Experimental demonstration of the relationship between CHSH and Instrumental inequalities

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Instrumental and CHSH
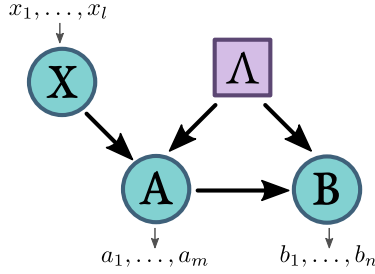

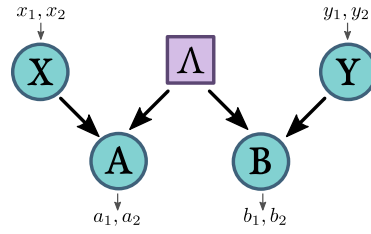Figure 1: The DAG representing a general Instrumental scenario.

Figure 2: The DAG representing a general Instrumental scenario.

The so-called "Instrumental" causal structure is a widely studied structure both for its simplicity and commonness. It can be described by four random variables Λ, X, A, B where X and Λ are independent of the others and B depends on X only through A, as graphically depicted in fig. 1.

The consequences of these causal relations on the distributions of the four variables have been throughly studied in previous works [1, 2]. Suppose that X, A, B can take l, m, n different values and call $P(a_i b_j | x_k)$ the probability that A and B take values a_i and b_j respectively, given that X takes value x_k. Bonet [2] pointed out that the polytope of allowed correlation is characterized not only by the *Pearl's inequalities*:

$$\sum_{j=0}^{n} P(a_i b_j | x_{k(i,j)}) \leq 1 \tag{1}$$

for all $i \in 1, \ldots, m$ and for all the possible functions $k(i,j)$. For example for $(l, m, n) = (3, 2, 2)$ the inequality:

$$P(a_1 b_1 | x_1) + P(a_2 b_2 | x_1) + P(a_1 b_1 | x_2) + P(a_2 b_1 | x_2) + P(a_1 b_2 | x_3) \leq 2 \tag{2}$$

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must also be satisfied, along with other similar inequalities.

These inequalities are closely related to the one presented in [3]:

$$-\langle B\rangle_{x_1} + 2\langle B\rangle_{x_2} + \langle A\rangle_{x_1} - \langle AB\rangle_{x_1} + 2\langle AB\rangle_{x_3} \leq 3 \tag{3}$$

which, as was recently experimentally violated in the same work, and admits a quantum violation of 3.79 ± 0.013. As one can expect, with the same data we can also violate inequality (2) with a value of 2.159.

A similar analisys can be done in the CHSH scenario, depicted in fig. 2, where the variables all have two possible outcomes. Here instead we have the notorious inequality:

$$\langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle \leq 2 \tag{4}$$

As was pointed out in [4] these two scenarios (CHSH and the Instrumental) are somewhat related, and indeed, in a sense, the two quantum violations can be seen as a consequence of one another. To see this we must can think of the Instrumental scenario as a restricted CHSH, where we postselect in the case of $Y = A$, and we augment the variable X with an additional value x_3 as described in [4]. From this analisis the maximum quantum violation would be $(3 + \sqrt{2})/2$.
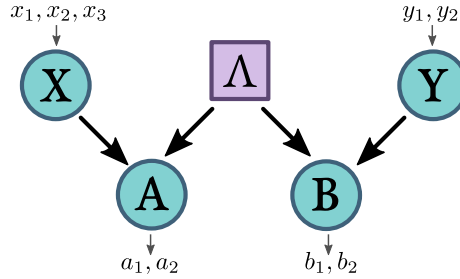

Figure 3: A generalization of the CHSH DAG.

This can be verified experimentally provided we have an apparatus capable of reproducing the causal structure depicted in fig. 3 i.e. a CHSH scenario where X can take 3 possible values, as the apparatus presented in [3]. Taking advantage of this we can violate all three inequalities (2), (3) and (4) with the same set of data:

1. To violate the Instrumental inequalities we postselect in the case $Y = A$, obtaining values 3.79 ± 0.013 and 2.159 for (3) and (2) respectively.

2. To violate the CHSH inequality we just ignore the case $X = x_3$, obtaining the value 2.619 ± 0.005.

Instrumental and contextuality

Interestingly the inequality (2) can also be obtained using the methods presented in [5]. Indeed, following [5], the exclusivity structure of the events in (2), form
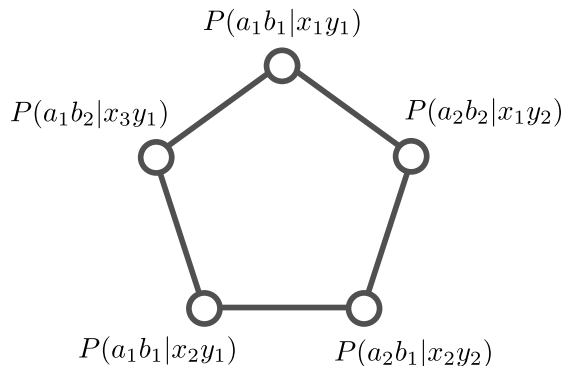
Figure 4: The exclusivity graph of the bonet inequality.

a C_5 cyclic graph, i.e. the pentagon depicted in fig. 4, so that the classical limit would be 2. In this framework (2) is analogous to the KCBS contextual inequality. Nonetheless the quantum limit for the KCBS is $\sqrt{2}$ [5], different from the limit expected for the Bonet inequality found in [4]. This is probably due to the fact that the KCBS limit is attained with a particular configuration of projectors in 3D space, while in the Instrumental scenario the only possible projectors are necessarily tensor product of projectors in two 2D space. It would be interesting to see if the same limit $\sqrt{5}$ can be obtained in the Instrumental scenario, maybe in a generalized context where X is not tricothomic, and if the other inequalities for the Instrumental scenario presented in [2] can be obtained as contextual inequalities.

References

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