

Probabilité et Simulation

PolyTech INFO4 (Grenoble) – 2024-2025 – Practical Sessions

Last updated: 2024-09-19. References: [1], [2]

1. Theory recap 11.9.24

- *Jet set* Ω = finite set of possible outcomes ω
- *Probability* on Ω = set of weights $P(\omega) \in \mathbb{R}$ on each $\omega \in \Omega$ such that
 - $P(\omega) > 0 \forall \omega \in \Omega$
 - $\sum_{\omega \in \Omega} P(\omega) = 1$
- *Event* $A \subseteq \Omega$ = subset of the jet set
- *Complementary event* $A^c = \Omega/A$
- The cardinality of a set S is denoted by $|S|$
- *Uniform probability of the event* A

$$P(A) = \sum_{\omega \in A} P(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|} \quad (1)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (2)$$

- *Binomial theorem*

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j \quad (3)$$

1.1. Counting

- Number of *permutations* of k elements:
 - Number of ways to *order* k elements
 - **Only order matters**
- $$P_k = k! \quad (4)$$
- Number of *dispositions* of k elements out of n ($k \leq n$):
 - Number of ways to *choose and order* k elements out of n
 - **Order and elements** matter
 - Number of injections $f : \{1, \dots, k\} \rightarrow \{1, \dots, n\}$

$$D_{n,k} = \underbrace{n(n-1)\dots}_{k \text{ times}} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!} \quad (5)$$

- Number of *combinations* of k elements out of n ($k \leq n$):
 - Number of ways to *choose* k elements out of n
 - **Only elements** matter
 - Number of subsets of cardinality k of a set of cardinality n
 - Number of dispositions modulo number of permutations

$$C_{n,k} = \frac{D_{n,k}}{P_k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \text{choose}(n, k) \quad (6)$$

1.2. Exercises

1.2.1. Handshakes and kisses

There are f girls and g boys in a room. Boys exchange handshakes, girls exchange kisses, boys and girls exchange kisses. How many kisses in total?

The number of kisses exchanged among girls is the number of subsets of cardinality 2 of a set of cardinality f , that is $\binom{f}{2} = \frac{f(f-1)}{2}$. Or, think that each girl gives $f - 1$ kisses, and one needs a factor of one half to avoid double counting.

For the number of kisses exchanged between boys and girls: the first girl gives g kisses, the second girl gives g kisses, and so on, so we have fg in total.

$$\text{number of kisses} = \frac{f(f-1)}{2} + fg \quad (7)$$

1.2.2. Throwing a dice

Throw a fair dice with f faces n times. What's the prob to never get the same result twice?

Let $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{F} = \{1, \dots, f\}$. The jet set is

$$\Omega = \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \mathcal{F} \text{ for all } i \in \mathcal{N}\} = \mathcal{F}^n \quad (8)$$

with cardinality

$$|\Omega| = |\mathcal{F}^n| = |\mathcal{F}|^n = f^n \quad (9)$$

The event we're looking at is

$$A = \{\omega \in \Omega : \omega_i \neq \omega_j \text{ for all } i \neq j \in \mathcal{N}\} \quad (10)$$

Clearly if $n > f$ then $P(A) = 0$. Let $n \leq f$. The (uniform) probability of the event A is $P(A) = \frac{|A|}{|\Omega|}$, with

$$\begin{aligned} |A| &= \# \text{ of ways to choose and order } n \text{ elements out of } f \\ &= \underbrace{f(f-1)\dots}_n = f(f-1)\dots(f-n+1) = \frac{f!}{(f-n)!} \end{aligned} \quad (11)$$

$$P(A) = \frac{f!}{f^n(f-n)!} \quad (12)$$

1.2.3. Birthday paradox

What is the probability that at least 2 people out of n have the same birthday? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned} P(A) &= 1 - P\left(\underbrace{\text{no two people have the same birthday}}_{\text{Ex. 2}}\right) \\ &= 1 - \frac{y!}{y^n(y-n)!} \end{aligned} \quad (13)$$

Formal solution Let $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{Y} = \{1, \dots, y\}$ with $n \leq y$. The jet set is

$$\begin{aligned}\Omega &= \text{distributions of possible birthdays of } n \text{ people} \\ &= \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \mathcal{Y} \text{ for all } i \in \mathcal{N}\} = \mathcal{Y}^n\end{aligned}\quad (14)$$

where ω_i is the birthday of the i -th person. The cardinality of the jet set is

$$|\Omega| = |\mathcal{Y}^n| = |\mathcal{Y}|^n = y^n \quad (15)$$

The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \neq j \in \mathcal{N} : \omega_i = \omega_j\} \quad (16)$$

Note that this is the complementary event to the event defined in Equation 10 of Exercise 2. Thus we can compute its probability as

$$P(A) = 1 - P(A^c) \quad (17)$$

in agreement with Equation 13.

