Investigating Centers of Triangles: The Fermat Point

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by

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ABSTRACT

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Somewhere along their journey through their math classes, many students develop a fear of mathematics. They begin to view their math courses as the study of tricks and often seemingly unsolvable puzzles. There is a demand for teachers to make mathematics more useful and believable by providing their students with problems applicable to life outside of the classroom with the intention of building upon the mathematics content taught in the classroom.

This paper discusses how to integrate one specific problem, involving the Fermat Point, into a high school geometry curriculum. It also calls educators to integrate interesting and challenging problems into the mathematics classes they teach. In doing so, a teacher may show their students how to apply the mathematics skills taught in the classroom to solve problems that, at first, may not seem directly applicable to mathematics. The purpose of this paper is to inspire other educators to pursue similar problems and investigations in the classroom in order to help students view mathematics through a more useful lens. After a discussion of the Fermat Point, this paper takes the reader on a brief tour of other useful centers of a triangle to provide future researchers and educators a starting point in order to create relevant problems for their students.

Investigating Centers of Triangles: The Fermat Point
by Katie Strauss

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An Introduction to Investigating Centers of Triangles: The Fermat Point

If someone were to ask you to discuss the centers of a triangle, what comes to mind? You might quickly think of the centroid and distribution of equal areas or the circumcenter and a point equidistant from each vertex of the triangle. With a little more time you might eventually come up with the incenter and a point equidistant from each side of a given triangle, in addition to the orthocenter and the intersection of a triangle's altitudes. Or better yet, what would you have to say about the advances of Pierre de Fermat in geometry? Have you ever learned (or even heard…) about the Fermat point? The NCTM (2000) Standards for Geometry for grades 9-12 state that students must be able to apply their studies of theorems and constructions to solve "real-life" problems. Students should:

- explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them;
- use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

This indicates a need to provide students with specific problems applying to the mathematics taught in a general high school geometry classroom. With this demand in mind, consider the following problem inspired by discussions of the Fermat point in the introduction to *Minimal Networks: The Steiner Problem and Its Generalizations* by Alexandr O. Ivanov and Alexei A. Tuzhilin (1994):

New Business Problem

A company has established three new businesses; one in Columbus, one in Cleveland, and one in Pittsburgh (see *Figure 1*). They want to figure out how to connect the power lines between the three cities so that they can minimize the total cost of the power lines (that is, use the least amount of wiring possible between cities). Approximately where should the wiring from each city meet?

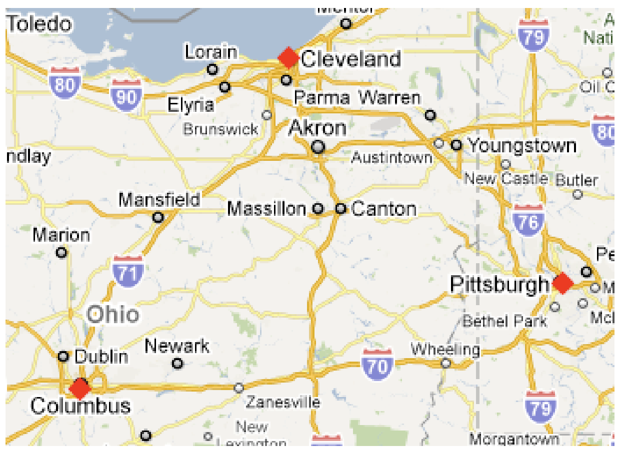

Figure 1

While not immediately obvious, the solution to this problem is one of the lesser-known centers of a triangle, called the *Fermat point*. Most high school curricula teach the four previously mentioned centers of a triangle: centroid, circumcenter, orthocenter, and incenter, but there is likely no mention of the Fermat point or other points of concurrency in a triangle. In fact, I had never heard of the Fermat point until my senior year of college as I scanned the chapter project ideas in the textbook, *Mathematics for High School Teachers: An Advanced Perspective* (2003). In Chapter 8, the following project is presented (p. 429):

