Randomized Bounded Arithmetic

Technical report

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In this technical report, we prove that the class $\mathcal{POR}$ is precisely the class of functions which are Σ^b_1-representable in RS^1_3. In doing so, we first show that the class $\mathcal{POR}$ corresponds to that of the quoted Σ^b_1-representable functions. There are two main steps to be established:

1. In Section 1 we prove that all functions in $\mathcal{POR}$ are Σ^b_1-representable in RS^1_3. The proof is by induction on the structure of probabilistic oracle functions and is inspired by the encoding machinery as presented in [2, 6].
2. In Section 2 we show that all functions which are Σ^b_1-representable in RS^1_3 are in $\mathcal{POR}$ by way of realizability techniques similar to Cook and Urquhart's one [5].

1 All Functions in $\mathcal{POR}$ are Σ^b_1-Representable in RS^1_3

In Section 1.1, we introduce the class $\mathcal{POR}$ of polynomial-time oracle functions, as inspired by both Antonelli et al. [1] and Cobham [4] and Ferreira [6]. Then, in Section 1.2, we introduce our language, $\mathcal{L}$, obtained as extending $\mathcal{L}_{\mathbb{W}}$ by Ferreira and Oitavem [8] with a special predicate symbol $\texttt{Flip}(\cdot)$ and a bounded theory RS^1_3, very close to Ferreira's one [7, 6]. In Section 1.3 we provide a non-standard, quantitative semantics for terms and formulas in $\mathcal{L}$. Actually, for clarity's sake, we also introduce a more familiar, qualitative interpretation for formulas of $\mathcal{L}$, to be compared with the quantitative one. Finally, in Section 1.4, we present our main result here, showing that each polynomial-time oracle function is Σ^b_1-representable in our theory RS^1_3.

1.1 The Class $\mathcal{POR}$

Let $\mathbb{B} = \{\mathbf{0}, \mathbf{1}\}$ denote the set of bits, $\mathbb{S} = \mathbb{B}^*$ be the set of binary strings of finite length, and $\mathbb{O} = \mathbb{B}^{\mathbb{S}}$ indicate the set of binary strings of infinite length. Metavariables $\omega', \omega'', \ldots$ are used to denote the elements $\mathbb{O}$. Given two binary strings x, y, we will use $x \subseteq y$ to express that x is an *initial* or *prefix substring* of y. Let $|\cdot|$ denote the length-map of a string, i.e. for any string x, $|x|$ indicates the length of x. Given two strings x, y, one can define a new string by concatenation, indicated as $x \frown y$ (which will be always abbreviated as simply xy), or by binary product, denoted

$x \times y$ and obtained by self-concatenating x for $|y|$-times. Given an infinite string of bits, ω, and a finite string, x, $\omega(x)$ denotes *one* specific bit of ω, the so-called x-th bit of ω.

Definition 1 (The Class $\mathcal{POR}$)**.** The *class* $\mathcal{POR}$ is the smallest class of functions from $\mathbb{S}^n \times \mathbb{O}$ to $\mathbb{S}$, containing:

- the empty function $E(x, \omega) = \boldsymbol{\epsilon}$
- the projection functions $P_i^n(x_1, \ldots, x_n, \omega) = x_i$, for $n \in \mathbb{N}$ and $1 \leq i \leq n$
- the word-successor $S_{\mathbb{b}}(x, \omega) = x\mathbb{b}$, for every $\mathbb{b} \in \mathbb{B}$
- the conditional function

$$C(\boldsymbol{\epsilon}, y, z_{\mathbf{0}}, z_{\mathbf{1}}, \omega) = y$$
$$C(x\mathbb{b}, y, z_{\mathbf{0}}, z_{\mathbf{1}}, \omega) = z_{\mathbb{b}},$$

  where $\mathbb{b} \in \mathbb{B}$
- the query function $Q(x, \omega) = \omega(x)$

and closed under:

- composition, such that f is defined from $g, h_1, \ldots, h_k$ as

$$f(\vec{x}, \omega) = g(h_1(\vec{x}, \omega), \ldots, h_k(\vec{x}, \omega), \omega)$$

- bounded recursion, such that f is defined from g, h_0, h_1 as

$$f(\vec{x}, \boldsymbol{\epsilon}, \omega) = g(\vec{x}, \omega)$$
$$f(\vec{x}, y\mathbf{0}, \omega) = h_0\big(\vec{x}, y, f(\vec{x}, y, \omega), \omega\big)|_{t(\vec{x},y)}$$
$$f(\vec{x}, y\mathbf{1}, \omega) = h_1\big(\vec{x}, y, f(\vec{x}, y, \omega), \omega\big)|_{t(\vec{x},y)}$$

  where t is defined from $\boldsymbol{\epsilon}, \mathbf{0}, \mathbf{1}, \frown, \times$ by explicit definition.

Remark 1. *Notice that Ferreira's characterization [6], not only does not include the query function* Q*, but also the conditional is not used. Instead, it contains the "substring-conditional" function:*

$$\mathrm{S}(x, y, \omega) = \begin{cases} \mathbf{1} & \text{if } x \subseteq y \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Nevertheless, we can define it due by bounded recursion. First, let $f_{\mathsf{Tail}}(x, \omega)$ *be defined as follows:*

$$\mathrm{Tail}(\boldsymbol{\epsilon}, \omega) = \boldsymbol{\epsilon}$$
$$\mathrm{Tail}(x\mathbb{b}, \omega) = x|_x.$$

Then, let $Eq(x, y, \omega)$ *be:*

$$\mathrm{Eq}(x, \boldsymbol{\epsilon}, \omega) = \mathrm{C}(x, \mathbf{1}, \mathtt{0}, \mathtt{0}, \omega)$$
$$\mathrm{Eq}(x, y\mathbf{0}, \omega) = \mathrm{C}(x, \mathtt{0}, \mathrm{Eq}(\mathrm{Tail}(x), y, \omega), \mathtt{0}, \omega)|_{\mathtt{1}}$$
$$\mathrm{Eq}(x, y\mathbf{1}, \omega) = \mathrm{C}(x, \mathtt{0}, \mathtt{0}, \mathrm{Eq}(\mathrm{Tail}(x), y, \omega), \omega)|_{\mathtt{1}}.$$

Finally, we can define $\mathrm{S}(x,y,\omega)$ *as:*

$$\begin{aligned}
\mathrm{S}(x,\boldsymbol{\epsilon},\omega) &= \mathrm{C}(x,\mathbf{1},\mathbf{0},\mathbf{0},\omega)\\
\mathrm{S}(x,y\mathbf{0},\omega) &= \mathrm{C}\big(x,\mathbf{1},\mathrm{C}\big(\mathrm{Eq}(x,y\mathbf{0},\omega),\mathrm{S}(x,y,\omega),\mathbf{1},\mathbf{1},\omega\big),\mathrm{S}(x,y,\omega),\omega\big)\\
\mathrm{S}(x,y\mathbf{1},\omega) &= \mathrm{C}\big(x,\mathbf{1},\mathrm{S}(x,y,\omega),\mathrm{C}\big(\mathrm{Eq}(x,y\mathbf{1},\omega),\mathrm{S}(x,y,\omega),\mathbf{1},\mathbf{1},\omega\big)\big),\omega\big)
\end{aligned}$$

Actually even the conditional function C could be defined by bounded recursion as follows:

$$\begin{aligned}
C(\epsilon,y,z_0,z_1,\omega) &= y\\
C(x\mathbf{0},y,z_0,z_1,\omega) &= z_0|_{z_0}\\
C(x\mathbf{1},y,z_0,z_1,\omega) &= z_1|_{z_1}.
\end{aligned}$$

but we will take it as a primitive function of $\mathcal{POR}$ to make the the realizability interpretation of Section 2 better readable.

1.2 The Theory RS^1_3

The Language $\mathcal{L}$. The language $\mathcal{L}$ is the first-order language with equality defined in [8], augmented by a predicate symbol $\mathtt{Flip}(\cdot)$, as described below:

Definition 2 (Terms)**.** Let $x,y,\ldots$ denote variables. Terms are defined by the following grammar:

$$t,s ::= x \;\big|\; \epsilon \;\big|\; \mathtt{0} \;\big|\; \mathtt{1} \;\big|\; t \frown s \;\big|\; t \times s.$$

Notation 1. The symbol $\frown$ is usually omitted: $t \frown s$ is abbreviated as ts.

Definition 3 (Formulas)**.** Let $x,y,\ldots$ denote variables and $t,s,\ldots$ terms. Formulas are defined by the following grammar:

$$F,G ::= \mathtt{Flip}(t) \;\big|\; t=s \;\big|\; t \subseteq s \;\big|\; \neg F \;\big|\; F \wedge G \;\big|\; F \vee G \;\big|\; F \to G \;\big|\; (\exists x)F \;\big|\; (\forall x)F.$$

Notation 2. Given a term t, the abbreviation $\mathtt{1}^t$ stands for the term $\mathtt{1} \times t$. Given two terms t,s, the formula $t \preceq s$ is syntactic sugar for $\mathtt{1}^t \subseteq \mathtt{1}^s$, meaning that the length of t is less than or equal to that of s. Given three terms t,r,s, the abbreviation $t|_r = s$ denotes the following formula,

$$(\mathtt{1}^r \subseteq \mathtt{1}^t \wedge s \subseteq t \wedge \mathtt{1}^r = \mathtt{1}^s) \vee (\mathtt{1}^t \subseteq \mathtt{1}^r \wedge s = t)$$

saying that s is the *truncation* of t at the length of r.

Notation 3 (Bounded and Subword Quantification)**.** In $\mathcal{L}$, *bounded quantification* is quantification in the form $(\forall x \preceq t)F$, which abbreviates $(\forall x)(\mathtt{1}^x \subseteq \mathtt{1}^t \to F)$, or $(\exists x \preceq t)F$. *Subword quantification* is quantification in the form $\forall x \subseteq^* t$ and $\exists x \subseteq^* t$, such that $(\forall x \subseteq^* t)F$ and $(\exists x \subseteq^* t)F$ abbreviate (resp.) $\forall x(\exists w \subseteq t(wx \subseteq t) \to F$ and $\exists x(\exists w \subseteq t(wx \subseteq t) \wedge F)$. For readability's sake, in the following, we also abbreviates so-called *initial subword quantifications* $(\forall x)(x \subseteq t \to F)$ as $(\forall x \subseteq t)F$ and $(\exists x)(x \subseteq t \wedge F)$ as $(\exists x \subseteq t)F$.

The Theory RS^1_3. The theory RS^1_3 we are going to define is basically made of the the axioms by [6] as expressed in $\mathcal{L}$.

Definition 4 (Σ^b_1-Formula)**.** A Σ^b_0-formula is a formula is a subword quantified formula, i.e. a formula belonging to the smallest class of $\mathcal{L}$ containing atomic formulas and closed under Boolean operations and subword quantifications. A Σ^b_1*-formula* in $\mathcal{L}$ is a formula of the form $(\exists x)\big(x \preceq t(\vec{z}) \wedge F(\vec{z}, x)\big)$, where F is a subword quantified formula.

Every string $s \in \mathbb{S}$ can be seen as a term $\overline{s}$ of $\mathcal{L}$, such that $\overline{\epsilon} = \epsilon, \overline{s\mathbf{0}} = \overline{s} \frown \mathtt{0}$ and $\overline{s\mathbf{1}} = \overline{s} \frown \mathtt{1}$, e.g. $\overline{\mathbf{001}} = \mathtt{0} \frown \mathtt{0} \frown \mathtt{1}$.

Definition 5 (Theory RS^1_3)**.** The theory RS^1_3 is defined by the axioms below:

- Basic axioms:
  1. $x\epsilon = x$
  2. $x(y\mathtt{0}) = (xy)\mathtt{0}$
  3. $x(y\mathtt{1}) = (xy)\mathtt{1}$
  4. $x \times \epsilon = \epsilon$
  5. $x \times y\mathtt{0} = (x \times y)x$
  6. $x \times y\mathtt{1} = (x \times y)x$
  7. $x \subseteq \epsilon \leftrightarrow x = \epsilon$
  8. $x \subseteq y\mathtt{0} \leftrightarrow x \subseteq y \vee x = y\mathtt{0}$
  9. $x \subseteq y\mathtt{1} \leftrightarrow x \subseteq y \vee x = y\mathtt{1}$
  10. $x\mathtt{0} = y\mathtt{0} \rightarrow x = y$
  11. $x\mathtt{1} = y\mathtt{1} \rightarrow x = y$
  12. $x\mathtt{0} \neq y\mathtt{1}$
  13. $x\mathtt{0} \neq \epsilon$
  14. $x\mathtt{1} \neq \epsilon$
- Axiom scheme for induction on notation:

$$B(\epsilon) \wedge (\forall x)\big(B(x) \rightarrow B(x\mathtt{0}) \wedge B(x\mathtt{1})\big) \rightarrow (\forall x)B(x),$$

  where B is a Σ^b_1-formula in $\mathcal{L}$.

We will call Σ^b_1 the class containing all and only the Σ^b_1-formulas. An *extended* Σ^b_1*-formula* is any formula of $\mathcal{L}$ that can be constructed in a finite number of steps by starting with subword quantifications and bounded existential quantifications.

Proposition 1 ([6])**.** *In RS^1_3 any extended Σ^b_1-formula is logically equivalent to a Σ^b_1-formula.*[1]

[1] Actually, Ferreira proves this result for his theory Σ^b_1-NIA [7, pp. 148-149], but it clearly holds for RS^1_3 as well.

1.3 Semantics for Formulas in $\mathcal{L}$

Qualitative Interpretation of $\mathcal{L}$. The standard, *qualitative* model for terms and formulas of $\mathcal{L}$ consists in $\mathscr{W} = (\mathbb{S}, \frown, \times)$. In this case logical operators are interpreted in the canonical way and $\texttt{Flip}(\cdot)$ is treated as a standard, unary predicate of first-order logic, which is interpreted as a subset of $\mathbb{S}$.

Definition 6 (Interpretation for Terms)**.** An environment $\xi : \mathcal{G} \mapsto \mathbb{S}$, where $\mathcal{G}$ is the set of term variables, is a mapping that assigns to each variable a string. Given a term t in $\mathcal{L}$ and an environment ξ, the *interpretation of t in ξ* is the string $[\![t]\!]_\xi \in \mathbb{S}$ inductively defined as follows:

$$\begin{aligned} [\![\epsilon]\!]_\xi &= \boldsymbol{\epsilon} & [\![x]\!]_\xi &:= \xi(x) \in \mathbb{S} \\ [\![\texttt{0}]\!]_\xi &= \mathbf{0} & [\![t \frown s]\!]_\xi &= [\![t]\!]_\xi [\![s]\!]_\xi \\ [\![\texttt{1}]\!]_\xi &= \mathbf{1} & [\![t \times s]\!]_\xi &= [\![t]\!]_\xi \times [\![s]\!]_\xi. \end{aligned}$$

Definition 7 (Qualitative Semantics for $\mathcal{L}$-Formulas)**.** Given a formula F in $\mathcal{L}$ and an interpretation $\rho = (\xi, \omega^{\texttt{FLIP}})$, where $\xi : \mathcal{G} \to \mathbb{S}$ and $\omega^{\texttt{FLIP}} \subseteq \mathbb{O}$, the *interpretation of F in ρ*, $[\![F]\!]_\rho$, is inductively defined as follows:

$$[\![\texttt{Flip}(t)]\!]_\rho := \begin{cases} 1 & \text{if } \omega^{\texttt{FLIP}}([\![t]\!]_\rho) = \mathbf{1} \\ 0 & \text{otherwise} \end{cases}$$

$$[\![t = s]\!]_\rho := \begin{cases} 1 & \text{if } [\![t]\!]_\rho = [\![s]\!]_\rho \\ 0 & \text{otherwise} \end{cases}$$

$$[\![t \subseteq s]\!]_\rho := \begin{cases} 1 & \text{if } [\![t]\!]_\rho \subseteq [\![s]\!]_\rho \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} [\![\neg G]\!]_\rho &:= 1 - [\![G]\!]_\rho \\ [\![G \wedge H]\!]_\rho &:= min\{[\![G]\!]_\rho, [\![H]\!]_\rho\} \\ [\![G \vee H]\!]_\rho &:= max\{[\![G]\!]_\rho, [\![H]\!]_\rho\} \\ [\![G \to H]\!]_\rho &:= max\{(1 - [\![G]\!]_\rho), [\![H]\!]_\rho\} \\ [\![(\forall x)G]\!]_\rho &:= min\{[\![G]\!]_{\rho\{x \leftarrow s\}} \mid s \in \mathbb{S}\} \\ [\![(\exists x)G]\!]_\rho &:= max\{[\![G]\!]_{\rho\{x \leftarrow s\}} \mid s \in \mathbb{S}\}. \end{aligned}$$

Quantitative Interpretation of $\mathcal{L}$. Inspired by [1], we introduce the alternative, *quantitative* semantics for $\mathcal{L}$-terms and -formulas. Actually, terms are defined in the standard way, exactly as in the Definition 6 above. On the other hand, the semantics for formulas is inherently quantitative, as any formula is associated with a (measurable) set.

Definition 8 (Quantitative Semantics)**.** Given a formula F and an environment $\xi : \mathcal{G} \to \mathbb{S}$, where $\mathcal{G}$ is the set of term variables, the *interpretation of F in ξ*, $[\![F]\!]_\xi$, is the (measurable) set of sequences inductively defined as follows:

$$[\![\texttt{Flip}(t)]\!]_\xi := \{\omega \mid \omega([\![t]\!]_\xi) = \mathbf{1}\}$$

$$[\![t = s]\!]_\xi := \begin{cases} \mathbb{O} & \text{if } [\![t]\!]_\xi = [\![s]\!]_\xi \\ \emptyset & \text{otherwise} \end{cases}$$

$$[\![t \subseteq s]\!]_\xi := \begin{cases} \mathbb{O} & \text{if } [\![t]\!]_\xi \subseteq [\![s]\!]_\xi \\ \emptyset & \text{otherwise} \end{cases}$$

$$\begin{aligned} [\![\neg G]\!]_\xi &:= \mathbb{O} - [\![G]\!]_\xi \\ [\![G \vee H]\!]_\xi &:= [\![G]\!]_\xi \cup [\![H]\!]_\xi \\ [\![G \wedge H]\!]_\xi &:= [\![G]\!]_\xi \cap [\![H]\!]_\xi \\ [\![G \to H]\!]_\xi &:= (\mathbb{O} - [\![G]\!]_\xi) \cup [\![H]\!]_\xi \\ [\![(\exists x)G]\!]_\xi &:= \bigcup_{i \in \mathbb{S}} [\![G]\!]_{\xi\{x \leftarrow i\}} \\ [\![(\forall x)G]\!]_\xi &:= \bigcap_{i \in \mathbb{S}} [\![G]\!]_{\xi\{x \leftarrow i\}}. \end{aligned}$$

Notation 4. For readability's sake, in what follows, we may abbreviate the former interpretation as $[\![\cdot]\!]_\omega$ and the quantitative interpretation $[\![\cdot]\!]_\xi$ as simply $[\![\cdot]\!]$.

1.4 $\mathcal{POR}$ is Σ_1^b-representable in RS_3^1

First, we modify the standard definition of Σ_1^b-representability so to fit our peculiar class of string (oracle) functions and our probabilistic word language.

Definition 9 (Σ_1^b-Representability)**.** A function $f : \mathbb{S}^j \times \mathbb{O} \to \mathbb{S}$ is Σ_1^b-*representable* in RS_3^1 if and only if there is a Σ_1^b-formula $G(\vec{x}, y)$ in $\mathcal{L}$ such that:

1. $RS_3^1 \vdash (\forall \vec{x})(\exists y)G(\vec{x}, y)$
2. $RS_3^1 \vdash (\forall \vec{x})(\forall y)(\forall z)\big(G(\vec{x}, y) \wedge G(\vec{x}, z) \to y = z\big)$
3. for all $n_1, \ldots, n_j, m \in \mathbb{S}$, and $\omega \in \mathbb{O}$, $f(n_1, \ldots, n_j, \omega) = m$ if and only if $\omega \in [\![G(\overline{n_1}, \ldots, \overline{n_j}, \overline{m})]\!]$.

Notice that, as seen, the language $\mathcal{L}$ can be seen a fragment of both the language $\mathcal{L}^*_{\mathsf{FOL}}$, that is a first-order word language $\mathcal{L}$ including the unary predicate $\mathtt{Flip}(\cdot)$, and of (a slight variation of) $\mathcal{L}_{\mathsf{MQPA}}$, as defined in [1]. Indeed, in Theorem 1.1 below the desired $\mathcal{L}$-formula $G(x, y)$ is treated as a formula of FOL when dealing with conditions 1. and 2., but as a formula of MQPA in 3.

The following proof relies on a well-known result by Parikh.

Proposition 2 ("Parikh" [10])**.** *Let $F(\vec{x}, y)$ be a bounded formula in $\mathcal{L}$ such that $RS_3^1 \vdash (\forall \vec{x})(\exists y)F(\vec{x}, y)$. Then, there is a term t such that $RS_3^1 \vdash (\forall \vec{x})(\exists y \preceq t(\vec{x}))F(\vec{x}, y)$.*

Actually, the theorem is usually presented in the context of Buss' bounded theories, as stating that, given a bounded formula B (in $\mathcal{L}_{\mathbb{N}}$), such that $S_2^i \vdash (\forall \vec{x})(\exists y)B$, then there is a term $t(\vec{x})$ such that also $S_2^i \vdash (\forall \vec{x})\big(\exists y \leq t(\vec{x})\big)B(\vec{x}, y)$, [2, 3]. However, due to [8], Buss' *syntactic* proof would hold for Ferreira's Σ_1^b-NIA [7] as well. The same result holds for RS_3^1, which does not contain any specific rule concerning $\mathtt{Flip}(\cdot)$ and so is defined by the same peculiar axioms as Ferreira's Σ_1^b-NIA [6].

Theorem 1.1. *Every $f \in \mathcal{POR}$ is Σ_1^b-representable in RS_3^1.*

Proof Sketch. The proof is by induction on the structure of functions in $\mathcal{POR}$:[2]

Base case. Each basic function is Σ_1^b-representable in RS_3^1.

- $f = E$ is Σ_1^b-represented in RS_3^1 by the formula:

$$G_E(x, y) := x = x \wedge y = \epsilon.$$

1. Given x, existence is proved by considering $y = \epsilon$. By the reflexivity of identity both $RS_3^1 \vdash x = x$ and $RS_3^1 \vdash \epsilon = \epsilon$ hold. So, for the definition of $\wedge$, also $RS_3^1 \vdash x = x \wedge \epsilon = \epsilon$. We conclude $RS_3^1 \vdash (\forall x)(\exists y)(x = x \wedge y = \epsilon)$.
2. Uniqueness is proved assuming $RS_3^1 \vdash x = x \wedge z = \epsilon$. So, for the definition of $\wedge$, in particular $RS_3^1 \vdash z = \epsilon$. Since, as shown, $RS_3^1 \vdash y = \epsilon$, by the transitivity of identity, we conclude $RS_3^1 \vdash y = z$.

[2] Full details are given in Appendix 3.1.2.

3. Assume $E(n, \omega^*) = m$. If $m = \boldsymbol{\epsilon}$, then

$$\begin{aligned}[\![\overline{n} = \overline{n} \wedge \overline{m} = \epsilon]\!] &= [\![\overline{n} = \overline{n}]\!] \cap [\![\overline{m} = \epsilon]\!] \\ &= \mathbb{O} \cap \mathbb{O} \\ &= \mathbb{O}.\end{aligned}$$

So, in this case, for any ω^*, $\omega^* \in [\![\overline{n} = \overline{n} \vee \overline{m} = \epsilon]\!]$, as clearly $\omega^* \in \mathbb{O}$.
If $m \neq \boldsymbol{\epsilon}$, then

$$\begin{aligned}[\![\overline{n} = \overline{n} \vee \overline{m} = \epsilon]\!] &= [\![\overline{n} = \overline{n}]\!] \cap [\![\overline{m} = \epsilon]\!] \\ &= \mathbb{O} \cap \emptyset \\ &= \emptyset.\end{aligned}$$

So, for any ω^*, $\omega^* \notin [\![\overline{n} = \overline{n} \vee \overline{m} = \epsilon]\!]$, as clearly $\omega^* \notin \emptyset$.

$f = P_i^n$, for $1 \leq i \leq n$, is Σ_1^b-represented in RS_3^1 by the formula:

$$G_{P_i^n}(x, y) := \bigwedge_{i \in J} (x_j = x_j) \wedge y = x_i,$$

where $J = \{1, ..., n\} \backslash \{i\}$.

$f = S_b$ is Σ_1^b-represented in RS_3^1 by the formula:

$$G_S(x, y) := y = x\mathtt{b},$$

where $\mathtt{b} = \mathtt{0}$ if $b = \mathbf{0}$ and $\mathtt{b} = \mathtt{1}$ if $b = \mathbf{1}$.

$f = C$ is Σ_1^b-represented in RS_3^1 by the formula:

$$G_C(x_1, v, z_0, z_1, y) := (x = \epsilon \wedge y = v) \vee (\exists x' \preceq x)(x = x'\mathtt{0} \wedge y = z_0) \vee (\exists x' \preceq x)(x = x'\mathtt{1} \wedge y = z_1).$$

$f = Q$ is Σ_1^b-represented in RS_3^1 by the formula:

$$G_Q(x, y) := \big(\mathtt{Flip}(x) \wedge y = \mathtt{1}\big) \vee \big(\neg\mathtt{Flip}(x) \wedge y = \mathtt{0}\big).$$

Notice that this proof relies on the fact that every $f \in \mathcal{POR}$ is capable of invoking exactly *one* oracle.

1. Existence is proved by cases. Intuitively, if $RS_3^1 \vdash \mathtt{Flip}(x)$, let $y = \mathtt{1}$. By the reflexivity of identity $RS_3^1 \vdash \mathtt{1} = \mathtt{1}$ holds, so also $RS_3^1 \vdash \mathtt{Flip}(x) \wedge \mathtt{1} = \mathtt{1}$. In this case we conclude, for the definition of $\vee$, we conclude $RS_3^1 \vdash \big(\mathtt{Flip}(x) \wedge \mathtt{1} = \mathtt{1}\big) \vee \big(\neg\mathtt{Flip}(x) \wedge \mathtt{1} = \mathtt{0}\big)$, that is $RS_3^1 \vdash (\exists y)((\mathtt{Flip}(x) \wedge y = \mathtt{1}) \vee (\neg\mathtt{Flip}(x) \wedge y = \mathtt{0}))$. If $RS_3^1 \vdash \neg\mathtt{Flip}(x)$, let $y = \mathtt{0}$. By the reflexivity of identity $RS_3^1 \vdash \mathtt{0} = \mathtt{0}$ holds. Thus, for the definition of $\wedge$, $RS_3^1 \vdash \neg\mathtt{Flip}(x) \wedge \mathtt{0} = \mathtt{0}$ and, for that of $\vee$, we conclude again $RS_3^1 \vdash \big(\mathtt{Flip}(x) \wedge \mathtt{0} = \mathtt{1}\big) \vee \big(\neg\mathtt{Flip}(x) \wedge \mathtt{0} = \mathtt{0}\big)$. that is $RS_3^1 \vdash (\exists y)((\mathtt{Flip}(x) \wedge y = \mathtt{1}) \vee (\neg\mathtt{Flip}(x) \wedge y = \mathtt{0}))$.

2. Uniqueness is established relying on the transitivity of identity.
3. Finally, it is shown that, for every $n, m \in \mathbb{S}$ and $\omega^* \in \mathbb{O}$, $Q(n, \omega^*) = m$ if and only if $\omega^* \in [\![G_Q(\overline{n}, \overline{m})]\!]$.
   Assume $m = \mathbf{1}$. Then $Q(n, \omega^*) = \mathbf{1}$, which is $\omega^*(n) = \mathbf{1}$,

$$\begin{aligned}
[\![\big(\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{1}\big) \vee \big(\neg\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{0}\big)]\!] &= [\![\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{1}]\!] \cup [\![\neg\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{0}]\!] \\
&= \big([\![\mathtt{Flip}(\overline{n})]\!] \cap [\![\mathtt{1} = \mathtt{1}]\!]\big) \cup \big([\![\neg\mathtt{Flip}(\overline{n})]\!] \cap [\![\mathtt{1} = \mathtt{0}]\!]\big) \\
&= \big([\![\mathtt{Flip}(\overline{n})]\!] \cap \mathbb{O}\big) \cup \big([\![\neg\mathtt{Flip}(\overline{n})]\!] \cap \emptyset\big) \\
&= [\![\mathtt{Flip}(\overline{n})]\!] \\
&= \{\omega \mid \omega(n) = \mathbf{1}\}.
\end{aligned}$$

   Clearly, $\omega^* \in [\![\big(\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{1}\big) \vee \big(\neg\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{0}\big)]\!]$. The case $m = \mathbf{0}$ and the opposite direction are proved in a similar way.

Inductive case. If f is defined by composition or bounded recursion from Σ^b_1-representable functions, then f is Σ^b_1-representable in RS^1_3:

- *Composition.* Assume that f is defined by composition from functions $g, h_1, \ldots, h_k$, so that

$$f(\vec{x}, \omega) = g(h_1(\vec{x}, \omega), \ldots, h_k(\vec{x}, \omega), \omega)$$

  and that $g, h_1, \ldots, h_k$ are represented in RS^1_3 by (resp.) the Σ^b_1-formulas $G_g, G_{h_1}, \ldots, G_{h_k}$. By Proposition 2, there exist suitable terms $t_g, t_{h1}, \ldots, t_{h_k}$ such that condition 1. of Definition 9 can be strengthened to $RS^1_3 \vdash (\forall\vec{x})(\exists y \preceq t_i)G_i(\vec{x}, y)$, for each $i \in \{g, h_1, \ldots, h_k\}$. We conclude that $f(\vec{x}, \omega)$ is Σ^b_1-represented in RS^1_3 by the following formula:

$$G(x, y) := \big(\exists z_1 \preceq t_{h_1}(\vec{x})\big) \ldots \big(\exists z_k \preceq t_{h_k}(\vec{x})\big)\big(G_{h_1}(\vec{x}, z_1) \wedge \cdots \wedge G_{h_k}(\vec{x}, z_k) \wedge G_g(z_1, \ldots, z_k, y)\big).$$

  Indeed, by IH, $G_g, G_{h_1}, \ldots, G_{h_k}$ are Σ^b_1-formulas, so also G is is Σ^b_1. Moreover, conditions 1.-3. are proved to hold by slight variations of the standard proofs.

- *Bounded Recursion.* Assume that f is defined by bounded recursion from f, h_0, h_1 so that:

$$\begin{aligned}
f(\vec{x}, \boldsymbol{\epsilon}, \omega) &= g(\vec{x}, \omega) \\
f(\vec{x}, y\mathbf{i}, \omega) &= h_i(\vec{x}, y, f(\vec{x}, y, \omega), \omega)|_{t(\vec{x}, y)}
\end{aligned}$$

  where $\mathbf{i} \in \{\mathbf{0}, \mathbf{1}\}$ and $i \in \{0, 1\}$, so that $\mathbf{i} = \mathbf{0}$ when $i = 0$ and $\mathbf{i} = \mathbf{1}$ when $i = 1$. In addition, let g, h_0, h_1 be represented in RS^1_3 by (resp.) the following Σ^b_1-formulas G_g, G_{h_0}, G_{h_1}. Moreover, by Proposition 2, there exist some suitable terms t_g, t_{h_0}, t_{h_1} such that condition 1. of Definition 9 can be strengthened to its "bounded version". Then, it can be proved that $f(\vec{x}, y)$ is Σ^b_1-defined in RS^1_3 by the formula below:

$$\begin{aligned}
G(x, y) := \big(\exists v \preceq t_g(\vec{x})t_g(\vec{x})(y \times t(\vec{x}, y)t(\vec{x}, y)\mathtt{11})\big)\Big(&F_{lh}(v, \mathtt{1} \times y\mathtt{1}) \\
&\wedge (\exists z \preceq t_g(\vec{x}))\big(F_{eval}(v, \epsilon, z) \wedge G_g(\vec{x}, z)\big) \\
&\wedge (\forall u \subset y)(\exists z)\Big(\tilde{z} \preceq t(\vec{x}, y)\Big(F_{eval}(v, \mathtt{1} \times u, z) \wedge F_{eval}(v, \mathtt{1} \times u\mathtt{1}, \tilde{z}) \\
&\wedge \big(u\mathtt{0} \subseteq y \rightarrow (\exists z_0 \preceq t_{h_0}(\vec{x}, u, z))\big(G_{h_0}(\vec{x}, u, z, z_0) \wedge z_0|_{t(\vec{x}, u)} = \tilde{z}\big)\big) \\
&\wedge \big(u\mathtt{1} \subseteq y \rightarrow \big(\exists z_1 \preceq t_{h_1}(\vec{x}, u, z)\big)\big(G_{h_1}(\vec{x}, u, z, z_1) \wedge z_1|_{t(\vec{x}, u)} = \tilde{z}\big)\big)\Big)\Big)\Big)
\end{aligned}$$

where F_{lh} and F_{eval} are Σ^b_1-formulas defined as in [7]. Roughly, $F_{lh}(x,y)$ states that the number of $\mathtt{1}$s in the (s-)encoding of x is yy, while $F_{eval}(x,y,z)$ is a "decoding" formula (strongly resembling the acting of Gödel's β-formula), expressing that the "bit" encoded in x at position y is z. Moreover, $x \subset y$ is an abbreviation for $x \subseteq y \wedge x \neq y$. This formula G satisfies all the desired requirements to Σ^b_1-represent in RS^1_3 the function f, obtained by bounded recursion from g, h_0 and h_1. In particular, conditions 1. and 2., concerning existence and uniqueness, have already been proved to hold in Ferreira's [7]. Moreover, G basically expresses that, given the desired encoding sequence v, (i) the ϵ-th bit of v is (the encoding of) z', such that $G_g(\vec{x}, z')$ holds, where (for IH) G_g is the Σ^b_1-formula representing the function g, and (ii) given that for each $u \subset y$, z denotes the "bit" encoded in v at position $1 \times u$ and, similarly, $\tilde{z}$ is the next "bit", encoded in v at position $\mathtt{1} \times u\mathtt{1}$, then if $u\mathtt{b} \subseteq y$ (that is, if we are considering an initial substring of y the last bit of which is b) then there is a z_b, such that $G_{h_b}(\vec{x}, y, z, z_b)$, where G_{h_b} Σ^b_1-represents the function f_{h_b} and the truncation of z_b at $t(\vec{x}, u)$ is precisely $\tilde{z}$, with $b = 0$ when $\mathtt{b} = \mathtt{0}$ and $b = 1$ when $\mathtt{b} = \mathtt{1}$.[3]

□

2 All Functions which are Σ^b_1-Representable in RS^1_3 are in $\mathcal{POR}$

In this Section, we show that if a function is Σ^b_1-representable in RS^1_3, then it is in $\mathcal{POR}$. The proof basically adapts the strategy used by Cook and Urquhart's for system IPV$^\omega$ [5] and is structured as follows:

1. First, in Section 2.1, we define a basic equational theory $\mathcal{POR}^\lambda$ for a simply typed λ-calculus endowed with primitives corresponding to functions of $\mathcal{POR}$.
2. Then, in Section 2.2, we define a first-order *intuitinistic* theory $I\mathcal{POR}^\lambda$, which extends $\mathcal{POR}^\lambda$ with usual predicate calculus as well as an **NP**-induction schema. It is also shown that $I\mathcal{POR}^\lambda$ is strong enough to prove all theorems of IRS^1_3, the intuitionistic version of RS^1_3.
3. In Section 2.3 we develop a realizability interpretation of $I\mathcal{POR}^\lambda$ (inside itself), showing that from any derivation of $(\forall x)(\exists y)A(x,y)$, where A is a Σ^b_0-formula, one can extract a λ-term $\mathtt{t}$ of $\mathcal{POR}^\lambda$ such that $(\forall x)A(x, \mathtt{t}x)$ is provable in $I\mathcal{POR}^\lambda$. From this we deduce that every function which is Σ^b_1-representable in IRS^1_3 is in $\mathcal{POR}$.
4. Finally, in Section 2.4, we extend this result to classical RS^1_3, by showing that any Σ^b_1-formula provable in $I\mathcal{POR}^\lambda$ + Excluded Middle is already provable in $I\mathcal{POR}^\lambda$.

2.1 The System $\mathcal{POR}^\lambda$

We define an equational theory for a simply typed λ-calculus augmented with primitives for functions of $\mathcal{POR}$. Actually, these primitives do not exactly correspond to the ones in the definition

[3] Otherwise said, if $u\mathtt{0} \subseteq y$, there is a z_0 such that the Σ^b_1-formula $G_{h_0}(\vec{x}, u, z, z_0)$ represents the function h_0 and, in this case, $\tilde{z}$ corresponds to the truncation of z_0 at $t(\vec{x}, u)$, that is the "bit" encoded by v at the position $\mathtt{1} \times u\mathtt{1}$ (i.e. corresponding to $u\mathtt{0} \subseteq y$) is precisely such $\tilde{z}$. Equally, if $u\mathtt{1} \subseteq y$, there is a z_0 such that the Σ^b_1-formula $G_{h_1}(\vec{x}, u, z, z_1)$ represents now the function h_1 and $\tilde{z}$ corresponds to the truncation of z_1 at $t(\vec{x}, u)$, that is the "bit" encoded by v at position $\mathtt{1} \times u\mathtt{1}$ (i.e. corresponding to $u\mathtt{1} \subseteq y$) is precisely such $\tilde{z}$.

of $\mathcal{POR}$, although the resulting function algebra is proved equivalent. The choice of the primitives follows the principle that the defining equations for the functions different from the recursion operator should not depend on it, whereas its defining equations should depend on it and on functions defined without it.

2.1.1 The Syntax of $\mathcal{POR}^\lambda$

Definition 10 (Types of $\mathcal{POR}^\lambda$)**.** *Types of* $\mathcal{POR}^\lambda$ are defined by the grammar below:

$$\tau := \mathsf{s} \mid \tau \Rightarrow \tau.$$

Definition 11 (Terms of $\mathcal{POR}^\lambda$)**.** *Terms of* $\mathcal{POR}^\lambda$ are standard, simply typed λ-terms plus the constants from the signature below:

$$\begin{aligned}
0, 1, \epsilon &: \mathsf{s} \\
\circ &: \mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s} \\
\mathsf{Tail} &: \mathsf{s} \Rightarrow \mathsf{s} \\
\mathsf{Trunc} &: \mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s} \\
\mathsf{Cond} &: \mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s} \\
\mathsf{Flipcoin} &: \mathsf{s} \Rightarrow \mathsf{s} \\
\mathsf{Rec} &: \mathsf{s} \Rightarrow (\mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s}) \Rightarrow (\mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s}) \Rightarrow (\mathsf{s} \Rightarrow \mathsf{s}) \Rightarrow \mathsf{s} \Rightarrow \mathsf{s}.
\end{aligned}$$

Intuitively, $\mathsf{Tail}(x)$ computes the string obtained by deleting the first digit of x; $\mathsf{Trunc}(x, y)$ computes the string obtained by truncating x at the length of y; $\mathsf{Cond}(x, y, z, w)$ computes the function that yields y when $x = \boldsymbol{\epsilon}$, z when $x = x'\mathbf{0}$, and w when $x = x'\mathbf{1}$; $\mathsf{Flipcoin}(x)$ indicates a random $\mathbf{0}/\mathbf{1}$ generator; Rec is the operator for bounded recursion on notation.

