<u>Probabilité et Simulation</u>

PolyTech INFO4 (Grenoble) – 2024-2025 – Practical Sessions

Last updated: 2024-09-19. References: [1], [2]

1. Theory recap 11.9.24

- *Jet set* $\Omega =$ finite set of possible outcomes ω
- *Probability* on $\Omega =$ set of weights $P(\omega) \in \mathbb{R}$ on each $\omega \in \Omega$ such that
  - $P(\omega) > 0 \forall \omega \in \Omega$
  - $\sum_{\omega \in \Omega} P(\omega) = 1$
- *Event* $A \subseteq \Omega =$ subset of the jet set
- *Complementary event* $A^c = \Omega / A$
- The cardinality of a set S is denoted by $|S|$
- *Uniform probability of the event* A

$$P(A) = \sum_{\omega \in A} P(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|} \tag{1}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{2}$$

- *Binomial theorem*

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \tag{3}$$

1.1. Counting

- Number of *permutations* of k elements:
  - Number of ways to *order* k elements
  - **Only order matters**

$$P_k = k! \tag{4}$$

- Number of *dispositions* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose and order* k elements out of n
  - **Order and elements** matter
  - Number of injections $f : \{1, ..., k\} \to \{1, ..., n\}$

$$D_{n,k} = \underbrace{n(n-1)...}_{k \text{ times}} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!} \tag{5}$$

- Number of *combinations* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose* k elements out of n
  - **Only elements** matter
  - Number of subsets of cardinality k of a set of cardinality n
  - Number of dispositions modulo number of permutations

$$C_{n,k} = \frac{D_{n,k}}{P_k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \text{choose}(n,k) \tag{6}$$

1.2. Exercises

1.2.1. Handshakes and kisses

There are f girls and g boys in a room. Boys exchange handshakes, girs exchange kisses, boys and girls exchange kisses. How many kisses in total?

The number of kisses exchanegs among girls is the number of subsets of cardinality 2 of a set of cardinality f, that is $\binom{f}{2} = \frac{f(f-1)}{2}$. Or, think that each girl gives $f-1$ kisses, and one needs a factor of one half to avoid double counting.

For the number of kisses exchanged between boys and girls: the first girl gives g kisses, the second girl gives g kisses, and so on, so we have fg in total.

$$\text{number of kisses} = \frac{f(f-1)}{2} + fg \tag{7}$$

1.2.2. Throwing a dice

Throw a fair dice with f faces n times. What's the prob to never get the same result twice?

Let $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{F} = \{1, ..., f\}$. The jet set is

$$\Omega = \{\omega = (\omega_1, ..., \omega_n) : \omega_i \in \mathcal{F} \text{ for all } i \in \mathcal{N}\} = \mathcal{F}^n \tag{8}$$

with cardinality

$$|\Omega| = |\mathcal{F}^n| = |\mathcal{F}|^n = f^n \tag{9}$$

The event we're looking at is

$$A = \{\omega \in \Omega : \omega_i \neq \omega_j \text{ for all } i \neq j \in \mathcal{N}\} \tag{10}$$

Clearly if $n > f$ then $P(A) = 0$. Let $n \leq f$. The (uniform) probability of the event A is $P(A) = \frac{|A|}{|\Omega|}$, with

$$\begin{aligned} |A| &= \#\text{ of ways to choose and order } n \text{ elements out of } f \\ &= \underbrace{f(f-1)...}_{n} = f(f-1)...(f-n+1) = \frac{f!}{(f-n)!} \end{aligned} \tag{11}$$

$$P(A) = \frac{f!}{f^n(f-n)!} \tag{12}$$

1.2.3. Birthday paradox

What is the probability that at least 2 people out of n have the same birthday? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned} P(A) &= 1 - P\left(\underbrace{\text{no two people have the same birthday}}_{\text{Ex. 2}}\right) \\ &= 1 - \frac{y!}{y^n(y-n)!} \end{aligned} \tag{13}$$

Formal solution Let $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{Y} = \{1, ..., y\}$ with $n \leq y$. The jet set is

$$\begin{aligned}\Omega &= \text{distributions of possible birthdays of } n \text{ people} \\ &= \{\omega = (\omega_1, ..., \omega_n) : \omega_i \in \mathcal{Y} \text{ for all } i \in \mathcal{N}\} = \mathcal{Y}^n\end{aligned} \tag{14}$$

where ω_i is the birthday of the i-th person. The cardinality of the jet set is

$$|\Omega| = |\mathcal{Y}^n| = |\mathcal{Y}|^n = y^n \tag{15}$$

The event we're looking at is

$$A = \left\{\omega \in \Omega : \exists i \neq j \in \mathcal{N} : \omega_i = \omega_j\right\} \tag{16}$$

Note that this is the complementary event to the event defined in Equation 10 of Exercise 2. Thus we can compute its probability as

$$P(A) = 1 - P(A^c) \tag{17}$$

in agreement with Equation 13.

