Automatic Control Project

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1 Question 1

Question: Linearize the equations of dynamics (1) with respect to the origin x = 0 and write the linearized system in the state space representation x' = Ax + Bu. Is the system asymptotically stable, stable or unstable? Is the system controllable?

1.1 Solution

1.1.1 Linearization

Given the non-linear system described by the equations of dynamics,

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + f_v(\dot{\mathbf{q}}) + G(\mathbf{q}) = \tau$$

we aim to linearize these equations with respect to the origin $q = [0, 0, 0, 0]^T$. In order to perform this operation, we need to determine the equation of the dynamics of our system. The state vector of our system is:

$$\mathbf{q} = \begin{pmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{pmatrix}^T$$

where θ and α are the values of the angles, while $\dot{\theta}$ and $\dot{\alpha}$ are the corresponding angular velocities.
To complete the linearization, we need to find the derivative of our state variables. $\dot{\alpha}$ and $\dot{\theta}$ can be computed trivially as the time derivative of the first 2 variable of $\mathbf{q}$, while $\ddot{\alpha}$ and $\ddot{\theta}$ must be retrieved by solving for $\ddot{q}$ the equation f.

$$\ddot{\mathbf{q}} = M(\mathbf{q})^{-1} \left(\tau - C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} - f_v(\dot{\mathbf{q}}) - G(\mathbf{q})\right)$$

Substituting the values of the all the parameters we get a new rearranged equation of motion:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \frac{14000\,\sigma_1}{45\cos(\alpha)^2-203} + \frac{6000\cos(\alpha)\,\sigma_2}{45\cos(\alpha)^2-203} \\ \frac{2000\left(9\cos(\alpha)^2-58\right)\sigma_2}{3(45\cos(\alpha)^2-203)} - \frac{6000\cos(\alpha)\,\sigma_1}{45\cos(\alpha)^2-203} \end{bmatrix}$$

where

$$\sigma_1 = \frac{-9\sin(\alpha)\,\dot{\alpha}^2}{4000} + \frac{9\,\dot{\theta}\cos(\alpha)\sin(\alpha)\,\alpha}{2000} + \frac{\dot{\theta}}{5} - u$$

$$\sigma_2 = \frac{9\cos(\alpha)\sin(\alpha)\,\dot{\theta}^2}{4000} - \frac{2\,\dot{\alpha}}{5} + \frac{2943\sin(\alpha)}{20000}$$

Since we are computing the linearization of the system, we can neglect terms of order greater than one.

$$\begin{bmatrix}\dot{\theta}\\ \dot{\alpha}\\ \ddot{\theta}\\ \ddot{\alpha}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \dot{\alpha}\\ \frac{1200\,\dot{\alpha}}{79} - \frac{8829\,\alpha}{1580} - \frac{1400\,\dot{\theta}}{79} + \frac{7000\,u}{79}\\ \frac{48069\,\alpha}{1580} - \frac{19600\,\dot{\alpha}}{237} + \frac{600\,\dot{\theta}}{79} - \frac{3000\,u}{79}\end{bmatrix}$$

To formulate the Linearized State-Space Representation, we need to compute the Jacobian matrix A and B as:

$$A = \left.\frac{\partial f}{\partial \mathbf{q}}\right|_{\mathbf{q}=0} \qquad B = \left.\frac{\partial f}{\partial \mathbf{u}}\right|_{\mathbf{q}=0}$$

The linearized system in the state space representation is therefore given by:

$$\dot{\mathbf{q}} = A\mathbf{q} + B\mathbf{u}$$

$$\begin{bmatrix}\dot{\theta}\\ \dot{\alpha}\\ \ddot{\theta}\\ \ddot{\alpha}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1.0000 & 0\\ 0 & 0 & 0 & 1.0000\\ 0 & -5.5880 & -17.7215 & 15.1899\\ 0 & 30.4234 & 7.5949 & -82.7004\end{bmatrix}\begin{bmatrix}\theta\\ \alpha\\ \dot{\theta}\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 88.6076\\ -37.9747\end{bmatrix} u$$

1.1.2 Internal Stability:

To check the stability of the open loop system, we need to compute the eigenvalues of the matrix A. The system is exponentially stable if the eigenvalues are strictly less than zero. The eigenvalues of the matrix are:

$$\lambda_1 = 0, \quad \lambda_2 = -84.7869, \quad \lambda_3 = 0.36612, \quad \lambda_4 = -16.0011.$$

Since we have an eigenvalue with a positive real part, the open loop system must be considered as instable.

It is possible to check the internal stability of the system using a Linear-Matrix-Inequality (LMI) approach. To do so, we need to find a symmetric matrix P such that the quadratic function $V(x) := x^T P x$ decreases strictly along the solution, except the origin. This requires finding a symmetric matrix P that satisfies the following conditions:

$$\begin{cases} P \geq 0 \\ A^T P + PA < 0 \end{cases}$$

The solver could not find a proper P satisfying this condition, confirming the previous answer.

