Topics in Quantum Field Theory

Magnetic Monopoles

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Context

These notes contain an extended version of the material that I will discuss in today's presentation on magnetic monopoles.

Sec. [1] gives some hints on differential forms and their application in electromagnetism; we will define here the notation that we use through the rest of the notes, however this is not generally crucial in order to understand the physics behind the laws. Anyway, we will stick to this notation, especially for the quantum mechanical part.

Sec. [2] is about magnetic monopoles in relativistic Quantum Mechanics. This section aims to give an idea of the **consequences** of possibly having magnetic monopoles around in our Universe.
For both Sec. [1] and Sec. [2], I strictly refer to [Zee (2010)], specifically to Chap. IV.4. In Sec. [2] I often refer to [Weinberg (2012)] Chap. 5 as well.

Finally, both Sec. [3] and Appendix [A] aim to give some intuition on the **reasons** why magnetic monopoles are expected to be a general prediction of several – and as it's known, needed – extensions of the Standard Model of Particle Physics. The main references for these sections are [Vilenkin and Shellard (2000)] and [Weinberg (2012)].

Of these last two sections, only Sec. [3] will be presented today; nevertheless, the Appendix can be consulted by anyone interested in further technical details.

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1 Differential forms

In the following section we discuss the language of differential forms, by specifically focusing on the way they allow to write down the laws of electromagnetism in an elegant and compact way. Moreover, we will readily recognise some properties of Maxwell's equations, as arising from the formal consequences of the definition of this formalism.
For the sake of generality, we will work on a D-dimensional space [1].

Einstein's summation convention over repeated indices is assumed from now on, unless otherwise stated.

1.1 Generalities

Let us define x^μ to be our (real) coordinates on some physical space and $A_\mu(x)$ [2] to be some functions of the above coordinates ($\mu = 1, \ldots, D$).
Let us also introduce the differentials dx^μ; these are objects that, under any change of coordinates of the form,

$$x^\mu \longrightarrow x'^\mu(x^\alpha) \tag{1}$$

transform as

$$dx^\mu \longrightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

according to the so-called *chain rule.*

Then the following combination

$$A := A_\mu dx^\mu$$

is defined as a 1-form.

We see that this object is coordinate independent, as long as our functions A_μ transform as *covariant* vector components under the above transformation rule (1)

$$A_\mu \longrightarrow A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu$$

More generally, a p-form is defined in the following way

$$H_{(p)} := \frac{1}{p\,!} H_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} \tag{2}$$

[1] Even though we use the Greek-index notation, we'll better avoid restricting our discussion to D-dimensional **spacetimes**; of course, this will be our natural setting as soon as we turn to the physics of electromagnetism, but we shall bear in mind that the definitions we are about to introduce here happen to be metric-independent.

[2] Again, these are just (possibly nice) **general** functions of our x^μ variables. However, the symbols we use to name these functions readily anticipate what our sudden interpretation of them will be once we turn to electromagnetism.

and for the 0-form, we just pick an ordinary function: $H_{(0)} \equiv \Lambda(x)$ ("no dx around").

Consider now what we could interpret as an area element in our physical space [3] (let us play with 2-forms in D = 2 for simplicity) and apply the coordinate transformation above; we easily get

$$dx\,dy \equiv \left(\frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial y'}dy'\right)\left(\frac{\partial y}{\partial x'}dx' + \frac{\partial y}{\partial y'}dy'\right)$$

But now we also realize that we can get an expression of the form

$$dx\,dy = J(x, y; x', y')\,dx'\,dy' \tag{3}$$

as long as we assume our differentials to behave as anti-commuting objects (Grassmann variables), namely

$$\begin{aligned} dx\,dy &\equiv -dy\,dx \\ \Longrightarrow dx\,dx &= dy\,dy = 0 \end{aligned}$$

and the request (3) is exactly what we want to implement for our physical purposes: under this prescription, the above factored-out quantity is just the *Jacobian* of the coordinate transformation (1)

$$J(x, y; x', y') \equiv \left(\frac{\partial x}{\partial x'}\frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'}\frac{\partial y}{\partial x'}\right)$$

Moreover, this extra property we implement has a nice geometrical interpretation: if we now assume (for instance) that our 2-form is embedded in a D=3 space, then the anti-commuting property of our differential gives our surfaces the possibility to have a **sign**, and this is interpreted as the **orientation** of our surface in space.

The whole discussion above is generalizable to the case of p-forms in D-dimensional spaces. Take 3-forms in D=3 dimensions as an example: the word "surface" can be replaced by "volume", and all the analogue results follow from the algebra.