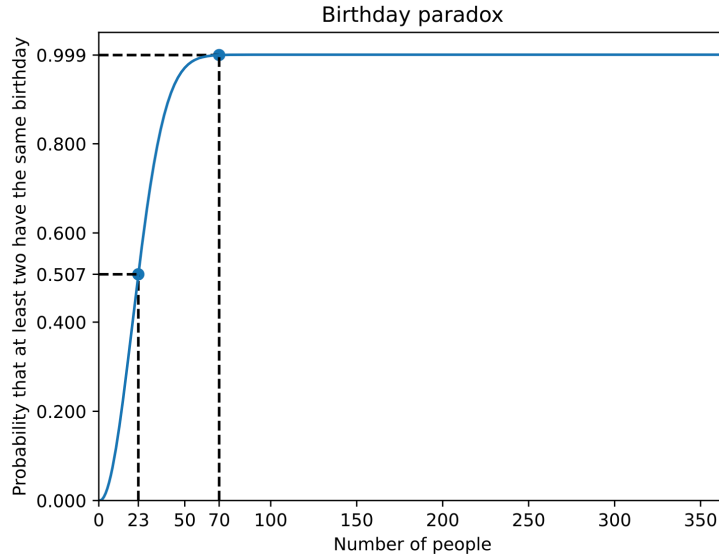


Figure 1: Birthday paradox probability.

1.2.4. Same birthday as the prof

What is the probability that at least 1 student out of n has the same birthday of the prof? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned}P(A) &= 1 - P(\underbrace{\text{nobody has the prescribed birth date}}_{A^c}) \\ &= 1 - \left(\frac{y-1}{y}\right)^n\end{aligned}\quad (18)$$

Formal solution 1 As above $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{Y} = \{1, \dots, y\}$ with $n \leq y$. The jet set is $\Omega = \mathcal{Y}^{n+1}$, that is the set of possible birthdays of $n+1$ people, the $(n+1)$ -th being the prof. Its cardinality is $|\Omega| = y^{n+1}$. The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \in \mathcal{N} : \omega_i = \omega_{n+1}\} \quad (19)$$

with complementary event

$$A^c = \{\omega \in \Omega : \omega_i \neq \omega_{n+1} \forall i \in \mathcal{N}\} \quad (20)$$

As usual $P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|}$, with

$$|A^c| = \underbrace{y}_{\text{prof}} \cdot \underbrace{(y-1)^n}_{\text{students}} \quad (21)$$

So, $P(A) = 1 - \frac{y(y-1)^n}{y^{n+1}} = 1 - \left(\frac{y-1}{y}\right)^n$, in agreement with Equation 18.

Formal solution 2 Using the probability of the complementary event is often the smartest way to proceed, but for the sake of completeness let's see how to get the same result directly. Let A_j be the event “exactly j students out of n have the same birthday as the prof”. The event we look at then is

$$A = \sqcup_{j \in \mathcal{N}} A_j \quad (22)$$

with probability (cf Equation 2)

$$P(A) = \sum_{j \in \mathcal{N}} P(A_j) = \frac{\sum_{j \in \mathcal{N}} |A_j|}{|\Omega|} \quad (23)$$

The cardinality of A_j is

$$\begin{aligned} |A_j| &= \underbrace{1 \dots 1}_{j \text{ times}} \cdot \underbrace{(y-1) \dots (y-1)}_{n-j \text{ times}} \cdot \underbrace{y}_{\text{prof}} \cdot \underbrace{\binom{n}{j}}_{\text{number of ways to choose } j \text{ elements out of } n} \\ &= y(y-1)^{n-j} \binom{n}{j} \end{aligned} \quad (24)$$

By an application of the binomial theorem (Equation 3) and a short manipulation,

$$\sum_{j=1}^n |A_j| = y(y^n - (y-1)^n) \quad (25)$$

which leads back to Equation 18.

