Automatic Control course project

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1 Project rules

In this document you can find the task for the final project of the Automatic Control course. The project is an **individual work**, and it is worth **4 points** out of the 30 total points of the exam.

You are asked to answer the questions presented in this document, properly motivating them, in a **written report** in PDF format. **Please add your personal ID number on top of the first page of the report**. There is no layout requirement and it can be written in any way you prefer. You will be given 3 questions, worth a total of 4 points. Additionally, writing the report in LaTeX will give you an extra 0.5 points. The extra points can be used to reach the 4/4 grade in case some answers are wrong, but **it will not be possible to obtain more than 4 points in total for the final grade**. Note, additionally, that there is a **strict upper limit** to the number of pages of the report of **6 pages maximum**. You are not required to fill all the 6 pages; answering correctly to the questions is enough to get the full points. Together with the written report, you must submit your **Matlab code** (with comments on the key steps).

When you want to deliver your project, send an e-mail to riccardo.ballaben@unitn.it. Please put as object of the e-mail *'Project Delivery - [name] [surname]'* and attach a **ZIP file** containing the Matlab code and the report, named *'[year].[month].[day] - [surname] [name]'*, where year, month, and day correspond to the delivery date in number.

The project has to be delivered **by the end of the first day of the month when you want to verbalize the exam** (note that it is possible to send the e-mail at any day of the previous months). For example, if you want to verbalize your grade in July, you have until 23:59 of the 1st of July to send the e-mail. The grade you will get for the project is valid until February 2024, so you do not need to send your file multiple times. If you are unhappy with the grade of the project you can submit it again. Note, however, that the project deliveries are accepeted only **once in the summer/autumn session** and **once in the winter session**. This means that you can redo the project at most once.

2 System description: Rotary inverted pendulum

The system we are going to consider is the rotary inverted pendulum shown in Figure 1, which consists of two main components: the inverted pendulum and rotary arm, connected by a revolute joint, and a servo motor, which controls the angular position of the rotary arm.

The system is controlled through the current given to the motor. The relation between the current and the torque generated by the motor can be expressed as a dynamical system. In this project, we are going to neglect the dynamics of the motor and assume to be able to control directly the torque generated by it.

We denote with θ and α, respectively, the angles describing the rotation of the rotary arm in the plane and the rotation of the inverted pendulum about the revolute joint connecting it to the rotary arm. More precisely, we are going to identify with $\alpha = 0$ the configurations in which the inverted pendulum points up.

Then, for $q := [\theta \ \alpha]^\top \in \mathbb{R}^2$, the equations describing the time evolution of the angular coordinates are

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + f_v(\dot{q}) + G(q) = \tau, \tag{1}$$

where $M(q) \in \mathbb{R}^{2\times 2}$ is the symmetric and positive definite generalized mass matrix, $C(q,\dot{q}) \in \mathbb{R}^{2\times 2}$ is the matrix describing the effect of the Coriolis and centripetal forces, $f_v(\dot{q}) \in \mathbb{R}^2$ models the effect of the viscous friction, $G(q) \in \mathbb{R}^2$ is the vector of the gravitational forces and $\tau \in \mathbb{R}^2$ is the input vector.

Given the parameters of the system and their values,

- The mass of the pendulum, $m_p = 0.1[kg]$,
- The length of the pendulum, $L_p = 0.3[m]$,
- The position of the center of mass of the pendulum, $l_p = \frac{L_p}{2} = 0.15[m]$,
- The moment of inertia of the pendulum $J_p = 0.003[kgm^2]$,
- The length of the rotary arm, $L_r = 0.15[m]$,
- The moment of inertia of the rotary arm, $J_r = 0.01[kgm^2]$,
- The viscous friction coefficients of the pendulum and rotary arm, respectively, $B_p = 0.4[Nms/rad]$ and $B_r = 0.2[Nms/rad]$,
- The gravity constant $g = 9.81[m/s^2]$,