1.2 Differential operation and the link to electromagnetism

We now define the differential operator d to act on a p-form H (see (2)) in the following way

$$dH := \frac{1}{p!}\partial_\nu H_{\mu_1\mu_2\ldots\mu_p}dx^\nu dx^{\mu_1}dx^{\mu_2}\ldots dx^{\mu_p} \tag{4}$$

where the coloured symbols enhance the precise action of this operator on H.
Thus, the differential operation takes a p-form as input and gives a (p+1)-form as a result. This is something we were already used to if we think for a moment about it: by applying d to a 0-form (a function) we get

[3] The reason behind this resides in the fact that we would like to integrate our p-forms and establish a connection to the integrals we use in quantum mechanics and in field theory.

$$d\Lambda(x) = \partial_{x^\mu}\Lambda(x)\, dx^\mu \longrightarrow d\Lambda \equiv \partial_\mu\Lambda\, dx^\mu$$

as we already know from the definition of the *differential of a function*. The result is nothing but a 1-form!
Similarly, a 1-form A becomes a 2-form under the action of d

$$dA := \partial_\mu A_\nu\, dx^\mu dx^\nu$$

But now we get something interesting: let's massage the above expression and apply the anti-commuting property of our differentials dx^α

$$\begin{aligned} dA := \partial_\mu A_\nu\, dx^\mu dx^\nu &\equiv \frac{1}{2}\left(\partial_\mu A_\nu\, dx^\mu dx^\nu + \partial_\nu A_\mu\, dx^\nu dx^\mu\right) \\ &= \frac{1}{2}\underbrace{\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)}_{F_{\mu\nu}} dx^\mu dx^\nu \\ &= \frac{1}{2}F_{\mu\nu}\, dx^\mu dx^\nu := F \end{aligned}$$

we recover the Maxwell field tensor $F_{\mu\nu}$! Thus we see that, the usual definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ can be equivalently stated in the compact language of differential forms as

$$F = dA \tag{5}$$

where F (the *electromagnetic field*) is the 2-form defined just slightly above, and A (the *electromagnetic potential*) is the 1-form that gives it under the action of the differential operation d.

The great thing about this notation is that it seems to *hide* the coordinate-dependent part of our objects and definitions. Even if this might seem to be a mere matter of aesthetics, it often turns out to be useful in more complicated contexts [4].

Now we wish to show that another important identity of electromagnetism can be obtained in the language of differential forms.
Let's start by differentiating further the definition (4); we get

$$ddH \equiv \frac{1}{p!}\partial_\rho\partial_\nu H_{\mu_1\mu_2\ldots\mu_p} dx^\rho dx^\nu dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p}$$

but now, since $dx^\rho dx^\nu = -dx^\nu dx^\rho$ and $\partial_\rho\partial_\nu = \partial_\nu\partial_\rho$ [5], we immediately get

$$ddH = 0$$

[4] An example is given by String Theory, where a 2-form potential B arises in the closed string sector, giving rise to a 3-form strength tensor $H = dB$ (the so-called Kalb-Ramond field). Higher-p forms are usually involved as well in this framework.

[5] This is true only if our functions $H_{\mu_1\mu_2\ldots\mu_p}(x^\alpha)$ have continuous second derivatives in x^α everywhere in space, and this we will always assume.

and since H is just a generic p-form, we conclude that on any differential form, the above operation is just given by a null operator ($dd = 0$).
Since $F = dA$, this means that

$$dF = 0 \tag{6}$$

which can be also written down in its explicit form as

$$(\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu})\, dx^\rho dx^\mu dx^\nu = 0$$

or equivalently

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$$

the well known Bianchi identity.

As it's known, this identity already contains two of the four Maxwell's equations [6], namely

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\partial_t \vec{B} \end{aligned} \tag{7}$$

The remaining two equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho \\ \vec{\nabla} \times \vec{B} &= \vec{J} + \partial_t \vec{E} \end{aligned} \tag{8}$$

can be put together in the following compact expression

$$d * F = J \tag{9}$$

which is often referred to as the *equations of motion* of the gauge field A_μ.

1.3 Another quick mathematical notion: closed and exact forms

Let's write down the following definitions:

- a p-form α is said to be **closed** if $d\alpha = 0$
- a p-form α is said to be **exact** if there exists a (p - 1)-form β such that $\alpha = d\beta$

From the above result $dd = 0$, we readily get

$$\alpha \text{ is exact} \implies \alpha \text{ is closed}$$

however the inverse is not generally true. Actually, it can be shown that in general

$$\alpha \text{ is closed} \implies \alpha \text{ is } \textbf{locally} \text{ exact}$$

[6] These are the two equations that are usually not affected by the presence of sources.

2 Quantum mechanics likes magnetic monopoles

In the present section we will try to understand what are the consequences of assuming the existence of magnetic monopoles, from the point of view of relativistic Quantum Mechanics.
First, let's mention a curious fact about classical field theory.