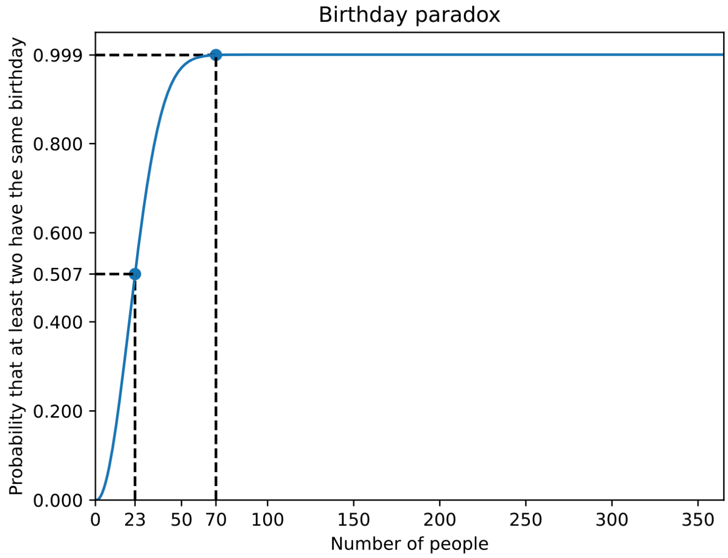

Figure 1: Birthday paradox probability.

1.2.4. Same birthday as the prof

What is the probability that at least 1 student out of n has the same birthday of the prof? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned}P(A) &= 1 - P\left(\underbrace{\text{nobody has the prescribed birth date}}\right) \\ &= 1 - \left(\frac{y-1}{y}\right)^n\end{aligned} \tag{18}$$

Formal solution 1 As above $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{Y} = \{1, ..., y\}$ with $n \leq y$. The jet set is $\Omega = \mathcal{Y}^{n+1}$, that is the set of possible birthdays of $n + 1$ people, the $(n + 1)$-th being the prof. Its cardinality is $|\Omega| = y^{n+1}$. The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \in \mathcal{N} : \omega_i = \omega_{n+1}\} \tag{19}$$

with complementary event

$$A^c = \{\omega \in \Omega : \omega_i \neq \omega_{n+1} \forall i \in \mathcal{N}\} \tag{20}$$

As usual $P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|}$, with

$$|A^c| = \underbrace{y}_{\text{prof}} \cdot \underbrace{(y-1)^n}_{\text{students}} \tag{21}$$

So, $P(A) = 1 - \frac{y(y-1)^n}{y^{n+1}} = 1 - \left(\frac{y-1}{y}\right)^n$, in agreement with Equation 18.

Formal solution 2 Using the probability of the complementary event is often the smartest way to proceed, but for the sake of completeness let's see how to get the same result directly. Let A_j be the event "exactly j students out of n have the same birthday as the prof". The event we look at then is

$$A = \sqcup_{j \in \mathcal{N}} A_j \tag{22}$$

with probability (cf Equation 2)

$$P(A) = \sum_{j \in \mathcal{N}} P(A_j) = \frac{\sum_{j \in \mathcal{N}} |A_j|}{|\Omega|} \tag{23}$$

The cardinality of A_j is

$$\begin{aligned} |A_j| &= \underbrace{1 ... 1}_{j \text{ times}} \cdot \underbrace{(y-1)...(y-1)}_{n-j \text{ times}} \cdot \underbrace{y}_{\text{prof}} \cdot \underbrace{\binom{n}{j}}_{\text{number of ways to choose } j \text{ elements out of } n} \\ &= y(y-1)^{n-j} \binom{n}{j} \end{aligned} \tag{24}$$

By an application of the binomial theorem (Equation 3) and a short manipulation,

$$\sum_{j=1}^{n} |A_j| = y(y^n - (y-1)^n) \tag{25}$$

which leads back to Equation 18.

2. Theory recap 18.9.24

2.1. Conditional probability

- *Conditional probability*

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0 \tag{26}$$

- not reallty defined if $P(B) = 0$, cf [2] pag. 427.
- often used as

$$P(A \cap B) = P(A \mid B) P(B) \tag{27}$$

- Conditional probability and complementary event (proof: simple exercise.)

$$P(A \mid B) + P(A^c \mid B) = 1 \tag{28}$$

- *Total probability theorem*

$$P(A) = P(A \mid B) P(B) + P(A \mid B^c) P(B^c) \tag{29}$$

- *Bayes theorem*

$$P(A \cap B) = P(B \cap A) \Rightarrow P(A \mid B) P(B) = P(B \mid A) P(A) \tag{30}$$

2.2. Independent events

Let Ω be equipped with a probability P.

