<u>Probabilité et Simulation</u>

PolyTech INFO4 (Grenoble) – 2025-2026 – Practical Sessions

Last updated: 2025-09-11. Info: davide.legacci@univ-grenoble-alpes.fr

Contents

1. Theory recap 2025-09-12 - Probability on finite set

1.1. Probability space

- *Random process*: one random outcome out of finitely many
- *Sample space* $\Omega =$ finite set of all possible outcomes ω
- *Probability* on $\Omega =$ set of weights $\mathbb{P}(\omega) \in \mathbb{R}$ on each $\omega \in \Omega$ such that
  - $\mathbb{P}(\omega) > 0 \forall \omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$
- *Event* $A \subseteq \Omega =$ subset of the sample space

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) \tag{1}$$

- *Complementary event* $A^c = \Omega / A$ (pronounced "*not* A")
- "A *and* B" $= A \cap B$
- "A *or* B" $= A \cup B$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \tag{2}$$

- *indicator function* of event $A \subseteq \Omega$

$$\mathbb{1}_A : \Omega \to \mathbb{R}, \quad \mathbb{1}_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases} \tag{3}$$

- Denoted by $|S|$ the *cardinality* of a set S
- Denoted by S^n the *cartesian product* of S with itself n times

$$S^n = S \times S \times ... \times S = \{(s_1, ..., s_n) : s_i \in S \text{ for all } i = 1, ..., n\} \tag{4}$$

- *Cardinality of cartesian product*

$$|S^n| = |S|^n \tag{5}$$

1.2. Uniform probability

- *Every outcome* $\omega \in \Omega$ *has the same weight*

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|} \tag{6}$$

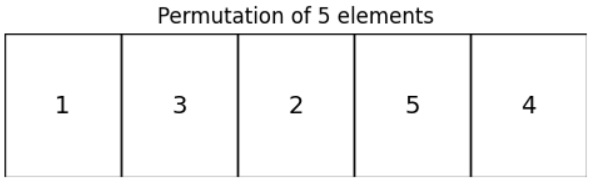
- *Uniform probability of the event* $A \subseteq \Omega$

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|} \tag{7}$$

1.3. Counting

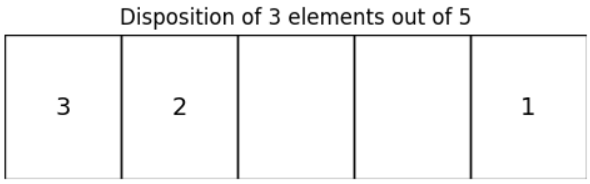
- Number of *permutations* of k elements:
  - Number of ways to *order* k elements
  - **Only order matters**

$$P_k = k! \tag{8}$$



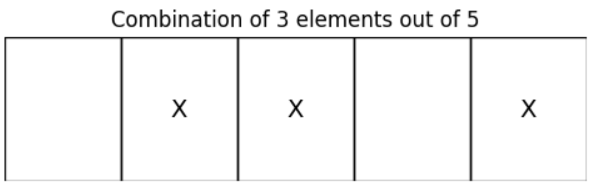
- Number of *dispositions* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose and order* k elements out of n
  - **Order and elements** matter
  - Number of injections : $\{1, ..., k\} \to \{1, ..., n\}$

$$D_{n,k} = \underbrace{n(n-1)...}_{k \text{ times}} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!} \tag{9}$$



- Number of *combinations* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose* k elements out of n
  - **Only elements** matter
  - Number of subsets of cardinality k of a set of cardinality n
  - Number of dispositions modulo number of permutations

$$C_{n,k} = \frac{D_{n,k}}{P_k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \text{choose}(n, k) \tag{10}$$



- *Binomial theorem*

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \tag{11}$$

Exercises

Exercise 1.1 *(Handshakes and kisses)*

There are f girls and g boys in a room. Boys exchange handshakes, girs exchange kisses, boys and girls exchange kisses. How many kisses in total?

The number of kisses exchanegs among girls is the number of subsets of cardinality 2 of a set of cardinality f, that is $\binom{f}{2} = \frac{f(f-1)}{2}$. Or, think that each girl gives $f - 1$ kisses, and one needs a factor of one half to avoid double counting.

For the number of kisses exchanged between boys and girls: the first girl gives g kisses, the second girl gives g kisses, and so on, so we have fg in total.

$$\text{number of kisses} = \frac{f(f-1)}{2} + fg \tag{12}$$

Exercise 1.2 *(Throwing a dice)* Throw a fair dice with f faces n times. What's the prob to never get the same result twice?

General strategy

- Identify sample space Ω (write in set-theoretic notation!) and its cardinality $|\Omega|$
- Identify event $A \subseteq \Omega$ (write in set-theoretic notation!) and its cardinality $|A|$
- Uniform probability? If so, use $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$

Let $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{F} = \{1, ..., f\}$. The sample space is

$$\Omega = \{\omega = (\omega_1, ..., \omega_n) : \omega_i \in \mathcal{F} \text{ for all } i \in \mathcal{N}\} = \mathcal{F}^n \tag{13}$$

with cardinality

$$|\Omega| = |\mathcal{F}^n| = |\mathcal{F}|^n = f^n \tag{14}$$

Endow the sample space with the uniform probability (since every outcome of the experiment is equiprobable).

