The Fundamental Law of Memory Recall

Quantitative Life Science course, Final Presentation

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Purposes

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Presentation, analysis and results reproduction of the paper of Naim et.al. "Fundamental Law of Memory Recall"

- Theory
  - Theory mention from the paper
  - Study and exploration of the underlying theoretical framework, from the paper by Romani et.al. "Scaling Laws of Associative Memory Retrieval"
- Computational Results
  - Results from the paper
  - Reproduction of the results
  - Some additional analysis and ideas
- Results from experiments and conclusion

Results of Recall experiments: Study of Recall:
Experiments!

Typical experiments involve recalling randomly assembled lists of words in an arbitrary order after a brief exposure

- When the presented list becomes longer, the average number of recalled words grows but in a sublinear way
- In some studies, recall performance was found to exhibit a power-law relation to the number of presented words, but parameters of this relation were extremely variable across different experimental conditions.

? Impossible to describe memory recall with a universal mathematical law

NO!

In this paper it is shown that:

- Most of the **variability** in recall can be accounted for by measuring the **acquisition** and **maintenance** of information during the presentation phase of the experiment, and when that is controlled, recall itself is much more predictable
- Relation between the number of **items in memory** and the average fraction of it that can be successfully **recalled** is described by a **parameter-free analytical expression**

Items

1. Memory items are represented in the brain by overlapping random sparse neuronal ensembles in dedicated memory networks

$$\vec{\xi} = (0, 1, 1, \dots, 0, 1)$$

$$\xi_i = \begin{cases} 1 & \text{w.p. } f \\ 0 & \text{w.p. } 1 - f \end{cases}$$

Sparseness: $f \ll 1$

N: number of neurons of each item

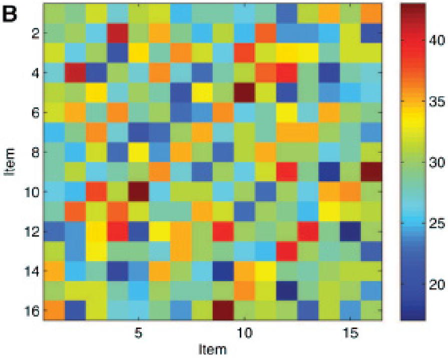
2. The next item to be recalled is the one with the largest **overlap** to the current one

Overlap (k,m) $= \sum_{i=1}^{N} \xi_i^k \xi_i^m = \vec{\xi^m} \cdot \vec{\xi^k}$

The Similarity Matrix

L: Number of items

$$S_{km} = \sum_{i=1}^{N} \xi_i^k \xi_i^m$$



Elements of the same row have positive correlation! After some algebra one finds

$$\langle S_{km} \rangle = N f^2$$
$$var(S_{km}) = N f^2 (1 - f^2)$$
$$cov(S_{km}, S_{kn}) = N f^3 (1 - f)$$
$$\varrho(S_{km}, S_{kn}) = \frac{f}{1+f} \tag{1}$$

In the limit of sparse encoding $(f \ll 1)$ we have that

$$\varrho(S_{km}, S_{kn}) \xrightarrow{f \ll 1} 0$$

That allows to replace the similarity matrix S_{km} with a **Random Symmetric Similarity Matrix**

The Model: The Recall Dynamic

Ingredients: The Similarity Matrix (SM) and the first recalled item

Dynamic: The next element being recalled is the one with the largest overlap with the current one

As soon as an element is recalled twice, we are in a loop, and no new elements can be recalled

Simulation: By repeating the process, one can calculate the average number of recalled items $\langle k \rangle$

Theory: As we are going to see now, a theoretical expression for $\langle k \rangle$ as a function of L can be found

