

Probabilité et Simulation

PolyTech INFO4 (Grenoble) – TD

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References

- Fundamentals: [1] B. Jourdain, *Probabilités et statistique pour l'ingénieur*. 2018.
- Further reading: [2] P. Billingsley, *Probability and Measure*. John Wiley & Sons, 2012.

Websites

- CM: <https://github.com/jonatha-anselmi/INFO4-PS>
- TD: <https://github.com/davidelegacci/probasim24>

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1. Theory recap - Probability on finite set

Reminder: a set is

- *finite* (*fini*), if it has a finite number of elements
- *countable* (*dénombrable*), if either it is finite, or it can be made in one to one correspondence with the set of natural numbers

1.1. Finite probability space

- *Random process*: one random outcome out of finitely many
 - *Sample space* Ω = *finite* set of possible *outcomes* ω
 - *Probability* on Ω = set of weights $\mathbb{P}(\omega) \in \mathbb{R}$ on each $\omega \in \Omega$ such that
 - $\mathbb{P}(\omega) > 0 \forall \omega \in \Omega$
 - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$
-

- *Event* $A \subseteq \Omega$ = any subset of the sample space

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) \quad (1)$$

- *Complementary event* $A^c = \Omega/A$ (pronounced “*not A*”)
- “*A and B*” = $A \cap B$
- “*A or B*” = $A \cup B$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad (2)$$

- Probability of complementary event

$$\Omega = A^c \sqcup A \Rightarrow 1 = \mathbb{P}(A^c) + \mathbb{P}(A) \quad (3)$$

- *Indicator function* of event $A \subseteq \Omega$

$$\mathbb{1}_A : \Omega \rightarrow \mathbb{R}, \quad \mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \quad (4)$$

- Denoted by $|S|$ the *cardinality* of a set S
- Denoted by S^n the *cartesian product* of S with itself n times

$$S^n = S \times S \times \dots \times S = \{(s_1, \dots, s_n) : s_i \in S \text{ for all } i = 1, \dots, n\} \quad (5)$$

- *Cardinality of cartesian product*

$$|S^n| = |S|^n \quad (6)$$

1.2. Uniform probability

- *Every outcome* $\omega \in \Omega$ *has the same weight*

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|} \quad (7)$$

- *Uniform probability of the event* $A \subseteq \Omega$

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|} \quad (8)$$

1.3. Counting

- Number of *permutations* of k elements:
 - Number of ways to *order* k elements
 - **Only order matters**

$$P_k = k! \quad (9)$$

Permutation of 5 elements

1	5	4	3	2
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- Number of *dispositions* of k elements out of n ($k \leq n$):
 - Number of ways to *choose and order* k elements out of n
 - **Order and elements matter**
 - Number of injections : $\{1, \dots, k\} \rightarrow \{1, \dots, n\}$

$$D_{n,k} = \underbrace{n(n-1)\dots}_{k \text{ times}} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!} \quad (10)$$

Disposition of 3 elements out of 5

	2	1	3	
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- Number of *combinations* of k elements out of n ($k \leq n$):
 - Number of ways to *choose* k elements out of n
 - **Only elements matter**
 - Number of subsets of cardinality k of a set of cardinality n
 - Number of dispositions modulo number of permutations

$$C_{n,k} = \frac{D_{n,k}}{P_k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \text{choose}(n, k) \quad (11)$$

Combination of 3 elements out of 5

		X	X	X
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- *Binomial theorem*

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j \quad (12)$$

Exercises

Exercise 1.1 (*Handshakes and kisses*)

There are f girls and g boys in a room. Boys exchange handshakes, girls exchange kisses, boys and girls exchange kisses. How many kisses in total?

The number of kisses exchanged among girls is the number of subsets of cardinality 2 of a set of cardinality f , that is $\binom{f}{2} = \frac{f(f-1)}{2}$. Or, think that each girl gives $f - 1$ kisses, and one needs a factor of one half to avoid double counting.

For the number of kisses exchanged between boys and girls: the first girl gives g kisses, the second girl gives g kisses, and so on, so we have fg in total.

$$\text{number of kisses} = \frac{f(f-1)}{2} + fg \quad (13)$$

Exercise 1.2 (*Throwing a dice*) Throw a fair dice with f faces n times. What's the prob to never get the same result twice?

General strategy

- Identify sample space Ω (write in set-theoretic notation!) and its cardinality $|\Omega|$
- Identify event $A \subseteq \Omega$ (write in set-theoretic notation!) and its cardinality $|A|$
- Uniform probability? If so, use $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$

Let $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{F} = \{1, \dots, f\}$. The sample space is

$$\Omega = \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \mathcal{F} \text{ for all } i \in \mathcal{N}\} = \mathcal{F}^n \quad (14)$$

with cardinality

$$|\Omega| = |\mathcal{F}^n| = |\mathcal{F}|^n = f^n \quad (15)$$

Endow the sample space with the uniform probability (since every outcome of the experiment is equiprobable).

The event we're looking at is

$$A = \{\omega \in \Omega : \omega_i \neq \omega_j \text{ for all } i \neq j \in \mathcal{N}\} \quad (16)$$

Clearly if $n > f$ then $\mathbb{P}(A) = 0$. Let $n \leq f$. The (uniform) probability of the event A is $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$, with

$$\begin{aligned} |A| &= \# \text{ of ways to choose and order } n \text{ elements out of } f \\ &= \underbrace{f(f-1)\dots}_n = f(f-1)\dots(f-n+1) = \frac{f!}{(f-n)!} \end{aligned} \quad (17)$$

$$\mathbb{P}(A) = \frac{f!}{f^n(f-n)!} \quad (18)$$

Exercise 1.3 (*Birthday paradox*) What is the probability that at least 2 people out of n have the same birthday? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}\left(\underbrace{\text{no two people have the same birthday}}_{\text{Ex. 2}}\right) \\
&= 1 - \frac{y!}{y^n(y-n)!}
\end{aligned}
\tag{19}$$

Formal solution Let $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{Y} = \{1, \dots, y\}$ with $n \leq y$. The sample space is

$$\begin{aligned}
\Omega &= \text{distributions of possible birthdays of } n \text{ people} \\
&= \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \mathcal{Y} \text{ for all } i \in \mathcal{N}\} = \mathcal{Y}^n
\end{aligned}
\tag{20}$$

where ω_i is the birthday of the i -th person. The cardinality of the sample space is

$$|\Omega| = |\mathcal{Y}^n| = |\mathcal{Y}|^n = y^n \tag{21}$$

The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \neq j \in \mathcal{N} : \omega_i = \omega_j\} \tag{22}$$

Note that this is the complementary event to the event defined in Equation 16 of Exercise 2. Thus we can compute its probability as

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) \tag{23}$$

in agreement with Equation 19.

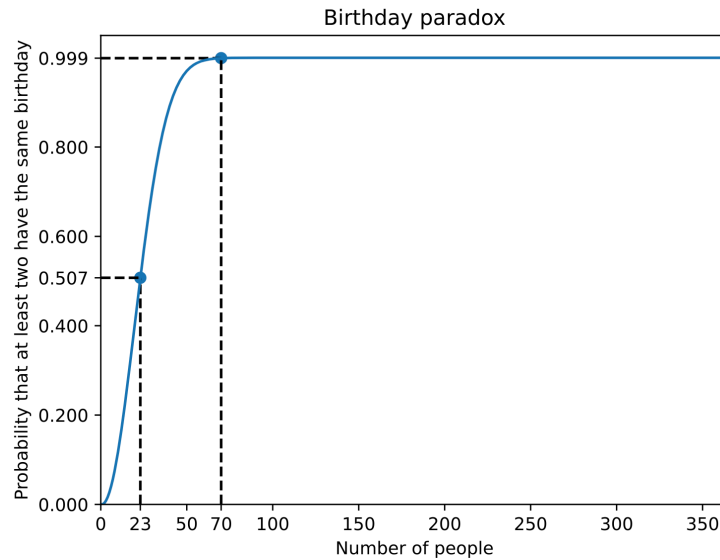


Figure 4: Birthday paradox probability. [Code available.](#)

Exercise 1.4 (*Same birthday as the prof*) What is the probability that at least 1 student out of n has the same birthday of the prof? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}\left(\underbrace{\text{nobody has the prescribed birth date}}\right) \\
&= 1 - \left(\frac{y-1}{y}\right)^n
\end{aligned}
\tag{24}$$

Formal solution 1 As above $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{Y} = \{1, \dots, y\}$ with $n \leq y$. The sample space is $\Omega = \mathcal{Y}^{n+1}$, that is the set of possible birthdays of $n + 1$ people, the $(n + 1)$ -th being the prof. Its cardinality is $|\Omega| = y^{n+1}$. The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \in \mathcal{N} : \omega_i = \omega_{n+1}\} \quad (25)$$

with complementary event

$$A^c = \{\omega \in \Omega : \omega_i \neq \omega_{n+1} \forall i \in \mathcal{N}\} \quad (26)$$

As usual $\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{|A^c|}{|\Omega|}$, with

$$|A^c| = \underbrace{y}_{\text{prof}} \cdot \underbrace{(y-1)^n}_{\text{students}} \quad (27)$$

So, $\mathbb{P}(A) = 1 - \frac{y(y-1)^n}{y^{n+1}} = 1 - \left(\frac{y-1}{y}\right)^n$, in agreement with Equation 24.