the terms in (1) can be written as

$$
\begin{aligned}
M(q) &= \begin{bmatrix} J_r + m_p(L_r^2 + l_p^2(1-\cos(\alpha)^2)) & m_p l_p L_r \cos(\alpha) \\ m_p l_p L_r \cos(\alpha) & J_p + m_p l_p^2 \end{bmatrix}, \\
C(q,\dot{q}) &= \begin{bmatrix} 2m_p l_p^2 \dot{\alpha} \sin(\alpha)\cos(\alpha) & -m_p l_p L_r \dot{\alpha} \sin(\alpha) \\ -m_p l_p^2 \dot{\theta} \sin(\alpha)\cos(\alpha) & 0 \end{bmatrix}, \\
f_v(\dot{q}) &= \begin{bmatrix} B_r \dot{\theta} \\ B_p \dot{\alpha} \end{bmatrix}, \\
G(q) &= \begin{bmatrix} 0 \\ -m_p l_p g \sin(\alpha) \end{bmatrix}, \\
\tau &= \begin{bmatrix} u \\ 0 \end{bmatrix}.
\end{aligned}
$$

Let us call with x the vector $x = [\theta\ \alpha\ \dot{\theta}\ \dot{\alpha}]^\top \in \mathbb{R}^4$. Our goal is to stabilize the zero position and zero velocity configuration, $x = [0\ 0\ 0\ 0]^\top$, using the torque u generated by the motor as control input.

3 Questions

1. **Question**: Linearize the equations of dynamics (1) with respect to the origin $x = 0$ and write the linearized system in the state space representation $\dot{x} = Ax + Bu$. Is the system asymptotically stable, stable or unstable? Is the system controllable? (1pt)

   Hint: Remember that, when computing the linearization, you should neglect terms of order greater than one, such as $\dot{x}x, x^2$, and so on.

2. **Question**: Compute a gain matrix K_1 such that the closed-loop system $\dot{x} = A + BK_1$ has a convergence rate $\alpha = 2$. Compute a second gain matrix K_2 ensuring the same convergence rate of the closed-loop system such that K_2 has minimum norm. Report the two gain matrices and simulate the two nonlinear closed-loop systems obtained by substituting in the nonlinear dynamics (1) the input selection $u = K_i x, i \in \{1,2\}$, for 6 seconds, starting from the initial configuration

$$
x(0) = \begin{bmatrix} \theta(0) \\ \alpha(0) \\ \dot{\theta}(0) \\ \dot{\alpha}(0) \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0 \\ 0.06 \\ 0 \end{bmatrix}.
$$

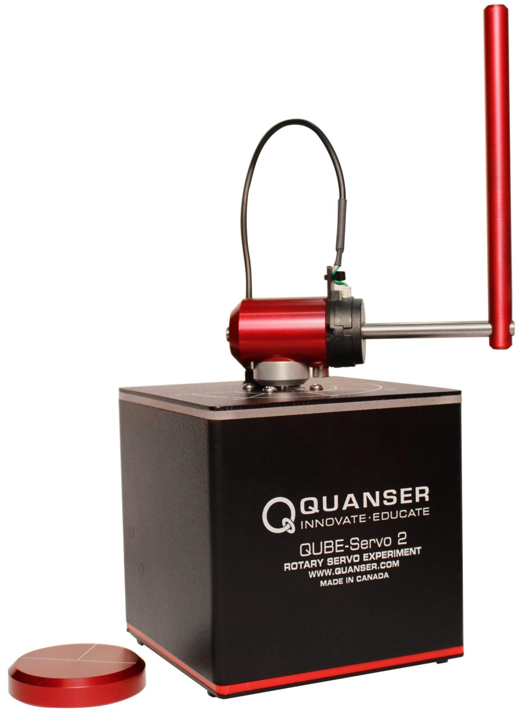

Figure 1: Rotary inverted pendulum by Quanser.

Based on the simulation results, which of the two controllers would you implement and why? (2pt)

Hint: Remember that to minimize the norm of the gain matrix K you need to include in the constraints of the optimization problem the LMIs

$$W > I_n,$$
$$\begin{bmatrix} kI_n & X^\top \\ X & kI_p \end{bmatrix} > 0.$$

3. **Question**: Compute the overshoot M corresponding to the closed-loop matrix $A + BK$ for the two gain matrices K computed in the previous step. Plot the exponential bound on the trajectories of the previous point. Does the exponential bound hold also for the solutions of the nonlinear system? (1pt)