2.1 A would-be symmetry

Let's go back to Maxwell's equations for a moment (eqs.(9) and (6)); if we set $J = 0$ in equation (9), then all four Maxwell's equations can be combined into two complex equations as follows

$$\begin{aligned} &\vec{\nabla} \times (\vec{E} + i\vec{B}) - i\frac{\partial}{\partial t}(\vec{E} + i\vec{B}) = 0 \\ &\vec{\nabla} \cdot (\vec{E} + i\vec{B}) = 0 \end{aligned} \tag{10}$$

and we can see that these equations are invariant under the following nice global transformation,

$$\vec{E} + i\vec{B} \longrightarrow e^{i\alpha}(\vec{E} + i\vec{B})$$

However, we also see that this very elegant symmetry is broken as soon as we add again the source terms encoded in the 1-form J. The reason why this happens is that the above complex combination of all Maxwell's equations is no longer working in the $J \neq 0$ case.
Notice that J is the spacetime current associated to **electrically**-charged objects; with a little effort, one can show that the nice phase-shift symmetry above is restored once we introduce magnetic monopole sources into the model ($J_m \neq 0$). In this way, our coupled equations would read as follows

$$\begin{aligned} &\vec{\nabla} \times (\vec{E} + i\vec{B}) - i\frac{\partial}{\partial t}(\vec{E} + i\vec{B}) = \vec{J} + i\,\vec{J}_m \\ &\vec{\nabla} \cdot (\vec{E} + i\vec{B}) = \rho + i\,\rho_m \end{aligned} \tag{11}$$

and the symmetry of the model would be

$$\begin{aligned} &\vec{E} + i\vec{B} \longrightarrow e^{i\alpha}(\vec{E} + i\vec{B}) \\ &J^\mu + i\,J^\mu_m \longrightarrow e^{i\alpha}(J^\mu + i\,J^\mu_m) \end{aligned}$$

A comment on this.
The way Maxwell's equations have been historically formulated, excluded magnetic charges (J^μ_m terms) because these had never been observed. However, nothing stops us from guessing what the consequences of having (even a few) such charges around us would be: it might also be that, for instance, these magnetic charges have been swept away by some mechanism during the evolution of our Universe, causing the formulation of classical electrodynamics by human beings to be magnetic-monopole-less.

Indeed, there are nowadays some arguments suggesting that the above statements might be realistic, and this we will discuss in the next sections, where a possible mechanism that produces magnetic charges in the early Universe will be presented, on the grounds of Field Theory (see Sec [3]) and Topology (see Appendix [A]).

But now let us go back to our main task for this section: what does Quantum Mechanics think about the existence of such point-like magnetic charges?

2.2 The charge quantization condition

In what follows, we will assume that Maxwell's equations could include magnetic charges, by only relying on the argument of the previous paragraph.
An immediate consequence of this will be revealed in a second; the inclusion of magnetic charges breaks the standard Bianchi identity (7); note that, in the language of differential forms, that identity arose as a consequence of the general definition $F = dA$. We have to throw away this definition in a way that will become clear soon.

This might make us unhappy: are we modifying the whole structure of the theory, based on "nice phase shifts" only?
A further justification to this will be given in the next sections. We will see that the way magnetic monopoles arise in Field Theory has to do with the definition of a new gauge-invariant field tensor $\mathcal{F}_{\mu\nu}$, that is related to the usual tensor $F_{\mu\nu}$, but not generally coincident with it.

The argument that follows was proposed by Dirac in 1931, and reserves stunning consequences.
Let's consider the following results from relativistic Quantum Mechanics: under a local gauge transformation, we get

$$\begin{aligned}\psi(x) &\longrightarrow e^{i\Lambda(x)}\psi(x)\\ A_\mu(x) &\longrightarrow A_\mu(x) + \frac{1}{ie}e^{-i\Lambda(x)}\partial_\mu e^{i\Lambda(x)}\end{aligned}$$

where $\psi(x)$ is the spinor denoting the *electron field* of electric charge e, which we would not specifically care about here. We note that the gauge transformation of the potential can be written down in the following way, by adopting the language of differential forms

$$A \longrightarrow A + \frac{1}{ie}e^{-i\Lambda}d\,e^{i\Lambda}$$

where $A = A_\mu(x)dx^\mu$ is our 1-form, and $\Lambda = \Lambda(x)$ is a 0-form.