Note that a factor $\frac{y}{y}$, corresponding to the prof's birthday, simplifies in the last step. Alternatively, you can fix the birthday of the prof and exclude it from the analysis from the beginning. In this case $|\Omega| = y^n$ and $|A^c| = (y-1)^n$, leading to the same result.

Formal solution 2 Using the probability of the complementary event is often the smartest way to proceed, but for the sake of completeness let's see how to get the same result directly. Let A_j be the event “*exactly j students out of n have the same birthday as the prof*”. The event we look at then is

$$A = \sqcup_{j \in \mathcal{N}} A_j \quad (28)$$

with probability (cf Equation 2)

$$\mathbb{P}(A) = \sum_{j \in \mathcal{N}} \mathbb{P}(A_j) = \frac{\sum_{j \in \mathcal{N}} |A_j|}{|\Omega|} \quad (29)$$

The cardinality of A_j is

$$\begin{aligned} |A_j| &= \underbrace{1 \dots 1}_{j \text{ times}} \cdot \underbrace{(y-1) \dots (y-1)}_{n-j \text{ times}} \cdot \underbrace{y}_{\text{prof}} \cdot \underbrace{\binom{n}{j}}_{\text{number of ways to choose } j \text{ elements out of } n} \\ &= y(y-1)^{n-j} \binom{n}{j} \end{aligned} \quad (30)$$

By an application of the binomial theorem (Equation 12) and a short manipulation,

$$\sum_{j=1}^n |A_j| = y(y^n - (y-1)^n) \quad (31)$$

which leads back to Equation 24.

2. Theory recap - Conditional probability and independence

2.1. Conditional probability

- Let \mathbb{P} be a probability on Ω , and consider the events $A, B \subseteq \Omega$.
- *Conditional probability*: probability of A given B

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \text{ if } \mathbb{P}(B) \neq 0 \quad (32)$$

- not really defined if $\mathbb{P}(B) = 0$, cf [2] pag. 427.
- often used as

$$\mathbb{P}(A \cap B) = \mathbb{P}(A | B)\mathbb{P}(B) \quad (33)$$

- Conditional probability and complementary event (proof: simple exercise.)

$$\mathbb{P}(A | B) + \mathbb{P}(A^c | B) = 1 \quad (34)$$

- *Total Probability Theorem*

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A | B)\mathbb{P}(B) + \mathbb{P}(A | B^c)\mathbb{P}(B^c) \quad (35)$$

- *Bayes theorem*

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A) \Rightarrow \mathbb{P}(A | B)\mathbb{P}(B) = \mathbb{P}(B | A)\mathbb{P}(A) \quad (36)$$

See [this notebook](#) for an example of Bayes theorem in action.

2.2. Independent events

Let Ω be equipped with a probability \mathbb{P} .

- two events $A, B \subseteq \Omega$ are said *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \quad (37)$$

- equivalently, by definition of conditional probability, A and B are independent if

$$\mathbb{P}(A | B) = \mathbb{P}(A) \quad (38)$$

- n events A_1, \dots, A_n are said *independent* if

$$\mathbb{P}(\cap_{i \in I} A_i) = \prod_{i \in I} \mathbb{P}(A_i) \text{ for all } I \subseteq \{1, \dots, n\} \quad (39)$$

- pairwise independence does not imply independence of all events!

Exercises

Exercise 2.1 (*Pile ou Face*) Jet de 2 pieces, $\Omega = \{PP, PF, FP, FF\}$. Cet espace est muni de la probabilité uniforme. Soient les événements:

- $A =$ 1ere piece donne P
- $B =$ 2eme piece donne F
- $C =$ les deux pieces donnent le meme resultat

Questions:

- A et B sont indépendantes?
- A, B et C sont indépendants?

$$\begin{array}{ll} A = \{PP, PF\} & \mathbb{P}(A) = 1/2 \\ B = \{PF, FF\} & \mathbb{P}(B) = 1/2 \\ C = \{PP, FF\} & \mathbb{P}(C) = 1/2 \\ A \cap B = \{PF\} & \mathbb{P}(A \cap B) = 1/4 = \mathbb{P}(A)\mathbb{P}(B) \\ A \cap C = \{PP\} & \mathbb{P}(A \cap C) = 1/4 = \mathbb{P}(A)\mathbb{P}(C) \\ B \cap C = \{FF\} & \mathbb{P}(B \cap C) = 1/4 = \mathbb{P}(B)\mathbb{P}(C) \\ A \cap B \cap C = \emptyset & \mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C). \end{array}$$

Ainsi les événements A, B et C sont 2 à 2 indépendants mais pas indépendants.

Figure 5: Pairwise independence does not imply independence of all events!

Exercise 2.2 (*Pieces mecaniques defectueuses*) Parmi 10 pièces mécaniques, 4 sont défectueuses. On prend successivement deux pièces au hasard dans le lot (sans remise). Quelle est la probabilité pour que les deux pièces soient correctes?

Solution 1 Let A_i be the event *the i -th drawn piece is good*, with $i \in \{1, 2\}$. We need the probability of the event $A_2 \cap A_1$. By definition of conditional probability,

$$\mathbb{P}(A_2 \cap A_1) = \underbrace{\mathbb{P}(A_2 \mid A_1)}_{\frac{5}{9}} \underbrace{\mathbb{P}(A_1)}_{\frac{6}{10}} = \frac{1}{3}. \quad (40)$$

Solution 2 The sample space is the set of subsets of cardinality 2 of a set of cardinality 10, so $|\Omega| = \binom{10}{2}$. The event we consider is the set of subsets of cardinality 2 of a set of cardinality 6, so $|A| = \binom{6}{2}$. Then

$$\mathbb{P}(A) = \frac{\binom{6}{2}}{\binom{10}{2}} = \frac{6 \cdot 5}{10 \cdot 9} = \frac{1}{3}. \quad (41)$$

Exercise 2.3 (*Betting on cards*) We have three cards:

- a *red* card with both faces red;
- a *white* card with both faces white;
- a *mixed* card with a red face and a white face.

One of the three cards is drawn at random and one of the faces of this card (also chosen at random) is exposed. This face is red. You are asked to bet on the color of the hidden face. Do you choose red or white?

Intuitive solution Since we observe a red face, the white card cannot be on the table. There are three possibilities left: 1. we're observing a face of the red card (in which case the hidden face is red); 2. we are observing the other face of the red card (in which case the hidden face is red); 3. we are observing the red face of the mixed card (in which case the hidden face is white). So the hidden face is red 2 out of 3 times.

