Laboratory Work on
Vibration of Ships and Platforms

Report of **Group 12,** composed by
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1 Introduction

This report documents the procedures performed for experimental analysis of a three degree of freedom dynamic system. The objective is to obtain experience with this type of system, improve the skills for experimental work and learn to analyze experimental data, comparing it with theoretical models to draw conclusions. [1]

1.1 Characterization of the system

As stated on project assignment [1], the experimental apparatus of the system is constituted by a *"dynamic control system for education purposes made of three subsystems. The first is the electro-mechanical installation composed of masses, damper and springs, the actuator (electric DC motor) and motion sensors or encoders. The second is the control unit in real time, the interface, the actuator amplifier and the auxiliary power units. The third sub-system is the software program which runs in the Windows system"*. Effectively, we can think that we are considering a system of three degrees of freedom, in which we have three masses, the first connected to the wall and the second and third connected to each other by springs; the damping is assumed as a consequence of the friction between the masses and the electro mechanical plant; and it is possible to use the actuator to generate a force of the type $F_0 sin(wt)$.

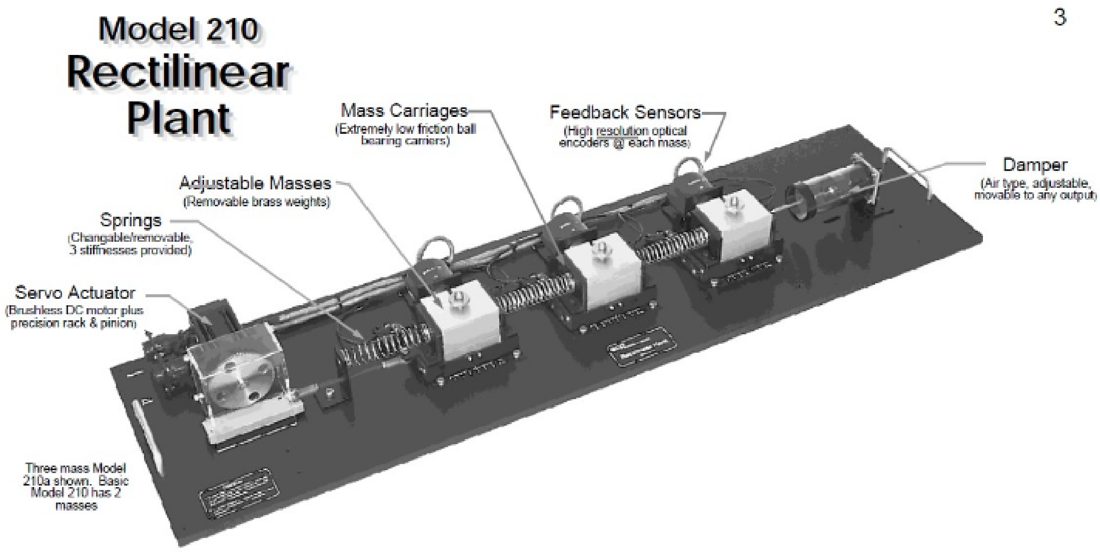


Figure 1 - Electro Mechanical plant representing the studied three-degree-of-freedom system with damping

The system can be more clearly described in terms of the coordinates, as shown in Figure 2:

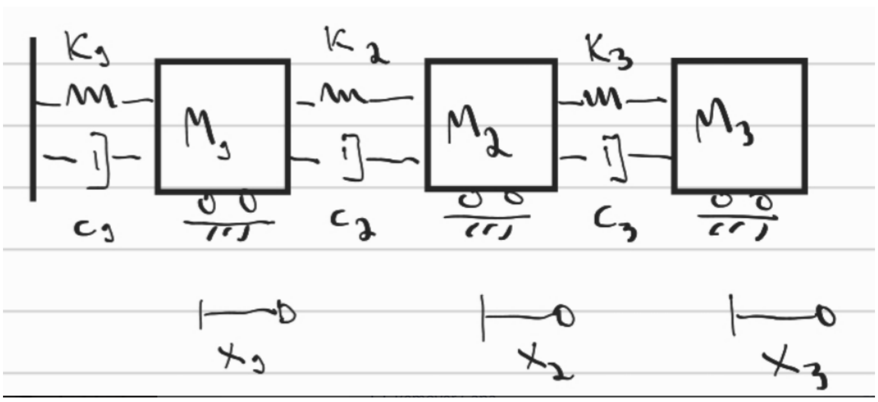


Figure 2 - Description of the studied system in terms of masses, dampers and coordinates

In the subsequent section, Lagrangian Mechanics it's used to find the differential equations of motion that describe the behavior of the masses presented in Figure 2.

1.2 Differential Equations of Motion

In many problems of classical mechanics, deducing equations of motion from direct application of fundamental principle of dynamics (Newton's second law) can become complex and laborious, since it is necessary to clearly identify forces considering their vectorial character.

Lagrangian mechanics is a reformulation of classical mechanics that provides a simpler way of deducing the equations of motion by considering the energy of bodies in the system. This means that it is possible to derive the equations of motion by thinking holistically about the problem (without having to individually consider the force acting on each body) and by thinking in terms of scalars (without having to deal with the complexities involved in the correct vectorial description of forces).

The systematic method for finding the equations of motion from the Lagrangian formulation consists of:

1) Find the generalized coordinates to describe the problem;
2) Define the Lagrangian of the system through the generalized coordinates;
3) Apply the Euler-Lagrange equations.

The Lagrangian L for a body is defined as:

$$L = K - U$$

where K is the kinetic energy and U is the potential energy. For the entire system, the Lagrangian can be given by:

$$L = \sum_{i}^{n} K_i - \sum_{i}^{n} U_i$$

where $\sum_{i}^{n} K_i$ is the sum of the kinetic energies and $\sum_{i}^{n} U_i$ is the sum of the potential energies. The Euler-Lagrange equations are defined as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x'_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

where x_i and x'_i are the generalized coordinates and their time derivatives.

To find the correct equations of motion, it is necessary to include forces that cannot be derived from potential analysis. For a system subjected to a linear damping force, the incorporation of the damping force into the Lagrangian formalism can be achieved through Rayleigh's dissipation function, given by:

$$F = \frac{1}{2}\sum_{i}^{n} c_i(x'_i)$$

Then, the modified Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x'_i}\right) - \frac{\partial L}{\partial x_i} + \frac{\partial F}{\partial x'_i} = 0$$

Applying the described formalism to the studied system, we define the generalized coordinates considering $x_1,\ x_2\ and\ x_3$ having the origin placed in the equilibrium positions of $M_1,\ M_2\ and\ M_3$, respectively. Then we define the Lagrangian of the system as:

$$L = \frac{1}{2}[(M_1 {x'_1}^2 + M_2 {x'_2}^2 + M_3 {x'_3}^2) - (k_1 {x_1}^2 + k_2 (x_2 - x_1)^2 + k_3 (x_3 - x_2)^2 + k_3 {x_3}^2)]$$

with a Rayleigh dissipation function given by:

$$F = \frac{1}{2}(c_1 {x'_1}^2 + c_2({x'_2}^2 - {x'_1}^2) + c_3({x'_3}^2 - {x'_2}^2) + c_3 {x'_3}^2)$$

and, for the forced response, an external force given by:

$$F(t) = \begin{bmatrix} F_1(t) \\ 0 \\ 0 \end{bmatrix}$$

Applying the Euler-Lagrange equations for the respective degrees of freedom and putting them it in matrix form, we get:

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} x''_1 \\ x''_2 \\ x''_3 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_i(t) \\ 0 \\ 0 \end{bmatrix}$$

It's important to note that in this case $M_1,\ M_2\ and\ M_3$ represent the whole mass of the respective bodies. Later, when we consider the addition of masses $m_1,\ m_2$ and

m_3, the only change in the presented equation of motion is to add the respective masses to the mass matrix.

2 Experimental Work

It is important to note that the data used to describe the experimental procedures in this work was not measured by the authors of this report, but sent by the professor Shan Wang.

In general terms, the experimental procedure consists primarily of calibrating the equipment and finding the relations between data provided by the experimental apparatus and physical units; the free decay test is then applied to obtain the masses, spring constants and damping coefficients; finally, the motion of each mass is measured for free response (initially displacing the body from its equilibrium position) and forced response (by applying a harmonic excitation force in the first mass).

2.1 Relation between displacements and encoder position

We seek to find the relation between displacement and encoder position to translate measured numbers in the experimental apparatus into valid units that can be analyzed. To find this relation, one has to apply the following steps:

1) Make sure the ruler it's in the zero position before any measurement;
2) Move the body to a certain ruler position (a displacement) and hold this position for a certain period of time in order to obtain a reliable corresponding encoder position value;
3) Repeat procedure 2) for a sufficiently large sample of displacements and corresponding encoder positions;
4) Find the general equation that expresses the relationship between encoder position and displacement.

Applying this procedure by taking a step size of 5mm on the encoder position, repeated in three independent ways, the following graphs are obtained:

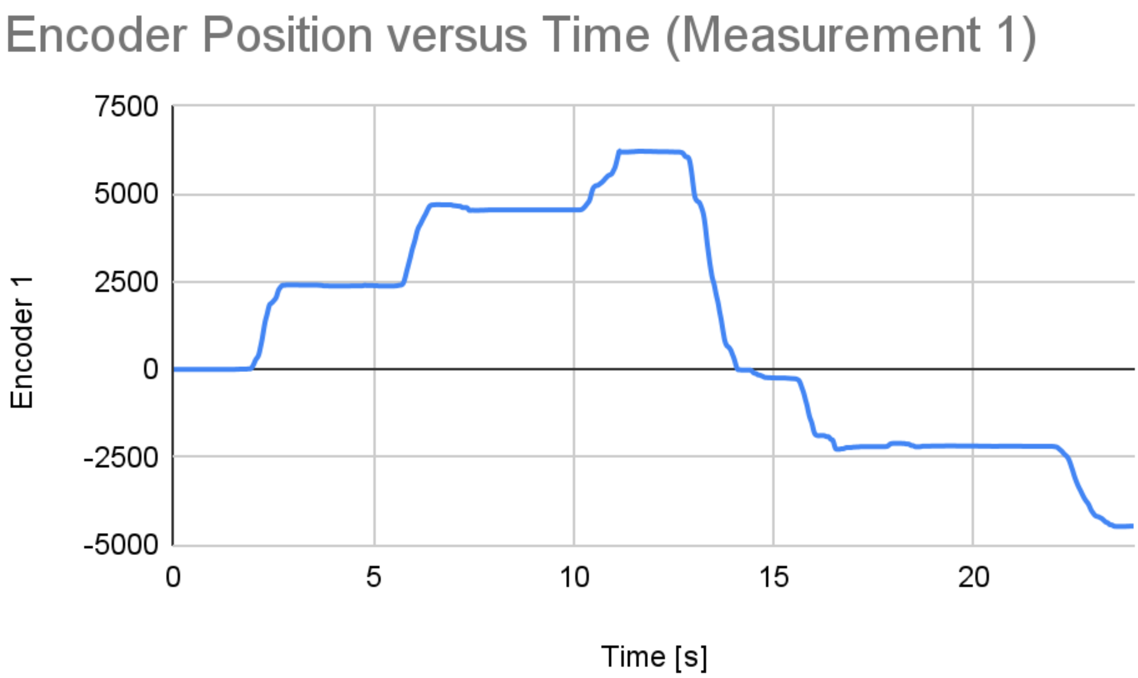


Figure 3 - Measurement 1 of Displacement versus Time for specified encoder position step intervals

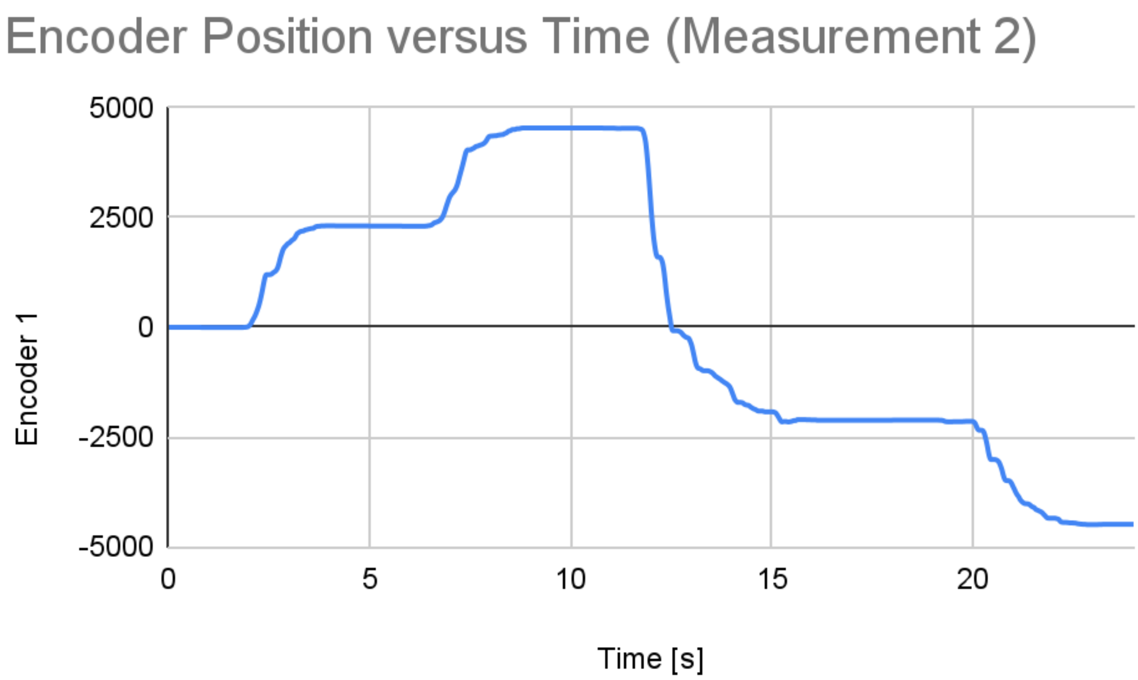


Figure 4 - Measurement 2 of Displacement versus Time for specified encoder position step intervals

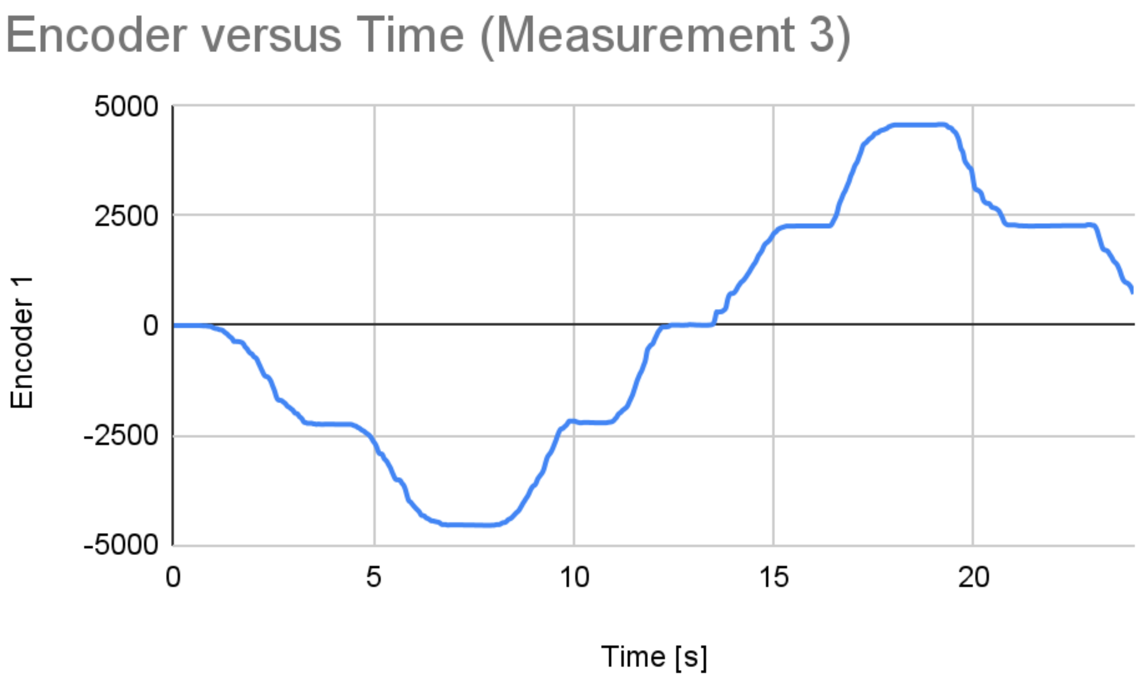


Figure 5 - Measurement 3 of Displacement versus Time for specified encoder position step intervals

aking the values from each displacement for each measurement, we get the following table containing data relating displacement and encoder position for each measurement:

