<u>Probabilité et Simulation</u>

PolyTech INFO4 (Grenoble) – 2024-2025 – Practical Sessions

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1. Theory recap

- *Jet set* $\Omega =$ finite set of possible outcomes ω
- *Probability* on $\Omega =$ set of weights $P(\omega) \in \mathbb{R}$ on each $\omega \in \Omega$ such that
  - $P(\omega) > 0 \forall \omega \in \Omega$
  - $\sum_{\omega \in \Omega} P(\omega) = 1$
- *Event* $A \subseteq \Omega =$ subset of the jet set
- *Complementary event* $A^c = \Omega / A$
- The cardinality of a set S is denoted by $|S|$
- *Uniform probability of the event* A

$$P(A) = \sum_{\omega \in A} P(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|} \tag{1}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{2}$$

- *Binomial theorem*

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \tag{3}$$

1.1. Counting

- Number of *permutations* of k elements:
  - Number of ways to *order* k elements
  - **Only order matters**

$$P_k = k! \tag{4}$$

- Number of *dispositions* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose and order* k elements out of n
  - **Order and elements** matter
  - Number of injections $f : \{1, ..., k\} \to \{1, ..., n\}$

$$D_{n,k} = \underbrace{n(n-1)...}_{k \text{ times}} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!} \tag{5}$$

- Number of *combinations* of k elements out of n $(k \leq n)$:
  - Number of ways to *choose* k elements out of n
  - **Only elements** matter
  - Number of subsets of cardinality k of a set of cardinality n
  - Number of dispositions modulo number of permutations

$$C_{n,k} = \frac{D_{n,k}}{P_k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \text{choose}(n,k) \tag{6}$$

1.2. Exercises

1.2.1. Handshakes and kisses

There are f girls and g boys in a room. Boys exchange handshakes, girs exchange kisses, boys and girls exchange kisses. How many kisses in total?

The number of kisses exchanegs among girls is the number of subsets of cardinality 2 of a set of cardinality f, that is $\binom{f}{2} = \frac{f(f-1)}{2}$. Or, think that each girl gives $f-1$ kisses, and one needs a factor of one half to avoid double counting.

For the number of kisses exchanged between boys and girls: the first girl gives g kisses, the second girl gives g kisses, and so on, so we have fg in total.

$$\text{number of kisses} = \frac{f(f-1)}{2} + fg \tag{7}$$

1.2.2. Throwing a dice

Throw a fair dice with f faces n times. What's the prob to never get the same result twice?

Let $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{F} = \{1, ..., f\}$. The jet set is

$$\Omega = \{\omega = (\omega_1, ..., \omega_n) : \omega_i \in \mathcal{F} \text{ for all } i \in \mathcal{N}\} = \mathcal{F}^n \tag{8}$$

with cardinality

$$|\Omega| = |\mathcal{F}^n| = |\mathcal{F}|^n = f^n \tag{9}$$

The event we're looking at is

$$A = \left\{\omega \in \Omega : \omega_i \neq \omega_j \text{ for all } i \neq j \in \mathcal{N}\right\} \tag{10}$$

Clearly if $n > f$ then $P(A) = 0$. Let $n \leq f$. The (uniform) probability of the event A is $P(A) = \frac{|A|}{|\Omega|}$, with

$$\begin{aligned} |A| &= \#\text{ of ways to choose and order } n \text{ elements out of } f \\ &= \underbrace{f(f-1)...}_{n} = f(f-1)...(f-n+1) = \frac{f!}{(f-n)!} \end{aligned} \tag{11}$$

$$P(A) = \frac{f!}{f^n(f-n)!} \tag{12}$$

1.2.3. Birthday paradox

What is the probability that at least 2 people out of n have the same birthday? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$\begin{aligned} P(A) &= 1 - P\left(\underbrace{\text{no two people have the same birthday}}_{\text{Ex. 2}}\right) \\ &= 1 - \frac{y!}{y^n(y-n)!} \end{aligned} \tag{13}$$

Formal solution Let $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{Y} = \{1, ..., y\}$ with $n \leq y$. The jet set is

$$
\begin{aligned}
\Omega &= \text{distributions of possible birthdays of } n \text{ people} \\
&= \{\omega = (\omega_1, \ldots, \omega_n) : \omega_i \in \mathcal{Y} \text{ for all } i \in \mathcal{N}\} = \mathcal{Y}^n
\end{aligned} \tag{14}
$$

where ω_i is the birthday of the i-th person. The cardinality of the jet set is

$$
|\Omega| = |\mathcal{Y}^n| = |\mathcal{Y}|^n = y^n \tag{15}
$$

The event we're looking at is

$$
A = \left\{\omega \in \Omega : \exists i \neq j \in \mathcal{N} : \omega_i = \omega_j\right\} \tag{16}
$$

Note that this is the complementary event to the event defined in Equation 10 of Exercise 2. Thus we can compute its probability as

$$
P(A) = 1 - P(A^c) \tag{17}
$$

in agreement with Equation 13.

