict symmetric monoidal [? ]. Although we are not aware of a characterisation of copyable relations for this case, it is easy to show that the discardable relations are precisely those  $a \subseteq X \times Y$  such that  $\forall x \in X. \{y \in Y \mid (x, y) \in a\} \neq Y$ .

We close now with an observation from [? , Prop. 2.10&2.11].

**Lemma 2.29.** *Let  $(\mathcal{C}, \otimes, I)$  be a gs-monoidal category. Then  $\mathcal{C}\text{-Fun}$ ,  $\mathcal{C}\text{-Tot}$ , and  $\mathcal{C}\text{-TFun}$  are gs-monoidal sub-categories of  $\mathcal{C}$ .*

As for functors between symmetric monoidal categories, also functors between gs-monoidal categories come in several variants.

**Definition 2.30.** *For gs-monoidal categories  $\mathcal{C}$  and  $\mathcal{D}$ , a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  equipped with a lax symmetric monoidal structure*

$$\psi: \otimes \circ (F \times F) \rightarrow F \circ \otimes, \quad \psi_0: I \rightarrow F(I)$$

*is **gs-monoidal** if it is both relevant and affine.*

## 2.4 Recurring examples: spans and (weighted) relations

Recall from [?] that the category  $\mathbf{Span}(\mathcal{A})$  of *spans* associated with a category with finite limits  $\mathcal{A}$  is a gs-monoidal category. The  $\mathbf{Span}(\mathcal{A})$  has the same objects as  $\mathcal{A}$ , and an arrow from  $X$  to  $Y$  is a *span*, i.e. an equivalence class of diagrams of the form  $(X \xleftarrow{a_X} A \xrightarrow{a_Y} Y)$  of  $\mathcal{A}$ , where  $(A, a_X, a_Y) \sim (B, b_X, b_Y)$  if there exists an isomorphism  $i : A \rightarrow B$  such that  $b_X \circ i = a_X$  and  $b_Y \circ i = a_Y$ . Identities and composition are defined as follows

- the identity of  $X$  is the span  $X \xleftarrow{\text{id}_X} X \xrightarrow{\text{id}_X} X$ ;
- the composition of spans  $X \leftarrow A \xrightarrow{f} Y$  and  $Y \xleftarrow{g} B \rightarrow Z$  is given by the span  $X \leftarrow A \times_Y B \rightarrow Z$  obtained through the pullback of  $f$  and  $g$

$$\begin{array}{ccccc}
 & & A \times_Y B & & \\
 & \swarrow & \downarrow & \searrow & \\
 & A & & B & \\
 \swarrow & & \searrow & \swarrow & \searrow \\
 X & & Y & & Z
 \end{array}$$

(Note: The diagram shows a pullback square with  $A \times_Y B$  at the top,  $A$  on the left,  $B$  on the right, and  $Y$  at the bottom. Arrows  $f: A \rightarrow Y$  and  $g: B \rightarrow Y$  are shown. Arrows  $X \leftarrow A$  and  $B \rightarrow Z$  are also shown.)

The tensor product is given via the categorical product  $\times$  of  $\mathcal{A}$  and the gs-monoidal structure is given by the arrows

$$\nabla_X = (X \xleftarrow{\text{id}} X \xrightarrow{\nabla_X} X \times X), \quad !_X = (X \xleftarrow{\text{id}} X \xrightarrow{!_X} 1).$$

Although the duplicator  $\nabla_X$  and the discharger  $!_X$  are natural in  $\mathcal{A}$ , in general they are not in  $\mathbf{Span}(\mathcal{A})$  as observed first in [?] and, in an equivalent way, in [?, Ex. 2.1.4(10)] where it is observed that  $\mathbf{Span}(\mathcal{A})$  is not a restriction category (see also Proposition 2.48). We will refer to this gs-monoidal category as  $(\mathbf{Span}(\mathcal{A}), \times, 1)$ .

**Remark 2.31.** It is worth to observe that the sub-category  $\mathbf{Span}_m(\mathcal{A})$  of  $\mathbf{Span}(\mathcal{A})$  whose arrows are spans whose left leg is a mono is gs-monoidal as well (since monos are stable under pullbacks) and in fact it has diagonals. This category is a particular instance of a *category of partial maps*  $\mathbf{Par}(\mathcal{A}, \mathcal{M})$  associated with a stable system of monics  $\mathcal{M}$  presented in [?, Sec. 3.1].

Moreover, the sub-category  $\mathbf{Span}_e(\mathcal{A})$  of  $\mathbf{Span}(\mathcal{A})$  of spans whose left leg is a split epimorphism<sup>2</sup> has projections, see [?, Prop. 5.4].

If  $\mathcal{A}$  is an extensive category (see [?]) with finite limits, then the category  $\mathbf{Span}(\mathcal{A})$  has another gs-monoidal structure. The tensor product is given by the categorical sum  $+$  and gs-monoidal structure is given by the arrows

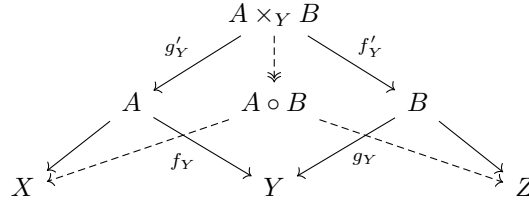
$$\nabla_X = (X \xleftarrow{(\text{id}, \text{id})} X + X \xrightarrow{\text{id}} X + X), \quad !_X = (X \xleftarrow{!} 0 \xrightarrow{\text{id}} 0).$$

This category is actually cartesian and we will refer to it as  $(\mathbf{Span}(\mathcal{A}), +, 0)$ .

<sup>2</sup>An arrow  $f : A \rightarrow B$  splits if it has a *section*, i.e. an arrow  $s : B \rightarrow A$  such that  $f \circ s = \text{id}_B$ .

### 2.4.1 From spans to relations

It is well-known that the notion of category of relations **Rel** of Example 2.27 can be generalised to regular categories (see [?, Ex. 1.4]) or, more generally, to categories equipped with a proper, stable factorization system (see [?]). Indeed, let  $\mathcal{A}$  be a regular category, then consider the category of relation **Rel**( $\mathcal{A}$ ) whose objects are those of  $\mathcal{A}$  and arrows are given by jointly monic spans. The composition of relations  $X \xleftarrow{f_X} A \xrightarrow{f_Y} Y$  and  $Y \xleftarrow{g_Y} B \xrightarrow{g_Z} Z$  is given by first considering the span  $X \leftarrow A \times_Y B \rightarrow Z$  obtained by taking the pullback of  $f$  and  $g$ , and then considering the regular-epi/mono factorization of the induced arrow  $\langle f_X \circ g'_Y, g_Z \circ f'_Y \rangle : A \times_Y B \rightarrow X \times Z$ .

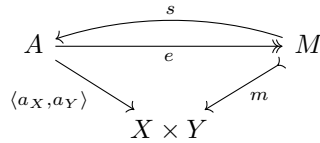


The categorical product  $\times$  induces a gs-monoidal structure as in  $(\mathbf{Span}(\mathcal{A}), \times, 1)$ , and this gs-monoidal category will be denoted as  $(\mathbf{Rel}(\mathcal{A}), \times, 1)$ . From a logical perspective, the main reason why the **Rel**-construction can be generalised in this setting is that regular categories are able to properly deal with the  $(\exists, =, \wedge, \top)$ -fragment of first-order logic (the *regular fragment*). In particular, they have an “internal” notion of existential quantifier, which allows us to mimic the usual composition of relations (which is defined via the existential quantifier). Similarly, if  $\mathcal{A}$  is also extensive, then the coproduct induces a cartesian monoidal structure as in  $(\mathbf{Span}(\mathcal{A}), +, 0)$ , which we will denote as  $(\mathbf{Rel}(\mathcal{A}), +, 0)$ .

**Remark 2.32.** For a regular category  $\mathcal{A}$  in which every regular epi splits, the category of relations is equivalent to a suitable category of spans. Namely, **Rel**( $\mathcal{A}$ ) is equivalent to the category whose objects are those of  $\mathcal{A}$  and whose arrows are equivalence classes of diagrams of the form  $(X \xleftarrow{a_X} A \xrightarrow{a_Y} Y)$  of  $\mathcal{A}$ , where  $(A, a_X, a_Y) \sim (B, b_X, b_Y)$  if there exist two arrows  $h : A \rightarrow B$  and  $k : B \rightarrow A$  such that

$$\begin{cases} b_X \circ h = a_X \\ b_Y \circ h = a_Y \end{cases} \quad \begin{cases} a_X \circ k = b_X \\ a_Y \circ k = b_Y \end{cases}$$

Indeed, every span  $(X \xleftarrow{a_X} A \xrightarrow{a_Y} Y)$  is in the same equivalence class of  $(X \xleftarrow{m_X} M \xrightarrow{m_Y} Y)$ , where  $m := \langle m_X, m_Y \rangle$  is the image of the factorization of  $\langle a_X, a_Y \rangle$



In particular, the above observation holds for  $\mathcal{A} = \mathbf{Set}$ , where this kind of spans are referred to as *garbage equivalent* ones [?].

#### 2.4.2 And now, weighted relations

While the previous section generalised relations by changing the base category, an alternative approach is to equip each relation with a *weight*. Consider a semiring  $(M, \oplus, \odot, 0, 1)$ . It is easy to check that there exists a functor  $\mathcal{M} : \mathbf{Set} \rightarrow \mathbf{Set}$  that sends every set  $X$  to

$$\mathcal{M}(X) = \{h : X \rightarrow M \mid h \text{ has finite support}\}$$

where *finite support* means that  $h(x) \neq 0$  for a finite number of elements  $x \in X$ , and every function  $f : X \rightarrow Y$  to the function  $\tilde{f} : \mathcal{M}(X) \rightarrow \mathcal{M}(Y)$  which sends every  $M$ -valued function  $h : X \rightarrow M$  with finite support to

$$\tilde{f}(h)(y) = \bigoplus_{x \in f^{-1}(y)} h(x)$$

Explicitly, the image of the composition  $g \circ f$  of  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  is

$$\widetilde{g \circ f}(h)(z) = \tilde{g}\left(\bigoplus_{x \in f^{-1}(y)} h(x)\right) = \bigoplus_{y \in g^{-1}(z)} \left(\bigoplus_{x \in f^{-1}(y)} h(x)\right)$$

Recall that  $(\mathbf{Set}, \times, \{\bullet\})$  is cartesian monoidal with respect to the direct product. The above functor is lax symmetric monoidal with respect to that monoidal structure, with the obvious coherence arrows

$$\psi_{X,Y} : \mathcal{M}(X) \times \mathcal{M}(Y) \rightarrow \mathcal{M}(X \times Y) \quad \psi_0 : \{\bullet\} \rightarrow \mathcal{M}(\{\bullet\})$$

given by  $\psi_{X,Y}(h,k)(x,y) = h(x) \odot k(y)$  and  $\psi_0(\bullet)(\bullet) = 1$ .

