Combinatorial Optimization: homework 1

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1 Graphs

1.1 Definitions

A symmetric graph is *d-regular* if all of its vertices have the same degree d. A 3-regular graph is also called *cubic*.

Theorem 1 *Any bridgeless cubic graph G admits a perfect matching.*

Definition 1 *A d-regular graph $G = (V, E)$ is* factorable *into $G_1 = (V, E_1), \ldots, G_p = (V, E_p)$ if the G_i's are regular and $E = E_1 \cup \ldots \cup E_p$. If $p = d$ then we say that G is 1-factorable (because all of its factors G_i are 1-regular).*

Definition 2 *An* edge-colouring *of a symmetric graph G assigns a colour to each edge of G in such a way that no two adjacent edges are coloured the same.*

The least number of colours required for an edge-colouring of a graph G is called the *chromatic index* of G.

An *interval graph* draws the binary relation of non-empty intersection between intervals of $\mathbb{R}$. A *split graph* is a graph whose vertex set can be partitioned into a clique and a stable set. A *chord* in a cycle C is an edge whose endpoints are not adjacent in C.

Definition 3 *A symmetric graph G is* chordal *(or* triangulated*) if every induced cycle of G has a chord.*

1.2 Exercises

Exercise 1 *Draw a cubic graph with chromatic index 3 and one with chromatic index 4.*

Exercise 2 *Show that any cubic graph has chromatic index ≤ 4.*

Exercise 3 *Show that any interval graph is chordal.*

Exercise 4 *Show that any split graph is chordal.*

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2 0-1 LP formulations

2.1 Problems

Problem 1 (EQUICUT) *Given a symmetric graph $G = (V, E)$, $|V|$ even, and a weight function $c : E \to I\!R$, find a partition of V into equally sized subsets V_1, V_2 such that the total weight of the set of edges with one endpoint in V_1 and the other in V_2 is minimized.*

Problem 2 (COMPLETE BALANCED BIPARTITE SUBGRAPH) *Given a symmetric graph $G = (V, E)$, find a (not necessarily induced) complete bipartite subgraph $B = (V_1 \cup V_2, V_1 \times V_2)$ of G such that $|V_1| = |V_2|$ is maximum.*

Problem 3 (SET COVERING) *Given a discrete finite set S (ground set) and subsets $S_1, \ldots, S_m$ of S, find a minimum cardinality set C^* of indexes such that $\bigcup_{i \in C^*} S_i = S$.*

Problem 4 (SET PACKING) *Given a discrete finite set S (ground set) and subsets $S_1, \ldots, S_m$ of S, find a maximum cardinality set P^* of indexes such that $S_i \cap S_j = \emptyset$ for all $i, j \in P^* : i \neq j$.*

Problem 5 (SCHEDULING COMMON OPERATIONS) *Given a discrete finite set S (ground set), subsets $S_1, \ldots, S_m$ of S and positive integers $d_1, \ldots, d_m$, find a linear order $\sigma : S \to I\!N$ of the elements of S such that the number of indexes i for which $\sigma(j) > d_i$ for some $j \in S_i$ is minimized.*

2.2 Exercises

Exercise 5 *Formulate Problem k as 0-1 linear programming, $k = 1, \ldots, 5$.*

Exercise 6 *Can Problem 4 be formulated as* STABLE SET? And as VERTEX COVERING*? On which graph?*

Exercise 7 *Can Problem 5 be formulated as* STABLE SET? And as VERTEX COVERING*? On which graph?*