

Probabilité et Simulation

PolyTech INFO4 (Grenoble) – TD

Last updated: 2025-09-26

Contact: davide.legacci@univ-grenoble-alpes.fr

References

- Fundamentals: [1] B. Jourdain, *Probabilités et statistique pour l'ingénieur*. 2018.
- Further reading: [2] P. Billingsley, *Probability and Measure*. John Wiley & Sons, 2012.

Websites

- CM: <https://github.com/jonatha-anselmi/INFO4-PS>
- TD: <https://github.com/davidelegacci/probasim24>

Contents

1. Theory recap - Probability on finite set	2
2. Theory recap - Conditional probability and independence	6
Bibliography	10

1. Theory recap - Probability on finite set

1.1. Probability space

- *Random process*: one random outcome out of finitely many
 - *Sample space* Ω = finite set of all possible outcomes ω
 - *Probability on Ω* = set of weights $\mathbb{P}(\omega) \in \mathbb{R}$ on each $\omega \in \Omega$ such that
 - $\mathbb{P}(\omega) > 0 \forall \omega \in \Omega$
 - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$
-

- *Event* $A \subseteq \Omega$ = subset of the sample space

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) \quad (1)$$

- *Complementary event* $A^c = \Omega/A$ (pronounced “not A ”)
- “ A and B ” = $A \cap B$
- “ A or B ” = $A \cup B$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad (2)$$

- Probability of complementary event

$$\Omega = A^c \sqcup A \Rightarrow 1 = \mathbb{P}(A^c) + \mathbb{P}(A) \quad (3)$$

- *Indicator function* of event $A \subseteq \Omega$

$$\mathbb{1}_A : \Omega \rightarrow \mathbb{R}, \quad \mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \quad (4)$$

- Denoted by $|S|$ the *cardinality* of a set S
- Denoted by S^n the *cartesian product* of S with itself n times

$$S^n = S \times S \times \dots \times S = \{(s_1, \dots, s_n) : s_i \in S \text{ for all } i = 1, \dots, n\} \quad (5)$$

- *Cardinality of cartesian product*

$$|S^n| = |S|^n \quad (6)$$

1.2. Uniform probability

- *Every outcome $\omega \in \Omega$ has the same weight*

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|} \quad (7)$$

- *Uniform probability of the event $A \subseteq \Omega$*

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|} \quad (8)$$

1.3. Counting

- Number of *permutations* of k elements:
 - Number of ways to *order* k elements
 - **Only order matters**

$$P_k = k! \quad (9)$$

Permutation of 5 elements

1	5	4	3	2
---	---	---	---	---

- Number of *dispositions* of k elements out of n ($k \leq n$):
 - Number of ways to *choose and order* k elements out of n
 - **Order and elements** matter
 - Number of injections : $\{1, \dots, k\} \rightarrow \{1, \dots, n\}$

$$D_{n,k} = \underbrace{n(n-1)\dots}_{k \text{ times}} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!} \quad (10)$$

Disposition of 3 elements out of 5

	2	1	3	
--	---	---	---	--

- Number of *combinations* of k elements out of n ($k \leq n$):
 - Number of ways to *choose* k elements out of n
 - **Only elements** matter
 - Number of subsets of cardinality k of a set of cardinality n
 - Number of dispositions modulo number of permutations

$$C_{n,k} = \frac{D_{n,k}}{P_k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \text{choose}(n, k) \quad (11)$$

Combination of 3 elements out of 5

		X	X	X
--	--	---	---	---

- *Binomial theorem*

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j \quad (12)$$

Exercises

Exercise 1.1 (Handshakes and kisses)

There are f girls and g boys in a room. Boys exchange handshakes, girls exchange kisses, boys and girls exchange kisses. How many kisses in total?

The number of kisses exchanged among girls is the number of subsets of cardinality 2 of a set of cardinality f , that is $\binom{f}{2} = \frac{f(f-1)}{2}$. Or, think that each girl gives $f - 1$ kisses, and one needs a factor of one half to avoid double counting.