Displacement [mm]	Encoder (1)	Encoder (2)	Encoder (3)	Average Encoder
-10,00	-4 467,00	-4 491,00	-4 551,00	-4 503,00
-5,00	-2 190,00	-2 120,00	-2 251,00	-2 187,00
0,00	0,00	0,00	0,00	0,00
5,00	2 381,00	2 308,00	2 270,00	2 319,67
10,00	4 690,00	4 534,00	4 574,00	4 599,33
15,00	6 212,00	-	-	6 212,00

Table 1 - Data relating Displacement and Encoder Position for each Measurement

Taking the average value of the measured encoder position for the corresponding displacement, we find the graph:

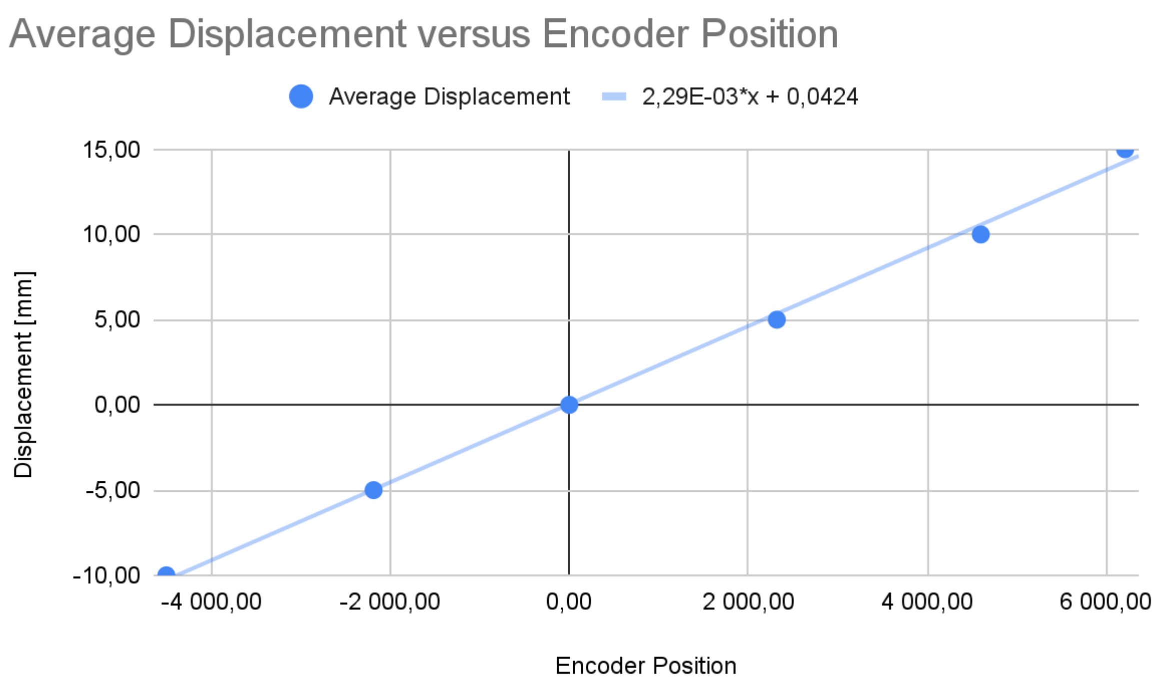


Figure 6 - Average Displacement versus Encoder Position for different measurements

From where it's derived the following relationship between displacement and encoder position:

Displacement/Encoder position	
0,002289	mm/count

2.2 Free Decay Test

Before proceeding with the measurement of motion for free and forced responses, free-decay tests are performed in order to, ultimately, obtain experimental values for the masses, spring constants and damping coefficients. The data is obtained from the relationship between amplitudes in peaks of oscillation for the studied system with a mass m added.

More precisely, we know that for the system with only the mass M of the analyzed body, we have that:

$$\delta = \frac{1}{n} ln(\frac{x_i}{x_{i+1}})$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

The value of δ, and consequently of ζ can be experimentally obtained by measuring the amplitudes at peaks x_i and x_{i+1} for n given cycles. We can also find the damped frequency w_D by measuring the damped period T_D for a time difference between peaks:

$$w_D = \frac{2\pi}{T_D}$$

These relationships are visually expressed in Figure 7:

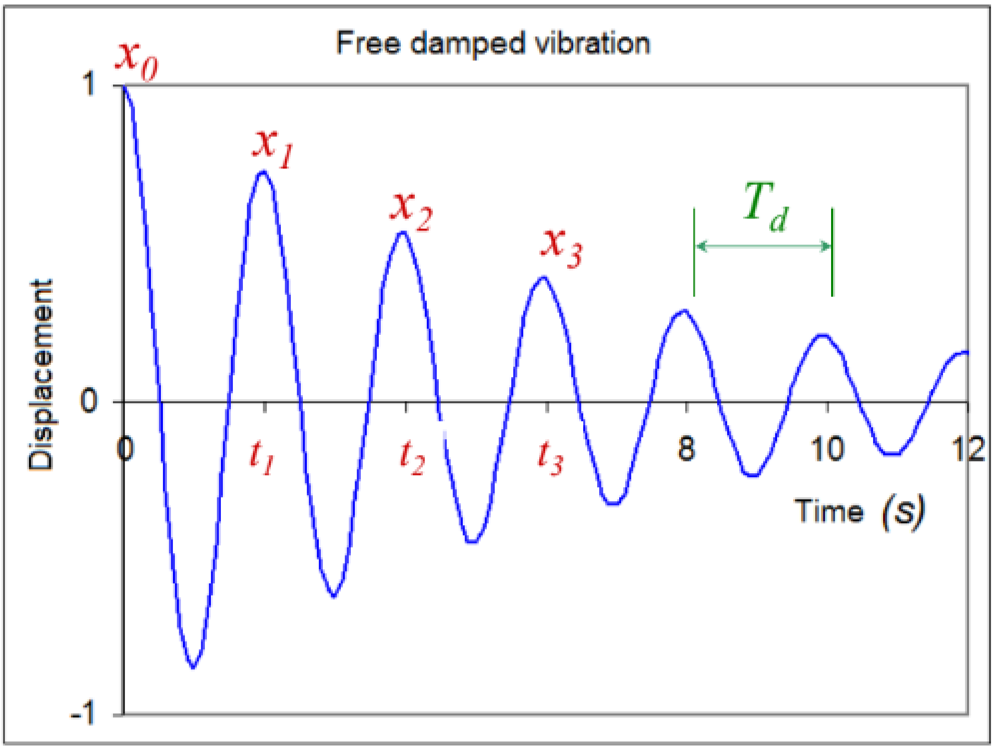


Figure 7 - Description of relevant parameters to be measured on free decay test

We know that the natural frequency is given by:

$$w_n = \sqrt{\frac{k}{M}}$$

and that the damped frequency relates to the natural frequency as:

$$w_D = w_n\sqrt{1 - \zeta^2}$$

Finally, the damping coefficient can be calculated with the following expression:

$$c = 2\zeta w_n M$$

However, experimental data from measurements with only the mass M it's not enough to find a given set of k, M, or c. But if we add a mass m, which can be experimentally measured, and coupled this mass with mass M, and then measure w_D (now expressed as w_{D2}), we get that:

$$w_n = \frac{w_D}{\sqrt{1-\zeta^2}} = \sqrt{\frac{k}{M}}$$

$$w_{n2} = \frac{w_{D2}}{\sqrt{1-\zeta_2^{\ 2}}} = \sqrt{\frac{k}{(M+m)}}$$

From where we can derive the relationship:

$$\frac{w_{n2}}{w_n} = \frac{\sqrt{\frac{k}{(M+m)}}}{\sqrt{\frac{k}{M}}}$$

If we define $\left(\frac{w_{n2}}{w_n}\right)^2 = A$, which is a constant, we find:

$$M = m\left(\frac{1}{A} - 1\right)$$

Having m and A, we have M. From M it is possible to subsequently calculate k and c

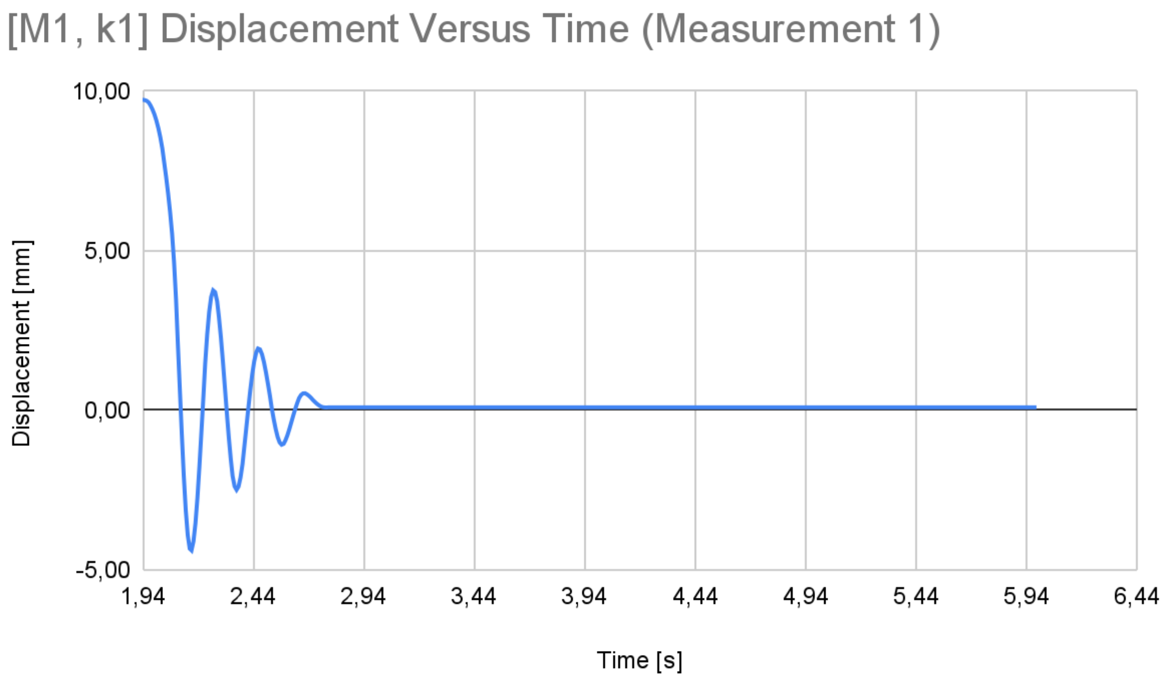
$$k = Mw_n^{\ 2}$$

$$c = 2\zeta_2 w_{n2}(M + m)$$

For each mass ($M1,\ M2,\ M3$), three independent measurements are made with and without the added mass ($m1,\ m2,\ m3$, respectively).

2.2.1 M1 and m1

From the data and graphs, it is possible to find the values of the peak amplitudes (x_i) and corresponding times for the free decay test with and without the added mass m for each measurement:



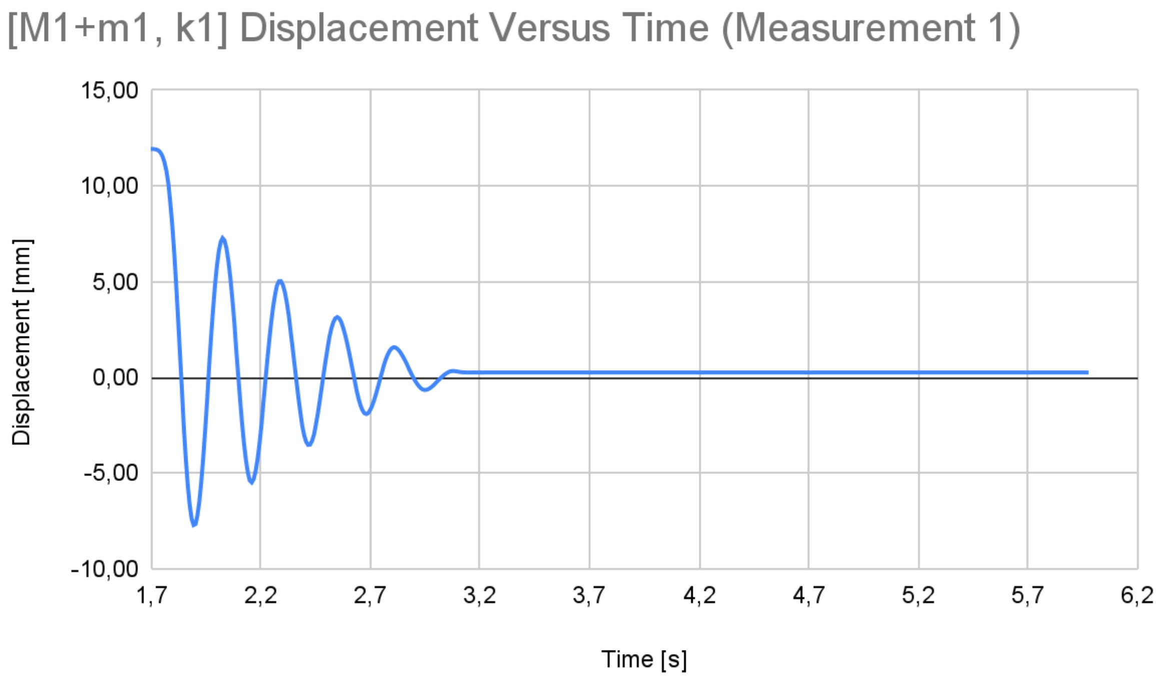


Figure 8 - M1, m1, k1 Displacement versus Time for free decay test (Measurement 1)

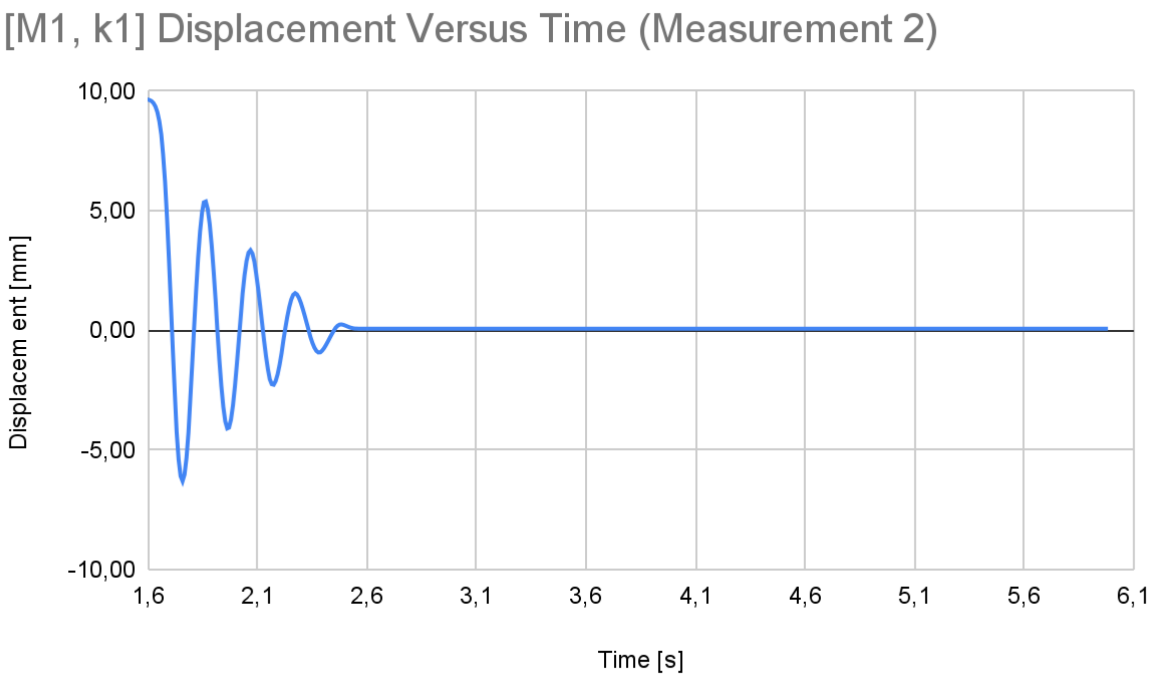
which has the following peaks and corresponding times:

[M1, k1] Peaks (Measurement

1)	
t [s]	x [mm]
1,939	9,73
2,258	3,759147349
2,461	1,936807952

and, for the case with the added mass:

[M1+m1, k1] Peaks (Measurement 1)	
t [s]	x [mm]
1,709	11,93
2,036	7,28
2,302	5,03