Formal solution Denote by R a red face and by W a white face, and denote by a subscript o the observed face, and by a subscript h the hidden face. The possible outcomes then are

$$\Omega = \{R_h \cap R_o, R_h \cap W_o, W_h \cap R_o, W_h \cap W_o\}. \quad (42)$$

For example, $W_h \cap R_o$ is the event “*the hidden face is white and the observed face is red*”. We are given the information that the observed face is red, so we need to find the probability that the hidden face has a certain color *given* the fact that the observed face is red. In other words, we need to find the probabilities

$$\mathbb{P}(W_h \mid R_o) \text{ and } \mathbb{P}(R_h \mid R_o). \quad (43)$$

Clearly it suffices to find one of them, since by Equation 34 they sum to 1. By definition of conditional probability (Equation 32)

$$\mathbb{P}(R_h \mid R_o) = \frac{\mathbb{P}(R_h \cap R_o)}{\mathbb{P}(R_o)}. \quad (44)$$

The numerator of Equation 44 corresponds to the probability of drawing the red card (event denoted by r), which is $\frac{1}{3}$:

$$\begin{aligned}
\mathbb{P}(R_h \cap R_o) &= \mathbb{P}(R_o \cap r) \\
&= \underbrace{P(R_o \mid r)}_1 \underbrace{P(r)}_{\frac{1}{3}} = \frac{1}{3}.
\end{aligned} \tag{45}$$

The denominator of Equation 44 is the probability of drawing the red card, plus the probability of drawing the mixed card (event denoted by m) and observing the red face, that is $\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}$. This follows formally from the Total Probability Theorem (Equation 35) and the definition of conditional probability (Equation 32):

$$\begin{aligned}
\mathbb{P}(R_o) &= \mathbb{P}(R_o \cap R_h) + \mathbb{P}(R_o \cap W_h) \\
&= \frac{1}{3} + \mathbb{P}(R_o \cap m) \\
&= \frac{1}{3} + \underbrace{P(R_o \mid m)}_{\frac{1}{2}} \underbrace{P(m)}_{\frac{1}{3}} = \frac{1}{2}
\end{aligned} \tag{46}$$

In conclusion, given that the observed face is red, the hidden face is red with probability $2/3$:

$$\mathbb{P}(R_h \mid R_o) = \frac{1/3}{1/2} = \frac{2}{3}. \tag{47}$$

Exercise 2.4 (Russian roulette) You are playing two-person Russian roulette with a revolver featuring a rotating cylinder with six bullet slots. Each time the gun is triggered, the cylinder rotates by one slot. Two bullets are inserted one next to the other into the cylinder, which is then randomly positioned. Your opponent is the first to place the revolver against her temple. She presses the trigger and... she stays alive. With great display of honor, she offers you to rotate the barrel again at random before firing in turn. What do you decide?

The bullets occupy the positions x and $x + 1 \bmod 6$:

$$\Omega = \{12, 23, 34, 45, 56, 61\} \tag{48}$$

Say the revolver shots from position 1. The event “*the first player dies*” is

$$\text{die}_1 = \{12, 61\} \tag{49}$$

so $\mathbb{P}(\text{die}_1) = \frac{1}{3}$ and $\mathbb{P}(\text{live}_1) = \frac{2}{3}$. We need to compute

$$\mathbb{P}(\text{die}_2 \mid \text{live}_1) = \frac{\mathbb{P}(\text{die}_2 \cap \text{live}_1)}{\mathbb{P}(\text{live}_1)} \tag{50}$$

Since the cylinder rotates after being triggered we have $\text{die}_2 = \{56, 61\}$ and $\text{die}_2 \cap \text{live}_1 = \{61\}$, so $\mathbb{P}(\text{die}_2 \mid \text{live}_1) = \frac{1/6}{2/3} = \frac{1}{4} < P(\text{die}_1)$. So you don't shuffle the barrel.

3. Theory recap - Probability space, discrete random variables, and distributions

3.1. Probability measure

The relevant references are [1] pag. 11 and [2], pag. 22 and 160.

Definition 3.1.1 (sigma-field) Let Ω be any set. A σ -field \mathcal{A} on Ω is a collection of subsets of Ω that¹

1. is closed under complement: if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$;
2. contains the whole set: $\Omega \in \mathcal{A}$;
3. is closed under countable union: if A_1, A_2, \dots is a countable family of sets of \mathcal{A} then their union $\bigcup_{i=1}^{\infty} A_i$ is also in \mathcal{A} .

A subset of Ω that is in \mathcal{A} is called *event*.

Definition 3.1.2 (Measure) Given a set Ω and a σ -algebra \mathcal{A} on Ω , a *measure* μ is a function

$$\mu : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \quad (51)$$

such that

1. $\mu(\emptyset) = 0$
2. *countable additivity* (also called σ -additivity) is fulfilled, namely the measure of a *disjoint* countable union of sets in \mathcal{A} is the sum of their measures:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \quad (52)$$

Definition 3.1.3 (Probability measure) Given a set Ω and a σ -algebra \mathcal{A} on Ω , a *probability measure* \mathbb{P} is a measure (in the sense above) with the additional requirement that

$$\mathbb{P}(\Omega) = 1. \quad (53)$$

- Note that this implies that $\mathbb{P}(A) \leq 1$ for all events $A \in \mathcal{A}$.
- A triple $(\Omega, \mathcal{A}, \mathbb{P})$ where \mathcal{A} is a σ -algebra on Ω and \mathbb{P} is a probability measure is called *probability space*.

Tip Putting all together, a probability measure $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$ on a space Ω is a function from a “well-behaved” collection of subsets of Ω (the σ -field) to $[0, 1]$, such that $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$, and fulfilling countable additivity.

3.2. Discrete random variables

Definition 3.2.1 (Discrete random variable) Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a *discrete random variable* X is a function $X : \Omega \rightarrow F$ such that

1. F is a *countable* set; we call it the space of *values* of the random variable, and we say that any $x \in F$ is a *value*.
2. *the level sets of X are events*, that is

$$\{X = x\} := \{\omega \in \Omega : X(\omega) = x\} \in \mathcal{A} \text{ for all } x \in F \quad (54)$$

- clearly, $\{X = x\} = \emptyset \in \mathcal{A}$ for all $x \in F \setminus \text{Im}(X)$
- property 2. guarantees that $\mathbb{P}\{X = x\}$ is well-defined for all $x \in F$, which allows for the following definition:

Definition 3.2.2 (Distribution of a discrete random variable) The *distribution* (or *law*) of a random variable X is the function $\mu_X : F \rightarrow [0, 1]$ defined by

$$\mu_X(x) = \mathbb{P}\{X = x\} \text{ for all } x \in F. \quad (55)$$

In words: for any value, the law of a random variable returns the probability that the random variable takes the given value.

¹In french, this set is called *tribu* on Ω . The term σ -algebra is also used – and is more common in the context of pure analysis, c.f. [3] – whereas the term σ -field is more common in the context of probability theory, c.f. [2].

- two discrete random variables X and Y taking values resp. in F and G are *independent* if

$$\mathbb{P}\{X = x, Y = y\} = \mathbb{P}\{X = x\}\mathbb{P}\{Y = y\} \text{ for all } x \in F, y \in G \quad (56)$$

- it is understood that $\{X = x, Y = y\}$ is a shorthand for the event

$$\{\omega \in \Omega : X(\omega) = x\} \cap \{\omega \in \Omega : Y(\omega) = y\} \in \mathcal{A}. \quad (57)$$

- the definition generalises to collections of DRVs, see Section 2.2.3 in [1].

Tip A discrete random variable is a *function on Ω* with *countably many values*. Think of it as an experiment with countably many random outcomes. Its *law, or distribution*, gives the probability to observe each of the possible (countable) values of the random variable.

Tip

- $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$ and $\mathbb{P}(\Omega) = 1$
- $X : \Omega \rightarrow F$ countable values space, with $\{X = x\} \in \mathcal{A}$ is an event for all $x \in F$
- $\mu_X : F \rightarrow [0, 1]$ such that $\mu_X(x) = \mathbb{P}\{X = x\}$ is the distribution of X

3.3. Standard discrete distributions

3.3.1. Bernoulli $\mathcal{B}(p)$

- The Bernoulli distribution models a random experiment which has two possible outcomes.
- More precisely, the Bernoulli distribution is the distribution of a discrete random variable X that can assume only values in $F = \{0, 1\}$.
- The distribution is parametrized by $p \in [0, 1]$, the probability to observe 1. We then have

$$\begin{aligned} \mu : \{0, 1\} &\rightarrow [0, 1] \\ 1 &\mapsto p \\ 0 &\mapsto 1 - p \\ x &\mapsto p^x(1 - p)^{1-x} \end{aligned} \quad (58)$$

- When a random variable X follows the Bernoulli distribution we write $X \sim \mathcal{B}(p)$.

3.3.2. Binomial $\mathcal{B}(n, p)$

- Distribution of the discrete random variable $X = X_1 + \dots + X_n$, where the X_i -s are independent Bernoulli variables of parameter $p \in [0, 1]$.
- $F = \{0, \dots, n\}$; $k \in F$ is value of the sum

$$\begin{aligned} \mu : F &\rightarrow [0, 1] \\ k &\mapsto \binom{n}{k} p^k (1 - p)^{n-k} \end{aligned} \quad (59)$$

3.3.3. Poisson $\mathcal{P}(\lambda)$

- probability of observing a given number of independent events occurring at constant rate $\lambda > 0$
- $F = \mathbb{N}$; $n \in F$ is number of observed events

$$\begin{aligned} \mu : F &\rightarrow [0, 1] \\ n &\mapsto e^{-\lambda} \frac{\lambda^n}{n!} \end{aligned} \quad (60)$$

3.3.4. Geometric $\mathcal{G}(p)$

- First successful event from a sequence of independent p -Bernoulli events.