2. Theory recap 18.9.24

2.1. Conditional probability

- *Conditional probability*

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0 \quad (26)$$

- not really defined if $P(B) = 0$, cf [2] pag. 427.
- often used as

$$P(A \cap B) = P(A | B)P(B) \quad (27)$$

- *Total probability theorem*

$$P(A) = P(A | B)P(B) + P(A | B^c)P(B^c) \quad (28)$$

- *Bayes theorem*

$$P(A | B)P(B) = P(B | A)P(A) \quad (29)$$

2.2. Independent events

Let Ω be equipped with a probability P .

- two events $A, B \subseteq \Omega$ are said *independent* if

$$P(A \cap B) = P(A)P(B) \quad (30)$$

- n events A_1, \dots, A_n are said *independent* if

$$P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i) \text{ for all } I \subseteq \{1, \dots, n\} \quad (31)$$

- pairwise independence does not imply independence of all events!

2.3. Exercises

2.3.1. Pile ou Face

Jet de 2 pieces, $\Omega = \{PP, PF, FP, FF\}$. Cet espace est muni de la probabilité uniforme. Soient les événements:

- $A =$ 1ere piece donne P
- $B =$ 2eme piece donne F
- $C =$ les deux pieces donnent le meme resultat

Questions:

- A et B sont indépendantes?
- A, B et C sont indépendants?

$$\begin{array}{ll} A = \{PP, PF\} & \mathbb{P}(A) = 1/2 \\ B = \{PF, FF\} & \mathbb{P}(B) = 1/2 \\ C = \{PP, FF\} & \mathbb{P}(C) = 1/2 \\ A \cap B = \{PF\} & \mathbb{P}(A \cap B) = 1/4 = \mathbb{P}(A)\mathbb{P}(B) \\ A \cap C = \{PP\} & \mathbb{P}(A \cap C) = 1/4 = \mathbb{P}(A)\mathbb{P}(C) \\ B \cap C = \{FF\} & \mathbb{P}(B \cap C) = 1/4 = \mathbb{P}(B)\mathbb{P}(C) \\ A \cap B \cap C = \emptyset & \mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C). \end{array}$$

Ainsi les événements A, B et C sont 2 à 2 indépendants mais pas indépendants.

Figure 2: Pairwise independence does not imply independence of all events!

2.3.2. Pieces mecaniques defectueuses

Parmi 10 pièces mécaniques, 4 sont défectueuses. On prend successivement deux pièces au hasard dans le lot (sans remise). Quelle est la probabilité pour que les deux pièces soient correctes?

Solution 1 Let A_i be the event *the i -th drawn piece is good*, with $i \in \{1, 2\}$. We need the probability of the event $A_2 \cap A_1$. By definition of conditional probability,

$$P(A_2 \cap A_1) = \underbrace{P(A_2 | A_1)}_{\frac{5}{9}} \underbrace{P(A_1)}_{\frac{6}{10}} = \frac{1}{3}. \quad (32)$$

Solution 2 The jet set is the set of subsets of cardinality 2 of a set of cardinality 10, so $|\Omega| = \binom{10}{2}$. The event we consider is the set of subsets of cardinality 2 of a set of cardinality 6, so $|A| = \binom{6}{2}$. Then

$$P(A) = \frac{\binom{6}{2}}{\binom{10}{2}} = \frac{6 \cdot 5}{10 \cdot 9} = \frac{1}{3}. \quad (33)$$

2.3.3. Betting on cards

We have 3 cards:

- 1st: 2 red faces

- 2nd: 2 white faces
- 3rd: 1 red face and 1 white face

One of the three cards is drawn at random and one of the faces of this card (also chosen at random) is exposed. This face is red. You are asked to bet on the color of the hidden face. Do you choose red or white?

The jet set is composed of the possible outcomes of the sequences of two events: draw a card (out of three), and expose a face (out of two). The possible outcomes are

$$\Omega = \{RR, RW, WR, WW\} \quad (34)$$

with

1. first entry for hidden face,
2. second entry for observed face.