- two events $A, B \subseteq \Omega$ are said *independent* if

$$P(A \cap B) = P(A)P(B) \tag{31}$$

- n events $A_1, \ldots, A_n$ are said *independent* if

$$P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i) \text{ for all } I \subseteq \{1, \ldots, n\} \tag{32}$$

- pairwise independence does not imply independence of all events!

2.3. Exercises

2.3.1. Pile ou Face

Jet de 2 pieces, $\Omega = \{\mathrm{PP}, \mathrm{PF}, \mathrm{FP}, \mathrm{FF}\}$. Cet espace est muni de la probabilite uniforme. Soient les evenements:

- $A = \text{1ere piece donne P}$
- $B = \text{2eme piece donne F}$
- $C = \text{le deux pieces donnent le meme resultat}$

Questions:

- A et B sont indépendantes?
- A, B et C sont indépendants?

$$\begin{aligned}
A &= \{PP, PF\} & \mathbb{P}(A) &= 1/2 & & \\
B &= \{PF, FF\} & \mathbb{P}(B) &= 1/2 & & \\
C &= \{PP, FF\} & \mathbb{P}(C) &= 1/2 & & \\
A \cap B &= \{PF\} & \mathbb{P}(A \cap B) &= 1/4 & &= \mathbb{P}(A)\mathbb{P}(B) \\
A \cap C &= \{PP\} & \mathbb{P}(A \cap C) &= 1/4 & &= \mathbb{P}(A)\mathbb{P}(C) \\
B \cap C &= \{FF\} & \mathbb{P}(B \cap C) &= 1/4 & &= \mathbb{P}(B)\mathbb{P}(C) \\
A \cap B \cap C &= \emptyset & \mathbb{P}(A \cap B \cap C) &= 0 & &\neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).
\end{aligned}$$

Ainsi les événements A, B et C sont 2 à 2 indépendants mais pas indépendants.

Figure 2: Pairwise independence does not imply independence of all events!

2.3.2. Pieces mecaniques defectueuses

Parmi 10 pièces mécaniques, 4 sont déefectueuses. On prend successivement deux pièces au hasard dans le lot (sans remise). Quelle est la probabilité pour que les deux pièces soient correctes?

Solution 1 Let A_i be the event *the i-th drawn piece is good*, with $i \in \{1, 2\}$. We need the probability of the event $A_2 \cap A_1$. By definition of conditional probavility,

$$P(A_2 \cap A_1) = \underbrace{P(A_2 \mid A_1)}_{\frac{5}{9}} \underbrace{P(A_1)}_{\frac{6}{10}} = \frac{1}{3}. \tag{33}$$

Solution 2 The jet set is the set of subsets of cardinality 2 of a set of cardinality 10, so $|\Omega| = \binom{10}{2}$. The event we consider is the set of subsets of cardinality 2 of a set of cardinality 6, so $|A| = \binom{6}{2}$. Then

$$P(A) = \frac{\binom{6}{2}}{\binom{10}{2}} = \frac{6 \cdot 5}{10 \cdot 9} = \frac{1}{3}. \tag{34}$$

2.3.3. Betting on cards

We have three cards:

- a *red* card with both faces red;
- a *white* card with both faces white;
- a *mixed* card with a red face and a white face.

One of the three cards is drawn at random and one of the faces of this card (also chosen at random) is exposed. This face is red. You are asked to bet on the color of the hidden face. Do you choose red or white?

Intuitive solution The cards are RR, RW, WW with W for white and R for red. Call RR the "red" card, WW the "white" card, and WR the "mixed" card. Since we observe a red face, the white card cannot be on the table. There are three possibilities left: 1. we're observing a face of the red card (in which case the hidden face is red); 2. we are observing the other face of the red card (in which case the hidden face is red); 3. we are observing the red face of the mixed card (in which case the hidden face is white). So the hidden face is red 2 out of 3 times.

Formal solution The jet set contains the possible outcomes of a sequence of two events: 1. draw a card (out of three), and 2. observe a face (out of two). Denote by R a red face and by W a white face, and denote by a subscript o the observed face, and by a subscript h the hidden face. The possible outcomes then are

$$\Omega = \{R_h \cap R_o, R_h \cap W_o, W_h \cap R_o, W_h \cap W_o\} \tag{35}$$

where the first entry indicates the hidden face, and the second entry indicates the observed face. For example, $W_h \cap R_o$ is the event "*the hidden face is white and the observed face is red*", and similarly for the others.