Finally, note that there are two additional endofunctors  $\mathcal{M}_e$  and  $\mathcal{M}_u$  on  $\mathbf{Set}$  that are both lax symmetric monoidal with respect to the same monoidal structure  $(\mathbf{Set}, \times, \{\bullet\})$ . These are defined as follows

$$\mathcal{M}_e(X) = \{h : X \rightarrow M \mid h \text{ has support at most one and is idempotent}\}$$

$$\mathcal{M}_u(X) = \{h : X \rightarrow M \mid h \text{ has finite support and is normalised}\}$$

where idempotent means that  $\forall x \in X. h(x) = h(x) \odot h(x)$  and normalised that  $\bigoplus_{x \in X} h(x) = 1$ . What is noteworthy is that, besides being both lax symmetric monoidal functors,  $\mathcal{M}_e$  is relevant and  $\mathcal{M}_u$  is affine.

**Remark 2.33.** The setting above is general enough to recover various relational structures defined in the literature. If e.g.  $M$  is the Boolean semiring  $\{0,1\}$ , then  $\mathcal{M}$  is the lax symmetric monoidal functor  $\mathcal{P}$  associating to  $X$  its finite subsets, and it is neither relevant nor affine. The relevant functor  $\mathcal{P}_e$  is restricted to subsets of at most one element, while the affine functor  $\mathcal{P}_u$  is restricted to subsets with at least one element.

## 2.5 On duality

All the definitions above about the gs-monoidal structure can be easily dualised.

**Definition 2.34.** A *cogs-monoidal category* is a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  such that its dual category  $\mathcal{C}^{op}$  is a gs-monoidal category.

In words, each object  $X$  is equipped with a monoid structure  $\text{match}_X : X \otimes X \rightarrow X$  and  $\text{new}_X : I \rightarrow X$ , namely there exists two arrows

$$\text{match}_X = \begin{array}{c} X \\ | \\ \bullet \\ \text{---} \end{array} \quad \text{new}_X = \begin{array}{c} X \\ | \\ \bullet \end{array}$$

satisfying the obvious axioms. Symbolically, we also write  $\Delta_X : X \otimes X \rightarrow X$  for the arrow  $\text{match}_X$  and  $i_X : I \rightarrow X$  for the arrow  $\text{new}_X$ .

**Remark 2.35.** We assume, without repeating the statements, that all the remarks in the previous sections concerning sharing and garbage can be dualised, so that e.g.  $\Delta_X : X \otimes X \rightarrow X$  and  $i_X : I \rightarrow X$  can be described as suitable monoidal transformations, and that their naturality boils down to require that a cogs-monoidal category is actually cocartesian monoidal. See also Appendix C.

**Example 2.36.** The categories of spans introduced in Section 2.4 are also cogs-monoidal. Indeed, if  $\mathcal{A}$  has finite limits,  $(\mathbf{Span}(\mathcal{A}), \times, 1)$  has a cogs-monoidal structure given by the arrows

$$\Delta_X = (X \times X \xleftarrow{\nabla_X} X \xrightarrow{\text{id}} X), \quad i_X = (1 \xleftarrow{!} X \xrightarrow{\text{id}} X).$$

If  $\mathcal{A}$  is also an extensive category,  $(\mathbf{Span}(\mathcal{A}), +, 0)$  has cogs-monoidal structure (and it is in fact cocartesian) given by the arrows

$$\Delta_X = (X + X \xleftarrow{\text{id}} X + X \xrightarrow{(\text{id}, \text{id})} X), \quad i_X = (0 \xleftarrow{\text{id}} 0 \xrightarrow{i} X).$$

The same considerations apply for the category  $\mathbf{Rel}(\mathcal{A})$  and the gs-monoidal structures presented in Examples 2.4.1.

**Definition 2.37.** A *big-monoidal category* is a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  that is both a gs-monoidal and a cogs-monoidal category.

Notice that the definition of a big-monoidal category does not include any requirements regarding the interaction between the monoidal and the comonoidal structures. The rest of this section is devoted to studying some relevant axioms that establish the possible ways these structures interact with each other.

**Definition 2.38.** A big-monoidal category  $(\mathcal{C}, \otimes, I)$  is *special* if the law below on the left holds, while it is *connected* if the law below on the right holds<sup>3</sup>

<sup>3</sup>See <https://ncatlab.org/nlab/show/bimonoid>.

We will further elaborate on bigs-monoidality later in this paper, and we start by presenting two straightforward instances.

**Definition 2.39.** A *bialgebraic category* is a bigs-monoidal category  $(\mathcal{C}, \otimes, I)$  such that the following equalities hold

**Remark 2.40.** Bialgebraic categories have been investigated in the flownomial calculus for flowchart description proposed by G. Ștefănescu, see e.g. the axioms for angelic branching in [?] and the references therein.

Hence, in a bialgebraic category each object is a (bicommutative) bimonoid.

**Remark 2.41.** If a bialgebraic category satisfies the law

then each object carries the structure of an Hopf algebra, where the antipode for an object  $X$  is given by  $\text{id}_X$ <sup>4</sup>. Note that this property does not imply that such a category is special or, in this case equivalently, connected. In fact, if a bialgebraic category is connected then both  $\text{id}_X \circ !_X = \text{id}_I$  and  $!_X \circ \text{id}_X = \text{id}_I$  hold, that is, each object is isomorphic to  $I$ .

<sup>4</sup>See <https://ncatlab.org/nlab/show/Hopf+algebra>.

**Example 2.42.** Looking again at Example 2.36, we have that the bigs-monoidal categories of spans ( $\mathbf{Span}(\mathcal{A}), +, 0$ ) and relations ( $\mathbf{Rel}(\mathcal{A}), +, 0$ ) are bialgebraic categories, and in fact they are bicartesian (i.e. both cartesian and cocartesian) monoidal. Neither is connected, but the latter is special, the former is not.

**Definition 2.43.** A *Frobenius category* is a bigs-monoidal category  $(\mathcal{C}, \otimes, I)$  such that the Frobenius law holds, i.e.

**Remark 2.44.** The Frobenius law implies [? ]

and if moreover an object is special [? ]

**Example 2.45.** Looking once more at Example 2.36, we have that the bigs-monoidal categories of spans ( $\mathbf{Span}(\mathcal{A}), \times, 1$ ) and relations ( $\mathbf{Rel}(\mathcal{A}), \times, 1$ ) are Frobenius categories. Neither is connected, but both are special.

**Remark 2.46.** Frobenius categories are also compact-closed (sometimes called rigid symmetric monoidal categories), where the dual of an object  $X$  is the object itself and the unit and counit arrows are defined as  $\nabla_A \circ i_A: I \rightarrow A \otimes A$  and  $\Delta_A \circ !_A: A \otimes A \rightarrow I$ , respectively, while the triangle identities trivially hold.

**Remark 2.47.** The Frobenius law is incompatible with the bialgebraic structure: A category that is both Frobenius and bialgebraic is also connected, thus each object  $X$  is isomorphic to  $I$  (see Remark 2.41). Indeed, we have

where the first equality is obtained through the bigs-monoidal structure, the second through the Frobenius law and the last two through bialgebra equalities.

## 2.6 On Markov and restriction categories

Two categorical notions have come to the forefront in recent years: Markov categories [?] as models for probabilistic systems, and restriction categories [? ?] as abstraction of partial functions. This section establishes their correspondence with the notions introduced in Sections 2.1, 2.2, and 2.3.

**Proposition 2.48.** *Gs-monoidal categories with diagonals correspond exactly to restriction categories with restriction products.*

*Proof.* Moving from gs-monoidal categories to restriction categories, the core of the proof occurs in [?, Rem. 2.15], where gs-monoidal categories are shown to have enough structure to define the *domain* of an arrow  $f : X \rightarrow Y$  as

Then, the axioms (R.2) and (R.3) of Definition B.1 actually hold for any gs-monoidal category, while (R.1) and (R.4) hold for the presence of diagonals. The proof that they have restriction products is a straightforward check. We refer to [? ?] and the references therein for all the details.  $\square$

**Remark 2.49.** Notice that the previous proposition can be dualised, providing the characterisation of *corestriction categories*, see [?, Ex. 2.1.3(12)], in terms of cogs-monoidal categories.

**Remark 2.50.** An alternative proof of Proposition 2.48 can be already found in [?, Ex. 2.1.3(6-7)] and in [?, Sec. 4]. Note that restriction categories with restriction products have been proved equivalent to several other notions, such as p-categories with one-element object [?], partial cartesian categories in the sense of Curien and Obtulowicz [?], and they are a special case of the bicategories of partial maps of Carboni [?]. See also Proposition 4.10 later.

**Proposition 2.51.** *Gs-monoidal categories with projections correspond exactly to Markov categories.*



*Proof.* Straightforward, since the presence of projections boils down to the unit  $I$  being the terminal object (see Remark 2.22), which in fact is a precise characterisation of Markov categories.<sup>5</sup>  $\square$

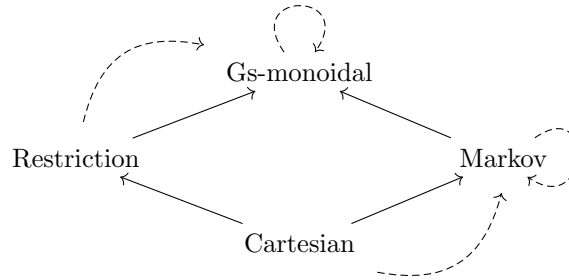
**Remark 2.52.** In [? ], the authors introduced independently *CD-categories* (where “CD” stands for copy/discard) which are the same of gs-monoidal categories and use the term *affine* CD-categories for what are today known as Markov categories. The term Markov category was introduced in [? ]. In the context of Markov categories, functional arrows are referred to as *deterministic*.

### 3 A taxonomy of Kleisli categories

Our taxonomy of Kleisli categories is done for symmetric monoidal monads, which are equivalent to commutative monads, see [? ] and now [? , Appendix C]. We discuss three classes of such monads: affine and relevant monads, first considered in [? ? ], and gs-monoidal monads (which are both affine and relevant).

We then study the properties of their Kleisli categories depending on the categorical structures we previously considered. Indeed, rephrasing [? , Thm. 4.3], the Kleisli category of a relevant/affine monad on a cartesian category is respectively a retraction/Markov category. Hence, we extend this result considering affine and relevant monads on our one-dimensional taxonomy.

For instance, in the following diagram we represent for an affine monad the corresponding nature of its Kleisli category using a dashed arrow. Hence, we prove that the Kleisli category of an affine monad on a Markov category is a Markov category, while the Kleisli category of an affine monad on a restriction category is just gs-monoidal.



A similar taxonomy for the enriched case is going to be provided in Section 4, where we consider the Kleisli categories of enriched monads that are *colax relevant* and *colax affine*, as well as *colax gs-monoidal*.