Notation 5. We will usually denote $x \circ y$ simply as xy. Moreover, to enhance readability, let T be any constant $\mathsf{Tail}, \mathsf{Trunc}, \mathsf{Cond}, \mathsf{Flipcoin}, \mathsf{Rec}$ of arity n, we indicate $\mathsf{T}\mathsf{u}_1 \dots \mathsf{u}_n$ as $\mathsf{T}(\mathsf{u}_1, \dots, \mathsf{u}_n)$.

$\mathcal{POR}^\lambda$ is reminiscent of PV^ω by Cook and Urquhart [5] without the induction rule (R5) that we do not need. The main difference being the constant $\mathsf{Flipcoin}$, which, as said, intuitively denotes a function which randomly generates either $\mathbf{0}$ or $\mathbf{1}$ when it reads a string. These interpretations will be made clearer by Definition 14 below.

Remark 2. *In the following, we often define terms implicitly using bounded recursion on notation. Otherwise said, we define new terms as* $\mathsf{F} : \mathsf{s} \Rightarrow \dots \Rightarrow \mathsf{s}$ *by equations of the form:*

$$\begin{aligned}
\mathsf{F}\vec{x}\epsilon &:= \mathsf{G}\vec{x} \\
\mathsf{F}\vec{x}(y\mathsf{0}) &:= \mathsf{H}_0\vec{x}y(\mathsf{F}\vec{x}y) \\
\mathsf{F}\vec{x}(y\mathsf{1}) &:= \mathsf{H}_1\vec{x}y(\mathsf{F}\vec{x}y)
\end{aligned}$$

where $\mathsf{G}, \mathsf{H}_0, \mathsf{H}_1$ *are already-defined terms, and the second and third equations satisfy a length bound given by some term* K *(which is usually simply* $\lambda\vec{x}.\lambda y.\mathsf{0}$*). The term* F *can be explicitly defined as follows:*

$$\mathsf{F} := \lambda\vec{x}.\lambda y.\mathsf{Rec}(\mathsf{G}\vec{x}, \lambda yy'.\mathsf{H}_0\vec{x}yy', \lambda yy'.\mathsf{H}_1\vec{x}yy', \mathsf{K}\vec{x}, y).$$

We now introduce some abbreviations for composed functions:

- $\mathsf{B}(x) := \mathsf{Cond}(x, \epsilon, 0, 1)$ indicates the function that computes the last digit of x, i.e. coerces x to a Boolean value.
- $\mathsf{BNeg}(x) := \mathsf{Cond}(x, \epsilon, 1, 0)$ indicates the function that computes the Boolean negation of $\mathsf{B}(x)$.
- $\mathsf{BOr}(x, y) := \mathsf{Cond}(\mathsf{B}(x), \mathsf{B}(y), \mathsf{B}(y), 1)$ indicates the function that coerces x, y to Booleans and then performs the OR-operation.
- $\mathsf{BAnd}(x, y) := \mathsf{Cond}\big(\mathsf{B}(x), \epsilon, 0, \mathsf{B}(y)\big)$ indicates the function that coerces x, y to Booleans and then performs the AND-operation.
- $\mathsf{Eps}(x) := \mathsf{Cond}(x, 1, 0, 0)$ indicates the characteristic function of the predicate "$x = \epsilon$".
- $\mathsf{Bool}(x) := \mathsf{BAnd}\big(\mathsf{Eps}(\mathsf{Tail}(x)), \mathsf{BNeg}(\mathsf{Eps}(x))\big)$ indicates the characteristic function of the predicate "$x = 0 \vee x = 1$".
- $\mathsf{Zero}(x) := \mathsf{Cond}\big(\mathsf{Bool}(x), 0, \mathsf{Cond}(x, 0, 0, 1), 0\big)$ indicates the characteristic function of the predicate "$x = 0$".
- $\mathsf{Conc}(x, y)$ indicates the concatenation function defined by the equations below:

$$\begin{aligned} \mathsf{Conc}(x, \epsilon) &:= x \\ \mathsf{Conc}(x, y0) &:= \mathsf{Conc}(x, y)0 \\ \mathsf{Conc}(x, y1) &:= \mathsf{Conc}(x, y)1. \end{aligned}$$

- $\mathsf{Eq}(x, y)$ indicates the characteristic function of the predicate "$x = y$" and is defined by double recursion by the equations below:

$$\begin{aligned} \mathsf{Eq}(\epsilon, \epsilon) &:= 1 \\ \mathsf{Eq}(\epsilon, y0) = \mathsf{Eq}(\epsilon, y1) &:= 0 \end{aligned}$$

$$\begin{aligned} \mathsf{Eq}(x0, \epsilon) &:= 0 \\ \mathsf{Eq}(x0, y0) &:= \mathsf{Eq}(x, y) \\ \mathsf{Eq}(x0, y1) &:= 0 \end{aligned}$$

$$\begin{aligned} \mathsf{Eq}(x1, \epsilon) &:= 0 \\ \mathsf{Eq}(x1, y0) &:= 0 \\ \mathsf{Eq}(x1, y1) &:= \mathsf{Eq}(x, y). \end{aligned}$$

- $\mathsf{Times}(x, y)$ indicates the function for self-concatenation, $x, y \mapsto x \times y$, and is defined by the equations below:

$$\begin{aligned} \mathsf{Times}(x, \epsilon) &:= \epsilon \\ \mathsf{Times}(x, y\mathsf{b}) &:= \mathsf{Conc}(\mathsf{Times}(x, y), x), \end{aligned}$$

  where $\mathsf{b} \in \{0, 1\}$.

- $\mathsf{Sub}(x, y)$ indicates the initial-substring function $x, y \mapsto S(x, y)$, and is defined by bounded recursion as follows:

$$\begin{aligned}\mathsf{Sub}(x, \epsilon) &:= \mathsf{Eps}(x)\\ \mathsf{Sub}(x, y0) &:= \mathsf{BOr}\big(\mathsf{Sub}(x, y), \mathsf{Eq}(x, y0)\big)\\ \mathsf{Sub}(x, y1) &:= \mathsf{BOr}\big(\mathsf{Sub}(x, y), \mathsf{Eq}(x, y1)\big).\end{aligned}$$

Definition 12 (Formulas of $\mathcal{POR}^\lambda$)**.** *Formulas of* $\mathcal{POR}^\lambda$ are all equations $\mathsf{t} = \mathsf{u}$, where t, u are terms of type s.

Definition 13 (The Theory $\mathcal{POR}^\lambda$)**.** *Axioms of* $\mathcal{POR}^\lambda$ are the following ones:

- Defining equations for the constants of $\mathcal{POR}^\lambda$:

$$\begin{aligned}\epsilon x = x\epsilon &= x\\ x(y\mathsf{b}) &= (xy)\mathsf{b}\end{aligned}$$

$$\begin{aligned}\mathsf{Tail}(\epsilon) &= \epsilon\\ \mathsf{Tail}(x\mathsf{b}) &= x\end{aligned}$$

$$\begin{aligned}\mathsf{Trunc}(x, \epsilon) &= \epsilon\\ \mathsf{Trunc}(\epsilon, y) &= \epsilon\\ \mathsf{Trunc}(x\mathsf{b}, y0) = \mathsf{Trunc}(x\mathsf{b}, y1) &= \mathsf{Trunc}(x, y)\mathsf{b}\end{aligned}$$

$$\begin{aligned}\mathsf{Cond}(\epsilon, y, z, w) &= y\\ \mathsf{Cond}(x0, y, z, w) &= z\\ \mathsf{Cond}(x1, y, z, w) &= w\end{aligned}$$

$$\mathsf{Bool}\big(\mathsf{Flipcoin}(x)\big) = 1$$

$$\begin{aligned}\mathsf{Rec}(x, h_0, h_1, k, \epsilon) &= x\\ \mathsf{Rec}(x, h_0, h_1, k, y0) &= \mathsf{Trunc}\big(h_0 y(\mathsf{Rec}(x, h_0, h_1, k, y)), ky\big)\\ \mathsf{Rec}(x, h_0, h_1, k, y1) &= \mathsf{Trunc}\big(h_1 y(\mathsf{Rec}(x, h_0, h_1, k, y)), ky\big)\end{aligned}$$

  where $\mathsf{b} \in \{0, 1\}$.

- The (β)- and (ν)-axioms:

$$\mathsf{C}\big[(\lambda x.\mathsf{t})\mathsf{u}\big] = \mathsf{C}\big[\mathsf{t}\{\mathsf{u}/x\}\big] \qquad (\beta)$$

$$\mathsf{C}\big[\lambda x.\mathsf{t}x\big] = \mathsf{C}\big[\mathsf{t}\big] \qquad (\nu)$$

  where $\mathsf{C}\big[\,\cdot\,\big]$ indicates a context with a unique occurrence of the hole $\big[\ \big]$, so that $\mathsf{C}\big[\mathsf{t}\big]$ denotes the variable-capturing replacement of $\big[\ \big]$ by t in $\mathsf{C}\big[\ \big]$.

The inference rules of $\mathcal{POR}^\lambda$ are the following ones:

$$\mathsf{t} = \mathsf{u} \vdash \mathsf{u} = \mathsf{t} \tag{R1}$$

$$\mathsf{t} = \mathsf{u}, \mathsf{u} = \mathsf{v} \vdash \mathsf{t} = \mathsf{v} \tag{R2}$$

$$\mathsf{t} = \mathsf{u} \vdash \mathsf{v}\{\mathsf{t}/x\} = \mathsf{v}\{\mathsf{u}/x\} \tag{R3}$$

$$\mathsf{t} = \mathsf{u} \vdash \mathsf{t}\{\mathsf{v}/x\} = \mathsf{u}\{\mathsf{v}/x\}. \tag{R4}$$

Moreover, let $\vdash_{\mathcal{POR}^\lambda} \mathsf{t} = \mathsf{u}$ indicate that the equation $\mathsf{t} = \mathsf{u}$ is deducible using instances of the axioms and inference rules above. Given any set T of equations, we let $T \vdash_{\mathcal{POR}^\lambda} \mathsf{t} = \mathsf{u}$ indicate that the equation $\mathsf{t} = \mathsf{u}$ is deducible using instances of the axioms and inference rules of $\mathcal{POR}^\lambda$ together with equations from T.

2.1.2 Relating $\mathcal{POR}$ and $\mathcal{POR}^\lambda$

For any string $s \in \mathbb{S}$, let $\overline{\overline{s}} : \mathsf{s}$ denote the term of $\mathcal{POR}^\lambda$ corresponding to it, i.e. $\overline{\overline{\epsilon}} = \epsilon$, $\overline{\overline{s\mathbf{0}}} = \overline{\overline{s}}0$, and $\overline{\overline{s\mathbf{1}}} = \overline{\overline{s}}1$. For any $\omega \in \mathbb{O}$, let T_ω be the set of all equations of the form $\mathsf{Flipcoin}(\overline{\overline{s}}) = \overline{\overline{\omega(s)}}$.

Definition 14 (Provable Representability)**.** Let $f : \mathbb{O} \times \mathbb{S}^j \to \mathbb{S}$. A term $\mathsf{t} : \mathsf{s} \Rightarrow \ldots \Rightarrow \mathsf{s}$ of $\mathcal{POR}^\lambda$ *provably represents* f if and only if for all strings $s_1, \ldots, s_j, s \in \mathbb{S}$, and $\omega \in \mathbb{O}$,

$$f(s_1, \ldots, s_n, \omega) = s \quad \Leftrightarrow \quad T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{t}\overline{\overline{s_1}} \ldots \overline{\overline{s_j}} = \overline{\overline{s}}.$$

Example 1. The term $\mathsf{Flipcoin} : \mathsf{s} \Rightarrow \mathsf{s}$ provably represents the query function $Q(x, \omega) = \omega(x)$ of $\mathcal{POR}$, since for any $s \in \mathbb{S}$ and $\omega \in \mathbb{O}$,

$$\mathsf{Flipcoin}(\overline{\overline{s}}) = \overline{\overline{\omega(s)}} \vdash_{\mathcal{POR}^\lambda} \mathsf{Flipcoin}(\overline{\overline{s}}) = \overline{\overline{Q(s, \omega)}}.$$

Let us show that some of the terms described above provably represent the intended functions. Let $f_{\mathsf{Tail}}(s, \omega)$ indicate the string obtained by chopping the first digit of s (with $E(s, \omega) = \boldsymbol{\epsilon}$) and $f_{\mathsf{Trunc}}(s_1, s_2, \omega) = s_1|_{s_2}$

Lemma 1. *The terms* $\mathsf{Tail}, \mathsf{Trunc}, \mathsf{Cond}$ *provably represent the functions* $f_{\mathsf{Tail}}, f_{\mathsf{Trunc}}$ *and* C*, respectively.*

Proof Sketch. For Tail and Cond, the claim follows immediately from the defining axioms of the corresponding constants. For example, if $s_1 = s_2\mathbf{0}$, then $f_{\mathsf{Tail}}(s_1, \omega) = s_2$ and $\overline{\overline{s_1}} = \overline{\overline{s_2\mathbf{0}}} = \overline{\overline{s_2}}0$. Using the defining axioms of Tail:

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{Tail}(\overline{\overline{s_1}}) = \mathsf{Tail}(\overline{\overline{s_2}}0) = \overline{\overline{s_2}}.$$

For Trunc by double induction on two strings $s_1, s_2 \in \mathbb{S}$ we conclude that:

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{Trunc}(\overline{\overline{s_1}}, \overline{\overline{s_2}}) = \overline{\overline{s_1|_{s_2}}}.$$

□

Theorem 1. *1. Any function $f \in \mathcal{POR}$ is provably represented by a term $\mathsf{t} \in \mathcal{POR}^\lambda$.*

2. For any term $\mathsf{t} \in \mathcal{POR}^\lambda$, there is a function $f \in \mathcal{POR}$ such that f is provably represented by t.

Proof Sketch. ($\Rightarrow$) The proof is by induction on the structure of $f \in \mathcal{POR}$.

Base case. Each base function is provably represented. Let us consider just two examples:

- E is provably represented by the term $\lambda x.\epsilon$. Indeed for any string $s \in \mathbb{S}$, $\overline{\overline{E(s,\omega)}} = \overline{\overline{\boldsymbol{\epsilon}}} = \epsilon$ holds and $\vdash_{\mathcal{POR}^\lambda} (\lambda x.\epsilon)\overline{\overline{s}} = \epsilon$ is an instance of the (β)-axiom. So, we conclude:

$$\vdash_{\mathcal{POR}^\lambda} (\lambda x.\epsilon)\overline{\overline{s}} = \overline{\overline{E(s,\omega)}}.$$

- Q is provably represented by $\mathsf{Flipcoin}$, as observed in Example 1

Inductive case. Each function defined by composition or bounded recursion from provably represented functions, is provably represented by a term of $\mathcal{POR}^\lambda$ as well. Let us consider the case of bounded recursion. Let f be defined by bounded recursion, i.e.

$$\begin{aligned} f(s_1,\ldots,s_n,\boldsymbol{\epsilon},\omega) &= g(s_1,\ldots,s_n,\omega)\\ f(s_1,\ldots,s_n,s\mathbf{0},\omega) &= h_0\big(s_1,\ldots,s_n,s,f\big(s_1,\ldots,s_n,s,\omega\big),\omega\big)|_{k(s_1,\ldots,s_n,s)}\\ f(s_1,\ldots,s_n,s\mathbf{1},\omega) &= h_1\big(s_1,\ldots,s_n,s,f\big(s_1,\ldots,s_n,s,\omega\big),\omega\big)|_{k(s_1,\ldots,s_n,s)}. \end{aligned}$$

By IH, g, h_0, h_1, k are provably represented by the corresponding terms $\mathsf{G}, \mathsf{H}_0, \mathsf{H}_1, \mathsf{K}$ (resp.). So, for any $s_1,\ldots,s_{n+2}, s \in \mathbb{S}$ and $\omega \in \mathbb{O}$, we derive:

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{G}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}} = \overline{\overline{g(s_1,\ldots,s_n,\omega)}} \tag{G}$$

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{H}_0\overline{\overline{s_1}}\ldots\overline{\overline{s_{n+2}}} = \overline{\overline{h_0(s_1,\ldots,s_{n+2},\omega)}} \tag{H_0}$$

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{H}_1\overline{\overline{s_1}}\ldots\overline{\overline{s_{n+2}}} = \overline{\overline{h_1(s_1,\ldots,s_{n+2},\omega)}} \tag{H_1}$$

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{K}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}} = \overline{\overline{k(s_1,\ldots,s_n,\omega)}}. \tag{K}$$

Let $\omega \in \mathbb{O}$ and $s_1,\ldots,s_n,s \in \mathbb{S}$. We can prove by induction on s that

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{F}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s}} = \overline{\overline{f(s_1,\ldots,s_n,\omega)}},$$

where

$$\mathsf{F} := \lambda x_1\ldots\lambda x_n.\lambda x.\mathsf{Rec}(\mathsf{G}x_1\ldots x_n, \mathsf{H}_0x_1\ldots x_n, \mathsf{H}_1x_1,\ldots,x_n,\mathsf{K}x_1\ldots x_n, x).$$

- If $s = \boldsymbol{\epsilon}$, then $f(s_1,\ldots,s_n,s,\omega) = g(s_1,\ldots,s_n,\omega)$. Using the (β)-axiom, we deduce

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{F}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s}} = \mathsf{Rec}(\mathsf{G}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \mathsf{H}_0\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \mathsf{H}_1\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \mathsf{K}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \overline{\overline{s}})$$

  and, using the axiom $\mathsf{Rec}(\mathsf{G}x_1\ldots x_n, \mathsf{H}_0x_1\ldots x_n, \mathsf{H}_1x_1\ldots x_n, \mathsf{K}x_1\ldots x_n, \epsilon) = \mathsf{G}x_1\ldots x_n$, we also obtain:

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{F}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s}} = \mathsf{G}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}},$$

  by $(R2)$ and $(R3)$. We conclude using (G) together with $(R2)$.

- If $s = s_m\mathbf{0}$, then $f(s_1,\ldots,s_n,s,\omega) = h_0(s_1,\ldots,s_n,s_m,f\big(s_1,\ldots,s_n,s,\omega\big),\omega)|_{k(s_1,\ldots,s_n,s_m)}$. By IH, we can suppose that:

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{F}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s_m}} = \overline{\overline{f(s_1,\ldots,s_n,s',\omega)}}.$$

Then, using the (β)-axiom $\mathsf{F}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s}} = \mathsf{Rec}(\mathsf{G}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \mathsf{H}_0\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \mathsf{H}_1\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \mathsf{K}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}, \overline{\overline{s}})$, the axiom $\mathsf{Rec}(g, h_0, h_1, k, x\mathsf{0}) = \mathsf{Trunc}(h_0 x(\mathsf{Rec}(g, h_0, h_1, k, \mathsf{0})), kx)$ and IH we deduce,

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{F}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s}} = \mathsf{Trunc}(\mathsf{H}_0\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s_m}}\ \overline{\overline{f(s_1,\ldots,s_n,s_m,\omega)}}, \mathsf{K}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}})$$

by ($R2$) and ($R3$). Using (H_0) and (K), we finally conclude using ($R3$) and ($R2$):

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{F}\overline{\overline{s_1}}\ldots\overline{\overline{s_n}}\ \overline{\overline{s}} = \overline{\overline{h_0\big(s_1,\ldots,s_n,s_m,f(s_1,\ldots,s_n,s_m,\omega)\big)|_{k(s_1,\ldots,s_n,s_m)}}}.$$

- The case $s = s_m\mathbf{1}$ is proved in a similar way.

($\Leftarrow$) It is a consequence of the normalization property for the simply typed λ-calculus: a β-normal term $\mathsf{t} : \mathsf{s} \Rightarrow \ldots \Rightarrow \mathsf{s}$ cannot contain variables of higher types. By enumerating all possible normal forms one can check that these all represent functions in $\mathcal{POR}$. □

Corollary 1. *For any function $f : \mathbb{S}^j \times \mathbb{O} \to \mathbb{S}$, $f \in \mathcal{POR}$ if and only if f is provably represented by some term $\mathsf{t} : \mathsf{s} \Rightarrow \ldots \Rightarrow \mathsf{s} \in \mathcal{POR}^\lambda$.*

2.2 The Theory $I\mathcal{POR}^\lambda$

In this section we introduce a first-order intuitionistic theory, called $I\mathcal{POR}^\lambda$, which extends $\mathcal{POR}^\lambda$ with basic predicate calculus as well as a restricted induction principle. We also define IRS^1_3 as a variant of RS^1_3 having the intuitionistic predicate calculus as its logical basis, instead of the classical one. All theorems of both $\mathcal{POR}^\lambda$ and IRS^1_3 are provable in $I\mathcal{POR}^\lambda$. In particular, $I\mathcal{POR}^\lambda$ can be seen as an extension of $\mathcal{POR}^\lambda$. The theory $I\mathcal{POR}^\lambda$ provides the language to associate derivations in IRS^1_3 with polytime computable functions (corresponding to terms of $I\mathcal{POR}^\lambda$) and, thus, construct a polytime realizability interpretation of RS^1_3.

2.2.1 The Syntax of $I\mathcal{POR}^\lambda$

Indeed, the equational theory $\mathcal{POR}^\lambda$ is rather weak. In particular, one cannot prove even simple equations, as $x = \mathsf{Tail}(x)\mathsf{B}(x)$. The only viable way to prove it is by some form of induction:

$$\begin{aligned} &\vdash_{\mathcal{POR}^\lambda} \epsilon = \epsilon\epsilon = \mathsf{Tail}(\epsilon)\mathsf{B}(\epsilon) \\ x = \mathsf{Tail}(x)\mathsf{B}(x) &\vdash_{\mathcal{POR}^\lambda} x\mathsf{0} = \mathsf{Tail}(x\mathsf{0})\mathsf{B}(x\mathsf{0}) \\ x = \mathsf{Tail}(x)\mathsf{B}(x) &\vdash_{\mathcal{POR}^\lambda} x\mathsf{1} = \mathsf{Tail}(x\mathsf{1})\mathsf{B}(x\mathsf{1}). \end{aligned}$$

From this we would like to deduce, by induction, that $x = \mathsf{Tail}(x)\mathsf{B}(x)$. So, we introduce $I\mathcal{POR}^\lambda$, the language of which is basically an extension of that of $\mathcal{POR}^\lambda$ including (a translation for) any expression of RS^1_3. In particular, the grammar for terms of $I\mathcal{POR}^\lambda$ is precisely the same as that of Definition 11.

Definition 15 (Formulas of $I\mathcal{POR}^\lambda$)**.** *Formulas of $I\mathcal{POR}^\lambda$* are defined as follows:

- All equations $\mathsf{t} = \mathsf{u}$ of $\mathcal{POR}^\lambda$ are formulas of $I\mathcal{POR}^\lambda$
- For all (possibly open) terms $\mathsf{t}, \mathsf{u} : \mathsf{s}$ of $\mathcal{POR}^\lambda$, $\mathsf{t} \subseteq \mathsf{u}$ and $\mathtt{Flip}(\mathsf{t})$ are formulas of $I\mathcal{POR}^\lambda$
- Formulas of $I\mathcal{POR}^\lambda$ are closed under $\wedge, \vee, \to, \forall, \exists$, where $\mathsf{t} : \mathsf{s}$ is a possibly open term of $\mathcal{POR}^\lambda$.

Notation 6. We define $\bot := \mathsf{0} = \mathsf{1}$ and $\neg A := A \to \bot$.

The notion of Σ_0^b- and Σ_1^b-formula of $I\mathcal{POR}^\lambda$ is defined as for RS_3^1, Definition 4.

Remark 3. *Any formula of RS_3^1 can be seen as a formula of $I\mathcal{POR}^\lambda$, where each occurrence of the symbol $\mathtt{0}$ is replaced by $\mathsf{0}$, $\mathtt{1}$ by $\mathsf{1}$, $\frown$ by $\circ$ (usually abbreviated as xy), $\times$ by Times. In the following we will suppose that any formula of RS_3^1 is a formula of $I\mathcal{POR}^\lambda$.*

Definition 16 (The Theory $I\mathcal{POR}^\lambda$)**.** The axioms and inference rules of $I\mathcal{POR}^\lambda$ include the standard rules of the intuitionistic first-order predicate calculus, usual rules for the equality symbol, together with the axioms below:

1. All axioms of $\mathcal{POR}^\lambda$
2. $x \subseteq y \leftrightarrow \mathsf{Sub}(x, y) = \mathsf{1}$
3. $x = \epsilon \vee x = \mathsf{Tail}(x)\mathsf{0} \vee x = \mathsf{Tail}(x)\mathsf{1}$
4. $\mathsf{0} = \mathsf{1} \to x = \epsilon$
5. $\mathsf{Cond}(x, y, z, w) = w' \leftrightarrow (x = \epsilon \wedge w' = y) \vee (x = \mathsf{Tail}(x)\mathsf{0} \wedge w' = z) \vee (x = \mathsf{Tail}(x)\mathsf{1} \wedge w' = w)$
6. $\mathtt{Flip}(x) \leftrightarrow \mathsf{Flipcoin}(x) = \mathsf{1}$
7. Any formula of the form

$$\big(A(\epsilon) \wedge (\forall x)(A(x) \to A(x\mathsf{0})) \wedge (\forall x)(A(x) \to A(x\mathsf{1}))\big) \to (\forall y)A(y)$$

where A is of the form $(\exists z \preceq \mathsf{t})\mathsf{u} = \mathsf{v}$, with t containing only first-order open variables.

Notation 7. In what follows, we will refer to a formula of the form $(\exists z \preceq \mathsf{t})\mathsf{u} = \mathsf{v}$, with t containing only first-order open variables, as an **NP**-*predicate*.

Observe that, in $I\mathcal{POR}^\lambda$, for any $s \in \mathbb{S}$ and $\omega \in \mathbb{O}$, the theory T_ω is equivalent to the theory generated by all formulas $\mathtt{Flip}(\overline{s})$ if $\omega(s) = \mathbf{1}$, and $\neg\mathtt{Flip}(\overline{s})$ if $\omega(s) = \mathbf{0}$.

2.2.2 Relating $I\mathcal{POR}^\lambda$ with $\mathcal{POR}^\lambda$ and IRS_3^1

Now that we've formally introduced $I\mathcal{POR}^\lambda$, it can be shown that all theorems of both $\mathcal{POR}^\lambda$ and the intuitionistic version of RS_3^1 are derivable in it. In particular, Proposition 3 can be easily established by inspecting all rules of $\mathcal{POR}^\lambda$.

Proposition 3. *Any theorem of $\mathcal{POR}^\lambda$ is a theorem of $I\mathcal{POR}^\lambda$.*

Let us now consider the intuitionistic version of RS_3^1, i.e. IRS_3^1. In order to prove that every theorem of IRS_3^1 is derivable in $I\mathcal{POR}^\lambda$ we first need to establish a few useful properties concerning $I\mathcal{POR}^\lambda$.[4] Notice, in particular, that the recursion schema of $I\mathcal{POR}^\lambda$ differs from that of IRS_3^1 as dealing with formulas of the form $(\exists y \preceq \mathsf{t})\mathsf{u} = \mathsf{v}$ and not with all the Σ_1^b-ones. The two schemas can be related due to Proposition 14 below, proved by induction on the structure of formulas.

Proposition 4. *For any Σ_0^b-formula $A(x_1, \ldots, x_n)$ of $\mathcal{L}$, there exists a term $\mathsf{t}_A(x_1, \ldots, x_n)$ of $\mathcal{POR}^\lambda$ such that:*

[4] Full details can be found in Appendix 3.1.4.

1. $\vdash_{I\mathcal{POR}^\lambda} A \leftrightarrow \mathsf{t}_A = \mathsf{0}$
2. $\vdash_{I\mathcal{POR}^\lambda} \mathsf{t}_A = \mathsf{0} \vee \mathsf{t}_A = \mathsf{1}$.

This leads us to the following immediate corollary and allows us to prove Theorem 2 relating $I\mathcal{POR}^\lambda$ and IRS^1_3.

Corollary 2. *i For any Σ^b_0-formula A, $\vdash_{I\mathcal{POR}} A \vee \neg A$.*

ii For any closed Σ^b_0-formula A in $\mathcal{L}$ and $\omega \in \mathbb{O}$, either $T_\omega \vdash_{I\mathcal{POR}^\lambda} A$ or $T_\omega \vdash_{I\mathcal{POR}^\lambda} \neg A$.

Theorem 2. *Any theorem of IRS^1_3 is a theorem of $I\mathcal{POR}^\lambda$.*

Proof. First, observe that, as a consequence of Proposition 14, for any Σ^b_1-formula $A = (\exists x_1 \preceq t_1) \dots (\exists x_n \preceq t_n)B$ of IRS^1_3,

$$\vdash_{I\mathcal{POR}^\lambda} A \leftrightarrow (\exists x_1 \preceq \mathsf{t}_1) \dots (\exists x_n \preceq \mathsf{t}_n)\mathsf{t}_B = \mathsf{0}.$$

Therefore, any instance of the Σ^b_1-recursion schema of IRS^1_3 is derivable in $I\mathcal{POR}^\lambda$ from the **NP**-induction schema. Then, to prove that $I\mathcal{POR}^\lambda$ extends IRS^1_3 it suffices to check that all basic axioms of IRS^1_3 are provable in $I\mathcal{POR}^\lambda$. □

Due to Corollary 11, we can even establish the following result.

Lemma 2. *Let A be a closed Σ^b_0-formula of IRS^1_3 and $\omega \in \mathbb{O}$, then:*

$$T_\omega \vdash_{I\mathcal{POR}^\lambda} A \quad \Leftrightarrow \quad \omega \in [\![A]\!].$$

Proof. ($\Rightarrow$) This soundness result is easily established by induction on the rules of $I\mathcal{POR}^\lambda$.

($\Leftarrow$) From Corollary 11, we know that either $T_\omega \vdash_{I\mathcal{POR}^\lambda} A$ or $T_\omega \vdash_{I\mathcal{POR}^\lambda} \neg A$. Hence, if $\omega \in [\![A]\!]$, then it cannot be $T_\omega \vdash_{I\mathcal{POR}^\lambda} \neg A$ (by soundness). So, we conclude $T_\omega \vdash_{I\mathcal{POR}^\lambda} A$. □

2.3 Realizability

In this section, we introduce realizability as internal to $I\mathcal{POR}^\lambda$. As a corollary, we obtain that from any derivation in IRS^1_3 (actually, in $I\mathcal{POR}^\lambda$) of a formula in the form $(\forall x)(\exists y)A(x,y)$, one can extract a functional term $\mathsf{f} : \mathsf{s} \Rightarrow \mathsf{s}$ of $\mathcal{POR}^\lambda$, such that $\vdash_{I\mathcal{POR}^\lambda} (\forall x)A(x, \mathsf{f}x)$. This allows us to conclude that if a function f is Σ^b_1-representable in IRS^1_3, then $f \in \mathcal{POR}$.

Notation 8. Let $\mathbf{x}, \mathbf{y}$ denote finite sequences of term variables (resp.) $x_1, \dots, x_n$ and $y_1, \dots, y_k$ and $\mathbf{x}(\mathbf{y})$ be an abbreviation for $y_1(\mathbf{x}), \dots, y_k(\mathbf{x})$. Let Λ be a shorthand for the empty sequence and $y(\Lambda) := y$.

Definition 17. Formulas $x \circledR A$ are defined by induction on the structure of A of I$\mathcal{POR}^\lambda$:

$$\begin{aligned}
\Lambda \circledR A &:= A \qquad (A \text{ atomic}) \\
\mathbf{x}, \mathbf{y} \circledR (B \wedge C) &:= (\mathbf{x} \circledR B) \wedge (\mathbf{y} \circledR C) \\
z, \mathbf{x}, \mathbf{y} \circledR (B \vee C) &:= (z = \mathsf{0} \wedge \mathbf{x} \circledR B) \vee (z \neq \mathsf{0} \wedge \mathbf{y} \circledR C) \\
\mathbf{y} \circledR (B \to C) &:= (\forall \mathbf{x})\big((\mathbf{x} \circledR B) \to \mathbf{y}(\mathbf{x}) \circledR C\big) \wedge (B \to C) \\
z, \mathbf{x} \circledR (\exists y)B &:= \mathbf{x} \circledR B\{z/y\} \\
\mathbf{x} \circledR (\forall y)B &:= (\forall y)\big(\mathbf{x}(y) \circledR B\big),
\end{aligned}$$

where no variable in $\mathbf{x}$ occurs free in A. Given terms $\mathbf{t} = \mathtt{t}_1, \ldots, \mathtt{t}_n$, we let:

$$\mathbf{t} \circledR A := (\mathbf{x} \circledR A)\{\mathbf{t}/\mathbf{x}\}.$$

We can also link the derivability of such new formulas with that of formulas of $I\mathcal{POR}$. Both results are established by induction (resp., on the structure of formulas of $I\mathcal{POR}$ and on the height of derivations).

Theorem 3 (Soundness)**.** *If* $\vdash_{I\mathcal{POR}^\lambda} \mathbf{t} \circledR A$*, then* $\vdash_{I\mathcal{POR}^\lambda} A$*.*

Notation 9. Given $\Gamma = A_1, \ldots, A_n$, let $\mathbf{x} \circledR \Gamma$ be a shorthand for $\mathbf{x}_1 \circledR A_1, \ldots, \mathbf{x}_n \circledR A_n$.

Theorem 4 (Completeness)**.** *If* $\vdash_{I\mathcal{POR}^\lambda} A$*, then there exist terms* $\mathbf{t}$*, such that* $\vdash_{I\mathcal{POR}^\lambda} \mathbf{t} \circledR A$*.*

Proof Sketch. We prove that if $\Gamma \vdash_{I\mathcal{POR}^\lambda} A$, then there exist terms $\mathbf{t}$ such that $\mathbf{x} \circledR \Gamma \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}\mathbf{x}_1 \ldots \mathbf{x}_n \circledR A$. The proof is by induction on the derivation of $\Gamma \vdash_{I\mathcal{POR}^\lambda} A$. Let us here consider just the case of rule $\vee R_1$ as an example:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash A\end{array}}{\Gamma \vdash B \vee C} \vee R_1$$

By IH, there exist terms $\mathbf{u}$, such that $\mathbf{x} \circledR \Gamma \vdash_{I\mathcal{POR}^\lambda} \mathbf{u}\mathbf{x} \circledR B$. Since $x, y \circledR B \vee C$ is defined as $(x = \mathsf{0} \wedge y \circledR B) \vee (x \neq \mathsf{0} \wedge y \circledR C)$, we can take $\mathbf{t} = \mathsf{0}, \mathbf{u}$. □

Corollary 3. *Let* $(\forall x)(\exists y)A(x, y)$ *be a closed theorem of* $I\mathcal{POR}^\lambda$*, where* A *is a* Σ^b_1*-formula. Then, there exists a closed term* $\mathtt{t} : \mathtt{s} \Rightarrow \mathtt{s}$ *of* $\mathcal{POR}^\lambda$ *such that:*

$$\vdash_{I\mathcal{POR}^\lambda} (\forall x)A(x, \mathtt{t}x).$$

Proof. By Theorem 6, there exist $\mathbf{w} = \mathtt{t}, w$ such that $\vdash_{I\mathcal{POR}^\lambda} \mathbf{w} \circledR (\forall x)(\exists y)A(x, y)$, that is, by Definition 17,

$$\begin{aligned}\mathbf{w} \circledR (\forall x)(\exists y)A(x, y) &\equiv (\forall x)(\mathbf{w}(x) \circledR (\exists y)A(x, y))\\ &\equiv (\forall x)(w(x) \circledR A(x, \mathtt{t}x)).\end{aligned}$$

From this we deduce, by Theorem 3, that

$$\vdash_{I\mathcal{POR}^\lambda} (\forall x)A(x, \mathtt{t}x).$$

□

Now, we have all the ingredients to prove that if a function is Σ^b_1-representable in IRS^1_3, in the sense of Definition 4, then it is in $\mathcal{POR}$.

Corollary 4. *For any function* $f : \mathbb{O} \times \mathbb{S} \to \mathbb{S}$*, if there is a closed formula* Σ^b_1*-formula* $A(x, y)$ *in* $\mathcal{L}$ *such that*

1. $IRS^1_3 \vdash (\forall x)(\exists! y)A(x, y)$
2. $[\![A(\overline{\overline{s_1}}, \overline{\overline{s_2}})]\!] = \{\omega \mid f(\omega, s_1) = s_2\}$,

then $f \in \mathcal{POR}$.

Proof. Since $\vdash_{IRS^1_3} (\forall x)(\exists! y)A(x,y)$, by Theorem 2, also $\vdash_{I\mathcal{POR}} (\forall x)(\exists! y)A(x,y)$. From $\vdash_{I\mathcal{POR}^\lambda} (\forall x)(\exists y)A(x,y)$ we deduce $\vdash_{I\mathcal{POR}^\lambda} (\forall x)A(x,\mathsf{g}x)$ for some closed term $\mathsf{g} : \mathsf{s} \Rightarrow \mathsf{s}$ of $\mathcal{POR}^\lambda$, by Corollary 3 and, by Theorem 5, there is a function $g \in \mathcal{POR}$ such that for any $\omega \in \mathbb{O}, s_1, s_2 \in \mathbb{S}$,

$$T_\omega \vdash_{I\mathcal{POR}^\lambda} A(\overline{\overline{s_1}}, \overline{\overline{s_2}}) \;\Leftrightarrow\; g(s_1,\omega) = s_2.$$

From this we conclude:

$$\begin{aligned} g(s_1,\omega) = s_2 &\;\Leftrightarrow\; T_\omega \vdash_{I\mathcal{POR}^\lambda} A(\overline{\overline{s_1}}, \overline{\overline{s_2}}) \\ &\overset{L}{\Leftrightarrow}{}^2\; \omega \in [\![A(\overline{\overline{s_1}}, \overline{\overline{s_2}})]\!] \\ &\;\Leftrightarrow\; f(s_1,\omega) = s_2. \end{aligned}$$

So, $f = g$ and, thus, $f \in \mathcal{POR}$. □

2.4 ∀NP-Conservativity of $I\mathcal{POR}^\lambda + (EM)$ over $I\mathcal{POR}^\lambda$

Corollary 4 is already very close to the result we are looking for. The remaining step to conclude our proof is its extension from intuitionistic IRS^1_3 to classical RS^1_3, showing that any function which is Σ^b_1-representable in RS^1_3 is also in $\mathcal{POR}$. The proof is obtained by adapting the method from [**CoquardHofmann**]. In doing so, we start by considering an extension of $I\mathcal{POR}^\lambda$ via the excluded middle (EM). Then, we show that the realizability interpretation extends to such $I\mathcal{POR}^\lambda$ + (EM), so that for any of its closed theorems $(\forall x)(\exists y \preceq \mathsf{t})A(x,y)$, with A in Σ^b_1, there is a closed term $\mathsf{t} : \mathsf{s} \Rightarrow \mathsf{s}$ of $\mathcal{POR}^\lambda$ such that $\vdash_{I\mathcal{POR}} (\forall x)A(x,\mathsf{t}x)$.