Random Asymmetric SM: p_0

$\mathbf{p_0}$: Transition probability between any two items

$$p_0 = \frac{1}{L-1} \sim \frac{1}{L}$$

Prob. of coming back to any of the $m-1$ prev. recalled items:

$$(m-1)p_0$$

Prob. of recalling exactly k items

$$p(k;L) = (1-p_0)(1-2p_0)\dots(1-(k-2)p_0)\,(k-1)p_0$$

Let's go to the limit

$$1 \ll k \ll L$$

This allows us to write

$$1 - kp_0 \sim e^{-kp_0}$$

$$\begin{aligned} p(k;L) &= e^{-p_0} e^{-2p_0} \dots e^{-(k-2)p_0 (k-1)p_0} = \\ &= e^{-\frac{1}{2}(k-2)(k-1)p_0} (k-1)p_0 \sim \\ &\sim kp_0 e^{-\frac{k^2 p_0}{2}} = \\ &= \frac{k}{L} e^{-\frac{k^2}{2L}} \end{aligned}$$

And defining

$$x = k/\sqrt{L}$$

We can obtain the normalized PDF

$$p(x;L) = xe^{-\frac{x^2}{2}}$$

Random Asymmetric SM: $\langle k \rangle$

By definition

$$\langle x \rangle = \int_0^\infty x p(x; L) = \sqrt{\frac{\pi}{2}}$$

And therefore

$$\langle k \rangle = \sqrt{\frac{\pi L}{2}} \tag{2}$$

And by calculating

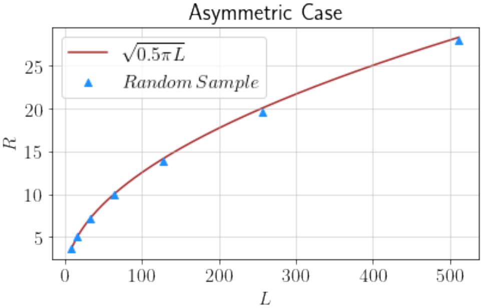
$$\langle k^2 \rangle = L \langle x^2 \rangle = 2L$$

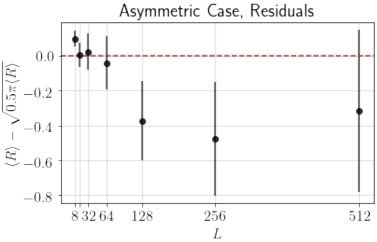
One can calculate the standard deviation also

$$\sigma(k) = \sqrt{\left(2 - \frac{\pi}{2}\right) L} \tag{3}$$

Random Asymmetric SM: Results

N = 10000





Problem: Due to the symmetry, the retrieval process quickly enters into a two-item loop

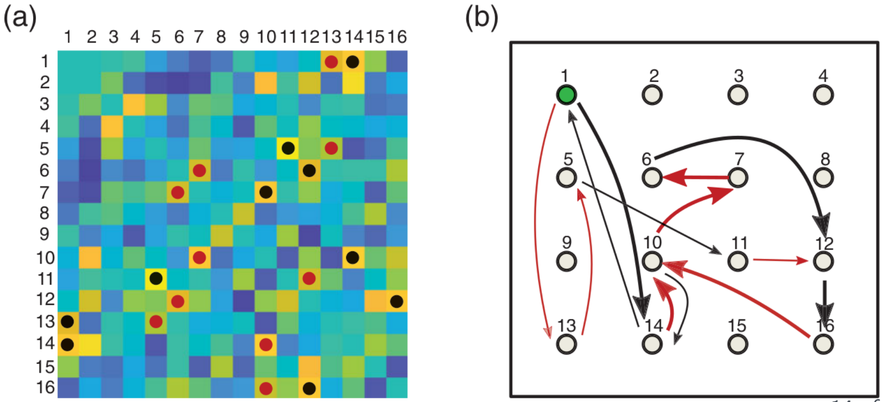
1. Let's consider the rows $\vec{S_n}$ and $\vec{S_k}$, and assume $S_{nk} = \max \vec{S_n}$
2. We have a two-item loop if $S_{nk} = S_{kn} = \max \vec{S_k}$
3. Let's define $\vec{S}_k^-$ as $\vec{S_k}$ without the S_{kn} element
4. We have the loop if and only if $S_{nk} > \max \vec{S}_k^-$ i.e. $\max \vec{S}_n > \max \vec{S}_k^-$
5. $\max \vec{S}_n$ random variable is the maximum of $L-1$ elements and $\max \vec{S}_k^-$ random variable is the maximum of $L-2$ elements

