Orthogonal Projections Notes

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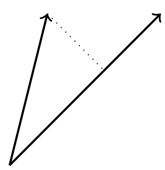

Goal: if W is a subspace in $\mathbb{R}^n$, how do we get $\hat{y}$, the projection in W of $\vec{y}$?

$$\hat{y} = \text{proj}_W(\vec{y}) \tag{1}$$

Theorem (Orthogonal Decomposition Theorem)
Let W be a subspace with orthogonal basis $\{\vec{u_1} \cdots \vec{u_p}\}$. Let $\vec{y} \in \mathbb{R}^n$.

$$\hat{y} = \left(\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\right)\vec{u}_1 \cdots \left(\frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p}\right)\vec{u}_p \tag{2}$$

decompose...

$$\vec{y} = \hat{y} + (\vec{y} - \hat{y}) \tag{3}$$

$$\hat{y} \in W \tag{4}$$

$$(\vec{y} - \hat{y}) \in W^{\perp} \tag{5}$$

Theorem (Best Approximation Theorem)
Let W be a subspace of $\mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$, $\hat{y} = proj_W(\vec{y})$

$$\textit{Then } \hat{y} \textit{ is the closest point in } W \textit{ to } \vec{y}. \tag{6}$$

and...

$$\textit{If } \vec{y} \in W \textit{ then } \hat{y} = \vec{y} \tag{7}$$

Theorem (Projection is a Linear Transform)
Let W be a subspace in $\mathbb{R}^n$
Let $\{\vec{u}_1 \cdots \vec{u}_p\}$ be an orthonormal basis of W

Orthonormal basis means $\vec{u}_i$ are all unit vectors, so:

$$|\vec{u}_1| = 1 \textit{ and } \vec{u}_1 \cdot \vec{u}_1 = 1 \tag{8}$$