From $I\mathcal{POR}^\lambda$ to $I\mathcal{POR}^\lambda$ + (Markov). Let (EM) be the Excluded-Middle schema, $A \vee \neg A$ and *Markov's principle* be defined as follows.

$$\neg\neg(\exists x)A \to (\exists x)A, \tag{Markov}$$

where A is a Σ^b_1-formula.

Proposition 5. *For any Σ^b_1-formula A, if $\vdash_{I\mathcal{POR}^\lambda + (EM)} A$, then $\vdash_{I\mathcal{POR}^\lambda + (Markov)} A$.*

Proof Sketch. The claim is proved by applying the double negation translation, with the following two remarks. First, for any Σ^b_0-formula A, $\vdash_{I\mathcal{POR}^\lambda} \neg\neg A \to A$. Using (Markov), the double negation of an instance of the **NP**-induction can be shown equivalent to an instance of the **NP**-induction schema. □

So, we basically need to show that the realizability interpretation defined in Section 2.3 extends to $I\mathcal{POR}^\lambda$ + (Markov), that is for any of its closed theorems $(\forall x)(\exists y \preceq \mathsf{t})A(x,y)$, with A in Σ^b_1, there is a closed term $\mathsf{t} : \mathsf{s} \Rightarrow \mathsf{s}$ of $\mathcal{POR}^\lambda$ such that $\vdash_{I\mathcal{POR}^\lambda} (\forall x)A(x,\mathsf{t}x)$.

From $I\mathcal{POR}^\lambda$ to $(I\mathcal{POR}^\lambda)^*$ Let us assume given a subjective encoding $\sharp : (\mathsf{s} \Rightarrow \mathsf{s}) \Rightarrow \mathsf{s}$ in $I\mathcal{POR}^\lambda$ of first-order unary functions as strings, together with a "decoding" function $\mathsf{app} : \mathsf{s} \Rightarrow \mathsf{s} \Rightarrow \mathsf{s}$ satisfying:

$$\vdash_{I\mathcal{POR}^\lambda} \mathsf{app}(\sharp \mathsf{f}, x) = \mathsf{f}x.$$

Moreover, let

$$x * y := \sharp\big(\lambda z.\mathsf{BAnd}\big(\mathsf{app}(x,z), \mathsf{app}(y,z)\big)\big)$$

and

$$T(x) := (\exists y)\big(\mathsf{B}(\mathsf{app}(x,y)) = \mathsf{0}\big).$$

There is a *meet semi-lattice* structure on the set of terms of type s defined by $\mathsf{t} \sqsubseteq \mathsf{u}$ iff $\vdash_{I\mathcal{POR}^\lambda} T(\mathsf{u}) \to T(\mathsf{t})$ with top element $\mathbb{1} = \sharp(\lambda x.1)$ and meet given by $x * y$. Indeed, from $T(x * 1) \leftrightarrow T(x)$, $x \sqsubseteq \mathbb{1}$ follows. Moreover, from $\mathsf{B}(\mathsf{app}(x,\mathsf{u})) = \mathsf{0}$, we obtain $\mathsf{B}\big(\mathsf{app}(x * y, \mathsf{u})\big) = \mathsf{BAnd}\big(\mathsf{app}(x,\mathsf{u}), \mathsf{app}(y,\mathsf{u})\big) = \mathsf{0}$, whence $T(x) \to T(x * y)$, i.e. $x * y \sqsubseteq x$. One can similarly prove $x * y \sqsubseteq y$. Finally, from $T(x) \to T(v)$ and $T(y) \to T(v)$, we deduce $T(x * y) \to T(v)$, by observing that $\vdash_{I\mathcal{POR}^\lambda} T(x*y) \to T(y)$. Notice that the formula $T(x)$ is *not* in Σ^b_1, as its existential quantifier is not bounded.

Definition 18. For any formula A of $I\mathcal{POR}^\lambda$ and fresh variable x, we define formulas $x \Vdash A$ inductively, as follows:

$$\begin{aligned}
x \Vdash A &:= A \vee T(x) \qquad (A \textit{ atomic})\\
x \Vdash B \wedge C &:= x \Vdash B \wedge x \Vdash C\\
x \Vdash B \vee C &:= x \Vdash B \vee x \Vdash C\\
x \Vdash B \to C &:= (\forall y)\big(y \Vdash B \to x * y \Vdash C\big)\\
x \Vdash (\exists y)B &:= (\exists y)x \Vdash B\\
x \Vdash (\forall y)B &:= (\forall y)x \Vdash B.
\end{aligned}$$

The following Lemma 3 is then established by induction on the structure of formulas in $I\mathcal{POR}^\lambda$.

Lemma 3. *If A is provable in $I\mathcal{POR}^\lambda$ without using* **NP***-induction, then $x \Vdash A$ is provable in $I\mathcal{POR}^\lambda$.*

Lemma 4. *Let $A = (\exists x \preceq \mathsf{t})B$, where B is a Σ^b_0-formula. Then, there exists a term $\mathsf{u} : \mathsf{s}$, with $FV(\mathsf{u}_A) = FV(B)$, such that:*

$$\vdash_{I\mathcal{POR}^\lambda} A \leftrightarrow T(\mathsf{u}_A).$$

Proof. Since $B(x)$ is a Σ^b_0-formula, for all terms $\mathsf{v} : \mathsf{s}$, $\vdash_{I\mathcal{POR}^\lambda} B(x) \leftrightarrow \mathsf{t}_{x \preceq \mathsf{t} \wedge B}(x) = \mathsf{0}$ (where $t_{x \preceq t \wedge B}$ has the free variables of t and B). Let $C(x)$ be a Σ^b_0-formula, one can show by induction on its structure that for all term $\mathsf{v} : \mathsf{s}$, $\mathsf{t}_{C(\mathsf{v})} = \mathsf{t}_C(\mathsf{v})$. Then,

$$\vdash_{I\mathcal{POR}^\lambda} A \leftrightarrow (\exists x)\mathsf{t}_{x \preceq \mathsf{t} \wedge B}(x) = \mathsf{0} \leftrightarrow (\exists x)T(\sharp(\lambda x.\mathsf{t}_{x \preceq \mathsf{t} \wedge B}(x))).$$

So, we let $\mathsf{u}_A = \sharp(\lambda x.t_{x \preceq \mathsf{t} \wedge B}(x))$. □

From which we obtain the following three properties:

i. $\vdash_{I\mathcal{POR}^\lambda} (x \Vdash A) \leftrightarrow (A \vee T(x))$

ii. $\vdash_{I\mathcal{POR}^\lambda} (x \Vdash \neg A) \leftrightarrow (A \to T(x))$

iii. $\vdash_{I\mathcal{POR}^\lambda} (x \Vdash \neg\neg A) \leftrightarrow (A \vee T(x))$.

where A is a Σ^b_1-formula.

Corollary 5 (Markov's Principle)**.** *If A is a Σ^b_1-formula, then*

$$\vdash_{I\mathcal{POR}^\lambda} x \Vdash \neg\neg A \to A.$$

Then, to define the extension $(I\mathcal{POR}^\lambda)^*$ of $I\mathcal{POR}$, we start by formally introducing PIND.

Definition 19 (PIND)**.** Let PIND(A) indicate the formula:

$$\big(A(\epsilon) \wedge \big((\forall x)(A(x) \to A(x\mathsf{0})) \wedge (\forall x)(A(x) \to A(x\mathsf{1}))\big) \to (\forall x)A(x).$$

Observe that if $A(x)$ is a formula of the form $(\exists y \preceq \mathsf{t})\mathsf{u} = \mathsf{v}$, then the formula $z \Vdash \mathrm{PIND}(A)$ is of the form $\mathrm{PIND}\big(A(x) \vee T(z)\big)$, which is *not* an instance of the **NP**-induction schema (as the formula $T(z) = (\exists x)\mathsf{B}\big(\mathsf{app}(z, x)\big) = \mathsf{0}$ is not bounded).

Definition 20 (The Theory $(I\mathcal{POR}^\lambda)^*$)**.** Let $(I\mathcal{POR}^\lambda)^*$ indicate the theory extending $I\mathcal{POR}^\lambda$ with all instances of the induction schema $\mathrm{PIND}\big(A(x) \vee B\big)$, where $A(x)$ is of the form $(\exists y \preceq \mathsf{t})\mathsf{u} = \mathsf{v}$, and B is an arbitrary formula with $x \notin FV(B)$.

From the discussion above we deduce:

Proposition 6. *For any Σ^b_1-formula A, if $\vdash_{I\mathcal{POR}^\lambda} A$, then $\vdash_{(I\mathcal{POR}^\lambda)^*} x \Vdash A$.*

Then, we can extend the realizability interpretation of Section 2.3 to $(I\mathcal{POR}^\lambda)$ by simply constructing a realizer for $\mathrm{PIND}(A(x) \vee B)$.

Lemma 5. *Let $A(x) = (\exists y \preceq \mathsf{t})\mathsf{u} = \mathsf{0}$ and B be any formula not containing free occurrences of x. Then, there exist terms t such that:*

$$\vdash_{I\mathcal{POR}} \mathsf{t} \circledR \mathrm{PIND}(A(x) \vee B).$$

So, by Theorem 3, we obtain immediately, for any Σ^b_1-formula A and formula B, with$x \notin FV(A)$,

$$\vdash_{I\mathcal{POR}^\lambda} \mathit{PIND}(A(x) \vee B).$$

Corollary 6 ($\forall$**NP**-Conservativity of $I\mathcal{POR}^\lambda$ + (EM) over $I\mathcal{POR}^\lambda$)**.** *Let A be a Σ^b_1-formula, if $\vdash_{I\mathcal{POR}^\lambda + (EM)} (\forall x)(\exists y \preceq \mathsf{t})A(x, y)$ then $\vdash_{I\mathcal{POR}^\lambda} (\forall x)(\exists y \preceq \mathsf{t})A(x, y)$.*

Concluding the Proof. We can finally conclude proving Proposition 7.

Proposition 7. *Let $(\forall x)(\exists y \preceq \mathsf{t})A(x, y)$ be a closed theorem of $I\mathcal{POR}^\lambda$ + (Markov), where A is a Σ^b_1-formula. Then, there exists a closed term $\mathsf{t} : \mathsf{s} \Rightarrow \mathsf{s}$ of $\mathcal{POR}^\lambda$ such that:*

$$\vdash_{I\mathcal{POR}^\lambda} (\forall x)A(x, \mathsf{t}x).$$

Proof. If $I\mathcal{POR}^\lambda$ + (Markov) proves $(\forall x)(\exists y)A(x,y)$, then by Parikh's Proposition 2, it also proves $(\exists y \preceq \mathsf{t})A(x,y)$ and $(I\mathcal{POR}^\lambda)^*$ proves $z \Vdash (\exists y \preceq \mathsf{t})A(x,y)$. Let $B := (\exists y \preceq \mathsf{t})A(x,y)$. By taking $\mathsf{z} = \mathsf{u}_C$, using Lemma 4, we deduce $\vdash_{(I\mathcal{POR}^\lambda)^*} B$ and thus, by Lemma 3 and Lemma 29, we deduce that there exist $\mathbf{t}, \mathbf{u}$ such that $\vdash_{I\mathcal{POR}^\lambda} \mathbf{t}, \mathbf{u}$ ® B, which implies $\vdash_{I\mathcal{POR}^\lambda} A(x, \mathbf{t}x)$. Thus, $\vdash_{I\mathcal{POR}^\lambda} (\forall x)(A(x), \mathbf{t}x)$. □

So, by Proposition 5, if $\vdash_{I\mathcal{POR}^\lambda + (EM)} (\forall x)(\exists y \preceq \mathsf{t})A(x,y)$, where A is a closed Σ^b_1-formula, then there is a closed term $\mathsf{t} : \mathsf{s} \Rightarrow \mathsf{s}$ of $\mathcal{POR}^\lambda$ such that $\vdash_{I\mathcal{POR}} (\forall x)A(x, \mathsf{t}x)$, From this we conclude our proof arguing as for Corollary 4.

Corollary 7. *Let $RS^1_3 \vdash (\forall x)(\exists y \preceq t)A(x,y)$, where A is a Σ^b_1-formula with only x, y free. For any function $f : \mathbb{S} \times \mathbb{O} \to \mathbb{S}$, if $(\forall x)(\exists y \preceq t)A(x,y)$ represents f so that:*

1. *$RS^1_3 \vdash (\forall x)(\exists! y)A(x,y)$*
2. *$[\![A(\overline{\overline{s_1}}, \overline{\overline{s_2}})]\!] = \{\omega \mid f(s_1, \omega) = s_2\}$,*

then $f \in \mathcal{POR}$.

3 On the Equivalence between $\mathcal{POR}$ and SFP

Appendix

3.1 Section 1

3.1.1 A Sequent Calculus for RS_3^1

Theorem 1.1 can be proved using any appropriate deductive system. Following [9], we extend standard **G3c** (with equality) with some specific rules.

G3cS$_3^1$

Logical Axioms

$$\frac{}{A, \Gamma \vdash \Delta, A}\, Ax \qquad \frac{}{\bot, \Gamma \vdash \Delta}\, \bot L$$

Logical Rules

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \wedge R$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \vee L \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \vee R$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta} \to L \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B} \to R$$

$$\frac{A(t/x), (\forall x)A, \Gamma \vdash \Delta}{(\forall x)A, \Gamma \vdash \Delta} \forall L \qquad \frac{\Gamma \vdash \Delta, A(y/x)}{\Gamma \vdash \Delta, (\forall x)A} \forall R$$

where y is an *eigenvariable*.

$$\frac{A(y/x), \Gamma \vdash \Delta}{(\exists x)A, \Gamma \vdash \Delta} \exists L \qquad \frac{\Gamma \vdash \Delta, (\exists x)A, A(t/x)}{\Gamma \vdash \Delta, (\exists x)A} \exists R$$

where y is an *eigenvariable*.

Equality Rules

$$\frac{x = x, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\, Ref \qquad \frac{P(z/x), y = z, P(y/x), \Gamma \vdash \Delta}{y = z, P(y/x)\Gamma \vdash \Delta}\, Repl$$

Word Rules

$$\frac{x\epsilon = x, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\, F1 \qquad \frac{x(y\mathfrak{b}) = (xy)\mathfrak{b}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\, F2$$

$$\frac{x\epsilon = x, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\, F4 \qquad \frac{x \times y\mathfrak{b} = (x \times y)x, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\, F5$$

$$\frac{x \subseteq \epsilon, x = \epsilon, \Gamma \vdash \Delta}{x \subseteq \epsilon, \Gamma \vdash \Delta}\, F7a \qquad \frac{x = \epsilon, x \subseteq \epsilon, \Gamma \vdash \Delta}{x = \epsilon, \Gamma \vdash \Delta}\, F7b$$

$$\frac{x \subseteq y, x \subseteq y\mathfrak{b}, \Gamma \vdash \Delta \qquad x = y\mathfrak{b}, x \subseteq y\mathfrak{b}, \Delta \vdash \Gamma}{x \subseteq y\mathfrak{b}, \Gamma \vdash \Delta}\, F8a$$

$$\frac{x \subseteq y\mathtt{b}, x \subseteq y, \Gamma \vdash \Delta}{x \subseteq y, \Gamma \vdash \Delta}\, F8b \qquad \frac{x \subseteq y\mathtt{b}, x = y\mathtt{b}, \Gamma \vdash \Delta}{x = y\mathtt{b}, \Gamma \vdash \Delta}\, F8c$$

$$\frac{x = y, x\mathtt{b} = y\mathtt{b}, \Gamma \vdash \Delta}{x\mathtt{b} = y\mathtt{b}, \Gamma \vdash \Delta}\, F10$$

$$\frac{}{x\mathtt{0} = y\mathtt{1}, \Gamma \vdash \Delta}\, F12 \qquad \frac{}{x\mathtt{b} = \epsilon, \Gamma \vdash \Delta}\, F13$$

Induction Rule

$$\frac{\Gamma \vdash \Delta, B(\epsilon) \qquad B(y), \Gamma \vdash \Delta, B(y\mathtt{b})}{\Gamma \vdash \Delta, B(x)}\, \Sigma^b_1\text{-}NIA$$

where $B(x)$ is a Σ^b_1-formula.

Notation 10. Let us denote as $F10^*$ the following instance of *Repl*:

$$\frac{x\mathtt{b} = y\mathtt{b}, x = y, \Gamma \vdash \Delta}{x = y, \Gamma \vdash \Delta}\, F10^*$$

Indeed,

$$\frac{\dfrac{x\mathtt{b} = y\mathtt{b}, x\mathtt{b} = x\mathtt{b}, x = y, \Gamma \vdash \Delta}{x\mathtt{b} = x\mathtt{b}, x = y, \Gamma \vdash \Delta}\, Repl}{x = y, \Gamma \vdash \Delta}\, Ref$$

3.1.2 The Proof of Theorem 1.1

Preliminaries. For simplicity's sake, we will introduce some abbreviation and notational conventions.

Notation 11. Let $t \subset s$ be an abbreviation for $t \subseteq s \wedge t \neq s$. Let $t \equiv s$ be an abbreviation for $\mathtt{1}^t = \mathtt{1}^s$.

Moreover, le tus use $(\forall x \subseteq y)A$ and $(\exists x \subseteq y)A$ as shorthands of (resp.) $(\forall x)(x \subseteq y \to A)$ and $(\exists x)(x \subseteq y \to A)$.

For readability, we will also use the following compact rules:

$$\frac{\Gamma, z \preceq y \vdash \Delta, A(z/x)}{\Gamma \vdash \Delta, (\forall x \preceq y)A}\, \forall_{\preceq} R \qquad \frac{\Gamma \vdash \Delta, (\exists x \preceq y)A, t \preceq y \qquad \Gamma \vdash \Delta, (\exists x \preceq y)A, A(t/x)}{\Gamma \vdash \Delta, (\exists x \preceq y)A}\, \exists_{\preceq} R$$

where z is an *eigenvariable*

$$\frac{(\forall x \preceq y)A, \Gamma \vdash \Delta, x \preceq t \qquad A(t/x), (\forall x \preceq y)A, \Gamma \vdash \Delta}{(\forall x \preceq y)A, \Gamma \vdash \Delta}\, \forall_{\preceq} L \qquad \frac{z \preceq y, A(z/x), \Gamma \vdash \Delta}{(\exists x \preceq y)A, \Gamma \vdash \Delta}\, \exists_{\preceq} L$$

where z is an *eigenvariable*.

$$\frac{z \subseteq y, \Gamma \vdash \Delta, A(z/x)}{\Gamma \vdash \Delta, (\forall x \subseteq y)A}\, \forall_{\subseteq} R \qquad \frac{\Gamma \vdash \Delta, (\exists x \subseteq y)A, t \subseteq y \qquad \Gamma \vdash \Delta, (\exists x \subseteq y)A, A(t/x)}{\Gamma \vdash \Delta, (\exists x \subseteq y)A}\, \exists_{\subseteq} R$$

where z is an *eigenvariable*.

$$\frac{(\forall x \subseteq y)A, \Gamma \vdash \Delta, x \subseteq t \qquad A(t/x), (\forall x \subseteq y)A, \Gamma \vdash \Delta}{(\forall x \subseteq y)A, \Gamma \vdash \Delta} \forall_{\subseteq} L \qquad \frac{z \subseteq y, A(z/x), \Gamma \vdash \Delta}{(\exists x \subseteq y)A, \Gamma \vdash \Delta} \exists_{\subseteq} L$$

where z is an *eigenvariable*.

Notice that if $y = \epsilon$, we have:[5]

$$\frac{\epsilon \preceq \epsilon, \Gamma \vdash \Delta, A(\epsilon/x)}{\Gamma \vdash \Delta, (\forall x \preceq \epsilon)A} \forall_{\preceq} R^* \qquad \frac{\epsilon \preceq \epsilon, A(\epsilon/x), \Gamma \vdash \Delta}{(\exists x \preceq \epsilon)A, \Gamma \vdash \Delta} \exists_{\preceq} L^*$$

$$\frac{\epsilon \subseteq \epsilon, \Gamma \vdash \Delta, A(\epsilon/x)}{\Gamma \vdash \Delta, (\forall x \subseteq \epsilon)A} \forall_{\subseteq} R^* \qquad \frac{\epsilon \subseteq \epsilon, A(\epsilon/x), \Gamma \vdash \Delta}{(\exists x \subseteq \epsilon)A, \Gamma \vdash \Delta} \exists_{\subseteq} L^*$$

Proving Theorem 1.1.

Theorem 1.1. *Every $f \in \mathcal{POR}$ is Σ^b_1-representable in RS^1_3.*

Proof. The proof is by induction on the structure of $f \in \mathcal{POR}$.[6]

Base case. If f is a basic function, then there are five possible sub-cases:

i. $E(x, \omega) = \boldsymbol{\epsilon}$. The function is Σ^b_1-represented in RS^1_3 by the formula:

$$x = x \wedge y = \epsilon.$$

1. (Existence.)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z = z \vdash (\exists y)(z = z \wedge y = \epsilon), z = z}\, Ax}{\vdash (\exists y)(z = z \wedge y = \epsilon), z = z}\, Ref \qquad \dfrac{\dfrac{}{\epsilon = \epsilon \vdash (\exists y)(z = z \wedge y = \epsilon), \epsilon = \epsilon}\, Ax}{\vdash (\exists y)(z = z \wedge y = \epsilon), \epsilon = \epsilon}\, Ref}{\vdash (\exists y)(z = z \wedge y = \epsilon), z = z \wedge \epsilon = \epsilon}\, \wedge R}{\vdash (\exists y)(z = z \wedge y = \epsilon)}\, \exists R}{\vdash (\forall x)(\exists y)(x = x \wedge y = \epsilon)}\, \forall R$$

2. (Uniqueness.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x_2 = x_3, x_1 = x_1, x_2 = \epsilon, x_3 = \epsilon \vdash x_2 = x_3}\, Ax}{x_1 = x_1, x_2 = \epsilon, x_1 = x_1, x_3 = \epsilon \vdash x_2 = x_3}\, Repl}{x_1 = x_1 \wedge x_2 = \epsilon, x_1 = x_1 \wedge x_2 = \epsilon \vdash x_2 = x_3}\, \wedge Ls}{(x_1 = x_1 \wedge x_2 = \epsilon) \wedge (x_1 = x_1 \wedge x_2 = \epsilon) \vdash x_2 = x_3}\, \wedge L}{\vdash (x_1 = x_1 \wedge x_2 = \epsilon) \wedge (x_1 = x_1 \wedge x_2 = \epsilon) \rightarrow x_2 = x_3}\, \rightarrow R}{\vdash (\forall x)(\forall y)(\forall z)((x = x \wedge y = \epsilon) \wedge (x = x \wedge z = \epsilon) \rightarrow y = z)}\, \forall Rs$$

3. For every $n, m \in \mathbb{S}$ and $\omega^* \in \mathbb{O}$, $E(n, \omega^*) = m$ if and only if $\omega \in [\![\overline{n} = \overline{n} \wedge \overline{m} = \epsilon]\!]$.

[5] Derivability is proved in Proposition 8.

[6] For readability's sake, in what follows, we will avoid to use the s-word notation whenever there is no ambiguity between terms in $\mathcal{L}$ and strings, i.e. in the syntactical conditions 1. and 2.

($\Rightarrow$.) Assume $m = \boldsymbol{\epsilon}$. So,

$$\begin{aligned}\llbracket \overline{n} = \overline{n} \wedge \overline{m} = \epsilon \rrbracket &= \llbracket \overline{n} = \overline{n} \wedge \overline{\boldsymbol{\epsilon}} = \epsilon \rrbracket \\ &= \llbracket \overline{n} = \overline{n} \wedge \epsilon = \epsilon \rrbracket \\ &= \llbracket \overline{n} = \overline{n} \rrbracket \cap \llbracket \epsilon = \epsilon \rrbracket \\ &= \mathbb{O} \cap \mathbb{O} \\ &= \mathbb{O}.\end{aligned}$$

As for any ω^*, $\omega^* \in \mathbb{O}$, which is $\omega^* \in \llbracket \overline{n} = \overline{n} \wedge \overline{m} = \epsilon \rrbracket$.
($\Leftarrow$.) Assume (by contraposition) $m \neq \boldsymbol{\epsilon}$. So,

$$\begin{aligned}\llbracket \overline{n} = \overline{n} \wedge \overline{m} = \epsilon \rrbracket &= \llbracket \overline{n} = \overline{n} \rrbracket \cap \llbracket \overline{m} = \epsilon \rrbracket \\ &= \mathbb{O} \cap \emptyset \\ &= \emptyset.\end{aligned}$$

Clearly, for any ω^*, $\omega^* \notin \emptyset$, so $\omega^* \notin \llbracket \overline{n} = \overline{n} \wedge \overline{m} = \emptyset \rrbracket$.

- $P_i^n(x_1, \ldots, x_n, \omega) = x_i$. The function is Σ_1^b-represented in RS_3^1 by the formula:

$$\bigwedge_{j \in J} (x_j = x_j) \wedge y = x_i$$

where $J = \{1, \ldots, n\} \setminus i$. The proof is similar to the one above.

- $S_b(x, \omega) = x\mathbf{b}$. the function is Σ_1^b-representable in RS_3^1 by the formula:

$$y = x\mathtt{b}$$

where $\mathtt{b} \in \{\mathtt{0}, \mathtt{1}\}$.

1. (Existence.)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z\mathtt{b} = z\mathtt{b} \vdash (\exists y)(y = z\mathtt{b}), z\mathtt{b} = z\mathtt{b}}\, Ax}{\vdash (\exists y)(y = z\mathtt{b}), z\mathtt{b} = z\mathtt{b}}\, Ref}{\vdash (\exists y)(y = z\mathtt{b})}\, \exists R}{\vdash (\forall x)(\exists y)(y = x\mathtt{b})}\, \forall R$$

2. (Uniqueness.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x_2 = x_3, x_2 = x_1\mathtt{b}, x_3 = x_1\mathtt{b} \vdash x_2 = x_3}\, Ax}{x_2 = x_3, x_2 = x_1\mathtt{b}, x_3 = x_1\mathtt{b} \vdash x_2 = x_3}\, Repl}{x_2 = x_1\mathtt{b}, x_3 = x_1\mathtt{b} \vdash x_2 = x_3}\, \wedge L}{x_2 = x_1\mathtt{b} \wedge x_3 = x_1\mathtt{b} \vdash x_2 = x_3}\, \to R}{\vdash (x_2 = x_1\mathtt{b}) \wedge (x_3 = x_1\mathtt{b}) \to x_2 = x_3}\, \forall R\mathrm{s}$$

3. For every $n, m \in \mathbb{S}$ and $\omega^* \in \mathbb{O}$, $S_b(n, \omega^*) = m$ if and only if $\omega^* \in [\![\overline{m} = \overline{n}\mathtt{b}]\!]$.
   ($\Rightarrow$) Assume $b = 0$, so $m = n\mathbf{0}$ ($\mathtt{b} = \mathtt{0}$). Then,

$$\begin{aligned}[\![\overline{m} = \overline{n}\mathtt{0}]\!] &= [\![\overline{n\mathbf{0}} = \overline{n}\mathtt{0}]\!] \\ &= [\![\overline{n}\mathtt{0} = \overline{n}\mathtt{0}]\!] \\ &= \mathbb{O}.\end{aligned}$$

   So, for any $\omega^*, \omega^* \in \mathbb{O}$, which is $\omega^* \in [\![\overline{m} = \overline{n}\mathtt{0}]\!]$.
   Analogously, if $b = 1$ and $m = n\mathbf{1}$ ($\mathtt{b} = \mathtt{1}$).

$$\begin{aligned}[\![\overline{m} = \overline{n}\mathtt{1}]\!] &= [\![\overline{n\mathbf{1}} = \overline{n}\mathtt{1}]\!] \\ &= [\![\overline{n}\mathtt{1} = \overline{n}\mathtt{1}]\!] \\ &= \mathbb{O}.\end{aligned}$$

   ($\Leftarrow$) Assume (by contraposition), $b = 0$ and $m \neq n\mathbf{0}$ ($\mathtt{b} = \mathtt{0}$). Then clearly $[\![\overline{m} = \overline{n}\mathtt{0}]\!] = \emptyset$, and clearly, for any $\omega^* \in \mathbb{O}$, $\omega^* \notin \emptyset$. The case of $b = 0$ and $m \neq n\mathbf{1}$ ($\mathtt{b} = \mathtt{0}$) is equivalent.

- $C(x, z, \omega) = y$. The function is Σ^b_1-representable in RS^1_3 by the formula:

$$(x \subseteq z \wedge y = \mathtt{1}) \vee (\neg(x \subseteq z) \wedge y = \mathtt{0}).$$

1. (Existence.) Let,

$$\mathcal{D}_C$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x_1 \subseteq x_2 \vdash ..., x_1 \subseteq x_2 \wedge 0 = 1, x_1 \subseteq x_2}Ax}{\vdash ..., x_1 \subseteq x_2, x_1 \subseteq x_2 \wedge 0 = 1, \neg(x_1 \subseteq x_2}\neg R \quad \dfrac{}{0 = 0 \vdash ..., x_1 \subseteq x_2 \wedge 0 = 1, 0 = 0}Ax}{\vdash ..., x_1 \subseteq x_2, x_1 \subseteq x_2 \wedge 0 = 1, \neg(x_1 \subseteq x_2) \wedge 0 = 0}\wedge R}{\vdash ..., x_1 \subseteq x_2, (x_1 \subseteq x_2 \wedge 0 = 1) \vee (\neg(x_1 \subseteq x_2) \wedge 0 = 0)}\vee R}{\vdash (\exists y)((x_1 \subseteq x_2 \wedge y = 1) \vee (\neg(x_1 \subseteq x_2) \wedge y = 0)), x_1 \subseteq x_2, 1 = 0}\exists R$$

   So,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x_1 \subseteq x_2 \vdash ..., x_1 \subseteq x_2}Ax}{\vdash ..., x_1 \subseteq x_2, \neg(x_1 \subseteq x_2)}\neg R \quad \mathcal{D}_C}{\vdash ..., x_1 \subseteq x_2, \neg(x_1 \subseteq x_2) \wedge 1 = 0}\wedge R \quad \dfrac{\dfrac{}{1 = 1 \vdash ..., 1 = 1}Ax}{\vdash ..., 1 = 1, \neg(x_1 \subseteq x_2) \wedge 1 = 0}Ref}{\vdash ..., x_1 \subseteq x_2 \wedge 1 = 1, \neg(x_1 \subseteq x_2) \wedge 1 = 0}\wedge R}{\vdash ..., (x_1 \subseteq x_2 \wedge 1 = 1) \vee (\neg(x_1 \subseteq x_2) \wedge y = 0)}\vee R}{\vdash (\exists y)((x_1 \subseteq x_2 \wedge y = 1) \vee (\neg(x_1 \subseteq x_2) \wedge y = 0))}\exists R}{\vdash (\forall x)(\forall z)(\exists y)((x \subseteq z \wedge y = 1) \vee (\neg(x \subseteq z) \wedge y = 0))}\forall Rs$$

2. (Uniqueness.) Let $F(x, v, y)$ be an abbreviation for $(x \subseteq v \wedge y = \mathtt{1}) \vee (\neg(x \subseteq v) \wedge y = \mathtt{0})$ and

$$\mathcal{D}_{C1}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x_3 = x_4, x_3 = \mathtt{1}, x_1 \subseteq x_2, x_4 = \mathtt{1} \vdash x_3 = x_4}\,Ax}{x_1 \subseteq x_2, x_3 = \mathtt{1}, x_1 \subseteq x_2, x_4 = \mathtt{1} \vdash x_3 = x_4}\,Repl}{x_1 \subseteq x_2, x_3 = \mathtt{1}, x_1 \subseteq x_2 \wedge x_4 = \mathtt{1} \vdash x_3 = x_4}\,\wedge L \quad \dfrac{\dfrac{\dfrac{}{x_1 \subseteq x_2, x_3 = \mathtt{1}, x_4 = \mathtt{0} \vdash x_3 = x_4, x_1 \subseteq x_2}\,Ax}{x_1 \subseteq x_2, x_3 = \mathtt{1}, \neg(x_1 \subseteq x_2), x_4 = \mathtt{0} \vdash x_3 = x_4}\,\neg L}{x_1 \subseteq x_2, x_3 = \mathtt{1}, \neg(x_1 \subseteq x_2) \wedge x_4 = \mathtt{0} \vdash x_3 = x_4}\,\wedge L}{x_1 \subseteq x_2, x_3 = \mathtt{1}, F(x_1, x_2, x_4) \vdash x_3 = x_4}\,\vee L$$

$$\mathcal{D}_{C2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x_1 \subseteq x_2, x_3 = \mathtt{0}, x_4 = \mathtt{1} \vdash x_3 = x_4, x_1 \subseteq x_2}\,Ax}{\neg(x_1 \subseteq x_2), x_3 = \mathtt{0}, x_1 \subseteq x_2, x_4 = \mathtt{1} \vdash x_3 = x_4}\,\neg L}{\neg(x_1 \subseteq x_2), x_3 = \mathtt{1}, x_1 \subseteq x_2 \wedge x_4 = \mathtt{1} \vdash x_3 = x_4}\,\wedge L \quad \dfrac{\dfrac{\dfrac{}{x_3 = x_4, \neg(x_1 \subseteq x_2), x_3 = \mathtt{0}, x_4 = \mathtt{0} \vdash x_3 = x_4}\,Ax}{\neg(x_1 \subseteq x_3), x_3 = \mathtt{0}, \neg(x_1 \subseteq x_2), x_4 = \mathtt{0} \vdash x_3 = x_4}\,\neg L}{\neg(x_1 \subseteq x_2), x_3 = \mathtt{0}, \neg(x_1 \subseteq x_2) \wedge x_4 = \mathtt{0} \vdash x_3 = x_4}\,\wedge L}{\neg(x_1 \subseteq x_2), x_3 = \mathtt{0}, F(x_1, x_2, x_4) \vdash x_3 = x_4}\,\vee L$$

Then,

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_{C1}}{x_1 \subseteq x_2, x_3 = \mathtt{1}, F(x_1, x_2, x_4) \vdash x_3 = x_4}\,\wedge L}{x_1 \subseteq x_2 \wedge x_3 = \mathtt{1}, F(x_1, x_2, x_4) \vdash x_3 = x_4}\,\wedge L \quad \dfrac{\dfrac{\mathcal{D}_{C2}}{\neg(x_1 \subseteq x_2), x_3 = \mathtt{0}, F(x_1, x_2, x_4) \vdash x_3 = x_4}\,\wedge L}{\neg(x_1 \subseteq x_2) \wedge x_3 = \mathtt{0}, F(x_1, x_2, x_4) \vdash x_3 = x_4}\,\wedge L}{F(x_1, x_2, x_3), F(x_1, x_2, x_4) \vdash x_3 = x_4}\,\vee L$$
$$\dfrac{F(x_1, x_2, x_3) \wedge F(x_1, x_2, x_4) \rightarrow x_3 = x_4}{\vdash (\forall x)(\forall v)(\forall y)(\forall z)((F(x, v, y) \wedge F(x, v, z) \rightarrow y = z)}\,\wedge L \quad \forall R\mathrm{s}$$

3. ($\Rightarrow$) For every $m, n, l \in \mathbb{S}$ and $\omega^* \in \mathbb{O}$, $C(m, n, \omega^*) = l$ if and only if $\omega^* \in [\![(\overline{m} \subseteq \overline{n} \wedge \overline{l} = \mathtt{1}) \vee (\neg(\overline{m} \subseteq \overline{n}) \vee (\neg(\overline{m} \subseteq \overline{n}) \wedge \overline{l} = \mathtt{0})]\!]$.
($\Leftarrow$) Assume
$$l = \begin{cases} \mathbf{1} & \text{if } m \subseteq n \\ \mathbf{0} & \text{otherwise.} \end{cases}$$
So,
$$\begin{aligned}
[\![(\overline{m} \subseteq \overline{n} \wedge \overline{l} = \mathtt{1}) \vee (\neg(\overline{m} \subseteq \overline{n}) \wedge \overline{l} = \mathtt{0})]\!] &= [\![\overline{m} \subseteq \overline{n} \wedge \overline{l} = \mathtt{1}]\!] \cup [\![\neg(\overline{m} \subseteq \overline{n}) \wedge \overline{l} = \mathtt{0}]\!] \\
&= \big([\![\overline{m} \subseteq \overline{n}]\!] \cap [\![\overline{l} = \mathtt{1}]\!]\big) \cup \big((\mathbb{O} - [\![\overline{m} \subseteq \overline{n}]\!]) \cap [\![\overline{l} = \mathtt{0}]\!]\big) \\
&= \begin{cases} \big(\mathbb{O} \cap [\![\overline{l} = \mathtt{1}]\!]\big) \cup \big(\emptyset \cap [\![\overline{l} = \mathtt{0}]\!]\big) & \text{if } m \subseteq n \\ \big(\emptyset \cap [\![\overline{l} = \mathtt{1}]\!]\big) \cup \big(\mathbb{O} \cap [\![\overline{l} = \mathtt{0}]\!]\big) & \text{otherwise} \end{cases} \\
&= \begin{cases} \big(\mathbb{O} \cap [\![\overline{\mathbf{1}} = \mathtt{1}]\!]\big) \cup \big(\emptyset \cap [\![\overline{\mathbf{1}} = \mathtt{0}]\!]\big) & \text{if } m \subseteq n \\ \big(\emptyset \cap [\![\overline{\mathbf{0}} = \mathtt{1}]\!]\big) \cup \big(\mathbb{O} \cap [\![\overline{\mathbf{0}} = \mathtt{0}]\!]\big) & \text{otherwise} \end{cases} \\
&= \begin{cases} \big(\mathbb{O} \cap [\![\mathtt{1} = \mathtt{1}]\!]\big) \cup \big(\emptyset \cap [\![\mathtt{1} = \mathtt{0}]\!]\big) & \text{if } m \subseteq n \\ \big(\emptyset \cap [\![\mathtt{0} = \mathtt{1}]\!]\big) \cup \big(\mathbb{O} \cap [\![\mathtt{0} = \mathtt{0}]\!]\big) & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{O} & \text{if } m \subseteq n \\ \mathbb{O} & \text{otherwise} \end{cases} \\
&= \mathbb{O}.
\end{aligned}$$
We conclude that for any $\omega^* \in \mathbb{O}$ such that $C(m, n, \omega^*) = l$, then $\omega^* \in [\![(\overline{m} \subseteq \overline{n} \wedge \overline{l} = \mathtt{1}) \vee (\neg(\overline{m} \subseteq \overline{n}) \wedge \overline{l} = \mathtt{0})]\!]$.
($\Leftarrow$) The other direction is proved ina. similar way.

