Theory recap 2025-09-12 - Probability on finite set

Probability space

- *Random process*: one random outcome out of finitely many
- *Sample space* Ω = finite set of all possible outcomes ω
- *Probability* on Ω = set of weights $\mathbb{P}(\omega) \in \mathbb{R}$ on each $\omega \in \Omega$ such that
  - $\mathbb{P}(\omega) > 0 \forall \omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

- *Event* $A \subseteq \Omega$ = subset of the sample space

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega)$$

- *Complementary event* $A^c = \Omega / A$ (pronounced "*not* A")
- "A *and* B" $= A \cap B$
- "A *or* B" $= A \cup B$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- Probability of complementary event

$$\Omega = A^c \sqcup A \Rightarrow 1 = \mathbb{P}(A^c) + \mathbb{P}(A)$$

- *Indicator function* of event $A \subseteq \Omega$

$$\mathbb{1}_A : \Omega \to \mathbb{R}, \quad \mathbb{1}_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}$$

- Denoted by $|S|$ the *cardinality* of a set S
- Denoted by S^n the *cartesian product* of S with itself n times

$$S^n = S \times S \times ... \times S = \{(s_1, ..., s_n) : s_i \in S \text{ for all } i = 1, ..., n\}$$

- *Cardinality of cartesian product*

$$|S^n| = |S|^n$$

Uniform probability

- *Every outcome $\omega \in \Omega$ has the same weight*

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

- *Uniform probability of the event $A \subseteq \Omega$*

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

Counting

- Number of *permutations* of k elements:
  - Number of ways to *order* k elements
  - **Only order matters**

$$P_k = k!$$

Permutation of 5 elements

1	5	4	3	2

- Number of *dispositions* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose and order* k elements out of n
  - **Order and elements** matter
  - Number of injections : $\{1, ..., k\} \to \{1, ..., n\}$

$$D_{n,k} = \underbrace{n(n-1)...}_{k \text{ times}} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$$

Disposition of 3 elements out of 5

	2	1	3	

- Number of *combinations* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose* k elements out of n
  - **Only elements** matter
  - Number of subsets of cardinality k of a set of cardinality n
  - Number of dispositions modulo number of permutations

$$C_{n,k} = \frac{D_{n,k}}{P_k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \text{choose}(n,k)$$

Combination of 3 elements out of 5

		X	X	X

- *Binomial theorem*

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$