For the number of kisses exchanged between boys and girls: the first girl gives g kisses, the second girl gives g kisses, and so on, so we have fg in total.

$$\text{number of kisses} = \frac{f(f-1)}{2} + fg \quad (13)$$

Exercise 1.2 (Throwing a dice) Throw a fair dice with f faces n times. What's the prob to never get the same result twice?

General strategy

- Identify sample space Ω (write in set-theoretic notation!) and its cardinality $|\Omega|$
- Identify event $A \subseteq \Omega$ (write in set-theoretic notation!) and its cardinality $|A|$
- Uniform probability? If so, use $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$

Let $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{F} = \{1, \dots, f\}$. The sample space is

$$\Omega = \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \mathcal{F} \text{ for all } i \in \mathcal{N}\} = \mathcal{F}^n \quad (14)$$

with cardinality

$$|\Omega| = |\mathcal{F}^n| = |\mathcal{F}|^n = f^n \quad (15)$$

Endow the sample space with the uniform probability (since every outcome of the experiment is equiprobable).

The event we're looking at is

$$A = \{\omega \in \Omega : \omega_i \neq \omega_j \text{ for all } i \neq j \in \mathcal{N}\} \quad (16)$$

Clearly if $n > f$ then $\mathbb{P}(A) = 0$. Let $n \leq f$. The (uniform) probability of the event A is $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$, with

$$\begin{aligned} |A| &= \# \text{ of ways to choose and order } n \text{ elements out of } f \\ &= \underbrace{f(f-1)\dots}_n = f(f-1)\dots(f-n+1) = \frac{f!}{(f-n)!} \end{aligned} \quad (17)$$

$$\mathbb{P}(A) = \frac{f!}{f^n(f-n)!} \quad (18)$$

Exercise 1.3 (Birthday paradox) What is the probability that at least 2 people out of n have the same birthday? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}\left(\underbrace{\text{no two people have the same birthday}}_{\text{Ex. 2}}\right) \\
&= 1 - \frac{y!}{y^n(y-n)!}
\end{aligned}
\tag{19}$$

Formal solution Let $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{Y} = \{1, \dots, y\}$ with $n \leq y$. The sample space is

$$\begin{aligned}
\Omega &= \text{distributions of possible birthdays of } n \text{ people} \\
&= \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \mathcal{Y} \text{ for all } i \in \mathcal{N}\} = \mathcal{Y}^n
\end{aligned}
\tag{20}$$

where ω_i is the birthday of the i -th person. The cardinality of the sample space is

$$|\Omega| = |\mathcal{Y}^n| = |\mathcal{Y}|^n = y^n \tag{21}$$

The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \neq j \in \mathcal{N} : \omega_i = \omega_j\} \tag{22}$$

Note that this is the complementary event to the event defined in Equation 16 of Exercise 2. Thus we can compute its probability as

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) \tag{23}$$

in agreement with Equation 19.

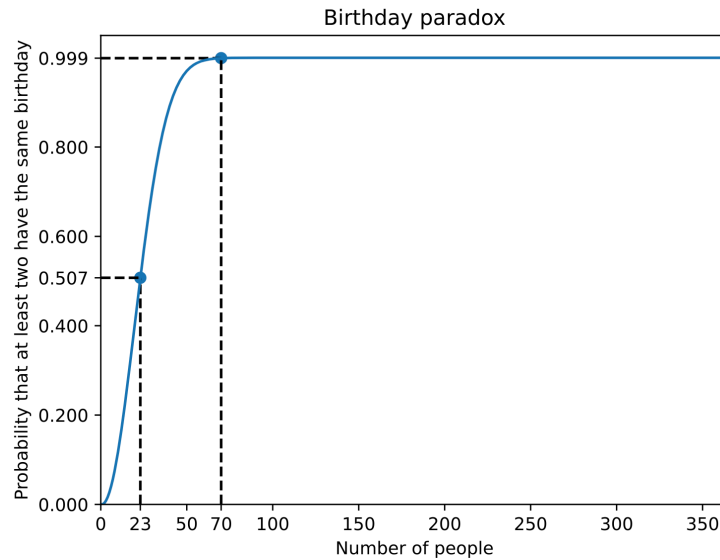


Figure 4: Birthday paradox probability. [Code available.](#)