**Definition 3.1** (Symmetric monoidal monad). *Let  $\mathcal{C}$  be a symmetric monoidal category. Let  $T : \mathcal{C} \rightarrow \mathcal{C}$  be a monad carrying the structure of a lax symmetric*

<sup>5</sup>A complete analysis of Markov categories and their equivalent presentations, together with relevant references, can be found in <https://ncatlab.org/nlab/show/Markov+category>.

monoidal functor with structure maps  $c: \otimes \circ (T \times T) \rightarrow T \circ \otimes$  and  $u: I \rightarrow TI$ . Then  $T$  is a **symmetric monoidal monad** if  $u = \eta_I$  and the following two diagrams commute

$$\begin{array}{ccc}
 & X \otimes Y & \\
 \eta \otimes \eta \swarrow & & \searrow \eta \\
 TX \otimes TY & \xrightarrow{c} & T(X \otimes Y)
 \end{array} \tag{3}$$

$$\begin{array}{ccccc}
 TTX \otimes TTY & \xrightarrow{c} & T(TX \otimes TY) & \xrightarrow{Tc} & TT(X \otimes Y) \\
 \mu \otimes \mu \downarrow & & & & \downarrow \mu \\
 TX \otimes TX & \xrightarrow{c} & & & T(X \otimes Y)
 \end{array} \tag{4}$$

**Remark 3.2.** It is well-known that on a symmetric monoidal category, symmetric monoidal monads are equivalent to commutative monads, see [?, Th. 2.3] and [?, Th. 3.2]. Definition 3.1 corresponds to the notion of monad internal to the 2-category of symmetric monoidal categories, lax functors and monoidal natural transformations. The commutativity of diagrams (3) and (4) is equivalent to requiring that  $\mu$  and  $\eta$  are monoidal natural transformations.

**Example 3.3.** Let us consider the lax symmetric monoidal functor  $\mathcal{M}$  introduced in Section 2.4.2. This functor defines a symmetric monoidal monad on  $(\mathbf{Set}, \times, \{\bullet\})$ , sometimes called the *semiring monad*, with natural transformations  $\eta_X: X \rightarrow \mathcal{M}(X)$  and  $\mu_X: \mathcal{M}(\mathcal{M}(X)) \rightarrow \mathcal{M}(X)$  given by

$$\begin{aligned}
 - \eta_X(x_0)(x) &= \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} \\
 - \mu_X(\lambda)(x) &= \bigoplus_{h \in \mathcal{M}(X)} \lambda(h) \cdot h(x)
 \end{aligned}$$

**Remark 3.4.** Notice that, in the previous example, if  $M$  is the semiring of Booleans, then  $\mathcal{M}$  is the finite subsets monad  $\mathcal{P}$ , while if  $M$  is the semiring of natural numbers  $\mathbb{N}$ , then  $\mathcal{M}$  is the monad of *finite multisets*, see [?]. Moreover, if  $M$  is the semiring of positive real numbers  $\mathbb{R}^+$ , then  $\mathcal{M}$  is the monad of *unnormalised probability distributions*, see [?], while if  $M$  is either the semiring  $([0, 1], \max, \min, 0, 1)$  or the semiring  $([0, 1], \max, \cdot, 0, 1)$  one obtains Golubtsov's categories of *fuzzy information transformer*, see [?].

**Definition 3.5.** A symmetric monoidal monad  $(T, \mu, \eta, c, u)$  on a gs-monoidal category  $\mathcal{C}$  is

- **affine** if  $(T, c, u)$  is an affine functor;
- **relevant** if  $(T, c, u)$  is a relevant functor;
- **gs-monoidal** if  $(T, c, u)$  is a gs-monoidal functor.

**Remark 3.6.** Affine and relevant monads were introduced in [?] in the context of cartesian monoidal categories. While the term “affine” is used in [?], the term “relevant” appears in [?] due to a connection with *relevant logic*. In particular, in [?, Th. 2.1] the author considers as definition of affine monad on a cartesian monoidal category one of the two equivalent conditions

- the unit of the monad  $\eta_I : I \rightarrow T(I)$  is an isomorphism
- the following diagram commutes

$$\begin{array}{ccc} T(X) \otimes T(Y) & \xrightarrow{\psi_{X,Y}} & T(X \otimes Y) \\ & \searrow id & \downarrow \langle T(\pi_1), T(\pi_2) \rangle \\ & & T(X) \otimes T(Y) \end{array}$$

Indeed, if the category  $\mathcal{C}$  has projections,  $I$  is terminal and it gives an equivalence between the second condition above, which relies on that of affine functor (diagram (2) of Definition 2.19), and the fact that  $\eta_I$  is an isomorphism. Hence, Definition 3.5 is more general since it makes sense also in contexts in which  $I$  is not terminal, such as in gs-monoidal categories.

For relevant monads the author of [?] proves that the condition for relevant functors (diagram (1) of Definition 2.7) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} T(X \otimes Y) & \xrightarrow{\langle T(\pi_1), T(\pi_2) \rangle} & T(X) \otimes T(Y) \\ & \searrow id & \downarrow \psi_{X,Y} \\ & & T(X \otimes Y) \end{array}$$

see [?, Prop. 2.2] and also [?, Lem. 4.2].

**Remark 3.7.** Observe that for any object  $X$  of a gs-monoidal category, the hom-set  $\mathcal{C}(X, I)$  has canonically the structure of a commutative monoid. This simple observation can be used to generalise the ordinary notion of affine categories and monads, by requiring additional algebraic properties for such a monoid. Indeed, the more general notions *weakly Markov* category and *weakly affine* monad has been introduced in [?]: Weakly Markov categories are gs-monoidal categories such that the monoid  $\mathcal{C}(X, I)$  is a group. For the corresponding weakly affine monads, instead of assuming  $T(I)$  to be isomorphic to  $I$ , it is just required that the commutative monoid structure of  $T(I)$  is a group. We are not aware of a diagrammatic characterisation as for affine categories and monads, making it applicable to gs-monoidal and relevant categories.

**Remark 3.8.** It is well-known that a monad  $(T, \mu, \eta, c, u)$  on a symmetric monoidal category  $\mathcal{C}$  induces a symmetric monoidal structure on  $\mathcal{C}_T$  if the monad is symmetric monoidal. In particular, the tensor product on  $\mathcal{C}_T$  is defined as  $X \otimes^\sharp Y := X \otimes Y$  and  $f \otimes^\sharp g := c_{X', Y'} \circ (f \otimes g)$ , where  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$  are arrows of  $\mathcal{C}_T$ . The obvious functor  $\mathcal{K} : \mathcal{C} \rightarrow \mathcal{C}_T$ , which is the

identity on objects and acts by post-composition with  $\eta$  on the arrows, is strict symmetric monoidal

$$\otimes^\sharp \circ (F \times F) = F \circ \otimes$$

and the arrows defining the symmetric monoidal structure on  $\mathcal{C}_T$  are obtained as the image of those defining the structure of  $\mathcal{C}$  through the functor  $\mathcal{K}$ .

Hence, it is possible to conclude that  $\mathcal{K}$  preserves equalities of arrows  $f = g$  of  $\mathcal{C}$ , where  $f$  and  $g$  are obtained through compositions and products of the structural arrows of  $\mathcal{C}$ .

**Lemma 3.9.** *Let  $(T, \mu, \eta, c, u)$  be a symmetric monoidal monad on a symmetric monoidal category  $\mathcal{C}$ . If an object has a comonoid structure in  $\mathcal{C}$  then it does so in  $\mathcal{C}_T$ .*

The above argument was already observed in [?, Prop. 4.4] (see also [?, Lem. 2]) and it is the key step in proving the following result.

**Proposition 3.10.** *Let  $(T, \mu, \eta, c, u)$  be a symmetric monoidal monad on a gs-monoidal category  $\mathcal{C}$ . Then the Kleisli category  $\mathcal{C}_T$  is a gs-monoidal category with  $\text{copy}_X$  and  $\text{dis}_X$  given for every object  $X$  by*

$$\text{copy}_X^\sharp := \eta_{X \otimes X} \circ \text{copy}_X, \quad \text{dis}_X^\sharp := \eta_I \circ \text{dis}_X.$$

□

In other words,  $\mathcal{K} : \mathcal{C} \rightarrow \mathcal{C}_T$  is a (strict) gs-monoidal functor.

**Example 3.11.** The Kleisli category  $\mathbf{Set}_{\mathcal{M}}$  of the semiring monad  $\mathcal{M}$  introduced in Section 2.4.2 is the category of sets and *weighted relations*, which is a gs-monoidal category. Some instances for different semirings  $M$  are presented in Remark 3.4.

We also obtain a further instance of the above result.

**Proposition 3.12.** *Let  $(T, \mu, \eta, c, u)$  be an affine monad on a gs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  has projections then so does the Kleisli category  $\mathcal{C}_T$ .*

*Proof.* If  $I$  is terminal and the monad is affine, then  $T(I) \cong I$ . Hence,  $I$  is terminal also in  $\mathcal{C}_T$ . □

**Example 3.13.** As shown in Section 2.4.2, the functor  $\mathcal{M}_u$  is affine, hence the Kleisli category  $\mathbf{Set}_{\mathcal{M}_u}$  has projections. For the Boolean semiring, the Kleisli category of  $\mathcal{P}_u$  discussed in Remark 2.33 is the category whose objects are sets and whose arrows are total relations.

**Remark 3.14.** As an instance of Proposition 3.12, we have that if  $\mathcal{C}$  is a cartesian monoidal category and  $T$  is affine, then the Kleisli category  $\mathcal{C}_T$  has projections. Note instead that if  $\mathcal{C}$  either has diagonals or it is just a gs-monoidal category, then  $\mathcal{C}_T$  is a gs-monoidal category. The taxonomy is strict since the identity functor is an affine monad.

**Proposition 3.15.** *Let  $(T, \mu, \eta, c, u)$  be a relevant monad on a gs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  has diagonals then so does the Kleisli category  $\mathcal{C}_T$ .*

*Proof.* We need to prove the naturality of  $\nabla^\sharp$ , i.e. for every arrow  $f : X \rightarrow Y$  in  $\mathcal{C}_T$ , which corresponds to an arrow  $f : X \rightarrow T(Y)$  in  $\mathcal{C}$ , we show that

$$\nabla_Y^\sharp \circ^\sharp f = (f \otimes^\sharp f) \circ^\sharp \nabla_X^\sharp$$

in  $\mathcal{C}_T$ . Since the composition of two arrows  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  in  $\mathcal{C}_T$  is obtained as  $\mu_Z \circ T(g) \circ f$ , then we obtain

$$\begin{aligned} (f \otimes^\sharp f) \circ^\sharp \nabla_X^\sharp &= \mu_{Y \otimes Y} \circ T(c) \circ T(f \otimes f) \circ \eta_{X \otimes X} \circ \nabla_X \\ &= \mu_{Y \otimes Y} \circ \eta_{T(Y \otimes Y)} \circ c_{Y, Y} \circ (f \otimes f) \circ \nabla_X && \text{(naturality of } \eta) \\ &= \mu_{Y \otimes Y} \circ \eta_{T(Y \otimes Y)} \circ c_{Y, Y} \circ \nabla_{TY} \circ f && \text{(naturality of } \nabla) \\ &= \mu_{Y \otimes Y} \circ \eta_{T(Y \otimes Y)} \circ T(\nabla_Y) \circ f && \text{(relevant monad)} \\ &= \nabla_Y^\sharp \circ^\sharp f. \end{aligned}$$

□

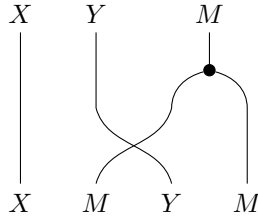
**Example 3.16.** The Kleisli category of the relevant monad  $\mathcal{P}_e$  discussed in Remark 2.33 is the category of sets and partial functions, which has diagonals.