- $Q(x, \omega) = y$. The function is Σ_1^b-representable in RS_3^1 by the formula, $G_Q(x, y)$,

$$\big(\texttt{Flip}(x) \wedge y = \texttt{1}\big) \vee \big(\neg\texttt{Flip}(x) \wedge y = \texttt{0}\big).$$

Notice that the query function always invokes (precisely) *one* oracle, whose choice is basically enucleated by the interpretation of the predicate $\texttt{Flip}(\cdot)$.

1. (Existence.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{Flip}(x) \vdash (\exists y)G_Q(z), \texttt{Flip}(z)}\,Ax}{\vdash (\exists y)F_Q, \texttt{Flip}(z), \neg\texttt{Flip}(z)}\,\neg R \quad \dfrac{\dfrac{\dfrac{}{\texttt{Flip}(z) \vdash \ldots, \texttt{Flip}(z)}\,Ax}{\vdash \ldots\texttt{Flip}(z), \neg\texttt{Flip}(z)}\,\neg R \quad \dfrac{}{0 = 0 \vdash \ldots, \texttt{Flip}(z), 0 = 0}\,Ref}{\vdash \ldots, \texttt{Flip}(z), \texttt{Flip}(z) \wedge 0 = 1, \neg\texttt{Flip}(z) \wedge 0 = 0}\,\wedge R}{\vdash \ldots, \texttt{Flip}(z), (\texttt{Flip}(z) \wedge 0 = 1) \vee (\neg\texttt{Flip}(z) \wedge 0 = 0)}\,\vee R}{\vdash (\exists y)G_Q(z), \texttt{Flip}(z), 1 = 0}\,\exists R \quad \dfrac{}{1 = 1 \vdash \ldots, 1 = 1}\,Ref}{\vdash (\exists y)G_Q(z), \texttt{Flip}(z), \neg\texttt{Flip}(z) \wedge 1 = 0}\,\wedge R}{\vdash (\exists y)G_Q(z), \texttt{Flip}(z) \wedge 1 = 1, \neg\texttt{Flip}(z) \wedge 1 = 0}\,\wedge R}{\vdash G_Q(z), (\texttt{Flip}(z) \wedge 1 = \text{œ}) \vee (\neg\texttt{Flip}(z) \wedge 1 = 0)}\,\vee R}{\vdash (\exists y)((\texttt{Flip}(z) \wedge y = 1) \vee (\neg\texttt{Flip}(z) \wedge y = 0))}\,\exists R}{\vdash (\forall x)(\exists y)((\texttt{Flip}(x) \wedge y = 1) \vee (\neg\texttt{Flip}(x) \wedge y = 0))}\,\forall R$$

2. (Uniqueness.) First,

$$\mathcal{D}_{Q1}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x_2 = x_3, \texttt{Flip}(x_1), x_2 = \texttt{1}, x_3 = \texttt{1} \vdash x_2 = x_3}\,Ax}{\texttt{Flip}(x_1), x_2 = \texttt{1}, \texttt{Flip}(x_1), x_3 = \texttt{1} \vdash x_2 = x_3}\,Repl}{\texttt{Flip}(x_1), x_2 = \texttt{1}, \texttt{Flip}(x_1) \wedge x_3 = \texttt{1} \vdash x_2 = x_3}\,\wedge L \quad \dfrac{\dfrac{}{\texttt{Flip}(x_1), x_2 = \texttt{1}, x_3 = \texttt{0} \vdash x_2 = x_3, \texttt{Flip}(x_1)}\,Ax}{\texttt{Flip}(x_1), x_2 = \texttt{1}, \neg\texttt{Flip}(x_1), x_3 = \texttt{0} \vdash x_2 = x_3}\,\neg L}{\texttt{Flip}(x_1), x_2 = \texttt{1}, \neg\texttt{Flip}(x_1) \wedge x_3 = \texttt{0} \vdash x_2 = x_3}\,\wedge L}{\texttt{Flip}(x_1), x_2 = \texttt{1}, G_Q(x_1, x_3) \vdash x_2 = x_3}\,\vee L$$

$$\dfrac{\texttt{Flip}(x_1), x_2 = \texttt{1}, G_Q(x_1, x_3) \vdash x_2 = x_3}{\texttt{Flip}(x_1) \wedge x_2 = \texttt{1}, G_Q(x_1, x_3) \vdash x_2 = x_3}\,\wedge L$$

$$\mathcal{D}_{Q2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{Flip}(x_1), x_3 = \texttt{1}, y = \texttt{0} \vdash x_2 = x_3, \texttt{Flip}(x_1)}\,Ax}{\neg\texttt{Flip}(x_1), y = \texttt{0}, \texttt{Flip}(x_1), x_3 = \texttt{1} \vdash x_2 = x_3}\,\neg L}{\neg\texttt{Flip}(x_1), y = \texttt{0}, \texttt{Flip}(x_1) \wedge x_3 = \texttt{1} \vdash x_2 = x_3}\,\wedge L \quad \dfrac{\dfrac{}{x_2 = x_3, \neg\texttt{Flip}(x_1), x_2 = \texttt{0}, x_3 = \texttt{0} \vdash x_2 = x_3}\,Ax}{\neg\texttt{Flip}(x_1), x_2 = \texttt{0}, \neg\texttt{Flip}(x_1), x_3 = \texttt{0} \vdash x_2 = x_3}\,Repl}{\neg\texttt{Flip}(x_1), x_2 = \texttt{0}, \neg\texttt{Flip}(x_1) \wedge x_3 = \texttt{0} \vdash x_2 = x_3}\,\wedge L}{\neg\texttt{Flip}(x_1), y = \texttt{0}, G_Q(x_1, x_3) \vdash x_2 = x_3}\,\vee L$$

$$\dfrac{\neg\texttt{Flip}(x_1), y = \texttt{0}, G_Q(x_1, x_3) \vdash x_2 = x_3}{\neg\texttt{Flip}(x_1) \wedge y = \texttt{0}, G_Q(x_1, x_3) \vdash x_2 = x_3}\,\wedge L$$

We conclude as follows,

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_{Q1} \qquad \mathcal{D}_{Q2}}{\texttt{Flip}(x_1) \wedge x_2 = \texttt{1}, G_Q(x_1, x_2) \vdash x_2 = x_3 \qquad \neg\texttt{Flip}(x_1) \wedge y = \texttt{0}, G_Q(x_1, x_3) \vdash x_2 = x_3}\,\vee L}{G_Q(x_1, x_2), G_Q(x_1, x_3) \vdash x_2 = x_3}\,\wedge L}{G_Q(x_1, x_2) \wedge G_Q(x_1, x_3) \vdash x_2 = x_3}}{\vdash G_Q(x_1, x_2) \wedge G_Q(x_1, x_3) \to x_2 = x_3}$$

$$\vdash (\forall x)(\forall y)(\forall z)(G_Q(x, y) \wedge G_Q(x, z) \to y = z)\,\forall R\text{s}$$

3. For every $n, m \in \mathbb{S}$ and $\omega^* \in \mathbb{O}$, $Q(n, \omega^*) = m$ if and only if $\omega^* \in [\![(\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{0})]\!]$.
($\Rightarrow$) Assume $Q(n, \omega^*) = m$ and $m = \mathbf{1}$, which is $\omega^*(n) = \mathbf{1}$. So,

$$\begin{aligned}
[\![(\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{0})]\!] &= [\![(\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{1}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{1}} = \mathtt{0})]\!] \\
&= [\![(\mathtt{Flip}(\overline{n}) \wedge \mathtt{1} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \mathtt{1} = \mathtt{0})]\!] \\
&= [\![\mathtt{Flip}(\overline{n}) \wedge \mathtt{1} = \mathtt{1}]\!] \cup [\![\neg\mathtt{Flip}(\overline{n}) \wedge \mathtt{1} = \mathtt{0}]\!] \\
&= \big([\![\mathtt{Flip}(\overline{n})]\!] \cap [\![\mathtt{1} = \mathtt{1}]\!]\big) \cup \big([\![\neg\mathtt{Flip}(\overline{n})]\!] \cap [\![\mathtt{1} = \mathtt{0}]\!]\big) \\
&= \big([\![\mathtt{Flip}(\overline{n})]\!] \cap \mathbb{O}\big) \cup \big([\![\neg\mathtt{Flip}(\overline{n})]\!] \cap \emptyset\big) \\
&= [\![\mathtt{Flip}(\overline{n})]\!] \cup \emptyset \\
&= [\![\mathtt{Flip}(\overline{n})]\!] \\
&= \{\omega \mid \omega(n) = \mathbf{1}\}.
\end{aligned}$$

Since, as seen, $\omega^*(n) = \mathbf{1}$, clearly $\omega^* \in \{\omega \mid \omega(n) = \mathbf{1}\} = [\![(\mathtt{Flip}(\overline{n} \wedge \overline{\mathbf{1}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n} \wedge \overline{\mathbf{1}} = \mathtt{0})]\!]$.
Assume $Q(n, \omega^*) = m$ and $m = \mathbf{0}$, which is $\omega^*(n) = \mathbf{0}$. So,

$$\begin{aligned}
[\![(\mathtt{Flip}(\overline{n} \wedge \overline{m} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{m} = \mathtt{0})]\!] &= [\![(\mathtt{Flip}(\overline{n} \wedge \overline{\mathbf{0}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{1}} = \mathtt{0})]\!] \\
&= [\![(\mathtt{Flip}(\overline{n} \wedge \mathtt{0} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \mathtt{0} = \mathtt{0})]\!] \\
&= [\![(\mathtt{Flip}(\overline{n}) \wedge \mathtt{0} = \mathtt{1}]\!] \cup [\![\neg\mathtt{Flip}(\overline{n}) \wedge \mathtt{0} = \mathtt{0}]\!] \\
&= \big([\![\mathtt{Flip}(\overline{n}) \cap [\![\mathtt{0} = \mathtt{1}]\!]\big) \cup \big([\![\neg\mathtt{Flip}(\overline{n})]\!] \cap [\![\mathtt{0} = \mathtt{0}]\!]\big) \\
&= \big([\![\mathtt{Flip}(\overline{n} \cap \emptyset\big) \cup \big([\![\neg\mathtt{Flip}(\overline{n}]\!] \cap \mathbb{O}\big) \\
&= \emptyset \cup [\![\neg\mathtt{Flip}(\overline{n})]\!] \\
&= [\![\neg\mathtt{Flip}(\overline{n})]\!] \\
&= \mathbb{O} - [\![\mathtt{Flip}(\overline{n})]\!] \\
&= \{\omega \mid \omega(n) = \mathbf{0}\}.
\end{aligned}$$

Since, as seen $\omega^*(n) = \mathbf{0}$, clearly $\omega^* \in \{\omega \mid \omega(n) = \mathbf{0}\} = [\![(\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{0}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{0}} = \mathtt{0})]\!]$.
($\Leftarrow$) Assume (by contraposition) $\omega^*(n) = \mathbf{1}$, so $Q(n, \omega^*) = \mathbf{1}$ and $m = \mathbf{0}$. As shown above, in this case $[\![(\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{0}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{0}} = \mathtt{0})]\!] = \{\omega \mid \omega(n) = \mathbf{0}\}$. For assumption, ω^* is such that $\omega^*(n) = \mathbf{1}$, so $\omega^* \notin \{\omega \mid \omega(n) = \mathbf{0}\}$, i.e. $\omega^* \notin [\![(\mathtt{Flip}(\overline{n} \wedge \overline{\mathbf{0}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{0}} = \mathtt{0})]\!]$.
Assume (by contraposition) $\omega^*(n) = \mathbf{0}$, so $Q(n, \omega^*) = \mathbf{0}$ and $m = \mathbf{1}$. As shown above, in this case $[\![(\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{1}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{1}} = \mathtt{0})]\!] = \{\omega \mid \omega(n) = \mathbf{1}\}$. For assumption, ω^* is such that $\omega^*(n) = \mathbf{0}$, so $\omega^* \notin \{\omega \mid \omega(n) = \mathbf{1}\}$, i.e. $\omega^* \notin [\![(\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{1}} = \mathtt{1}) \vee (\neg\mathtt{Flip}(\overline{n}) \wedge \overline{\mathbf{1}} = \mathtt{0})]\!]$.

Inductive Case. Let $f \in \mathcal{POR}$ be obtained from Σ_1^b-representable functions by either composition of bounded reduction:

- *Composition.* Let $f \in \mathcal{POR}$ be defined by composition from functions $g, h_1, \ldots, h_k$, so that

$$f(\vec{x}, \omega) = g(h_1(\vec{x}, \omega), \ldots, h_k(\vec{x}, \omega), \omega)$$

and $g, h_1, \ldots, h_k$ are represented in RS^1_3 by (resp.) the Σ^b_1-formulas $G_g, G_{h_1}, \ldots, G_{h_k}$. By Proposition 2, there is some suitable terms $t_g, t_{h_1}, \ldots, t_{h_k}$, such that $RS^1_3 \vdash (\forall \vec{x})(\exists y \preceq t_i)F_i(\vec{x}, y)$, with $i \in \{g, h_1, \ldots, h_k\}$. We conclude that $f(\vec{x}, \omega)$ is Σ^b_1-represented inn RS^1_3 by the following formula:

$$G_f := \big(\exists z_1 \preceq t_{h_1}(\vec{x})\big) \ldots \big(\exists z_k \preceq t_{h_k}(\vec{x})\big)\big(G_{h_1}(\vec{x}, z_k) \wedge \cdots \wedge G_{h_k}(\vec{x}, z_k) \wedge G_g(z_1, \ldots, z_k, y)\big).$$

Indeed, since for assumption $G_g, G_{h_1}, \ldots, G_{h_k}$ are Σ^b_1-formulas, also G_f is a Σ^b_1-formula. For simplicity's sake, let us prove the claim for the simple case $f(x, \omega) = g(h(x, \omega), \omega)$. The proof can be extended to the general case with $h_1, \ldots, h_k$ in the straightforward way.

1. (Existstence.) For hypothesis h, g are represented in RS^1_3 by the Σ^b_1-formulas (resp.) G_h and G_g. So, by Condition 1 (resp.) $RS^1_3 \vdash (\forall x)(\exists y)G_h$ and $RS^1_3 \vdash (\forall x)(\exists y)G_g(x, y)$. By Proposition 2, in particular, there are two $\mathcal{L}$-term, t_h and t_g, such that:

$$RS^1_3 \vdash (\forall x)(\exists y \preceq t_h)G_h(x, y)$$
$$RS^1_3 \vdash (\forall x)(\exists y \preceq t_g)G_g(x, y),$$

where, as said $(\exists y \preceq t)F$ is a shorthand for $(\exists y)(\mathtt{1}^y \subseteq \mathtt{1}^t \wedge F)$. We can prove that:

$$RS^1_3 \vdash (\forall x)\big(\exists y \preceq t_g(t_h/x)\big)(\exists z \preceq t_h)\big(G_h(x, y) \wedge G_g(z, y)\big).$$

For readability's sake, the us use F_h to abbreviate $(\forall x)\big(\exists y \preceq t_h(x)\big)G_h(x, y)$ and F_g to abbreviate $(\forall x)\big(\exists \preceq t_g(x)\big)G_g(x, y)$. So,

$$\mathcal{D}_{comp.e}$$

$$\dfrac{\dfrac{}{z_1 \preceq t_h(x_1), \ldots \vdash z_1 \preceq t_h(y_1)}\,Ax \qquad \dfrac{\dfrac{}{G_h(x_1, z_1), \ldots \vdash G_h(x_1, z_1)}\,Ax \qquad \dfrac{}{G_g(z_1, y_1), \ldots \vdash G_g(z_1, y_1)}\,Ax}{G_h(x_1, z_1), G_g(z_1, y_1), \ldots \vdash G_h(x_1, z_1) \wedge G_g(z_1, y_1)}\,\wedge R}{z_1 \preceq t_h(x_1), G_h(x_1, z_1), G_g(z_1, y_1), \ldots \vdash \big(\exists z \preceq t_h(y_1)\big)\big(G_h(x_1, z) \wedge G_g(z, y_1)\big)}\,\exists_{\preceq} R$$

We conclude,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y_1 \preceq t_g(t_h/x_1) \ldots \vdash y_1 \preceq t_g(t_h/x_1)}\,Ax}{y_1 \preceq t_g(z_1), z_1 \preceq t_h(x_1) \ldots \vdash y_1 \preceq t_g(t_h/x_1)}\,\mathrm{L?} \qquad \mathcal{D}_{comp.e}}{z_1 \preceq t_h(x_1), G_h(x_1, z_1), y_1 \preceq t_g(z_1), G_g(z_1, y_1) \ldots \vdash (\exists y \preceq t_g(t_h/z_1))(\exists z \preceq t_h(y))(G_h(x_1, z) \wedge G_g(z, y))}\,\exists_{\preceq} R}{z_1 \preceq t_h(x_1), G_h(x_1, z_1), (\exists y \preceq t_g(z_1))G_g(z_1, y) \ldots \vdash (\exists y \preceq t_g(t_h/x_1))(\exists z \preceq t_h(y))(G_h(x_1, z) \wedge G_g(z, h))}\,\exists_{\preceq} L}{z_1 \preceq t_h(x_1), G_h(x_1, z_1), F_f, F_g \vdash (\exists y \preceq t_g(t_h/x_1))(\exists z \preceq t_h(y))(G_h(x_1, z) \wedge G_g(z, h))}\,\forall L}{(\exists z \preceq t_h(x_1))G_h(x_1, z)F_f, F_g \vdash (\exists y \preceq t_g(t_h/x_1))(\exists z \preceq t_h(y))(G_h(x_1, z) \wedge G_g(z, h))}\,\exists_{\preceq} L}{F_f, F_g \vdash (\exists y \preceq t_g(t_h/x_1))(\exists z \preceq t_h(y))(G_h(x_1, z) \wedge G_g(z, h))}\,\forall L}{F_f, F_g \vdash (\forall x)(\exists y \preceq t_g(t_h/x))(\exists z \preceq t_h(y))(G_h(x, z) \wedge G_g(z, h))}\,\forall R}{F_f \wedge F_g \vdash (\forall x)(\exists y \preceq t_g(t_h/x))(\exists z \preceq t_h(y))(G_h(x, z) \wedge G_g(z, h))}\,\wedge L$$

2. (Uniqueness.) Uniqueness is proved relying on the uniqueness conditions for G_g and G_h. Let F_g, F_h and F_f be abbreviations for (resp.) $(\forall x)(\forall y)(\forall z)(G_g(x,y) \wedge G_g(x,z) \rightarrow y = z)$, $(\forall x)(\forall y)(\forall z)(G_h(x,y) \wedge G_h(x,z) \rightarrow y = z)$ and $(\forall x)(\forall y)(\forall z)(G_f(x,y) \wedge G_f(x,z) \rightarrow y = z)$. Then,

$$\mathcal{D}_{comp.u}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y_1 = z_1, v \preceq t_h(x_1)... \vdash y_1 = z_1}\,Ax \quad \dfrac{\dfrac{}{G_g(v,y_1)... \vdash G_g(v,y_1)}\,Ax \quad \dfrac{}{G_g(v,z_1)... \vdash G_g(v,z_1)}\,Ax}{v \preceq t_h(x_1)... \vdash y_1 = z_1, G_g(v,y_1) \wedge G_g(v,z_1)}\,\wedge R}{G_g(v,y_1) \wedge G_g(v,z_1) \rightarrow y_1 = z_1, v \preceq t_h(x_1), G_h(x_1,v), G_g(v,y_1), G_h(x_1,v), G_g(v,z_1), F_g, F_h \vdash y_1 = z_1}\,\rightarrow L}{v \preceq t_h(x_1), G_h(x_1,v), G_g(v,y_1), G_g(v,y_1), v \preceq t_h, G_h(x_1,v), G_g(v,z_1), F_g, F_h \vdash y_1 = z_1}\,\forall Ls}{v = w, v \preceq t_h(x_1), G_h(x_1,v), G_g(v,y_1), w \preceq t_h, G_h(x_1,w), G_g(w,z_1), F_g, F_h \vdash y_1 = z_1}\,Repl$$

We conclude,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F_g, F_h, G_f(x_1,y_1), G_f(x_1,z_1) \vdash G_f(y_1,z_1)}}{F_g, F_h, G_f(x_1,y_1) \wedge G_f(x_1,z_1) \vdash y_1 = z_1}\,\wedge L}{F_g, F_h \vdash G_f(x_1,y_1) \wedge G_f(x_1,z_1) \rightarrow y_1 = z_1}\,\rightarrow R}{F_g, F_h \vdash F_f}\,\forall Rs}{F_g \wedge F_h \vdash F_f}\,\wedge L}{v \preceq t_h(x_1), G_h(x_1,v) \wedge G_g(v,y_1), w \preceq t_h, G_h(x_1,w) \wedge G_g(w,z_1), F_g, F_h \vdash y_1 = z_1}\,\exists L_{\preceq}{}^{s}}{v \preceq t_h(x_1), G_h(x_1,v), G_g(v,y_1), w \preceq t_h, G_h(x_1,w), G_g(w,z_1), F_g, F_h \vdash y_1 = z_1}\,\wedge Ls}{G_h(x_1,v) \wedge G_h(x_1,w) \rightarrow v = w, v \preceq t_h(x_1), G_h(x_1,v), G_g(v,y_1), w \preceq t_h, G_h(x_1,w), G_g(w,z_1), F_g, F_h \vdash y_1 = z_1}\,\forall Ls \quad \rightarrow L$$

$$\dfrac{\mathcal{D}_{comp.u} \qquad \dfrac{\dfrac{}{G_h(x_1,v)... \vdash G_h(x_1,v)}\,Ax \quad \dfrac{}{G_h(x_1,w)... \vdash G_h(x_1,w)}\,Ax}{v \preceq t_h(x_1)... \vdash y_1 = z_1, y_1 = z_1, G_h(x_1,v) \wedge G_h(x_1,w)}\,\wedge R}{v = w, v \preceq t_h(x_1)... \vdash y_1 = z_1} \quad \rightarrow L$$

3. For every $n, m \in \mathbb{S}$ and $\omega^* \in \mathbb{O}$, $f(n,\omega^*) = m$ if and only if $\omega^* \in [\![(\exists z \preceq t_h)(G_h(\overline{n},z) \wedge G_g(z,\overline{m}))]\!]$.

   ($\Rightarrow$) Assume that $g(h(n,\omega^*),\omega^*) = m$. Let $h(n,\omega^*) = s$, for some $s \in \mathbb{S}$. Since $s \in \mathbb{S}$, there is a $t \in \mathbb{S}$ such that $s \subseteq t$ (e.g. $t = s\mathbf{1}$). Then, for IH, $\omega^* \in [\![G_h(\overline{n},\overline{s})]\!]$. For assumption $g(h(n,\omega^*),\omega^*)$, so $g(s,\omega^*) = m$. Thus, for IH, $\omega^* \in [\![G_g(\overline{s},\overline{m})]\!]$. Since $\omega^* \in [\![G_h(,\overline{s})]\!]$, for basic set theory also $\omega^* \in [\![G_h(\overline{n},\overline{s})]\!] \cap [\![G_g(\overline{s},\overline{m})]\!] = [\![G_h(\overline{n},\overline{s} \wedge G_g(\overline{s},\overline{m})]\!]$ and clearly, also $\omega^* \in [\![(\exists z)(G_h(\overline{n},z) \wedge G_g(z,\overline{m}))]\!] = \bigcup_{i \in \mathbb{S}} [\![G_h(\overline{n},z) \wedge G_g(z,\overline{m})]\!]_{z \leftarrow i}$. Furthermore, as seen, $[\![\overline{s} \subseteq \overline{t}]\!] = \mathbb{O}$. We conclude that $\omega^* \in [\![(\exists z \preceq t)(G_h(\overline{n},z) \wedge G_g(z,\overline{m})]\!] = \bigcup_{i \in \mathbb{S}} [\![z \subseteq \overline{t} \wedge (G_h(\overline{n},z) \wedge G_g(z,\overline{m}))]\!]_{z \leftarrow i}$.

   ($\Leftarrow$) Similarly proved, due to Proposition 2.

- *Bounded Recursion.* In order to deal with bounded recursion some preliminary work is needed, [7].

Counting within RS^1_3. Given a string, we need to count within RS^1_3 the number of its `1`s. This can be formalized due to some auxiliary formulas. The intuitive idea is to assign to each "string" x another string $z = z_0 z_1 \dots z_n$ such that the length of x is equal to n for each z_i, the length of z_i is equal to i, and each z_{k+1} with $1 \leq k \leq n$ is obtained from z_k so that: (i) if the $(k+1)$-th element of x is `0`, then $z_k + 1 = z_k\mathtt{0}$, (ii) otherwise, $z_{k+1} = \mathtt{1}z_k$. Let us call the

string z so defined, the *x-ordered string*. This construction is formalized due to the following formulas:

$$\begin{aligned}
F_{o1}(x,y) &:= (\exists u)((u\mathtt{0} = x \wedge y = x) \vee u\mathtt{1} = x \wedge y = \mathtt{1}u)) \\
F_{o2}(x) &:= (\exists u)(\exists v)(u = \mathtt{1} \times u \wedge v = \mathtt{0} \times v \wedge x = uv) \\
F_{o3}(x,z) &:= zz \equiv x \times x\mathtt{1} \wedge (\forall x' \subseteq x)(\exists z' \subseteq z)(z'z' \equiv x' \times x'\mathtt{1} \\
&\quad \wedge (\forall z'')(\forall y')\big(y' \equiv x' \wedge z''y' = z' \rightarrow (\exists y)\big(z'y \subseteq z \\
&\quad \wedge \big((x'\mathtt{0} \subseteq x \rightarrow F_{o1}(y'\mathtt{0}, y)\big) \wedge (x'\mathtt{1} \subseteq x \rightarrow F_{o1}(y'\mathtt{1}, y))))\big).
\end{aligned}$$

Intuitively, $F_{o1}(x,y)$ says that: if the last bit of x is $\mathtt{0}$, then $y = x$; if the last bit of x is $\mathtt{1}$, then $y = \mathtt{1}u$, where $u\mathtt{1} = x$, which is where u is equal to x after removing its last bit, $\mathtt{1}$. $F_{o2}(x)$ basically expresses that x is made of a (possibly empty) sequence of $\mathtt{1}$s concatenated with a sequence of $\mathtt{0}$s. According to $x = uv$, where u is a (possibly empty) sequence of $\mathtt{1}$s and v of $\mathtt{0}$s, i.e. roughly x is in the form $\mathtt{11}\ldots\mathtt{00}\ldots$. $F_{o3}(x,z)$ enucleates the construction described above, saying that z is the x-ordered string. Indeed, $F_{o3}(x,z)$ says that (1) the length of zz is equal to that of $x \times x\mathtt{1}$, (2) for each $x' \subseteq x$, there is a $z' \subseteq z$, such that (i) the length of $z'z'$ is equal to the length of $x' \times x'\mathtt{1}$ and (ii) for each z'' and y', with $|y'| = |x'|$ and $z''y' = z'$, there is a y such that (a) $z'y \subseteq z$ and (b) if $x'\mathtt{0} \subseteq x$, then $F_{o1}(y'\mathtt{0}, y)$; if $x'\mathtt{1} \subseteq x$, then $F_{o1}(y'\mathtt{1}, y)$. Basically, (2) ensures that each (initial) substring of z is defined in the proper way.

It is now possible to define the desired $F_{ones}(\cdot,\cdot)$:[7]

$$\begin{aligned}
F_{ones}(x,u) &:= u = \mathtt{1} \times u \wedge (\exists z)(F_{o3}(x,z) \wedge (\forall x' \subseteq x)(\forall z') \\
&\quad (((x = \epsilon) \vee (x'\mathtt{1} \equiv x \wedge F_{o3}(x',z'))) \rightarrow (\exists y)(y = \mathtt{0} \times y \wedge z = z'uy))).
\end{aligned}$$

Intuitively, $F_{ones}(x,u)$ expresses that u is a sequence of $\mathtt{1}$s the length of which corresponds to the number of $\mathtt{1}$s in x. Indeed, $F_{ones}(x,u)$ says that (1) u is a sequence of $\mathtt{1}$s of length u (as $u = \mathtt{1} \times u$) and (2) that there is a z such that (i) z is the x-ordered string (as $F_{o3}(x,z)$), (ii) for all x', z' with $x' \subseteq x$: if $x = \epsilon$ or the length of x is equal to that of x plus one and z' is the x'-ordered string, then there is a sequence of $\mathtt{0}$s, y, such that $z = z'uy$. Notice that, as said, z is the x-ordered string, so uy is a string of $\mathtt{1}$s, u, concatenated with by a string of $\mathtt{0}$s, y, so that the number of $\mathtt{1}$s in uy is the same as the number of $\mathtt{1}$s in x.

It is also intuitively clear that the following properties hold: (i) the number of $\mathtt{1}$s in the empty string is 0, (ii) given a string x such that it contains a number of $\mathtt{1}$s equal to the length of y, then $x\mathtt{0}$ contains the same number of $\mathtt{1}$s, while $x\mathtt{1}$ contains a number of $\mathtt{1}$s equal to $|y| + 1$, (iii) given a string x, its number of $\mathtt{1}$s is defined by a unique string of $\mathtt{1}$s (the length of which corresponds to the number of $\mathtt{1}$s in x).[8] Formally:

i. $RS_3^1 \vdash F_{ones}(\epsilon, \epsilon)$
ii. $RS_3^1 \vdash F_{ones}(x,y) \rightarrow F_{ones}(x\mathtt{0}, y) \wedge F_{ones}(x\mathtt{1}, y\mathtt{1})$
iii. $RS_3^1 \vdash F_{ones}(x,y) \wedge F_{ones}(x,z) \rightarrow y = z$.

[7] Notice that this formula is slightly different from the corresponding one defined in [7, p. 97]. Indeed, Ferreira's $Ones(\cdot,\cdot)$ is such that $Ones(\epsilon, u)$ is derivable for every $u = \mathtt{1} \times u$ in contrast to the required condition 3 of Proposition 9. For further details, see Section 3.1.4.

[8] For further details, see Proposition 9 in Section 3.1.3.

Encoding. It is possible to formally encode sequences of $\epsilon, \mathtt{0}, \mathtt{1}$, separated by commas as strings. Following Ferreira [7, p. 97], we encode $\mathtt{0}$ as $\mathtt{01}$, $\mathtt{1}$ as $\mathtt{11}$. First, we introduce the formula $F_{seq}(\cdot)$ below:

$$F_{seq}(x) := (\exists y \subseteq x)(x \equiv yy \wedge (\forall z \subseteq y)\neg(x|_{zz}\mathtt{00} \subseteq x)).$$

The formula $F_{seq}(x)$ basically says that x is in the form of an encoding sequence. Indeed, it expresses that x is a string made of initial-couple substrings (indeed, its length corresponds to an even number) and does not contain any initial-couple substrings in the form $\mathtt{00}$, where given a string $y = y_1y_2 \dots y_n$, we call its *initial-couple substring* each strinng y_iy_{i+1}, for $i \in \{1, 3, \dots, n-1\}$. Then, $F_s(\cdot, \cdot)$ is defined as follows:

$$\begin{aligned} F_s(x, y) := y \equiv x \wedge F_{seq}(x) \wedge (\forall z)(zz \preceq x \rightarrow (x|_{zz}\mathtt{01} \subseteq x \rightarrow y|_{zz}\mathtt{00} \subseteq y) \\ \wedge (x|_{zz}\mathtt{10} \subseteq x \rightarrow y|_{zz}\mathtt{00} \subseteq y) \\ \wedge (x|_{zz}\mathtt{11} \subseteq x \rightarrow y|_{zz}\mathtt{11} \subseteq y)). \end{aligned}$$

The formula $F_s(x, y)$ expresses that (1) $|x| = |y|$, (2) x is in the form of an encoding sequence, as $F_{seq}(x)$ and (3) y is such that each initial-couple substring in x, which does not encode a separation mark, corresponds to the initial-couple substring $\mathtt{00}$ in y (i.e. $x|_{zz}\mathtt{01} \subseteq x \rightarrow y|_{zz}\mathtt{00} \subseteq y$ and $x|_{zz}\mathtt{10} \subseteq x \rightarrow y|_{zz}\mathtt{00} \subseteq y$), while initial-couple substrings in x encoding the separation mark, $\mathtt{11}$, correspond to the same substring $\mathtt{11}$ in y (i.e. $x|_{zz}\mathtt{11} \subseteq x \rightarrow y|_{zz}\mathtt{11} \subseteq y$). Notice that, in this way, y somehow indicates the number of "bits" (either $\mathtt{0}$ or $\mathtt{1}$) and of separation marks of the sequences encoded by x. For simplicity's sake, let us call y the *s-encoding* of x.

Decoding. In order to measure the length of a sequence, let us introduce the following formula:

$$F_{lh}(x, u) := (\exists y \equiv x)(F_s(x, y) \wedge F_{ones}(y, uu))$$

where clearly $(\exists y \equiv x)G$ is a shorthand for $(\exists y)(y \equiv x \wedge G)$. The formula $F_{lh}(x, u)$ intuitively says that there is a y, of the same length of x, such that (1) y is the s-encoding of x and (as $F_s(x, y)$) and (2) the number of $\mathtt{1}$s in y is uu. Clearly, as $\mathtt{11}$ represents the encoding in y of initial-couples $\mathtt{01}$ or $\mathtt{10}$ in x (i.e. the encoding in x of a "bit" in the sequence) u is precisely the number of "bits" in the original sequence, encoded by x (so, excluding the number of separation marks in it).

Then, $F_{word}(\cdot)$ is defined as below:

$$F_{word}(x) := (\exists y \subseteq x)(x \equiv yy \wedge (\forall z \subset y)(x|_{zz}\mathtt{01} \subseteq x \vee x|_{zz}\mathtt{10} \subseteq x))$$

where, as predictable, $(\forall z \subset y)G$ is an abbreviation for $(\forall z)(z \subset y \rightarrow G)$. $F_{word}(x)$ basically says that x is a string made of initial-couple substrings in either the form $\mathtt{01}$ or $\mathtt{10}$, which is x is a string encoding a sequence of (possibly alternated) $\mathtt{0}$s and $\mathtt{1}$s.

In order to define the desired "decoding" formula, two other auxiliary formulas need to be

introduced:

$$\begin{aligned} F_{e1}(s,r,v) := {} & F_{seq}(s) \wedge F_{word}(v) \wedge (\exists y)(\exists u)(F_{lh}(s,u) \wedge r \subset u \wedge F_s(s,y) \\ & \wedge (\forall y' \subseteq y)(F_{ones}(y', rr) \rightarrow s|_{y'}v \equiv s \vee s|_{y'}v\mathtt{11} \subseteq s)) \\ F_{e2}(v,x) := {} & F_{word}(v) \wedge v \equiv xx \wedge (\forall x' \subseteq x)((x'\mathtt{0} \subseteq x \rightarrow v|_{x'x'}\mathtt{01} \subseteq v) \\ & \wedge (x'\mathtt{1} \subseteq x \rightarrow v|_{x'x'}\mathtt{10} \subseteq v)). \end{aligned}$$

Given a string s encoding a sequence and a string v of initial-couple substrings either in the form `01` or `10`, $F_{e1}(s,r,v)$ allows one to "select" the *encoding for a bit* in s. The formula $F_{e2}(v,x)$ says (1) that v is a string, the initial-couple substrings of which are either in the form `01` or `10`, (2) that the length of w is equal to that of xx, (3) that for each (initial) substring x' of x: (3.i) if $x'\mathtt{0} \subseteq x$, then the concatenation of v at the length of $x'x'$ is a(n initial-) substring of v, (3.ii) if $x'\mathtt{1} \subseteq x$, then the concatenation of the truncation of v at the length of $x'x'$ with `10` is a(n initial-) substring of v. Otherwise said, $F_{e2}(w,x)$ expresses that, given a string x of (possibly alternated) `0`s and `1`s, v, the length of which is double that of x, such that each occurrence of `0` in x corresponds to a couple-substring `01` in (the corresponding position of) v and each occurrence of `1` in x corresponds to a couple-substring `10` in (the corresponding position of) v.

We can now define F_{eval} as follows:

$$F_{eval}(x,y,z) := (\exists u \preceq x)(F_{e1}(x,y,u) \wedge F_{e2}(u,z)).$$

Intuitively, this formula allows one to "decode" each encoding couple-substring of x. Indeed, it says that x is a string encoding a sequence of $\mathtt{0}, \mathtt{1}, \epsilon$s, separated by commas, defined as described above (as $F_{seq}(x)$). Moreover, there is a string u, which is a string of initial couple-substrings either in the form `01` or `10` (as $F_{word}(u)$). When the length of y indicate a couple-substring encoding a "bit", z is a string such that each of its "bit" is `0` if the corresponding couple-substring in u is `01` and is `1` if the corresponding couple-substring in u is `10`. Therefore, $F_{eval}(x,y,z)$ expresses that the "bit" encoded in x at position y is z.