Napoleon triangles and concurrencies. Create a dynamic geometry drawing that begins with the vertices of a ΔABC as free points.

a. Construct equilateral triangles *LBC*, *MAC*, and *NAB* such that *L*, *M*, and *N* are exterior to ΔABC. Prove that $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ are concurrent at the Fermat point of ΔABC.
b. Let *P*, *Q*, and *R* be the centers of the equilateral triangles *LBC*, *MAC*, and *NAB*. Verify dynamically that $\overline{AP}$, $\overline{BQ}$, and $\overline{CR}$ are concurrent at a point (the **Napoleon point** or **Torricelli point** of ΔABC).
c. Construct equilateral triangles *L'BC*, *M'AC*, *N'AB* that overlap ΔABC. Then construct segments corresponding to those constructed in parts **a** and **b**. Do there exist corresponding points of concurrency?

Having no previous knowledge of the *Fermat*, *Napoleon*, or *Torricelli points*, I undertook an investigation of the different centers of triangles. After an overwhelming exploration of the different centers (Clark Kimberling's *Encyclopedia of Triangle Centers* contains over 3,000 entries!), I decided to further investigate the history and application of this so-called "Fermat point" and consider how it might find a place in a typical high school geometry curriculum.

History: The Initial Problem

In the 1600's, Pierre de Fermat posed the initial problem of interest, whose solution is the *Fermat point*:

> In a given acute-angled triangle *ABC*, locate a point *P* whose Euclidean distances from *A*, *B*, and *C* have the smallest possible sum.
>
> *(Usiskin et. al., 2003, p. 372)*

In other words, in any triangle with all angles less than 90°, find a point *P*, that minimizes the total distance of *AP* + *BP* + *CP* (see *Figure 2*). There are two well-known solutions to this problem, which will help solve the *New Business Problem*.

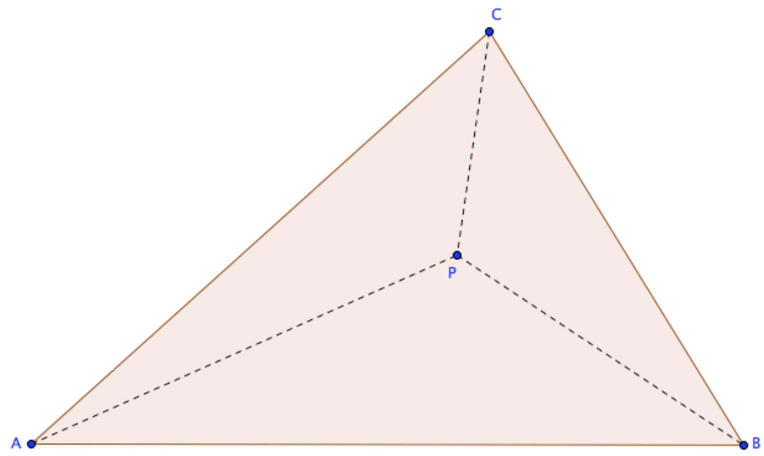

Figure 2

History: The Initial Solutions

By 1640, Torricelli proposed the first solution. First, consider the given acute-angled ΔABC. Assuming previous knowledge of common constructions (e.g. equilateral triangles, circumcircles, etc.), construct an equilateral triangle on each side of the triangle. Then, construct the circumcircles around the three equilateral triangles, as shown in *Figure 3*. The intersection of the three circles is the solution, called the *Torricelli point* (Ivanov & Tuzhilin, 1994, p. *xi-xii*).

In 1750, Simpson proposed an alternative solution, considering the same ΔABC, with all angles less than 120°. As before, construct an equilateral triangle on each side of the triangle. Connect the outermost vertex of each equilateral triangle to the opposite vertex on ΔABC, as shown in *Figure 4*. These three lines are called the *Simpson lines* and their intersection is the same as the *Torricelli point*, also called the *Fermat point* (Ivanov & Tuzhilin, 1994, p. *xiii*).