**Remark 3.17.** As an instance of Proposition 3.15, we have that if  $\mathcal{C}$  is a cartesian monoidal category and  $T$  is relevant, then the Kleisli category  $\mathcal{C}_T$  has diagonals. Note instead that if  $\mathcal{C}$  either has projections or it is just a gs-monoidal category, then  $\mathcal{C}_T$  is a gs-monoidal category. The taxonomy is strict since the identity functor is a relevant monad.

This result has been proved in [?, Lem. 4.11], and it is now a consequence of the two propositions given above.

**Corollary 3.18.** *Let  $(T, \mu, \eta, c, u)$  be a gs-monoidal monad on a gs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  is cartesian monoidal then so is the Kleisli category  $\mathcal{C}_T$ .*

**Example 3.19.** Let  $(\mathcal{C}, \otimes, I)$  be a gs-monoidal category and  $M \in \mathcal{C}$  a monoid. Then the *action monad*  $(-) \otimes M : \mathcal{C} \rightarrow \mathcal{C}$  is a symmetric monoidal monad with  $u = i_M$  and  $c_{X, Y} : (X \otimes M) \otimes (Y \otimes M) \rightarrow (X \otimes Y) \otimes M$  given by



If  $M$  is connected (see Definition 2.38), then the monad is affine. If  $M$  is special (see again Definition 2.38), then the monad is relevant.

We close by dualising Proposition 3.10 and its consequences.

**Proposition 3.20.** *Let  $(T, \mu, \eta, c, u)$  be a symmetric monoidal monad on a cogs-monoidal category  $\mathcal{C}$ . Then the Kleisli category  $\mathcal{C}_T$  is a cogs-monoidal category with  $\text{match}_X$  and  $\text{new}_X$  given for every object  $X$  by*

$$\text{match}_X^\sharp := \eta_X \circ \text{match}_X, \quad \text{new}_X^\sharp := \eta_X \circ \text{new}_X.$$

□

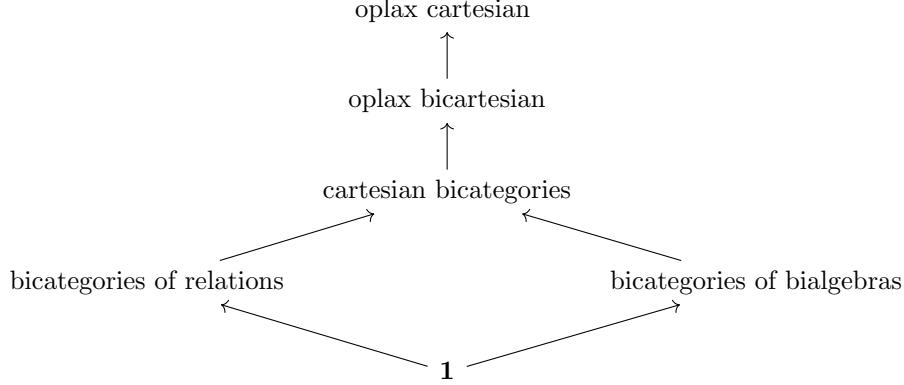
**Corollary 3.21.** *Let  $(T, \mu, \eta, c, u)$  be a symmetric monoidal monad on a bigs-monoidal/bialgebraic/Frobenius category  $\mathcal{C}$ . Then so is the Kleisli category  $\mathcal{C}_T$ .*

**Remark 3.22.** As noted, a cogs-monoidal category is the dualisation of a gs-monoidal category. The same dualisation can be exploited by introducing co-projections and codiagonals, as well as coaffine and corelevant functors. The results about Kleisli categories in this section can be simply restated for these notions. We leave them as well as their enriched versions in Appendix C, which also describes the dual of the results obtained in forthcoming Section 4.2.

## 4 A taxonomy of the enriched context

The 2-categorical formalism is a generalisation of the classical one, where the hom-sets are replaced by categories. This has been particular useful in the context of term and graph rewriting, where rewriting sequences are modelled as 2-cells (see e.g. [?] and the references therein). In this work we consider as 2-categorical context the one of preorder-enriched categories, where the hom-sets are preorders. This approach is well-suited in the context of categories for relations, which have a natural two-dimensional structure, given by set-theoretic inclusion. It is important to note that this enrichment becomes trivial if we consider functions instead of relations. In other words, functions are inherently one-dimensional structures, while relations provide a more complex view, which can be described through a richer categorical structure, such as a bicategory or a 2-category. We can find two relevant families of 2-categories that abstract the poset-enriched structure of **Rel**: Cartesian bicategories [?] by Carboni and Walters and allegories [?] by Freyd and Scedrov.

Hence, in the enriched context, the notion of gs-monoidal category is replaced by the notion of *preorder-enriched gs-monoidal category*. Requiring the oplax naturality of comultiplication  $\nabla_X$  and of discharger  $!_X$ , one obtains the notion of *oplax cartesian category*. If one requires both a comonoidal and monoidal structure on the objects and their oplax naturality, one obtains the notion of *oplax bicartesian category*. Cartesian bicategories can be then presented as oplax bicartesian categories in which the enrichment is posetal and satisfy three additional conditions. Special cases of cartesian bicategories are bicategories of relations (those satisfying *Frobenius law*) and bicategories of bialgebras. Notice that these two structures are incompatible in the sense that a bicategory of relations that is also a bicategory of bialgebras has to be the trivial one.



#### 4.1 Order-enriched categories

We have seen that gs-monoidal categories enjoy some features of **Rel** with respect to total and functional arrows. Our next step is to recall how to build on the notion of gs-monoidality to account for the usual poset-enrichment of **Rel**.

**Definition 4.1.** A *preorder-enriched gs-monoidal category*  $\mathcal{C}$  is a gs-monoidal category  $\mathcal{C}$  that is also a preorder-enriched monoidal category.

In the following we will often consider also poset-enriched categories. Recall that a preorder-enriched monoidal category consists of a preorder-enriched category  $\mathcal{C}$ , an object  $I$  of  $\mathcal{C}$ , a preorder-enriched functor  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and enriched natural monoidal structure isomorphisms

$$\lambda: I \otimes - \rightarrow \text{id}_{\mathcal{C}}, \quad \rho: - \otimes I \rightarrow \text{id}_{\mathcal{C}}, \quad \alpha: (- \otimes -) \otimes - \rightarrow - \otimes (- \otimes -)$$

such that the underlying category equipped with the underlying functor  $\otimes$ , the object  $I$ , and the natural isomorphisms  $\lambda$ ,  $\rho$ , and  $\alpha$  is a monoidal category (see [?] for details). Since a preorder-enriched functor is just an ordinary functor that is in addition monotone, the preorder structure and the monoidal structure are required to interact by the monotonicity of the tensor product  $\otimes$ ; the preorder-enrichment of the structure isomorphisms  $\lambda$ ,  $\rho$ , and  $\alpha$  does not add any additional condition since preorder-enrichment for natural transformations between preorder-enriched functors is trivial.

**Example 4.2.** Let us consider the category of spans  $\mathbf{Span}(\mathcal{A})$  presented in Example 2.4. It is well-known that this category has a natural 2-categorical structure, where the 2-cells are defined as follows

- a 2-cell  $\alpha: (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$  is an arrow  $\alpha: A \rightarrow B$  in  $\mathcal{A}$  such that the following diagram commutes

$$\begin{array}{ccc}
 & A & \\
 \swarrow & \downarrow \alpha & \searrow \\
 X & & Y \\
 \swarrow & \downarrow & \searrow \\
 & B & 
 \end{array}$$

- vertical composition of 2-cells is given by composition in  $\mathcal{A}$ ;
- horizontal composition of 2-cells as well as associators and unitors are induced by the universal property of pullbacks.

We will refer to  $\mathbf{PSpan}(\mathcal{A})$  as the preorder-enriched category obtained by considering the preorder-reflection of the previous 2-categorical structure, and to  $(\mathbf{PSpan}(\mathcal{A}), \times, 1)$  and  $(\mathbf{PSpan}(\mathcal{A}), +, 0)$  as its gs-monoidal counterparts. Given the correspondence presented in Remark 2.32, for a regular category  $\mathcal{A}$  in which every regular epi splits the poset-reflection of the previous 2-category boils down to what we call  $\mathbf{PRel}(\mathcal{A})$ , i.e. the usual poset-enrichment of  $\mathbf{Rel}(\mathcal{A})$ .

**Example 4.3.** Recall that a semiring  $(M, +, \cdot, 0, 1)$  comes equipped with a canonical preorder  $\leq_M$ , namely  $a \leq_M b$  if there exists  $c$  such that  $a + c = b$ , so the Kleisli category of the semiring monad  $\mathcal{M}$  introduced in Example 2.4.2 is canonically a preorder-enriched gs-monoidal category. Indeed, it is gs-monoidal thanks to Proposition 3.10 and is preorder-enriched, assuming that for two arrows  $f, g : X \rightarrow Y$  in  $\mathbf{Set}_{\mathcal{M}}$ ,  $f \leq g$  if  $\forall x \in X \forall y \in Y f(x)(y) \leq_M g(x)(y)$ .

Note that the same construction occurs if we have a general ordered semiring  $(M, \leq, +, \cdot, 0, 1)$ , i.e. where  $(M, \leq)$  is a preorder and  $+$  and  $\cdot$  are functions that are order-preserving on both arguments.

**Example 4.4.** Consider now the sub-category  $\mathbf{PreOrd}$  of  $\mathbf{Set}$  whose objects are preorders and whose arrows are monotone functions. It is cartesian monoidal, inheriting the structure from the direct product in  $\mathbf{Set}$ , and it comes equipped with a canonical preorder-enrichment, assuming that for two arrows  $f, g : X \rightarrow Y$  in  $\mathbf{PreOrd}$ ,  $f \leq g$  if  $\forall x \in X f(x) \leq g(x)$  in  $Y$ .

In a general preorder-enriched gs-monoidal category, no further compatibility with a gs-monoidal structure is required. However, most often additional axioms hold. We recall the notion of *oplax cartesian category* [? , Def. 3.2].