- $F = \mathbb{N}^*$; $k \in F$ is first succesful event

$$\begin{aligned}\mu : F &\rightarrow [0, 1] \\ k &\mapsto p(1-p)^{k-1}\end{aligned}\tag{61}$$

3.4. Useful stuff

- Vandermonde's identity

$$\sum_{k_1=0}^k \binom{n_1}{k_1} \binom{n_2}{k-k_1} = \binom{n_1+n_2}{k}\tag{62}$$

Exercises

Exercise 3.1 (*Sum of independent binomial distributions*) Let $X_i \sim \mathcal{B}(n_i, p)$ with $i \in \{1, 2\}$ be independent discrete random variables following the Bernoulli law. Find the law of $X_1 + X_2$.

Hint: c.f. derivation of binomial distribution [1] pag. 16.

The laws $\mu_i : F_i = \{0, \dots, n_i\} \rightarrow [0, 1]$ of the two variables are given by

$$\mu_i(k_i) = \mathbb{P}(X_i = k_i) = \binom{n_i}{k_i} p^{k_i} (1-p)^{n_i-k_i}\tag{63}$$

The law of $X_1 + X_2$ takes value in $F = \{0, \dots, n_1 + n_2\}$ and for all $k \in F$ is given by

$$\begin{aligned}\mu(k) &= \mathbb{P}(X_1 + X_2 = k) \\ &= \mathbb{P}\left(\bigsqcup_{\substack{k_i \in F_i \\ k_1+k_2=k}} \{X_1 = k_1, X_2 = k_2\}\right) \\ &= \sum_{\substack{k_i \in F_i \\ k_1+k_2=k}} \mathbb{P}(X_1 = k_1) \mathbb{P}(X_2 = k_2) && \text{by c. add and indep.} \\ &= \sum_{\substack{k_i \in F_i \\ k_1+k_2=k}} \mu(k_1) \mu(k_2) && \text{by def of law} \\ &= \sum_{\substack{k_i \in F_i \\ k_1+k_2=k}} \binom{n_1}{k_1} \binom{n_2}{k_2} p^{k_1+k_2} (1-p)^{n_1+n_2-k_1-k_2} \\ &= p^k (1-p)^{n_1+n_2-k} \sum_{\substack{k_i \in F_i \\ k_1+k_2=k}} \binom{n_1}{k_1} \binom{n_2}{k_2}\end{aligned}\tag{64}$$

Let's focus on the sum. For each fixed $k_1 \in F_1$, k_2 is constrained to be $k - k_1$. Furthermore, in order for k_2 to be ≥ 0 , k_1 can be at most equal to k . So the constraints

$$\begin{aligned}k_1 &\in \{0, \dots, n_1\} \\ k_2 &\in \{0, \dots, n_2\} \\ k_1 + k_2 &= k\end{aligned}\tag{65}$$

can be replaced by the constraints

$$\begin{aligned} k_1 &\in \{0, \dots, k\} \\ k_2 &= k - k_1 \end{aligned} \tag{66}$$

namely

$$\sum_{\substack{k_i \in F_i \\ k_1 + k_2 = k}} \binom{n_1}{k_1} \binom{n_2}{k_2} = \sum_{k_1=0}^k \binom{n_1}{k_1} \binom{n_2}{k - k_1} = \binom{n_1 + n_2}{k} \tag{67}$$

where the second step follows by Vandermonde's identity.

Remark 3.4.1 Note that it is correct to have k_1 running from 0 to k :

- If $k \leq n_1$, k_1 can be at most k so that $k_2 = k - k_1 \geq 0$, so we have directly Equation 67.
- If $k > n_1$, we have

$$\begin{aligned} \sum_{\substack{k_i \in F_i \\ k_1 + k_2 = k}} \binom{n_1}{k_1} \binom{n_2}{k_2} &= \sum_{k_1=0}^{n_1} \binom{n_1}{k_1} \binom{n_2}{k - k_1} \\ &= \sum_{k_1=0}^{n_1} \binom{n_1}{k_1} \binom{n_2}{k - k_1} + \sum_{k_1=n_1+1}^k \binom{n_1}{k_1} \binom{n_2}{k - k_1} \end{aligned} \tag{68}$$

since each summand in the the second sum is zero², and we get again Equation 67.

So in conclusion

$$\mu(k) = \binom{n_1 + n_2}{k} p^k (1 - p)^{n_1 + n_2 - k} \tag{69}$$

namely $X_1 + X_2 \sim \mathcal{B}(n_1 + n_2, p)$.

Exercise 3.2 (*Sum of independent Poisson distributions*) Let $X_i \sim \mathcal{P}(\lambda_i)$ with $i \in \{1, 2\}$ be independent discrete random variables following the Poisson law. Find the law of $X_1 + X_2$.

Hint: c.f. previous exercise and binomial theorem.

Analogously to before, with $i \in \{1, 2\}$, we look for the law $\mu : \mathbb{N} \rightarrow [0, 1]$ given by

$$\begin{aligned} \mu(n) &= \mathbb{P}(X_1 + X_2 = n) = \sum_{\substack{n_i \in \mathbb{N} \\ n_1 + n_2 = n}} \mu_1(n_1) \mu_2(n_2) \\ &= \sum_{\substack{n_i \in \mathbb{N} \\ n_1 + n_2 = n}} e^{-\lambda_1} \frac{\lambda_1^{n_1}}{n_1!} e^{-\lambda_2} \frac{\lambda_2^{n_2}}{n_2!} \end{aligned} \tag{70}$$

As before we replace the constraint by $n_1 \in \{0, \dots, n\}$ and $n_2 = n - n_1$, so

²Recall that $\binom{a}{b} = 0$ if $b > a$.

$$\begin{aligned}
\mu(n) &= e^{-(\lambda_1 + \lambda_2)} \sum_{n_1=0}^n \frac{\lambda_1^{n_1}}{n_1!} \frac{\lambda_2^{n-n_1}}{(n-n_1)!} \cdot \frac{n!}{n!} \\
&= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \sum_{n_1=0}^n \binom{n}{n_1} \lambda_1^{n_1} \lambda_2^{n-n_1} \\
&= e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!}
\end{aligned} \tag{71}$$

So $X_1 + X_2 \sim \mathcal{P}(\lambda_1 + \lambda_2)$.

Exercise 3.3 (*Min of independent geometric distributions*) Let $X_i \sim \mathcal{G}(p_i)$ with $i \in \{1, 2\}$ be independent DRVs following the geometric law. Find the law of $\min\{X_1 + X_2\}$.

Hint: set up the problem in terms of inequalities. Recall that the geometric series $\sum_{i=0}^{\infty} r^i$ with $r \in [0, 1)$ converges to $\frac{1}{1-r}$.

Let $Z = \min\{X_1 + X_2\}$. We look for the law $\mu : \mathbb{N}^* \rightarrow [0, 1]$ such that

$$\begin{aligned}
\mu(k) &= \mathbb{P}(Z = k) \\
&= \mathbb{P}(Z \geq k) - \mathbb{P}(Z \geq k+1) \\
&= \mathbb{P}(X_1 \geq k, X_2 \geq k) - \mathbb{P}(X_1 \geq k+1, X_2 \geq k+1) \\
&= \mathbb{P}(X_1 \geq k)\mathbb{P}(X_2 \geq k) - \mathbb{P}(X_1 \geq k+1)\mathbb{P}(X_2 \geq k+1)
\end{aligned} \tag{72}$$

Let's drop the subscript for a moment. For a DRV $X \sim \mathcal{G}(p)$ and for $k \in \mathbb{N}^*$ we need

$$\begin{aligned}
\mathbb{P}(X \geq k) &= \mathbb{P}\left(\bigsqcup_{i \geq k} (X = i)\right) \\
&= \sum_{i \geq k} \mathbb{P}(X = i) \\
&= \sum_{i \geq k} p(1-p)^{i-1} \\
&= p(1-p)^{k-1} + p(1-p)^k + p(1-p)^{k+1} + \dots \\
&= p(1-p)^{k-1} (1 + (1-p) + (1-p)^2 + \dots) \\
&= p(1-p)^{k-1} \sum_{j=0}^{\infty} (1-p)^j \\
&= (1-p)^{k-1}
\end{aligned} \tag{73}$$

Plugging in Equation 72 we get

$$\begin{aligned}
\mu(k) &= \mathbb{P}(X_1 \geq k)\mathbb{P}(X_2 \geq k) - \mathbb{P}(X_1 \geq k+1)\mathbb{P}(X_2 \geq k+1) \\
&= (1-p_1)^{k-1}(1-p_2)^{k-1} - (1-p_1)^k(1-p_2)^k \\
&= (1-p_1)^{k-1}(1-p_2)^{k-1} [1 - (1-p_1)(1-p_2)]
\end{aligned} \tag{74}$$

Let $\alpha = (1-p_1)(1-p_2)$ and $\beta = 1 - \alpha = p_1 + p_2 - p_1p_2$, so $\mu(k) = \beta(1-\beta)^{k-1}$, i.e. $Z \sim \mathcal{G}(\beta)$.