The event we're looking at is

$$A = \left\{\omega \in \Omega : \omega_i \neq \omega_j \text{ for all } i \neq j \in \mathcal{N}\right\} \tag{15}$$

Clearly if $n > f$ then $\mathbb{P}(A) = 0$. Let $n \leq f$. The (uniform) probability of the event A is $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$, with

$$\begin{aligned} |A| &= \# \text{ of ways to choose and order } n \text{ elements out of } f \\ &= \underbrace{f(f-1)...}_{n} = f(f-1)...(f-n+1) = \frac{f!}{(f-n)!} \end{aligned} \tag{16}$$

$$\mathbb{P}(A) = \frac{f!}{f^n(f-n)!} \tag{17}$$

Exercise 1.3 *(Birthday paradox)* What is the probability that at least 2 people out of n have the same birthday? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$
\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}\Bigg(\underbrace{\text{no two people have the same birthday}}_{\text{Ex. 2}}\Bigg) \\
&= 1 - \frac{y!}{y^n(y-n)!}
\end{aligned} \tag{18}
$$

Formal solution Let $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{Y} = \{1, ..., y\}$ with $n \leq y$. The sample space is

$$
\begin{aligned}
\Omega &= \text{distributions of possible birthdays of } n \text{ people} \\
&= \{\omega = (\omega_1, ..., \omega_n) : \omega_i \in \mathcal{Y} \text{ for all } i \in \mathcal{N}\} = \mathcal{Y}^n
\end{aligned} \tag{19}
$$

where ω_i is the birthday of the i-th person. The cardinality of the sample space is

$$
|\Omega| = |\mathcal{Y}^n| = |\mathcal{Y}|^n = y^n \tag{20}
$$

The event we're looking at is

$$
A = \left\{\omega \in \Omega : \exists i \neq j \in \mathcal{N} : \omega_i = \omega_j\right\} \tag{21}
$$

Note that this is the complementary event to the event defined in Equation 15 of Exercise 2. Thus we can compute its probability as

$$
\mathbb{P}(A) = 1 - \mathbb{P}(A^c) \tag{22}
$$

in agreement with Equation 18.

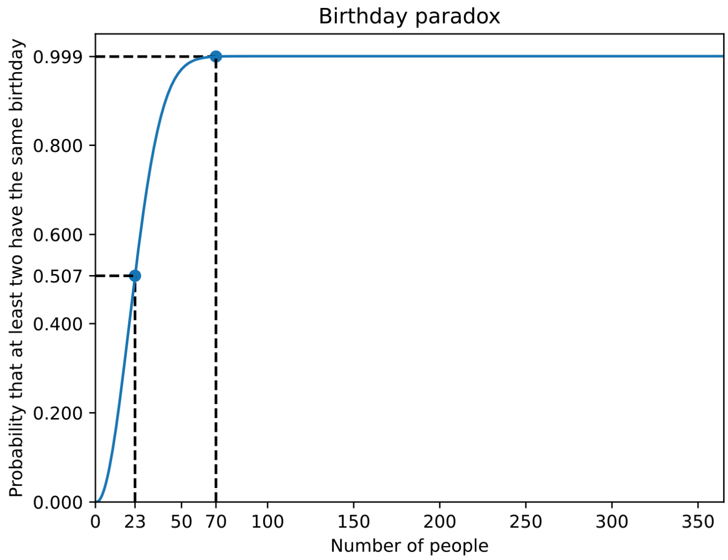


Figure 4: Birthday paradox probability. Code available.

Exercise 1.4 *(Same birthday as the prof)* What is the probability that at least 1 student out of n has the same birthday of the prof? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$
\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}\Big(\underbrace{\text{nobody has the prescribed birth date}}\Big) \\
&= 1 - \left(\frac{y-1}{y}\right)^n
\end{aligned} \tag{23}
$$

Formal solution 1 As above $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{Y} = \{1, ..., y\}$ with $n \leq y$. The sample space is $\Omega = \mathcal{Y}^{n+1}$, that is the set of possible birthdays of $n + 1$ people, the $(n + 1)$-th being the prof. Its cardinality is $|\Omega| = y^{n+1}$. The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \in \mathcal{N} : \omega_i = \omega_{n+1}\} \tag{24}$$

with complementary event

$$A^c = \{\omega \in \Omega : \omega_i \neq \omega_{n+1} \forall i \in \mathcal{N}\} \tag{25}$$

As usual $\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{|A^c|}{|\Omega|}$, with

$$|A^c| = \underbrace{y}_{\text{prof}} \cdot \underbrace{(y-1)^n}_{\text{students}} \tag{26}$$

So, $\mathbb{P}(A) = 1 - \frac{y(y-1)^n}{y^{n+1}} = 1 - \left(\frac{y-1}{y}\right)^n$, in agreement with Equation 23.

Formal solution 2 Using the probability of the complementary event is often the smartest way to proceed, but for the sake of completeness let's see how to get the same result directly. Let A_j be the event *"exactly j students out of n have the same birthday as the prof"*. The event we look at then is

$$A = \sqcup_{j \in \mathcal{N}} A_j \tag{27}$$

with probability (cf Equation 2)

$$\mathbb{P}(A) = \sum_{j \in \mathcal{N}} \mathbb{P}(A_j) = \frac{\sum_{j \in \mathcal{N}} |A_j|}{|\Omega|} \tag{28}$$

The cardinality of A_j is

$$\begin{aligned} |A_j| &= \underbrace{1...1}_{j \text{ times}} \cdot \underbrace{(y-1)...(y-1)}_{n-j \text{ times}} \cdot \underbrace{y}_{\text{prof}} \cdot \underbrace{\binom{n}{j}}_{\text{number of ways to choose } j \text{ elements out of } n} \\ &= y(y-1)^{n-j}\binom{n}{j} \end{aligned} \tag{29}$$

By an application of the binomial theorem (Equation 11) and a short manipulation,

$$\sum_{j=1}^{n} |A_j| = y(y^n - (y-1)^n) \tag{30}$$

which leads back to Equation 23.

Bibliography