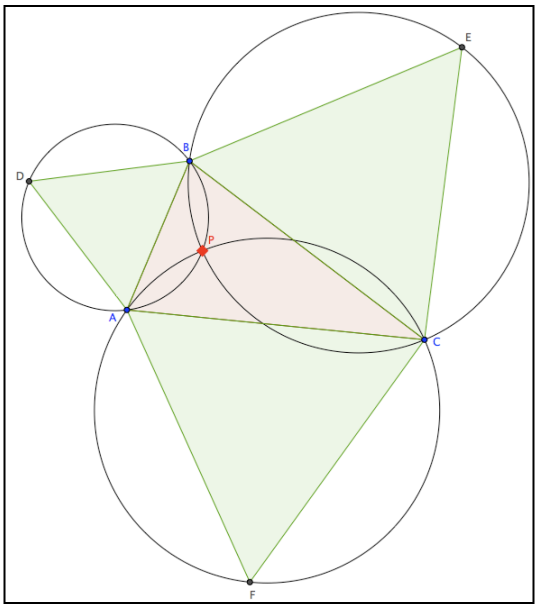

Figure 3

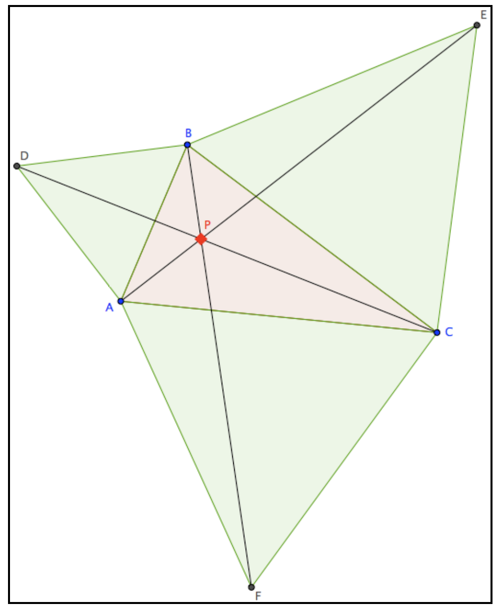

Figure 4

But, what if a triangle, ΔDEF, has one angle over 120°, say $\angle DEF$ (as shown below in *Figure 5*)? In this case, the point that minimizes the total distances would simply be the vertex, *E*, of the obtuse angle (Ivanov & Tuzhilin, 1994, p. *xiii*), though this is not immediately obvious.

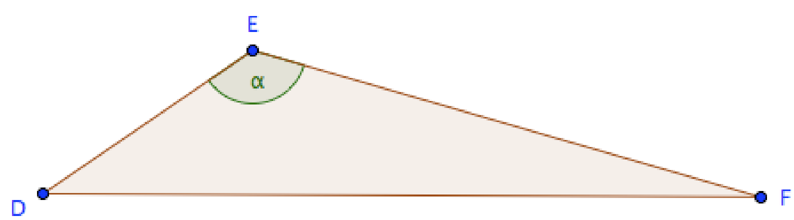

Figure 5

In order to see this, first consider the same constructions as proposed above, by Torricelli and Simpson. The solution using either of these constructions yields a point outside ΔDEF (see *Figure 6* and *Figure 7*). The construction using the Simpson lines (*Figure 7*) illustrates that the point *P* does not solve the problem posed by Fermat. First, extend $\overline{EI}$, since it appears (from the other two Simpson lines) that the point of interest, *P*, lies outside ΔDEF. Notice that $\overline{DP} > \overline{DE}$ and $\overline{FP} > \overline{EF}$. Then, it is clear that the point *P* is not the correct solution. Therefore, the point *E* is the solution for this case.