Let's turn to spherical coordinates; the 2-form strength field F associated to a point-like magnetic charge must fit the general expression

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$$

By assuming that our particle is only magnetically charged, we must require $F_{0i} = 0$, $\forall j = 1, 2, 3$ (no electric field), thus we get

$$F \equiv \frac{1}{2} F_{ij} dx^i dx^j$$

and since in spherical components we have $dx^1 = dr$, $dx^2 = r\, d\cos(\theta)$, $dx^3 = r\, d\varphi$, and we expect $F_{ij} \sim B_r \sim \frac{e_m}{4\pi r^2}$, we immediately get [7] [8]

$$F = \frac{e_m}{4\pi}\, d\cos(\theta)\, d\varphi \tag{12}$$

How to determine the 1-form potential now? Since $dd = 0$, one might guess that the answer is

$$A = \frac{e_m}{4\pi} \cos(\theta) d\varphi$$

The problem is that this 1-form is not well-defined at $\theta = 0$ (north pole) because $d\varphi$ has no meaning there!
But here's a trick: we can find a function to put in front of $d\varphi$, that vanishes when $\theta = 0$. This task is easily solved in the following way

$$A_N = \frac{e_m}{4\pi} (\cos(\theta) - 1) d\varphi$$

However, this is not enough, as this solution still has a singular behaviour at $\theta = \pi$ (south pole), where the expression doesn't vanish and $d\varphi$ still has no sense to be defined.
Thus we can define **another** gauge field of the form

$$A_S = \frac{e_m}{4\pi} (\cos(\theta) + 1) d\varphi$$

which has a good behaviour in all points in space, except $\theta = 0$.

"Our gauge field is said to be defined **locally**, but not globally."

Here is an important point.
This result comes from the fact that our *ansatz* (12) automatically satisfies $dF = 0$ *per se*: F is **closed** by construction, and hence it is **locally** exact. If F happened to be **globally** exact - as it is in standard electromagnetism, where $F = dA$ is given as a *global* definition - then Stoke's theorem would immediately imply that $e_m = 0$.

Back to our task, we understand that in order to have a complete description of this system, we need to use both of these solutions, by matching two *coordinate patches* together. Let's go this way, by overlapping our two descriptions along the equator of the sphere ($\theta = \frac{\pi}{2}$): unfortunately, along the equator, the two fields do not match naturally

[7] Because of spherical symmetry, B_r will be the only non-vanishing component in F_{ij}; here e_m is some magnetic charge we introduce as a source for B_r. The factor of $\frac{1}{2}$ is cancelled by the commutation properties of dx^μs.

[8] Also note that this expression for F correctly gives $\int_{S^2} F = e_m$, i.e., our magnetic field is not divergence-less and satisfies its own Gauss's law.

$$A_S\left(\theta=\frac{\pi}{2}\right)-A_N\left(\theta=\frac{\pi}{2}\right)=2\frac{e_m}{4\pi}d\varphi$$

But what if these two solutions are locally related by a gauge transformation? Then Physics should match naturally, as it doesn't care about gauge transformations at all. Let's see what is the condition that guarantees our matching: we impose

$$2\frac{e_m}{4\pi}d\varphi \overset{!}{=} \frac{1}{ie}\, e^{-i\Lambda}d\, e^{i\Lambda}$$

and get

$$d\Lambda=2\left(\frac{e_m\, e}{4\pi}\right)d\varphi \implies \Lambda=2\left(\frac{e_m\, e}{4\pi}\right)\varphi \implies e^{i\Lambda}\equiv e^{2i(\frac{e_m\, e}{4\pi})\varphi}$$

and since $\varphi=0$ and $\varphi=2\pi$ indicate the same point in space, we must have

$$e^{2i(\frac{e_m\, e}{4\pi})2\pi}\overset{!}{=}e^0=1$$

leading to

$$e_m=\frac{2\pi}{e}n \qquad \forall n\in \mathrm{Z} \tag{13}$$

also known as the **Dirac quantization condition**.

The immediate result is that

> " If a magnetic charge e_m exists, then it should be quantized in units of $\frac{2\pi}{e}$ "

But this is not the end of the story, as this statement can also be put in the following *dual* version

> " If a magnetic charge e_m exists, then **any electric charge** e should be quantized in units of $\frac{2\pi}{e_m}$ "

This second statement is quite important: this whole argument seems to give a quantum mechanical proof of the quantization of the **electric** charge, provided that magnetic monopoles exist!

Another comment on this result.
Eq. (13) also gives us hints on what to expect from any QFT that admits the existence of monopoles. The elementary charges $(e, g_m=e_m(n=1))$ of the whole theory must satisfy

$$g_m\, e=2\pi$$

and this can be summed-up in the following statement

> " If a quantum field theory of electric charges is weakly coupled, then the corresponding quantum field theory of magnetic charges is strongly coupled "

or equivalently, the other way around.
This goes under the name of **electromagnetic duality**.

2.3 Aharonov-Bohm effect and the Dirac string

In this last paragraph, we comment on a *Gedankenexperiment* originally proposed by A. Coleman, that connects the Aharonov-Bohm effect to the Dirac quantization argument discussed above.