Exercise 3.4 (**Lost messages*) If there is extra time, do [Exercise 4.1](#), else do in next session.

4. Theory recap - Expectation and variance

Recall from last week that

Tip

- $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$ and $\mathbb{P}(\Omega) = 1$
- $X : \Omega \rightarrow F$ countable, with $\{X = x\} \in \mathcal{A}$ for all $x \in F$
- $\mu : F \rightarrow [0, 1]$ such that $\mu(x) = \mathbb{P}\{X = x\}$

4.1. Expected value of a discrete random variable

In this section when we say “ X is a RV” we mean “ $X : \Omega \rightarrow F \subset \mathbb{R}$ is a discrete random variable with real values.”

Definition 4.1.1 (*Expected value*) A RV is *integrable* if $\sum_{x \in F} |x| \mathbb{P}(X = x) < +\infty$, and in this case its *expected value* $\mathbb{E}(X)$ is the real number

$$\mathbb{E}(X) := \sum_{x \in F} x \mathbb{P}(X = x). \quad (75)$$

Proposition 4.1.2 (*Linearity of expectation*)

$$\mathbb{E}(X + aY) = \mathbb{E}(X) + a\mathbb{E}(Y) \quad (76)$$

Some **properties** of the expected value of a RV:

- $\mathbb{E}(\text{constant}) = \text{constant}$
- Sufficient condition, positivity, monotonicity: see [1] pag. 20.

Some **examples**:

- $X \sim \mathcal{B}(p) \Rightarrow \mathbb{E}(X) = p$
- $X \sim \mathcal{B}(n, p) \Rightarrow \mathbb{E}(X) = np$ (immediate by linearity from the above)
- $X \sim \mathcal{P}(\lambda) \Rightarrow \mathbb{E}(X) = \lambda$
- $X \sim \mathcal{G}(p) \Rightarrow \mathbb{E}(X) = \frac{1}{p}$

Theorem 4.1.3 (*Expectation of function / Théorème de transfert, ou de transport*) Let $X : \Omega \rightarrow F \subset \mathbb{R}$ be RV, and consider some function $f : F \rightarrow \mathbb{R}$. Then

$$\mathbb{E}(f(X)) = \sum_{x \in F} f(x) \mathbb{P}(X = x) \quad (77)$$

whenever defined (see [1] Th. 2.3.6 for details).

Proposition 4.1.4 (*Expectation and independence*) Let X, Y be independent RV and f, g two functions on their values such that all the expectations are well-defined (i.e. all the random variables are integrable). Then

$$\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y)) \quad (78)$$

4.2. Variance of a discrete random variable

Definition 4.2.1 (*Variance*) A RV X is called *square integrable* if X^2 is integrable, that is if $\sum_{x \in F} x^2 \mathbb{P}(X = x) < +\infty$, and in this case

$$\text{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2] \quad (79)$$

- If X is square integrable then the variance is well defined, cf [1] Remark 2.3.11

- The variance is a measure of the spreading, dispersion, of a random variable around its expected value

The two following properties of the variance are very useful for concrete calculations:

Lemma 4.2.2 (*Variance as difference of expectations*)

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \quad (80)$$

The variance is in general **not linear**:

Lemma 4.2.3 (*Variance after scaling and shifting*)

$$\text{Var}(aX + b) = a^2 \text{Var}(X) \text{ for all } a, b \in \mathbb{R} \quad (81)$$

Proof Exercise; both lemmas follow by linearity of the expectation. \square

Proposition 4.2.4 (*Variance and independence*) Let $(X_i)_{i \in \{1, \dots, n\}}$ be a family of square integrable random variables. Then their sum is square integrable, and **if the X_i are independent** then

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) \quad (82)$$

Some **examples**:

- $X \sim \mathcal{B}(p) \Rightarrow \text{Var}(X) = p(1 - p)$, see [Exercise 4.3](#) and [proof](#).
- $X \sim \mathcal{B}(n, p) \Rightarrow \text{Var}(X) = np(1 - p)$ (immediate by [Proposition 4.2.4](#))
- $X \sim \mathcal{P}(\lambda) \Rightarrow \text{Var}(X) = \lambda$, see [proof](#).

Exercises

Exercise 4.1 (*Lost messages*) On a telecommunication channel, it has been estimated that in T time units there arrives a number of messages that can be estimated by a DRV $\sim \mathcal{P}(\lambda T)$. Each message has a loss probability equal to p , independent of the other messages. Find the probability that the number of lost message in T units of time is equal to l .

Without loss of generality rescale $\lambda \leftarrow \lambda T$. We need to find the discrete random variable L whose range $\{0, 1, 2, \dots\} \ni l$ contains the possible numbers l of lost messages in one time unit. The probability $\mathbb{P}(L = l)$ to lose l message is then by definition by the law of L .

Let X_i be the DRV for the event “the i -th message is lost”. Since each message is lost with probability p , $X_i \sim \mathcal{B}(p)$ for all $i \in \{1, 2, \dots\}$.

Let $L_a = \sum_{i=1}^a X_i$ be the DRV whose range $\text{Im}(L_a) = \{0, 1, \dots, a\} \ni l$ contains the numbers l of possible lost messages out of a arrived. Since L_a is the sum of a independent p -Bernoulli DRVs, L_a follows the binomial distribution:

$$L_a \sim \mathcal{B}(a, p). \quad (83)$$

Finally, let A be the DRV estimating the number of arrived messages $a \in \{0, 1, \dots\}$ in one time unit; we are given that $A \sim \mathcal{P}(\lambda)$.

The law of L is given by

$$\mathbb{P}(L = l) = \mathbb{P}\left[\bigcup_{a=l}^{\infty} \{L_a = l \cap A = a\}\right], \quad (84)$$

that is, we look at the disjoint union of all the events in which, given a arrived messages, l are lost. By countable additivity and independence,

$$\mathbb{P}(L = l) = \sum_{a=l}^{\infty} \mathbb{P}(L_a = l) \mathbb{P}(A = a) \quad (85)$$

Now L_a follows a binomial distribution and A follows a Poisson distribution, so

$$\begin{aligned} \mathbb{P}(L = l) &= \sum_{a=l}^{\infty} \binom{a}{l} p^l (1-p)^{a-l} e^{-\lambda} \frac{\lambda^a}{a!} \cdot \frac{\lambda^l}{\lambda^l} \\ &= \lambda^l p^l \frac{1}{l!} e^{-\lambda} \sum_{a=l}^{\infty} \frac{1}{(a-l)!} (1-p)^{a-l} \lambda^{a-l} \\ &= \lambda^l p^l \frac{1}{l!} e^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda - \lambda p)^j}{j!} \\ &= \lambda^l p^l \frac{1}{l!} e^{-\lambda} e^{\lambda - \lambda p} \\ &= \frac{(\lambda p)^l}{l!} e^{-\lambda p}. \end{aligned} \quad (86)$$

So $V \sim \mathcal{P}(\lambda p)$.

Exercise 4.2 (Poisson expectation) Let $N \sim \mathcal{P}(\lambda)$. Find $\mathbb{E}\left(X := \frac{1}{N+1}\right)$

By [Theorem 4.1.3](#) we have

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} \frac{1}{n+1} e^{-\lambda} \frac{\lambda^n}{n!} \quad (87)$$

Multiply and divide by λ and shift the running index to get $\mathbb{E}(X) = \frac{1-e^{-\lambda}}{\lambda}$.

Exercise 4.3 (Archery) An archer shoots n arrows at a target. The shots are independent, and each shot hits the target with probability p . Let X be the random variable “*number of times the target is hit*”.

1. What is the law of X ?
2. What is the expectation of X ?
3. What is the value of p that maximises the variance of X ?

The archer bets on his result. He gets g euros when he hits the target, and loses l euros when he misses it. Let Y be the random variable that represent the net gain of the archer at the end of the n shots.