**Definition 4.5.** An *oplax cartesian category*  $\mathcal{C}$  is a preorder-enriched gs-monoidal category  $\mathcal{C}$  such that every arrow is *oplax copyable* and *oplax discardable*, i.e. the following inequalities hold for every arrow  $f : X \rightarrow Y$

$$\begin{array}{ccc}
 \begin{array}{c} \text{ } \\ \text{ } \\ \bullet \\ \text{ } \\ Y \\ | \\ \boxed{f} \\ | \\ X \end{array} & \leq & \begin{array}{c} Y \quad Y \\ | \quad | \\ \boxed{f} \quad \boxed{f} \\ \diagdown \quad \diagup \\ \bullet \\ | \\ X \end{array} \\
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \bullet \\ | \\ \boxed{f} \\ | \\ X \end{array} & \leq & \begin{array}{c} \bullet \\ | \\ \text{ } \\ | \\ X \end{array}
 \end{array}$$

The notion of *oplax cocartesian category* works as expected, starting from a cogs-monoidal category and preserving the direction of inequalities, and one gets the notion of *oplax bicartesian category* if both previous notions hold. In the following, we premise the adjective *strict* if the underlying order is a poset.

**Example 4.6.** The category  $(\mathbf{PSpan}(\mathcal{A}), \times, 1)$ , introduced in Example 4.2, is oplax cartesian [? , Prop. 5.3], while  $(\mathbf{PRel}(\mathcal{A}), \times, 1)$  is strictly so. In fact,



both are actually oplax bicartesian. As for **PreOrd** with the direct product, all arrows are actually copyable and discardable, since it is cartesian monoidal, hence it is oplax cartesian.

**Remark 4.7.** The notion of oplax cartesian category could be split in two, considering separately the arrows that are *oplax copyable* and those that are *oplax discardable*, as in the previous sections. Here we preferred to keep these two notions together, leaving implicit the obvious partition and generalisation of the forthcoming results, such as Proposition 4.16.

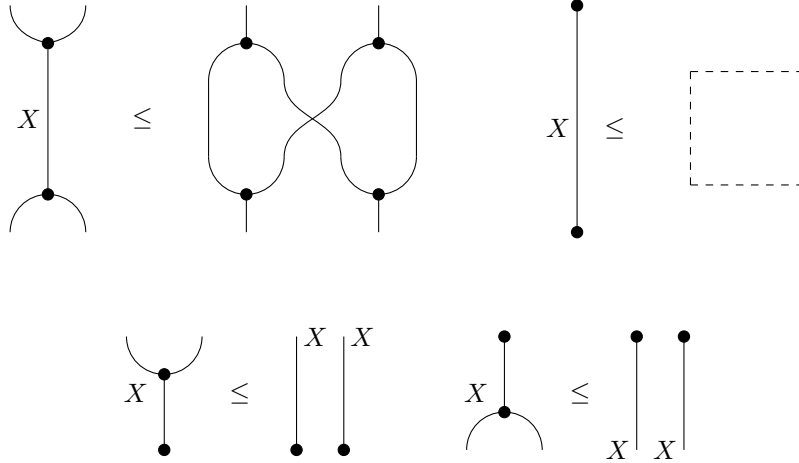
**Example 4.8.** Let  $\mathcal{M} : \mathbf{Set} \rightarrow \mathbf{Set}$  be the functor introduced in Section 2.4.2 for semiring  $(M, \oplus, \odot, 0, 1)$  equipped with a preorder such as in Example 4.3. Consider the following slight modifications of  $\mathcal{M}$

$$\mathcal{M}_e^s(X) = \{h : X \rightarrow M \mid h \text{ has support at most one and is sub-idempotent}\}$$

$$\mathcal{M}_u^s(X) = \{h : X \rightarrow M \mid h \text{ has finite support and is sub-normalised}\}$$

where sub-idempotent means that  $\forall x \in X. h(x) \leq h(x) \odot h(x)$  and sub-normalised that  $\bigoplus_{x \in X} h(x) \leq 1$ . In the Kleisli category  $\mathbf{Set}_{\mathcal{M}_e^s}$  every arrow is oplax copyable, while in  $\mathbf{Set}_{\mathcal{M}_u^s}$  every arrow is oplax discardable.

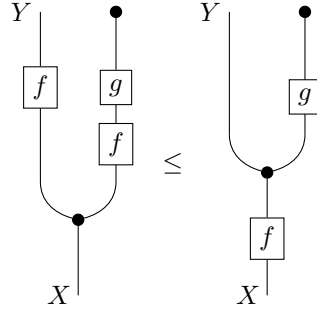
**Remark 4.9.** Note that in every bigs-monoidal category that is either oplax cartesian or oplax cocartesian the following inequalities hold



In other terms, each object is a *lax* (bicommutative) bimonoid or, equivalently, such categories have a *lax* bialgebraic structure.

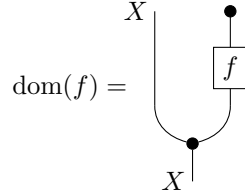
In the following proposition we observe that the notion of oplax cartesian category may subsume that of restriction category.

**Proposition 4.10.** *Let  $\mathcal{C}$  be a strict oplax cartesian category such that the following inequality holds for every pair of arrows  $f : X \rightarrow Y$  and  $g : Y \rightarrow W$*



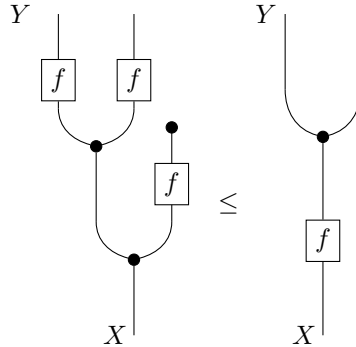
Then  $\mathcal{C}$  is a restriction category.

*Proof.* We already noted in the proof of Proposition 2.48 that the axioms (R.2) and (R.3) of Definition B.1 hold for any gs-monoidal category when we consider the restriction structure given by the domain of an arrow



Now, the proof that (R.1) also holds follows from [?, Prop. 3.6]. As for (R.4), it is enough to employ the first axiom of oplax cartesian categories and the additional requirement holding for  $\mathcal{C}$ .  $\square$

**Remark 4.11.** The previous proposition shows that the notions of strict oplax cartesian category and restriction category are very closely related, since the former satisfies the first three requirements (R.1)-(R.3) of the latter, and the last condition (R.4) is recovered by the additional inequality in Proposition 4.10. Assuming such inequality, the restriction structure induced by an oplax cartesian category satisfies all the necessary conditions for having restriction products in the sense of [?, p. 20], except for an inequality of the final condition, namely



It is a simple check, and indeed a sanity check, that in strict oplax cartesian categories this condition corresponds via (R.1) to the naturality of duplicators.

We close the thread with a result concerning positivity (see Definition 2.12).

**Corollary 4.12.** *Let  $\mathcal{C}$  be a strict oplax cartesian category. If it is positive, then it is a restriction category.*

*Proof.* It is easy to note that in a strict oplax cartesian category for any arrow  $h : X \rightarrow W$  we have that  $!_W \circ h$  is functional. Combined with positivity, it implies the inequality in the statement of Proposition 4.10, hence (R.4).  $\square$

On functors between oplax cartesian categories, one often has additional inequalities, which take the following form.

**Definition 4.13.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be preorder-enriched gs-monoidal categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$  a preorder-enriched lax symmetric monoidal functor with structure arrows  $\psi, \psi_0$ . Then  $F$  is called **colax affine** if the following inequality holds*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(!_X)} & F(I) \\ & \searrow \scriptstyle !_X & \nearrow \scriptstyle \psi_0 \\ & I & \end{array} \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$$

and it is called **colax relevant** if the following inequality holds

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\nabla_X)} & F(X \otimes X) \\ & \searrow \scriptstyle \nabla_{F(X)} & \nearrow \scriptstyle \psi_{X,X} \\ & F(X) \otimes F(X) & \end{array} \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$$

If  $F$  is both colax affine and colax relevant it is called **colax gs-monoidal**<sup>6</sup>.

**Example 4.14.** Let  $\mathcal{C}$  be a locally small oplax cartesian category. Then, for every object  $A$  of  $\mathcal{C}$ , the representable functor  $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{PreOrd}$  has a canonical colax gs-monoidal structure given by

$$\psi_{X,Y} : \mathcal{C}(A, X) \times \mathcal{C}(A, Y) \rightarrow \mathcal{C}(A, X \otimes Y)$$

sending  $f : A \rightarrow X$  and  $g : A \rightarrow Y$  to  $(f \otimes g) \circ \nabla_A$ , and

$$\psi_0 : I \rightarrow \mathcal{C}(A, I)$$

sending the unique element of  $I$  to  $!_A$ . We refer to [? , Thm. 6.3] for details.

## 4.2 An enriched taxonomy of Kleisli categories

**Definition 4.15.** *A symmetric monoidal monad  $(T, \mu, \eta, c, u)$  on a preorder-enriched gs-monoidal category is said to be a **colax gs-monoidal monad** if  $T$  is a colax gs-monoidal functor.*

<sup>6</sup>The use of “colax” refers to the direction of the 2-cell, namely from  $F(\nabla_A)$  to  $\psi_{A,A} \circ \nabla_{FA}$ . Note also that in [? ] colax gs-monoidal functors were called colax cartesian.

**Proposition 4.16.** *Let  $(T, \mu, \eta, c, u)$  be a colax gs-monoidal monad on a preorder-enriched gs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  is oplax cartesian then so is the Kleisli category  $\mathcal{C}_T$ .*

*Proof.* Recall that  $\mathcal{C}_T$  is gs-monoidal thanks to Proposition 3.10 and inherits the preorder-enrichment from  $\mathcal{C}$ . We now prove that it is oplax cartesian.

We first show that for every arrow  $f : X \rightarrow Y$  in  $\mathcal{C}_T$ , i.e.  $f : X \rightarrow TY$  in  $\mathcal{C}$ , it holds that  $!_Y^\sharp \circ^\sharp f \leq !_X^\sharp$  in  $\mathcal{C}_T$ , i.e. that

$$T(!_Y) \circ f \leq \eta_I \circ !_X$$

in  $\mathcal{C}$ . Applying first the assumption that  $T$  is colax affine (i.e. that  $T(!_Y) \leq \eta_I \circ !_T(Y)$ ) and then the fact that  $\mathcal{C}$  is oplax cartesian (in particular, that  $!_{T(Y)} \circ f \leq !_T(X)$ ), we obtain

$$T(!_Y) \circ f \leq \eta_I \circ !_T(Y) \circ f \leq \eta_I \circ !_T(X).$$

It remains to prove that for every  $f : X \rightarrow Y$  in  $\mathcal{C}_T$  it holds that  $\nabla_Y^\sharp \circ^\sharp f \leq (f \otimes^\sharp f) \circ^\sharp \nabla_X^\sharp$  in  $\mathcal{C}_T$ , i.e. that

$$T(\nabla_Y) \circ f \leq (\mu_{Y \otimes Y} \circ T(c_{Y,Y}) \circ T(f \otimes f)) \circ (\eta_{X \otimes X} \circ \nabla_X)$$

in  $\mathcal{C}$ . Notice that, using the naturality of  $\eta$  and  $\mu$ , the previous inequality happens to be equivalent to

$$T(\nabla_Y) \circ f \leq c_{Y,Y} \circ (f \otimes f) \circ \nabla_X$$

in  $\mathcal{C}$ . But this can be easily derived as follows

$$T(\nabla_Y) \circ f \leq c_{Y,Y} \circ \nabla_{T(Y)} \circ f \leq c_{Y,Y} \circ (f \otimes f) \circ \nabla_X$$

by first using the fact that  $T$  is colax relevant (i.e.  $T(\nabla_Y) \leq c_{Y,Y} \circ \nabla_{T(Y)}$ ), and then the oplax cartesianity of  $\mathcal{C}$  (in particular, that  $\nabla_{T(Y)} \circ f \leq (f \otimes f) \circ \nabla_X$ ).  $\square$

**Example 4.17.** Given a preorder  $(X, \leq)$ , the Hoare preorder on its subsets is given by  $U \leq_d V$  if for any  $x \in U$  there exists  $y \in V$  such that  $x \leq y$ . Or, equivalently, if  $\downarrow U \subseteq \downarrow V$ , for  $\downarrow U = \{x \in X \mid \exists u \in U. x \leq u\}$  the *downward-closure* of  $X$ .