$$\max \vec{S}_n > \max \vec{S}_k^- \quad \text{w.p.} \sim 0.5$$

Random Symmetric SM

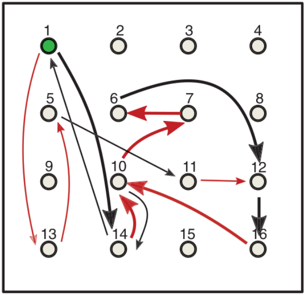
New dynamic!

We prevent the return to the just-retrieved item: when the most similar item is the one just retrieved, the second most similar item is chosen instead



We need to calculate three probabilities

- p_0 transition probability between two items
- p_1 the probability that retrieval proceeds along the previous trajectory in the opposite direction
- p_2 probability of retrieving a new item after an old one.



Calculation of p0, intuitive idea

In the Asymmetric case

$$p_0 \approx \frac{1}{L}$$

But here we prevent to go back to the just previously recalled item, that would happen with probability $1/2$

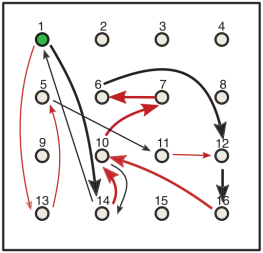
We therefore have

$$p_0 = \frac{1}{2L}$$

Calculation of p1, intuitive idea

We want $S_{nj} = S_{jn} = \max \vec{S}_j$ to be the maximum element of $\vec{S}_n$, i.e. that

$$\max \vec{S}_j > \max \vec{S}_n$$



$$p_1 = P(\max \vec{S}_j > \max \vec{S}_n \mid \max \vec{S}_n > \max \vec{S}_k)$$

$$p_1 = 1/3 \tag{4}$$

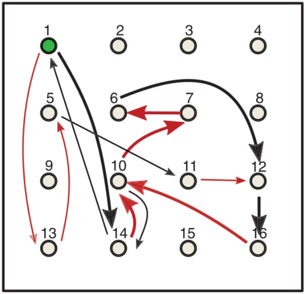
p_2 probability of retrieving a new item after an old one.

Every time we have two possibilities

1. Again go back to the previous item i (1)
2. Open a new path (with probability p_2)

After some algebra one finds

$$p_2 \approx 0.3$$



Random Symmetric SM: p_0, p_1, p_2

Summarizing:

$$p_0 = \frac{1}{2L} \quad p_1 = \frac{1}{3} \quad p_2 \approx 0.3$$

Average number of items recalled before the 1st repetition:

$$\langle \mathbf{k} \rangle_{\mathbf{1st}} \sim \sqrt{\frac{1}{p_0}} = \sqrt{\pi L}$$

Average number of opened cycles:

$$P(l) = \frac{2}{3}\left(\frac{1}{3}\right)^{l-1} \quad \longrightarrow \quad \langle \mathbf{l} \rangle = \frac{1}{1-p_1} = \frac{3}{2}$$

Average number of old items visited before a new cycle is opened

$$\frac{\mathbf{1}}{\mathbf{p_2}} \approx 3$$

In the Asymmetric case we found

$$\langle k \rangle = \sqrt{\frac{\pi}{2p_0}}$$

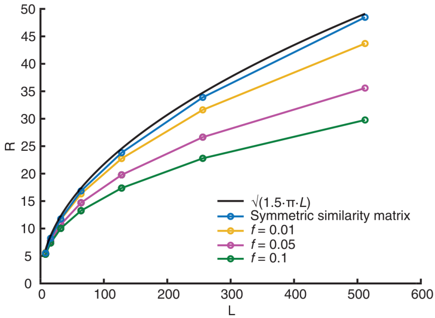
where $p_0 = 1/L$ was the probability to close the cycle (by going to a specific previous item). Here instead we have