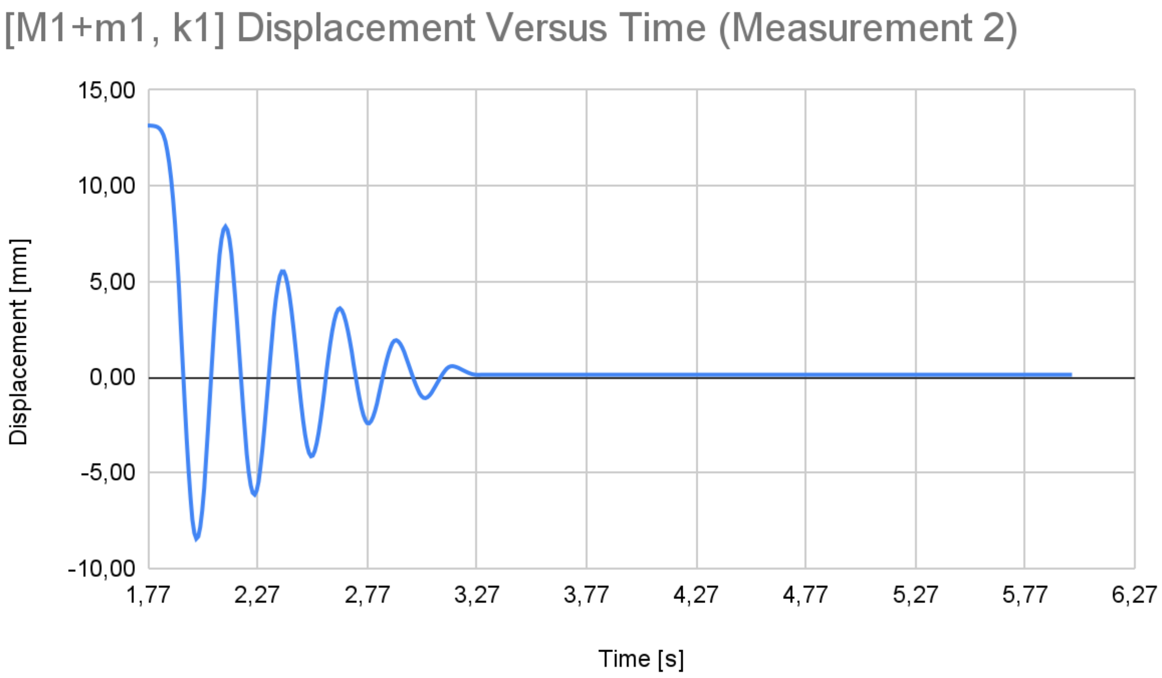


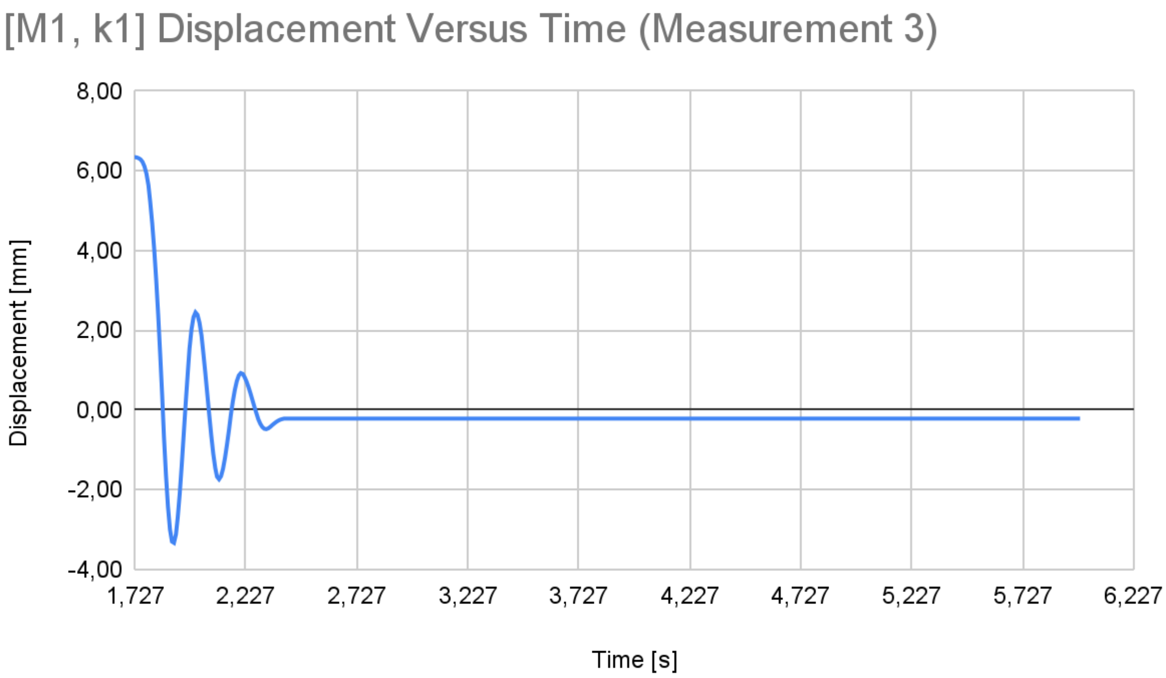
Figure 9 - M1, m1, k1 Displacement versus Time for free decay test (Measurement 2)

which has the following peaks and corresponding times:

[M1, k1] Peaks (Measurement 2)	
t [s]	**x [mm]**
1,603	9,65
1,868	5,386890202
2,072	3,349349922

and, for the case with the added mass:

[M1+m1, k1] Peaks (Measurement 2)	
t [s]	**x [mm]**
1,771	13,15
2,125	7,87
2,382	5,53



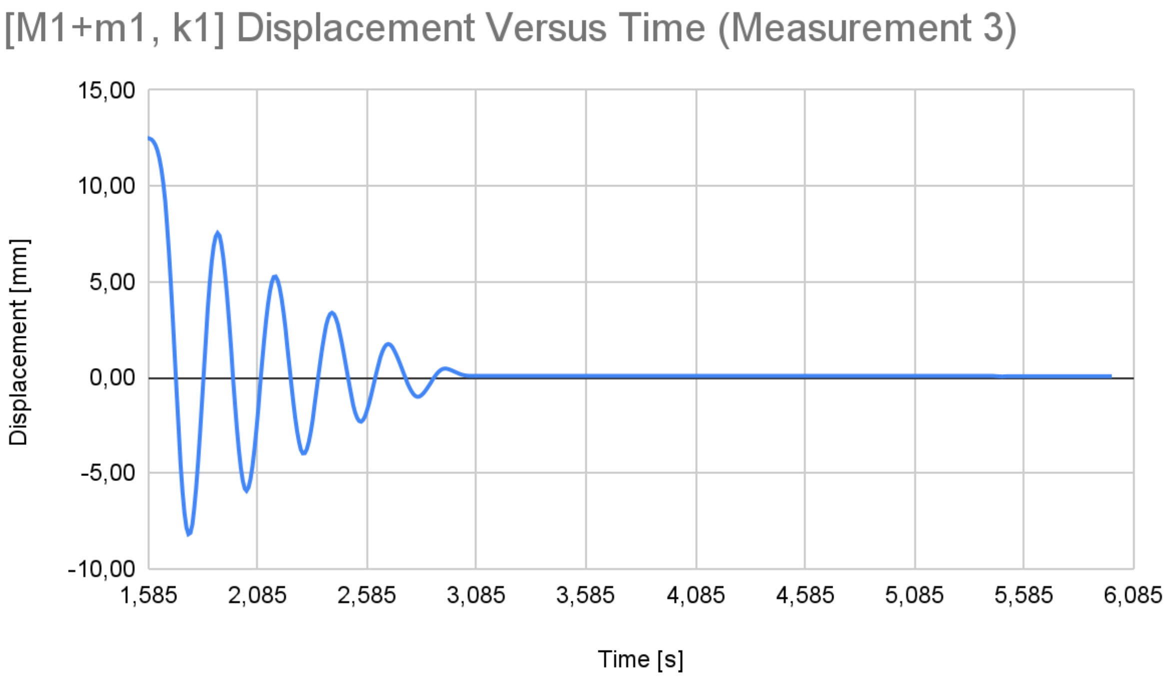


Figure 10 - M1, m1, k1 Displacement versus Time for free decay test (Measurement 3)

which has the following peaks and corresponding times:

[M1, k1] Peaks (Measurement 3)	
t [s]	**x [mm]**
1,727	6,35
2,001	2,45
2,205	0,93

and, for the case with the added mass:

[M1+m1, k1] Peaks (Measurement 3)	
t [s]	**x [mm]**
1,585	12,50
1,904	7,538899038
2,169	5,270132276

Considering an added mass $m1$ with a measured value of $0.478\ kg$, with this data it is possible to calculate T_D, w_d, σ, ζ, w_n, for different consecutive peaks (x_i and x_{i+1}) and different measurements to find out a sample of values of $M1$, $k1$ and $c1$:

M1, k1, c1											
										m1	**0,478**
Measurement 1											
i	**Td**	**Wd**	**x_i**	**x_i+1**	**sigma**	**zeta**	**wn**	**A**	**M1**	**k1**	**c1**
1 (m)	0,319	19,697	9,73	3,76	0,951	0,209	20,143	0,921	0,041	16,581	2,220
1 (m+M)	0,327	19,215	11,93	7,28	0,495	0,111	19,333				
2 (m)	0,319	19,697	9,73	1,94	0,807	0,179	20,019	0,930	0,036	14,424	1,921
2 (m+M)	0,327	19,215	11,93	5,03	0,432	0,097	19,305				
Measurement 2											
i	**Td**	**Wd**	**x_i**	**x_i+1**	**sigma**	**zeta**	**wn**	**A**	**M1**	**k1**	**c1**

1 (m)	0,265	23,710	9,65	5,39	0,583	0,130	23,914	0,558	0,378	216,293	1,553
1 (m+M)	0,354	17,749	13,15	7,87	0,514	0,115	17,867				
2 (m)	0,265	23,710	9,65	3,35	0,529	0,118	23,878	0,558	0,379	216,048	2,967
2 (m+M)	0,354	17,749	13,15	5,53	0,433	0,097	17,833				
Measurement 3											
i	**Td**	**Wd**	**x_i**	**x_i+1**	**sigma**	**zeta**	**wn**	**A**	**M1**	**k1**	**c1**
1 (m)	0,274	22,931	6,35	2,45	0,952	0,210	23,452	0,715	0,191	105,035	0,856
1 (m+M)	0,319	19,697	12,50	7,54	0,506	0,113	19,824				
2 (m)	0,274	22,931	6,35	0,93	0,962	0,212	23,462	0,711	0,194	106,745	0,742
2 (m+M)	0,319	19,697	12,50	5,27	0,432	0,097	19,789				

Table 2 - Calculated values from Free decay test for M1, m1, k1 for different peak ratios and measurements

Taking the average values, we find:

Averages		
M1	**k1**	**c1**
0,203	112,521	1,710

2.2.2 M2 and m2

Applying the same procedure for mass $M2$ (adding mass $m2$) and spring with constant $k2$, we get:

M2, k2, c2											
										m2	**0,475**
Measurement 1											

i	Td	Wd	x_i	x_i+1	sigma	zeta	wn	A	M2	k2	c2
1 (m)	0,319	19,697	11,67	5,44	0,763	0,169	19,985	0,608	0,306	122,404	3,288
1 (m+M)	0,407	15,438	10,65	5,81	0,605	0,135	15,581				
2 (m)	0,319	19,697	11,67	1,89	0,911	0,201	20,106	0,607	0,307	124,240	4,178
2 (m+M)	0,407	15,438	10,65	2,29	0,768	0,170	15,667				

Measurement 2

i	Td	Wd	x_i	x_i+1	sigma	zeta	wn	A	M2	k2	c2
1 (m)	0,318	19,758	11,30	5,94	0,644	0,143	19,965	0,666	0,238	94,997	3,038
1 (m+M)	0,389	16,152	11,41	6,35	0,586	0,131	16,292				
2 (m)	0,318	19,758	11,30	2,29	0,798	0,177	20,075	0,665	0,239	96,481	3,794
2 (m+M)	0,389	16,152	11,41	2,65	0,730	0,162	16,369				

Measurement 3

i	Td	Wd	x_i	x_i+1	sigma	zeta	wn	A	M2	k2	c2
1 (m)	0,345	18,212	11,79	5,54	0,756	0,168	18,474	0,711	0,193	65,906	3,126
1 (m+M)	0,408	15,400	9,60	4,89	0,675	0,150	15,577				
2 (m)	0,345	18,212	11,79	1,96	0,896	0,198	18,579	0,715	0,189	65,363	4,122
2 (m+M)	0,408	15,400	9,60	1,60	0,895	0,197	15,709				

Table 3 - Calculated values from Free decay test for M2, m2, k2 for different peak ratios and measurements

with averages of:

Averages		
M2	**k2**	**c2**
0,246	94,899	3,591

2.2.3 M3 and m3

And, for $M3$, $m3$, $k3$, we get:

M3, k3, c3

										m3	**0,477**

Measurement 1

i	Td	Wd	x_i	x_i+1	sigma	zeta	wn	A	M3	k3	c3
1 (m)	0,301	20,874	10,28	4,94	0,733	0,163	21,156	0,907	0,049	21,820	4,479
1 (m+M)	0,319	19,697	14,02	5,36	0,961	0,211	20,152				
2 (m)	0,301	20,874	10,28	1,89	0,847	0,187	21,250	0,896	0,056	25,131	4,320
2 (m+M)	0,319	19,697	14,02	2,25	0,915	0,202	20,110				

Measurement 2

i	Td	Wd	x_i	x_i+1	sigma	zeta	wn	A	M3	k3	c3
1 (m)	0,292	21,518	12,02	7,32	0,496	0,111	21,652	0,536	0,413	193,721	3,187
1 (m+M)	0,399	15,747	13,00	7,85	0,505	0,113	15,849				
2 (m)	0,292	21,518	12,02	3,66	0,595	0,133	21,710	0,535	0,415	195,741	3,559
2 (m+M)	0,399	15,747	13,00	4,22	0,563	0,126	15,873				

Measurement 3

i	Td	Wd	x_i	x_i+1	sigma	zeta	wn	A	M3	k3	c3
1 (m)	0,293	21,444	10,73	6,13	0,560	0,125	21,614	0,647	0,261	121,722	2,342
1 (m+M)	0,363	17,309	14,13	9,40	0,407	0,091	17,382				
2 (m)	0,293	21,444	10,73	2,75	0,680	0,151	21,694	0,644	0,264	124,330	2,690
2 (m+M)	0,363	17,309	14,13	5,57	0,466	0,104	17,404				

Table 4 - Calculated values from Free decay test for M3, m3, k3 for different peak ratios and measurements

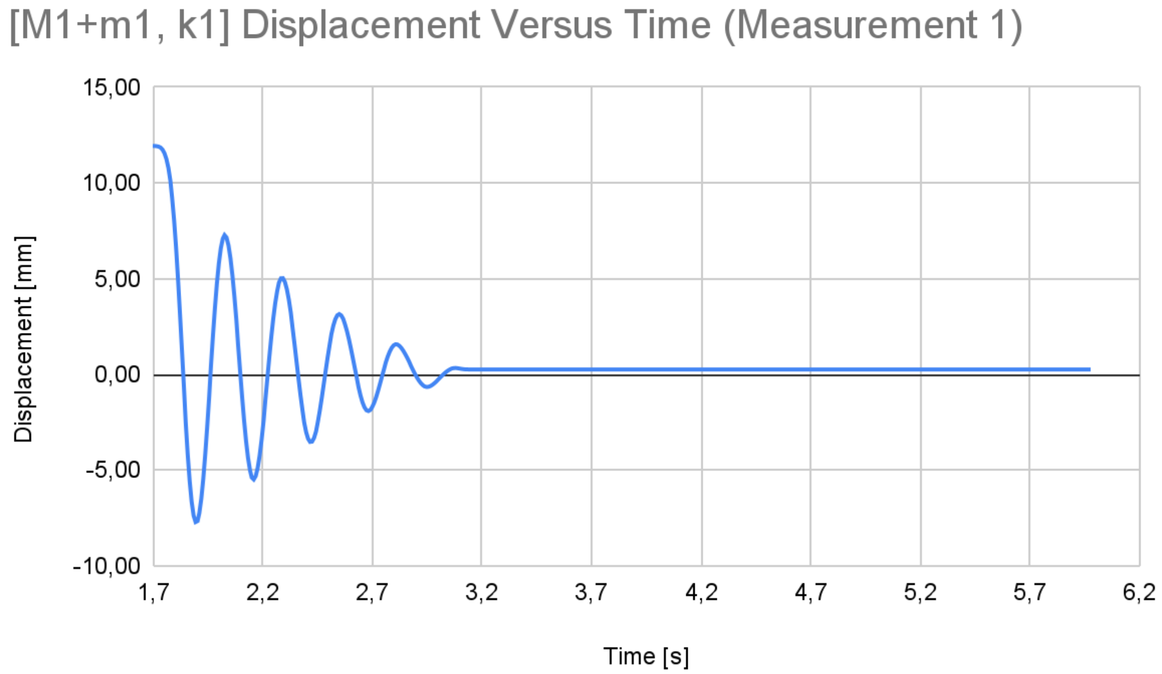
with averages of:

Averages		
M3	k3	c3
0,243	113,744	3,429

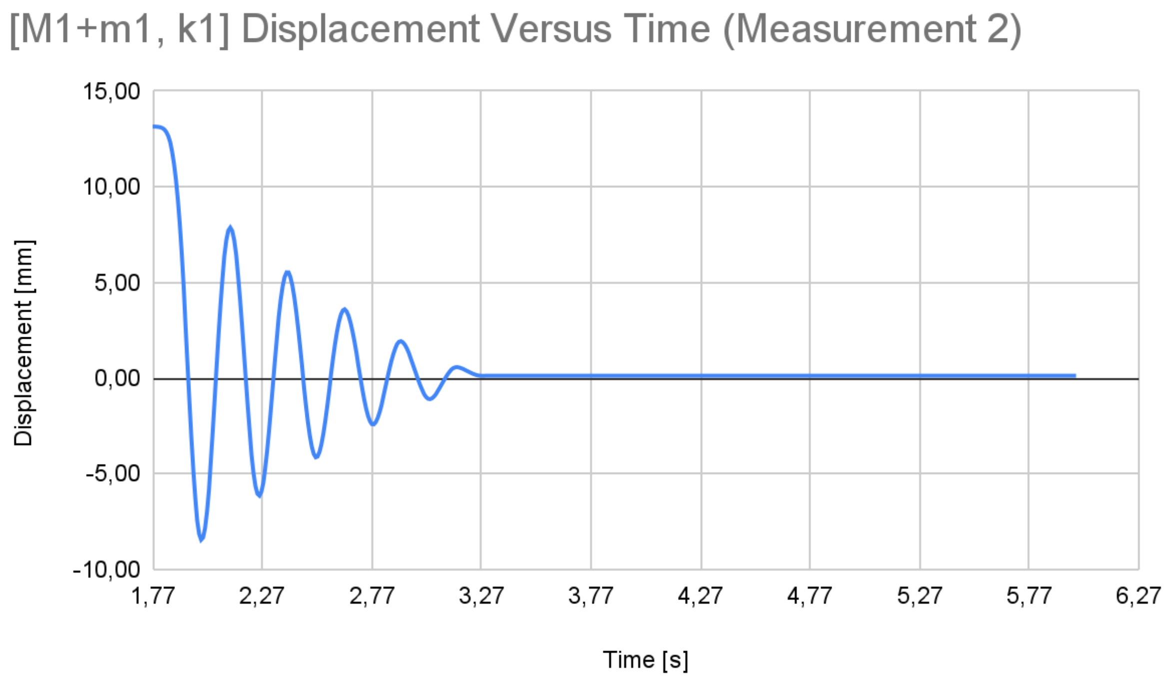
2.3 1DOF Free Response

The experimental results for the free response are those obtained in the free decay test with the added masses of $m1,\ m2$ and $m3$. The plots for each mass and each measurement it’s given below:

2.3.1 M1 and m1



[M1+m1, k1] Displacement Versus Time (Measurement 1)
Displacement [mm]
15,00
10,00
5,00
0,00
-5,00
-10,00
1,7
2,2
2,7
3,2
3,7
4,2
4,7
5,2
5,7
6,2
Time [s]



[M1+m1, k1] Displacement Versus Time (Measurement 2)
Displacement [mm]
15,00
10,00
5,00
0,00
-5,00
-10,00
1,77
2,27
2,77
3,27
3,77
4,27
4,77
5,27
5,77
6,27
Time [s]

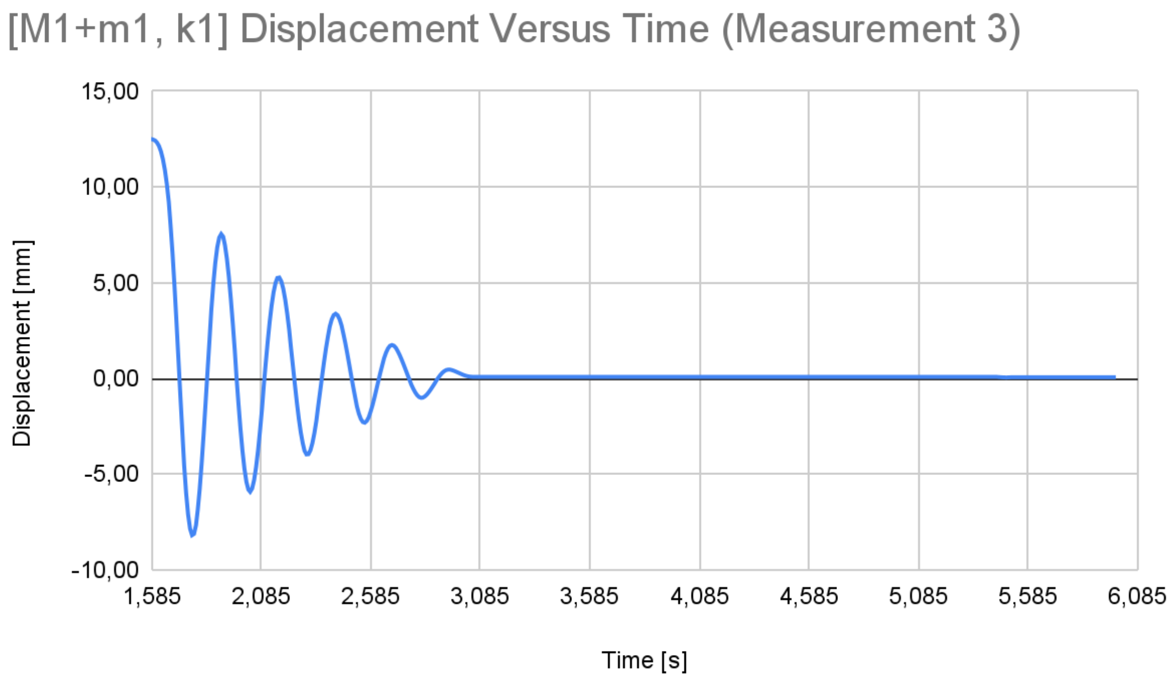
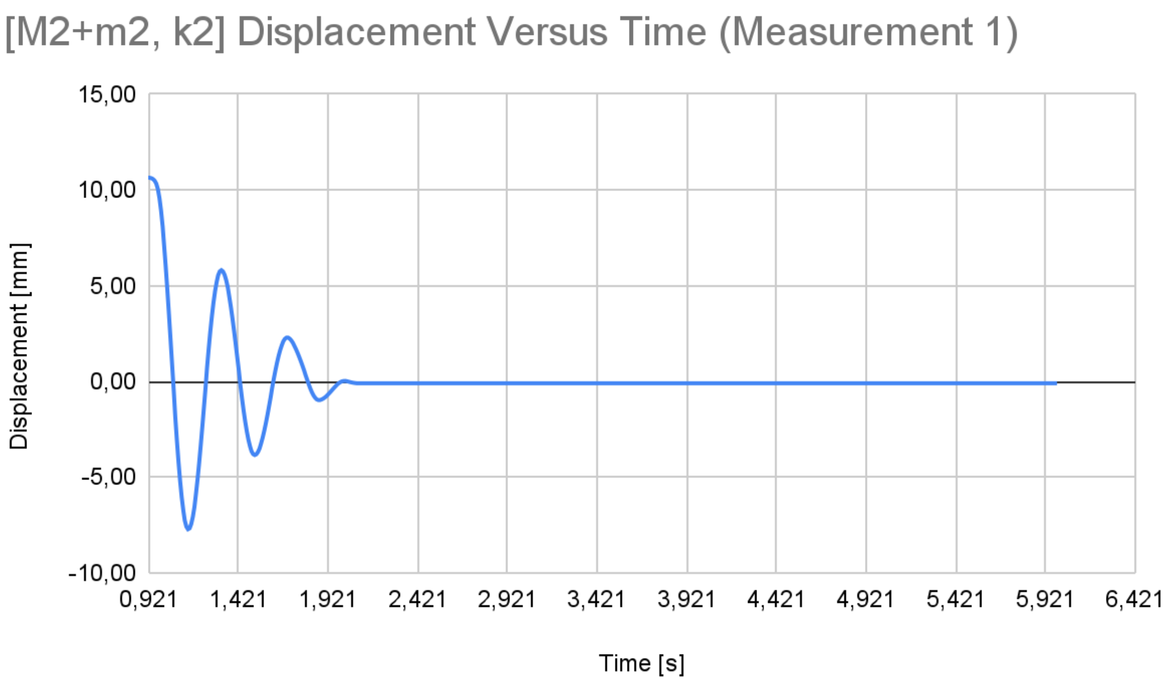
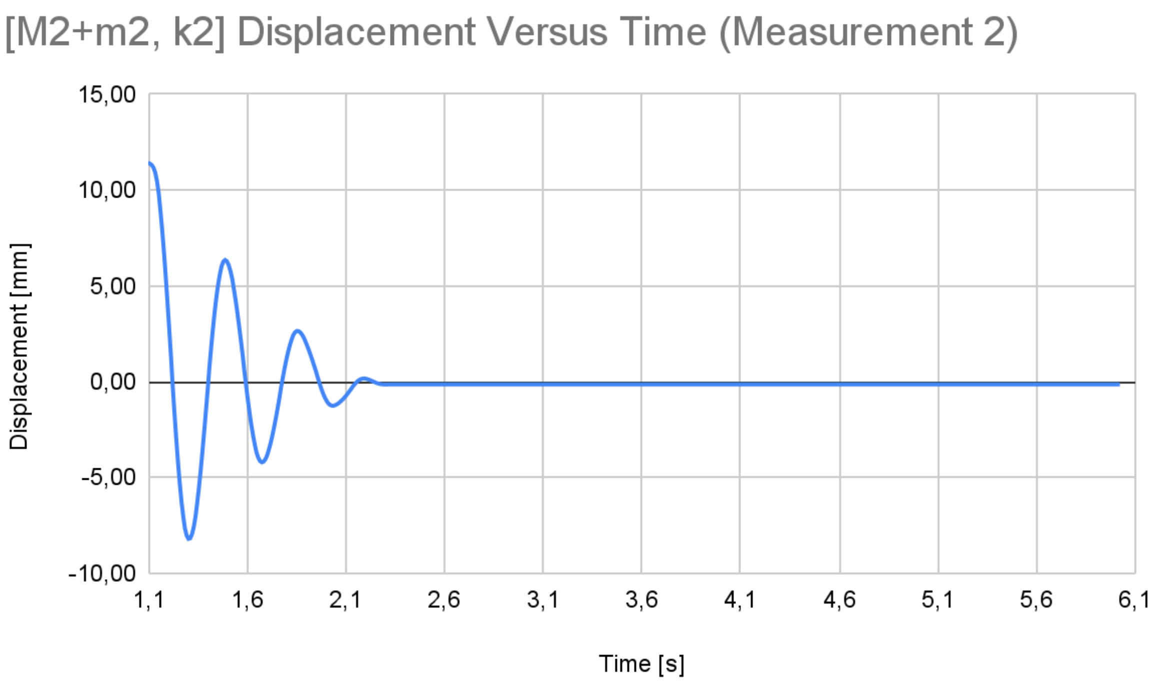


Figure 11 - M1, m1, k1 Displacement versus time for free decay test for each measurement

2.3.2 M2 and m2


[M2+m2, k2] Displacement Versus Time (Measurement 1)
Displacement [mm]
15,00
10,00
5,00
0,00
-5,00
-10,00
0,921
1,421
1,921
2,421
2,921
3,421
3,921
4,421
4,921
5,421
5,921
6,421
Time [s]


[M2+m2, k2] Displacement Versus Time (Measurement 2)
Displacement [mm]
15,00
10,00
5,00
0,00
-5,00
-10,00
1,1
1,6
2,1
2,6
3,1
3,6
4,1
4,6
5,1
5,6
6,1
Time [s]

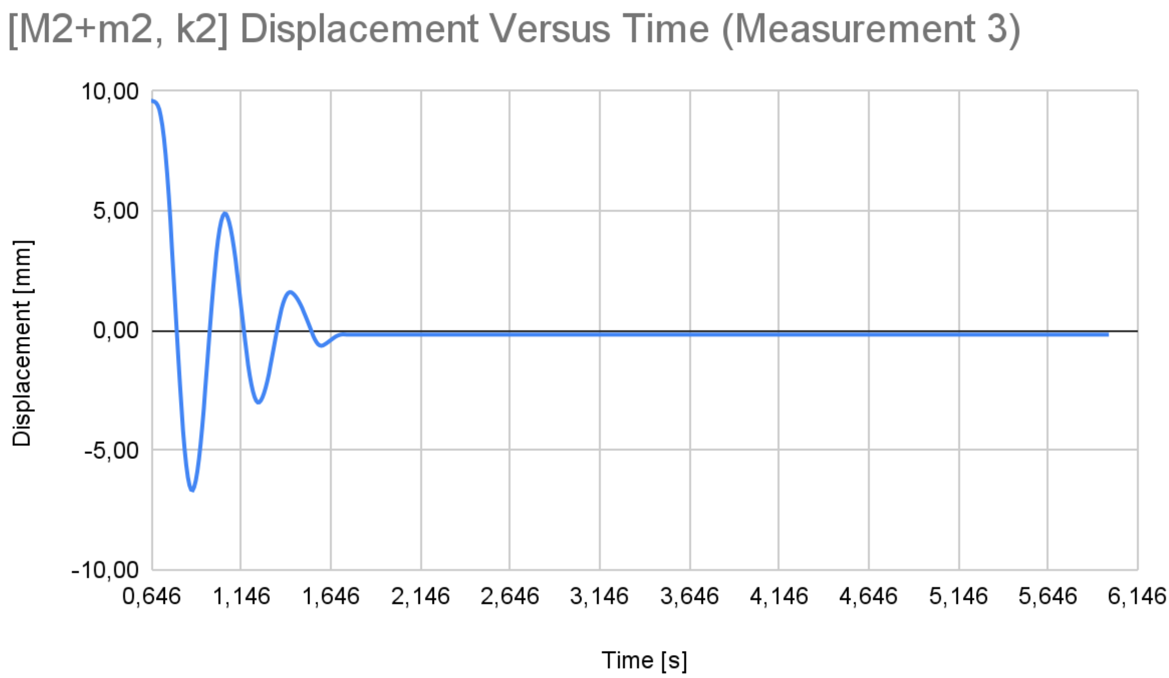
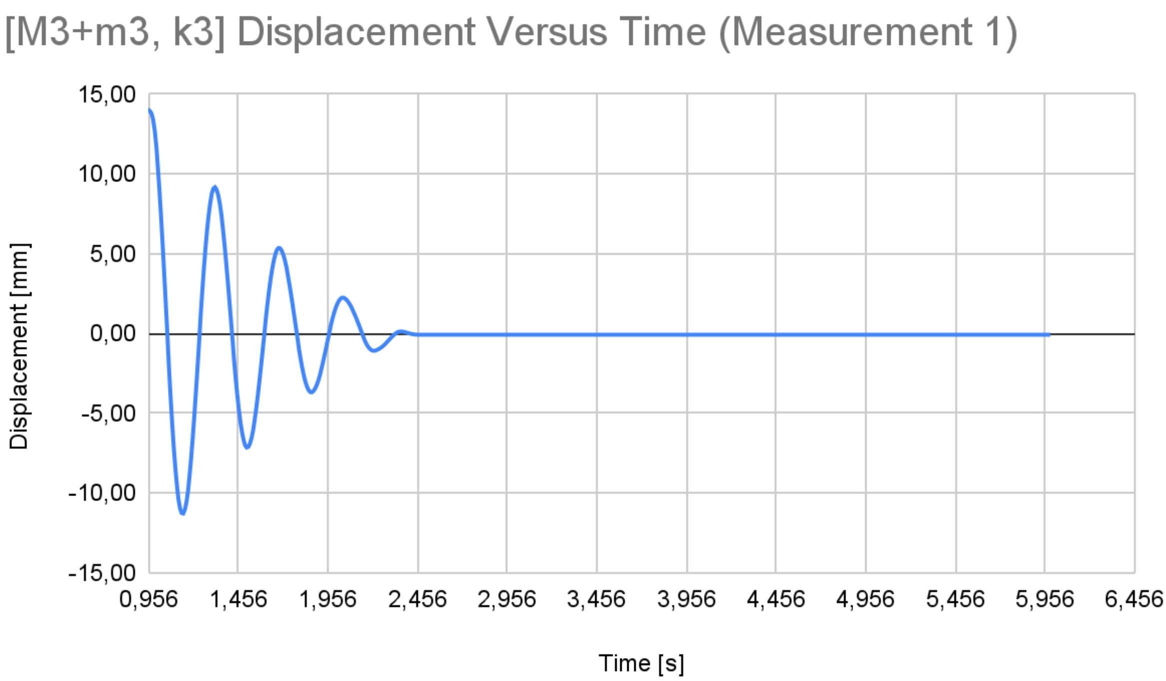
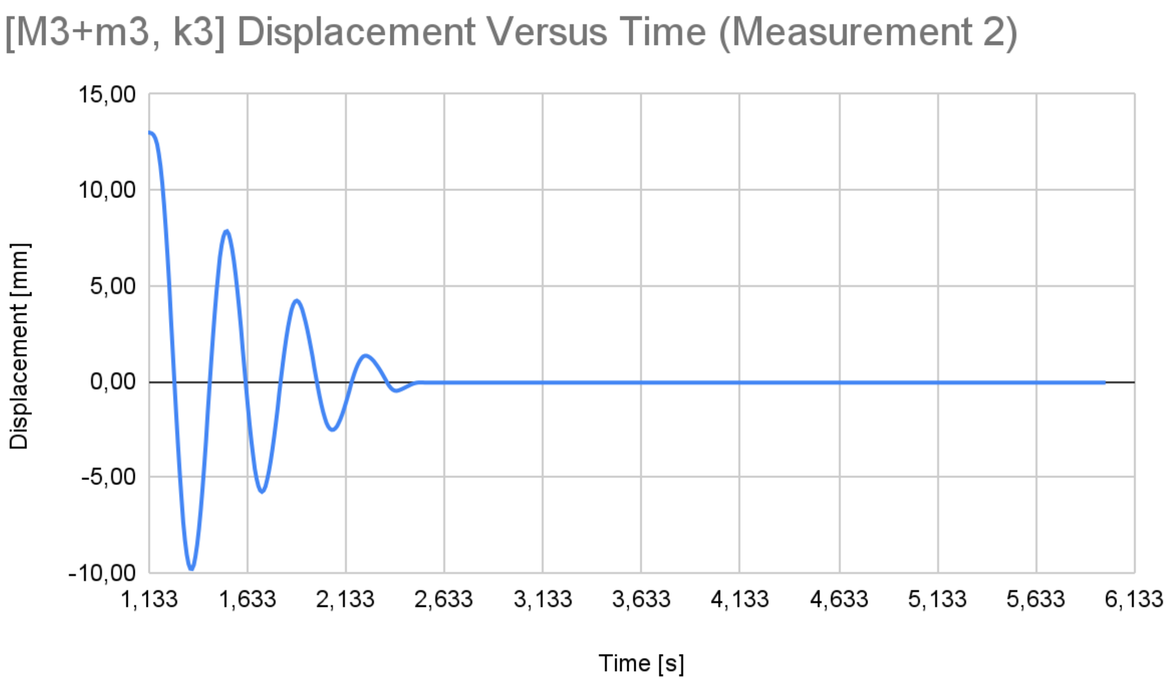