Consider now the finite subset functor  $\mathcal{P}$  on **Set**, and recall that the subcategory **PreOrd** is preorder-enriched (see Example 4.4). Now,  $\mathcal{P}$  can be extended to a preorder-enriched functor **PreOrd**  $\rightarrow$  **PreOrd** with the preorder  $\leq_d$  on subsets given above. Indeed, if  $f \leq g : (X, \leq) \rightarrow (Y, \leq)$ , then  $\mathcal{P}(f) \leq \mathcal{P}(g)$  since  $\mathcal{P}(f)(U) = \bigcup_{x \in U} f(x) \leq_d \bigcup_{x \in U} g(x) = \mathcal{P}(g)(U)$  for all  $U \subseteq X$ .

In fact,  $\mathcal{P}$  is a colax gs-monoidal monad and it becomes affine or relevant if one restricts respectively to subsets with at least or at most one element, as done in Remark 2.33.

**Remark 4.18.** Consider now the full sub-category **PO** of **PreOrd** whose objects are posets, which is poset-enriched. The functor  $\mathcal{P}^\downarrow : \mathbf{PO} \rightarrow \mathbf{PO}$  assigns to each poset  $(X, \leq)$  its (possibly infinite) downward-closed subsets: the preorder  $\leq_d$  in Example 4.17 is a partial order, since it coincides with subset inclusion. As for functions, if  $f : (X, \leq) \rightarrow (Y, \leq)$ , then  $\mathcal{P}^\downarrow(f)$  sends a downward-closed subset  $U$  of  $X$  into the downward-closed subset  $\downarrow \bigcup_{x \in U} f(x)$  of  $Y$ .

It is easy to check that  $\mathcal{P}^\downarrow$  is a poset-enriched functor. It is a colax gs-monoidal monad, with essentially the same structure arrows as  $\mathcal{P}$ , so that e.g.  $\eta_X : (X, \leq) \rightarrow (\mathcal{P}^\downarrow(X), \subseteq)$  sends  $x \in X$  to  $\downarrow \{x\}$ . This monad becomes affine when restricted to non-empty downward-closed subsets.

The monad  $\mathcal{P}^\downarrow$  coincides with the *lower subset monad* in [?] Also, as shown in [?], the algebras of  $\mathcal{P}^\downarrow$  are posets with arbitrary sups, and the free algebras are supercoherent posets, i.e. posets with arbitrary suprema and where every element is a union of supercompact elements. Similar considerations can be made for the category **PO**<sub>⊥</sub> of pointed posets [?].

**Example 4.19.** Let  $\mathcal{C}$  be a preorder-enriched gs-monoidal category and  $M \in \mathcal{C}$  a monoid that is *lax* special and *lax* connected, i.e. such that it holds

$$\begin{array}{ccc} \begin{array}{c} | \\ M \end{array} & \leq & \begin{array}{c} \bullet \\ \circlearrowleft \\ \bullet \\ | \\ M \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} | \\ M \end{array} & \leq & \begin{array}{c} \bullet \\ \bullet \\ | \\ M \end{array} \end{array}$$

Then *action monad*  $(-) \otimes M : \mathcal{C} \rightarrow \mathcal{C}$  is colax gs-monoidal with the same structure maps as in Example 3.19.

Looking at Example 2.45 and Example 4.6, we have that  $(\mathbf{PSpan}(\mathcal{A}), \times, 1)$  is a special lax connected oplax bicartesian category (i.e. it is oplax bicartesian and each object is special and lax connected), yet not connected.

**Example 4.20.** Given a preorder  $(X, \leq)$ , we define the *upward-closure* of a subset  $U \subseteq X$  as  $\uparrow U = \{x \in X \mid \exists u \in U. u \leq x\}$ . Consider also a semiring  $(M, \oplus, \odot, 0, 1)$  and the functor  $\mathcal{M} : \mathbf{Set} \rightarrow \mathbf{Set}$  discussed in Example 2.4.2. If  $M$  is preorder-enriched and 0 is a minimal element for such preorder, then  $\mathcal{M}(X)$  can be equipped with a preorder given as  $h \leq_u k : X \rightarrow M$  whenever for every upward-closed subset  $U \subseteq X$

$$\bigoplus_{x \in U} h(x) \leq \bigoplus_{x \in U} k(x)$$

We can then extend  $\mathcal{M}$  to a preorder-enriched functor  $\mathcal{M} : \mathbf{PreOrd} \rightarrow \mathbf{PreOrd}$ . Let  $f : (X, \leq) \rightarrow (Y, \leq)$  be a monotone function and note that for every upward-closed subset  $V \subseteq Y$  it holds that  $f^{-1}(V)$  is upward-closed. To prove that  $\tilde{f}$  is monotone it suffices to see that for  $h \leq_u k$  and  $V$  upward-closed we have

$$\bigoplus_{y \in V} \bigoplus_{x \in f^{-1}(y)} h(x) = \bigoplus_{x \in f^{-1}(V)} h(x) \leq \bigoplus_{x \in f^{-1}(V)} k(x) = \bigoplus_{y \in V} \bigoplus_{x \in f^{-1}(y)} k(x)$$

To prove that it is preorder-enriched we need to show that if  $f \leq g : X \rightarrow Y$  then for every  $h : X \rightarrow M$  it holds  $\tilde{f}(h) \leq_u \tilde{g}(h) : Y \rightarrow M$ . Indeed, for every  $V \subseteq Y$  upward-closed, if  $y \in V$  and  $x \in f^{-1}(y)$  then  $y \leq g(x)$  and  $g(x) \in V$ . Hence

$$\bigoplus_{y \in V} \bigoplus_{x \in f^{-1}(y)} h(x) \leq \bigoplus_{y \in V} \bigoplus_{x \in g^{-1}(y)} h(x)$$

since 0 is a minimal element for the preorder on  $\mathcal{M}$  and thus  $a \leq a \oplus b$  always holds. Moreover,  $\mathcal{M}$  is a symmetric monoidal monad with the usual structure arrows

$$\begin{aligned} - \eta_X(x_0)(x) &= \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} \\ - \mu_X(\lambda)(x) &= \bigoplus_{h \in \mathcal{M}(X)} \lambda(h) \odot h(x) \end{aligned}$$

Recall now Example 4.8. The monad  $\mathcal{M}$  just defined becomes colax affine if one restricts  $\mathcal{M}(X)$  to functions with finite support and sub-normalised; it is affine if one further restricts to normalised functions. Moreover, it is colax relevant if one restricts  $\mathcal{M}(X)$  to sub-idempotent functions (thanks to 0 being minimal we may drop the requirement that the support is at most one for  $\mathcal{M}_e^s(X)$  in Example 4.8); it is relevant if one further restricts to idempotent functions with support at most one. If we consider the semiring of positive real numbers and restrict  $\mathcal{M}(X)$  to normalised functions, then the above monad can be viewed as a discrete version of the preordered Kantorovich monad, as defined in [? ].

**Example 4.21.** Let us follow up on the previous example and assume that  $M$  is a *dioid*, i.e. that the addition is idempotent. It is well-known that the canonical preorder  $\leq_M$  is now a partial order, and it can be equivalently defined as  $a \leq_M b$  if  $a \oplus b = b$ . In fact,  $\leq_M$  is a join-semilattice, since  $a \vee b = a \oplus b$  and  $\perp = 0$ . Let us further assume that such a partial order is actually total. Then, the preorder  $\leq_u$  can be described as  $h \leq_u k$  if for any  $x \in \text{supp}(h)$  there exists  $y \in \text{supp}(k)$  such that  $x \leq y$  and  $h(x) \leq_M k(y)$ . For  $\mathcal{M}$  the Boolean semiring, we recover the Hoare preorder of Example 4.17.

### 4.3 Some properties of maps

Following the usual notation adopted in the context of cartesian bicategories [? ], we recall the notion of *adjoint* and *map*.

**Definition 4.22.** Let  $\mathcal{C}$  a poset-enriched category. An arrow  $f : X \rightarrow Y$  is left adjoint to an arrow  $f^* : Y \rightarrow X$  if

$$\begin{array}{c} X \\ | \\ X \end{array} \leq \begin{array}{c} X \\ | \\ \boxed{f^*} \\ | \\ \boxed{f} \\ | \\ X \end{array} \quad \begin{array}{c} Y \\ | \\ \boxed{f} \\ | \\ \boxed{f^*} \\ | \\ Y \end{array} \leq \begin{array}{c} Y \\ | \\ Y \end{array}$$

Equivalently,  $f^*$  is right adjoint to  $f$ . In symbols  $f \dashv f^*$ .

**Definition 4.23.** Let  $\mathcal{C}$  be a strict oplax cartesian category. An arrow  $f: X \rightarrow Y$  is a **map** if it has a right adjoint  $f \dashv f^*$ .

We denote by  $\mathbf{Map}(\mathcal{C})$  the poset-enriched sub-category of  $\mathcal{C}$  whose objects are those of  $\mathcal{C}$ , and whose arrows are maps.

**Lemma 4.24.** Let  $\mathcal{C}$  be a strict oplax cartesian category. Then every map is  $\mathcal{C}$ -functional and  $\mathcal{C}$ -total.