The singularities we found in the above definitions of A_N and A_S are usually referred to as *Dirac strings*. These are regions in space where the magnetic field is squeezed in the direction of the corresponding line that identifies the singularity of A. This can be better seen if we write down our two potentials in Cartesian coordinates (where $n^i = x^i/r$ and $r = \sqrt{x_i x^i}$)

$$A_N^i = -\epsilon^{ij3} n^i \frac{e_m}{z+r}$$
$$A_s^i = -\epsilon^{ij3} n^i \frac{e_m}{z-r}$$

From here we see that A_N represents a monopole with a Dirac string attached to the south pole, while A_S consists of a Dirac string attached to the north pole. As we now, these two solutions are gauge-equivalent.

Let's match these two cases together, by linking a monopole and an anti-monopole via a Dirac string on the direction that separates them; it turns out that this potential is also a good description of an infinitely thin solenoid that connects the two magnetic monopoles placed at its ends.

Coleman's argument is the following: as we know from the Aharanov-Bohm effect, if a particle of elementary electric charge e travels around a tiny closed circle C that contains our *stringy solenoid*, its quantum mechanical amplitude gets a gauge-invariant phase factor

$$U[C] = \exp\left(ie \oint_C d\vec{l} \cdot \vec{A}\right)$$

Now, if we want magnetic monopoles to exist, this stringy solenoid (the Dirac string) should be **undetectable** by the charged particle that goes around it.
This is equivalent to state that, for this specific experimental setup, we expect to get

$$U[C] \stackrel{!}{=} 1$$

From Stoke's theorem we also know that

$$\oint_{C=\partial(S^2/P_{S,N})} d\vec{l} \cdot \vec{A}_{N,S} = \int_C A_{N,S} \stackrel{Stokes}{=} \int_{S^2/P_{S,N}} dA_{N,S} \simeq \int_{S^2} F \equiv e_m$$

where $C = \partial(S^2/P_{S,N})$ is taken to be a small loop around the south pole if we integrate A_N, and vice-versa for the other solution (the result is obviously the same). Thus, C represents the boundary of a sphere with a point removed (the south pole P_S for the integration of A_N or the north pole P_N for A_S), and so in the **limit** of a small loop [9], Stoke's theorem can be applied, and the result of the integral is approximatively the flux around the whole sphere S^2 that surrounds the monopole.

The result is again

$$e_m = \frac{2\pi}{e} n \qquad \forall n \in \mathrm{Z}$$

Thus, this argument leads exactly to the Dirac quantization condition (13) we found by a quite independent argument.

[9] So we are assuming that the two monopoles are well separated, and this ensures that the Dirac string tends to be indefinitely thin.

3 Field theory likes magnetic monopoles

Many extensions of the Standard Model of Particle Physics suggest the existence of particles carrying magnetic charges. Generally, all extensions of the SM are grouped under the name of Grand Unified Theories (GUTs), and break down to the former after a spontaneous symmetry breaking mechanism (SSB) ; monopole configurations happen to be allowed during this mechanism, for many types of such GUTs.

In this section we will introduce the idea of topological defects, keeping aside the term *topological*, for which I refer to the Appendix [A] if anyone is interested in further details.

3.1 SSB, vacuum manifolds and the Kibble Mechanism

As we know from the previous presentations, the SSB of a global symmetry consists of the **choice** of a specific field configuration by the vacuum of the theory, among all the ones which are allowed by the structure of the so-called *vacuum manifold*. This happens because all such vacua are **physically equivalent**, as they are all characterized by the same energy density.

During cosmological phase transitions, it might happen that some regions of space that are *causally disconnected* from each other develop different vacuum expectation values (vev_s).
Naturally, the gradient energy between such regions is physically relevant, as it can be **localized** on the boundary between them: this we will call a **soliton** or, more specifically, a **defect** configuration.

The mechanism described here is the one that gives rise to the simplest class of topological defects, known as **domain walls** (if the field of the theory is real-valued) or **textures** (if the field is complex-valued).
Further convoluted mechanisms can also give rise to more complicated defects [10], like **cosmic strings** and **monopoles**.
The general idea behind all such defect formations goes under the name of *Kibble mechanism.*