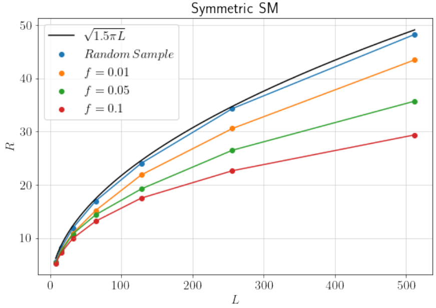
$$p_0(1 - p_1) = \frac{1}{2L}\frac{2}{3} = \frac{1}{3L}$$

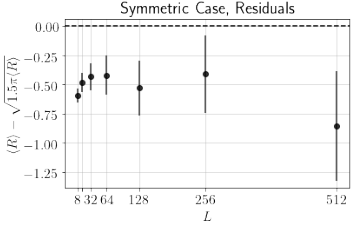
From which

$$R \equiv \langle k \rangle = \sqrt{\frac{3\pi L}{2}} \tag{5}$$

Random Symmetric SM: Results







Slow process. We want to build $\widetilde{S}_{km}$ with the same first and second order statistics of S_{km}. We define the normalized size of the k-th item as

$$M^k = \frac{1}{N} \sum_{i=1}^{N} \xi_i^k$$

that for large N can become

$$M^k \sim \mathcal{N}\left(f, \sqrt{\frac{f(1-f)}{N}}\right)$$

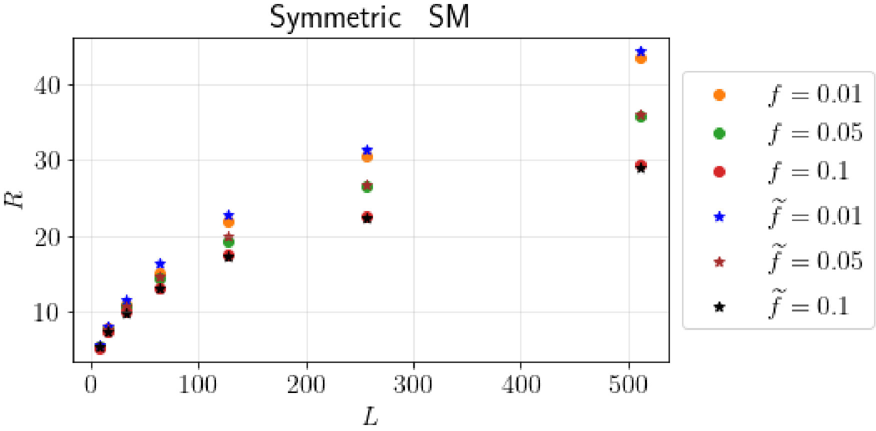
And in order to find the same 2nd order statistics one can define