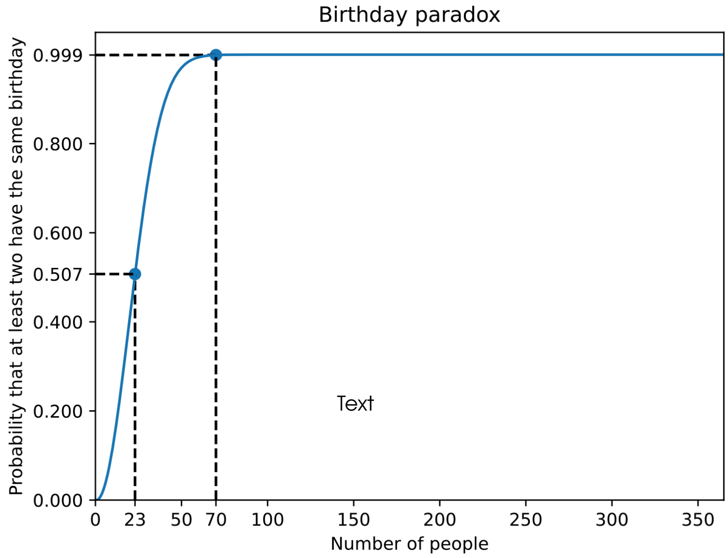

Figure 1: Birthday paradox probability.

1.2.4. Same birthday as the prof

What is the probability that at least 1 student out of n has the same birthday of the prof? (Assume: uniform birth probability and year with y number of days).

Quick solution

$$
\begin{aligned}
P(A) &= 1 - P\left(\underbrace{\text{nobody has the prescribed birth date}}\right) \\
&= 1 - \left(\frac{y-1}{y}\right)^n
\end{aligned} \tag{18}
$$

Formal solution 1 As above $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{Y} = \{1, \ldots, y\}$ with $n \leq y$. The jet set is $\Omega = \mathcal{Y}^{n+1}$, that is the set of possible birthdays of $n+1$ people, the $(n+1)$-th being the prof. Its cardinality is $|\Omega| = y^{n+1}$. The event we're looking at is

$$
A = \{\omega \in \Omega : \exists i \in \mathcal{N} : \omega_i = \omega_{n+1}\} \tag{19}
$$

with complementary event

$$
A^c = \{\omega \in \Omega : \omega_i \neq \omega_{n+1} \forall i \in \mathcal{N}\} \tag{20}
$$

As usual $P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|}$, with

$$|A^c| = \underbrace{y}_{\text{prof}} \cdot \underbrace{(y-1)^n}_{\text{students}} \tag{21}$$

So, $P(A) = 1 - \frac{y(y-1)^n}{y^{n+1}} = 1 - \left(\frac{y-1}{y}\right)^n$, in agreement with Equation 18.

Formal solution 2 Using the probability of the complementary event is often the smartest way to proceed, but for the sake of completeness let's see how to get the same result directly. Let A_j be the event "exactly j students out of n have the same birthday as the prof". The event we look at then is

$$A = \sqcup_{j \in \mathcal{N}} A_j \tag{22}$$

with probability (cf Equation 2)

$$P(A) = \sum_{j \in \mathcal{N}} P(A_j) = \frac{\sum_{j \in \mathcal{N}} |A_j|}{|\Omega|} \tag{23}$$

The cardinality of A_j is

$$\begin{aligned} |A_j| &= \underbrace{1...1}_{j \text{ times}} \cdot \underbrace{(y-1)...(y-1)}_{n-j \text{ times}} \cdot \underbrace{y}_{\text{prof}} \cdot \underbrace{\binom{n}{j}}_{\text{number of ways to choose } j \text{ elements out of } n} \\ &= y(y-1)^{n-j}\binom{n}{j} \end{aligned} \tag{24}$$

By an application of the binomial theorem (Equation 3) and a short manipulation,

$$\sum_{j=1}^{n} |A_j| = y(y^n - (y-1)^n) \tag{25}$$

which leads back to Equation 18.