Bounded Recursion. These auxiliary functions make it possible to conclude the proof. Let f be defined from g, h_0, h_1 by bounded recursion as below:

$$\begin{aligned} f(\vec{x}, \boldsymbol{\epsilon}) &= g(\vec{x}) \\ f(\vec{x}, y\mathbf{0}) &= h_0(\vec{x}, y, f(\vec{x}, y))|_{t(\vec{x},y)} \\ f(\vec{x}, y\mathbf{1}) &= h_1(\vec{x}, y, f(\vec{x}, y))|_{t(\vec{x},y)} \end{aligned}$$

where $\mathbf{i} \in \{\mathbf{0}, \mathbf{1}\}$ and $i = 0$ if $\mathbf{i} = \mathbf{0}$ and $i = 1$ if $\mathbf{i} = \mathbf{1}$, and t is an $\mathcal{L}$-term. For hypothesis g, h_0, h_1 are Σ_1^b-represented by (resp.) F_g, F_{h_0}, F_{h_1} and by Proposition 2, there are some terms t_g, t_{h_0}, t_{h_1} so that the existential conditions can be strengthened. Then, f is Σ_1^b-represented

in RS^1_3 by the following formula:

$$\begin{aligned} G_f(\vec{x},y) := (\exists s \preceq t_g(\vec{x})(y \times t(\vec{x},y)t(\vec{x},y)\mathtt{11}))(F_{lh}(s,\mathtt{1}\times y\mathtt{1}) \\ \wedge (\exists z \preceq t_g(\vec{x}))(F_{eval}(s,\epsilon,z) \wedge F_g(\vec{x},z)) \\ \wedge (\forall u \subset y)(\exists v)(\tilde{v} \preceq t(\vec{x},y) \wedge F_{eval}(s,\mathtt{1}\times u,v) \wedge F_{eval}(s,\mathtt{1}\times u\mathtt{1},\tilde{v}) \\ \wedge (u\mathtt{0} \subseteq y \rightarrow (\exists v_0 \preceq t_{h_0}(\vec{x},u,v))(G_{h_0}(\vec{x},y,v,v_0) \wedge v_0|_{t(\vec{x},u)} = \tilde{v})) \\ \wedge (u\mathtt{1} \subseteq y \rightarrow (\exists v_1 \preceq t_{h_1}(\vec{x},u,v))(G_{h_1}(\vec{x},u,v,v_1) \wedge v_1|_{t(\vec{x},u)} = \tilde{v})))). \end{aligned}$$

Uniqueness and existence can be proved due to a quite convoluted machinery, already defined by Ferreira. Semantically, this formula intuitively says that given the sequences $\vec{x}$, y is so defined to precisely correspond to the output obtained from g, h_0, h_1 by bounded recursion. Indeed, G_f, intuitively says that there is a string s (with the proper term bound), such that there is a $\mathtt{1}\times y\mathtt{1}$ is the number of "bits" in the sequence encoded by s, as $F_{lh}(s,\mathtt{1}\times y\mathtt{1})$. Then, y is so constructed that, there is a z, which corresponds to the ϵ-th "bit" in the sequence encoded by s and such that $G_g(s,z)$, where G_g is the Σ^b_1-formula representing g (for IH). Moreover, for each $u \subset y$, there are $v, \tilde{v}$ which are respectively the $|u|$-th and the $|u|+1$-th "bits" of s. If $u\mathtt{0} \subseteq y$, then there is a (properly bounded) v_0 such that such that $G_{h_0}(\vec{x},y,v,v_0)$, where G_{h_0} represents the function f_{h_0} (for IH) and its truncation at $t(\vec{x},u)$ is precisely $\tilde{v}$.[9] Similarly, if $u\mathtt{1} \subseteq y$, then there is a (properly bounded) v_1 such that $G_{h_0}(\vec{x},y,v,v_1)$, where G_{h_1} represents the function G_{h_1} (for IH) and its truncation at $t(\vec{x},u)$ is precisely $\tilde{v}$.

□

3.1.3 On Bounded Recursion

Auxiliary Lemmas.

Lemma 6. *The sequent* $\vdash (\forall x)(\mathtt{1}\times x = \epsilon \rightarrow x = \epsilon)$ *is derivable in* $\mathbf{G3cS}^1_3$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{}{\epsilon=\epsilon, \mathtt{1}\times\epsilon=\epsilon\vdash\epsilon=\epsilon}\,Ax}{\mathtt{1}\times\epsilon=\epsilon\vdash\epsilon=\epsilon}\,Ref}{\vdash \mathtt{1}\times\epsilon=\epsilon\rightarrow\epsilon=\epsilon}\,\rightarrow R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(\mathtt{1}\times y)\mathtt{b}=\epsilon, \mathtt{1}\times y\mathtt{b}=(\mathtt{1}\times y)\mathtt{b}, \mathtt{1}\times y\mathtt{b}=\epsilon, \mathtt{1}\times y=\epsilon\rightarrow y=\epsilon\vdash y\mathtt{b}=\epsilon}\,F13}{(\mathtt{1}\times y)\mathtt{b}=\epsilon, \mathtt{1}\times y\mathtt{b}=(\mathtt{1}\times y)\mathtt{b}, \mathtt{1}\times y\mathtt{b}=\epsilon, \mathtt{1}\times y=\epsilon\rightarrow y=\epsilon\vdash y\mathtt{b}=\epsilon}\,Repl}{\mathtt{1}\times y\mathtt{b}=(\mathtt{1}\times y)\mathtt{b}, \mathtt{1}\times y\mathtt{b}=\epsilon, \mathtt{1}\times y=\epsilon\rightarrow y=\epsilon\vdash y\mathtt{b}=\epsilon}\,F5}{\mathtt{1}\times y\mathtt{b}=\epsilon, \mathtt{1}\times y=\epsilon\rightarrow y=\epsilon\vdash \mathtt{1}\times y\mathtt{b}=\epsilon\rightarrow y\mathtt{b}=\epsilon}\,\rightarrow R}{\mathtt{1}\times y=\epsilon\rightarrow y=\epsilon\vdash \mathtt{1}\times y\mathtt{b}=\epsilon\rightarrow y\mathtt{b}=\epsilon}\,\Sigma^b_1\text{-}NIA$$

$$\dfrac{\vdash \mathtt{1}\times z=\epsilon\rightarrow z=\epsilon}{\vdash (\forall x)(\mathtt{1}\times x=\epsilon\rightarrow x=\epsilon)}\,\forall R$$

□

Proposition 8. *The rules* $\forall_\preceq R^*$, $\exists_\preceq L^*$, $\forall_\subseteq R^*$ *and* $\exists_\subseteq L^*$ *are derivable in* $\mathbf{G3cS}^1_3$.

Proof. The proofs are equivalent.

[9] So, if we are considering an (encoded) sub-sequence in which the last bit is **0**, then $\tilde{v}$ (which, as said, is the initial-substring $\tilde{v} = vb$, where b is either $\mathtt{0}$ or $\mathtt{1}$) roughly corresponds to the output of $h_0(\vec{x},y,v)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{z=\epsilon,\ldots\vdash\Gamma,A(z/x)}{1\times z=\epsilon,\ldots\vdash\Gamma,A(z/x)}\,L\,6}{1^z\subseteq\epsilon,\ldots\vdash\Gamma,A(z/x)}\,F7a}{1\times\epsilon=\epsilon,z\preceq\epsilon,\Gamma\vdash\Delta,A(z/x)}\,Repl}{\dfrac{z\preceq\epsilon,\Gamma\vdash\Delta,A(z/x)}{\Gamma\vdash\Delta,(\forall x\preceq\epsilon)A}\,\forall_{\preceq}R}\,F4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{z=\epsilon,A(z/x),\ldots\vdash\Delta}{1\times z=\epsilon,A(z/x),\ldots\vdash\Delta}\,L\,6}{1^z\subseteq\epsilon,A(z/x),\ldots\vdash\Delta}\,F7a}{1\times\epsilon=\epsilon,z\preceq\epsilon,A(z/x)\Gamma\vdash\Delta}\,Repl}{\dfrac{z\preceq\epsilon,A(z/\epsilon),\Gamma\vdash\Delta}{(\exists x\preceq\epsilon)A,\Gamma\vdash\Delta}\,\exists_{\preceq}L}\,F4$$

$$\dfrac{\dfrac{z=\epsilon,z\subseteq\epsilon,\Gamma\vdash\Delta,A(z/x)}{z\subseteq\epsilon,\Gamma\vdash\Delta,A(z/x)}\,F7a}{\Gamma\vdash\Delta,(\forall x\subseteq\epsilon)A}\,\forall_{\subseteq}R \qquad \dfrac{\dfrac{z=\epsilon,z\subseteq\epsilon,\Gamma\vdash\Delta,A(z/x)}{z\subseteq\epsilon,A(z/x),\Gamma\vdash\Delta}\,F7a}{(\exists x\subseteq\epsilon)A,\Gamma\vdash\Delta}\,\exists_{\subseteq}L$$

□

Lemma 7. *The following rule is derivable in* $\mathbf{G3cS}^1_3$,

$$\dfrac{x=\epsilon\vdash A(\epsilon/x)\qquad x=\mathtt{b}\vdash A(\mathtt{b}/x)}{\vdash(\forall x\subseteq\mathtt{b})A}\,\forall_{\subseteq}R^{**}$$

Proof. Indeed,

$$\dfrac{\dfrac{x_1=\epsilon,x_1\subseteq\epsilon,x_1\subseteq\mathtt{b}\vdash A(x_1/\mathtt{b})}{x_1\subseteq\epsilon,x_1\subseteq\mathtt{b}\vdash A(x_1/\mathtt{b})}\,F7a\qquad x_1=\mathtt{b},x_1\subseteq\mathtt{b}\vdash A(x_1/\mathtt{b})}{\dfrac{x_1\subseteq\mathtt{b}\vdash A(x_1/\mathtt{b})}{\vdash(\forall x\subseteq\mathtt{b})A}\,\forall_{\subseteq}R^{*}}\,F8a$$

□

Lemma 8. $RS^1_3\vdash(\forall x)(\forall y)(\forall z)\big((xy)z=x(yz)\big)$.

Proof. Indeed,

$$\mathcal{D}_{\epsilon}$$

$$\dfrac{\dfrac{}{(x_1x_2)\epsilon=x_1(x_2\epsilon),x_2\epsilon=x_2,(x_1x_2)\epsilon=x_1x_2\vdash(x_1x_2)\epsilon=x_1(x_2\epsilon)}\,Ax}{x_2\epsilon=x_2,(x_1x_2)\epsilon=x_1x_2\vdash(x_1x_2)\epsilon=x_1(x_2\epsilon)}\,Repl}{\vdash(x_1y_1)\epsilon=x_1(y_1\epsilon)}\,F1\mathrm{s}$$

$$\mathcal{D}_{ind}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b}))\ldots\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b}))}\,Ax}{(y_1z_2)\mathtt{b}=y_1(z_2\mathtt{b}),(x_1y_1)(z_2\mathtt{b})=x_1((y_1z_2)\mathtt{b})\ldots\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b}))}\,Repl}{(x_1y_1)(z_2\mathtt{b})=x_1((y_1z_2)\mathtt{b})\ldots\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b})}\,F2}{(x_1(y_1z_2))\mathtt{b}=x_1((y_1z_2)\mathtt{b}),(x_1y_1)(z_2\mathtt{b})=(x_1(y_1z_2))\mathtt{b}\ldots\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b}))}\,Repl}{(x_1y_1)(z_2\mathtt{b})=(x_1(y_1z_2))\mathtt{b}\ldots\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b})}\,F2}{(x_1y_1)z_2=x_1(y_1z_2),(x_1y_1)(z_2\mathtt{b})=((x_1y_1)z_2)\mathtt{b}\ldots\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b}))}\,Repl}{(x_1y_1)(z_2\mathtt{b})=((x_1y_1)z_2)\mathtt{b},(\forall x)(\forall y)((xy)z_2=x(yz_2))\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b}))}\,\forall L}{(\forall x)(\forall y)((xy)z_2=x(yz_2))\vdash(x_1y_1)(z_2\mathtt{b})=x_1(y_1(z_2\mathtt{b}))}\,F2$$

We conclude as follows:

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_{\epsilon} & \mathcal{D}_{ind}\\ \vdash (x_1y_1)\epsilon = x_1(y_1\epsilon) & (\forall x)(\forall y)\big((xy)z_2 = x(yx_2)\big) \vdash (x_1y_1)z_2\mathtt{b} = x_1(y_1(x_2\mathtt{b}))\end{array}}{\vdash (x_1y_1)z_1 = x_1(y_1z_1)}\,\Sigma^b_1\text{-}NIA}{\vdash (\forall x)(\forall y)(\forall z)\big((xy)z = x(yz)\big)}\,\forall R\mathrm{s}$$

□

Lemma 9. $RS^1_3 \vdash (\forall x)(\epsilon x = x)$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{}{\epsilon\epsilon = \epsilon \vdash \epsilon\epsilon = \epsilon}\,Ax}{\vdash \epsilon\epsilon = \epsilon}\,F1 \qquad \dfrac{\dfrac{\dfrac{}{\epsilon(y_1\mathtt{b}) = y_1\mathtt{b}, \epsilon(y_1\mathtt{b}) = (\epsilon y_1)\mathtt{b}, \epsilon y_1 = y_1 \vdash \epsilon(y_1\mathtt{b}) = y_1\mathtt{b}}\,Ax}{\epsilon(y_1\mathtt{b}) = (\epsilon y_1)\mathtt{b}, \epsilon y_1 = y_1 \vdash \epsilon(y_1\mathtt{b}) = y_1\mathtt{b}}\,Repl}{\epsilon y_1 = y_1 \vdash \epsilon(y_1\mathtt{b}) = y_1\mathtt{b}}\,F2}{\vdash \epsilon y = y}\,\Sigma^b_1\text{-}NIA}{\vdash (\forall x)(\epsilon x = x)}\,\forall R$$

□

Lemma 10. $RS^1_3 \vdash (\forall x)\big(\neg(x\mathtt{b} = x)\big)$.

Proof. Assume $\mathtt{b} = \mathtt{0}$,

$$\dfrac{\dfrac{\dfrac{}{\epsilon\mathtt{0} = \mathtt{0} \vdash}\,F13}{\vdash \neg(\epsilon\mathtt{0} = \mathtt{0})}\,\neg R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{y_1\mathtt{0} = y_1, (y_1\mathtt{0})\mathtt{0} = y_1\mathtt{0} \vdash y_1\mathtt{0} = y_1}\,Ax}{(y_1\mathtt{0})\mathtt{0} = y_1\mathtt{0} \vdash y_1\mathtt{0} = y_1}\,F10}{(y_1\mathtt{0})\mathtt{0} = y_1\mathtt{0}, \neg(y_1\mathtt{0} = y_1) \vdash}\,\neg L}{\neg(y_1\mathtt{0} = y_1) \vdash \neg((y_1\mathtt{0})\mathtt{0} = y_1\mathtt{0})}\,\neg R \qquad \dfrac{\dfrac{}{(y_1\mathtt{1})\mathtt{0} = y_1\mathtt{1}, \neg(y_1\mathtt{0} = y_1) \vdash}\,F12}{\neg(y_1\mathtt{0} = y_1) \vdash \neg((y_1\mathtt{1})\mathtt{0} = y_1\mathtt{1})}\,\neg R}{\vdash \neg(y\mathtt{0} = y)}\,\Sigma^b_1\text{-}NIA}{\vdash (\forall x)\neg(x\mathtt{0} = x)}\,\forall R$$

Assume $\mathtt{b} = \mathtt{1}$,

$$\dfrac{\dfrac{\dfrac{}{\epsilon\mathtt{1} = \epsilon \vdash}\,F14}{\vdash \neg(\epsilon\mathtt{1} = \epsilon)}\,\neg R \qquad \dfrac{\dfrac{}{(y_1\mathtt{0})\mathtt{1} = y_1\mathtt{0}, \neg(y_1\mathtt{1} = y_1) \vdash}\,F12}{\neg(y_1\mathtt{1} = y_1) \vdash \neg((y_1\mathtt{0})\mathtt{1} = y_1\mathtt{0})}\,\neg R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{y_1\mathtt{1} = y_1, (y_1\mathtt{1})\mathtt{1} = y_1\mathtt{1} \vdash y_1\mathtt{1} = y_1}\,Ax}{(y_1\mathtt{1})\mathtt{1} = y_1\mathtt{1} \vdash y_1\mathtt{1} = y_1}\,F10}{(y_1\mathtt{1})\mathtt{1} = y_1\mathtt{1}, \neg(y_1\mathtt{1} = y_1) \vdash}\,\neg L}{\neg(y_1\mathtt{1} = y_1) \vdash \neg((y_1\mathtt{1})\mathtt{1} = y_1\mathtt{1})}\,\neg R}{\vdash \neg(y\mathtt{1} = y)}\,\Sigma^b_1\text{-}NIA}{\vdash (\forall x)\neg(x\mathtt{1} = x)}\,\forall R$$

□

Lemma 11. $RS^1_3 \vdash (\forall x)(\epsilon \times x = \epsilon)$.

Proof. Indeed,

$$\dfrac{\dfrac{}{\epsilon\times\epsilon=\epsilon\vdash\epsilon\times\epsilon=\epsilon}\,Ax}{\vdash\epsilon\times\epsilon=\epsilon}\,F4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon\times(y_1\mathtt{b})=\epsilon,\epsilon\epsilon=\epsilon,\epsilon\times(y_1\mathtt{b}),\epsilon\times(y_1\mathtt{b})=(\epsilon\times y_1)\epsilon,\epsilon\times y_1=\epsilon\vdash\epsilon\times(y_1\mathtt{b})=\epsilon}\,Repl}{\epsilon\epsilon=\epsilon,\epsilon\times(y_1\mathtt{b})=\epsilon\epsilon,\epsilon\times(y_1\mathtt{b})=(\epsilon\times y_1)\epsilon,\epsilon\times y_1=\epsilon\vdash\epsilon\times(y_1\mathtt{b})=\epsilon}\,F1}{\epsilon\times(y_1\mathtt{b})=\epsilon\epsilon,\epsilon\times(y_1\mathtt{b})=(\epsilon\times y_1)\epsilon,\epsilon\times y_1=\epsilon\vdash\epsilon\times(y_1\mathtt{b})=\epsilon}\,Repl}{\epsilon\times(y_1\mathtt{b})=(\epsilon\times y_1)\epsilon,\epsilon\times y_1=\epsilon\vdash\epsilon\times(y_1\mathtt{b})=\epsilon}\,F5}{\epsilon\times y_1=\epsilon\vdash\epsilon\times(y_1\mathtt{b})=\epsilon}\,\Sigma^b_1\text{-}NIA$$
$$\dfrac{\vdash\epsilon\times y=\epsilon}{\vdash(\forall x)(\epsilon\times x=\epsilon)}\,\forall R$$

□

Lemma 12. $RS^1_3\vdash 1\times(\epsilon\epsilon)=1\times\epsilon$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{1\times(\epsilon\epsilon)=1\times\epsilon,\epsilon=\epsilon\epsilon,1\times\epsilon=1\times\epsilon\vdash 1\times(\epsilon\epsilon)=1\times\epsilon}\,Ax}{\epsilon=\epsilon\epsilon,1\times\epsilon=1\times\epsilon\vdash 1\times(\epsilon\epsilon)=1\times\epsilon}\,Repl}{1\times\epsilon=1\times\epsilon\vdash 1\times(\epsilon\epsilon)=1\times\epsilon}\,F1}{\vdash 1\times(\epsilon\epsilon)=1\times\epsilon}\,Ref$$

□

Lemma 13. $RS^1_3\vdash\epsilon\epsilon=\epsilon\times\epsilon 1$

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{}{\epsilon\epsilon=\epsilon\times\epsilon 1,\epsilon\times\epsilon 1=\epsilon,\epsilon\epsilon=\epsilon\vdash\epsilon\epsilon=\epsilon\times\epsilon 1}\,Ax}{\epsilon\times\epsilon 1=\epsilon,\epsilon\epsilon=\epsilon\vdash\epsilon\epsilon=\epsilon\times\epsilon 1}\,Repl}{\epsilon\epsilon=\epsilon\vdash\epsilon\epsilon=\epsilon\times\epsilon 1}\,L\ 11}{\vdash\epsilon\epsilon=\epsilon\times\epsilon 1}\,F1$$

□

Corollary 8. $RS^1_3\vdash 1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1)$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1),\epsilon\times\epsilon 1=\epsilon,1\times\epsilon\epsilon=1\times\epsilon,\epsilon\epsilon=\epsilon,1\times\epsilon=1\times\epsilon\vdash 1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1)}\,Ax}{\epsilon\times\epsilon 1=\epsilon,1\times\epsilon\epsilon=1\times\epsilon,\epsilon\epsilon=\epsilon,1\times\epsilon=1\times\epsilon\vdash 1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1)}\,Repl}{1\times\epsilon\epsilon=1\times\epsilon,\epsilon\epsilon=\epsilon,1\times\epsilon=1\times\epsilon\vdash 1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1)}\,L\ 11}{\epsilon\epsilon=\epsilon,1\times\epsilon=1\times\epsilon\vdash 1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1)}\,Repl}{1\times\epsilon=1\times\epsilon\vdash 1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1)}\,F1}{\vdash 1\times\epsilon\epsilon=1\times(\epsilon\times\epsilon 1)}\,Ref$$

□

Lemma 14. $RS^1_3\vdash(\forall x)(1\times x=\epsilon\to x=\epsilon)$

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{}{\epsilon=\epsilon,1\times\epsilon=\epsilon\vdash\epsilon=\epsilon}\,Ax}{\dfrac{1\times\epsilon=\epsilon\vdash\epsilon=\epsilon}{\vdash 1\times\epsilon=\epsilon\to\epsilon=\epsilon}\to R}\,Ref \qquad \dfrac{\dfrac{\dfrac{}{(1\times y_1)1=\epsilon,1\times(y_1\mathtt{b})=(1\times y_1)1,1\times(y_1\mathtt{b})=\epsilon,1\times y_1=\epsilon\to y_1=\epsilon\vdash(y_1\mathtt{b})=\epsilon}F13}{1\times(y_1\mathtt{b})=(1\times y_1)1,1\times(y_1\mathtt{b})=\epsilon,1\times y_1=\epsilon\to y_1=\epsilon\vdash(y_1\mathtt{b})=\epsilon}\,Repl}{1\times(y_1\mathtt{b})=\epsilon,1\times y_1=\epsilon\to y_1=\epsilon\vdash(y_1\mathtt{b})=\epsilon}\,F5}{1\times y_1=\epsilon\to y_1=\epsilon\vdash 1\times(y_1\mathtt{b})=\epsilon\to(y_1\mathtt{b})=\epsilon}\to R}{\vdash 1\times\epsilon=\epsilon\to\epsilon=\epsilon}\,\Sigma^b_1\text{-}NIA}{\vdash 1\times y=\epsilon\to y=\epsilon}\,\forall R}{\vdash(\forall x)(1\times x=\epsilon\to x=\epsilon)}\,\forall R$$

□

Lemma 15. $RS^1_3\vdash(\forall x)(\forall y)(x=y\to 1\times x=1\times y)$

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1\times x_1=1\times y_1,1\times x_1=1\times x_1,x_1=y_1\vdash 1\times x_1=1\times y_1}\,Ax}{1\times x_1=1\times x_1,x_1=y_1\vdash 1\times x_1=1\times y_1}\,Repl}{x_1=y_1\vdash 1\times x_1=1\times y_1}\,Ref}{\vdash x_1=y_1\to 1\times x_1=1\times y_1}\to R}{\vdash(\forall x)(\forall y)(x=y\to 1\times x=1\times y)}\,\forall Rs$$

□

Lemma 16. $RS^1_3\vdash(\forall x)(xx=\epsilon\to x=\epsilon)$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{}{\epsilon=\epsilon,\epsilon\epsilon=\epsilon\vdash\epsilon=\epsilon}\,Ax}{\dfrac{\epsilon\epsilon=\epsilon\vdash\epsilon=\epsilon}{\vdash\epsilon=\epsilon\to\epsilon=\epsilon}\to R}\,Ref \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{((y\mathtt{b})y)\mathtt{b}=\epsilon,(y\mathtt{b})(y\mathtt{b})=((y\mathtt{b})y)\mathtt{b}...\vdash y\mathtt{b}=\epsilon}F13}{(y\mathtt{b})(y\mathtt{b})=((y\mathtt{b})y)\mathtt{b},(y\mathtt{b})(y\mathtt{b})=\epsilon,yy=\epsilon\to y=\epsilon\vdash y\mathtt{b}=\epsilon}\,Repl}{(y\mathtt{b})(y\mathtt{b})=\epsilon,yy=\epsilon\to y=\epsilon\vdash y\mathtt{b}=\epsilon}\,F2}{yy=\epsilon\to y=\epsilon\vdash(y\mathtt{b})(y\mathtt{b})=\epsilon\to y\mathtt{b}=\epsilon}\to R}{\vdash yy=\epsilon\to y=\epsilon}\,\Sigma^b_1\text{-}NIA}{\vdash(\forall x)(xx=\epsilon\to x=\epsilon)}\,\forall R$$

□

Corollary 9. $RS^1_3\vdash(\forall x)\big((xx=\epsilon\times\epsilon 1)\to x=\epsilon\big)$.

Proof. By putting together Lemma 16 and Lemma 11,

$$\dfrac{\dfrac{\dfrac{}{\epsilon=\epsilon,\epsilon\epsilon=\epsilon\times\epsilon 1\vdash\epsilon=\epsilon}\,Ax}{\dfrac{\epsilon\epsilon=\epsilon\times\epsilon 1\vdash\epsilon=\epsilon}{\vdash\epsilon\epsilon=\epsilon\times\epsilon 1\to\epsilon=\epsilon}\to R}\,Ref \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((y_1\mathtt{b})y_1)\mathtt{b}=\epsilon,(y_1\mathtt{b})(y_1\mathtt{b})=((y_1\mathtt{b})y_1)\mathtt{b}...\vdash y_1=\epsilon}F13}{(y_1\mathtt{b})(y_1\mathtt{b})=((y_1\mathtt{b})y_1)\mathtt{b},(y_1\mathtt{b})(y_1\mathtt{b})=\epsilon...\vdash y_1=\epsilon}\,Repl}{(y_1\mathtt{b})(y_1\mathtt{b})=\epsilon,\epsilon\times\epsilon 1,(y_1\mathtt{b})(y_1\mathtt{b})=\epsilon\times\epsilon 1...\vdash y_1=\epsilon}\,F2}{\epsilon\times\epsilon 1=\epsilon,(y_1\mathtt{b})(y_1\mathtt{b})=\epsilon\times\epsilon 1...\vdash y_1=\epsilon}\,Repl}{(y_1\mathtt{b})(y_1\mathtt{b})=\epsilon\times\epsilon 1,y_1y_1=\epsilon\times\epsilon 1\to y_1=\epsilon\vdash y_1=\epsilon}\,L\ 11}{y_1y_1=\epsilon\times\epsilon 1\to y_1=\epsilon\vdash(y_1\mathtt{b})(y_1\mathtt{b})=\epsilon\times\epsilon 1\to y_1=\epsilon}\to R}{\vdash yy=\epsilon\times\epsilon 1\to y=\epsilon}\,\Sigma^b_1\text{-}NIA}{\vdash(\forall x)((xx=\epsilon\times\epsilon 1)\to x=\epsilon)}\,\forall R$$

□

Corollary 10. $RS^1_3 \vdash (\forall x)\big(\big((1 \times (xx) = 1 \times (\epsilon \times \epsilon 1)\big) \to x = \epsilon\big)$.

Proof. By Lemma 9 and Lemma 15. □

More in general,

Lemma 17. $RS^1_3 \vdash (\forall x)(\forall y)\neg(x \mathbin{\mathtt{b}} y = \epsilon)$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x_1\mathtt{b} = \epsilon, x_1\mathtt{b}\epsilon = x_1\mathtt{b}, x_1\mathtt{b}\epsilon = \epsilon \vdash}\,F13}{x_1\mathtt{b}\epsilon = x_1\mathtt{b}, x_1\mathtt{b}\epsilon = \epsilon \vdash}\,Repl}{x_1\mathtt{b}\epsilon = \epsilon \vdash}\,F1}{\vdash \neg x_1\mathtt{b}\epsilon = \epsilon}\,\neg R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(x_1\mathtt{b}y_1)\mathtt{b} = \epsilon, x_1\mathtt{b}(y_1\mathtt{b}) = (x_1\mathtt{b}y_1)\mathtt{b}, x_1\mathtt{b}(y_1\mathtt{b}) = \epsilon, (\forall x)\neg(x\mathtt{b}y_1 = \epsilon) \vdash}\,F13}{x_1\mathtt{b}(y_1\mathtt{b}) = (x_1\mathtt{b}y_1)\mathtt{b}, x_1\mathtt{b}(y_1\mathtt{b}) = \epsilon, (\forall x)\neg(x\mathtt{b}y_1 = \epsilon) \vdash}\,Repl}{x_1\mathtt{b}(y_1\mathtt{b}) = \epsilon, (\forall x)\neg(x\mathtt{b}y_1 = \epsilon) \vdash}\,L\ 8}{(\forall x)\neg(x\mathtt{b}y_1 = \epsilon) \vdash \neg(x_1\mathtt{b}(y_1\mathtt{b}) = \epsilon)}\,\neg R}{\vdash \neg(x_1\mathtt{b}y_1 = \epsilon)}\,\Sigma^b_1\text{-}NIA}{\vdash (\forall x)(\forall y)\neg(x\mathtt{b}y = \epsilon)}\,\forall R\mathrm{s}$$

□

Lemma 18. $RS^1_3 \vdash (\forall x)(\forall y)(xy = \epsilon \to x = \epsilon \wedge y = \epsilon)$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon = \epsilon, \epsilon y_1 = \epsilon \vdash \epsilon = \epsilon}\,Ax}{\epsilon y_1 = \epsilon \vdash \epsilon = \epsilon}\,Ref \qquad \dfrac{\dfrac{}{y_1 = \epsilon\ldots \vdash y_1 = \epsilon}\,Ax}{\epsilon y_1 = y_1, \epsilon y_1 = \epsilon \vdash y_1 = \epsilon}\,Repl}{\epsilon y_1 = \epsilon \vdash y_1 = \epsilon}\,L\ 9}{\epsilon y_1 = \epsilon \vdash \epsilon = \epsilon \wedge y_1 = \epsilon}\,\wedge R}{\vdash \epsilon y_1 = \epsilon \to \epsilon = \epsilon \wedge y_1 = \epsilon}\,\to R \qquad \dfrac{\dfrac{\dfrac{}{(x_1\mathtt{b})y_1 = \epsilon, (\forall y)(x_1 y = \epsilon \to x_1 = \epsilon \wedge y = \epsilon) \vdash x_1\mathtt{b} = \epsilon \wedge y_1 = \epsilon}\,L\ 17}{(\forall y)(x_1 y = \epsilon \to x_1 = \epsilon \wedge y = \epsilon) \vdash (x_1\mathtt{b})y_1 = \epsilon \to x_1\mathtt{b} = \epsilon \wedge y_1 = \epsilon}\,\to R}{\vdash x_1 y_1 = \epsilon \to x_1 = \epsilon \wedge y_1 = \epsilon}\,\Sigma^b_1\text{-}NIA}{\vdash (\forall x)(\forall y)(xy = \epsilon \to x = \epsilon \wedge y = \epsilon)}\,\forall R\mathrm{s}$$

□

Lemma 19. *The following rule is derivable in* $\mathbf{G3cS}^1_3$,

$$\dfrac{y = \epsilon \vdash A(y/x)}{\vdash (\forall x)(xx \preceq \epsilon \to A)}\ R\ 19$$

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y = \epsilon\ldots \vdash A(y/x)}{yy = \epsilon, 1 \times yy = \epsilon\ldots \vdash A(y/x)}\,L\ 16}{1 \times yy = \epsilon, 1 \times yy \subseteq \epsilon\ldots \vdash A(y/x)}\,L\ 14}{1 \times yy \subseteq \epsilon\ldots \vdash A(y/x)}\,F7a}{1 \times \epsilon = \epsilon, 1 \times yy \subseteq 1 \times \epsilon \vdash A(y/x)}\,Repl}{\vdash yy \preceq \epsilon \to A(y/x)}\,F4}{\vdash (\forall x)(xx \preceq \epsilon \to A)}\,\forall R$$

□

Lemma 20. $RS^1_3 \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1$.

Proof. Indeed,

$$
\begin{array}{cl}
\hline
1 \times 00 = 1 \times (\epsilon 0 \times (\epsilon 0)1) \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Ax \\ \hline
1 \times (\epsilon 0 \times (\epsilon 0)1) = 11, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
\epsilon 11 = 11, 1 \times (\epsilon 0 \times (\epsilon 0)1) = \epsilon 11, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
1 \times (\epsilon 0 \times (\epsilon 0)1) = \epsilon 11, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & L\ 9 \\ \hline
1 \times \epsilon = \epsilon, 1 \times (\epsilon 0 \times (\epsilon 0)1) = (1 \times \epsilon)11, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
1 \times (\epsilon 0 \times (\epsilon 0)1) = (1 \times \epsilon)11, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F4 \\ \hline
(1 \times 0)1 = (1 \times \epsilon)11, 1 \times (\epsilon 0 \times (\epsilon 0)1) = (1 \times 0)1, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
1 \times (\epsilon 0 \times (\epsilon 0)1) = (1 \times 0)1, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F5 \\ \hline
1 \times (00) = (1 \times 0)1, 1 \times (\epsilon 0 \times (\epsilon 0)1) = 1 \times (00), 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
1 \times (\epsilon 0 \times (\epsilon 0)1) = 1 \times (00), 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F5 \\ \hline
1 \times (\epsilon 0 \times (\epsilon 0)1) = 1 \times (\epsilon 0 \times (\epsilon 0)1), \epsilon 0 \times (\epsilon 0)1 = 00, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
\epsilon 0 \times (\epsilon 0)1 = 00, 1 \times 00 = 11 \ldots \vdash 0 \equiv \epsilon 0 \times (\epsilon 0)1 & Ref \\ \hline
\epsilon 00 = 00, \epsilon 0 \times (\epsilon 0)1 = \epsilon 00, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
\epsilon 0 \times (\epsilon 0)1 = \epsilon 00, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & L\ 9 \\ \hline
0 \times \epsilon = \epsilon, \epsilon 0 \times (\epsilon 0)1 = (0 \times \epsilon)00, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
\epsilon 0 \times (\epsilon 0)1 = (0 \times \epsilon)00, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F4 \\ \hline
0 \times 0 = (0 \times \epsilon)0, \epsilon 0 \times (\epsilon 0)1 = (0 \times 0)0, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
\epsilon 0 \times (\epsilon 0)1 = (0 \times 0)0, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F5 \\ \hline
\epsilon 0 = 0, \epsilon 0 \times (\epsilon 0)1 = (\epsilon 0 \times \epsilon 0)\epsilon 0, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
\epsilon 0 \times (\epsilon 0)1 = (\epsilon 0 \times \epsilon 0)\epsilon 0, 1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & L\ 9 \\ \hline
1 \times 00 = 11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F4 \\ \hline
1 \times \epsilon = \epsilon, 1 \times 00 = (1 \times \epsilon)11 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
1 \times 00 = ((1 \times \epsilon)1)1 \ldots \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F4 \\ \hline
1 \times 00 = (1 \times 0)1, 1 \times 0 = (1 \times \epsilon)1 \vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & Repl \\ \hline
\vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1 & F5\mathrm{s}
\end{array}
$$

□

Lemma 21. *The following rule*

$$
\frac{}{1 \times 01 = 1 \times \epsilon 0 \vdash}\ R\ 21
$$

is derivable in $\mathbf{G3cS}^1_3$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon\mathtt{1}=\epsilon,\epsilon\mathtt{11}=\epsilon\mathtt{1},\mathtt{1}\times\epsilon=\epsilon,(\mathtt{1}\times\epsilon)\mathtt{11}=(\mathtt{1}\times\epsilon\mathtt{1})...\vdash}F13}{\epsilon\mathtt{11}=\epsilon\mathtt{1},\mathtt{1}\times\epsilon=\epsilon,(\mathtt{1}\times\epsilon)\mathtt{11}=(\mathtt{1}\times\epsilon\mathtt{1})...\vdash}F10}{\mathtt{1}\times\epsilon=\epsilon,(\mathtt{1}\times\epsilon)\mathtt{11}=(\mathtt{1}\times\epsilon)\mathtt{1}...\vdash}Repl\mathrm{s}}{(\mathtt{1}\times\epsilon)\mathtt{11}=(\mathtt{1}\times\epsilon)\mathtt{1},\mathtt{1}\times\mathtt{01}=\mathtt{1}\times\epsilon\mathtt{0}...\vdash}F4}{\mathtt{1}\times\epsilon\mathtt{0}=(\mathtt{1}\times\epsilon)\mathtt{1},\mathtt{1}\times\mathtt{01}=(\mathtt{1}\times\epsilon)\mathtt{11},\mathtt{1}\times\mathtt{01}=\mathtt{1}\times\epsilon\mathtt{0}...\vdash}Repl\mathrm{s}}{\mathtt{1}\times\mathtt{01}=(\mathtt{1}\times\epsilon)\mathtt{11},\mathtt{1}\times\mathtt{01}=\mathtt{1}\times\epsilon\mathtt{0}...\vdash}F5}{\mathtt{1}\times\mathtt{0}=(\mathtt{1}\times\epsilon)\mathtt{1},\mathtt{1}\times\mathtt{01}=(\mathtt{1}\times\mathtt{0})\mathtt{1},\mathtt{1}\times\mathtt{01}=\mathtt{1}\times\epsilon\mathtt{0}\vdash}Repl}{\mathtt{1}\times\mathtt{01}=(\mathtt{1}\times\mathtt{0})\mathtt{1},\mathtt{1}\times\mathtt{01}=\mathtt{1}\times\epsilon\mathtt{0}\vdash}F5}{\mathtt{1}\times\mathtt{01}=\mathtt{1}\times\epsilon\mathtt{0}\vdash}F5$$

□

Bounded Iteration. Auxiliary Formulas are defined as follows:

$$\begin{aligned}
F_{o1}(x,y) &:= (\exists u)\big((u\mathtt{0}=x\wedge y=x)\vee u\mathtt{1}=x\wedge y=\mathtt{1}u\big)\\
F_{o2}(x) &:= (\exists u)(\exists v)(u=\mathtt{1}\times u\wedge v=\mathtt{0}\times v\wedge x=uv)\\
F_{o3}(x,z) &:= zz\equiv x\times x\mathtt{1}\wedge(\forall x'\subseteq x)(\exists z'\subseteq z)\big(z'z'\equiv x'\times x'\mathtt{1}\wedge\\
&\qquad(\forall z'')(\forall y')\big(y'\equiv x'\wedge z''y'=z'\to(\exists y)\big(z'y\subseteq z\wedge\\
&\qquad\big((x'\mathtt{0}\subseteq x\to F_{o1}(y'\mathtt{0},y)\big)\wedge(x'\mathtt{1}\subseteq x\to F_{o1}(y'\mathtt{1},y))\big)\big)\big)\big)\big),
\end{aligned}$$

Lemma 22. $RS^1_3\vdash F_{o3}(\epsilon,\epsilon)$.