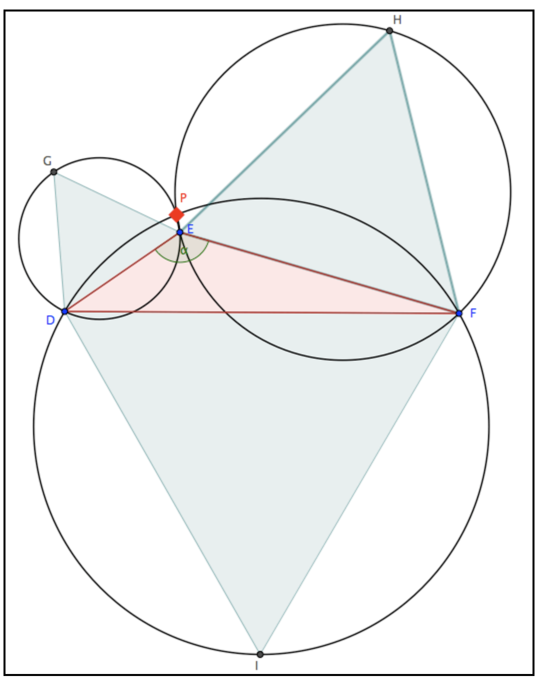

Figure 6

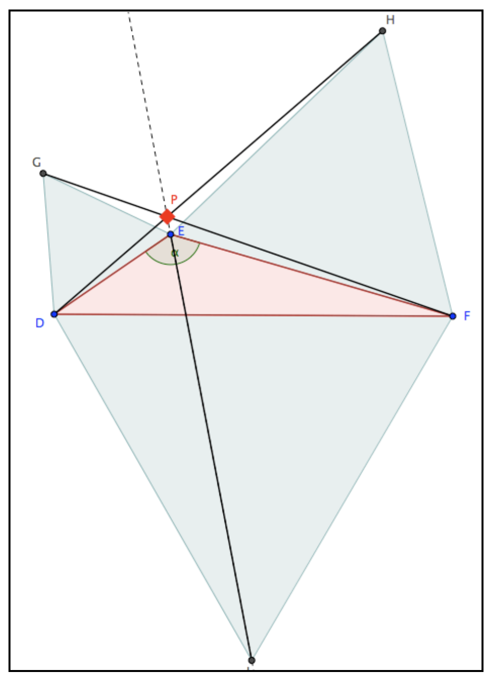

Figure 7

Solution to the *New Business Problem*

Returning to the original problem:

A company has established three new businesses; one in Columbus, one in Cleveland, and one in Pittsburgh (see *Figure 8*). They want to figure out how to connect the power lines between the three cities so that they can minimize the total cost of the power lines (that is, use the least amount of wiring possible between cities). Approximately where should the wiring from each city meet?

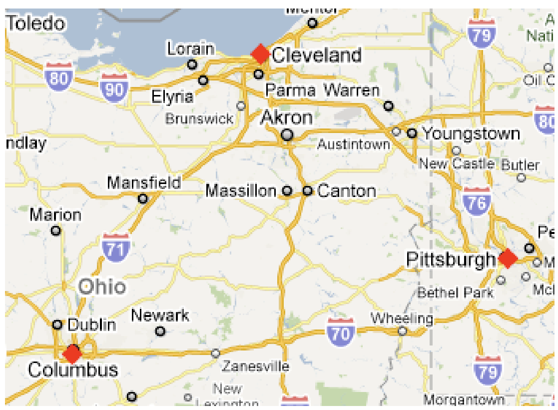

Figure 8

How might we solve this problem? We know that we want to find a point P so that the total distance from Columbus to P, Cleveland to P, and Pittsburgh to P is minimized, thus minimizing the total cost of the power lines between the three cities.