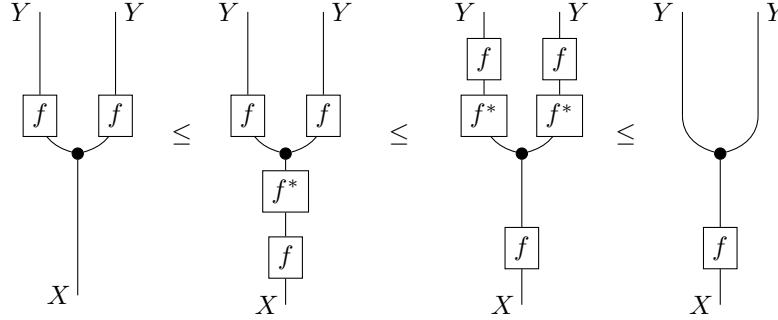
*Proof.* By definition of oplax cartesian category, we just need to prove that every map  $f: X \rightarrow Y$  satisfies

$$\begin{array}{c} Y \\ | \\ \boxed{f} \\ \bullet \\ \boxed{f} \\ | \\ X \end{array} \leq \begin{array}{c} Y \\ \cup \\ \bullet \\ | \\ \boxed{f} \\ | \\ X \end{array} \quad \begin{array}{c} \bullet \\ | \\ X \end{array} \leq \begin{array}{c} \bullet \\ | \\ \boxed{f} \\ | \\ X \end{array}$$

To prove that every map is  $\mathcal{C}$ -total we just need to observe that, by definition of left adjoint, we have that

$$\begin{array}{c} | \\ | \\ X \end{array} \leq \begin{array}{c} | \\ \boxed{f^*} \\ | \\ \boxed{f} \\ | \\ X \end{array} \Rightarrow \begin{array}{c} \bullet \\ | \\ X \end{array} \leq \begin{array}{c} \bullet \\ | \\ \boxed{f^*} \\ | \\ \boxed{f} \\ | \\ X \end{array} \leq \begin{array}{c} \bullet \\ | \\ \boxed{f} \\ | \\ X \end{array}$$

where the last inequality follows by the second axiom of oplax cartesian categories (applied to  $f^*$ ). Similarly, we can prove that every map is  $\mathcal{C}$ -functional as follows



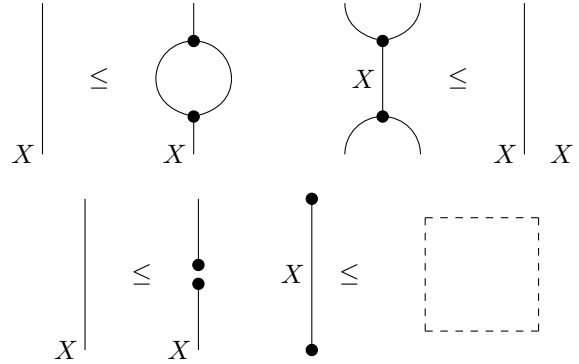
where the first and the last inequalities follow from the fact that  $f \dashv f^*$ , and the second one follows from the first inequality in the definition of oplax cartesian category.  $\square$

**Remark 4.25.** Notice that in general  $\mathbf{Map}(\mathcal{C})$  does not coincide with  $\mathcal{C}\text{-}\mathbf{TFun}$ , i.e. an arrow that is both  $\mathcal{C}$ -total and  $\mathcal{C}$ -functional is not necessarily a map. Indeed,  $\mathbf{Map}(\mathcal{C})$  is just a poset-enriched monoidal sub-category of  $\mathcal{C}\text{-}\mathbf{TFun}$ .

#### 4.4 Cartesian bicategories through a gs-monoidal lens

Now we recall the well-known notion of *cartesian bicategory* [?, Def. 1.2], presenting it in terms of oplax cartesian category.

**Definition 4.26.** A *cartesian bicategory*  $\mathcal{C}$  is a strict oplax cartesian category such that for every object  $X$ , the arrows  $\nabla_X : X \rightarrow X \otimes X$  and  $!_X : X \rightarrow I$  are maps, i.e. there are arrows  $\Delta_X : X \otimes X \rightarrow X$  and  $i_X : I \rightarrow X$  such that



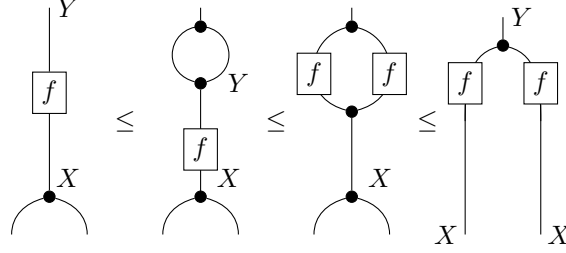
These inequalities act on orthogonal components, and thus hold for a large number of constructions. As partly shown later, they hold for bicategories of spans and cospans and coproduct on **Set**, sometimes with an equality [?].

The overloaded choice of the symbols  $\Delta_X$  and  $i_X$  is by no means by chance. Indeed, the following property holds.

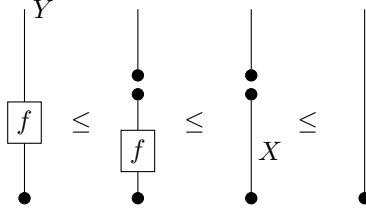
**Proposition 4.27.** Every cartesian bicategory  $\mathcal{C}$  is oplax cocartesian.



*Proof.* First, notice that  $\mathcal{C}$  is a cogs-monoidal category. Indeed, observe that all the equations that  $\Delta_X$  and  $i_X$  have to satisfy follow from the fact that the right adjoint is unique and from the right adjoint inequalities in Definition 4.26, see also [?, Rem. 1.3]. The two inequalities required in the definition of oplax cocartesian category can be easily proved as follows: for every arrow  $f : X \rightarrow Y$



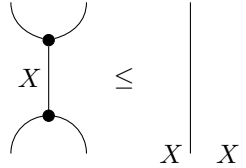
where the first and the last inequality follow by the properties of right adjoints, while the second one follows from the oplax cartesianity. Similarly, for every arrow  $f : X \rightarrow Y$



□

We can then sum up the relationship between cartesian bicategories and the gs-monoidal framework.

**Corollary 4.28.** *Cartesian bicategories correspond exactly to strict oplax bicartesian categories that are lax special and lax connected and satisfy*



*Proof.* The proposition above implies that a cartesian bicategory is an oplax bicartesian category, see Definition 4.5. Now, Remark 4.9 ensure that an oplax bicartesian category is also lax algebraic, hence the rightmost axiom on the bottom of Definition 4.26 holds. Then, being lax special and lax connected, see Example 4.19, the additional requirement ensures the correspondence with the remaining inequalities holding for cartesian bicategories. □

**Definition 4.29.** A **bicategory of bialgebras** is a cartesian bicategory that is also a bialgebraic category.

As before, a key difference are the additional laws for their interaction [? ].

**Definition 4.30.** A *bicategory of relations* is a cartesian bicategory that is also a Frobenius category.

**Example 4.31.** As expected,  $(\mathbf{PRel}(\mathcal{A}), \times, 1)$  is a bicategory of relations and  $(\mathbf{PRel}(\mathcal{A}), +, 0)$  a bicategory of bialgebras.

**Remark 4.32.** The notion of bicategory of relations is equivalent to the notion of *unitary pretabular allegory* in the sense of Freyd and Scedrov [? ]. Moreover, a bicategory of relations  $\mathcal{C}$  happens to be biequivalent to the regular category  $\mathbf{Map}(\mathcal{C})$  under the further assumption of being *functionally complete*, as shown in [? , Th. 3.5]. We refer to [? ] for more detail.

**Remark 4.33.** One of the useful consequence of the Frobenius law is that, as it happens in **Rel**, in every cartesian bicategory of relations, if  $f \leq g$  and both are maps then they are equal [? ], i.e. the order in  $\mathbf{Map}(\mathcal{C})$  is discrete.

**Remark 4.34.** We recall from [? ] that the category  $\mathbf{Map}(\mathcal{C})$  of a cartesian bicategory is actually cartesian. Moreover, combining Corollary 2.26 with it is a cartesian sub-category of  $\mathcal{C}\text{-}\mathbf{TFun}$ . However, the cartesian structure is not enough to conclude that every  $\mathcal{C}$ -function and  $\mathcal{C}$ -total arrow is a map. Indeed, this happens when  $\mathcal{C}$  is a bicategory of relations [? , Lem. 2.5].

**Proposition 4.35.** Let  $(T, \mu, \eta, c, u)$  be a colax gs-monoidal monad on a poset-enriched gs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  is a cartesian bicategory then so is the Kleisli category  $\mathcal{C}_T$ .

*Proof.* By Proposition 4.16, we have that  $\mathcal{C}_T$  is oplax cartesian. Moreover, by defining  $\Delta_X^\sharp : X \otimes X \rightarrow X$  and  $i_X^\sharp : I \rightarrow X$  in  $\mathcal{C}_T$  as the arrows  $\Delta_X^\sharp := \eta_X \circ \Delta_X$  and  $i_X^\sharp := \eta_X \circ i_X$  of  $\mathcal{C}$ , we obtain right adjoints to  $\nabla^\sharp$  and  $!^\sharp$  since the functor  $\mathcal{K} : \mathcal{C} \rightarrow \mathcal{C}_T$  is poset-enriched.  $\square$

**Corollary 4.36.** Let  $(T, \mu, \eta, c, u)$  be a colax gs-monoidal monad on a poset-enriched gs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  is a bicategory of bialgebras/of relations then so is the Kleisli category  $\mathcal{C}_T$ .

*Proof.* It follows from the fact that the functor  $\mathcal{K} : \mathcal{C} \rightarrow \mathcal{C}_T$  preserves the equalities of arrows obtained through composition and products of structural maps, see Remark 3.8.  $\square$

## 5 Conclusions and further work

The aim of the paper has been twofold. On the one side, we strove for putting some order in the array of those variants of symmetric monoidal categories, possibly poset-enriched, that has been proposed with computational and graphical aims in recent years. We hope that our streamlined presentation in terms of gs-monoidal categories could be beneficial to the interested community. On

the other side, we presented a series of results concerning the gs-monoidal and oplax cartesian structures of Kleisli categories, putting some order also on those variants of enrichments of commutative monads proposed in the literature.

Future threads of investigations include the combinatorial presentation of free gs-monoidal categories in terms of classes of graphs, as shown for string diagrams and hyper-graphs in e.g. [? ]. Another line of work concerns the extension of the taxonomy towards *traced* gs-monoidal categories [? ? ], in order to account for systems with feedback, and the lively area investigating alternative notions of Markov categories [? ? ] and affine monads [? ? ], aimed at distilling a categorical presentation of probability theory, see e.g. [? ]. And finally, the exploration of completeness theorems for functorial semantics, see e.g. [? ].

## Appendix A Lax monoidal functors

This section recalls the details of the definition of lax monoidal functor, see e.g. [? ]. Throughout,  $\mathcal{C}$  and  $\mathcal{D}$  are symmetric monoidal categories with tensor functor  $\otimes$  and monoidal unit  $I$ , and we assume that  $\otimes$  strictly associates without loss of generality in order to keep the diagrams simple. Left and right unitors are denoted by  $\lambda$  and  $\rho$ , respectively<sup>7</sup>, and braidings by  $\gamma$ .