3.2 Topological defects: from kinks to monopoles

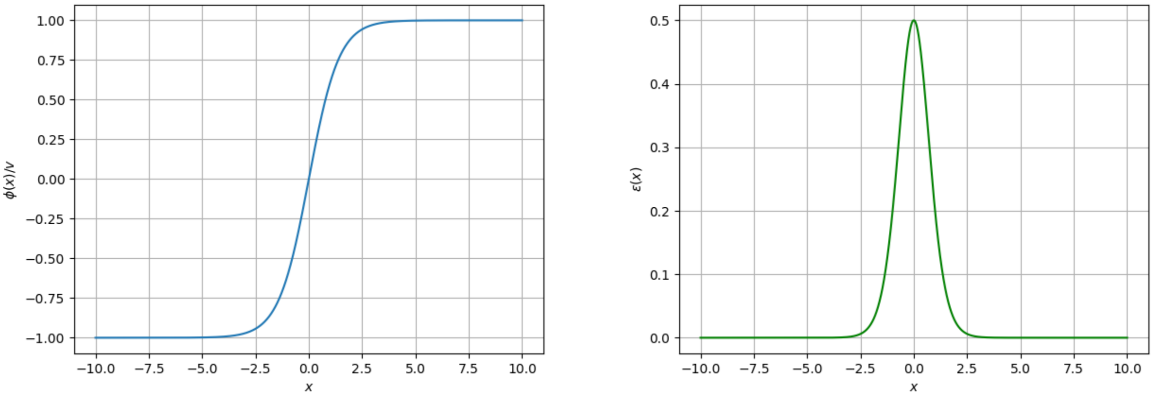
Let's give a concrete idea of how the statements above are realized.

3.2.1 Kinks and domain walls

The first defect worth analysing is the **kink**, a one dimensional analogue of the above mentioned *domain wall.*

The Lagrangian density we consider is

[10] Please see the Appendix [A] for more details on how such mechanisms work.



(a) Kink solution

(b) Energy density of the kink solution

Figure 1: (a) Example of kink solution $\phi_{kink}(x)$ for $x_0 = 0$, and for $\lambda = v = 1$. The anti-kink solution is just the opposite in sign of this one. (b) Energy density of the kink solution as a function of x.

$$\mathcal{L}_{kink} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}(\phi^2 - v^2)^2$$

where ϕ is a real scalar field living in a 1+1 dimensional spacetime.

As we know, the vacuum manifold is

$$\mathcal{M} = \{-v, v\}$$

and the SSB pattern is $\mathbb{Z}_2 \to I$.

It can be shown that indeed there exist **static** solutions that interpolate between the two allowed vacua; one such solution is

$$\phi_{kink}(x) = v\ \tanh\left(\frac{m}{\sqrt{2}}(x - x_0)\right)$$

where $m = \sqrt{\lambda}\,v$ and x_0 identifies the position of the kink in our 1-dimensional physical space. This solution interpolates between $\phi_0 = -v$ and $\phi_0 = v$.
There also exists a solution that interpolates between $\phi_0 = v$ and $\phi_0 = -v$, known as the *anti-kink* configuration

$$\phi_{anti-kink}(x) = -v\ \tanh\left(\frac{m}{\sqrt{2}}(x - x_0)\right)$$

Please see Fig. [1a] for a graphical visualization of this.

The kink solution identifies a topologically stable configuration of our scalar field in space: it can be transformed into one of the two trivial vacua only by means of an infinite amount

of energy!
Actually, the transition region between $-v$ and v has a width $\sim 1/m$ and can be identified as an extended *lump* of energy, but well localized in space (at $x = x_0$). This is clearly seen once one plots the static energy density as a function of x

$$\epsilon(x) = \frac{1}{2}(\partial_x \phi)^2 + V(\phi(x))$$

as you can see in Fig. [1b].

3.2.2 Vortices and cosmic strings

The next defect we could study would come from a generalization of the above Lagrangian to the case of a complex scalar field in one more spatial dimension. It can be shown that, similarly to the kink case, this model allows topologically stable field configurations: such objects are called **vortices** (in 2+1 dimensions) or **cosmic strings** (in 3+1 dimensions).

However, this goes beyond the scope of these notes, and thus we will jump to monopole models right away.

3.2.3 Monopoles

An example is given by the **'t Hooft-Polyakov monopoles**, arising from a $SU(2)$-invariant model [11] which is broken down spontaneously to $U(1)$

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - \frac{\lambda}{4}(\phi^\dagger \phi - v^2)^2 - \frac{1}{2} Tr\,(F_{\mu\nu} F^{\mu\nu})$$

where the vacuum manifold is $\mathcal{M} \simeq S^2$. Two out of three gauge bosons become massive, by getting a mass $m_v = ev$, and one remains always massless.

Both the field ϕ and the gauge boson field A_μ are in the adjoint representation, and the covariant derivative is defined as

$$D_\mu \phi := \partial_\mu \phi + ie[\phi, A_\mu]$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$$

In terms of the generators T^a of $SU(2)$, the above symbols read

$$\phi = \phi^a T^a \qquad A_\mu = A^a_\mu T^a \qquad F_{\mu\nu} = F^a_{\mu\nu} T^a$$

[11] Note that in the following model the symmetry is gauged, and the reasons of this go beyond what's intended to be discussed here. To be precise, also the vortex model needs to be gauged sometimes, and this is often the case in general, as the Higgs field is always gauged in the SM and in its extensions.