Figure 12 - M2, m2, k2 Displacement versus time for free decay test for each measurement

2.3.3 M3 and m3


[M3+m3, k3] Displacement Versus Time (Measurement 1)
Displacement [mm]
15,00
10,00
5,00
0,00
-5,00
-10,00
-15,00
0,956
1,456
1,956
2,456
2,956
3,456
3,956
4,456
4,956
5,456
5,956
6,456
Time [s]


[M3+m3, k3] Displacement Versus Time (Measurement 2)
Displacement [mm]
15,00
10,00
5,00
0,00
-5,00
-10,00
1,133
1,633
2,133
2,633
3,133
3,633
4,133
4,633
5,133
5,633
6,133
Time [s]

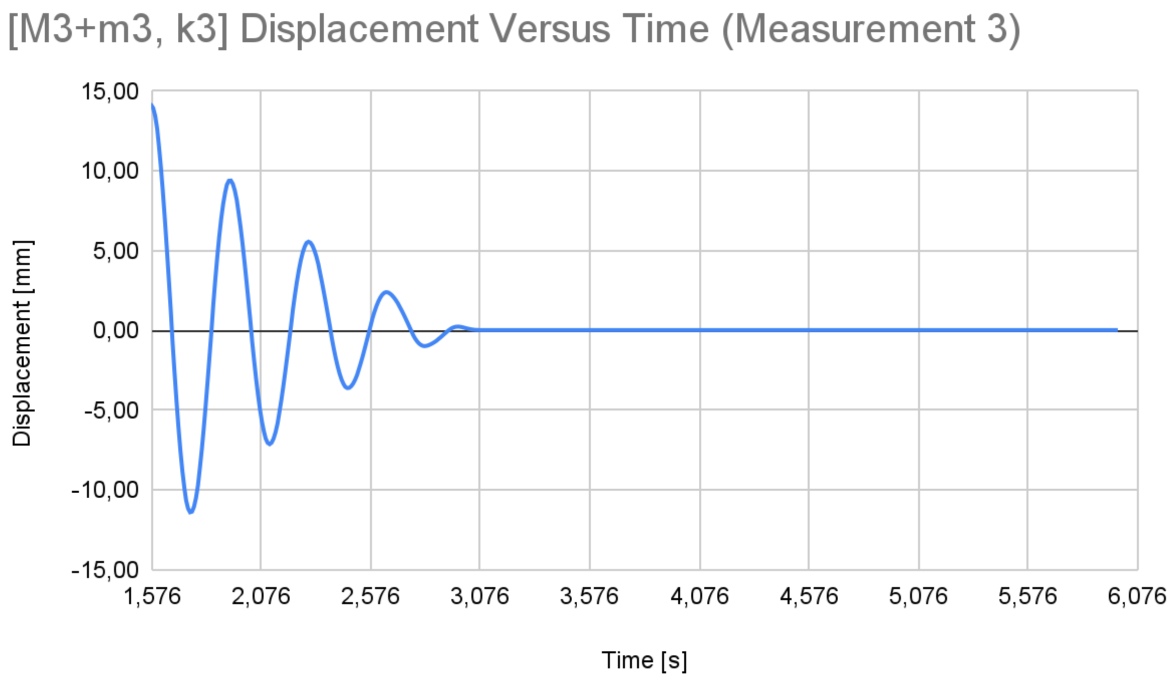


Figure 13 - M3, m3, k3 Displacement versus time for free decay test for each measurement

2.4 1DOF Forced Response

The experimental analysis of the one-degree-of-freedom forced response (from a harmonic excitation of the form $F_0 sin(wt)$) it's needed to obtain the relationship between voltage in the actuator and exciting force F_0. From the equation of motion, it follows that if the driven frequency w is low, the terms referring to velocity and acceleration tend to zero. Under these circumstances, it is possible to derive the following relationship:

$$kx = F_0 sin(wt)$$

if we choose a peak (where $sin(wt)$ equals 1), we find:

$$kx_i = F_0$$

and the value of k was obtained in the free decay test.

Selecting a sufficiently small driven angular frequency ($w = 0.1\, rad/s$), we obtain the graph of the movement of the mass M for different voltages:

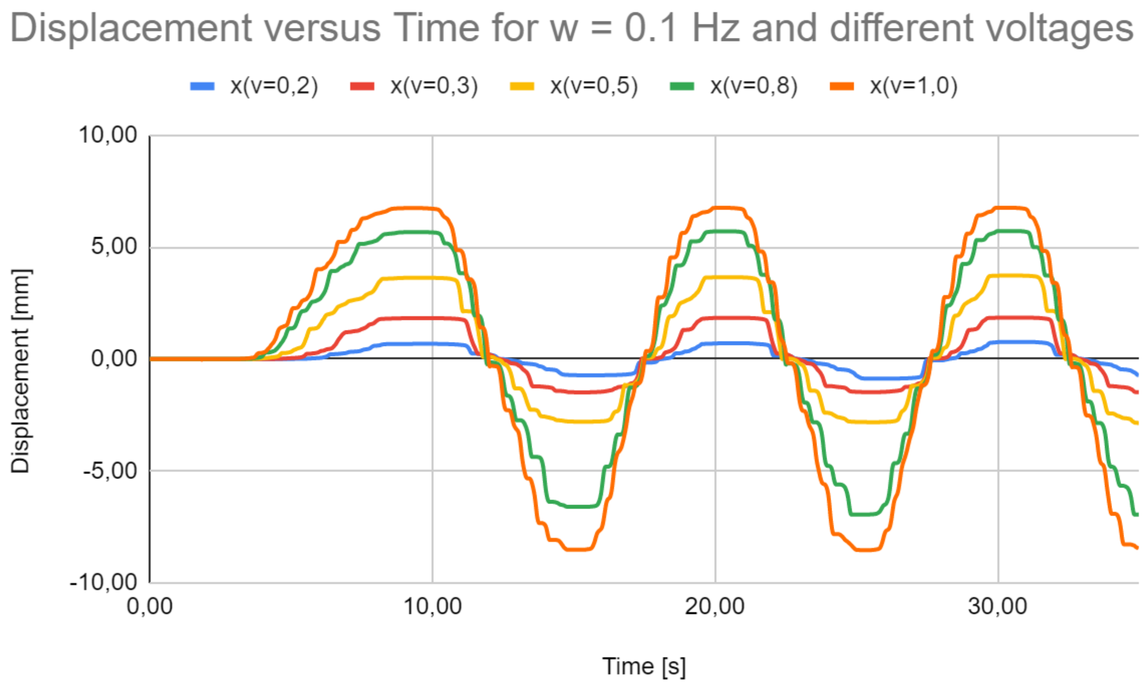


Figure 14 - Displacement versus time for small driven frequency and different voltages

From the data of the graph, we find the peak and corresponding exciting force F_0 for each voltage:

v [hz]	x0 [mm]	F0 [mN]
0,20	0,76	86,97
0,30	1,8	210,14
0,50	3,72	423,67
0,80	5,72	650,22
1,00	6,75	768,19

Table 5 - Relation between voltage and exciting force

Graphing the results:

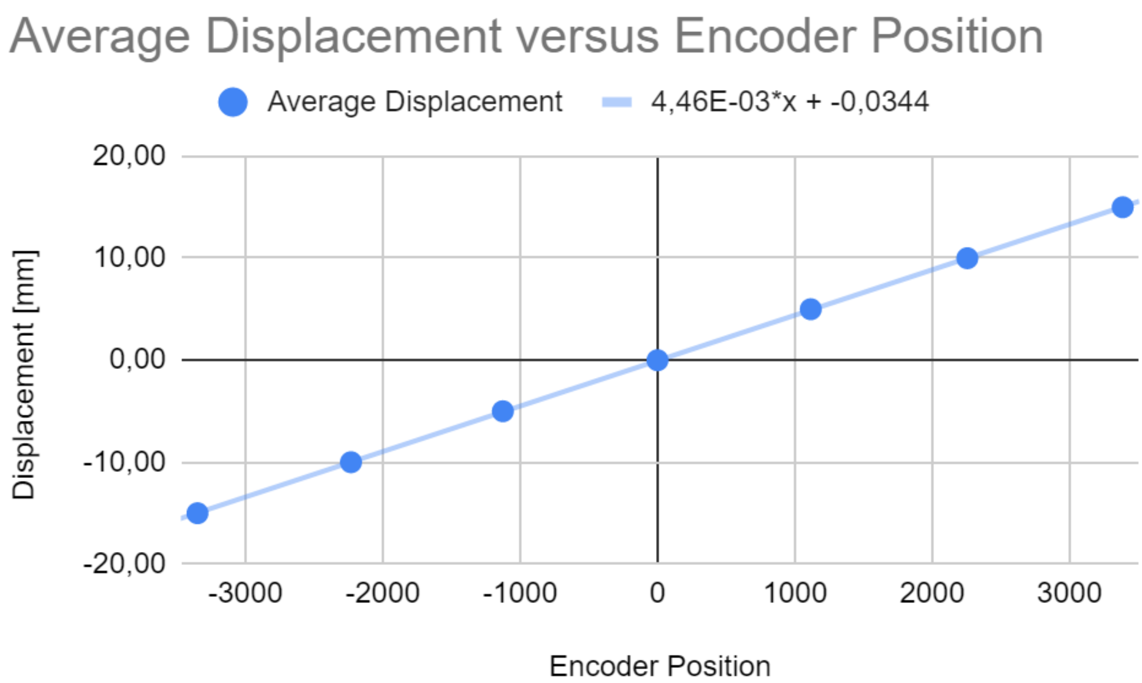


Figure 15 - Relation between voltage and exciting force

hence, we find the following relationship between voltage and exciting force:

Force/Voltage	
0,85	Newton/Hz

2.5 3DOF Free Response

Considering an initial displacement of the first mass, three measurements were made. Plotting the displacements for the corresponding time interval for each measurement, we find:

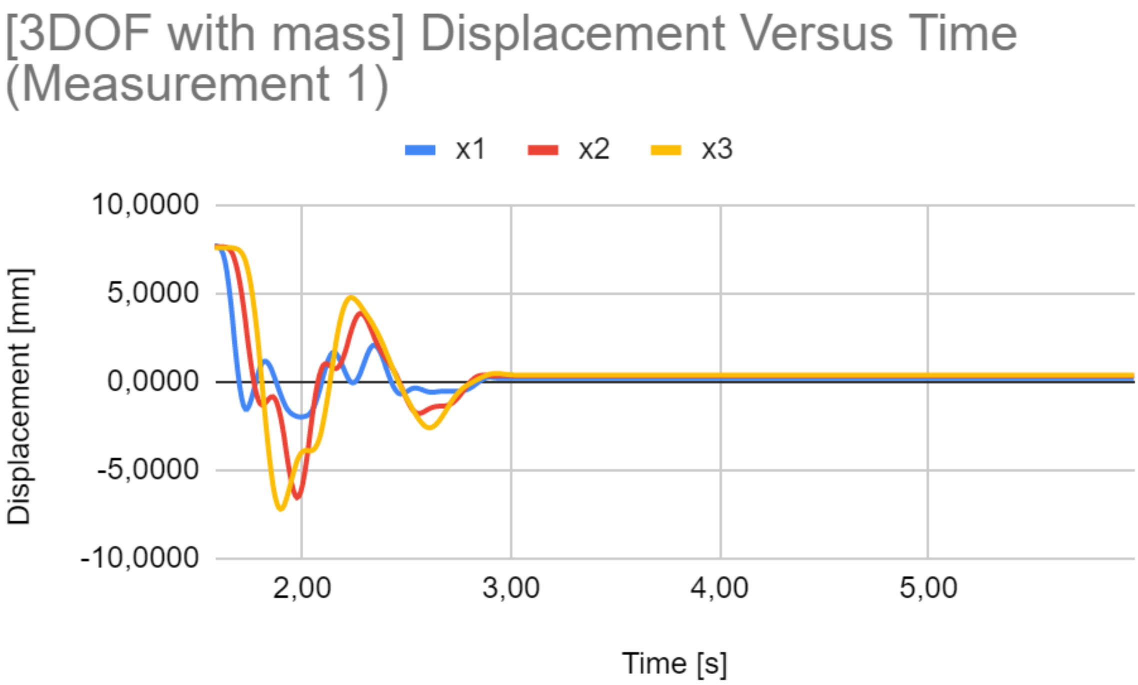


Figure 16 - Displacement versus Time for 3 DOF system with added masses (Measurement 1)