1.1.3 System Controllability:

To determine the controllability of a linear time-invariant (LTI), we need to compute the controllability matrix C.

$$C = \begin{bmatrix} B & AB & A^2B & A^3B & \cdots & A^{n-1}B \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0.0000 & -0.0002 & 0.0096 \\ 0 & -0.0000 & 0.0004 & -0.0333 \\ 0.0000 & -0.0002 & 0.0096 & -0.6782 \\ -0.0000 & 0.0004 & -0.0333 & 2.8373 \end{bmatrix}$$

To check the controllability of the system, we need to determine the rank of the controllability matrix C. The system is controllable if and only if the rank of C is equal to the number of states n.

The rank of the Matrix C is full and equal to 4, therefore the system is controllable.

2 Question 2

Compute a gain matrix K_1 such that the closed-loop system $\dot{x} = A + BK_1$ has a convergence rate $\alpha = 2$. Compute a second gain matrix K_2 ensuring the same convergence rate of the closed-loop system such that K_2 has minimum norm. Report the two gain matrices and simulate the two nonlinear closed-loop systems obtained by substituting in the nonlinear dynamics (1) the input selection $u = K_i x, i \in \{1, 2\}$, for 6 seconds, starting from the initial configuration

$$q(0) = \begin{bmatrix} \theta(0) & \alpha(0) & \dot{\theta}(0) & \dot{\alpha}(0) \end{bmatrix}^T = \begin{bmatrix} 0.05 & 0 & 0.06 & 0 \end{bmatrix}^T.$$

2.1 Solution

2.1.1 Computing K_1 given a convergence $\alpha = 2$:

To ensure a convergence rate of alpha using the close loop system, the following LMI system must hold:

$$\begin{cases} W \geq 0, \\ (AW + BX) + (AW + BX)^T \leq -2\alpha_c W, \end{cases} \tag{1}$$

where P is a symmetric matrix, $\alpha \geq 0$ is a scalar associated to the exponential convergence rate of the solution and K is the gain Matrix K we want to compute. Solving the optimization problem for $\alpha = 2$, the following K is found:

$$\begin{bmatrix} -5.0814 & -5.0814 & -2.3464 & -2.3464 \end{bmatrix}$$

The eigenvalue of the closed loop system $(A + BK)$ matrix are:

$$\mathrm{Re}(\lambda_1) = -5.081, \quad \mathrm{Re}(\lambda_2) = -5.081, \quad \mathrm{Re}(\lambda_3) = -2.346, \quad \mathrm{Re}(\lambda_4) = -2.346.$$

Given that $\lambda_i \leq 0, \quad \forall i$ the system is stable

2.1.2 Computing K_2 given a convergence $\alpha = 2$ and minimum control effort:

To ensure a convergence rate of alpha using the close loop system, the following LMI system must hold:

$$\begin{cases} W \geq I_n, \\ (AW + BX) + (AW + BX)^T \leq -2\alpha_c W, \\ \begin{bmatrix} kI_n & X^T \\ X & kI_p \end{bmatrix} \geq 0 \end{cases} \tag{2}$$

where W is a symmetric matrix, $\alpha \geq 0$ is a scalar associated to the exponential convergence rate of the solution and K is the gain Matrix K we want to compute. Solving the optimization problem for $\alpha = 2$, the following K is found:

$$\begin{bmatrix} 5.2553885 & 3059.3518 & 17.07226 & 39.953227 \end{bmatrix}$$

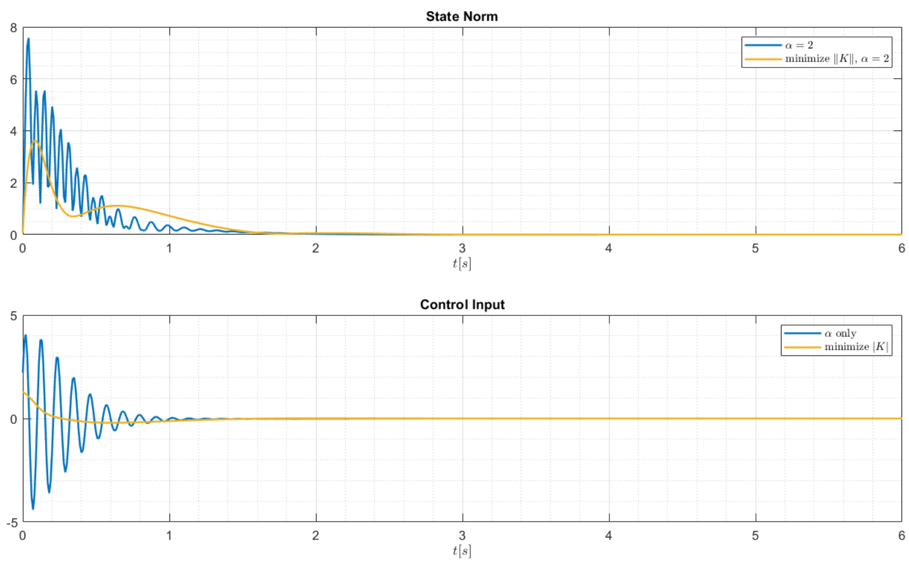
The eigenvalue of the closed loop system $(A + BK)$ matrix are:

$$\mathrm{Re}(\lambda_1) = -84.834, \quad \mathrm{Re}(\lambda_2) = -16.062, \quad \mathrm{Re}(\lambda_3) = -2.0024, \quad \mathrm{Re}(\lambda_4) = -2.0024.$$

Given that $\lambda_i \leq 0, \quad \forall i$ the system is also stable.