It can be shown that, similarly to the kink case, this model allows topologically stable static solutions.
The simplest solution is often known as the *hedgehog solution*

$$\phi^a_{mon}(\vec{r}) = v\, H(r)\frac{x^a}{r}$$
$$A^a_i(\vec{r}) = -(1-K(r))\epsilon^{aij}\frac{x^j}{e\, r^2} \qquad\qquad A^a_0 = 0$$

where $H(r)$ and $K(r)$ are *profile functions* (can be found numerically from the static equations of motions) and x^a/r is just the unit vector in polar coordinates, having its tail in the monopole centre

$$\frac{x^a}{r} = n^a \equiv \begin{pmatrix} \sin\theta\, \cos\varphi \\ \sin\theta\, \sin\varphi \\ \cos\theta \end{pmatrix}$$

More generally, all such solutions carry a $U(1)$ **magnetic** charge of the form

$$e_m \sim \frac{n}{e} \qquad \forall n \in \mathrm{Z}$$

where e is the gauge coupling of ϕ to A_μ.

As promised, a comment on this.
The $U(1)$ magnetic field associated to the monopole arises from a projection of the flavoured field tensor $F^a_{\mu\nu}$ along the direction of the ϕ field in flavour space; it can be shown that the following gauge-invariant field tensor can be defined ($|\phi| = \sqrt{\phi_a\phi^a}$)

$$\mathcal{F}_{\mu\nu} := \frac{\phi_a}{|\phi|}F^a_{\mu\nu} + \frac{1}{e|\phi|^3}\epsilon^{abc}\phi^a D_\mu\phi^b D_\nu\phi^c$$

and that it satisfies a modified version of Maxwell's equations, including magnetic monopole sources and radial magnetic fields that scale as $\frac{e_m}{r^2}$ far from the monopole centre; this is exactly the field we played with in the previous section!

We understand that this **unconfined magnetic charge** is just the result of having a gauge boson that remains massless after the SSB mechanism, and thus it is still able to mediate a long-range force of the Maxwell type.
This can be clearly seen in the so-called *unitary* or *combed-hedgehog* gauge

$$\phi^a_{comb} = v\, H(r)\delta^{3a}$$

where the new strength tensor simply reduces to

$$\mathcal{F}_{\mu\nu} \equiv H(r)(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu) \xrightarrow{r\to\infty} \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$$

and A^3_μ is the only field that remains massless in the SSB mechanism.

A Appendix: Homotopy Theory and GUTs

In this section we would like to understand **which** of the GUTs we mentioned above account for the existence of magnetic monopoles, and **how** this can be forecasted based on topological grounds.

A.1 Homotopy groups and topological stability

The first piece of the puzzle we need in order to understand the link between topology and monopoles resides in the concept of *homotopy group.*
Without stressing too much the rigorous mathematical statements, we adopt the following definition of the *n-th homotopy group*, which is enough for our purposes:

> " The n-th homotopy group $\pi_n(\mathcal{M})$ is the **group** of all **homotopically equivalent** n-dimensional **loops** on a manifold $\mathcal{M}$ "

In a more formal sense, what we call here a *n-dimensional loop* has to be intended as a map from a n-sphere S^n to the manifold $\mathcal{M}$ under consideration. These maps can be always composed under certain rules, and it can be shown that they indeed have a group structure, if we take these rules as a definition of product between the elements.
Moreover, any two loops are said to be *homotopically equivalent* if they can be continuously deformed into one another, i.e. by means of a continuous function that takes values on the manifold $\mathcal{M}$.
Hence, all these maps from S^n to $\mathcal{M}$ are always classified into **homotopy classes**, according to the above definition of equivalence!
This leads to the concept of topological stability, again, written down in a very informal way:

> " Any n-dimensional loop that belongs to a given homotopy class is topologically stable, meaning that it cannot be continuously deformed into another loop belonging to a different class "

That's all we need for the moment. Let's road to physics now.

A.2 Examples

Let's give some examples, in order to visualize the content of the previous shady paragraph.

$S^0 \to \mathcal{M}$:
All functions from the 0-sphere (a set consisting of two points) to a manifold $\mathcal{M}$ are grouped under the classes defined by the 0-th homotopy group $\pi_0(\mathcal{M})$. $\mathcal{M}$ is said to be **disconnected** if this homotopy group is non-trivial, i.e., if it doesn't contain functions (loops) that are all homotopically equivalent to a constant (all loops shrinking to a point). Briefly

$$\pi_0(\mathcal{M}) \neq I$$

reveals the existence of topologically stable loops, which in this case are just graphically represented by a couple of points on $\mathcal{M}$. We understand that the case $\mathcal{M} \simeq S^0$ is of immediate interest, because

$$\pi_0(\mathcal{M} \simeq S^0) = \mathbb{Z}_2$$

i.e., all loops can be classified by three possible values: -1, 0 and 1; we call these numbers **topological charges**.