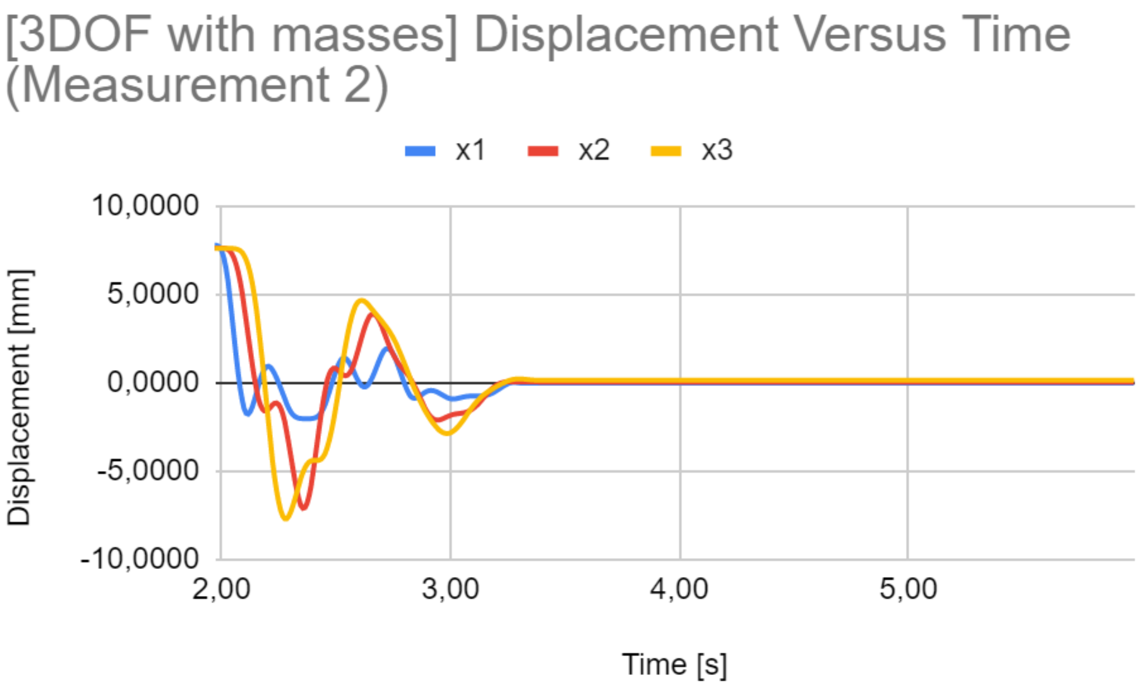


Figure 17 - Displacement versus Time for 3 DOF system with added masses (Measurement 2)

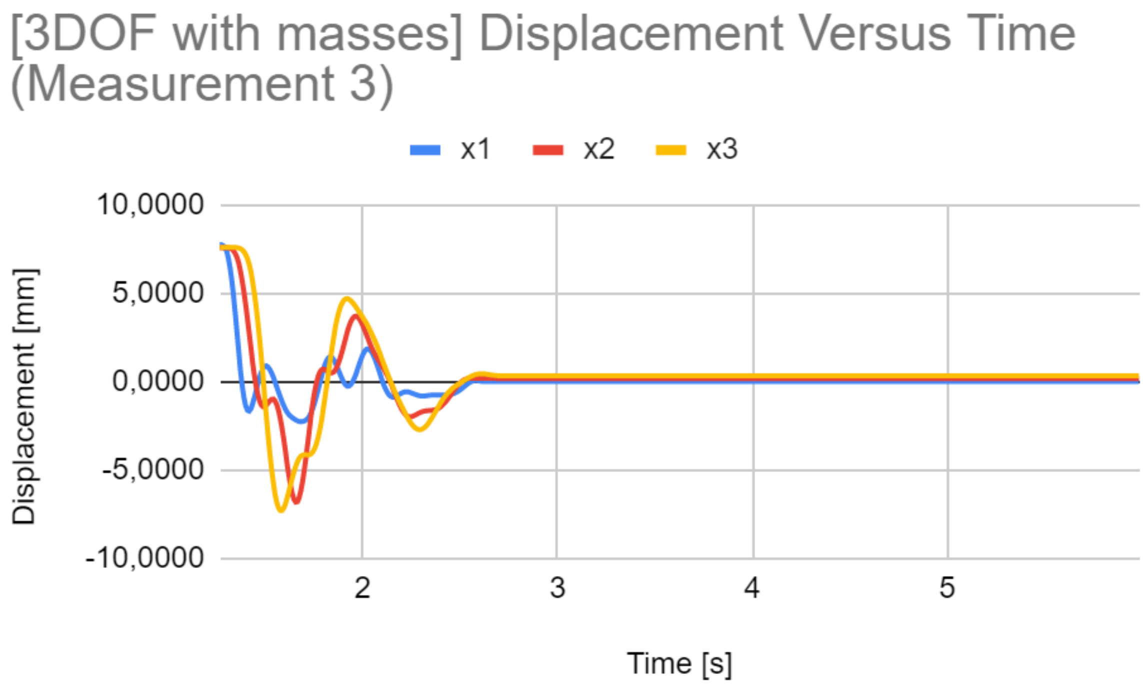


Figure 18 - Displacement versus Time for 3 DOF system with added masses (Measurement 3)

2.6 3DOF Forced Response

2.6.1 Estimation of Response Transfer Functions

A linear mechanical system's frequency response function is defined as the Fourier transform of the time domain response divided by the Fourier transform of the time domain input. The function of transformation, as defined, is supplied by the following relation: Omega becomes the frequency response function of the SDOF spring-mass-dashpot system when s =w i.

The frequency response function amplitude versus frequency of the SDOF can be obtained by running the MATLAB programming codes with a sine wave signal of fixed excitation amplitude as input, and the excitation frequency of the sine wave signal changing from low to high to cover the resonant frequency. The response amplitude is measured for each excitation frequency input of the sine wave signal.

The response amplitude is then plotted against the excitation frequency, displaying the frequency response of oscillator displacement vs excitation frequency. Plotting the modulus of the preceding equation vs frequency f in Hz, where s = i2f, yields the frequency response amplitude frequency curve.

The Frequency Response Function (FRF)

The FRF, represented by H (omega) or H(f) depending on whether it is written in rad/s or Hz, is simply the ratio of a system's steady-state response to an applied sinusoidal input, which can be a force, an imposed displacement, or nearly any other variable. The FRF is often expressed in complex form, although it may also be expressed in magnitude and phase. When computing a FRF, the sinusoidal input is assumed to have begun at - and any transient response is assumed to have gone away. As a result, the response is similarly sinusoidal, but it differs in amplitude and phase from the input.

As previously stated, FRFs of various types of single-DOF systems may be simply computed, or a FRF can be measured, by applying a sinusoidal input and monitoring the response.

The normal mode summation approach is commonly used to determine the FRFs of multi-DOF systems. This permits a multi-DOF system's reaction to be represented as the total of numerous single-DOF systems, one for each normal mode.

The FRF of a system is sometimes referred to as its 'transfer function,' although this word should be exclusively reserved for the system's reaction represented in Laplace notation. However, the two components are so closely connected that if the Laplace operator, s, in a system transfer function is substituted by I*omega the transfer function becomes the FRF, H(omega), in complex notation.

The simplest comparisons may include a FRFs measured using different excitation levels as a linearity check, then a FRFs measured or calculated switching the input and output points as a reciprocity check; after FRFs measured or calculated before and after a structural modification to show its effect on the system response; so FRFs calculated for different models and levels of damping; and at least a FRFs calculated before and after a data reduction that is intended to. It is common practice to utilize an overlay of all the FRFs, measured from all combinations of input and output coordinates, and count the resonance peaks as a preliminary estimate of the model order.

When comparing analytically derived FRF curves, three aspects must be considered. The first is how damping has been accounted for in the theoretical model; the second is that the analytical FRFs are usually synthesized from the structure's modal vectors and depend on the degree of modal truncation; and the third is that when the compared FRFs originate from two models, one obtained by a structural modification of the other, the comparison must take into account the frequency shift and the change in the scale factor in the FRF magnitude. For example, modifying the reference stiffness matrix by a factor of increases the frequencies in the modified model by a factor of alpha.

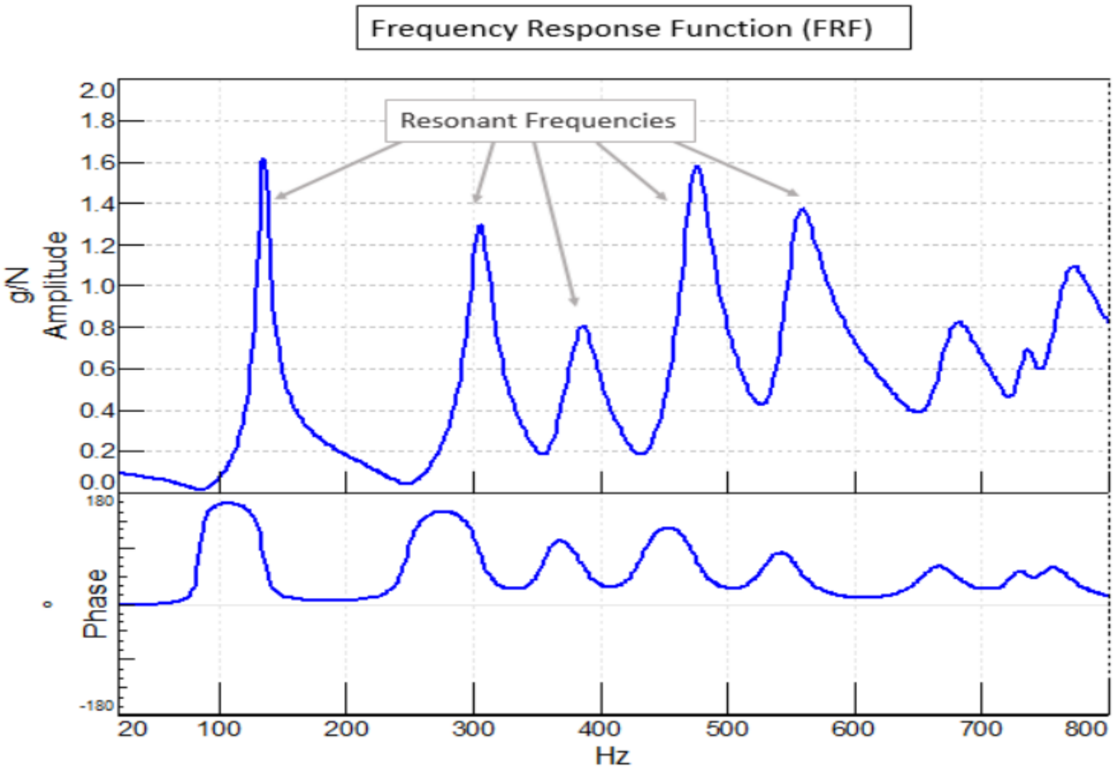


Figure 19 - Plot of Amplitude and Phase of a FRF function. Amplitude has peaks corresponding to natural frequencies/resonances of test objects. Phase has shifted at resonant frequency

Generally, the input force spectrum (X) should be flat versus frequency, exciting all frequencies uniformly. When viewing the response (Y), the peaks in the response indicate the natural/resonant frequencies of the structure under test.

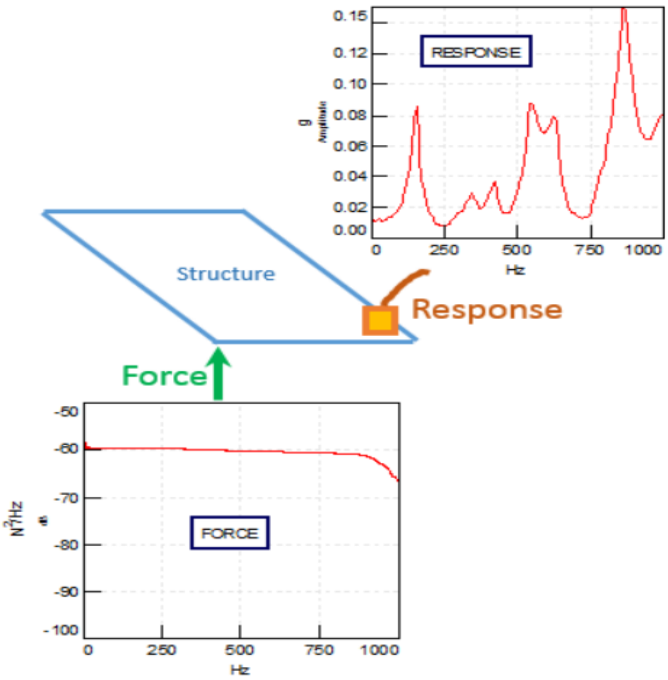


Figure 20 - A force with flat frequency response is applied to a structure to identify resonant frequencies in the response

2.6.2 Samples of Displacement for different frequencies

Given the relationship between exciting force and voltage, a voltage is selected (with a corresponding exciting force) to remain constant while varying the driven frequency. The selected voltage and corresponding exciting force is:

v [Hz]	F0 [N]
0,35	0,30

Then some displacement samples versus time for given driven frequencies are plotted:

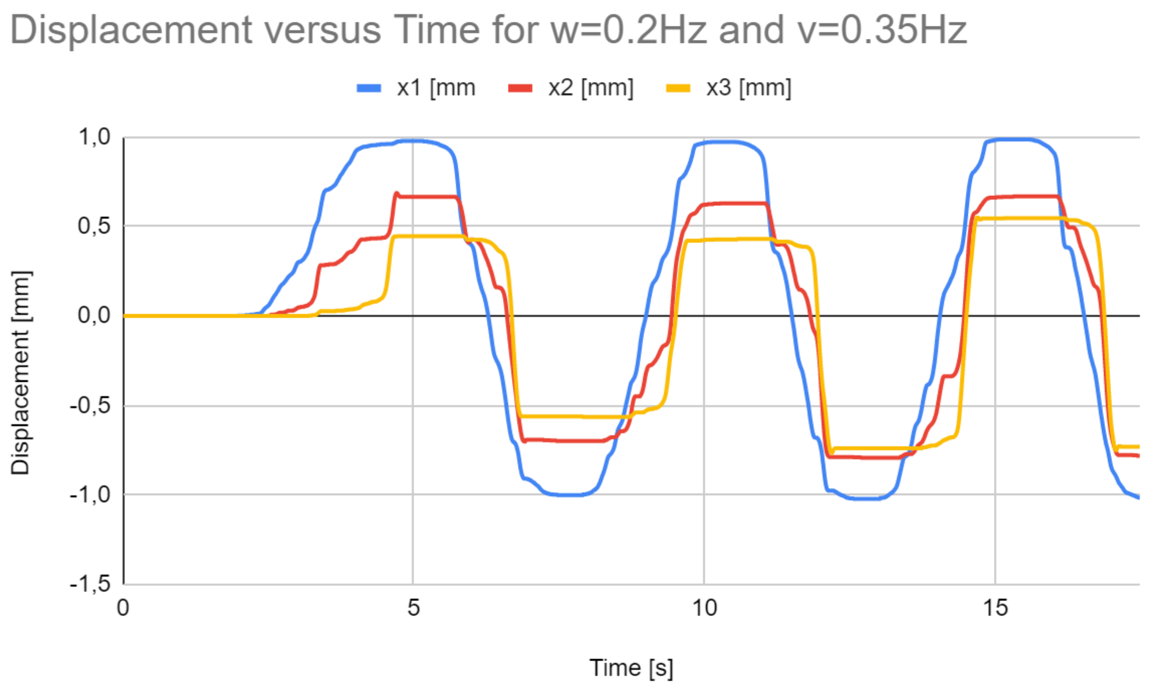


Figure 21 - Displacement versus time for a driven frequency of 0,2 hertz and an exciting force corresponding to an actuator voltage of 0.35Hz

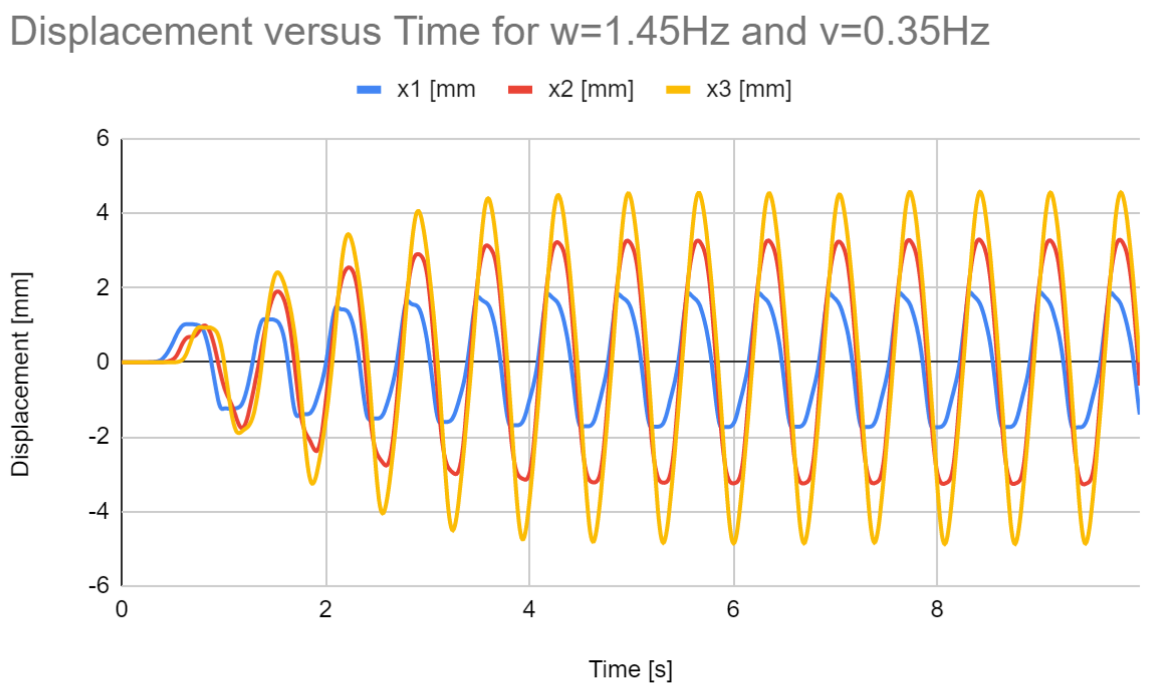


Figure 22 - Displacement versus time for a driven frequency of 1.45 hertz and an exciting force corresponding to an actuator voltage of 0.35Hz