Proof. For readability's sake, formulas are abbreviated in the obvious way.

$$\mathcal{D}_{o3\wedge}$$

$$\dfrac{\dfrac{\dfrac{}{\epsilon\mathtt{0}=\epsilon,\epsilon\mathtt{0}\subseteq\epsilon...\vdash F_{o1}(y_1\mathtt{0},\epsilon)}F13}{\epsilon\mathtt{0}\subseteq\epsilon,z_1=\epsilon,y_1=\epsilon...\vdash F_{o1}(y_1\mathtt{0},\epsilon)}F7a}{z_1=\epsilon,y_1=\epsilon...\vdash F_{o3\exists},\epsilon\mathtt{0}\subseteq\epsilon\to F_{o1}(y_1\mathtt{0},\epsilon)}\to R \qquad \dfrac{\dfrac{\dfrac{}{\epsilon\mathtt{1}=\epsilon,\epsilon\mathtt{1}\subseteq\epsilon...\vdash F_{o1}(\epsilon\mathtt{1},y_1)}F13}{\epsilon\mathtt{1}\subseteq\epsilon,z_1=\epsilon,y_1=\epsilon...\vdash Fo3\exists,F_{o1}(\epsilon\mathtt{1},y_1)}F7a}{z_1=\epsilon,y_1=\epsilon...\vdash F_{o3\exists},\epsilon\mathtt{1}\subseteq\epsilon\to F_{o1}(\epsilon\mathtt{1},y_1)}\to R$$
$$\dfrac{\qquad\qquad\qquad\qquad}{z_1=\epsilon,y_1=\epsilon...\vdash F_{o3\exists},(\epsilon\mathtt{0}\subseteq\epsilon\to F_{o1}(y_1\mathtt{0},\epsilon))\wedge(\epsilon\mathtt{1}\subseteq\epsilon\to F_{o1}(\epsilon\mathtt{1},y_1))}\wedge R$$

$$\mathcal{D}_{o3\forall}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon\epsilon\subseteq\epsilon...\vdash F_{o3\exists},\epsilon\epsilon\subseteq\epsilon}Ax}{\epsilon\epsilon=\epsilon...\vdash F_{o3\exists},\epsilon\epsilon\subseteq\epsilon}F7b}{z_1=\epsilon,y_1=\epsilon...\vdash F_{o3\exists},\epsilon\epsilon\subseteq\epsilon}F1 \qquad \mathcal{D}_{o3\wedge}}{z_1=\epsilon,y_1=\epsilon...\vdash F_{o3\exists},\epsilon\epsilon\subseteq\epsilon\wedge((\epsilon\mathtt{0}\subseteq\epsilon\to F_{o1}(y_1\mathtt{0},\epsilon))\wedge(\epsilon\mathtt{1}\subseteq\epsilon\to F_{o1}(\epsilon\mathtt{1},y_1)))}\wedge R}{z_1=\epsilon,y_1=\epsilon...\vdash F_{o3\exists}}\exists R}{z_1\epsilon=z_1,z_1\epsilon=\epsilon,y_1=\epsilon,z_1y_1=\epsilon\vdash F_{o3\exists}}Repl}{z_1\epsilon=\epsilon,y_1=\epsilon,z_1y_1=\epsilon\vdash F_{o3\exists}}F1}{y_1=\epsilon,z_1y_1=\epsilon\vdash F_{o3\exists}}Repl}{y_1\equiv\epsilon,z_1y_1=\epsilon\vdash F_{o3\exists}}L\ 14}{y_1\equiv\epsilon\wedge z_1y_1=\epsilon\vdash F_{o3\exists}}\wedge L}{\vdash y_1\equiv\epsilon\wedge z_1y_1=\epsilon\to F_{o3\exists}}\to R}{\vdash(\forall z'')(\forall y')(y'\equiv\epsilon\wedge z''y'=\epsilon\to F_{o3\exists})}\forall R\mathrm{s}$$

So, we conclude as follows:

$$\dfrac{\dfrac{}{\vdash \epsilon\epsilon \equiv \epsilon \times \epsilon\mathsf{1}}\, C\ 8 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon \subseteq \epsilon, \epsilon = \epsilon \vdash \epsilon \subseteq \epsilon}\, Ax}{\epsilon = \epsilon \vdash \epsilon \subseteq \epsilon}\, F7b}{\vdash \epsilon \subseteq \epsilon}\, Ref \qquad \dfrac{\dfrac{}{\vdash \epsilon\epsilon \equiv \epsilon \times \epsilon\mathsf{1}}\, C\ 8 \qquad \mathcal{D}_{o3\forall} \vdash F_{o3\forall}}{\vdash \epsilon\epsilon \equiv \epsilon \times \epsilon\mathsf{1} \wedge F_{o3\forall}}\, \wedge R}{\vdash (\exists z' \subseteq \epsilon) z'z' \equiv \epsilon \times \epsilon\mathsf{1} \wedge F_{o3\forall}}\, \exists_{\subseteq} R}{\vdash (\forall x' \subseteq \epsilon)(\exists z' \subseteq \epsilon)(z'z' \equiv x' \times x'\mathsf{1} \wedge F_{o3\forall})}\, \forall_{\subseteq} R^*}{\vdash F_{o3}(\epsilon, \epsilon)}\, \wedge R$$

□

Lemma 23. $RS^1_3 \vdash (\forall x' \subseteq \epsilon)(\forall z')(((\epsilon = \epsilon \vee x'\mathsf{1} \equiv \epsilon) \wedge F_{o3}(x', z')) \to (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z'\epsilon y))$.

Proof. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon = \mathsf{1} \times \epsilon \ldots \vdash \ldots \epsilon = \mathsf{1} \times \epsilon}\, Ax}{z_1 = \epsilon \ldots \vdash \ldots \epsilon = \mathsf{1} \times \epsilon}\, F4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon = z_1\epsilon\epsilon, z_1 = \epsilon \ldots \vdash \epsilon = z_1\epsilon\epsilon}\, Ax}{z_1\epsilon\epsilon = z_1, z_1 = \epsilon \ldots \vdash \epsilon = z_1\epsilon\epsilon}\, Repl}{z_1\epsilon = z_1, \epsilon = \epsilon\epsilon, z_1 = \epsilon \ldots \vdash \ldots \epsilon = z_1\epsilon\epsilon}\, Repl}{\epsilon = \epsilon\epsilon, z_1 = \epsilon \ldots \vdash \ldots \epsilon = z_1\epsilon\epsilon}\, F1}{z_1 = \epsilon \ldots \vdash \ldots \epsilon = z_1\epsilon\epsilon}\, F1}{z_1 = \epsilon, z_1 z_1 \equiv \epsilon \times \epsilon\mathsf{1}, F_{o3\forall\exists}, \epsilon = \epsilon \vee \epsilon\mathsf{1} \equiv \epsilon \vdash \ldots \epsilon = \mathsf{1} \times \epsilon \wedge \epsilon = z_1\epsilon\epsilon}\, \wedge R}{z_1 = \epsilon, z_1 z_1 \equiv \epsilon \times \epsilon\mathsf{1}, F_{o3\forall\exists}, \epsilon = \epsilon \vee \epsilon\mathsf{1} \equiv \epsilon \vdash (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z_1\epsilon y)}\, \exists R}{z_1 z_1 \equiv \epsilon \times \epsilon\mathsf{1}, F_{o3\forall\exists}, \epsilon = \epsilon \vee \epsilon\mathsf{1} \equiv \epsilon \vdash (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z_1\epsilon y)}\, C\ 10}{\epsilon = \epsilon \vee \epsilon\mathsf{1} \equiv \epsilon, F_{o3}(\epsilon, z_1) \vdash (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z_1\epsilon y)}\, \wedge R}{(\epsilon = \epsilon \vee \epsilon\mathsf{1} \equiv \epsilon) \wedge F_{o3}(\epsilon, z_1) \vdash (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z_1\epsilon y)}\, \wedge L}{\vdash ((\epsilon = \epsilon \vee \epsilon\mathsf{1} \equiv \epsilon) \wedge F_{o3}(\epsilon, z_1)) \to (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z_1\epsilon y)}\, \to R}{\vdash (\forall z')(((\epsilon = \epsilon \vee \epsilon\mathsf{1} \equiv \epsilon) \wedge F_{o3}(\epsilon, z')) \to (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z'\epsilon y))}\, \forall R}{\vdash (\forall x' \subseteq \epsilon)(\forall z')(((\epsilon = \epsilon \vee x'\mathsf{1} \equiv \epsilon) \wedge F_{o3}(x', z')) \to (\exists y)(y = \mathsf{0} \times y \wedge \epsilon = z'\epsilon y))}\, \forall_{\subseteq} R^*$$

□

Lemma 24. $RS^1_3 \vdash F_{o3}(\epsilon\mathsf{0}, \mathsf{0})$.

Proof. Indeed,

$$\mathcal{D}_{o3.ii}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon\mathsf{0} = y_1\mathsf{0}\ldots \vdash \epsilon\mathsf{0} = y_1\mathsf{0}}\, Ax}{\epsilon\mathsf{0} = \epsilon\mathsf{0}, y_1 = \epsilon\ldots \vdash \ldots\epsilon\mathsf{0} = y_1\mathsf{0}}\, Repl}{y_1 = \epsilon, z_2 = \epsilon\ldots \vdash \ldots\epsilon\mathsf{0} = y_1\mathsf{0}}\, Ref \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{0} = y_1\mathsf{0}\ldots \vdash \mathsf{0} = y_1\mathsf{0}}\, Ax}{\mathsf{0} = \epsilon\mathsf{0}, y_1 = \epsilon\ldots \vdash \mathsf{0} = y_1\mathsf{0}}\, Repl}{\epsilon\mathsf{0} = \epsilon, \mathsf{0} = \mathsf{0}, y_1 = \epsilon\ldots \vdash \mathsf{0} = y_1\mathsf{0}}\, Repl}{\mathsf{0} = \mathsf{0}, y_1 = \epsilon\ldots \vdash \mathsf{0} = y_1\mathsf{0}}\, L\ 9}{y_1 = \epsilon, z_2 = \epsilon\ldots \vdash \mathsf{0} = y_1\mathsf{0}}\, Ref}{y_1 = \epsilon, z_2 = \epsilon\ldots \vdash F_{o1}(y_1\mathsf{0}, \mathsf{0}), \epsilon\mathsf{0} = y_1\mathsf{0} \wedge \mathsf{0} = y_1\mathsf{0}, \epsilon\mathsf{1} = y_1\mathsf{0} \wedge \mathsf{0} = \mathsf{1}\epsilon}\, \wedge R}{y_1 = \epsilon, z_2 = \epsilon\ldots \vdash F_{o1}(y_1\mathsf{0}, \mathsf{0}), (\epsilon\mathsf{0} = y_1\mathsf{0} \wedge \mathsf{0} = y_1\mathsf{0}) \vee (\epsilon\mathsf{1} = y_1\mathsf{0} \wedge \mathsf{0} = \mathsf{1}\epsilon)}\, \vee R}{y_1 = \epsilon, z_2 = \epsilon, \epsilon\mathsf{0} \subseteq \epsilon\ldots \vdash F_{o1}(y_1\mathsf{0}, \mathsf{0})}\, \exists R}{y_1 = \epsilon, z_2 = \epsilon\ldots \vdash \epsilon\mathsf{0} \subseteq \mathsf{0} \to F_{o1}(y_1\mathsf{0}, \mathsf{0})}\, \to R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon\mathsf{1} = \epsilon\ldots \vdash F_{o1}(y_1\mathsf{1}, \mathsf{0})}\, F13}{\epsilon\mathsf{1} \subseteq \epsilon\ldots \vdash F_{o1}(y_1\mathsf{1}, \mathsf{0})}\, F7a \qquad \dfrac{}{\epsilon\mathsf{1} = \mathsf{0}\ldots \vdash F_{o1}(y_1\mathsf{1}, \mathsf{0})}\, F12}{\epsilon\mathsf{1} \subseteq \mathsf{0}, y_1 = \epsilon\ldots \vdash F_{o1}(y_1\mathsf{1}, \mathsf{0})}\, F8a}{y_1 = \epsilon, z_2 = \epsilon\ldots \vdash \epsilon\mathsf{1} \subseteq \mathsf{0} \to F_{o1}(y_1\mathsf{1}, \mathsf{0})}\, \to R$$

$$y_1 = \epsilon, z_2 = \epsilon\ldots \vdash (\epsilon\mathsf{0} \subseteq \mathsf{0} \to F_{o1}(y_1\mathsf{0}, \mathsf{0})) \wedge (\epsilon\mathsf{1} \subseteq \mathsf{0} \to F_{o1}(y_1\mathsf{1}, \mathsf{0}))\ \wedge R$$

$\mathcal{D}_{o3.i}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon 0 \subseteq 0, \epsilon 0 = 0... \vdash 0 \subseteq 0}\,Ax}{\epsilon 0 = 0... \vdash 0 \subseteq 0}\,F8c}{\epsilon 0 = 0, 0 = 0... \vdash 0 \subseteq 0}\,Repl}{0 = 0, y_1 = \epsilon... \vdash \epsilon 0 \subseteq 0}\,L\ 9}{y_1 = \epsilon... \vdash \epsilon 0 \subseteq 0}\,Ref \qquad \mathcal{D}_{o3.ii}}{\dfrac{y_1 = \epsilon... \vdash \big((\epsilon 0 \subseteq 0 \to F_{o1}(y_1 0, 0)) \wedge (\epsilon 1 \subseteq 0 \to F_{o1}(y_1 1, 0))\big)}{y_1 = \epsilon, z_2 = \epsilon... \vdash \epsilon 0 \subseteq 0 \wedge \big((\epsilon 0 \subseteq \epsilon 0 \to F_{o1}(y_1 0, 0)) \wedge (\epsilon 1 \subseteq 0 \to F_{o1}(y_1 1, 0))\big)}\,\exists R}\,\wedge R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y_1 = \epsilon, z_2 = \epsilon... \vdash F_{o3\exists}}{y_1 = \epsilon, z_2\epsilon = \epsilon... \vdash ...F_{o3\exists}}\,L\ 18}{y_1 = \epsilon, 1 \times y_1 = \epsilon, z_2 y_1 = \epsilon... \vdash ...F_{o3\exists}}\,Repl}{1 \times y_1 = \epsilon, y_1 \equiv \epsilon, 1 \times \epsilon = \epsilon, z_2 y_1 = \epsilon}\,L\ 14}{1 \times \epsilon = \epsilon, y_1 \equiv \epsilon, z_2 y_1 = \epsilon \vdash ...F_{o3\exists}}\,Repl}{y_1 \equiv \epsilon, z_2 y_1 = \epsilon \vdash ...F_{o3\exists}}\,F4}{y_1 \equiv \epsilon \wedge z_2 y_1 = \epsilon \vdash ...F_{o3\exists}}\,\wedge R}{\vdash ...y_1 \equiv \epsilon \wedge z_2 y_1 = \epsilon \to F_{o3\exists}}\,\to R}{\vdash F_{o3\forall\forall}(\epsilon 0, 0, \epsilon, \epsilon)}\,\forall Rs$$

$$\dfrac{\dfrac{}{\vdash ...\epsilon\epsilon \equiv \epsilon \times \epsilon 1}\,C\ 8 \qquad \vdash F_{o3\forall\forall}(\epsilon 0, 0, \epsilon, \epsilon)}{\vdash ...\epsilon\epsilon \equiv \epsilon \times \epsilon 1 \wedge F_{o3\forall\forall}}\,\wedge R$$

$\mathcal{D}_{o3\forall}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon \subseteq 0... \vdash \epsilon \subseteq \epsilon 0}\,Ax}{\epsilon 0 = 0, \epsilon \subseteq \epsilon 0... \vdash ...\epsilon \subseteq 0}\,Repl}{\epsilon \subseteq \epsilon 0... \vdash ...\epsilon \subseteq 0}\,L\ 9}{\epsilon \subseteq \epsilon... \vdash ...\epsilon \subseteq 0}\,F8b}{\epsilon = \epsilon \vdash ...\epsilon \subseteq 0}\,F7b}{\vdash ...\epsilon \subseteq 0}\,Ref \qquad \mathcal{D}_{o3.i}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \subseteq 0, 0 = 0 \vdash ...0 \subseteq 0}\,Ax}{0 = 0 \vdash ...0 \subseteq 0}\,F8c}{\vdash 0 \subseteq 0}\,Ref \qquad \mathcal{D}_{o3.ii}}{\vdash ...00 \equiv 0 \times 01 \wedge F_{o3\forall\forall}}\,\exists_{\subseteq} R^*}{\vdash (\exists z' \subseteq 0)(z'z' \equiv 0 \times 01 \wedge F_{o3\forall\forall})}\,L\ 7$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash ...\epsilon\epsilon \equiv \epsilon \times \epsilon 1 \wedge F_{o3\forall\forall}}\,\exists_{\subseteq} R \qquad \vdash (\exists z' \subseteq 0)(z'z' \equiv \epsilon \times \epsilon 1 \wedge F_{o3\forall\forall})}{\vdash F_{o3\forall\exists}(\epsilon 0, 0)}\,L\ 7}{\vdash F_{o3\forall\exists}(\epsilon 0, 0)}$$

We conclude as follows:

$$\dfrac{\dfrac{}{\vdash 00 \equiv \epsilon 0 \times (\epsilon 0)1}\,L\ 20 \qquad \vdash F_{o3\forall\exists}(\epsilon 0, 0)}{\vdash F_{o3}(\epsilon 0, 0)}\,\wedge R$$

□

Definition 21. The formula $F_{ones}(\cdot, \cdot)$ is defined as follows:

$$F_{ones}(x, u) := u \times 1 \times u \wedge (\exists z)\big(F_{o3}(x, z) \wedge (\forall x' \subseteq x)(\forall z') \big(\big((x = \epsilon \vee x'1 \equiv x) \wedge F_{o3}(x', z')\big) \to (\exists y)(y = 0 \times y \wedge z = z'uy)\big)\big).$$

Proposition 9. *1.* $RS_3^1 \vdash F_{ones}(\epsilon, \epsilon)$

2. $RS_3^1 \vdash F_{ones}(x, y) \to F_{ones}(x0, y) \wedge F_{ones}(x1, y1)$

3. $RS_3^1 \vdash F_{ones}(x, y) \wedge F_{ones}(x, z) \to y = z.$

Proof of 1. For readability's sake, formulas are abbreviated in the obvious way.

$$\dfrac{\dfrac{}{\epsilon = 1 \times \epsilon \vdash \epsilon = 1 \times \epsilon}\,Ax}{\vdash \epsilon = 1 \times \epsilon}\,F4 \qquad \dfrac{\dfrac{\dfrac{}{\vdash F_{ones\exists}1, F_{o3}(\epsilon,\epsilon)}\,L\ 22 \qquad \dfrac{}{\vdash F_{ones\forall}(\epsilon,\epsilon,\epsilon)}\,L\ 23}{\vdash F_{ones\exists}1, F_{o3}(\epsilon,\epsilon) \wedge F_{ones\forall}}\,\wedge R}{\vdash F_{ones\exists}1}\,\exists R}{\vdash F_{ones}(\epsilon,\epsilon)}\,\wedge R$$

□

Proof Sketch of 2. Let us use $H^*(x,y)$ as an abbreviation for the formula $(F_{ones}(x,y) \rightarrow F_{ones}(x\mathtt{0},y) \wedge F_{ones}(x\mathtt{1},y\mathtt{1})$. The overall structure of the proof derivation is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_{\epsilon\mathtt{0}}}{F_{ones}(\epsilon,y_1) \vdash F_{ones}(\epsilon\mathtt{0},y_1)} \qquad \dfrac{\mathcal{D}_{\epsilon\mathtt{1}}}{F_{ones}(\epsilon,y_1) \vdash F_{ones}(\epsilon\mathtt{1},y_1\mathtt{1})}\,\wedge R}{F_{ones}(\epsilon,y_1) \vdash F_{ones}(\epsilon\mathtt{0},y_1) \wedge F_{ones}(\epsilon\mathtt{1},y_1\mathtt{1})}\,\rightarrow R}{\vdash F_{ones}(\epsilon,y_1) \rightarrow F_{ones}(\epsilon\mathtt{0},y_1) \wedge F_{ones}(\epsilon\mathtt{1},y_1\mathtt{1})} \qquad \dfrac{\mathcal{D}_{x\mathtt{b}}}{(\forall u)\big(F^*(x_1,u)\big) \vdash F^*(x_1\mathtt{b},y_1)}\,\Sigma_1^b\text{-}NIA}{\vdash F_{ones}(x_1,y_1) \rightarrow F_{ones}(x_1\mathtt{1},y_1) \wedge F_{ones}(x_1\mathtt{1},y_1\mathtt{1})}\,\forall R\mathtt{s}}{\vdash (\forall x)(\forall y)\big(F_{ones}(x,y) \rightarrow F_{ones}(x\mathtt{0},y) \wedge F_{ones}(x\mathtt{1},y\mathtt{1})\big)}\,\forall R\mathtt{s}$$

For space reason we will prove $\mathcal{D}_{\epsilon\mathtt{0}}$ only.

$\mathcal{D}_{\epsilon\mathtt{0}.iiiL}$

$$\dfrac{\dfrac{}{\epsilon\mathtt{0} = \epsilon\ldots \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = \epsilon y_1 y)}\,F13 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 = 0 \times 0\ldots \vdash \ldots 0 = 0 \times 0}\,Ax}{\epsilon\mathtt{0} = 0, 0 = 0 \times 0 = \epsilon\ldots \vdash \ldots 0 = 0 \times 0}\,Repl}{0 \times 0 = \epsilon\mathtt{0}\ldots \vdash \ldots 0 = 0 \times 0}\,L\ 9}{0 \times \epsilon = \epsilon, 0 \times 0 = (0 \times \epsilon)\mathtt{0}\ldots \vdash \ldots 0 = 0 \times 0}\,Repl}{0 \times 0 = (0 \times \epsilon)\mathtt{0}\ldots \vdash \ldots 0 = 0 \times 0}\,F4}{z_2 = \epsilon\ldots \vdash \ldots 0 = 0 \times 0}\,F5 \qquad \dfrac{\dfrac{}{0 = z_2 y_1 \mathtt{0}\ldots \vdash \ldots 0 = z_2 y_1 \mathtt{0}}\,Ax}{0 = \epsilon\epsilon\mathtt{0}, z_2 = \epsilon, y_1 = \epsilon\ldots \vdash \ldots 0 = z_2 y_1 \mathtt{0}}\,Repls}{z_2 = \epsilon, y_1 = \epsilon\ldots \vdash \ldots 0 = z_2 y_1 \mathtt{0}}\,L\ 9\mathtt{s}}{z_2 = \epsilon, y_1 = \epsilon\ldots \vdash \ldots 0 = 0 \times 0 \wedge 0 = z_2 y_1 \mathtt{0}}\,\wedge R}{z_2 = \epsilon\ldots \vdash (\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\exists R}{z_2 z_2 \equiv \epsilon \times \epsilon 1, F_{\forall\exists o3}\ldots \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,C\ 9}{\epsilon 1 \equiv \epsilon 0, F_{o3}(\epsilon, z_2) \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\wedge L}{\epsilon 1 \equiv \epsilon 0 \wedge F_{o3}(\epsilon, z_2) \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\wedge L}{(\epsilon\mathtt{0} = \epsilon) \vee (\epsilon\mathtt{1} \equiv \epsilon\mathtt{0} \wedge F_{o3}(\epsilon, z_2)) \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\vee L}{y_1 = \epsilon, y_2 = \epsilon\ldots \vdash \ldots((\epsilon\mathtt{0} = \epsilon) \vee (\epsilon\mathtt{1} \equiv \epsilon\mathtt{0} \wedge F_{o3}(\epsilon, z_2))) \rightarrow (\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\rightarrow R}{y_1 = \epsilon, y_2 = \epsilon\ldots \vdash \ldots F_{ones\forall}(\epsilon\mathtt{0}, y_1, 0, \epsilon)}\,\forall R$$

$\mathcal{D}_{\epsilon\mathtt{0}.iiiR}$

$$\dfrac{\dfrac{}{\epsilon\mathtt{0} = \epsilon\ldots \vdash (\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,F13 \qquad \dfrac{\dfrac{\dfrac{}{\mathtt{01} \equiv \epsilon\mathtt{0}, F_{o3}(\epsilon, z_2), y_1 = \epsilon, y_2 = \epsilon\ldots \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,R\ 21}{\mathtt{01} \equiv \epsilon\mathtt{0} \wedge F_{o3}(\epsilon, z_2), y_1 = \epsilon, y_2 = \epsilon\ldots \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\wedge R}{\epsilon\mathtt{0} = \epsilon \vee (\mathtt{01} \equiv \epsilon\mathtt{0} \wedge F_{o3}(\epsilon, z_2)), y_1 = \epsilon, y_2 = \epsilon\ldots \vdash \ldots(\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\vee L}{y_1 = \epsilon, y_2 = \epsilon\ldots \vdash \ldots((\epsilon\mathtt{0} = \epsilon) \vee (\mathtt{01} \equiv \epsilon\mathtt{0} \wedge F_{o3}(\epsilon, z_2))) \rightarrow (\exists y)(y = 0 \times y \wedge 0 = z_2 y_1 y)}\,\rightarrow R}{y_1 = \epsilon, y_2 = \epsilon\ldots \vdash \ldots F_{ones\forall}(\epsilon\mathtt{0}, y_1, 0, 0)}\,\forall R$$

$\mathcal{D}_{\epsilon\mathtt{0}.ii}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_{\epsilon0.iiiL}}{y_1=\epsilon, y_2=\epsilon... \vdash ...F_{ones\forall}(\epsilon0, y_1, 0, \epsilon)} \quad \dfrac{\mathcal{D}_{\epsilon0.iiiR}}{y_1=\epsilon, y_2=\epsilon... \vdash ...F_{ones\forall}(\epsilon0, y_1, 0, 0)}\, L\ 7}{y_1=\epsilon, y_2=\epsilon... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}\wedge L}{y_1=\epsilon \wedge y_2=\epsilon... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}\, L\ 18}{\epsilon=y_1y_2... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}\, Repl}{\epsilon y_1y_2=y_1y_2, y_2=0\times y_2, \epsilon=\epsilon y_1y_2... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}\, L\ 9}{y_2=0\times y_2, \epsilon=\epsilon y_1y_2... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}$$

$\mathcal{D}_{\epsilon0.i}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon=\epsilon... \vdash ...\epsilon=\epsilon, \epsilon1\equiv\epsilon\wedge F_{o2}(\epsilon,\epsilon)}\, Ax}{z_1=\epsilon... \vdash ...\epsilon=\epsilon, \epsilon1\equiv\epsilon\wedge F_{o3}(\epsilon,\epsilon)}\, Ref}{z_1=\epsilon... \vdash ...(\epsilon=\epsilon)\vee(\epsilon1\equiv\epsilon\wedge F_{o3}(\epsilon,\epsilon))}\vee R \quad \dfrac{\dfrac{\mathcal{D}_{\epsilon0.ii}}{y_2=0\times y_2, \epsilon=\epsilon y_1y_2... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}\wedge L}{y_2=0\times y_2\wedge\epsilon=\epsilon y_1y_2... \vdash ...F_{ones\forall\forall}(\epsilon0, y_10)}\exists L}{(\exists y)(y=0\times y\wedge\epsilon=\epsilon y_1y)... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}\to L}{((\epsilon=\epsilon)\vee(\epsilon1\equiv\epsilon\wedge F_{o3}(\epsilon,\epsilon)))\to(\exists y)(y=0\times y\wedge\epsilon=\epsilon y_1y)... \vdash ...F_{ones\forall\forall}(\epsilon0, y_1, 0)}\forall L \quad \dfrac{\dfrac{\dfrac{}{\epsilon\subseteq\epsilon... \vdash \epsilon\subseteq\epsilon}\, Ax}{\epsilon=\epsilon... \vdash ...\epsilon\subseteq\epsilon}\, F7b}{F_{ones\forall\forall}(\epsilon, y_1, \epsilon)... \vdash ...\epsilon\subseteq\epsilon}\, Ref}{F_{ones\forall}(\epsilon, y_1, \epsilon, \epsilon)... \vdash F_{\exists ones1}(\epsilon0, y_1), F_{ones\forall\forall}(\epsilon0, y_1, 0)}\forall_{\subseteq} L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F_{ones\forall\forall}(\epsilon, y_1, \epsilon)... \vdash F_{\exists ones1}(\epsilon0, y_1), F_{ones\forall\forall}(\epsilon0, y_10)}{z_1=\epsilon, F_{ones\forall\forall}(\epsilon, y_1, z_1)... \vdash F_{\exists ones1}(\epsilon0, y_1), F_{ones\forall\forall}(\epsilon0, y_10)}\, Repl}{z_1z_1\equiv\epsilon\times\epsilon1, F_{\forall\exists o3}, F_{ones\forall\forall}(\epsilon, y_1, z_1), y_1=1\times y_1 \vdash F_{\exists ones1}(\epsilon0, y_1), F_{ones\forall\forall}(\epsilon0, y_10)}\, C\ 9}{F_{o3}(\epsilon, z_1), F_{ones\forall\forall}(\epsilon, y_1, z_1), y_1=1\times y_1 \vdash F_{\exists ones1}(\epsilon0, y_1), F_{ones\forall\forall}(\epsilon0, y_10)}\wedge L}{F_{o3}(\epsilon, z_1)\wedge F_{ones\forall\forall}(\epsilon, y_1, z_1), y_1=1\times y_1 \vdash F_{\exists ones1}(\epsilon0, y_1), F_{ones\forall\forall}(\epsilon0, y_10)}\wedge R}{F_{\exists ones1}(\epsilon, y_1), y_1=1\times y_1 \vdash F_{\exists ones1}(\epsilon0, y_1), F_{ones\forall\forall}(\epsilon0, y_10)}\exists R$$

We conclude as follows

$\mathcal{D}_{\epsilon0}$

$$\dfrac{\dfrac{}{y_1=1\times y_1, F_{\exists ones1}(\epsilon, y_1) \vdash y_1=1\times y_1}\, Ax \quad \dfrac{\dfrac{\dfrac{}{F_{\exists ones1}... \vdash F_{\exists ones1}, F_{o3}(\epsilon0, 0)}\, L\ 24 \quad \dfrac{\mathcal{D}_{\epsilon0.ii}}{F_{\exists ones1}... \vdash F_{\exists ones1}, F_{ones\forall\forall}(\epsilon0, 0)}}{y_1=1\times y_1, F_{\exists ones1}(\epsilon, y_1) \vdash F_{\exists ones1}, F_{o3}(\epsilon0, 0)\wedge F_{ones\forall\forall}(\epsilon0, 0)}\wedge R}{y_1=1\times y_1, F_{\exists ones1}(\epsilon, y_1) \vdash F_{\exists ones1}(\epsilon0, y_1)}\exists R}{y_1=1\times y_1, F_{\exists ones1}(\epsilon, y_1) \vdash F_{ones}(\epsilon0, y_1)}\wedge R$$

$$\dfrac{y_1=1\times y_1, F_{\exists ones1}(\epsilon, y_1) \vdash F_{ones}(\epsilon0, y_1)}{F_{ones}(\epsilon, y_1) \vdash F_{ones}(\epsilon0, y_1)}\wedge L$$

□

Remark 4. *Notice that Proposition 9.1 also holds for Ferreira's $Ones(\cdot,\cdot)$, but, differently from $F_{ones}(\epsilon, 11)$, also $Ones(\epsilon, 11)$ is derivable, which contradicts Proposition 9.3.*[10]

Sequences are coded by encoding 0 as 01, 1 as 10, and the separation mark as 11, [7, p. 97].

Definition 22. The formula $F_{seq}(\cdot)$ is defined as follows:

$$F_{seq}(x) := (\exists y\subseteq x)\big(x\equiv yy\wedge(\forall z\subseteq y)\neg(x|_{zz}00\subseteq x)\big).$$

The auxiliary formula $F_s(\cdot,\cdot)$ is the one below:

$$\begin{aligned}F_s(x,y) := y\equiv x\wedge F_{seq}(x)\wedge(\forall z)\big(zz\preceq x\to{}&(x|_{zz}01\subseteq x\to y|_{zz}00\subseteq y)\\&\wedge(x|_{zz}10\subseteq x\to y|_{zz}00\subseteq y)\\&\wedge(x|_{zz}11\subseteq x\to y|_{zz}11\subseteq y)\big).\end{aligned}$$

[10] For further details, see Appendix 3.1.4.

Definition 23. The formula $F_{lh}(\cdot,\cdot)$ is defined as follows:

$$F_{lh}(x,u) := (\exists y \equiv x)\big(F_s(x,y) \wedge F_{ones}(y,uu)\big).$$

Proposition 10. *1.* $RS^1_3 \vdash F_{seq}(\epsilon) \wedge F_{lh}(\epsilon,\epsilon)$

2. $RS^1_3 \vdash (\forall x)\big(F_{seq}(x) \to (\exists! y \preceq x)(y = \mathsf{1} \times y \wedge F_{lh}(x,y))\big)$.

Proof of 1. Let us prove the first clause only.

$$\mathcal{D}_{seq(\epsilon)}$$

$$\dfrac{\dfrac{\dfrac{}{\epsilon \subseteq \epsilon, \epsilon = \epsilon \vdash \epsilon \subseteq \epsilon}\,Ax}{\dfrac{\epsilon = \epsilon \vdash \epsilon \subseteq \epsilon}{\vdash \epsilon \subseteq \epsilon}\,Ref}\,F7b \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{1 \times \epsilon = 1 \times \epsilon\epsilon\ldots \vdash \epsilon \equiv \epsilon\epsilon}\,Ax}{\epsilon\epsilon = \epsilon, 1 \times \epsilon = 1 \times \epsilon \vdash \epsilon \equiv \epsilon\epsilon}\,Repl}{1 \times \epsilon = 1 \times \epsilon \vdash \epsilon \equiv \epsilon\epsilon}\,F1}{\vdash \epsilon \equiv \epsilon\epsilon}\,Ref \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon|_{\epsilon\epsilon}\mathsf{11} = \epsilon, \epsilon|_{\epsilon\epsilon}\mathsf{11} \subseteq \epsilon \vdash}\,F13}{\epsilon|_{\epsilon\epsilon}\mathsf{11} \subseteq \epsilon \vdash}\,F7a}{\vdash \neg\epsilon|_{\epsilon\epsilon}\mathsf{11} \subseteq \epsilon}\,\neg R}{\vdash (\forall z \subseteq \epsilon)\neg(\epsilon|_{zz}\mathsf{00} \subseteq \epsilon)}\,\forall_{\subseteq}R^*}{\vdash \epsilon \equiv \epsilon\epsilon \wedge (\forall \epsilon \subseteq \epsilon)\neg(x|_{zz}\mathsf{00} \subseteq \epsilon)}\,\wedge R}{\vdash F_{seq}(\epsilon)}\,\exists_{\subseteq}R^*$$

$$\mathcal{D}_{s(\epsilon,\epsilon)}$$

$$\dfrac{\dfrac{}{\epsilon \equiv \epsilon \vdash \epsilon \equiv \epsilon}\,Ax}{\vdash \ldots \epsilon \equiv \epsilon}\,Ref \qquad \dfrac{\mathcal{D}_{seq(\epsilon)}}{\vdash \ldots F_{seq}(\epsilon)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon|_{vv}\mathsf{01} = \epsilon \vdash \ldots \epsilon|_{vv}\mathsf{00} \subseteq \epsilon}\,F13}{\epsilon|_{vv}\mathsf{01} \subseteq \epsilon \vdash \ldots \epsilon|_{vv}\mathsf{00} \subseteq \epsilon}\,F7a}{\ldots \vdash \epsilon|_{vv}\mathsf{01} \subseteq \epsilon \to \epsilon|_{vv}\mathsf{00} \subseteq \epsilon}\,\to R \qquad \dfrac{\dfrac{\dfrac{}{\epsilon|_{vv}\mathsf{01} = \epsilon\ldots \vdash \epsilon|_{vv}\mathsf{00} \subseteq y}\,F13}{\epsilon|_{vv}\mathsf{01} \subseteq \epsilon\ldots \vdash \epsilon|_{vv}\mathsf{00} \subseteq y}\,F7a}{\ldots \vdash \epsilon|_{vv}\mathsf{10} \subseteq \epsilon \to \epsilon|_{vv}\mathsf{00} \subseteq \epsilon}\,\to R \qquad \dfrac{\dfrac{\dfrac{}{\epsilon|_{vv}\mathsf{11} = \epsilon\ldots \vdash \epsilon|_{vv} \subseteq \epsilon}\,F13}{\epsilon|_{vv}\mathsf{11} \subseteq \epsilon\ldots \vdash \epsilon|_{vv} \subseteq \epsilon}\,F7a}{\ldots \vdash \epsilon|_{vv}\mathsf{11} \subseteq \epsilon \to \epsilon|_{vv} \subseteq \epsilon}\,\to R$$
$$\dfrac{}{v = \epsilon \vdash \ldots F_{s2\wedge}}\,\wedge R$$
$$\dfrac{}{v = \epsilon \vdash \ldots F_{s1\wedge}(\epsilon,\epsilon,y)}\,R\ 19$$
$$\dfrac{}{\vdash \ldots F_{\forall s}(\epsilon,\epsilon)}$$
$$\dfrac{}{\vdash \ldots F_{seq}(\epsilon) \wedge F_{\forall s}(\epsilon,\epsilon)}\,\wedge R$$
$$\dfrac{}{\vdash \ldots F_s(\epsilon,\epsilon)}\,\wedge R$$

$$\mathcal{D}_{lh(\epsilon,\epsilon)}$$

$$\dfrac{\dfrac{\dfrac{}{1 \times \epsilon = 1 \times \epsilon, \vdash \epsilon \equiv \epsilon}\,Ax}{\vdash F_{lh}, \epsilon \equiv \epsilon}\,Ref \qquad \dfrac{\mathcal{D}_{s(\epsilon,\epsilon)}}{\vdash F_{lh}, F_s(\epsilon,\epsilon)} \qquad \dfrac{}{\vdash F_{lh}, F_{ones}(\epsilon,\epsilon\epsilon)}\,L\ 9.1}{\vdash F_{lh}, F_s(\epsilon,\epsilon) \wedge F_{ones}(\epsilon,\epsilon\epsilon)}\,\wedge R}{\vdash F_{lh}(\epsilon,\epsilon)}\,\exists_{\equiv}R$$

We conclude as follows:

$$\dfrac{\dfrac{\mathcal{D}_{seq(\epsilon)}}{\vdash F_{seq}(\epsilon)} \qquad \dfrac{\mathcal{D}_{lh(\epsilon,\epsilon)}}{\vdash F_{lh}(\epsilon,\epsilon)}}{\vdash F_{seq}(\epsilon) \wedge F_{lh}(\epsilon,\epsilon)}\,\wedge R$$

□

Due to some auxiliary formulas, it is defined a formula to "decode" the encoding strings.