4. What is the expectation of Y ?
5. What is the relation between g and l that guarantees the archer an expected gain of zero?

1. X is the sum of n independent p -Bernoulli variables, hence $X \sim \mathcal{B}(n, p)$ (binomial distribution).
2. We have to compute the expectation of a binomial random variable $X = X_1 + \dots + X_n$, where each X_i is a Bernoulli variable. Since expectations are linear we can compute the expectation of the Bernoulli variables, and sum them:

$$\mathbb{E}(X_i) = 1 \cdot p + 0 \cdot (1-p) = p \quad (88)$$

$$\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n) = np \quad (89)$$

For example, if $p = 0.5$ and $n = 10$, this means that the archer expects to hit the target 5 times.

3. Let's compute the variance of a Bernoulli and a binomial variable by Equation 80:

$$\mathbb{E}(X_i^2) = 1^2 \cdot p + 0^2 \cdot (1 - p) = p \quad (90)$$

$$\text{Var}(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 = p(1 - p) \quad (91)$$

By independence, $\text{Var}(X_1 + \dots + X_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n) = np(1 - p)$. To find the value of p that maximises the variance differentiate: $n(1 - 2p) = 0 \Rightarrow p = 0.5$.

4. Let Y_i = gain of i -th shot. Then $\mathbb{E}(Y_i) = gp - l(1 - p)$, and

$$\mathbb{E}(Y = Y_1 + \dots + Y_n) = n[gp - l(1 - p)] \quad (92)$$

For example if $n = 10$, $g = 1$, $l = 2$, we have $\mathbb{E}(Y) = 30p - 20$; and if furthermore $p = 0.5$ then $\mathbb{E}(Y) = -5$.

5. To find the value of relation between g and l required to have an expected gain of zero solve the equation $\mathbb{E}(Y) = 0$ to get

$$\frac{g}{l} = \frac{1 - p}{p}. \quad (93)$$

Thus as the probability p to hit the target goes to zero, a very big $\frac{g}{l}$ is required to guarantee an expected gain of zero; viceversa $\frac{g}{l}$ becomes infinitely small as $p \rightarrow 1$. At $p = 0.5$, as one would expect, $g = l$.

Exercise 4.4 (*Martingale doubling betting system) If time allows, do [Exercise 5.1](#), else do in next session.

5. Theory recap - Conditional law and conditional expectation

Tip All the random variables (RVs) we consider in the following are supposed to be integrable discrete random variables with real values. Recall that

- the *expected value* of a random variable $X : \Omega \rightarrow F$ is $\mathbb{E}[X] = \sum_{x \in F} x \mathbb{P}[X = x]$;
- the *law* of X is the function $F \rightarrow [0, 1]$ defined by $x \mapsto \mathbb{P}[X = x]$.

5.1. Marginal law

Let $X : \Omega \rightarrow F$ and $Y : \Omega \rightarrow G$ be random variables. Knowing the law of X and the law of Y is not enough to obtain the law of the *product random variable* (X, Y) with values in $F \times G$, unless we have further information on how X and Y are related. Let $(X = x \cap Y = y) \equiv \{X = x\} \cap \{Y = y\}$ denote the intersection of two events, so that (this is just notation)

$$\mathbb{P}[(X, Y) = (x, y)] \equiv \mathbb{P}[\{X = x\} \cap \{Y = y\}] \equiv \mathbb{P}[X = x \cap Y = y]. \quad (94)$$

Then

- If X and Y are independent, then $\mathbb{P}[(X, Y) = (x, y)] = \mathbb{P}[X = x]\mathbb{P}[Y = y]$, namely the law of the product random variable (X, Y) is the product of the laws of the variables X and Y .
- Conversely, knowing the law of (X, Y) one can recover the laws of X and Y by the *marginal law*:

$$\mathbb{P}[X = x] = \sum_{y \in G} \mathbb{P}[X = x \cap Y = y]. \quad (95)$$

- If the law of (X, Y) factors as a product,

$$\mathbb{P}[X = x \cap Y = y] = c\mu(x)\nu(y) \text{ for all } x \in F, y \in G, \quad (96)$$

then X and Y are independent (cf. [1] Rem. 2.2.12.)

One can play with these concepts, looking at the behavior of the expected value of functions of random variables for a fixed value of one of them.

5.2. Conditional law

- The random variable $(X \mid Y = y)$ with values in F for some fixed $y \in G$ has the following law:

Definition 5.2.1 The *conditional law* of $(X \mid Y = y)$ is the function from F to $[0, 1]$ given by

$$x \mapsto \mathbb{P}[X = x \mid Y = y] = \frac{\mathbb{P}[X = x \cap Y = y]}{\mathbb{P}[Y = y]} \quad (97)$$

where we use the definition of conditional probability (Equation 32).

5.3. Conditional expectation

- Let $X : \Omega \rightarrow F$ and $Y : \Omega \rightarrow G$ be random variables.
- Consider a function $f : F \times G \rightarrow \mathbb{R}$.
- $f(X, Y)$ is a random variable (assume it is integrable).
- It's expected value is

$$\begin{aligned} \mathbb{E}[f(X, Y)] &= \sum_{x \in F} \sum_{y \in G} f(x, y) \mathbb{P}[X = x \cap Y = y] \\ &= \sum_{y \in G} \left(\sum_{x \in F} f(x, y) \mathbb{P}[X = x \mid Y = y] \right) \mathbb{P}[Y = y] \end{aligned} \quad (98)$$

The term in brackets is a function $G \rightarrow \mathbb{R}$. Call it ψ . So we have $\psi : G \rightarrow \mathbb{R}$,

$$\psi(y) = \sum_{x \in F} f(x, y) \mathbb{P}[X = x \mid Y = y] \quad (99)$$

By the Transfert Theorem, Equation 98 is the expected value of the random variable $\psi(Y)$. We call this random variable *conditional expectation of $f(X, Y)$ given Y* .

Definition 5.3.1 The *conditional expectation of $f(X, Y)$ given Y* is the **discrete random variable** $\psi(Y)$. It is denoted by $\mathbb{E}[f(X, Y) \mid Y] := \psi(Y)$.

Special cases

- If $f(x, y) = x$ the conditional expectation of X given Y is the discrete random variable

$$\mathbb{E}[X \mid Y] = \psi(Y) \text{ with } \psi(y) = \sum_{x \in F} x \mathbb{P}[X = x \mid Y = y]. \quad (100)$$

- If X and Y are **independent** we have

$$\psi(y) = \sum_{x \in F} f(x, y) \mathbb{P}[X = x] = \mathbb{E}[f(X, y)]. \quad (101)$$

Expected value As mentioned above, by Equation 98 and Equation 99,

$$\mathbb{E}[\psi(Y)] = \mathbb{E}[f(X, Y)] \quad (102)$$

whenever all the RVs involved are integrable. More generally, we have the following:

- Let $X : \Omega \rightarrow F$ and $Y : \Omega \rightarrow G$ be random variables
- consider functions $f : F \times G \rightarrow \mathbb{R}$ and $g : G \rightarrow \mathbb{R}$
- consider $\psi(Y)$ as above

Proposition 5.3.2

$$\mathbb{E}[\psi(Y)g(Y)] = \mathbb{E}[f(X, Y)g(Y)] \quad (103)$$

whenever all the RVs involved are integrable.

By setting $g \equiv 1$ one recovers Equation 102. See Proposition 2.5.6 in [1].

Tip

- Let $X : \Omega \rightarrow F$ and $Y : \Omega \rightarrow G$ be random variables.
- The law of X is $x \mapsto \mathbb{P}[X = x]$
- The expectation of X is $\mathbb{E}[X] = \sum_{x \in F} x\mathbb{P}[X = x]$

Tip

- The random variable $(X \mid Y = y)$ is valued in F .
- Its law (called *conditional law of X given $(Y = y)$*) is by definition $x \mapsto \mathbb{P}[X = x \mid Y = y]$

Tip

- Define $\psi : G \rightarrow \mathbb{R}$ by Equation 100: $\psi(y) = \sum_{x \in F} x\mathbb{P}[X = x \mid Y = y]$
- The *conditional expectation of X given Y* is the random variable $\psi(Y)$ valued in \mathbb{R} .