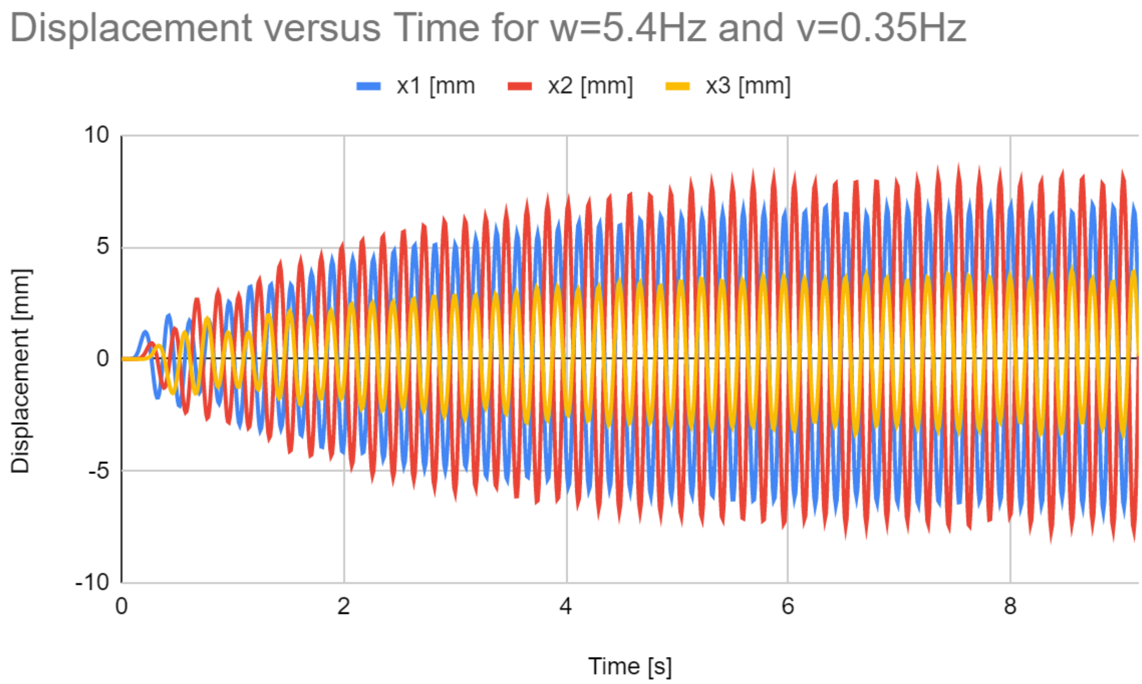


Figure 23 - Displacement versus time for a driven frequency of 5.4 hertz and an exciting force corresponding to an actuator voltage of 0.35Hz

2.6.3 Response Amplitude and driven frequency

From displacement versus time data, it is possible to find, for each driven frequency, the amplitude of the steady state solution (when the time is large enough) of each car considering their respective maximum displacement values. This yields the following pairs of frequency (converted to radians per second) and amplitude:

w [rad/s]	Car 1	Car 2	Car 3
1,256	0,99	0,69	0,55
3,768	1,10	1,21	1,17
6,28	1,27	1,36	1,52
8,792	1,86	2,85	3,73
8,9804	1,70	3,05	4,17
9,106	1,87	3,29	4,57
10,048	1,34	3,04	4,79
10,362	1,22	2,77	4,54
11,304	1,09	2,24	3,71
11,932	1,14	1,90	3,11

13,188	1,09	1,36	2,26
15,7	1,19	1,06	1,72
18,84	1,46	0,90	1,35
20,724	1,95	0,98	1,43
21,98	2,34	1,01	1,83
23,236	3,40	1,46	2,67
24,492	5,85	3,41	4,87
24,806	6,90	4,23	5,79
25,12	9,04	6,04	7,72
25,434	11,12	8,03	9,95
25,748	8,07	6,35	7,22
26,376	5,52	4,91	5,30
28,26	2,42	3,44	3,15
31,4	1,87	3,23	2,41
33,284	4,14	5,62	2,81
33,912	6,87	8,15	3,95
34,226	8,02	9,07	4,19
34,54	6,51	6,80	3,12
35,168	4,66	4,31	1,95
37,68	2,73	2,02	0,96

Table 6 - Relation between driven frequency and amplitude of steady state solution

which can be plotted to visually show the relationship:

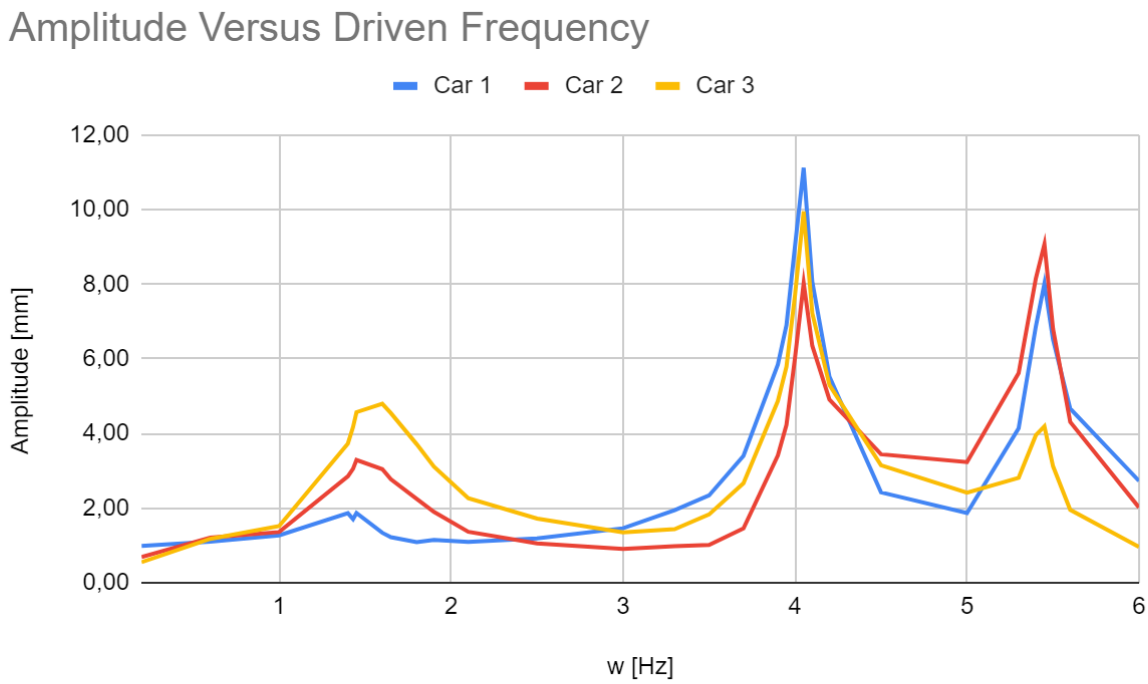


Figure 24 - Amplitude of steady state solution versus Driven Frequency for each car

From where the following natural frequencies can be inferred:

wn1 [rad/s]	wn2 [rad/s]	wn3 [rad/s]
9,11	25,43	34,23

3 Theoretical Model Considerations

3.1 Numerical model solution

Numerical methods provide a way to solve differential equations defined in continuous space and time by converting them to a large system of equations in discrete domains. For the studied case, the Method of Newmark was used to solve the equations of motion found using Lagrangian Mechanics; the numerical methods require the mass, stiffness and damping matrices as input, and produce the positions, velocities and accelerations for a corresponding time interval.

3.1.1 Method of Newmark

Newmark's method "gives a varying weight in the acceleration extremes of intervals in terms of velocity and acceleration" [2]. For a system with more than one degree of freedom, the displacement, acceleration and velocity can be obtained with:

$$x_{i+1} = K_{eff}^{-1} F_{eff}$$

$$x''_{i+1} = a_0(x_{i+1} - x_i) - a_2 x'_i - a_3 x''_i$$

$$x'_{i+1} = x'_i + a_7 * x''_i + a_6 * x''_{i+1}$$

Where:

$$K_{eff} = K + a_0 M + a_1 C$$

$$F_{eff} = F_{i+1} + M(a_0 x_i + a_2 x'_i + a_3 x''_i) + C(a_1 x_i + a_4 x'_i + a_5 x''_i)$$

And a_j are integrations constants given by:

$$a_0 = \frac{1}{\alpha dt^2};\ a_1 = \frac{\beta}{\alpha dt};\ a_2 = \frac{1}{\alpha dt};\ a_3 = \frac{1}{2\alpha} - 1;\ a_4 = \frac{\beta}{\alpha} - 1$$

$$a_5 = (\frac{\beta}{\alpha} - 2)\frac{\alpha dt}{2};\ a_6 = \beta * dt;\ a_7 = (1 - \beta)dt;$$

where F_i is the force matrix, M is the mass matrix, C is the damping matrix, α and β are integration parameters, effectively acting as weights for calculating the approximation of the acceleration, and may be adjusted to achieve accuracy and stability [3]. To start the iteration, the initial position x_0 and velocity x'_0 values must be specified and then used to calculate the initial acceleration x''_0.

3.2 Analytical model solution

3.2.1 Modal Analysis

Most vibrational multi degree freedom systems encountered in physical situations have distributed properties, such as mass and stiffness. Systems of this type are said to possess an infinite number of degrees of freedom, because the system is fully described only when the motion is known at every point of the system. The description of motion of discrete models requires only a finite number of degrees of coordinates. The modal analysis has the objective to discretize and analyze the total response of the system by considering it as a superposition of response of each of the natural modes of the system. For n DOFs we will have N modes. In general, any long slender object under tension will vibrate, such as it could be if we considered a guitar string.

We are therefore interested in the motion of a 3 DOFs system in the neighborhood of an equilibrium position and we want to study its response for a free oscillation condition. We assume that the generalized displacements from the equilibrium position are sufficiently small that the force/displacement and force/velocity relations are linear, so that the generalized coordinates and their time derivatives appear in the differential equations of motion at most to the first power. This can be addressed as the so-called “small/motion assumption”. Now, let us consider the three-degree-of-freedom system. For this system we will introduce all the variables available, the springs exhibit linear behavior and the dampers are viscous.

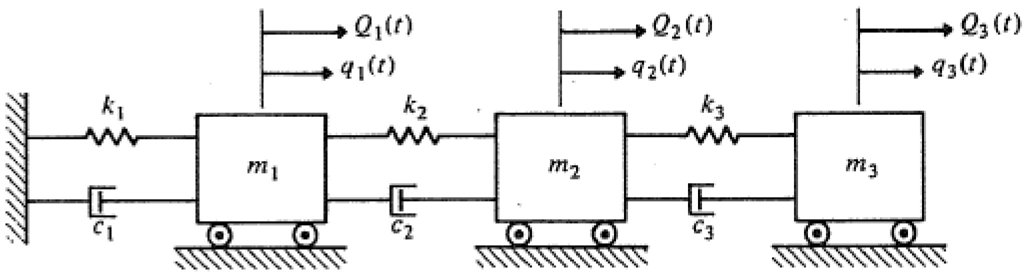


Figure 25 - System in generalized coordinates for modal analysis

In the figure the generalized coordinates q1(t), q2(t) and q3(t) represent the horizontal displacements of the masses m1, m2 and m3 respectively. Q1(t), Q2(t) and Q3(t) are the associated generalized externally applied forces. We want to derive the equations of motion. For this, it’s useful to better understand and analyze the free-body diagrams associated with the three masses.

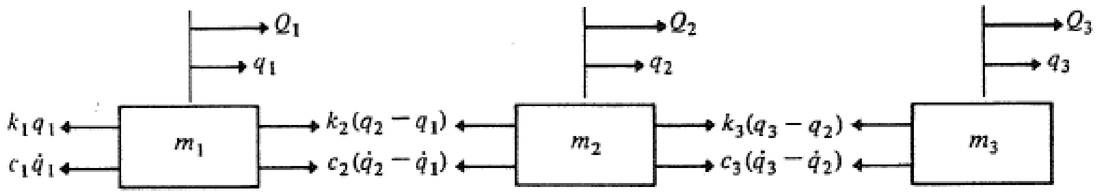


Figure 26 - Forces on generalized coordinates

From the body diagrams it’s now possible to obtain the generalized equations of motion for each masses:

1) $m_1 q_1'' + (c_1 + c_2) * q_1' - c_2 q_2' + (k_1 + k_2) * q_1 - k_2 q_2 = F_1(t)$
2) $m_2 q_2'' - c_2 q_1' + (c_2 + c_3) * q_2' - c_3 q_3' - k_2 q_1 + (k_2 + k_3) * q_2 - k_3 q_3 = F_2(t)$
3) $m_3 q_3'' - c_3 q_2' + c_3 q_3' - k_3 q_2 + k_3 q_3 = F_3(t)$

We can therefore represent the case as we did in the previous chapters by introducing the matrices of the masses:

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In many cases, the influence of damping upon the response of a vibratory system is minor and can be disregarded. However, it must be considered if the response of the system is required for a relatively long period of time compared to the natural periods of the system. Further, if the frequency of excitation (in the case of a periodic force and, therefore, an harmonic oscillation) is at or near one of the natural frequencies of the system, damping is of primary importance and must be taken into account. In general, since the effects are not known in advance, damping must be considered in the vibration analysis of any system. In this section, we shall consider the equations of motion of a damped multi degree-of-freedom system and their solution using Lagrange’s equations. In general, if we introduce:

$$[m]x'' + [c]x' + [k]x = F(\mathrm{t})$$

the solution of the equation corresponds to the undamped free vibration of the system. In this case, if the system is given some energy in the form of initial displacements or initial velocities or both, it vibrates indefinitely, because there is no dissipation of energy. Physically, it means that all coordinates have synchronous motions. The configuration of the system does not change its shape during motion, but its amplitude does. The configuration of the system given by the vector:

$$[X] = [X1\,;\ X2\,;\ \dots\,;\ Xn]$$

is known as the mode shape of the system. After a few peculiar simplifications and interactions with the generalized equation of motion the equation of the kind

$$T(t) = C_1 cos(\omega t + \phi)$$

where C1 and ϕ are constants known as the amplitude and the phase angle, respectively. The previous equation shows that all the coordinates can perform harmonic motion with the same frequency and the same phase angle. However the frequency ω cannot take an arbitrary value. For a nontrivial solution of the equation $\left[[k] - \omega^2[m]\right] * [X] = [0]$ the determinant Δ of the coefficient matrix must be zero. This case represents what is known as the eigenvalue or characteristic value problem. ω^2 is known as the eigenvalue and ω is called the natural frequency of the system. The lowest value of ω is called the fundamental or first natural frequency. In general, all the natural frequencies ω_i are distinct, although in some cases two natural frequencies might possess the same value.

There are several methods of determining the natural frequencies and mode shapes of a multi degree-of-freedom system. In our course we outlined specifically two of them: the Jacobi's method, the matrix iteration method and the Rayleigh's method. This last one is based on the rayleigh's principle and it gives an approximate value of the fundamental natural frequency, which is always larger than the exact value. In our case we have used the matrix iteration method to compute and calculate the natural frequencies and consequently we found all the eigenvalues and eigenvectors of real symmetric matrices.

The matrix iteration method assumes that the natural frequencies are distinct and well separated such that $\omega_1 < \omega_2 < \ \dots\ < \omega_n$. The iteration is started by selecting a trial vector $[X]_1$, which is then premultiplied by the dynamical matrix [D]. The resulting column vector is then normalized, usually by making one of its components equal to unity. The normalized column vector is premultiplied by [D] to obtain a third column vector, which is normalized in the same way as before and becomes still another trial column vector. The process is repeated until the successive normalized column vectors converge to a common vector: the fundamental eigenvector. The normalizing factor gives the largest value of $\lambda = 1/\omega^2$ that is, the smallest or the fundamental natural frequency.