Let's give a look to Field Theory; as we know from the previous section, a real scalar field described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}(\phi^2 - v^2)^2$$

exactly posses the vacuum manifold discussed now! It's a 0-sphere of radius v in field space.
It can be shown that the equations of motion coming from this model admit **topologically stable solutions**, which are classified by the topological charges founded above: 1 is the charge of the kink, -1 is the charge of the anti-kink and 0 is the charge of the two allowed vacua.

$S^1 \to \mathcal{M}$:
This case is interesting in Field Theory as well. For example, when we extend the above Lagrangian density to the case of a complex scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - \frac{\lambda}{4}(|\phi|^2 - v^2)^2$$

we know that we get a vacuum manifold $\mathcal{M} \simeq S^1$, and thus we see that

$$\pi_1(\mathcal{M} \simeq S^1) = \mathbb{Z}$$

As we can convince ourselves graphically, we indeed get an infinite number of topological charges this time, indicating an infinite number of topologically stable solutions.

Such solutions can be found, and go under the name of **vortices** or **cosmic strings**, again, depending on the number of spatial dimensions of the model.

$S^2 \to \mathcal{M}$:
Spoiler: we finally got to monopoles. As in the cases above, there are different possible models leading to the existence of topologically stable solutions of this type. We are now dealing with 2-spheres winding around our manifold $\mathcal{M}$.

An example is given by the **'t Hooft-Polyakov monopoles** discussed in the previous section

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - \frac{\lambda}{4}(\phi^\dagger \phi - v^2)^2 - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

where the vacuum manifold is $\mathcal{M} \simeq S^2$, and it can be shown that, again

$$\pi_2(\mathcal{M} \simeq S^2) = \mathbb{Z}$$

It can be seen that all these topologically stable solutions carry a $U(1)$ **magnetic** charge, which is proportional to the topological charge via the combination

$$e_m \sim \frac{n}{e}$$

where e is the usual gauge coupling of ϕ to A_μ.

But that's enough for us. Let's not stick to a specific model from now on: all we need to know is that whenever a gauged model has a vacuum manifold that admits

$$\pi_2(\mathcal{M}) \neq I$$

then we can have topologically stable solutions that carry a magnetic charge that is related to the fundamental electric charge defined in the model itself.

A.3 GUTs and Monopoles

We miss one last step now. Suppose we know our Grand Unified Symmetry Group, we call it G. Is there a way to deduce the vacuum manifold that breaks it down to the current SM of Particle Physics? The answer to this is affirmative and surprisingly simple: if we call $H = SU(3) \times U(1)_{em} \subset G$ our current unbroken symmetry group, then it can be shown that [12]

$$\mathcal{M} \simeq G/H$$

Some nice properties that are proven in textbooks are

$$\begin{aligned} &\pi_n(S^n) = Z \qquad \text{and} \qquad \pi_k(S^n) = I \quad \text{for } k < n \\ &\pi_n(G_1 \times G_2) = \pi_n(G_1) \times \pi_n(G_2) \\ &\pi_2(G/H) \simeq \pi_1(H) \qquad \text{for} \quad \pi_1(G) = 1 \end{aligned}$$

So now we want to prove the following

> " If the Grand Unified Group G is **simply connected** ($\pi_1(G) = I$), then the SSB mechanism
>
> $$G \to H = SU(3) \times U(1)_{em}$$
>
> allows topologically stable magnetic monopoles "

[12] This is not only true for SM of course; this result is general, and can be proven quite easily for any SSB pattern $G \to H \subseteq G$.

Let's see why this is true.[13]
We want to compute $\pi_2(\mathcal{M})$ and we hope that this is non-trivial. By using some of the properties above and by noting that $\pi_1(SU(3)) \simeq I$ [14], we get

$$\begin{aligned}\pi_2(\mathcal{M} \simeq G/H) &\simeq \pi_1(H) = \pi_1(SU(3) \times U(1)_{em}) \\ &= \pi_1(SU(3)) \times \pi_1(U(1)_{em}) \equiv \pi_1(U(1)_{em}) = Z\end{aligned}$$

thus

$$\pi_2(\mathcal{M} \simeq G/H) = Z \neq I$$

as we wanted to show.

Moreover, similarly to the 't Hooft-Polyakov case, we see that these monopoles are always expected to carry a $U(1)_{em}$ magnetic charge.

[13] There's a quite strong assumption concerning G: it must be simply connected, but this is the case for $SU(5)$ and $SU(10)$, for instance.

[14] Because $SU(3)$ is topologically equivalent to S^8.

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