2.1.3 Simulation of the non-linear closed-loop system

To simulate the evolution of our system, we numerically integrate the system's solution using MATLAB's ode45 function. The initial condition is specified by the problem statement.



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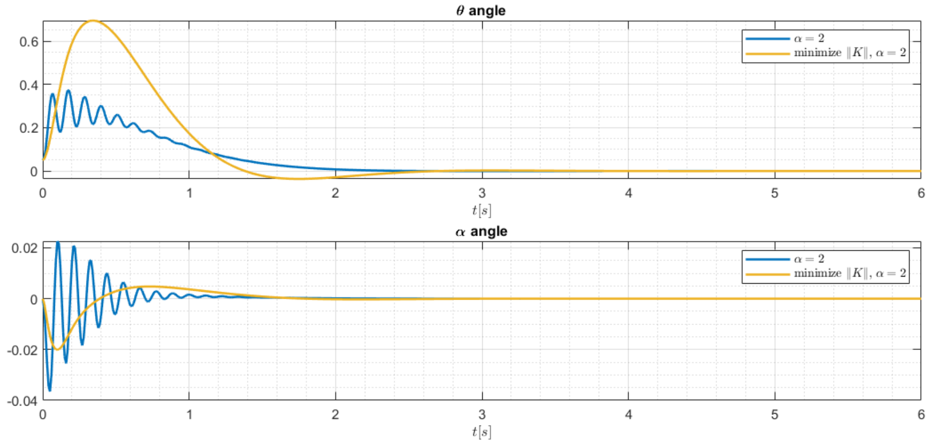


Based on the simulation results, which of the two controllers would you implement and why?

Looking at the simulation, the controller that minimizes effort while preserving a convergence rate is better. It eliminates higher frequency oscillations, leading to a smoother and more stable system response. This results in reduced wear and tear on system components and enhances overall the system longevity.

3 Question 3:

Compute the overshoot M corresponding to the closed-loop matrix $A+BK$ for the two gain matrices K computed in the previous step. Plot the exponential bound on the trajectories of the previous point. Does the exponential bound hold also for the solutions of the nonlinear system?

3.1 Solution

To compute the overshoot M_1 of the system obtained by imposing a convergence rate $\alpha = 2$, we use the following formula:

$$M_1 = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$$

where P is given by $P = Q^{-1}$, already computed in the previous answer.
The result is $M_1 = 36174$.

To compute the overshoot M_2 of the system obtained by imposing a convergence rate $\alpha = 2$ and minimum effort, we use the following formula:

$$M_2 = \sqrt{k_2}$$

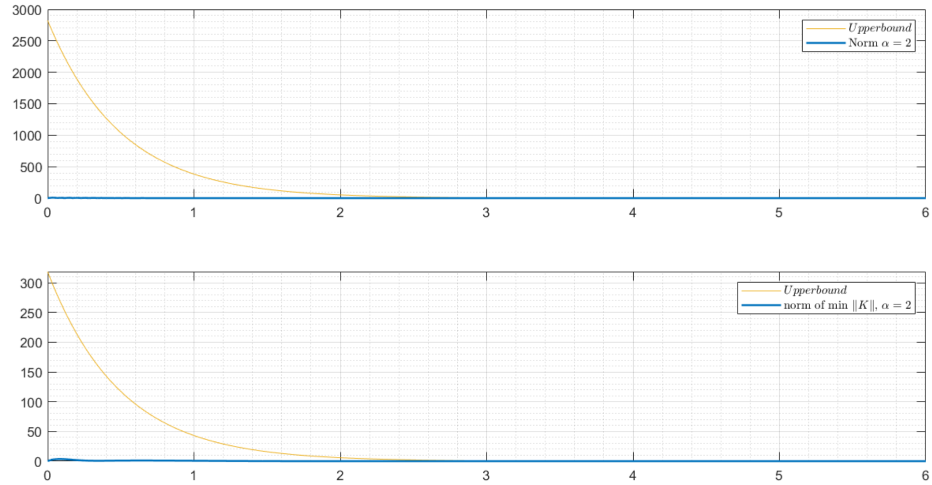
where k_2 is has been previously computed in the LMI 1.2.

The result is $M_1 = 4082$.

3.1.1 Plotting the Bounds

To plot the exponential bound with the M_i values found, we need to use the following formula:

$$|x(t)| \leq M_i|x(0)|e^{-\alpha t}$$



3.1.2 Conclusion

For non-linear system, if the starting point is not far from the equilibrium to which the linearization has been computed, the behaviour is expected to be similar to the linearized one. The exponential bound is therefore valid in our case, given the initial position and the small angle displacements. However, for larger initial displacements or if the system exhibits significant nonlinearities, deviations from the exponential bound may occur.