Definition 24. The formula $F_{word}(\cdot)$ is defined as follows:

$$F_{word}(x) := (\exists y \subseteq x)\big(x \equiv yy \wedge (\forall z \subset y)(x|_{zz}\mathtt{01} \subseteq x \vee x|_{zz}\mathtt{10} \subseteq w)\big).$$

$$\begin{aligned} F_{e1}(s,r,w) := {} & F_{seq}(s) \wedge F_{word}(w) \wedge (\exists y)(\exists u)\big(F_h(s,u) \wedge r \subset u \wedge F_s(s,y) \\ & \wedge (\forall y' \subseteq y)(F_{ones}(y', rr) \rightarrow s|_{y'}w \equiv s \vee s|_{y'}w\mathtt{11} \subseteq s)\big) \\ F_{e2}(w,x) := {} & F_{word}(w) \wedge w \equiv xx \wedge (\forall x' \subseteq x)\big((x'\mathtt{0} \subseteq x \rightarrow w|_{x'x'}\mathtt{01} \subseteq w) \\ & \wedge (x'\mathtt{1} \subseteq x \rightarrow w|_{x'x'}\mathtt{10} \subseteq w)\big) \end{aligned}$$

Definition 25. The formula $F_{eval}(x,y,z)$ is defined as follows:

$$F_{eval}(x,y,z) := (\exists u \preceq x)\big(F_{e1}(x,y,u) \wedge F_{e2}(u,z)\big).$$

Proposition 11. *1.* $RS^1_3 \vdash (\forall x)(\forall y)\big(F_{seq}(x) \wedge F_{lh}(x,y) \rightarrow (\forall z \subset y)(\exists! u \preceq x)F_{eval}(x,z,u)$

2. $RS^1_3 \vdash (\forall x)(\forall s)(\forall u)\big(F_{seq}(s) \wedge F_{lh}(s,u) \rightarrow (\exists \tilde{s} \preceq sxx\mathtt{11})\big(F_{lh}(\tilde{s}, u\mathtt{1}) \wedge F_{eval}(\tilde{s},u,x) \wedge (\forall r \subset u)(\forall y)\big(F_{eval}(\tilde{s},r,y) \leftrightarrow F_{eval}(s,r,y)\big)\big)\big)$.

3.1.4 Comparing $F_{ones}(\cdot,\cdot)$ and and Ferreira's $Ones(\cdot,\cdot)$

As anticipated, our definition of $F_{ones}(\cdot,\cdot)$ is slightly different from Ferreira's corresponding formula [7, p. 97]:

$$\begin{aligned} Ones(x,y) := {} & y = \mathtt{1} \times y \wedge (\exists z)\big(F_{o3}(x,z) \wedge (\forall x' \subseteq x)(\forall z')\big((x'\mathtt{1} = x \\ & \wedge F_{o3}(x',z')\big) \rightarrow (\exists u)(u = \mathtt{0} \times u \wedge z = z'yu)\big)\big). \end{aligned}$$

It seems that this formula satisfies the desired conditions 1 and 2 of Proposition 9, but not 3, as $RS^1_3 \vdash Ones(\epsilon,\epsilon)$, but also $RS^1_3 \vdash Ones(\epsilon,\mathtt{1})$.

Proposition 12. $RS^1_3 \vdash Ones(\epsilon,\epsilon)$.

Proof. Indeed,

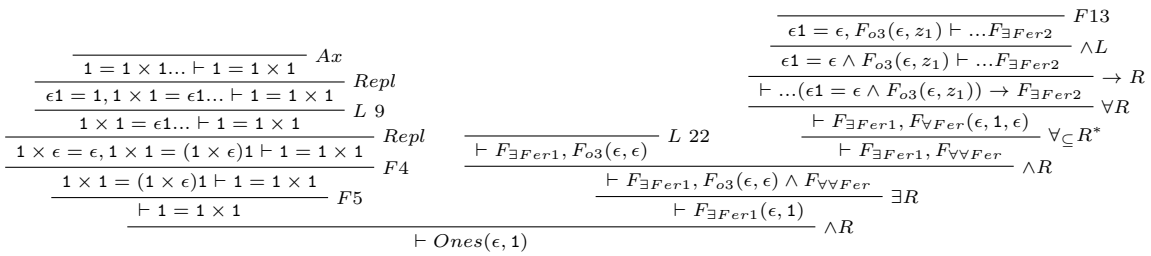
$$\dfrac{\dfrac{\dfrac{}{\epsilon = \mathtt{1}\times\epsilon \vdash \epsilon = \mathtt{1}\times\epsilon}\,Ax}{\vdash \epsilon = \mathtt{1}\times\epsilon}\,F4 \qquad \dfrac{\dfrac{\dfrac{}{\vdash F_{\exists Fer1}, F_{o3}(\epsilon)}\,L\ 22 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\epsilon\mathtt{1} = \epsilon, F_{o3}(\epsilon,z_1) \vdash \ldots(\exists u)(u = \mathtt{0}\times u \wedge \epsilon = z_1\epsilon u)}\,F13}{\epsilon\mathtt{1} = \epsilon \wedge F_{o3}(\epsilon,z_1) \vdash \ldots(\exists u)(u = \mathtt{0}\times u \wedge \epsilon = z_1\epsilon u)}\,\wedge L}{\vdash \ldots(\epsilon\mathtt{1} = \epsilon \wedge F_{o3}(\epsilon,z_1)) \rightarrow (\exists u)(u = \mathtt{0}\times u \wedge \epsilon = z_1\epsilon u)}\,\rightarrow R}{\vdash F_{\exists Fer1}, F_{\forall Fer}(\epsilon,\epsilon,\epsilon,\epsilon)}\,\forall R}{F_{\exists Fer1}, F_{\forall\forall Fer}(\epsilon,\epsilon,\epsilon,\epsilon)}\,\forall_{\subseteq}R^*}{\vdash F_{\exists Fer1}(\epsilon,\epsilon), F_{o3}(\epsilon,\epsilon) \wedge F_{\forall\forall Fer}(\epsilon,\epsilon,\epsilon,\epsilon)}\,\wedge R}{\vdash F_{\exists Fer1}(\epsilon,\epsilon)}\,\exists R}{\vdash Ones(\epsilon,\epsilon)}\,\wedge R$$

□

So, the property corresponding of Proposition 9.1 is proved to hold also for Ferreira's $Ones(x,y)$. However, in the specific case of x is ϵ, this formula seems not to satisfy uniqueness (Proposition 9.3) as, for example, $RS^1_3 \vdash Ones(\epsilon,\mathtt{1})$.

Proposition 13. $RS^1_3 \vdash Ones(\epsilon, \mathsf{1})$.

Proof. Indeed,



□

Remark 5. *Generally speaking, it seems that $RS^1_3 \vdash Ones(\epsilon, y)$ holds for each $y = \mathsf{1} \times y$.*

Section 2

Theorem 5. *For any function $f : \mathbb{S}^j \times \mathbb{O} \to \mathbb{S}$, $f \in \mathcal{POR}$ if and only if f is provably represented by some term $t : \mathsf{s} \to \cdots \to \mathsf{s} \to \mathsf{s}$.*

Proof. $\Rightarrow$ The proof is by induction on the structure of $\mathcal{POR}$-functions. Each basic function is provably represented:

- The function E is provably represented by the term $\lambda x.\epsilon$. Indeed for any string s, $\overline{\overline{E(s)}} = \overline{\overline{\epsilon}} = \epsilon$ and $\vdash_{\mathcal{POR}^\lambda} (\lambda x.\epsilon)\overline{\overline{s}} = \epsilon$ is an instance of the (β)-axiom.
- The function P^n_i is provably represented by the term $\lambda x_1 \dots \lambda x_n.x_i$. Indeed, given strings $s_1, \dots, s_n$, $\overline{\overline{P_i(s_1, \dots s_n)}} = \overline{\overline{s_i}}$ and $\vdash_{\mathcal{POR}^\lambda} (\lambda x_1 \dots \lambda x_n.x_i)\overline{\overline{s_1}}, \dots, \overline{\overline{s_n}} = \overline{\overline{s_i}}$ is an instance of the (β)-axiom.
- The functions S_0 and S_1 are provably represented by $\lambda x.x\mathsf{0}$ and $\lambda x.x\mathsf{1}$, respectively. Indeed, for any string s, $\overline{\overline{S_0(s)}} = \overline{\overline{s\mathbf{0}}} = \overline{\overline{s}}\mathsf{0}$ and $\vdash_{\mathcal{POR}^\lambda} (\lambda x.x\mathsf{0})\overline{\overline{s}} = \overline{\overline{s}}\mathsf{0}$ is an instance of the (β)-axiom. One can argue similarly for S_1.
- The function C is provably represented by Sub. The proof is based on the inductive cases below, following the definition of Sub.
- The function Q is provably represented by $\mathsf{Flipcoin}$, as observed in Example 1

Moreover, each function defined by composition or bounded recursion from provably represented functions, is provably represented by a term as well.

- If f is defined by composition, i.e. $f(s_1, \dots, s_n, \omega) = h\big(g_1(s_1, \dots, s_n, \omega), \dots g_k(s_1, \dots, s_n, \omega)\big)$, then by IH $h, g_1, \dots, g_k$ are represented by terms (resp.) $\mathsf{H}, \mathsf{G}_1, \dots, \mathsf{G}_n$. Hence, for all $\omega \in \mathbb{O}$ and strings $s_1, \dots, s_{max\{n,k\}}$,

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{H}\overline{\overline{s_1}}, \dots, \overline{\overline{s_k}} = \overline{\overline{h(s_1, \dots, s_k, \omega)}} \tag{H}$$

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{G}_i\overline{\overline{s_1}}, \dots, \overline{\overline{s_n}} = \overline{\overline{g_i(s_1, \dots, s_n, \omega)}} \tag{G_i}$$

Then, f is provably represented by $\mathsf{T} := \lambda x_1 \dots \lambda x_n.\mathsf{H}(\mathsf{G}_1 x_1 \dots x_n) \dots (\mathsf{G}_k x_1 \dots x_n)$. Indeed, using (G_i) and $(R3)$ we can derive

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{H}(\mathsf{G}_1\overline{\overline{s_1}}, \dots \overline{\overline{s_n}}) \dots (\mathsf{G}_k\overline{\overline{s_1}} \dots \overline{\overline{s_n}}) = \mathsf{H}\overline{\overline{g_1(s_1, \dots, s_n, \omega)}} \dots \overline{\overline{g_k(s_1, \dots, s_n, \omega)}}$$

and using (H) and $(R2)$ we can further derive

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{H}(\mathsf{G}_1\overline{\overline{s_1}}, \dots, \overline{\overline{s_n}}) \dots (\mathsf{G}_k\overline{\overline{s_1}}, \dots, \overline{\overline{s_n}}) = \overline{\overline{h\big(g_1(s_1, \dots, s_n, \omega), \dots, g_k(s_1, \dots, s_n, \omega)\big)}}.$$

Finally, using the equation above, the instance of (β)-axiom $\mathsf{T}\overline{\overline{s_1}} \dots \overline{\overline{s_n}} = \mathsf{H}(\mathsf{G}_1\overline{\overline{s_1}}, \dots \overline{\overline{s_n}}) \dots (\mathsf{G}_k\overline{\overline{s_1}}, \dots, \overline{\overline{s_n}})$ and $(R3)$ we can finally derive

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{T}\overline{\overline{s_1}} \dots \overline{\overline{s_n}} = \overline{\overline{h\big(g_1(s_1, \dots, s_n), \dots, g_k(s_1, \dots, s_n)\big)}}.$$

- if f is defined by bounded recursion, i.e.

$$\begin{aligned} f(s_1, \dots, s_n, \boldsymbol{\epsilon}, \omega) &= g(s_1, \dots, s_n, \omega) \\ f(s_1, \dots, s_n, s\mathbf{0}, \omega) &= h_0\big(s_1, \dots, s_n, s, f\big(s_1, \dots, s_n, s, \omega\big), \omega\big)|_{k(s_1,\dots,s_n,s)} \\ f(s_1, \dots, s_n, s\mathbf{1}, \omega) &= h_1\big(s_1, \dots, s_n, s, f\big(s_1, \dots, s_n, s, \omega\big), \omega\big)|_{k(s_1,\dots,s_n,s)}. \end{aligned}$$

Then, by IH, g, h_0, h_1, k are provably represented by (resp.) terms $\mathsf{G}, \mathsf{H}_0, \mathsf{H}_1, \mathsf{K}$. So, for any $\omega \in \mathbb{O}$ and strings $s_1, \dots, s_{n+2}, s$ we can derive

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{G}\overline{\overline{s_1}} \dots \overline{\overline{s_n}} = \overline{\overline{g(s_1, \dots, s_n, \omega)}} \tag{G}$$

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{H}_0\overline{\overline{s_1}} \dots \overline{\overline{s_{n+2}}} = \overline{\overline{h_0(s_1, \dots, s_{n+2}, \omega)}} \tag{H_0}$$

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{H}_1\overline{\overline{s_1}} \dots \overline{\overline{s_{n+2}}} = \overline{\overline{h_1(s_1, \dots, s_{n+2}, \omega)}} \tag{H_1}$$

$$T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{K}\overline{\overline{s_1}} \dots \overline{\overline{s_n}} = \overline{\overline{k(s_1, \dots, s_n, \omega)}}. \tag{K}$$

Let $\omega \in \mathbb{O}$ and $s_1, \dots, s_n, s \in \mathbb{S}$. We will prove by induction on s that $T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{T}\overline{\overline{s_1}} \dots \overline{\overline{s_n s}} = \overline{\overline{f(s_1, \dots, s_n, \omega)}}$, where

$$\mathsf{T} := \lambda x_1 \dots \lambda x_n.\lambda x.\mathsf{Rec}(\mathsf{G}x_1 \dots x_n, \mathsf{H}_0 x_1 \dots x_n, \mathsf{H}_1 x_1, \dots, x_n, \mathsf{K}x_1 \dots x_n, x).$$

- If $s = \boldsymbol{\epsilon}$, then $f(s_1, \dots, s_n, s, \omega) = g(s_1, \dots, s_n, \omega)$. So, using the (β)-axiom we deduce

  $$\vdash_{\mathcal{POR}^\lambda} \mathsf{T}\overline{\overline{s_1}} \dots \overline{\overline{s_n s}} = \mathsf{Rec}(\mathsf{G}\overline{\overline{s_1}} \dots \overline{\overline{s_n}}, \mathsf{H}_0\overline{\overline{s_1}} \dots \overline{\overline{s_n}}, \mathsf{H}_1\overline{\overline{s_1}} \dots \overline{\overline{s_n}}, \mathsf{K}\overline{\overline{s_1}} \dots \overline{\overline{s_n}}, \overline{s})$$

  and using the axiom $\mathsf{Rec}(\mathsf{G}x_1 \dots x_n, \mathsf{H}_0 x_1 \dots x_n, \mathsf{H}_1 x_1 \dots x_n, \mathsf{K}x_1 \dots x_n, \epsilon) = \mathsf{G}x_1 \dots x_n$ we deduce using $(R3)$ and $(R2)$

  $$\vdash_{\mathcal{POR}^\lambda} \mathsf{T}\overline{\overline{s_1}} \dots \overline{\overline{s_n s}} = \mathsf{G}\overline{\overline{s_1}} \dots \overline{\overline{s_n}}.$$

  We conclude using (G) together with $(R2)$.

– If $s = s'\mathbf{0}$, then $f(s_1,\dots,s_n,s,\omega) = h_0(s_1,\dots,s_n,s',f\big(s_1,\dots,s_n,s,\omega),\omega)\big|_{k(s_1,\dots,s_n,s')}$. By IH we can suppose that $T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{T}\overline{\overline{s_1}}\dots\overline{\overline{s_n s'}} = \overline{\overline{f(s_1,\dots,s_n,s',\omega)}}$. Then, using the (β)-axiom $\mathsf{T}\overline{\overline{s_1}}\dots\overline{\overline{s_n s}} = \mathsf{Rec}(\mathsf{G}\overline{\overline{s_1}}\dots\overline{\overline{s_n}},\mathsf{H}_0\overline{\overline{s_1}}\dots\overline{\overline{s_n}},\mathsf{H}_1\overline{\overline{s_1}}\dots\overline{\overline{s_n}},\mathsf{K}\overline{\overline{s_1}}\dots,\overline{\overline{s_n}},\overline{\overline{s}})$, the axiom $\mathsf{Rec}(g,h_0,h_1,k,x\mathsf{0}) = \mathsf{Trunc}(h_0 x(\mathsf{Rec}(g,h_0,h_1,k,\mathsf{0})),kx)$, and the IH we deduce, using $(R3)$ and $(R2)$,

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{T}\overline{\overline{s_1}}\dots\overline{\overline{s_n s}} = \mathsf{Trunc}(\mathsf{H}_0\overline{\overline{s_1}}\dots\overline{\overline{s_n s' f(s_1,\dots,s_n,s',\omega)}},\mathsf{K}\overline{\overline{s_1}}\dots\overline{\overline{s_n}}).$$

Now, using (H_0) and (K), we finally conclude using $(R3)$ and $(R2)$

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{T}\overline{\overline{s_1}}\dots\overline{\overline{s_n s}} = \overline{\overline{h_0\big(s_1,\dots,s_n,s',f(s_1,\dots,s_n,s',\omega)\big)\big|_{k(s_1,\dots,s_n)}}}$$

– The case $s = s'\mathbf{1}$ can be proved in a similar way.

$\Rightarrow$ It is a consequence of the normalization property for the simply-typed λ calculus: a β-normal term $t : \mathsf{s}\to\dots\to\mathsf{s}\to\mathsf{s}$ cannot contain variables of higher-types. By enumerating all possible normal forms one can check that these all represent functions in $\mathcal{POR}$. □

Lemma 25. *i.* $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Tail}(x) = \epsilon \leftrightarrow x = \mathsf{0}\vee x = \mathsf{1}\vee x = \epsilon)$

ii. $\vdash_{I\mathcal{POR}^\lambda} \neg(x=\epsilon)\to \mathsf{Cond}(x,y,z\mathsf{0},w\mathsf{0}) = \mathsf{Cond}(x,y,z,w)\mathsf{0}$

iii. $\vdash_{I\mathcal{POR}^\lambda} \neg(x=\epsilon)\to \mathsf{Cond}(x,y,z\mathsf{1},w\mathsf{1}) = \mathsf{Cond}(x,y,z,w)\mathsf{1}$

iv. $\vdash_{I\mathcal{POR}^\lambda} \big(\mathsf{B}(x)=\mathsf{0}\leftrightarrow x=\mathsf{Tail}(x)\mathsf{0}\big)\wedge\big(\mathsf{B}(x)=\mathsf{1}\leftrightarrow x=\mathsf{Tail}(x)\mathsf{1}\big)$

v. $\vdash_{I\mathcal{POR}^\lambda} \big(\mathsf{BNeg}(x)=\mathsf{0}\leftrightarrow x=\mathsf{Tail}(x)\mathsf{1}\big)\wedge\big(\mathsf{BNeg}(x)=\mathsf{1}\leftrightarrow x=\mathsf{Tail}(x)\mathsf{0}\big)$

vi. $x=\mathsf{0}\vee x=\mathsf{1}, y=\mathsf{0}\vee y=\mathsf{1}\vdash_{I\mathcal{POR}^\lambda}\mathsf{BOr}(x,y)=\mathsf{0}\leftrightarrow x=\mathsf{0}\wedge x=\mathsf{0}$,
$x=\mathsf{0}\vee x=\mathsf{1}, y=\mathsf{0}\vee y=\mathsf{1}\vdash_{I\mathcal{POR}^\lambda}\mathsf{BOr}(x,y)=\mathsf{1}\leftrightarrow x=\mathsf{1}\vee x=\mathsf{1}$

vii. $x=\mathsf{0}\vee x=\mathsf{1}, y=\mathsf{0}\vee y=\mathsf{1}\vdash_{I\mathcal{POR}^\lambda}\mathsf{BAnd}(x,y)=\mathsf{0}\leftrightarrow x=\mathsf{0}\vee x=\mathsf{0}$,
$x=\mathsf{0}\vee x=\mathsf{1}, y=\mathsf{0}\vee y=\mathsf{1}\vdash_{I\mathcal{POR}^\lambda}\mathsf{BAnd}(x,y)=\mathsf{1}\leftrightarrow x=\mathsf{1}\wedge x=\mathsf{1}$

viii. $\vdash_{I\mathcal{POR}^\lambda}\mathsf{Eps}(x)=\mathsf{1}\leftrightarrow x=\epsilon$

ix. $\vdash_{I\mathcal{POR}^\lambda}\mathsf{Flipcoin}(x)=\mathsf{0}\vee\mathsf{Flipcoin}(x)=\mathsf{1}$

x. $\vdash_{I\mathcal{POR}^\lambda}\mathtt{Flip}(x)\vee\neg\mathtt{Flip}(x)$

xi. $\vdash_{I\mathcal{POR}^\lambda}\mathsf{Eq}(x,y)=\mathsf{1}\leftrightarrow x=y$

xii. $\vdash_{I\mathcal{POR}^\lambda}\mathsf{Eq}(x,y)=\mathsf{0}\vee\mathsf{Eq}(x,y)=\mathsf{1}$

xiii. $\vdash_{I\mathcal{POR}^\lambda}\mathsf{Sub}(x,y)=\mathsf{0}\vee\mathsf{Sub}(x,y)=\mathsf{1}$

xiv. $\vdash_{I\mathcal{POR}^\lambda}\mathsf{Sub}(x,\epsilon)=\mathsf{1}\leftrightarrow x=\epsilon$,
$\vdash_{I\mathcal{POR}^\lambda}\mathsf{Sub}(x,y\mathsf{b})=\mathsf{1}\leftrightarrow\mathsf{Sub}(x,y)=\mathsf{1}\vee x=y\mathsf{b}$, *with* $\mathsf{b}\in\{\mathsf{0},\mathsf{1}\}$.

Proof. i. From the defining axioms of Tail, together with $\epsilon\mathsf{0}=\epsilon$ and $\epsilon\mathsf{1}=\epsilon$, it follows that

$$\vdash_{I\mathcal{POR}^\lambda}(x=\mathsf{0}\vee x=\mathsf{1}\vee x=\epsilon)\to\mathsf{Tail}(x)=\epsilon.$$

Conversely, from the axiom $(x=\epsilon)\vee(x=\mathsf{Tail}(x)\circ\mathsf{0})\vee(x=\mathsf{Tail}(x)\circ\mathsf{1})$ it follows that

$$\vdash_{I\mathcal{POR}^\lambda}\mathsf{Tail}(x)=\epsilon\to(x=\epsilon\vee x=\mathsf{0}\vee x=\mathsf{1}).$$

ii. Using Axiom 3 and Axiom 5 we deduce

$$\vdash_{I\mathcal{POR}^\lambda} \neg(x=\epsilon) \to \big(x = \mathsf{Tail}(x)0 \wedge \mathsf{Cond}(x,y,z0,w0) = z0 \wedge \mathsf{Cond}(x,y,z,w) = z\big) \\ \wedge \big(x = \mathsf{Tail}(x)1 \wedge \mathsf{Cond}(x,y,z0,w0) = w0 \wedge \mathsf{Cond}(x,y,z,w) = w\big)$$

from which the desired claim can be deduced.

iii. Similar to the previous case.

iv. Recall that $\mathsf{B}(x) = \mathsf{Cond}(x,\epsilon,0,1)$. From Axiom 3, it follows that

$$\vdash_{\mathcal{POR}^\lambda} \mathsf{B}(x) = 0 \leftrightarrow (x=\epsilon \wedge \epsilon = 0) \vee (x = \mathsf{Tail}(x)0 \wedge 0 = 0) \vee (x = \mathsf{Tail}(x)1 \wedge 0 = 1),$$

from which we deduce

$$x = \mathsf{Tail}(x)0 \vdash_{I\mathcal{POR}^\lambda} \mathsf{B}(x) = 0.$$

Moreover, we can deduce

$$\begin{aligned} x = \epsilon, \epsilon = 0 &\vdash_{I\mathcal{POR}^\lambda} x = \mathsf{Tail}(x)0 \\ x = \mathsf{Tail}(x), 0 = 0 &\vdash_{I\mathcal{POR}^\lambda} x = \mathsf{Tail}(x)0 \\ x = \mathsf{Tail}(x)1, 0 = 1 &\vdash_{I\mathcal{POR}^\lambda} x = \mathsf{Tail}(x)0 \end{aligned}$$

from which we conclude

$$\mathsf{B}(x) = 0 \vdash_{I\mathcal{POR}^\lambda} x = \mathsf{Tail}(x)0.$$

v. Similar to the previous case.

vi. Proved using the defining axioms of Cond together with b. and c..

vii. Similar to the previous one.

viii. Recall that $\mathsf{Eps}(x) = \mathsf{Cond}(x,1,0,0)$. From Axiom 5 it follows that

$$\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eps}(x) = 1 \leftrightarrow (x=\epsilon \wedge 1 = 1) \vee (x = \mathsf{Tail}(x)0 \wedge 0 = 1) \vee (x = \mathsf{Tail}(x)1 \wedge 0 = 1),$$

from which, using Axiom 4, we can deduce

$$\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eps}(x) = 0 \leftrightarrow x = \epsilon.$$

ix. From the axioms of Flipcoin and points iv., v., vii. and viii., we deduce

$$\vdash_{I\mathcal{POR}^\lambda} \mathsf{Tail}\big(\mathsf{Flipcoin}(x)\big) = \epsilon$$

and

$$\vdash_{I\mathcal{POR}^\lambda} \neg(\mathsf{Flipcoin}(x) = \epsilon).$$

Hence, from Axiom 3, we deduce

$$\vdash_{I\mathcal{POR}^\lambda} \big(\mathsf{Flipcoin}(x) = \mathsf{Tail}(\mathsf{Flipcoin}(x))0\big) \vee \big(\mathsf{Flipcoin}(x) = \mathsf{Tail}(\mathsf{Flipcoin}(x))1\big)$$

and, using $0 = \epsilon 0$ and $1 = \epsilon 1$, we finally deduce

$$\vdash_{I\mathcal{POR}^\lambda} \mathsf{Flipcoin}(x) = 0 \vee \mathsf{Flipcoin}(x) = 1.$$

x. From the previous point and Axiom 6.

xi. Using the axiom for Rec, we can prove all recursive equations of Eq. We show that $\vdash_{I\mathcal{POR}^\lambda} x = y \to \mathsf{Eq}(x, y) = 1$ as follows. Let $B(x) := \mathsf{Eq}(x, x) = 1$. Then, we have that $\vdash_{I\mathcal{POR}^\lambda} B(\epsilon, \epsilon)$. Moreover, we can prove $\vdash_{I\mathcal{POR}^\lambda} B(x) \to B(x0)$, using $\mathsf{Eq}(x0, x0) = \mathsf{Eq}(x, x)$ and, similarly, that $\vdash_{I\mathcal{POR}^\lambda} B(x) \to B(x1)$. We can conclude then by **NP**-induction that $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eq}(x, x) = 1$ holds and finall that $\vdash_{I\mathcal{POR}^\lambda} x = y \to \mathsf{Eq}(x, y) = 1$.

For the converse direction, let $B(x, y) := \mathsf{Cond}(\mathsf{Eq}(x, y), \epsilon, x, y) = x$. We will show by a double **NP**-induction that $\vdash_{I\mathcal{POR}^\lambda} B(x, y)$. From this, using the fact that $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eq}(x, y) = 1 \to \mathsf{Cond}(\mathsf{Eq}(x, y), \epsilon, x, y) = y$, we can deduce $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eq}(x, y) = 1 \to x = y$. The following equalities are all provable:

$$\begin{aligned}\mathsf{Cond}(\mathsf{Eq}(\epsilon, \epsilon), \epsilon, \epsilon, \epsilon) &= \epsilon\\ \mathsf{Cond}(\mathsf{Eq}(x0, \epsilon), \epsilon, x0, \epsilon) &= x0\\ \mathsf{Cond}(\mathsf{Eq}(x1, \epsilon), \epsilon, x1, \epsilon) &= x1.\end{aligned}$$

So, we can prove $B(\epsilon, \epsilon), B(x, \epsilon) \to B(x0, \epsilon)$ and $B(x, \epsilon) \to B(x1, \epsilon)$. Then, by applying **NP**-induction, we deduce that $B(x, \epsilon)$ is derivable. Suppose now that $B(x, y)$, i.e. $\mathsf{Cond}(\mathsf{Eq}(x, y), \epsilon, x, y) = x$ holds. The following equalities are provable:

$$\begin{aligned}\mathsf{Cond}(\mathsf{Eq}(\epsilon, y0), \epsilon, \epsilon, y0) &= \epsilon\\ \mathsf{Cond}(\mathsf{Eq}(x0, y0), \epsilon, x0, y0) &= \mathsf{Cond}(\mathsf{Eq}(x, y), \epsilon, x0, y0)\\ &\overset{ii.}{=} \mathsf{Cond}(\mathsf{Eq}(x, y), \epsilon, x, y)0\\ &= x0\\ \mathsf{Cond}(\mathsf{Eq}(x1, y0), \epsilon, x1, y0) &= x1\\ \mathsf{Cond}(\mathsf{Eq}(\epsilon, y1), \epsilon, \epsilon, y1) &= \epsilon\\ \mathsf{Cond}(\mathsf{Eq}(x0, y1), \epsilon, x0, y1) &= x0\\ \mathsf{Cond}(\mathsf{Eq}(x1, y1), \epsilon, x1, y1) &= \mathsf{Cond}(\mathsf{Eq}(x, y), \epsilon, x1, y1)\\ &\overset{iii.}{=} \mathsf{Cond}(\mathsf{Eq}(x, y), \epsilon, x, y)1\\ &= x1.\end{aligned}$$

So, we obtain that $B(x, y) \to B(x, y0)$ and $B(x, y) \to B(x, y1)$ are derivable. Then, by applying **NP**-induction, we conclude that $\vdash_{I\mathcal{POR}^\lambda} B(x, y)$.

xii. We will show that $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eps}\big(\mathsf{Eq}(x, y)\big) = 0 \wedge \mathsf{Tail}\big(\mathsf{Eq}(x, y)\big) = \epsilon$, from which the claim follows using the defining axioms of Eq and Cond, together with Axiom 3. $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eps}\big(\mathsf{Eq}(x, y)\big) = \mathsf{Cond}(\mathsf{Eq}(x, y), 1, 0, 0) = 0$ can be proved using a double **NP**-induction, similarly to what was done in the previous point. $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Tail}\big(\mathsf{Eq}(x, y)\big) = \epsilon$ is also proved in a similar manner.

xiii. Both $\mathsf{Eps}\big(\mathsf{Tail}(\mathsf{Sub}(x, y))\big) = 1$ and $\mathsf{Eps}\big(\mathsf{Sub}(x, y)\big) = 0$ can be proved by derivable **NP**-induction, similarly to the two cases above.

xiv. From the defining axioms of Rec and the definition of Sub we have that $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Sub}(x, \epsilon) = \mathsf{Eps}(x)$. So, the first claim follows from viii. Moreover, from the same axioms, using vi., we immediately deduce that $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Sub}(x, y\mathsf{b}) = 1 \leftrightarrow \mathsf{Sub}(x, y) = 1 \vee \mathsf{Eq}(x = y\mathsf{b})$. So, we conclude using xi. □

Lemma 26. *lemma6 For any formula A, $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} A$*

Proof.

$$\vdots$$

□

Proposition 14. *For any Σ^b_0-formula $A(x_1, \ldots, x_n)$, there exists a term $t_A(x_1, \ldots, x_n)$ of $\mathcal{POR}^\lambda$ such that:*

1. $\vdash_{I\mathcal{POR}^\lambda} A \leftrightarrow t_A = \mathsf{0}$
2. $\vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{0} \vee t_A = \mathsf{1}$.

Proof. The proof is by induction on the structure of A:

- If $A = \mathtt{Flip}(u)$, then $t_A = \mathsf{BNeg}\big(\mathsf{Flipcoin}(u)\big)$. Indeed, using the points from Lemma 3.1.4, we have

$$\vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{0} \overset{v.}{\leftrightarrow} \mathsf{Flipcoin}(u) = \mathsf{Tail}\big(\mathsf{Flipcoin}(u)\big)\mathsf{1}$$
$$\overset{ix.}{\leftrightarrow} \mathsf{Flipcoin}(u) = \mathsf{1}$$

  and we deduce claim 1. using Axiom 6. Claim 2. follows from Lemma 3.1.4.ix.

- if A is $u = v$, then $t_A = \mathsf{BNeg}\big(\mathsf{Eq}(u, v)\big)$. From Lemma .xi and Lemma 3.1.4.xii we have that $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eq}(u, v) = \mathsf{1} \leftrightarrow = v$ together with $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Tail}\big(\mathsf{Eq}(u, v)\big) = \epsilon$. Using Lemma 3.1.4.v we deduce then $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eq}(u, v) = \mathsf{1} \leftrightarrow \mathsf{BNeg}\big(\mathsf{Eq}(u, v)\big) = \mathsf{0}$ and finally $\vdash_{I\mathcal{POR}^\lambda} \mathsf{BNeg}\big(\mathsf{Eq}(u, v)\big) = \mathsf{0} \leftrightarrow u = v$. Claim 2. is Lemma 3.1.4.xii.

- If A is $u \subseteq v$, then $t_A = \mathsf{BNeg}\big(\mathsf{Sub}(u, v)\big)$. Claim 1. follows from Axiom 2. together with Lemma 3.1.4.v. Claim 2 is Lemma 3.1.4.v.

- If $A = \neg B$, then $t_A = \mathsf{Cond}(t_B, \epsilon, \mathsf{1}, \mathsf{0})$. Indeed, by the IH, we know that $\vdash_{I\mathcal{POR}^\lambda}$. Indeed, by IH, we know that $\vdash_{I\mathcal{POR}^\lambda} B \leftrightarrow t_B = \mathsf{0}$ and that $\vdash_{I\mathcal{POR}^\lambda} t_B = \mathsf{0} \vee t_B = \mathsf{1}$. Using these together with Axiom 5 and Axiom 7 we deduce that

$$\vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{0} \leftrightarrow t_B = \mathsf{Tail}(t_B)\mathsf{0} \overset{IH.2}{\leftrightarrow} \neg(t_B = \mathsf{0}) \overset{IH.1}{\leftrightarrow} \neg B.$$

  Moreover, for IH, $\vdash_{I\mathcal{POR}^\lambda} t_B = \mathsf{0} \vee t_B = \mathsf{1}$ it also follows that $\vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{0} \vee t_A = \mathsf{1}$ sing the defining axioms of Cond.

- If $A = B \wedge C$, then $t_A = \mathsf{BOr}(t_B, t_C)$. By IH, we have that $\vdash_{I\mathcal{POR}^\lambda} (B \leftrightarrow t_B = \mathsf{0}) \wedge (C \leftrightarrow t_C = \mathsf{0})$ and that $\vdash_{I\mathcal{POR}^\lambda} (t_B = \mathsf{0} \vee t_B = \mathsf{1}) \wedge (t_C = \mathsf{1} \vee t_C = \mathsf{1})$. Using these facts together with Lemma 3.1.4.vi and Lemma 3.1.4.vii and the axioms of Cond, it is shown that $\vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{0} \leftrightarrow B \wedge C$ and $\vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{0} \vee t_A = \mathsf{1}$.

- If $A = B \vee C$, then $t_A = \mathsf{BAnd}(t_B, t_C)$. We can argue similarly to the previous case.

- If $A = (\exists x \subseteq u)B$, then $t_A = (t_B)^\exists(\vec{x}, u)$, where for any function $f : \mathsf{s}^{n+1} \to \mathsf{s}$, we let $f^\exists : \mathsf{s}^{n+1} \to \mathsf{s}$ be defined by bounded recursion as follows:

$$f^\exists(\vec{x}, \epsilon) = f(\vec{x}, \epsilon)$$
$$f^\exists(\vec{x}, y\mathsf{0}) = \mathsf{BAnd}\big(f^\exists(\vec{x}, y, z), f(\vec{x}, y\mathsf{0})\big)$$
$$f^\exists(\vec{x}, y\mathsf{1}) = \mathsf{BAnd}\big(f^\exists(\vec{x}, y, z), f(\vec{x}, y\mathsf{1})\big).$$

Let $a = \mathsf{s}$ and $f : \mathsf{s} \to \mathsf{s}$ be fresh variables, and let $C(f, a, y)$ be the formula $\mathsf{Cond}\big(\mathsf{Sub}(a, y)\epsilon, \mathsf{0}, f^{\exists}(y)\big) = \mathsf{0}$. We will show that $f(a) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y)$. From this it follows that

$$\begin{aligned}\vdash_{I\mathcal{POR}^{\lambda}} \big(a \subseteq u \wedge B(a)\big) \leftrightarrow (a \subseteq u \wedge t_B(a)\mathsf{00}) \\ \to \mathsf{Sub}(a, u) = \mathsf{1} \wedge C(t_B, a, y) = \mathsf{0} \\ \to (t_B)^{\exists}(u) = \mathsf{0}\end{aligned}$$

from which we can deduce $\vdash_{I\mathcal{POR}^{\lambda}} (\exists x \subseteq u)B \to (t_B)^{\exists}(u) = \mathsf{0}$. We will prove that $f(a) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y)$ can be derived using **NP**-induction. Let us first show $f(a) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, \epsilon)$: from $\mathsf{Sub}(a, \epsilon) = \mathsf{1} \leftrightarrow a = \epsilon$ (Lemma 3.1.4.xiv), we deduce $C(f, a, \epsilon) \leftrightarrow \big(\mathsf{Sub}(a, \epsilon) = \mathsf{0}\big) \vee (f^{\exists}(\epsilon) = \mathsf{0}) \leftrightarrow \neg(a = \epsilon) \vee \big(f(\epsilon) = \mathsf{0}\big)$. Hence, using $f(a) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} \neg(a = \epsilon) \vee f(\epsilon) = \mathsf{0}$, which can be proved using Axiom 3, we can conclude $f(a) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, \epsilon)$. Let us now show that $f(a) = \mathsf{0}, C(f, a, y) \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y\mathsf{0})$. From $\mathsf{Sub}(a, y\mathsf{0}) = \mathsf{1} \leftrightarrow \mathsf{Sub}(a, y) = \mathsf{1} \vee a = y\mathsf{0}$ (Lemma 3.1.4.xiv) and $\mathsf{Sub}(a, y\mathsf{0}) = \mathsf{0} \vee \mathsf{Sub}(a, y\mathsf{0}) = \mathsf{1}$ (proved below), we deduce $C(f, a, y\mathsf{0}) \leftrightarrow \big(\mathsf{Sub}(a, y\mathsf{0}) = \mathsf{0}\big) \vee \big((\mathsf{Sub}(a, y) = \mathsf{1} \vee a = y\mathsf{0}) \wedge f^{\exists}(y\mathsf{0}) = \mathsf{0}\big)$. Using Lemma 3.1.4.vii we are reduced then to

$$\begin{aligned}C(f, a, y\mathsf{0}) \leftrightarrow \big((\mathsf{Sub}(a, y\mathsf{0}) = \mathsf{0}) \vee (\mathsf{Sub}(a, y) = \mathsf{0} \wedge f^{\exists}(y) = \mathsf{0}) \\ \vee (\mathsf{Sub}(a, y) = \mathsf{1} \wedge f(y\mathsf{0}) = \mathsf{0}) \vee (a = y\mathsf{0} \wedge f^{\exists}(y) = \mathsf{0}) \\ \vee (a = y\mathsf{0} \wedge f(y\mathsf{0}) = \mathsf{0})\big).\end{aligned} \tag{18}$$

Moreover, we also have that

$$C(f, a, y) \leftrightarrow \big((\mathsf{Sub}(a, y) = \mathsf{0}) \vee (\mathsf{Sub}(a, y) = \mathsf{1} \wedge f^{\exists}(y) = \mathsf{0})\big). \tag{19}$$

From the equivalence (18) we deduce

$$f(a) = \mathsf{0}, \mathsf{Sub}(a, y) = \mathsf{0}, \mathsf{Sub}(a, y\mathsf{0}) = \mathsf{0} \vdash_{I\mathcal{POR}} C(f, a, y\mathsf{0}).$$

Moreover, using the fact that $\vdash_{I\mathcal{POR}^{\lambda}} (\mathsf{Sub}(a, y\mathsf{0}) = \mathsf{1} \wedge \mathsf{Sub}(a, y) = \mathsf{0}) \to a = y\mathsf{0}$ we deduce $f(a) = \mathsf{0}, \mathsf{Sub}(a, y) = \mathsf{0}, \mathsf{Sub}(a, y\mathsf{0}) = \mathsf{1} \vdash_{I\mathcal{POR}^{\lambda}} f(y\mathsf{0}) = \mathsf{0}$ and, thus, using the equivalence (18),

$$f(a) = \mathsf{0}, \mathsf{Sub}(a, y) = \mathsf{0}, \mathsf{Sub}(a, y\mathsf{0}) = \mathsf{1} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y\mathsf{0}).$$

Since $\vdash_{I\mathcal{POR}^{\lambda}} \mathsf{Sub}(a, y\mathsf{0}) = \mathsf{0} \vee \mathsf{Sub}(a, y\mathsf{0}) = \mathsf{1}$, we can conclude that

$$f(a) = \mathsf{0}, \mathsf{Sub}(a, y) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y\mathsf{0}). \tag{20}$$

From the equivalence (18), it also follows that:

$$f(a) = \mathsf{0}, \mathsf{Sub}(a, y) = \mathsf{1}, f^{\exists}(y) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y\mathsf{0}). \tag{21}$$

Now, from (19), (20), and (21), we can deduce $f(a) = \mathsf{0}, C(f, a, y) \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y\mathsf{0})$. In a similar way, we can also deduce $f(a) = \mathsf{0}, C(f, a, y) \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y\mathsf{1})$. By applying the induction schema to $C(f, a, y)$, we finally conclude $f(a) = \mathsf{0} \vdash_{I\mathcal{POR}^{\lambda}} C(f, a, y)$.