Exercises

Exercise 5.1 (*Martingale doubling betting system*) Let $X_n \sim \mathcal{B}(p)$, $n = 1, 2, 3, 4, \dots$ be a sequence of independent Bernoulli variables (set $\mathbb{P}(X = 0) = q$ and $\mathbb{P}(X = 1) = p$ with $q + p = 1$, $q > 0$, $p > 0$). Fix a number $b_1 > 0$. Before each X_n is drawn, you bet on the outcome $X_n = 1$ the amount b_n defined recursively by $b_n = 2b_{n-1}$. If you win the bet, you receive $g_n = 2b_n$ and the game ends; else, you keep going.

1. What is your final gain?
2. What is the expected amount of money bet?

First, note that the amount placed on the n -th bet can be written as $b_n = 2^{n-1}b_1$. Let's then write down explicitly the process:

0. You bet b_1 on the event $X_1 = 1$;
 1. X_1 is drawn:
 - if $X_1 = 1$, you get g_1 and the game ends;
 - else, you bet b_2 on the event $X_2 = 1$.
 2. X_2 is drawn:
 - if $X_2 = 1$, you get g_2 and the game ends;
 - else, you bet b_3 on the event $X_3 = 1$.

...

n. X_n is drawn:

- if $X_n = 1$, you get g_n and the game ends;
- else, you bet b_{n+1} on the event $X_{n+1} = 1$.

...

Final gain

After placing n bets the invested amount is

$$B_n = \sum_{k=1}^n b_k = \sum_{k=1}^n b_1 2^{k-1} = b_1 \sum_{l=0}^{n-1} 2^l = b_1 \frac{1-2^n}{1-2} = b_1(2^n - 1). \quad (104)$$

by the [formula](#) for the sum of the first $n - 1$ terms of a geometric series. If the n -th placed bet is the winning one you receive $g_n = 2b_n$ and the game ends, so your final gain is always

$$2b_n - B_n = 2^n b_1 - (2^n - 1)b_1 = b_1. \quad (105)$$

This result is surprising: with this strategy (doubling your bet until you win), no matter how big n is (that is, how late you win) and how small the probability p to win is, *at the end of the day you go home with a net gain equal to your initial bet*. But before running to the closest casino, beware! If the probability to win is sufficiently small this well known strategy³ can lead you to invest colossal sums which may well bankrupt you before you get your win.

Expected bet amount The first occurrence of a winning bet is described by a random variable $X \sim \mathcal{G}(p)$ with values $F = \{1, 2, 3, \dots\} \ni n$. After placing n bets, the amount of invested money is the function $B : F \rightarrow \mathbb{R}$ with $B_n = b_1(2^n - 1)$ as per Equation 104. By [Theorem 4.1.3](#) and Equation 61 the expected amount of money bet is

$$\mathbb{E}(B(X)) = \sum_{n=1}^{\infty} B_n \mathbb{P}(X = n) = \sum_{n=1}^{\infty} B_n (1-p)^{n-1} p \quad (106)$$

For clarity let's write down more explicitly what is going on, recalling that $\mathbb{P}(X_i = 0) = q$ is the probability to lose each bet and $\mathbb{P}(X_i = 1) = p$ is the probability to win each bet, with $q + p = 1$.

$$\begin{aligned} \mathbb{E}(B(X)) &= b_1 \mathbb{P}(X_1 = 1) + \\ &+ (b_1 + b_2) \mathbb{P}(X_1 = 0, X_2 = 1) + \\ &+ (b_1 + b_2 + b_3) \mathbb{P}(X_1 = 0, X_2 = 0, X_3 = 1) + \\ &+ \dots + \\ &+ (b_1 + \dots + b_n) \mathbb{P}(X_1 = 0, \dots, X_{n-1} = 0, X_n = 1) + \dots \end{aligned} \quad (107)$$

Then

$$\begin{aligned} \mathbb{E}(B(X)) &= b_1 \sum_{n=1}^{\infty} (2^n - 1) q^{n-1} p \\ &= b_1 \frac{p}{q} \sum_{n=1}^{\infty} [(2q)^n - q^n] \end{aligned} \quad (108)$$

which is equivalent to Equation 106.

³[Martingale betting system](#) and [some random discussions](#) on the matter.

Recall that the geometric series $\sum_{i=0}^{\infty} r^i$ with $r \geq 0$ converges to $\frac{1}{1-r}$ iff $r < 1$, else it diverges to $+\infty$; and analogously for $\sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$. Since $q < 1$ the second series converges, whereas the first series converges iff $2q < 1$, that is if the probability p to win is strictly bigger than 0.5:

$$\text{Finite expected bet amount} \Leftrightarrow \text{prob. to win } p > \frac{1}{2}. \quad (109)$$

Thus, if $p \leq \frac{1}{2}$, the expected invested amount diverges to infinity. On the other hand, if $p > \frac{1}{2}$:

$$\begin{aligned} \mathbb{E}(B(X)) &= b_1 \frac{p}{q} \sum_{n=1}^{\infty} [(2q)^n - q^n] \\ &= b_1 \frac{p}{q} \left[\frac{2q}{1-2q} - \frac{q}{1-q} \right] \\ &= \frac{b_1}{2p-1}. \end{aligned} \quad (110)$$

Reasonably, the expected invested amount diverges to $+\infty$ as $p \rightarrow \frac{1}{2}$ from the right, and is equal to b_1 if there is certainty to win ($p = 1$). The graph of $(\frac{1}{2}, 1] \ni p \mapsto \frac{1}{2p-1}$ is shown below:



Figure 6: Horizontal: probability p of each winning Bernoulli event. Vertical: expected bet amount.

Tip Whenever a “special” number appears in your results, ask yourself whether it arises from the theory or whether it is imposed by the model at hand. Why is the probability $\frac{1}{2}$ special? Is it a consequence of our modeling choices, like the “doubling” betting strategy, or is it intrinsic to this problem (if there were an outcome more likely than the other, then you would bet on that one)? Try to generalize this scenario to different betting and reward policies, like $b_n = \alpha b_{n-1}$ and $g_n = \beta b_n$ for some $\alpha, \beta > 0$.

Exercise 5.2 (*Conditional law and expectation 1*) Let X_1, \dots, X_n be independent and identically distributed (I.I.D.) random variables following $\mathcal{B}(p)$ with $p \in (0, 1)$, and let $S = X_1 + \dots + X_n$. For $s \in [0, n]$,

1. find the conditional law of X_1 given $S = s$, and
2. find the conditional expectation of X_1 given S .

Hint: use [Definition 5.2.1](#) to find the conditional law, and Equation 100 to compute the conditional expectation.

1. By [Definition 5.2.1](#) the conditional law of X_1 given $S = s$ is

$$\begin{aligned}
 x &\mapsto \mathbb{P}[X_1 = x \mid S = s] \\
 &= \frac{\mathbb{P}[X_1 = x \cap S = s]}{\mathbb{P}[S = s]} \\
 &= \frac{\mathbb{P}[X_1 = x \cap X_2 + \dots + X_n = s - x]}{\mathbb{P}[S = s]} \\
 &= \mathbb{P}[X_1 = x] \frac{\mathbb{P}[X_2 + \dots + X_n = s - x]}{\mathbb{P}[S = s]}
 \end{aligned} \tag{111}$$

We have that $X_1 \sim \mathcal{B}(p)$, $X_2 + \dots + X_n \sim \mathcal{B}(n-1, p)$, and $S \sim \mathcal{B}(n, p)$, so

$$\mathbb{P}[X_1 = x \mid S = s] = \frac{p^x (1-p)^{1-x} \binom{n-1}{s-x} p^{s-x} (1-p)^{n-1-s+x}}{\binom{n}{s} p^s (1-p)^{n-s}} = \frac{\binom{n-1}{s-x}}{\binom{n}{s}} \tag{112}$$

Let's evaluate this for $x \in \{0, 1\}$. For $x = 0$ the binomial coefficients simplify to

$$\mathbb{P}[X_1 = 0 \mid S = s] = \frac{n-s}{n}, \tag{113}$$

while for $x = 1$ we get

$$\mathbb{P}[X_1 = 1 \mid S = s] = \frac{s}{n}. \tag{114}$$

For $x \in \{0, 1\}$ these can be packed into

$$\mathbb{P}[X_1 = x \mid S = s] = \left(\frac{s}{n}\right)^x \left(\frac{n-s}{n}\right)^{1-x} \tag{115}$$

meaning that the law of X_1 given $S = s$ is $\mathcal{B}(\frac{s}{n})$; note that this is independent of p !