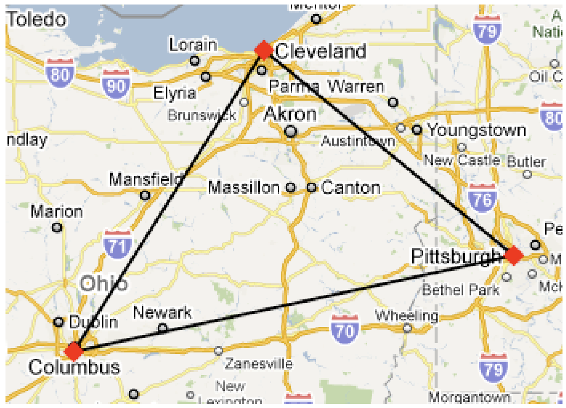

Figure 9

First, consider the map, shown in *Figure 9*, and note the shape of the triangle formed by “connecting” the three cities. None of the angles of the triangle formed exceeds 120°, therefore, one of the earlier solutions of Fermat’s problem will suffice.

Construct the equilateral triangles on each side of the triangle, as shown in *Figure 10*. Connect the outermost vertex of each equilateral triangle to the opposite vertex corresponding to one of the three cities.

Figure 10

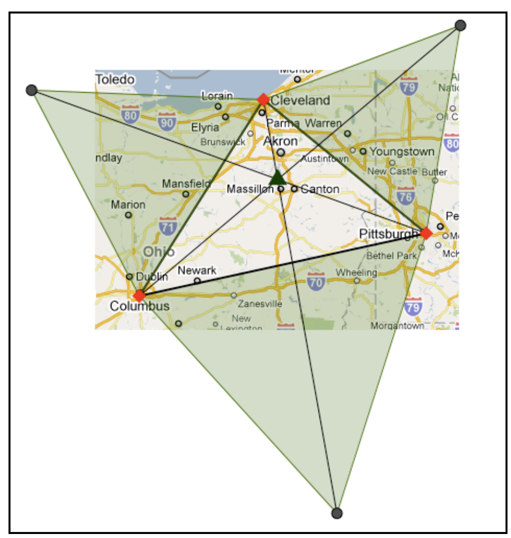

Figure 11

The intersection of these three segments yields a point that solves our problem (as shown in *Figure 11*), but further analysis of the map should lead us to make a best decision about where to place the meeting point of the power lines. The point that solves the problem does not appear to lie on a city on the map. This kind of reasoning is essential for high school geometry students. Once they have found the solution to the problem, they must decide whether or not it makes sense, and how to complete the problem to make the most sense in terms of the original statement of the problem.

Returning to the *New Business Problem*, we must consider that while this construction may yield the precise solution, one would need to make a decision about the location for the wires that makes the most sense geographically. The precise solution to this problem gives the leaders of the company a general location for the meeting point of the power lines. They would then need to choose the location that would be able to support the required power lines and not impede the topography of the chosen location. Considering how the mathematical solution would need to be altered to best fit the original problem would allow students to make connections between the mathematics that they have learned to real-world examples.

An Attempt to Generalize

Suppose the same company as before now wants to establish a new business in Toledo (see *Figure 12*). Again, they want to figure out how to connect the power lines between the <u>*four*</u> cities so that they can minimize the total cost of the power lines. Are the solutions to Fermat's problem able to be generalized beyond *three* points?

Figure 12

Perform the similar constructions as before by connecting each city in the shape of a quadrilateral and constructing equilateral triangles along each side of the quadrilateral. This construction does not yield a reasonable solution. In fact, there are multiple possible solutions (see *Figure 13*, *Figure 14*, *Figure 15*, and *Figure 16*). Each construction yields at least two points of concurrency, and it is unclear whether or not any of these constructions actually solves the four-point problem. The advanced mathematics required to solve this problem in its entirety lead into the “General Steiner Problem,” which requires an understanding of metric spaces and topology.