$$\widetilde{S}_{km} = M^k M^m Z^{km}$$

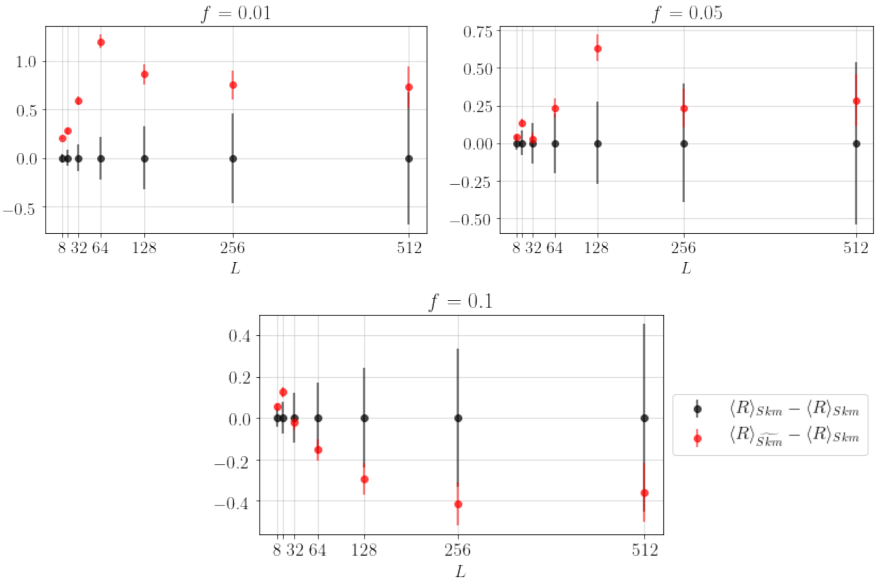
with the symmetric matrix

$$Z^{km} \sim \mathcal{N}\left(1, \frac{1-f}{f\sqrt{N}}\right)$$

One can directly sample $\widetilde{S}_{km}$ in place of building the items to build S_{km}



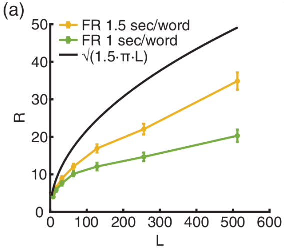
Random Symmetric SM: Equivalent Stat



Model Correction

$$R \approx \sqrt{\frac{3\pi}{2} L}$$

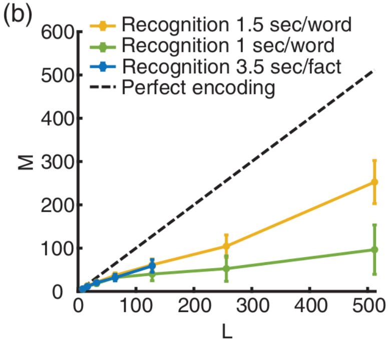
- **No free parameters** $\longrightarrow$ Result of the assumed recall mechanism
- **Universality** $\longrightarrow$ But people differ in terms of their memory effectiveness depending, e.g., on their age and experience.
- At odds with previous experimental studies showing that performance in a free recall task strongly depends on the **experimental protocol** (e.g. presentation rate during the acquisition stage)



Model Correction

It seems reasonable that *acquisition* depends on various factors, such as attention, age of participants, acquisition speed.
So, one should correct the Eq. replacing L with $\mathbf{M}$, **the number of words in memory after the whole list is presented**

$$R \approx \sqrt{\frac{3\pi}{2} M} \tag{6}$$



Model Correction: Recognition Experiment

Recognition experiment:

- Presentation of either words or short sentences expressing common knowledge facts
- At the end of presentation to each participant were shown a pair of words, one from the list just presented and one randomly chosen lure

c fraction of correctly recognized words
M average number of words remaining in memory

Number of errors:

