Exclusivity graph approach to Instrumental inequalities

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INSTRUMENTAL CAUSAL STRUCTURE

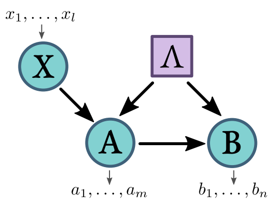

FIG. 1. The DAG representing a general Instrumental scenario.

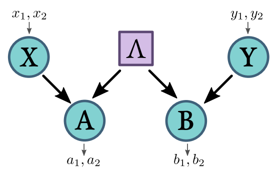

FIG. 2. The DAG representing the CHSH scenario.

The Instrumental causal structure is a widely studied structure both for its simplicity and commonness. It can be described by four random variables Λ, X, A, B where X and Λ are independent of the others and B depends on X only through A, as graphically depicted in fig. 1.

The consequences of these causal relations on the distributions of the four variables have been throughly studied in previous works [1, 2]. Suppose that X, A, B can take l, m, n different values and call $P(a_i b_j|x_k)$ the probability that A and B take values a_i and b_j respectively, given that X takes value x_k. It is known that such a system must respect the following inequalities, called *Pearl's inequalities* [1]:

$$\sum_{j=0}^{n} P(a_i b_j|x_{k(i,j)}) \leq 1 \tag{1}$$

for all $i \in 1, \dots, m$ and for all the possible functions $k(i, j)$.

In [2] Bonet showed that the polytope of allowed correlation is not tightly bounded by *Pearl's inequalities*. For instance for $(l, m, n) = (3, 2, 2)$ the following inequality must also be satisfied:

$$P(a_1b_1|x_1)+P(a_2b_2|x_1)+P(a_1b_1|x_2)+P(a_2b_1|x_2)+P(a_1b_2|x_3) \leq 2 \tag{2}$$

along with other similar inequalities obtained by permuting the labels of a_i, b_j and x_k.

These inequalities are closely related to the one presented in [3]:

$$-\langle B\rangle_{x_1} + 2\langle B\rangle_{x_2} + \langle A\rangle_{x_1} - \langle AB\rangle_{x_1} + 2\langle AB\rangle_{x_3} \leq 3 \tag{3}$$

which was experimentally violated in the same work, with a value of 3.79 ± 0.013. As one can expect, with the same data we can also violate inequality (2) with a value of 2.159 ± 0.005.

A similar analysis can be done in the CHSH scenario, depicted in fig. 2, where all the variables are dichotomic. Hence we have the notorious inequality:

$$\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle \leq 2 \tag{4}$$

As was pointed out in [4] these two scenarios (CHSH and the Instrumental) are related: the two quantum violations can be seen as a consequence of one another. Indeed if we think of the Instrumental scenario as a restricted CHSH, where we postselect in the case of $Y = A$, and we augment the variable X with an additional value x_3 as described in [4]. From this analysis the maximum quantum violation is $(3 + \sqrt{2})/2$.

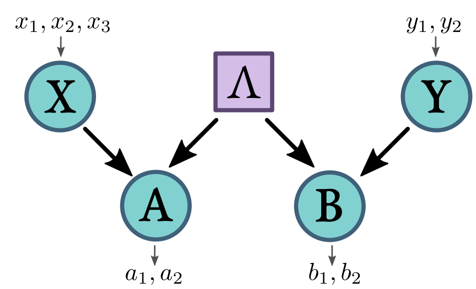

FIG. 3. A generalization of the CHSH DAG.

This can be verified experimentally provided we have an apparatus capable of reproducing the causal structure depicted in fig. 3 i.e. a CHSH scenario where X can take 3 possible values, as the apparatus presented in [3]. Taking advantage of this we can violate all three inequalities (2), (3) and (4) with the same set of data:

1. To violate the Instrumental inequalities we postselect in the case $Y = A$, obtaining values 3.787 ± 0.013 and 2.159 ± 0.005 for (3) and (2) respectively.
2. To violate the CHSH inequality we just ignore the case $X = x_3$, obtaining the value 2.619 ± 0.005.

INSTRUMENTAL AND EXCLUSIVITY GRAPHS

The exclusivity graph method was primarily developed to study non-contextual inequalities and obtain their quantum violation [5]. In this formalism every possible event, i.e. every possible set of outcomes and settings, is associated to a vertex in a (undirected) graph. Two vertices are then connected by an edge if and only if they

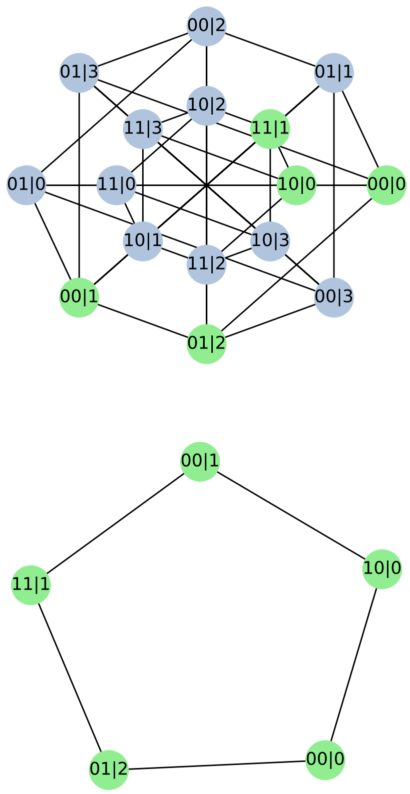

FIG. 4. The exclusivity graph of the bonet inequality.

are exclusive, i.e. if exists a measurement that can distinguish between them.