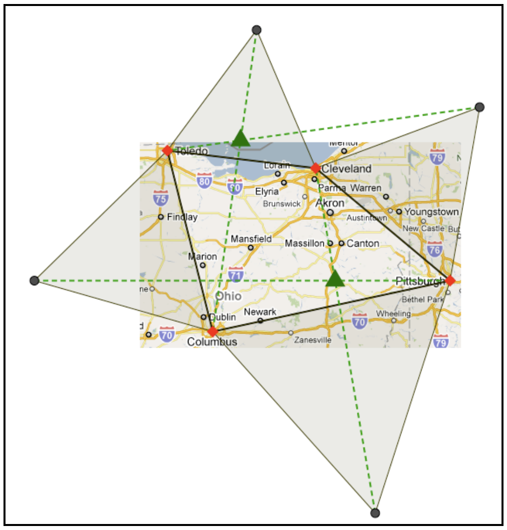

Figure 13

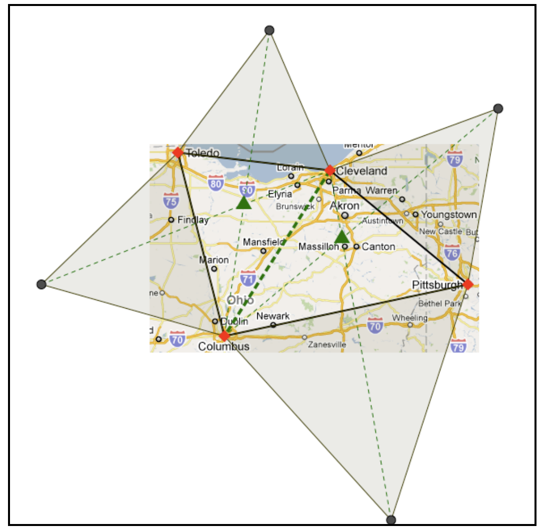

Figure 14

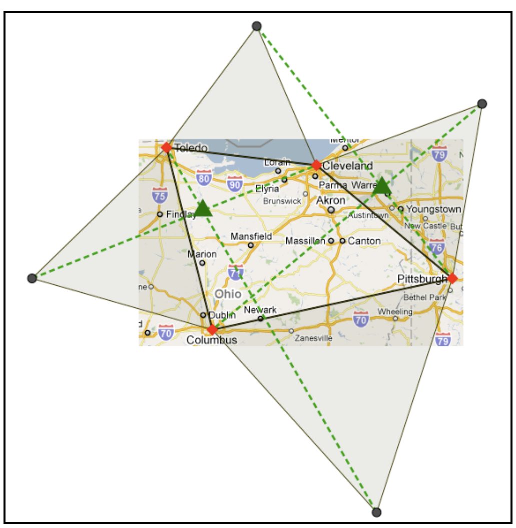

Figure 15

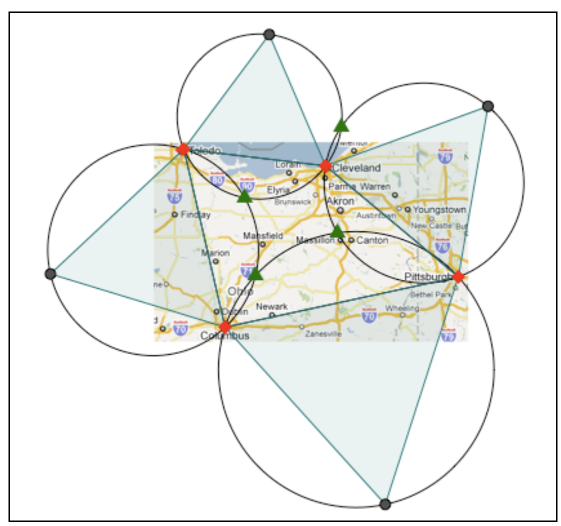

Figure 16

Extension

Much research has been done on the generalized version of this problem. It was made popular and attributed mainly to Jacob Steiner in the publication of the book, What is Mathematics? by Richard Courant and Herbert Robbins in 1941. The general problem, involving more than three points, is now referred to as the "General Steiner Problem," proposed by Jarnik and Kössler in 1934:

> *Find a shortest network spanning n points in the plane* (Ivanov & Tuzhilin, 1994, p. *xiv*).