3.2.1 Free Response

It can be proved that the analytical solution of the studied system is a linear combination of the natural modes of the system:

$$x(t) = [X]e^{-\lambda t}$$

where $[X]$ is the eigenvectors matrix - a matrix of eigenvectors corresponding to each normal mode, and λ is the corresponding eigenvalue. To fully solve the problem, one must substitute $x(t)$ into the differential equation of motion and solve a complex equation to obtain the eigenvectors and corresponding eigenvalues. If we, however, rearrange the equations of motion and make simplifications provided by the theory of discrete dynamic systems, we can find the eigenvectors matrix by doing:

$$\lambda[I] - [M]\,[K]^{-1} = 0$$

which, for our system, yields:

$$[X] = \begin{bmatrix} -0.2893 & -0.7503 & 0.5681 \\ -0.6053 & -0.3260 & -0.7369 \\ -0.7417 & 0.5752 & 0.3663 \end{bmatrix}$$

$$\lambda_1 = 0.034277; \quad \lambda_2 = 0.0040422; \quad \lambda_3 = 0.0021038;$$

If we consider that the damping matrix can be expressed as a linear combination of mass and stiffness matrix, that is:

$$[C] = \alpha[M] + \beta[K]$$

we can obtain the decoupled matrix by making assumptions based on the theory of discrete dynamic systems by considering:

$$[C] = [X]^T[c][X]$$

Neglecting non diagonal terms, we find:

$$[C] = \begin{bmatrix} 0.5657 & 0 & 0 \\ 0 & 4.3938 & 0 \\ 0 & 0 & 10.842 \end{bmatrix}$$

4 Comparison Between Experimental, Numerical, and Analytical Result

The laboratory work allows for a more in-depth knowledge of three-degree-of-freedom systems. It will also allow for the improvement of experimental abilities as well as the capacity to analyze and discuss experimental results. In this laboratory, a variety of studies were carried out with varying goals in mind. To begin, apply the theory's knowledge to the facts in order to determine the system's parameters (masses, damping coefficients and spring constants). Second, getting the free response of a three-dimensional (DOF) system to initial stimulation. Finally, achieving the replies but with a harmonic stimulation. The lab's final goal is to teach students how to analyze experimental data and compare it to theoretical models in order to derive conclusions.

Another approach to the problem is to look at it in terms of time. Using the formulation presented in the formulation of the theoretical method like

$m\ddot{x} + c\dot{x} + kx = F_0 sin(\omega t)\ or\ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{F_0 sin(\omega t)}{m}$ and selecting an exciting frequency, in this case w=2.1 Hz, which is the closest one to the natural frequency of the system calculated in previous chapters, the theoretical response can be obtained, and thus the following graphs can be obtained by comparing this result with the experimental one. It is crucial to note that this graph only depicts the steady-state condition, that is, when the homogeneous solution of the motion has no impact. Despite the amplitude disparity between the theoretical model and the practical data, the phase of both is very comparable. This is evident since both peaks occur at about the same time.

Using the theoretic model presented in previous sections and tables, the theoretic prediction in the time domain can be found, and therefore the following graph may be created. This image shows a crucial element connected to damping. Because of the model's limitations, we can't represent all of the friction involved in the experiment, therefore the theoretical damping is lower than the experimental data. This is demonstrated by the number of peaks in the curves, which is inversely related to the damping, i.e., the less damping, the more peaks the movement has.

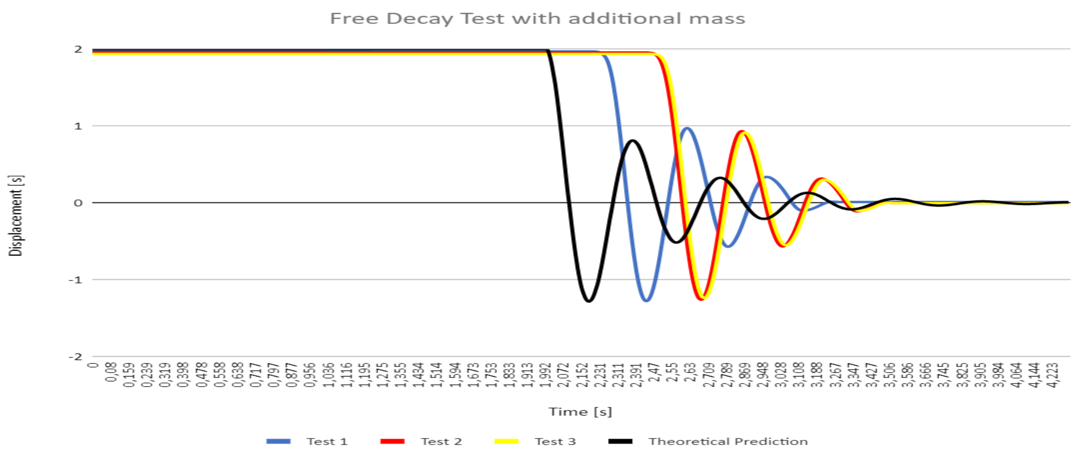


Figure 27 - Comparison between experimental results and theoretical prediction for free decay test with additional mass

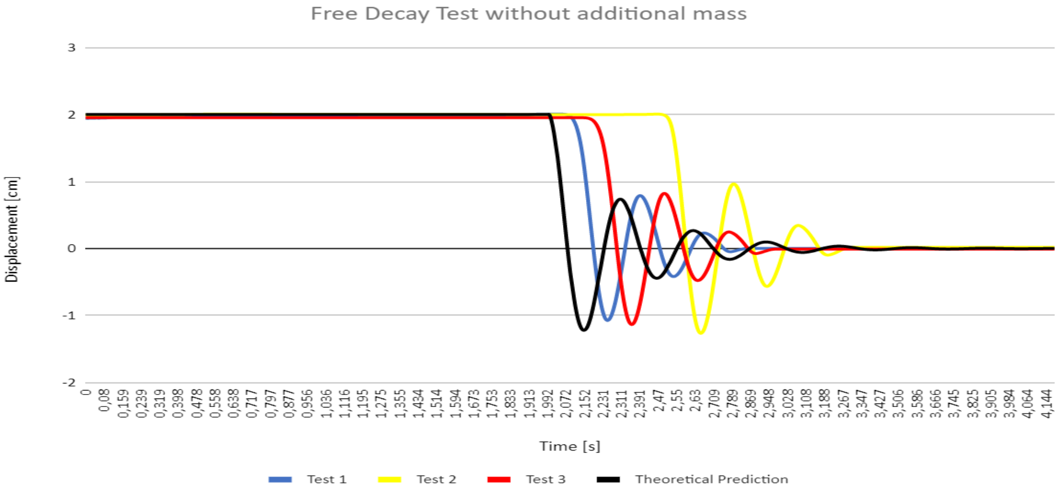


Figure 28 - Comparison between experimental results and theoretical prediction for free decay test without additional mass

For three reasons, the theoretical amplitude was predicted to be bigger than the observed amplitude. First, since the theoretical model does not account for all friction and damping. In this situation, the experimental data is higher than the theoretical data, most likely because the sensor only captured the transient regime because the automobile impacted the sensor so quickly. The amplitude in the transient regime differs from that in the steady state situation. Second, the theoretical model forecasts the amplitude for the steady-state situation, making comparing the amplitude date problematic. Finally, the machine that generates the force isn't the most calibrated machine, thus it can't guarantee a consistent force throughout the experiment.

On the other hand, this experimental data indicates that the system's natural frequency should be about 19 rad/s (2,7 Hz), which is close to the natural frequency obtained (average 20 rad/s). Because of the equipment utilized, it isn't precisely the same. So, despite the contradiction between expected and found in the forced response, we can conclude that the logarithmic decrement test is a good way to determine the different parameters in the system such as mass, damping coefficient, and spring coefficient without the need for sophisticated equipment. In terms of motion amplitude, the experiment was a success since it provided a suitable technique to demonstrate the natural frequency achieved in the 3DOF forced system.

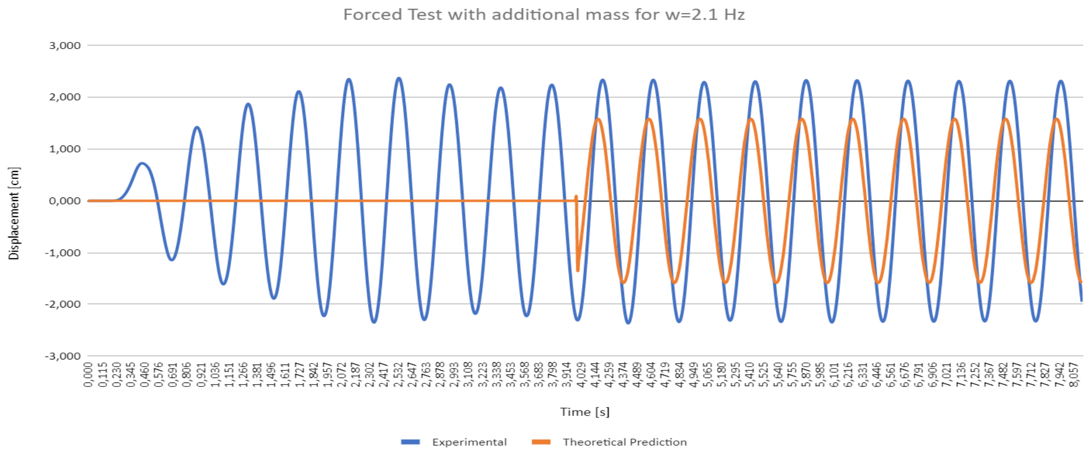


Figure 29 - Comparison between experimental results and theoretical prediction for forced test with additional mass for w=2.7 Hz and for steady state condition

5 Conclusions

The objectives of this project were to obtain experience with a three degree-of-freedom mass-spring-damper system, improve skills for experimental work and learn to analyze experimental data in light of what is predicted from theoretical models. To achieve the objectives, experimental data of the proposed system were obtained in the laboratory, considering a motion with and without harmonic excitation for one and three degrees of freedom.

The methodology to analyze the data consisted primarily of establishing relationships between data obtained from the experimental apparatus and physical quantities with experiments of systems with one degree of freedom. Then, results of free-decay tests for each car were used in order to calculate numerical values for the masses, spring constants and, ultimately, damping coefficients of the system.

Using these relationships and results, the free displacement response in the time domain of the three degrees of freedom system was compared with the one predicted analytically for the initial conditions observed. Then, forced response data (considering a harmonic excitation, in the three degrees of freedom system) were studied by varying the driven frequency for a constant excitation amplitude, from which the relationship between steady state amplitude and driven frequencies was inferred. This relationship was supposed to allow the calculation and subsequent comparison of the natural frequency with that calculated from the free-decay tests, but the data was so discrepant that we found it not worth putting in this report. The forced response was also compared with the predicted analytically for the initial conditions observed.

All aspects considered, it can be said that this laboratorial work helped us further our knowledge in the field of vibrating systems. Nevertheless, the importance of analyzing more deeply why the group was not capable of finding the proper analytical and numerical solutions that fit the data is highlighted, noting that in many subjects of naval engineering the vibration is considered to be a major problem and the comparison of experimental data with theoretical models and results of simulations is of vital importance.

List of Figure Captions

Figure 29 - Comparison between experimental results and theoretical prediction for forced test with additional mass for w=2.7 Hz and for steady state condition

Reference List

[1] Lab work/Dynamics and Hydrodynamics of Floating Bodies, https://fenix.tecnico.ulisboa.pt/disciplinas/DHN/2021-2022/2-semestre/assignment

[2] Introduction to Dynamics and Vibrations, Brown University https://www.brown.edu/Departments/Engineering/Courses/En4/Notes/Vibrations/Vibrations.htm#Sect56

[3] Mechanical Vibrations, S. S. Rao

Developed Software

Some calculations were performed using Google Sheets, with the extension of the language of Google App Script, and some were calculated using MATLAB. Below is the developed code and comments about its use in the project context:

Peaks

Helper function used to find a list of negative peaks from displacement data.

```
const RANGE = 10; // TODO: rename "RANGE" variable name

function FINDPEAKS(displacements) {

  displacements = displacements.map(x => Number(x));

  const peaks = [];

  for (let i = 1+RANGE; i < displacements.length-RANGE; i = i + 1) {

    const prevXs = displacements.slice(i-RANGE, i);

    const currentX = displacements[i];
```

```
    const nextXs = displacements.slice(i, i+RANGE);


    const allPrevXsAreSmaller = prevXs.every(prevX => prevX <= currentX);

    const allNextXsAreSmaller = nextXs.every(nextX => nextX <= currentX)

    const isPeak = allPrevXsAreSmaller && allNextXsAreSmaller;

    if (isPeak) {

      peaks.push(currentX)

    }

  }

  const uniquePeaks = Array(...new Set(peaks));

  const highestToLowestPeaks = uniquePeaks.sort((a,b) => b-a);

  return highestToLowestPeaks;

}
```

Newmark Integration

```
      /**

       * Newmark Integration Method

       *
---------------------------------------------------------------------------

       * Integrates a N-DOF system with a mass matrix "M", stiffness matrix
"K" and

       * damping matrix "C" subjected to an external force matrix F.
```

```
       * Returns the displacement of the system with respect to an inertial
frame of

       * reference.

       *

       * Input

       * ----------

       * @param {[n, 1]}     t - Time Vector

       * @param {[1, DOF]}   X0 - Initial Position Vector

       * @param {[1, DOF]}   Xdot0 - Initial Velocity Vector

       * @param {[DOF, DOF]} M - Mass Matrix

       * @param {[DOF, DOF]} K - Stiffness Matrix

       * @param {[DOF, DOF]} C - Damping Matrix

       * @param {[n, DOF]}   F - External Force Matrix

       *

       * Output

       * ----------

       * @return {[n, DOF]}  X - Displacement Response

       * @customfunction

       */


      function NEWMARK_INTEGRATE(t, X0, Xdot0, M, K, C, F) {

        // Transforming arguments to right type

        t = t.map(arr => Number(arr[0])), X0 = X0[0], Xdot0 = Xdot0[0];

        const numOfPoints = t.length, DOF = K.length;
```

```
if (!F) {
  F = zeros(numOfPoints, DOF);
}

// Initializing Variables
const alfa = 1/4, beta = 1/2;
const dt = t[2] - t[1];

// Initial Conditions
const X2dot0 = math.multiply(
  math.inv(M),
  math.add(
    F[0],
    math.unaryMinus(math.multiply(C, X0)),
    math.unaryMinus(math.multiply(K, Xdot0))
    )
  );
const X = [X0], Xdot = [Xdot0], X2dot = [X2dot0];

//  Integration Constants
const a0 = 1 / (alfa * (dt**2)),
      a1 = beta / (alfa * dt),
      a2 = 1 / (alfa * dt),
```

```
        a3 = ((1 / (2 * alfa)) - 1),

        a4 = (beta / alfa) - 1,

        a5 = ((alfa * dt) / 2) * ((beta / alfa) - 2),

        a6 = beta * dt,

        a7 = (1 - beta) * dt;


  // Form Effective Stiffness Matrix

  const Keff = math.add(K, math.multiply(M, a0), math.multiply(C,
a1));

  const inverseOfKeff = math.inv(Keff);


  for (let i = 0; i < t.length-1; i = i+1) {

    const A0 = math.multiply(X[i], a0),

          A1 = math.multiply(X[i], a1),

          A2 = math.multiply(Xdot[i], a2),

          A3 = math.multiply(X2dot[i], a3),

          A4 = math.multiply(Xdot[i], a4),

          A5 = math.multiply(X2dot[i], a5);


    const Feff = math.add(

                F[i+1],

                math.multiply(M, math.add(A0, A2, A3)),

                math.multiply(C, math.add(A1, A4, A5))

                );
```

```
    X[i+1] = math.multiply(inverseOfKeff, Feff);
    X2dot[i+1] = math.add(
                math.multiply(math.subtract(X[i+1], X[i]), a0),
                math.unaryMinus(math.multiply(Xdot[i], a2)),
                math.unaryMinus(math.multiply(X2dot[i], a3))
                );
    Xdot[i+1] = math.add(
                Xdot[i],
                math.multiply(X2dot[i], a7),
                math.multiply(X2dot[i+1], a6)
                );
  }

  return X;
}

function zeros(n, DOF) {
  return new Array(n).fill(new Array(DOF).fill(0));
}
```