**Definition A.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is ***lax monoidal*** if it is equipped with a natural transformation

$$\psi: \otimes \circ (F \times F) \rightarrow F \circ \otimes$$

and an arrow  $\psi_0: I \rightarrow F(I)$  such that the associativity diagrams

$$\begin{array}{ccc} F(A) \otimes F(B) \otimes F(C) & \xrightarrow{\text{id} \otimes \psi_{B,C}} & F(A) \otimes F(B \otimes C) \\ \psi_{A,B} \otimes \text{id} \downarrow & & \downarrow \psi_{A,B \otimes C} \\ F(A \otimes B) \otimes F(C) & \xrightarrow{\psi_{A \otimes B, C}} & F(A \otimes B \otimes C) \end{array}$$

and the unitality diagrams commute

$$\begin{array}{ccc} I \otimes F(A) & \xleftarrow{\lambda_{FA}} & F(A) \\ \psi_0 \otimes \text{id} \downarrow & & \downarrow F(\lambda_A) \\ F(I) \otimes F(A) & \xrightarrow{\psi_{I,A}} & F(I \otimes A) \end{array} \quad \begin{array}{ccc} F(A) \otimes I & \xleftarrow{\rho_{FA}} & F(A) \\ \text{id} \otimes \psi_0 \downarrow & & \downarrow F(\rho_A) \\ F(A) \otimes F(I) & \xrightarrow{\psi_{A,I}} & F(A \otimes I). \end{array} \quad (5)$$

$F$  is called ***lax symmetric monoidal*** if also the following diagram commutes

---

<sup>7</sup>Strict unitality could be assumed, but it would make some diagrams potentially confusing.

$$\begin{array}{ccc}
F(A) \otimes F(B) & \xrightarrow{\gamma_{F(A), F(B)}^{\mathcal{D}}} & F(B) \otimes F(A) \\
\downarrow \psi_{A,B} & & \downarrow \psi_{B,A} \\
F(A \otimes B) & \xrightarrow{F(\gamma_{A,B}^{\mathcal{C}})} & F(B \otimes A)
\end{array}$$

For example, if  $\mathcal{C}$  is the terminal monoidal category with only one object  $I$  and  $\text{id}_I$  as the only arrow, then  $F$  is simply a monoid in  $\mathcal{D}$ . We do not spell out the following dual version in full detail.

We also have **strong symmetric monoidal functors**, which are lax symmetric monoidal functors with invertible structure arrows; and **strict symmetric monoidal functors**, in which the structure arrows are identities.

**Definition A.2.** A *monoidal transformation* between lax monoidal functors  $\epsilon: (F, \psi_0, \psi) \rightarrow (F', \psi'_0, \psi') : \mathcal{C} \rightarrow \mathcal{D}$  is a family of arrows  $\epsilon_X : F(X) \rightarrow F'(X)$ , for  $X \in \mathcal{C}$ , satisfying

$$\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\epsilon_X \otimes \epsilon_Y} & F'(X) \otimes F'(Y) \\
\downarrow \psi & & \downarrow \psi' \\
F(X \otimes Y) & \xrightarrow{\epsilon_{X \otimes Y}} & F'(X \otimes Y)
\end{array}
\quad
\begin{array}{ccc}
I & \xrightarrow{\psi_0} & F(I) \\
\searrow \psi'_0 & & \swarrow \epsilon_I \\
& F'(I) &
\end{array}$$

When  $\epsilon$  is also a natural transformation between the underlying functors  $F, F'$  it is called **monoidal natural transformation**.

A monoid and comonoid structure on an object in a symmetric monoidal category often interact in a nice way, either such that they form a *bimonoid* or a *Frobenius monoid* (and sometimes both).

## Appendix B Restriction categories

**Definition B.1.** A category  $\mathcal{C}$  is a **restriction category** if there exists an assignment which sends every arrow  $f : X \rightarrow Y$  of  $\mathcal{C}$  to an arrow  $\bar{f} : X \rightarrow X$  such that the following conditions hold

- (R.1)  $f \circ \bar{f} = f$ ,
- (R.2)  $\bar{f} \circ \bar{g} = \bar{g} \circ \bar{f}$  for  $g : X \rightarrow W$ ,
- (R.3)  $\overline{g \circ f} = \bar{g} \circ \bar{f}$  for  $g : X \rightarrow W$ ,
- (R.4)  $\bar{g} \circ f = f \circ \overline{g \circ f}$  for  $g : Y \rightarrow W$ .

An arrow  $f : X \rightarrow Y$  in  $\mathcal{C}$  is **total** if  $\bar{f} = \text{id}_X$ .

A restriction category has **restriction terminal object** if there exists an object  $I$  such that for every  $X \in \mathcal{C}$  there is a total arrow  $t_X : X \rightarrow I$  such that  $t_I = \text{id}_I$  and for every  $f : X \rightarrow Y$  we have  $t_Y \circ f = t_X \circ \bar{f}$ .

A restriction category has **restriction binary products** if there exist a restriction functor  $- \times - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  (i.e. such that  $\overline{f \times g} = \overline{f} \times \overline{g}$ ) and total arrows  $\Delta : X \rightarrow X \times X$ ,  $p : X \times Y \rightarrow X$  and  $q : X \times Y \rightarrow Y$  satisfying

$$\begin{array}{ccc}
 & X & \\
 id \swarrow & \Delta \downarrow & \searrow id \\
 X & \xleftarrow{p} X \times X \xrightarrow{q} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 & X \times Y & \\
 \Delta \downarrow & & \searrow id \\
 X \times Y \times X \times Y & \xrightarrow{p \times q} & X \times Y
 \end{array}$$
  

$$\begin{array}{ccccc}
 X \times Y & \xleftarrow{\overline{f} \times \overline{g}} & X \times Y & \xrightarrow{\overline{f} \times \overline{g}} & X \times Y \\
 p \downarrow & & \downarrow f \times g & & \downarrow q \\
 X & & & & Y \\
 f \downarrow & & & & \downarrow g \\
 X' & \xleftarrow{p} X' \times Y' \xrightarrow{q} & Y' & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{\overline{f}} & X \\
 f \downarrow & & \downarrow \Delta \\
 X' & \xrightarrow{\Delta} & X' \times X' \\
 & & \downarrow f \times f
 \end{array}$$

A restriction category has **restriction products** if it has restriction terminal object and restriction binary products.

## Appendix C More on cogs-monoidal categories

**Definition C.1.** For cogs-monoidal categories  $\mathcal{C}$  and  $\mathcal{D}$ , a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  equipped with a lax symmetric monoidal structure

$$\psi : \otimes \circ (F \times F) \rightarrow F \circ \otimes, \quad \psi_0 : I \rightarrow F(I)$$

is **coaffine** if the following diagram commutes for all  $X$  in  $\mathcal{C}$

$$\begin{array}{ccc}
 F(X) & \xleftarrow{F(i_X)} & F(I) \\
 i_{FX} \swarrow & & \nearrow \psi_0 \\
 & I & 
 \end{array} \tag{6}$$

and it is **corelevant** if the following diagram commutes for all  $X$  in  $\mathcal{C}$

$$\begin{array}{ccc}
 F(X) & \xleftarrow{F(\Delta_X)} & F(X \otimes X) \\
 \Delta_{FX} \swarrow & & \nearrow \psi_{X,X} \\
 & F(X) \otimes F(X) & 
 \end{array} \tag{7}$$

A functor which is both coaffine and corelevant is **cogs-monoidal**.

**Proposition C.2.** Let  $(T, \mu, \eta, c, u)$  be a symmetric monoidal monad on a cogs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  has coprojections then so does the Kleisli category  $\mathcal{C}_T$ .

*Proof.* If  $I$  is initial then it is so also in  $\mathcal{C}_T$  since the functor  $\mathcal{C} \rightarrow \mathcal{C}_T$  preserves colimits.  $\square$

**Remark C.3.** In the above proposition it is not necessary to assume the monad to be coaffine. Indeed, if  $I$  is initial then any symmetric monoidal monad is trivially coaffine.

**Proposition C.4.** Let  $(T, \mu, \eta, c, u)$  be a corelevant monad on a cogs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  has codiagonals then so does the Kleisli category  $\mathcal{C}_T$ .

*Proof.* We have to prove that for every  $f : X \rightarrow Y$  in  $\mathcal{C}_T$ , which corresponds to an arrow  $f : X \rightarrow T(Y)$  in  $\mathcal{C}$ , it holds that  $f \circ^\# \Delta_X^\# = \Delta_Y^\# \circ^\# (f \otimes_T f)$  in  $\mathcal{C}_T$ . Indeed

$$\begin{aligned} f \circ^\# \Delta_X^\# &= f \circ \Delta_X \\ &= \Delta_{T(Y)} \circ (f \otimes f) && (\mathcal{C} \text{ codiagonals}) \\ &= T(\Delta_Y) \circ c_{Y,Y} \circ (f \otimes f) && (T \text{ corelevant}) \\ &= \Delta_Y^\# \circ^\# (f \otimes_T f) \end{aligned}$$

$\square$

**Definition C.5.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be oplax cocartesian categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$  a preorder-enriched lax symmetric monoidal functor with structure arrows  $\psi, \psi_0$ . The  $F$  is called **colax coaffine** if the following inequality holds

$$\begin{array}{ccc} F(X) & \xleftarrow{F(i_X)} & F(I) \\ & \swarrow \leq \searrow & \\ i_{F(X)} & I & \psi_0 \end{array}$$

and it is called **colax corelevant** if following inequality holds

$$\begin{array}{ccc} F(X) & \xleftarrow{F(\Delta_X)} & F(X \otimes X) \\ & \swarrow \leq \searrow & \\ \Delta_{F(X)} & F(X) \otimes F(X) & \psi_{X,X} \end{array}$$

If  $F$  is both colax affine and colax relevant it is called **colax cogs-monoidal**.

**Proposition C.6.** Let  $(T, \mu, \eta, c, u)$  be a colax cogs-monoidal monad on a preorder-enriched cogs-monoidal category  $\mathcal{C}$ . If  $\mathcal{C}$  is oplax cocartesian then so is the Kleisli category  $\mathcal{C}_T$ .

*Proof.*  $\mathcal{C}_T$  is cogs-monoidal thanks to Proposition 3.20 and inherits the preorder-enrichment from  $\mathcal{C}$ . We now prove that it is oplax cocartesian.

We first show that for every arrow  $f : I \rightarrow T(X)$  in  $\mathcal{C}_T$  it holds that  $f \leq i_X^\#$ . Since  $\mathcal{C}$  is oplax cocartesian we have that  $f \leq i_{T(X)}$ , and because  $T$  is colax coaffine it implies that  $i_{T(X)} \leq T(i_X) \circ \eta_I = i_X^\#$ .

It remains to prove that for every  $f : X \rightarrow Y$  in  $\mathcal{C}_T$ , which corresponds to an arrow  $f : X \rightarrow T(Y)$  in  $\mathcal{C}$ , it holds that  $f \circ^\# \Delta_X^\# \leq \Delta_Y^\# \circ^\# (f \otimes_T f)$  in  $\mathcal{C}_T$ . Indeed

$$\begin{aligned}
f \circ^\# \Delta_X^\# &= f \circ \Delta_X \\
&\leq \Delta_{T(Y)} \circ (f \otimes f) && (\mathcal{C} \text{ oplax cocartesian}) \\
&\leq T(\Delta_Y) \circ c_{Y,Y} \circ (f \otimes f) && (T \text{ colax corelevant}) \\
&= \Delta_Y^\# \circ^\# (f \otimes_T f)
\end{aligned}$$

□