Interestingly many known results about the instrumental scenario, including the inequality (2), can also be obtained using this formalism. Restricting for the moment to the case of of dichotomic observables ($n = m = 2$), we label as $p(ab|x)$ with $a, b \in \mathcal{A} = \mathcal{B} = \{0, 1\}$ and $x \in \mathcal{X} = \{0, \ldots, l\}$, the probability of having outcomes a and b with setting x. To obtain the exclusivity structure of the events in the instrumental scenario we follow [2], and we say that two events $ab|x$ and $a'b'|x'$ are *not* exclusive if there exists two functions $f : \mathcal{X} \to \mathcal{A}$ and $g : \mathcal{A} \to \mathcal{B}$ such that:

$$a = f(x) \quad \text{and} \quad a' = f(x') \tag{5}$$

$$b = g(a) \quad \text{and} \quad b' = g(a') \tag{6}$$

and exclusive if these two function do not exist. Using this method we can construct the exclusivity graph for various l, shown in figure 5, and use the methods described in [5] and [**?**] to obtain classical and quantum bounds for several inequalities in the instrumental scenario.

First, using the result in [5] we immediately obtain the result that there is no quantum violation in the case of $l = 2$, since the corresponding exclusivity graph (and its complement) does not contain any odd cyclic graph with more than 5 vertices.

For $l \geq 3$ instead we see that there is a violation for the inequality (2), since it forms a C_5 cyclic graph, i.e. the pentagon depicted in fig. 4, so that the classical limit would be 2. In this framework (2) is analogous to the KCBS contextual inequality. We must notice that the quantum limit for the KCBS is $\sqrt{5}$, different from the limit expected for the Bonet inequality found in [4].

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[2] B.Bonet, *Instrumentality tests revisited*, Proceedings of the Seventeenth conference on Uncertainty in artificial intelligence. Morgan Kaufmann Publishers Inc. (2001).

[3] R.Chaves, G.Carvacho, I.Agresti, V.Di Giulio, L.Aolita, S.Giacomini, F.Sciarrino, *Quantum violation of an instrumental test*, Nature Physics 14.3 291 (2018).

[4] T.Van Himbeeck, J.B. Brask, S.Pironio, R.Ramanathan, A.B.Sainz, E. Wolfe, *Quantum violations in the Instrumental scenario and their relations to the Bell scenario*, arXiv preprint arXiv:1804.04119 (2018). (2014). .

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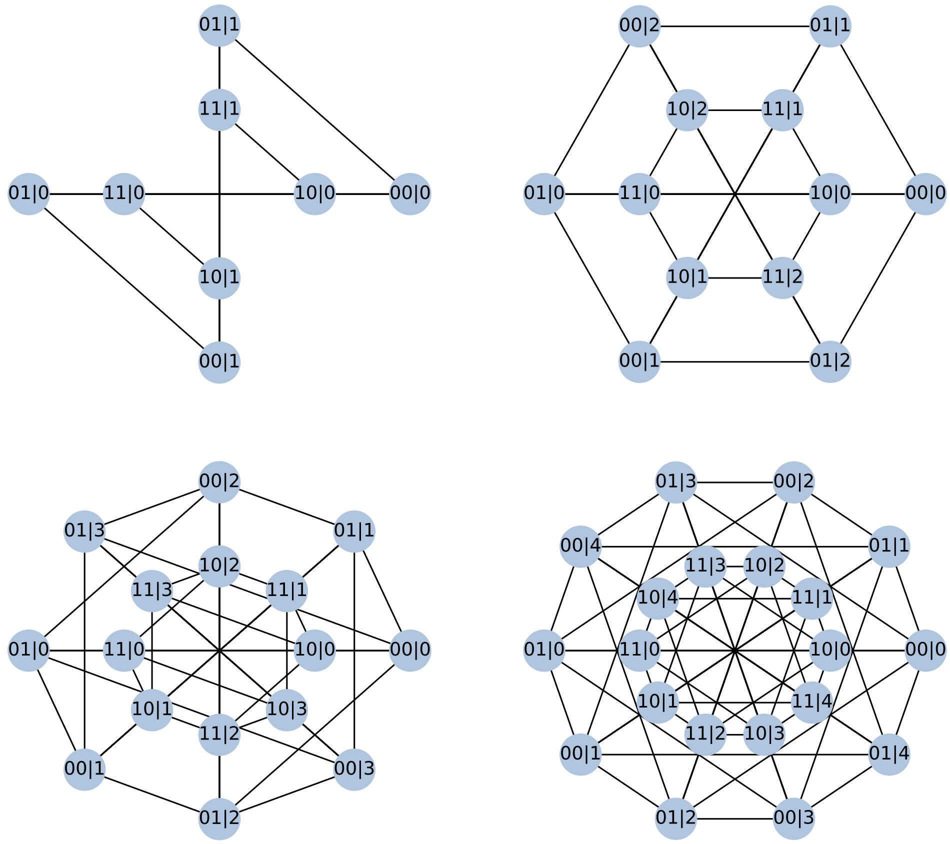

FIG. 5. The exclusivity graph for the instrumental scenario for $l = 2, 3, 4, 5$ from top left to bottom right.