$$L(1-c)$$

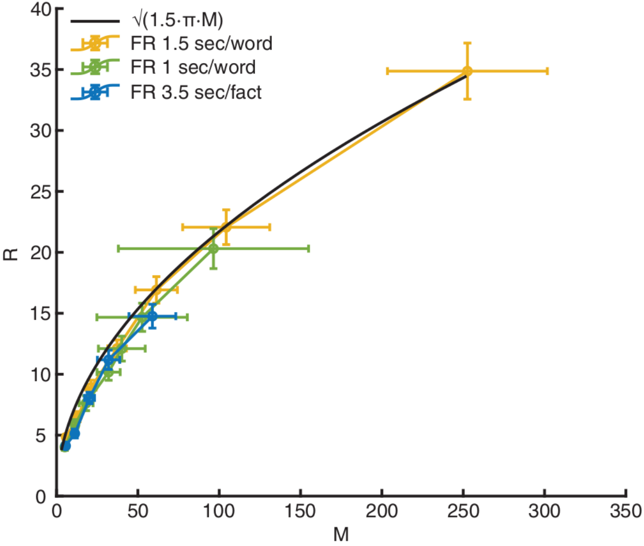
Words really not present in memory

$$2L(1-c)$$

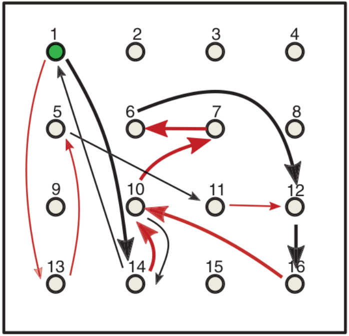
Words effectively remained in memory

$$M = L(2c-1) \tag{7}$$

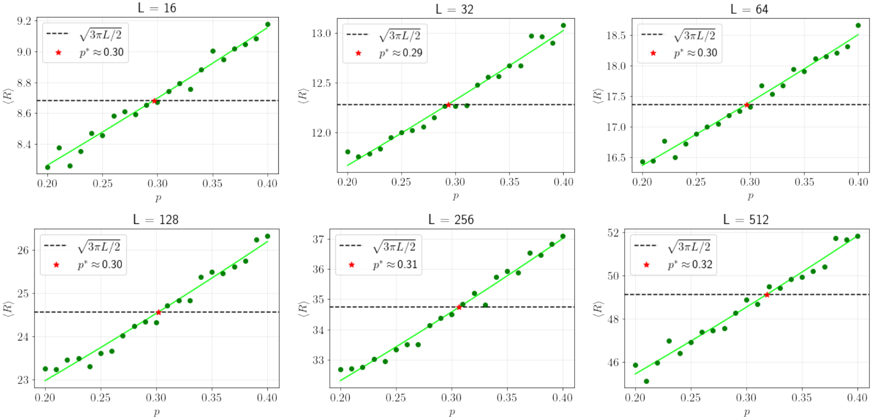
Model Correction: Final Result



1. Same dynamic as before until an element is reached for the second time
2. Jump to a never visited element with probability p^*
3. The process stops when either:
   - A jump fails
   - All the elements are retrieved



New Framework: Results



$$p* \lesssim p_1 = \frac{1}{3}$$

Conclusions

- The designed recall dynamic for a random symmetric SM of size x exhibits the relation
$$R \approx \sqrt{\frac{3\pi}{2}x}$$
- For large N, large L, sparse encoding $f \ll 1$, the SM son of our neuronal representation approximates previous case
- Data confirm with remarkable precision the analytical, parameter-free expression
$$R \approx \sqrt{\frac{3\pi}{2}M}$$
- Idea about the equivalence between paper's model and the jump model

APPENDIX

Random Symmetric SM: p_0

Calculation of p_0

Notation: now with $\vec{S}_n$ we indicate the *relevant elements* of the n-th row vector of the similarity matrix

Let's consider k current element and n an already visited element

$$
\begin{aligned}
p_0 &= P(S_{kn} = \max \vec{S}_k \mid S_{nk} < \max \vec{S}_n) = \\
&= P(S_{kn} = \max \vec{S}_k \mid S_{kn} < \max \vec{S}_n) = \\
&= \frac{P(S_{kn} = \max \vec{S}_k, S_{kn} < \max \vec{S}_n)}{P(S_{kn} < \max \vec{S}_n)} = \\
&= \frac{P(S_{kn} = \max \vec{S}_k) P(\max \vec{S}_k < \max \vec{S}_n)}{P(S_{kn} < \max \vec{S}_n)}
\end{aligned}
$$