Many mathematicians today continue to study this problem because of its many applications. Consider, for example, a problem similar to the *New Business Problem* mentioned earlier, except with ten or twenty cities. How does the problem change? What is required to finding this solution? Unfortunately, the advanced mathematics required to fully understand the general problem is very complicated and beyond the depth of the content already provided, but this sets up several more questions for further investigation. For example:

- How might this problem be extended beyond three points (or, how can this problem be extended to problems involving polygons other than triangles)? Is the solution exactly the same? If so, why? If not, how is it similar?
- What are other "basic" centers of a triangle? What are the uses of this center beyond its construction? How might knowing this center further the development of another branch of applied mathematics?
- What role does technology play in investigating Fermat's problem? What role

does it play in investigating the “General Steiner Problem”?

Other Useful Centers

As previously mentioned, there are *thousands* of other known centers of a triangle, and many of them provide useful and interesting investigations. For example, the following three centers of a triangle provide applications similar to those of the Fermat Point: First Napoleon Point, Spieker Center, and Nine-Point Center. I mention these briefly to spur interest in investigations of other centers of triangles for my colleagues.

The First Napoleon Point has a very similar construction to the Fermat Point. In fact, this point was mentioned in the chapter project of *Mathematics for High School Teachers: An Advanced Perspective* (2003) that initially inspired this study of the Fermat Point. To construct the Napoleon Point, first, construct equilateral triangles on each side of the original ΔABC and connect their centers to the opposite vertex on ΔABC. The point of concurrency, *P*, is the *First Napoleon Point* (see *Figure 17*). Interestingly, connecting these three centers yields an equilateral triangle. Future researchers might investigate why this construction forms another equilateral triangle and what purpose this point of concurrency serves beyond the initial construction.

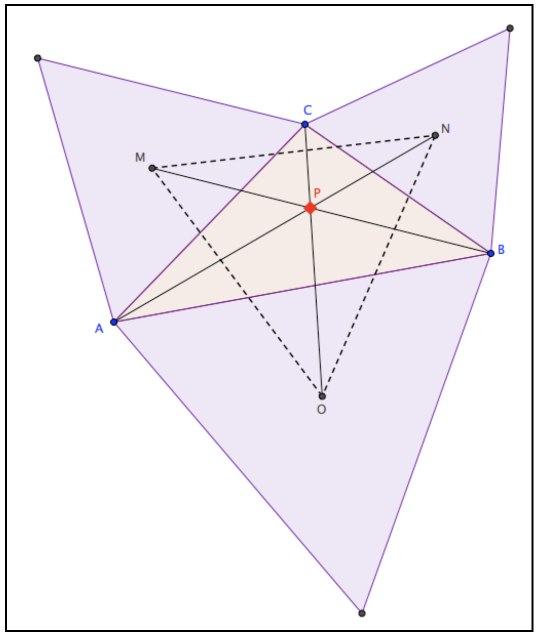

Figure 17

To construct the Spieker Center, construct the incircle of the median triangle (i.e. the triangle formed by connecting the midpoints of the original triangle). The center of the circle, *P*, is the *Spieker Center*, which is also the center of mass of the *perimeter* of the original ΔABC (see *Figure 18*). Again, future researchers might consider an investigation about the specific applications of this point of concurrency.