So for example WR is the event “*the hidden face is white and the observed face is red*”, and similarly for the others. This is equivalent to the event “*the mixed card is drawn, and the R face is observed.*” This second point of view helps in understanding that the probability on Ω is **not uniform**. Each outcome in Ω is the product of two *dependent* events of the type 1. *such card is drawn*, and 2. *such face is shown*. So the jet set is equivalently,

$$\Omega = \{rR, mW, mR, wW\} \quad (35)$$

where r is the red card, w the white card, m the mixed card; R the observed red face, W the observed white face.

The corresponding probabilities are computed by Equation 27:

$$\begin{aligned} P(RR) &= P(\text{observe R} \cap \text{draw red card}) \\ &= \underbrace{P(\text{observe R} \mid \text{draw red card})}_1 \underbrace{P(\text{draw red card})}_{\frac{1}{3}} = \frac{1}{3} \end{aligned} \quad (36)$$

$$\begin{aligned} P(RW) &= P(\text{observe W} \cap \text{draw mixed card}) \\ &= \underbrace{P(\text{observe W} \mid \text{draw mixed card})}_{\frac{1}{2}} \underbrace{P(\text{draw mixed card})}_{\frac{1}{3}} = \frac{1}{6} \end{aligned} \quad (37)$$

$$\begin{aligned} P(WR) &= P(\text{observe R} \cap \text{draw mixed card}) \\ &= \underbrace{P(\text{observe R} \mid \text{draw mixed card})}_{\frac{1}{2}} \underbrace{P(\text{draw mixed card})}_{\frac{1}{3}} = \frac{1}{6} \end{aligned} \quad (38)$$

$$\begin{aligned} P(WW) &= P(\text{observe W} \cap \text{draw white card}) \\ &= \underbrace{P(\text{observe W} \mid \text{draw white card})}_1 \underbrace{P(\text{draw white card})}_{\frac{1}{3}} = \frac{1}{3} \end{aligned} \quad (39)$$

Now by Equation 26

$$\begin{aligned}
P(\text{hidden is W} \mid \text{observe R}) &= \frac{P(\text{hidden is W} \cap \text{observe R})}{P(\text{observe R})} \\
&= \frac{P(WR)}{P(RR) + P(WR)} = \frac{1}{3}
\end{aligned} \tag{40}$$

$$\begin{aligned}
P(\text{hidden is R} \mid \text{observe R}) &= \frac{P(\text{hidden is R} \cap \text{observe R})}{P(\text{observe R})} \\
&= \frac{P(RR)}{P(RR) + P(WR)} = \frac{2}{3}
\end{aligned} \tag{41}$$

2.3.4. Russian roulette

You are playing two-person Russian roulette with a revolver featuring a rotating cylinder with six bullet slots. Each time the gun is triggered, the cylinder rotates by one slot. Two bullets are inserted one next to the other into the cylinder, which is then randomly positioned. Your opponent is the first to place the revolver against her temple. She presses the trigger and... she stays alive. With great display of honor, she offers you to rotate the barrel again at random before firing in turn. What do you decide?

The bullets occupy the positions x and $x + 1 \bmod 6$:

$$\Omega = \{12, 23, 34, 45, 56, 61\} \tag{42}$$

Say the revolver shots from position 1. The event “*the first player dies*” is

$$\text{die}_1 = \{12, 61\} \tag{43}$$

so $P(\text{die}_1) = \frac{1}{3}$ and $P(\text{live}_1) = \frac{2}{3}$. We need to compute

$$P(\text{die}_2 \mid \text{live}_1) = \frac{P(\text{die}_2 \cap \text{live}_1)}{P(\text{live}_1)} \tag{44}$$

Since the cylinder rotates after being triggered we have $\text{die}_2 = \{56, 61\}$ and $\text{die}_2 \cap \text{live}_1 = \{56\}$, so $P(\text{die}_2 \mid \text{live}_1) = \frac{1/6}{2/3} = \frac{1}{4} < P(\text{die}_1)$. So you don't shuffle the barrel.

Bibliography

- [1] B. Jourdain, *Probabilités et statistique pour l'ingénieur*. 2018.
- [2] P. Billingsley, *Probability and Measure*. John Wiley & Sons, 2012.