METODI RUNGE-KUTTA

DERIVAZIONE

$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$$t_0 < t_1 < \dots < t_M = T, t_{i+1} - t_i = h$$

$$\int_{t_i}^{t_{i+1}} \dot{y}(t)dt = y(t_{i+1}) - y(t_i)$$

$$= \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

$$= h \int_0^1 f(t_i + h_i s, y(t_i + sh))ds$$

DERIVAZIONE

$$
\begin{aligned}
&y(t_{i+1}) \\
&= y(t_i) + h\int_0^1 f(t_i + hs, y(t_i + sh))ds \\
&\approx y(t_i) + h\sum_{j=1}^{s} b_j f(t_i + hc_j, y(t_i + c_j h))
\end{aligned}
$$

$$\int_{t_i}^{t_i+c_j h} \dot{y}(t)dt = y(t_i + c_j h) - y(t_i)$$

DERIVAZIONE

$$\int_{t_i}^{t_i+c_j h} \dot{y}(t)dt = y(t_i + c_j h) - y(t_i)$$
$$= \int_{t_i}^{t_i+c_j h} f(t, y(t))dt =$$
$$= h \int_0^{c_j} f(t_i + sh, y(t_i + sh))ds$$
$$\approx h \sum_{k=1}^{s} a_{jk} f(t_i + c_k h, y(t_i + c_k h))$$

DERIVAZIONE

Riassumendo, abbiamo ottenuto la forma di un generico metodo Runge-Kutta, che è la seguente:

$$y_{i+1} = y_i + h \sum_{j=1}^{s} b_j f(t_i + c_j h, Y_j)$$

$$Y_j = y_i + h \sum_{k=1}^{s} a_{jk} f(t_i + c_k h, Y_k)$$

BUTCHER TABLEAU

$$y_{i+1} = y_i + h \sum_{j=1}^{s} b_j f(t_i + c_j h, Y_j)$$

$$Y_j = y_i + h \sum_{k=1}^{s} a_{jk} f(t_i + c_k h, Y_k)$$

$$\begin{array}{c|c}
c_1 & \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{bmatrix} \\
\vdots & \\
c_s & \\
\hline
 & b_1 \quad b_2 \quad \cdots \quad b_s
\end{array}$$

PUNTO MEDIO IMPLICITO

$$y_{i+1} = y_i + hf(t_i + \tfrac{h}{2}, Y_1)$$
$$Y_1 = y_i + \tfrac{h}{2}f(t_i + \tfrac{h}{2}, Y_1)$$

$$\begin{array}{c|c} \frac{1}{2} & \left[\frac{1}{2}\right] \\ \hline & 1 \end{array}$$

RK ESPLICITO

$$
\begin{array}{c|c}
\begin{matrix} 0 \\ c_2 \\ \vdots \\ \\ c_s \end{matrix} &
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
a_{21} & 0 & \ddots & \vdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 \\
\vdots & \ddots & \ddots & 0 & 0 \\
a_{s1} & a_{s2} & \cdots & a_{s,s-1} & 0
\end{bmatrix} \\
\hline
 & \begin{matrix} b_1 & b_2 & \cdots & b_s \end{matrix}
\end{array}
$$

RK ESPLICITO

$$y_{i+1} = y_i + h \sum_{j=1}^{s} b_j f(t_i + c_j h, Y_j)$$
$$Y_j = y_i + h \sum_{k=1}^{j-1} a_{jk} f(t_i + c_k h, Y_k)$$

$$Y_1 = y_i$$
$$Y_2 = y_i + h a_{21} f(t_i + c_1 h, y_i)$$
$$Y_3 = y_i + h a_{31} f(t_i + c_1 h, y_i) +$$
$$+ h a_{32} f(t_i + c_2 h, Y_2)$$

EULERO ESPLICITO

$$y_{i+1} = y_i + h \sum_{j=1}^{1} f(t_i, Y_1)$$

$$Y_1 = y_i$$

$$y_{i+1} = y_i + hf(t_i, y_i)$$

$$\begin{array}{c|c} 0 & [0] \\ \hline & 1 \end{array}$$

```python
def implicit_rk(f, time, y0, A, b, c):
    ys = np.zeros((len(y0),len(time)))
    ys[:,0] = y0
    for i in range(1, len(time)):
        h = time[i]-time[i-1]
        yVec = ys[:,i-1].repeat(len(c),axis=0)
        vec = lambda y : y.reshape(-1)
        func = lambda y : yVec + h * vec(np.einsum('ri,oi->or',A,f(y.reshape(len(y0),len(c))))) - y
        k = fsolve(func=func, x0=yVec, xtol=1e-14)
        k = k.reshape((len(y0),len(c)))
        ys[:,i] = ys[:,i-1] + h * np.einsum('i,oi->o',b,f(k))
    return xs, ys
```