Random Symmetric SM: p_0

$$p_0 = \frac{P(S_{kn} = \max \vec{S}_k) P(\max \vec{S}_k < \max \vec{S}_n)}{P(S_{kn} < \max \vec{S}_n)}$$

$$P(S_{kn} = \max \vec{S}_k) = \frac{1}{L-2} \approx \frac{1}{L}$$

$$P(\max \vec{S}_k < \max \vec{S}_n) = \frac{1}{2}$$

$$P(S_{kn} < \max \vec{S}_n) = 1 - P(S_{kn} = \max \vec{S}_n) \approx 1 - \frac{1}{L} \approx 1$$

And by putting together:

$$p_0 \approx \frac{1}{2L} \tag{8}$$

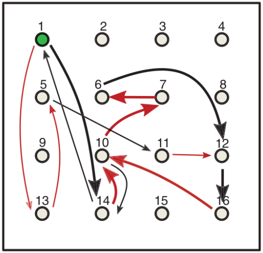
Random Symmetric SM: p_1

Calculation of p_1

$\mathbf{k}$ current element (16)
$\mathbf{n}$ old element, recalled after k (10) $\longrightarrow$ $S_{kn} = \max \vec{S}_k$
$\mathbf{j}$ element preceding n in the first path (14) $\longrightarrow$ $S_{jn} = \max \vec{S}_j$

We have two possibilities

1. Previous $n \to l$ was strong — Loop!
2. Previous $n \to l$ was weak — The strong is the one coming back!



Random Symmetric SM: p_1

We want $S_{nj} = S_{jn} = \max \vec{S}_j$ to be the maximum element of $\vec{S}_n$, i.e. that

$$\max \vec{S}_j > \max \vec{S}_n$$

(remember that S_{nj} is not a member of $\vec{S}_n$)!

We are moving from k to n, and therefore we must impose the condition that before we didn't move from n to k:

$$\max \vec{S}_n > \max \vec{S}_k$$

Therefore

$$p_1 = P(\max \vec{S}_j > \max \vec{S}_n \mid \max \vec{S}_n > \max \vec{S}_k)$$

$$p_1 = 1/3 \tag{9}$$

Random Symmetric SM: p_2

We have come back to n (10) and now we start retrieving old items: now we are at j (14). Every time we have two possibilities

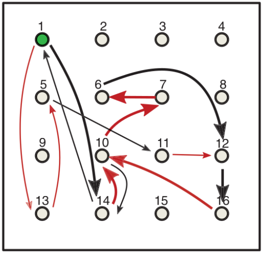
1. Again go back to the previous item i (1)
2. Open a new path (with probability p_2)

We have that

$$S_{jn} = \max \vec{S}_j \quad S_{ij} = \max \vec{S}_i$$

We can't go back to n; therefore our next element is either given by

$$\max{}_2 \vec{S}_j \quad \text{or} \quad S_{ji} = \max \vec{S}_i$$



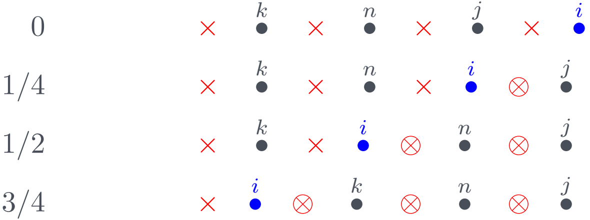
Random Symmetric SM: p_2

Since we want to open a new path, we *don't want* to go back to i, i.e. we want

$$\max_2 \vec{S}_j > \max \vec{S}_i = S_{ji}$$

With the conditions:

$$p_2 = P(\max_2 \vec{S}_j > \max \vec{S}_i \mid \max \vec{S}_j > \max \vec{S}_n, \max \vec{S}_n > \max \vec{S}_k)$$



$$p_2 = 1/4(0 + 1/4 + 1/2 + 3/4) = 3/8$$

$$p_2 \approx 0.3 \tag{10}$$

Notes about the Experiments

- To avoid practice effects, each participant performed a single free recall trial with a randomly assembled list of words of a given length
- Each participant performed one **recognition** and one **recall** trial with lists of the same number of words (but different words between recognition and recall)