Exercise 1.4 (*Same birthday as the prof*) What is the probability that at least 1 student out of n has the same birthday of the prof? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}\left(\underbrace{\text{nobody has the prescribed birth date}}\right) \\
&= 1 - \left(\frac{y-1}{y}\right)^n
\end{aligned}
\tag{24}$$

Formal solution 1 As above $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{Y} = \{1, \dots, y\}$ with $n \leq y$. The sample space is $\Omega = \mathcal{Y}^{n+1}$, that is the set of possible birthdays of $n + 1$ people, the $(n + 1)$ -th being the prof. Its cardinality is $|\Omega| = y^{n+1}$. The event we're looking at is

$$A = \{\omega \in \Omega : \exists i \in \mathcal{N} : \omega_i = \omega_{n+1}\} \quad (25)$$

with complementary event

$$A^c = \{\omega \in \Omega : \omega_i \neq \omega_{n+1} \forall i \in \mathcal{N}\} \quad (26)$$

As usual $\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{|A^c|}{|\Omega|}$, with

$$|A^c| = \underbrace{y}_{\text{prof}} \cdot \underbrace{(y-1)^n}_{\text{students}} \quad (27)$$

So, $\mathbb{P}(A) = 1 - \frac{y(y-1)^n}{y^{n+1}} = 1 - \left(\frac{y-1}{y}\right)^n$, in agreement with Equation 24.

Note that a factor $\frac{y}{y}$, corresponding to the prof's birthday, simplifies in the last step. Alternatively, you can fix the birthday of the prof and exclude it from the analysis from the beginning. In this case $|\Omega| = y^n$ and $|A^c| = (y-1)^n$, leading to the same result.

Formal solution 2 Using the probability of the complementary event is often the smartest way to proceed, but for the sake of completeness let's see how to get the same result directly. Let A_j be the event “*exactly j students out of n have the same birthday as the prof*”. The event we look at then is

$$A = \sqcup_{j \in \mathcal{N}} A_j \quad (28)$$

with probability (cf Equation 2)

$$\mathbb{P}(A) = \sum_{j \in \mathcal{N}} \mathbb{P}(A_j) = \frac{\sum_{j \in \mathcal{N}} |A_j|}{|\Omega|} \quad (29)$$

The cardinality of A_j is

$$\begin{aligned} |A_j| &= \underbrace{1 \dots 1}_{j \text{ times}} \cdot \underbrace{(y-1) \dots (y-1)}_{n-j \text{ times}} \cdot \underbrace{y}_{\text{prof}} \cdot \underbrace{\binom{n}{j}}_{\text{number of ways to choose } j \text{ elements out of } n} \\ &= y(y-1)^{n-j} \binom{n}{j} \end{aligned} \quad (30)$$

By an application of the binomial theorem (Equation 12) and a short manipulation,

$$\sum_{j=1}^n |A_j| = y(y^n - (y-1)^n) \quad (31)$$

which leads back to Equation 24.

2. Theory recap - Conditional probability and independence

2.1. Conditional probability

- Let \mathbb{P} be a probability on Ω , and consider the events $A, B \subseteq \Omega$.
- *Conditional probability*: probability of A given B

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \text{ if } \mathbb{P}(B) \neq 0 \quad (32)$$

- not really defined if $\mathbb{P}(B) = 0$, cf [2] pag. 427.
- often used as

$$\mathbb{P}(A \cap B) = \mathbb{P}(A | B)\mathbb{P}(B) \quad (33)$$

- Conditional probability and complementary event (proof: simple exercise.)

$$\mathbb{P}(A | B) + \mathbb{P}(A^c | B) = 1 \quad (34)$$

- *Total Probability Theorem*

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A | B)\mathbb{P}(B) + \mathbb{P}(A | B^c)\mathbb{P}(B^c) \quad (35)$$

- *Bayes theorem*

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A) \Rightarrow \mathbb{P}(A | B)\mathbb{P}(B) = \mathbb{P}(B | A)\mathbb{P}(A) \quad (36)$$

See [this notebook](#) for an example of Bayes theorem in action.