2. For the second part, by Equation 100 the conditional expectation of X_1 given S is

$$\mathbb{E}[X_1 \mid S] = \psi(S) \text{ with } \psi(s) = \sum_{x \in \{0,1\}} x \mathbb{P}[X_1 = x \mid S = s] = \frac{s}{n} \tag{116}$$

Hence, the conditional expectation of X_1 given S is the random variable

$$\mathbb{E}[X_1 \mid S] = \frac{S}{n}. \tag{117}$$

Exercise 5.3 (Conditional law and expectation 2) Let $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$ be independent random variables.

1. Find the conditional law of X given $X + Y = s$, and
2. Find the conditional expectation of X given $X + Y$.

1. Let $S = X + Y$. The conditional law of X given $S = s$ is

$$\{0, 1, \dots, s\} \ni x \mapsto \mathbb{P}[X = x \mid S = s] \quad (118)$$

Following the steps of the previous exercise we readily arrive to

$$\mathbb{P}[X = x \mid S = s] = \frac{\mathbb{P}[X = x]\mathbb{P}[Y = s - x]}{\mathbb{P}[S = s]}. \quad (119)$$

Recall by [Exercise 3.2](#) that $S \sim \mathcal{P}(\mu + \lambda)$, so

$$\mathbb{P}[X = x \mid S = s] = \frac{e^{-\lambda} \lambda^x}{x!} \frac{e^{-\mu} \mu^{s-x}}{(s-x)!} \frac{s!}{e^{-(\lambda+\mu)} (\lambda + \mu)^s} \quad (120)$$

which can be manipulated – multiply and divide by $(\lambda + \mu)^x$ – into

$$\mathbb{P}[X = x \mid S = s] = \binom{s}{x} p^x (1-p)^{s-x} \text{ with } p = \frac{\lambda}{\lambda + \mu}. \quad (121)$$

Hence the law of X given $S = s$ is binomial with s iterations and probability p .

2. To find the conditional expectation of X given S , solve

$$\mathbb{E}[X \mid S] = \psi(S) \text{ with } \psi(s) = \sum_{x=0}^s x \binom{s}{x} p^x (1-p)^{s-x} = sp \quad (122)$$

The series can be solved directly ([see for example this page](#)), but it's much more convenient to recognise this as $\mathbb{E}[\mathcal{B}(s, p)]$, the expected value of the binomial distribution, which by linearity (c.f. Section 4.1) is immediately concluded to be sp (where as above $p = \frac{\lambda}{\lambda + \mu}$). So in conclusion the conditional expectation of X given $X + Y$ is the random variable

$$\mathbb{E}[X \mid X + Y] = \frac{\lambda}{\lambda + \mu} (X + Y). \quad (123)$$

Exercise 5.4 (An electronic component has a lifetime...) Exercice corrigé 2.6.4. in [1].

Exercice corrigé 2.6.4. Un composant électronique a une durée de vie X qu'on mesure en nombre entier d'unités de temps. On fait l'hypothèse que, à chaque unité de temps, ce composant a une probabilité $p \in]0, 1[$ de tomber en panne, de sorte que $X \sim \mathcal{Geo}(p)$. On considère un autre composant dont la durée de vie Y est indépendante de X et de même loi. On pose

$$S = \min(X, Y) \text{ et } T = |X - Y|.$$

1. Que représentent S et T ?
2. Calculer $\mathbb{P}(S = s \text{ et } T = t)$ pour $s \geq 1$ et $t \geq 0$ (distinguer $t = 0$ de $t \geq 1$).
3. En déduire les lois de S et T puis $\mathbb{E}(T)$. Quel est le nom de la loi de S ?
4. Les variables aléatoires S et T sont-elles indépendantes ?

1. **Que représentent S et T ?**

S représente le premier temps de panne et T la durée qui sépare les deux temps de panne.

2. **Calculer $\mathbb{P}(S = s \text{ et } T = t)$ pour $s \geq 1$ et $t \geq 0$ (distinguer $t = 0$ et $t \geq 1$).**

Pour $s \geq 1$ et $t \geq 0$, on a toujours

$$\{S = s, T = t\} = \{X = s, Y = s + t\} \cup \{X = s + t, Y = s\}$$

mais l'union n'est disjointe que si $t \geq 1$ (sinon les deux événements sont égaux à $\{X = Y = s\}$). Donc pour $s, t \geq 1$, on a, en utilisant l'indépendance de X et Y ,

$$\begin{aligned} \mathbb{P}(S = s \text{ et } T = t) &= \mathbb{P}(X = s \text{ et } Y = s + t) + \mathbb{P}(Y = s \text{ et } X = s + t) \\ &= \mathbb{P}(X = s)\mathbb{P}(Y = s + t) + \mathbb{P}(Y = s)\mathbb{P}(X = s + t) \\ &= 2p^2(1 - p)^{2s+t-2}. \end{aligned}$$

Pour $s \geq 1$ et $t = 0$, on a, toujours par indépendance de X et Y ,

$$\begin{aligned} \mathbb{P}(S = s \text{ et } T = 0) &= \mathbb{P}(X = s \text{ et } Y = s) \\ &= \mathbb{P}(X = s)\mathbb{P}(Y = s) = p^2(1 - p)^{2s-2}. \end{aligned}$$

On conclut que

$$\forall s \geq 1, \forall t \geq 0, \mathbb{P}(S = s, T = t) = p^2 (1 + 1_{\{t>0\}}) (1 - p)^t (1 - p)^{2(s-1)}. \quad (10.1)$$

3. **En déduire les lois de S et T puis $\mathbb{E}(T)$. Quel est le nom de la loi de S ?**

Pour $s \geq 1$, on a par la formule des lois marginales

$$\begin{aligned} \mathbb{P}(S = s) &= \sum_{t=0}^{\infty} \mathbb{P}(S = s \text{ et } T = t) \\ &= p^2(1 - p)^{2s-2} + \sum_{t=1}^{\infty} 2p^2(1 - p)^{2s+t-2} \\ &= (1 - p)^{2(s-1)} \left(p^2 + 2p^2(1 - p) \sum_{u=0}^{\infty} (1 - p)^u \right) \text{ où } u = t - 1 \\ &= \left(p^2 + 2p^2(1 - p) \frac{1}{p} \right) (1 - p(2 - p))^{s-1} = p(2 - p)(1 - p(2 - p))^{s-1}. \end{aligned}$$

On reconnaît la loi géométrique de paramètre $p(2 - p)$.

Maintenant, toujours par la formule des lois marginales, pour $t \geq 0$,

$$\begin{aligned} \mathbb{P}(T = t) &= \sum_{s=1}^{\infty} \mathbb{P}(S = s \text{ et } T = t) = \sum_{s=1}^{\infty} (1 + 1_{\{t>0\}}) p^2(1 - p)^{2(s-1)+t} \\ &= (1 + 1_{\{t>0\}}) p^2(1 - p)^t \sum_{u=0}^{\infty} (1 - p(2 - p))^u = (1 + 1_{\{t>0\}}) \frac{p(1 - p)^t}{2 - p}. \end{aligned}$$

D'où

$$\begin{aligned} \mathbb{E}(T) &= \sum_{t=0}^{\infty} t \mathbb{P}(T = t) = \sum_{t=1}^{\infty} t \frac{2p(1 - p)^t}{2 - p} \\ &= \frac{2p(1 - p)}{2 - p} \sum_{t=1}^{\infty} t(1 - p)^{t-1} = \frac{2p(1 - p)}{2 - p} f'(1 - p), \end{aligned}$$

où $f(x) = \sum_{t=0}^{\infty} x^t = \frac{1}{1-x}$. Comme $f'(x) = \frac{1}{(1-x)^2}$, $f'(1 - p) = 1/p^2$ et $\mathbb{E}(T) = \frac{2(1-p)}{p(2-p)}$.

4. **Les variables aléatoires S et T sont-elles indépendantes ?**

On peut montrer l'indépendance en utilisant la définition. En effet, on a pour $s \geq 1$ et $t \geq 0$,

$$\begin{aligned} \mathbb{P}(S = s)\mathbb{P}(T = t) &= p(2 - p)[(1 - p)^2]^{s-1} (1 + 1_{\{t>0\}}) \frac{p(1 - p)^t}{2 - p} \\ &= p^2 (1 + 1_{\{t>0\}}) (1 - p)^{t+2(s-1)} = \mathbb{P}(S = s \text{ et } T = t), \end{aligned}$$

d'après (10.1). On peut aussi remarquer sur (10.1) que la loi du couple (S, T) se met sous forme produit $c\mu(s)\nu(t)$ et conclure que ces variables sont indépendantes en utilisant la remarque 2.2.12.