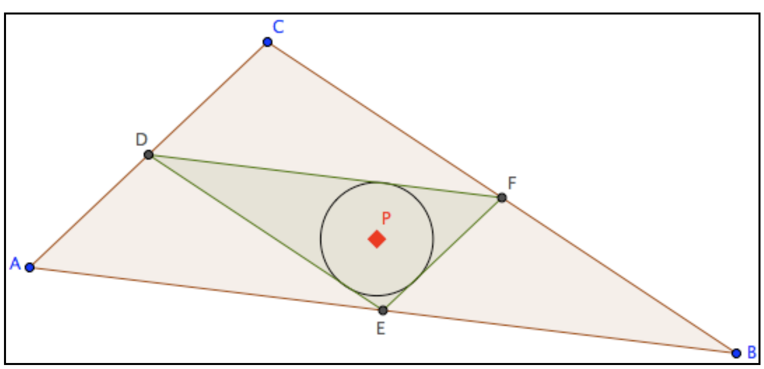

Figure 18

Finally, the Nine-Point Center is the center of the circle passing through nine critical points of the original ΔABC. First, construct the altitudes of AC, BC, and AB to obtain points D, E, and F, respectively. Connect these points to construct ΔDEF (see *Figure 19*). The circumcircle of ΔDEF passes through: 1) points D, E, and F, 2) the midpoints of AC, BC, and AB (points M, E, and K, respectively), and 3) the midpoints of AG, BG, and CG (points, N, P, and O, respectively), where G is the orthocenter of ΔABC. J is the *Nine-Point Center* (see *Figure 20*). Similarly, a study of this center and its applications beyond this construction would be an interesting topic of investigation.

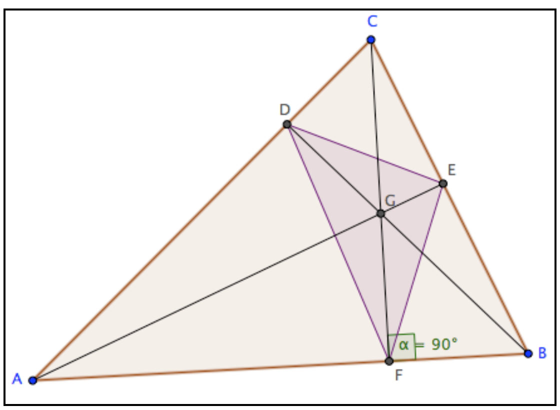

Figure 19

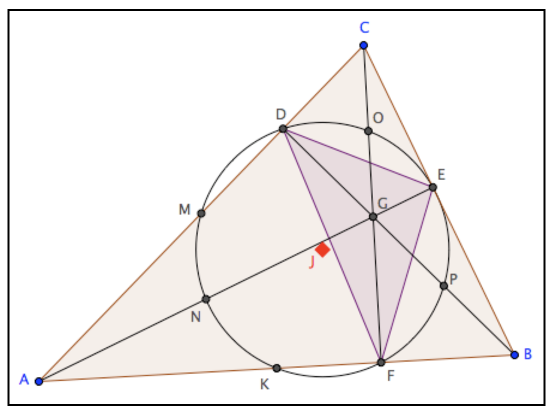

Figure 20

Why the Fermat Point?

At first glance, one might ask, “With thousands of other centers of triangles to study, why choose the Fermat point?” Honestly, there is no other reason beyond general interest in an initial problem. Additionally, upon further investigation, the usefulness of the Fermat point made it stand out above many of the other thousands of triangle centers.

Clearly, Fermat's problem is applicable to many real-life situations. Some variation of the problem, similar to the *New Business Problem*, could easily be integrated into the curriculum as a rich task during a study of relationships of triangles. This would enable students to understand the relevance of one of the thousands of triangle centers.

It is my hope that other teachers will take this idea and introduce similar topics in their classrooms; that someone will find the topic of centers of triangles equally fascinating and encourage their students to further investigate other lesser-known centers in a similar manner. Most schools do not extend their teaching of the relationships within triangles, specifically the importance and applications of the *different* centers of triangles. Many new advances can be made in applied mathematics as a result of the study of these relationships.

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