2.2. Independent events

Let Ω be equipped with a probability \mathbb{P} .

- two events $A, B \subseteq \Omega$ are said *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \quad (37)$$

- equivalently, by definition of conditional probability, A and B are independent if

$$\mathbb{P}(A | B) = \mathbb{P}(A) \quad (38)$$

- n events A_1, \dots, A_n are said *independent* if

$$\mathbb{P}(\cap_{i \in I} A_i) = \prod_{i \in I} \mathbb{P}(A_i) \text{ for all } I \subseteq \{1, \dots, n\} \quad (39)$$

- pairwise independence does not imply independence of all events!

Exercises

Exercise 2.1 (*Pile ou Face*) Jet de 2 pieces, $\Omega = \{PP, PF, FP, FF\}$. Cet espace est muni de la probabilité uniforme. Soient les événements:

- $A =$ 1ere piece donne P
- $B =$ 2eme piece donne F
- $C =$ les deux pieces donnent le meme resultat

Questions:

- A et B sont indépendantes?
- A, B et C sont indépendants?

$$\begin{array}{ll} A = \{PP, PF\} & \mathbb{P}(A) = 1/2 \\ B = \{PF, FF\} & \mathbb{P}(B) = 1/2 \\ C = \{PP, FF\} & \mathbb{P}(C) = 1/2 \\ A \cap B = \{PF\} & \mathbb{P}(A \cap B) = 1/4 = \mathbb{P}(A)\mathbb{P}(B) \\ A \cap C = \{PP\} & \mathbb{P}(A \cap C) = 1/4 = \mathbb{P}(A)\mathbb{P}(C) \\ B \cap C = \{FF\} & \mathbb{P}(B \cap C) = 1/4 = \mathbb{P}(B)\mathbb{P}(C) \\ A \cap B \cap C = \emptyset & \mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C). \end{array}$$

Ainsi les événements A, B et C sont 2 à 2 indépendants mais pas indépendants.

Figure 5: Pairwise independence does not imply independence of all events!

Exercise 2.2 (*Pieces mecaniques defectueuses*) Parmi 10 pièces mécaniques, 4 sont défectueuses. On prend successivement deux pièces au hasard dans le lot (sans remise). Quelle est la probabilité pour que les deux pièces soient correctes?

Solution 1 Let A_i be the event *the i -th drawn piece is good*, with $i \in \{1, 2\}$. We need the probability of the event $A_2 \cap A_1$. By definition of conditional probability,

$$\mathbb{P}(A_2 \cap A_1) = \underbrace{\mathbb{P}(A_2 \mid A_1)}_{\frac{5}{9}} \underbrace{\mathbb{P}(A_1)}_{\frac{6}{10}} = \frac{1}{3}. \quad (40)$$

Solution 2 The sample space is the set of subsets of cardinality 2 of a set of cardinality 10, so $|\Omega| = \binom{10}{2}$. The event we consider is the set of subsets of cardinality 2 of a set of cardinality 6, so $|A| = \binom{6}{2}$. Then

$$\mathbb{P}(A) = \frac{\binom{6}{2}}{\binom{10}{2}} = \frac{6 \cdot 5}{10 \cdot 9} = \frac{1}{3}. \quad (41)$$

Exercise 2.3 (*Betting on cards*) We have three cards:

- a *red* card with both faces red;
- a *white* card with both faces white;
- a *mixed* card with a red face and a white face.

One of the three cards is drawn at random and one of the faces of this card (also chosen at random) is exposed. This face is red. You are asked to bet on the color of the hidden face. Do you choose red or white?

Intuitive solution Since we observe a red face, the white card cannot be on the table. There are three possibilities left: 1. we're observing a face of the red card (in which case the hidden face is red); 2. we are observing the other face of the red card (in which case the hidden face is red); 3. we are observing the red face of the mixed card (in which case the hidden face is white). So the hidden face is red 2 out of 3 times.