To prove the converse direction of claim 1., let $D(y)$ be the formula:

$$D(y) := (\exists x \subseteq y)\big(\mathsf{Cond}((t_B)^{\exists}(y), \epsilon, t_B(x), \mathsf{0}) = \mathsf{0}\big).$$

We will show that $\vdash_{I\mathcal{POR}^\lambda} D(y)$ holds. This will suffice, since we can deduce $D(u) \vdash_{I\mathcal{POR}^\lambda} (t_B)^\exists(u) = 0 \to (\exists x \subseteq u)B$: from $\mathsf{Cond}\big((t_B)^\exists(y), \epsilon, t_B(a), 0) = 0\big)$ and $(t_B)^\exists(u) = 0$, it follows that $t_B(a) = 0$, so in particular we deduce:

$$\mathsf{Sub}(a, u) = 1, \mathsf{Cond}\big((t_B)^\exists(y), \epsilon, t_B(a), 0) = 0, (t_B)^\exists(u) = 0 \vdash_{I\mathcal{POR}^\lambda} a \subseteq u \wedge B(a)$$

and thus,

$$\mathsf{Sub}(a, u) = 1, \mathsf{Cond}\big((t_B)^\exists(y), \epsilon, t_B(a), 0 = 0\big), (t_B)^\exists(u) = 0 \vdash_{I\mathcal{POR}^\lambda} (\exists x \subseteq u)B(x).$$

Finally, $D(u), (t_B)^\exists(u) = 0 \vdash_{I\mathcal{POR}^\lambda} (\exists x \subseteq u)B(x)$, as desired. Let us show that $\vdash_{I\mathcal{POR}^\lambda} D(y)$, using **NP**-induction. First, we prove that $\vdash_{I\mathcal{POR}^\lambda} D(\epsilon)$. From $f^\exists(\epsilon) = f(\epsilon)$ we deduce that $\mathsf{Cond}\big((t_B)^\exists(\epsilon), \epsilon, t_B(\epsilon), 0\big) = \mathsf{Cond}(t_B(\epsilon), \epsilon, t_B(\epsilon), 0)$. Since, $\vdash_{I\mathcal{POR}^\lambda} t_B(\epsilon) = 0 \vee t_B(\epsilon) = 1$, we deduce $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Cond}\big((t_B)^\exists(\epsilon), \epsilon, t_B(\epsilon), 0\big) = 0$, and finally $\vdash_{I\mathcal{POR}^\lambda} D(\epsilon)$. We now prove that $D(y) \vdash_{I\mathcal{POR}^\lambda} D(y0)$: suppose $\mathsf{Cond}((t_B)^\exists(y), \epsilon, t_B(a), 0) = 0$ and $\mathsf{Sub}(a, y) = 1$ hold. This is equivalent to supposing

$$\mathsf{Sub}(a, y) = 1 \wedge \big(((t_B)^\exists(y) = 0 \wedge t_B(a) = 0) \vee (t_B)^\exists(y) = 1\big). \tag{22}$$

We can deduce

$$\mathsf{Sub}(a, y) = 1, (t_B)^\exists(y) = 0, t_B(a) = 0, (t_B)^\exists(y0) = 0 \vdash_{I\mathcal{POR}^\lambda} \mathsf{Sub}(a, y0) = 1 \wedge (t_B)^\exists(y0) = 0 \wedge t_B(a) = 0,$$

from which we deduce

$$\mathsf{Sub}(a, y) = 1, (t_B)^\exists(y) = 0, t_B(a) = 0, (t_B)^\exists(y0) = 0 \vdash_{I\mathcal{POR}^\lambda} D(y0).$$

Moreover, we can deduce

$$\mathsf{Sub}(a, y) = 1, (t_B)^\exists(y) = 0, t_B(a) = 0, (t_B)^\exists(y0) = 1 \vdash_{I\mathcal{POR}^\lambda} D(y0).$$

So, using point 2., $\vdash_{I\mathcal{POR}^\lambda} (t_B)^\exists(y0) = 0 \vee (t_B)^\exists(y0) = 1$, proved below, we can deduce:

$$\mathsf{Sub}(a, y) = 1, (t_B)^\exists(y) = 0, t_B(a) = 0 \vdash_{I\mathcal{POR}^\lambda} D(y0). \tag{23}$$

Now, using the fact that $\vdash_{I\mathcal{POR}^\lambda} (t_B)^\exists(y0) = 0 \leftrightarrow (t_B)^\exists(y) = 0 \vee t_B(y0) = 0$, we can deduce

$$\mathsf{Sub}(a, y) = 1, (t_B)^\exists(y) = 1, (t_B)^\exists(y0) = 0 \vdash_{I\mathcal{POR}^\lambda} t_B(y0) = 0$$

from which we obtain

$$\mathsf{Sub}(a, y) = 1, (t_B)^\exists(y) = 1 \vdash_{I\mathcal{POR}^\lambda} \mathsf{Cond}\big((t_B)^\exists(y0), \epsilon, t_B(y0), 0) = 0$$

and finally,

$$\mathsf{Sub}(a, y) = 1, (t_B)^\exists(y) = 1 \vdash_{I\mathcal{POR}^\lambda} D(y0). \tag{24}$$

From (22), (23) and (24), we now deduce $D(y) \vdash_{I\mathcal{POR}^\lambda} D(y0)$. We can deduce $D(y) \vdash_{I\mathcal{POR}^\lambda} D(y1)$ in a similar way. By applying the induction schema to $D(y)$, we conclude $\vdash_{I\mathcal{POR}^\lambda} D(y)$.

It remains to establish point 2. We will prove that

$$\vdash_{I\mathcal{POR}^\lambda} \mathsf{BOr}\big(\mathsf{Eq}((t_B)^\exists(y), 0), \mathsf{Eq}((t_B)^\exists(y), 1)\big) = 1$$

from which the claim follows using Lemma 3.1.4.vi, .xi, and .xii. Let $E(x)$ be the formula $\mathsf{BOr}\big(\mathsf{Eq}((t_B)^{\exists}(y),0),\mathsf{Eq}((t_B)^{\exists}(y),1)\big) = 1$. From $\vdash_{I\mathcal{POR}^{\lambda}} (t_B)^{\exists}(\epsilon) = t_B(\epsilon)$ and $\vdash_{I\mathcal{POR}^{\lambda}} t_B(\epsilon) = 0 \vee t_B(\epsilon) = 1$ we deduce $\vdash_{I\mathcal{POR}^{\lambda}} E(\epsilon)$ using Lemma 3.1.4.vi and .xi. We now show that $E(x) \vdash_{I\mathcal{POR}^{\lambda}} E(x0)$: using Lemma 3.1.4.vi and .xi, together with the definition of $f^{\exists}$, we deduce the following equivalences:

$$\begin{aligned} E(x) &\leftrightarrow (t_B)^{\exists}(x) = 0 \vee (t_B)^{\exists}(x) = 1 \\ E(x0) &\leftrightarrow \big((t_B)^{\exists}(x) = 0 \wedge t_B(x0) = 0\big) \vee (t_B)^{\exists}(x) = 1 \vee t_B(x0) = 1. \end{aligned} \tag{25}$$

Furthermore, we have the following deductions

$$\begin{aligned} (t_B)^{\exists}(x) = 0, t_B(x0) = 0 &\vdash_{I\mathcal{POR}^{\lambda}} E(x0) \\ (t_B)^{\exists}(x) = 1, t_B(x0) = 0 &\vdash_{I\mathcal{POR}^{\lambda}} E(x0) \\ (t_B)^{\exists}(x) = 0, t_B(x0) = 1 &\vdash_{I\mathcal{POR}^{\lambda}} E(x0) \\ (t_B)^{\exists}(x) = 1, t_B(x0) &\vdash_{I\mathcal{POR}^{\lambda}} E(x0) \end{aligned}$$

from which, using (25), we deduce

$$\begin{aligned} E(x), t_B(x0) = 0 &\vdash_{I\mathcal{POR}^{\lambda}} E(x0) \\ E(x), t_B(x0) = 1 &\vdash_{I\mathcal{POR}^{\lambda}} E(x0). \end{aligned}$$

Finally, using $\vdash_{I\mathcal{POR}^{\lambda}} t_B(x0) = 0 \vee t_B(x0) = 1, E(x) \vdash_{I\mathcal{POR}^{\lambda}} E(x0)$. In a similar way, it can be deduced $E(x) \vdash_{I\mathcal{POR}^{\lambda}} E(x1)$. By applying the induction schema to $E(x)$, we finally conclude $\vdash_{I\mathcal{POR}^{\lambda}} E(x)$.

- If $A = (\forall x \subseteq u)B$, then $t_A = (t_B)^{\forall}(\vec{x}, u)$, where for all functions f, $f^{\forall}(y)$ is defined by bounded recursion as follows:

$$\begin{aligned} f^{\forall}(\vec{x},\epsilon) &= f(\vec{x},\epsilon) \\ f^{\forall}(\vec{x},y0) &= \mathsf{BOr}\big(f^{\forall}(\vec{x},y), f(\vec{x},y0)\big) \\ f^{\forall}(\vec{x},y1) &= \mathsf{BOr}\big(f^{\forall}(\vec{x},y), f(\vec{x},y1)\big). \end{aligned}$$

  The arguments to establish claim 1. and 2. are analogous to those of the previous point.

- If $A = (\exists x \subseteq^* u)B$, then let $\mathtt{h}(y,z) := \mathsf{Cond}\big(\mathsf{Sub}(y,z),\epsilon,\epsilon,\mathtt{k}(y,z)\big)$, where $\mathtt{k}(y,z)$ is defined by bounded recursion as follows:

$$\begin{aligned} \mathtt{k}(y,\epsilon) &= y \\ \mathtt{k}(y,z0) &= \mathsf{Cond}\big(\mathsf{Sub}(y,z),\epsilon,\epsilon,\mathtt{k}(y,z)0\big) \\ \mathtt{k}(y,z1) &= \mathsf{Cond}\big(\mathsf{Sub}(y,z)\epsilon,\epsilon,\mathtt{k}(y,z)1\big) \end{aligned}$$

  using z as bound. Then, we can prove that

$$\vdash_{I\mathcal{POR}^{\lambda}} (\exists x \subseteq^* u)B \leftrightarrow (\exists w \subseteq u)\big(\exists x \subseteq \mathtt{h}(w,u)\big)B \tag{26}$$

  from which it will follow that we can let $t_A := t_{(\exists w\subseteq u)(\exists x\subseteq \mathtt{h}(w,u))B}$. Proving (27?) amounts at showing that $\vdash_{I\mathcal{POR}^{\lambda}} \big((\exists w \subseteq u)w \circ x \subseteq u\big) \leftrightarrow \big((\exists w \subseteq u)x \subseteq \mathtt{h}(w,u)\big)$, i.e. $w \subseteq u \vdash_{I\mathcal{POR}^{\lambda}} w \circ x \subseteq u \leftrightarrow x \subseteq \mathtt{h}(w,u)$. This can be proved by applying the induction schema to the formula $F(y) := \mathsf{Sub}(w \circ x, u) = \mathsf{Sub}\big(x, \mathtt{h}(w,u)\big)$.

- If $A = (\forall x \subseteq^* u)B$, then we can show that

$$\vdash_{I\mathcal{POR}^\lambda} (\forall x \subseteq^* u)B \leftrightarrow (\forall w \subseteq u)\big(\forall x \subseteq \mathtt{h}(w,u)\big)B \tag{27}$$

  by arguing similarly to the case above. Thus, let $t_A := t_{(\forall w \subseteq u)(\forall x \subseteq \mathtt{h}(w,u))}$.

□

Corollary 11. *For any closed Σ_0^b-formula A and $\omega \in \mathbb{O}$, either $T_\omega \vdash_{I\mathcal{POR}^\lambda} A$ or $T_\omega \vdash_{I\mathcal{POR}^\lambda} \neg A$.*

Proof. It suffices to show that for all closed A, $T_\omega \vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{0}$ or $T_\omega \vdash_{I\mathcal{POR}^\lambda} t_A = \mathsf{1}$. This can be proved by induction on A. Let us only consider the non-trivial cases:

- If $A = \mathtt{Flip}(u)$, then for hypothesis u is closed, so it corresponds to a string s_u. Then, we deduce $T_\omega \vdash_{I\mathcal{POR}^\lambda} t_A = \overline{\overline{\omega(s_u)}}$, depending on $\omega(s_u) = \mathsf{0}, \mathsf{1}$.
- If $A = (\exists x \subseteq u)B$, then, by IH, for any choice of a string s, we know that $T_\omega \vdash_{I\mathcal{POR}^\lambda} (t_B)^\exists(\overline{s}) = \mathsf{0}$ or $T_\omega \vdash_{I\mathcal{POR}^\lambda} (t_B)^\exists(\overline{s}) = \mathsf{1}$. The claim can be proved by induciton on s.

□

Lemma 27. *For any formula A there exists terms $\mathbf{t}_A^\perp$ such that $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_A^\perp \mathrel{®} A$.*

Proof. The proof is by induction on the structure of A:

- If A is $t = u$, by Axiom 4, $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} (\forall x)(x = \epsilon)$, from which we can deduce $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} t = u$. Since $\Lambda \mathrel{®} A \leftrightarrow A$, we are done.
- If A is $t \subseteq u$, from $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} t = u$ and $\vdash_{I\mathcal{POR}^\lambda} t \subseteq t$, we deduce $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} t \subseteq u$. Again, since $\Lambda \mathrel{®} A \leftrightarrow A$, we are done.
- If $A = \neg B$, since $\mathsf{0} = \mathsf{1}, \mathbf{x} \mathrel{®} B \vdash_{I\mathcal{POR}^\lambda} \mathsf{0} = \mathsf{1}$, we can take $\mathbf{t}_A^\perp := \Lambda$.
- If $A = B \wedge C$, by IH, $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_B^\perp \mathrel{®} B$ and $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_C^\perp \mathrel{®} C$. So, we let $\mathbf{t}_A^\perp := \mathbf{t}_B^\perp, \mathbf{t}_C^\perp$.
- If $A = B \vee C$, by IH, $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_B^\perp \mathrel{®} B$. So, we can take $\mathbf{t}_A := \mathsf{0}, \mathbf{t}_B^\perp$.
- If $A = B \to C$, by IH, $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_C^\perp \mathrel{®} C$. So, we can take $\mathbf{t}_A^\perp = \lambda \mathbf{x}.\mathbf{t}_C^\perp$.
- If $A = (\exists x)B$, by IH, $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_B^\perp \mathrel{®} B$. So, we can take $\mathbf{t}_A^\perp = x, \mathbf{t}_B^\perp$.
- If $A = (\forall x)B$, by IH, $\mathsf{0} = \mathsf{1} \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_B^\perp \mathrel{®} B$. So, we take $\mathbf{t}_A^\perp = \lambda x.\mathbf{t}_B^\perp$.

□

Theorem 6 (Completeness)**.** *If $\vdash_{I\mathcal{POR}^\lambda} A$, then there exist terms $\mathbf{t}$, such that $\vdash_{I\mathcal{POR}^\lambda} \mathbf{t} \mathrel{®} A$.*

Proof. We prove the following statement: if $\Gamma \vdash_{I\mathcal{POR}^\lambda} A$, then there exist terms $\mathbf{t}$ such that $\mathbf{x} \mathrel{®} \Gamma \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}\mathbf{x}_1 \dots \mathbf{x}_n \mathrel{®} A$. We argue by induction on the derivation of $\Gamma \vdash_{I\mathcal{POR}^\lambda} A$. For what concerns the rules of the intuitionistic predicate calculus, the cases of the identity rule and of the rules WL, CL (left weakening and left contraction), $\wedge R, \wedge L_i, \to R, \to L, \forall R, \forall L, \exists R, \exists L$ can be proved in the standard way. Let us consider below the cases of the rules $\vee R_1, \vee L$ and $\perp R$:

- For the rule

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee R_1$$

by IH, there exist terms $\mathbf{u}$, such that $\mathbf{x}$ ® $\Gamma \vdash_{I\mathcal{POR}^\lambda} \mathbf{ux}$ ® A. Since x, y ® $A \vee B \leftrightarrow (x = 0 \wedge y$ ® $A) \vee (x = 1 \wedge y$ ® $B)$, we can take $\mathbf{t} := 0, \mathbf{u}$.

The rule $\vee R_2$ can be treated similarly, by letting $\mathbf{t} := 1, \mathbf{u}$.

- For the rule

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee L$$

by IH, there exist terms $\mathbf{u}_1, \mathbf{u}_2$ such that $\mathbf{x}$ ® Γ, y ® $A \vdash_{I\mathcal{POR}^\lambda} \mathbf{u}_1\mathbf{x}y$ ® C and $\mathbf{x}$ ® Γ, y ® $B \vdash_{I\mathcal{POR}^\lambda} \mathbf{u}_2\mathbf{x}y$ ® C. Since y_1y_2 ® $A \vee B \leftrightarrow (y_1 = 0 \wedge y_2$ ® $A) \vee (y_1 = 1 \wedge y_2$ ® $B)$, we can take

$$\mathbf{t} := \lambda\mathbf{x}.\lambda y_1 y_2.\mathsf{Cond}(y_1, \epsilon, \mathbf{u}_1\mathbf{x}y_2, \mathbf{u}_2\mathbf{x}y_2).$$

- For the rule

$$\frac{\Gamma \vdash 0 = 1}{\Gamma \vdash A} \perp R$$

By IH, we have that $\mathbf{x}$ ® $\Gamma \vdash_{I\mathcal{POR}^\lambda} 0 = 1$. By Lemma 27, $0 = 1 \vdash_{I\mathcal{POR}^\lambda} \mathbf{t}_A^\perp$ ® A. So, we can take $\mathbf{t} := \lambda\mathbf{x}.\mathbf{t}_A^\perp$.

We now consider the realization of the axioms of $I\mathcal{POR}^\lambda$. All atomic axioms (hence all axioms of $\mathcal{POR}^\lambda$) are trivially realized. Let us consider the remaining ones:

- For Axiom 2 we have x, w ® $(x \subseteq y \to \mathsf{Sub}(x, y) = 1) \wedge (\mathsf{Sub}(x, y) = 1 \to x \subseteq y)$ if and only if u ® $(x \subseteq y) \to z(u)$ ® $\mathsf{Sub}(x, y) = 1), x \subseteq y \to \mathsf{Sub}(x, y) = 1)$, u ® $(\mathsf{Sub}(x, y) = 1 \to z(u)$ ® $x \subseteq y)$ and w ® $\big(\mathsf{Sub}(x, y) = 1 \to x \subseteq y\big)$. So, we can take $z := \lambda x.x$ and $w := \lambda x.x$.

- For Axiom 3 we have

$$\begin{aligned} y, z, w \text{ ® } (x = \epsilon) \vee \big((x = \mathsf{Tail}(x)0) \vee (x = \mathsf{Tail}(x)1)\big) \leftrightarrow (y = 0 \wedge z, w \text{ ® } x = \epsilon) \\ \vee \big((y = 1 \wedge z = 0 \wedge w \text{ ® } x = \mathsf{Tail}(x)0) \\ \vee (y = 1 \wedge z = 1 \wedge w \text{ ® } x = \mathsf{Tail}(x)1)\big). \end{aligned}$$

So, we can take $y := \mathsf{BNeg}\big(\mathsf{Eps}(x)\big)$, $z := \mathsf{BNeg}\big(\mathsf{Zero}(x)\big)$ and w any term.

- For Axiom 4, we have $(y$ ® $0 = 1 \to x = \epsilon) \leftrightarrow (\forall z)(y$ ® $0 = 1 \to yz$ ® $x = \epsilon)$. So, we can take $y := \lambda z.\mathsf{Eps}(x)$. Since $0 = 1 \vdash_{I\mathcal{POR}^\lambda} \mathsf{Eps}(x) = 1$ and $\vdash_{I\mathcal{POR}^\lambda} \mathsf{Eps}(x) = 1 \leftrightarrow x = \epsilon$ (Lemma 3.1.4.viii).

- For Axiom 5, we have

$$\begin{aligned} x'y' \text{ ® } \big(\mathsf{Cond}(x, y, z, w) = w' \to (x = \epsilon \wedge w' = y) \\ \vee (x = \mathsf{Tail}(x)0 \wedge w' = z) \\ \vee (x = \mathsf{Tail}(x)1 \wedge z = w)\big) \quad \leftrightarrow \ \mathsf{Cond}(x, y, z, w) = w' \to \big((x' = 0 \wedge x = \epsilon \wedge w' = y) \\ \vee (x' = 1 \wedge y' = 0 \wedge x = \mathsf{Tail}(x)0 \wedge w' = z) \\ \vee (x' = 1 \wedge y' = 1 \wedge x = \mathsf{Tail}(x)1 \wedge z = w)\big). \end{aligned}$$

So, we can let $x' = \mathsf{BNeg}\big(\mathsf{Eps}(x)\big)$ and $y' := \mathsf{BNeg}\big(\mathsf{Zero}(x)\big)$.

- For Axiom 6, we have

$$\Lambda \circledR \big(\mathtt{Flip}(x) \to \mathsf{Flipcoin}(x) = 1) \wedge (\mathsf{Flipcoin}(x) = 1 \to \mathtt{Flip}(x)\big) \leftrightarrow (\mathtt{Flip}(x) \to \mathsf{Flipcoin}(x) = 1) \wedge \big(\mathsf{Flipcoin}(x) = 1 \to \mathtt{Flip}(x)\big).$$

  So, the axiom is automatically realized.

- For the axiom schema 7, we have

$$\mathbf{y} \circledR \big(A(\epsilon) \wedge (\forall x.A(x) \to A(x\mathsf{0})) \wedge (\forall x.A(x) \to A(x\mathsf{1}))\big) \to (\forall x)A(x) \leftrightarrow (\forall x_1x_2x_3x_4)\big(x_1 \circledR A(\epsilon) \wedge ((\forall x')x' \circledR A(x) \to x_2x' \circledR A(x\mathsf{0})) \wedge ((\forall x')x' \circledR A(x) \to x_3x' \circledR A(x\mathsf{1}))) \to yx_1x_2x_3x_4 \circledR A(x_4).$$

  where $A(x) = (\exists x \preceq t)u = v$ and

$$\begin{aligned} x_1 \circledR A(\epsilon) &\leftrightarrow x_1 \subseteq t[\epsilon/z] \wedge y[\epsilon/z, x_1/x] = v[\epsilon/z, x_1/x] \\ x_2\mathbf{x}' \circledR A(x\mathsf{0}) &\leftrightarrow x_2x' \subseteq t[x\mathsf{0}/z] \wedge u[x\mathsf{0}/z, x_2/x] = v[x\mathsf{0}/z, x_2/x] \\ x_3\mathbf{x}' \circledR A(x\mathsf{1}) &\leftrightarrow x_3x' \subseteq t[x\mathsf{1}/z] \wedge u[x\mathsf{1}/z, x_3/x] = v[x\mathsf{1}/z, x_3/x]. \end{aligned}$$

  Now, y can be taken as the term T defined by bounded recursion as follows:

$$\begin{aligned} \mathsf{T}_1x_1x_2x_3\epsilon &= x_1 \\ \mathsf{T}_1x_1x_2x_3z\mathsf{1} &= x_2z(\mathsf{T}_1x_1x_2x_3z) \\ \mathsf{T}_1x_1x_2x_3z\mathsf{1} &= x_3z(\mathsf{T}_1x_1x_2x_3z) \end{aligned}$$

  where the bound is proved by $\lambda z.t$. To establish this claim, let $C(z) := \mathsf{T}x_1x_2x_3z \circledR A(z)$, we will show that $x_1 \circledR A(\epsilon), x_2 \circledR ((\forall x)(A(x) \to A(x\mathsf{0}))), x_3 \circledR ((\forall x)(A(x) \to A(x\mathsf{1}))) \vdash_{I\mathcal{POR}^\lambda} C(z)$. Let $\Gamma = x_1 \circledR A(\epsilon), x_2 \circledR ((\forall x)(A(x) \to A(x\mathsf{0}))), x_3 \circledR ((\forall x)(A(x) \to A(x\mathsf{1})))$. We argue by **NP**-induction on $C(z)$:

  - From $\vdash_{I\mathcal{POR}^\lambda} \mathsf{T}x_1x_2x_3\epsilon = x_1$, we deduce $\Gamma \vdash_{I\mathcal{POR}^\lambda} C(\epsilon)$
  - From $\vdash_{I\mathcal{POR}^\lambda} \mathsf{T}x_1x_2x_3(z\mathsf{0}) = x_2(\mathsf{T}x_1x_2x_3z)$ and $\Gamma, C(z) \vdash_{I\mathcal{POR}^\lambda} x_2z(\mathsf{T}x_1x_2x_3z) \circledR A(z\mathsf{0})$ we deduce $\Gamma, C(z) \vdash C(z\mathsf{0})$.
  - $\Gamma, C(z) \vdash C(z\mathsf{1})$ can be proved similarly to the pervious point.

  We conclude that $\Gamma \vdash C(z)$, i.e. that $\Gamma \vdash \mathsf{T}x_1x_2x_3x_4 \circledR A(x_4)$, which proves the claim.

□

Lemma 28. *Let A be a Σ^b_1-formula. Then,*

i. $\vdash_{I\mathcal{POR}^\lambda} (x \Vdash A) \leftrightarrow (A \vee T(x))$

ii. $\vdash_{I\mathcal{POR}^\lambda} (x \Vdash \neg A) \leftrightarrow (A \to T(x))$

iii. $\vdash_{I\mathcal{POR}^\lambda} (x \vDash \neg\neg A) \leftrightarrow (A \vee T(x))$.

Proof. i. $x \Vdash (\exists x \preceq t)B \leftrightarrow x \Vdash (\exists x)t_{x \subseteq t \wedge B}(x) = \mathsf{0} \leftrightarrow (\exists x)\big((x \subseteq t \wedge B) \vee T(x)\big) \leftrightarrow A \vee T(x)$.

ii. $x \Vdash \neg(\exists x \preceq t)B \leftrightarrow (\forall y)\big(y \Vdash A \to T(x * y)\big)$, which, by the previous point, is equivalent to $(\forall y)\big(A \vee T(y) \to T(x * y)\big)$, which is equivalent to $A \to T(x)$, using $\vdash_{I\mathcal{POR}^\lambda} T(y) \to T(x * y)$ and $\vdash_{I\mathcal{POR}^\lambda} T(x * \mathbb{1}) \leftrightarrow T(x)$.

iii. On the one hand, $x \Vdash A \to x \Vdash \neg\neg A$ is easy proved, using point i. On the other hand, using point ii., $x \Vdash \neg\neg A$ is equivalent to $(\forall y)\big((A \to T(y)) \to T(x * y)\big)$. By Lemma 4, given $\vdash_{I\mathcal{POR}^\lambda} A \leftrightarrow T(u_A)$, we deduce $(x \Vdash \neg\neg A) \to T(x * u_A)$. To conclude, we observe that $T(x * u_A)$ is equivalent to $A \vee T(x)$, as it can be deduced using Lemma 4 and $\vdash_{I\mathcal{POR}^\lambda} T(x * u_A) \leftrightarrow T(x) \vee T(u_A)$.

□

Lemma 29. *Let $A(x) = (\exists y \preceq t)u = \mathsf{0}$ and B be any formula not containing free occurrences of x. Then, there exists a term t such that:*

$$\vdash_{I\mathcal{POR}} t \circledR PIND(A(x) \vee B).$$

Proof. Let $D(x) = A(x) \vee B$. It suffices to construct a realizer of

$$\big(D(\epsilon) \wedge (\forall x)(A(x) \to D(x\mathsf{0})) \wedge (\forall x)(A(x) \to D(x\mathsf{1}))\big) \to (\forall x)(D(x)).$$

Let us assume that:

- $\mathbf{x}_1 \circledR D(\epsilon)$, that is $\mathbf{x}_1 = x_1^0 x_1^1 \mathbf{x}_1^2$, where
$$\big(x_1^0 = \mathsf{0} \wedge x_1^1 \preceq t[\epsilon/x] \wedge u[\epsilon/x, x_1^1/y] = \mathsf{0}\big) \vee \big(x_1^0 = \mathsf{1} \wedge \mathbf{x}_1^2 \circledR B\big).$$
- $\mathbf{x}_2 \circledR (\forall x)(A(x) \to D(x\mathsf{0}))$, where $\mathbf{x}_2 = x_2^0 x_2^1 \mathbf{x}_2^2$ and under the assumption $y \preceq t$ and $u = \mathsf{0}$,
$$\big(x_2^0 xy = \mathsf{0} \wedge x_2^1 xy \preceq t[y\mathsf{0}/x] \wedge u[y\mathsf{0}/x, x_2^1 xy/y] = \mathsf{0}\big) \vee \big(x_2^0 xy = \mathsf{1} \wedge \mathbf{x}_2^2 xy \circledR B\big).$$
- $\mathbf{x}_3 \circledR (\forall x)(A(x) \to D(x\mathsf{1}))$, where $\mathbf{x}_3 = x_3^0 x_3^1 \mathbf{x}_3^2$ and under the assumption $y \preceq t$ and $u = \mathsf{0}$,
$$\big(x_3^0 xy = \mathsf{0} \wedge x_3^1 xy \preceq t[y\mathsf{1}/x] \wedge u[y\mathsf{1}/x, x_3^1 xy/y] = \mathsf{0}\big) \vee \big(x_3^0 xy = \mathsf{1} \wedge \mathbf{x}_3^2 xy \circledR B\big).$$

We will construct a term $\mathtt{U} : \mathsf{s} \to \mathsf{s}$, satisfying the following specifications:

a. $\vdash_{IRS_3^1} \mathsf{Eps}(\mathtt{U}x) = \mathsf{0}$

b. If $\mathsf{B}(\mathtt{U}x) = \mathsf{0}$, then $\mathsf{Tail}(\mathtt{U}x) \circledR A(x)$

c. If $\mathsf{B}(\mathtt{U}x) = \mathsf{1}$, then
$$x_1^0 = \mathsf{1} \vee \big(y_0 \preceq t[x_0/y] \wedge u[x_0, y_0/x] = \mathsf{0} \wedge \mathsf{BOr}(\mathbf{x}_2^2 x_0 y_0, \mathbf{x}_3^2 x_0 y_0) = \mathsf{1}\big)$$
where $x_0 = \mathsf{Tail}(\mathtt{U}x)$ and $y_0 = \mathsf{Tail}(x_0)$.

Notice that the three conditions above can all be specified by way of an equation of the form $v[\mathtt{U}, x] = \mathsf{0}$. Indeed, if such a term exists, then we can define $t = t_1 t_2 t_3$ realizing $A(x) \vee B$ by

$$\begin{aligned} t_1 x &:= B(\mathtt{U}x) \\ t_2 x &:= \mathsf{Tail}(\mathtt{U}x) \\ t_3 x &:= \mathsf{Cond}\big(x_1^0, \epsilon, \mathbf{x}_1^2, \mathsf{BOr}(\mathbf{x}_2^2 x_0 y_0, \mathbf{x}_3^2 x_0 y_0)\big). \end{aligned}$$

We can define $\mathtt{U}$ by bounded recursion as follows:

$$\begin{aligned}
\mathtt{U}\epsilon &:= \mathsf{Cond}(x_1^0, \epsilon, x_1^1 0, 1)\\
\mathtt{U}(x0) &:= \mathsf{Cond}\big(\mathtt{U}x, \epsilon, \mathsf{Cond}(x_2^0 x(\mathsf{Tail}(\mathtt{U}x)), \epsilon, x_2^1 x(\mathsf{Tail}(\mathtt{U}x))1), \mathsf{Cond}(x_1^0, \epsilon, \epsilon, 1)1\big)\\
\mathtt{U}(x1) &:= \mathsf{Cond}\big(\mathtt{U}x, \epsilon, \mathsf{Cond}(x_2^0 x(\mathsf{Tail}(\mathtt{U}x)), \epsilon, x_2^1 x(\mathsf{Tail}(\mathtt{U}x))0, \mathbf{x}_2^2 x(\mathsf{Tail}(\mathtt{U}x))1), \mathsf{Cond}(x_1^0, \epsilon, \epsilon, 1)1\big).
\end{aligned}$$

We show that $\Gamma \vdash_{IRS_3^1} v[\mathtt{U}, x] = 0$, where $\Gamma = \mathbf{x}_1 \circledR D(\epsilon), \mathbf{x}_2 \circledR (\forall x)(A(x) \to D(x0)), \mathbf{x}_3 \circledR (\forall x)(A(x) \to D(x1))$, by **NP**-induction:

- $v[\mathtt{U}, \epsilon] = 0$:
  a. From $\mathbf{x}_1 \circledR D(\epsilon)$ it can be deduced that $x_1^0 = 0 \vee x_1^0 = 1$. Using this fact, together with $\mathsf{Eps}(\mathtt{U}\epsilon) = \mathsf{Eps}(\mathsf{Cond}(x_1^0, \epsilon, x_1^1 0, 1) = 0$, the claim can be proved using the defining axioms of Cond.
  b. $\mathsf{B}(\mathtt{U}\epsilon) = 0 \to \mathtt{U}\epsilon = x_1^1 0$, whence $\mathsf{Tail}(\mathtt{U}\epsilon) = x_1^1$ which, by hypothesis, realizes $A(\epsilon)$.
  c. $\mathsf{B}(\mathtt{U}\epsilon) = 1 \to x_1^0 = 1$.
- $v[\mathtt{U}, x] = 0 \to v[\mathtt{U}, x0] = 0$. Suppose $v[\mathtt{U}, x] = 0$ holds, i.e. a., b. and c. holds for x:
  a. From $x_1^0 = 0 \vee x_1^0 = 1$ and the fact that $x_1^0 = 0 \vdash_{I\mathcal{POR}^\lambda} \mathsf{B}(\mathtt{U}(x0)) = 0$ and $x_1^0 = 1 \vdash_{I\mathcal{POR}^\lambda} \mathsf{B}(\mathtt{U}(x0)) = 1$, we deduce $\mathsf{B}(\mathtt{U}(x0)) = 0 \vee \mathsf{B}(\mathtt{U}(x0)) = 1$, which implies $\mathsf{Eps}(\mathtt{U}(x0)) = 0$.
  b. $\mathsf{B}(\mathtt{U}0)) = 0$ implies $\mathtt{U}x = 0$ together with

  $$\mathsf{Tail}(\mathtt{U}(x0)) = \mathsf{Cond}(x_2^0 x(\mathsf{Tail}(\mathtt{U}x)), \epsilon, x_2^1 x(\mathsf{Tail}(\mathtt{U}x))0, \mathbf{x}_1^2 x(\mathsf{Tail}(\mathtt{U}x))1) x_2^0(\mathsf{Tail}(\mathtt{U}x)) = 0.$$

  From $v[\mathtt{U}, x]$ and $\mathtt{U}x = 0$, we deduce $\mathsf{Tail}(\mathtt{U}x) \circledR A(x)$ and, thus, by $\mathbf{x}_2 \circledR (\forall x)(A(x) \to D(x0))$, also $\mathbf{x}_2 x(\mathsf{Tail}(\mathtt{U}x)) \circledR D(x0)$. From $x_2^0 x(\mathsf{Tail}(\mathtt{U}x)) = 0$, we conclude $\mathsf{Tail}(\mathtt{U}(x0)) = x_2^1 x(\mathsf{Tail}(\mathtt{U}x)) \circledR A(x0)$.
  c. If $\mathsf{B}(\mathtt{U}(x0)) = 1$, then two possible cases arise:
     * $\mathsf{B}(\mathtt{U}x) = 1$ and $x_1^0 = 1$.
     * $\mathsf{B}(\mathtt{U}x) = 0$ and $x_2^0 x(\mathsf{Tail}(\mathtt{U}x)) = 1$. Then, by $v[\mathtt{U}, x]$, b. we deduce that $\mathsf{Tail}(\mathtt{U}x) \circledR A(x)$. So, by $\mathbf{x}_2 \circledR (\forall x)(A(x) \to D(x1))$, also $\mathbf{x}_2 x(\mathsf{Tail}(\mathtt{U}x)) \circledR D(x0)$. Moreover, from $x_2^0 x(\mathsf{Tail}(\mathtt{U}x)) = 1$, we conclude $\mathsf{Tail}(\mathtt{U}(x0)) = \mathbf{x}_2^2 x(\mathsf{Tail}(\mathtt{U}x)) \circledR B$.
- $v[\mathtt{U}, x] = 0 \to v[\mathtt{U}, x1] = 0$. This can be proved similarly to the previous case.

□

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