In this formulation, every element in the jet set is the intersection of two (dependent) events of the type 1. a face is hidden, and 2. a face is observed. Note that the event $W_h \cap R_o$ is equivalent to the event "*the mixed card is drawn, and the red face is observed.*" Under this second point of view, each outcome in Ω is the intersection of two (dependent) events of the type 1. a card is drawn, and 2. a face is observed. Denoting the event *"draw the red card"* by r, the event *"draw the white card"* by w, and the event *"draw the mixed card"* by m, the jet set is equivalently

$$\Omega = \{r \cap R_o, m \cap W_o, m \cap R_o, w \cap W_o\} \tag{36}$$

This formulation helps to understand that the probability on Ω **is not uniform**. The probabilities of the events in Ω are computed by Equation 27:

$$P(R_h \cap R_o) = P(r \cap R_o) = \frac{P(r \mid R_o)}{R_o} \tag{37}$$

However, we do not know the probabilities on the right hand side. As a simple trick, remember that $P(A \cap B) = P(B \cap A)$, so we can turn this around:

$$\begin{aligned} P(R_h \cap R_o) &= P(R_o \cap r) \\ &= \underbrace{P(R_o \mid r)}_{1} \underbrace{P(r)}_{\frac{1}{3}} = \frac{2}{6} \end{aligned} \tag{38}$$

$$
\begin{aligned}
P(R_h \cap W_o) &= P(W_o \cap m) \\
&= \underbrace{P(W_o \mid m)}_{\frac{1}{2}} \underbrace{P(m)}_{\frac{1}{3}} = \frac{1}{6}
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
P(W_h \cap R_o) &= P(R_o \cap m) \\
&= \underbrace{P(R_o \mid m)}_{\frac{1}{2}} \underbrace{P(m)}_{\frac{1}{3}} = \frac{1}{6}
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
P(W_h \cap W_o) &= P(W_o \cap w) \\
&= \underbrace{P(W_o \mid w)}_{1} \underbrace{P(w)}_{\frac{1}{3}} = \frac{2}{6}
\end{aligned} \tag{41}
$$

Now by Equation 26 and using these probabilities,

$$
\begin{aligned}
P(W_h \mid R_o) &= \frac{P(W_h \cap R_o)}{P(R_o)} \\
&= \frac{P(W_h \cap R_o)}{P(R_h \cap R_o) + P(W_h \cap R_o)} = \frac{1}{3}
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
P(R_h \mid R_o) &= \frac{P(R_h \cap R_o)}{P(R_o)} \\
&= \frac{P(R_h \cap R_o)}{P(R_h \cap R_o) + P(W_h \cap R_o)} = \frac{2}{3} \\
&= 1 - P(W_h \mid R_o)
\end{aligned} \tag{43}
$$

where the last line follows from Equation 28 and gives directly the answer. So in conclusion, given the fact that we observe a red face, the hidden face is also red with probability $2/3$.

2.3.4. Russian roulette

You are playing two-person Russian roulette with a revolver featuring a rotating cylinder with six bullet slots. Each time the gun is triggered, the cylinder rotates by one slot. Two bullets are inserted one next to the other into the cylinder, which is then randomly positioned. Your opponent is the first to place the revolver against her temple. She presses the trigger and… she stays alive. With great display of honor, she offers you to rotate the barrel again at random before firing in turn. What do you decide?

The bullets occupy the positions x and $x + 1 \bmod 6$:

$$
\Omega = \{12, 23, 34, 45, 56, 61\} \tag{44}
$$

Say the revolver shots from position 1. The event "*the first player dies*" is

$$
\mathrm{die}_1 = \{12, 61\} \tag{45}
$$

so $P(\mathrm{die}_1) = \frac{1}{3}$ and $P(\mathrm{live}_1) = \frac{2}{3}$. We need to compute

$$
P(\mathrm{die}_2 \mid \mathrm{live}_1) = \frac{P(\mathrm{die}_2 \cap \mathrm{live}_1)}{P(\mathrm{live}_1)} \tag{46}
$$

Since the cylynder rotates after being triggered we have $\mathrm{die}_2 = \{56, 61\}$ and $\mathrm{die}_2 \cap \mathrm{live}_1 = \{56\}$, so $P(\mathrm{die}_2 \mid \mathrm{live}_1) = \frac{1}{6} / \frac{2}{3} = \frac{1}{4} < P(\mathrm{die}_1)$. So you don't shuffle the barrel.

Bibliography

[1] B. Jourdain, *Probabilités et statistique pour l'ingénieur*. 2018.

[2] P. Billingsley, *Probability and Measure*. John Wiley & Sons, 2012.