Formal solution Denote by R a red face and by W a white face, and denote by a subscript o the observed face, and by a subscript h the hidden face. The possible outcomes then are

$$\Omega = \{R_h \cap R_o, R_h \cap W_o, W_h \cap R_o, W_h \cap W_o\}. \quad (42)$$

For example, $W_h \cap R_o$ is the event “*the hidden face is white and the observed face is red*”. We are given the information that the observed face is red, so we need to find the probability that the hidden face has a certain color *given* the fact that the observed face is red. In other words, we need to find the probabilities

$$\mathbb{P}(W_h \mid R_o) \text{ and } \mathbb{P}(R_h \mid R_o). \quad (43)$$

Clearly it suffices to find one of them, since by Equation 34 they sum to 1. By definition of conditional probability (Equation 32)

$$\mathbb{P}(R_h \mid R_o) = \frac{\mathbb{P}(R_h \cap R_o)}{\mathbb{P}(R_o)}. \quad (44)$$

The numerator of Equation 44 corresponds to the probability of drawing the red card (event denoted by r), which is $\frac{1}{3}$:

$$\begin{aligned}
\mathbb{P}(R_h \cap R_o) &= \mathbb{P}(R_o \cap r) \\
&= \underbrace{P(R_o \mid r)}_1 \underbrace{P(r)}_{\frac{1}{3}} = \frac{1}{3}.
\end{aligned} \tag{45}$$

The denominator of Equation 44 is the probability of drawing the red card, plus the probability of drawing the mixed card (event denoted by m) and observing the red face, that is $\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}$. This follows formally from the Total Probability Theorem (Equation 35) and the definition of conditional probability (Equation 32):

$$\begin{aligned}
\mathbb{P}(R_o) &= \mathbb{P}(R_o \cap R_h) + \mathbb{P}(R_o \cap W_h) \\
&= \frac{1}{3} + \mathbb{P}(R_o \cap m) \\
&= \frac{1}{3} + \underbrace{P(R_o \mid m)}_{\frac{1}{2}} \underbrace{P(m)}_{\frac{1}{3}} = \frac{1}{2}
\end{aligned} \tag{46}$$

In conclusion, given that the observed face is red, the hidden face is red with probability $2/3$:

$$\mathbb{P}(R_h \mid R_o) = \frac{1/3}{1/2} = \frac{2}{3}. \tag{47}$$

Exercise 2.4 (Russian roulette) You are playing two-person Russian roulette with a revolver featuring a rotating cylinder with six bullet slots. Each time the gun is triggered, the cylinder rotates by one slot. Two bullets are inserted one next to the other into the cylinder, which is then randomly positioned. Your opponent is the first to place the revolver against her temple. She presses the trigger and... she stays alive. With great display of honor, she offers you to rotate the barrel again at random before firing in turn. What do you decide?

The bullets occupy the positions x and $x + 1 \bmod 6$:

$$\Omega = \{12, 23, 34, 45, 56, 61\} \tag{48}$$

Say the revolver shots from position 1. The event “*the first player dies*” is

$$\text{die}_1 = \{12, 61\} \tag{49}$$

so $\mathbb{P}(\text{die}_1) = \frac{1}{3}$ and $\mathbb{P}(\text{live}_1) = \frac{2}{3}$. We need to compute

$$\mathbb{P}(\text{die}_2 \mid \text{live}_1) = \frac{\mathbb{P}(\text{die}_2 \cap \text{live}_1)}{\mathbb{P}(\text{live}_1)} \tag{50}$$

Since the cylinder rotates after being triggered we have $\text{die}_2 = \{56, 61\}$ and $\text{die}_2 \cap \text{live}_1 = \{56\}$, so $\mathbb{P}(\text{die}_2 \mid \text{live}_1) = \frac{1/6}{2/3} = \frac{1}{4} < P(\text{die}_1)$. So you don't shuffle the barrel.

Bibliography

- [1] B. Jourdain, *Probabilités et statistique pour l'ingénieur*. 2018.
- [2] P. Billingsley, *Probability and Measure*. John Wiley & Sons, 2012.