# HOMOGENEOUS DYNAMICS

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## OVERVIEW

**Contents.** The course consists in an introduction to the study of homogeneous flows, which are smooth flows on (quotients of) Lie groups obtained by multiplication by a 1-parameter subgroup. In particular, we will focus on the ergodic theory of Heisenberg nilflows and of geodesic and horocycle flows. We will also explore some connections with topics in number theory.

No prior knowledge in Lie group theory is assumed and all the relevant objects will be introduced during the lectures. However, some familiarity with basic notions in ergodic theory and in differential topology is recommended.

**Assessment.** There will be an oral examination. In order to pass the exam, the student should have developed a solid understanding and working knowledge of the material discussed in the lectures.

All the material covered in the lectures is subject of examination. The student should also be able to solve simple exercises, which will be assigned during the lectures.

**Material.** Lecture notes are available at `davideravotti.github.io` (the file will be updated throughout the semester). Additional useful material is:

[1] M. Einsiedler, T. Ward. Ergodic Theory with a view towards Number Theory. Graduate Texts in Mathematics 259, Springer, London, 2011.

[2] J.M. Lee Introduction to Smooth Manifolds. Graduate Texts in Mathematics 218, Springer, New York, NY, 2012.

[3] M.B. Bekka, M. Mayer. Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces. London Mathematical Society Lecture Note Series 269, Cambridge University Press, Cambridge, 2000.

## LESSON REGISTER

01 Mar: Introduction. Review of notions in ergodic theory (invariant measures, ergodicity, unique ergodicity).
Lecture notes: §§1.1, 1.2, 1.3.1 (except from Proposition 1.9 and Exercise 1.10), 1.3.2, 1.3.3.

08 Mar: Review of notions in ergodic theory (weak mixing, mixing). Review of concepts in differential topology (tangent vectors, vector fields, differentials of smooth maps).
Lecture notes: §§1.1, 1.4.

15 Mar: Definition of Lie groups, matrix Lie groups, and examples. The set of geometric tangent vectors at the identity is a vector space and is closed under the bracket operation. Example of $\mathrm{SL}(2,\mathbb{R})$. Definition of left-invariant vector fields and of Lie algebra. The set of geometric tangent vectors coincides with the tangent space

at the identity and with the Lie algebra.
Lecture notes: §§2.1, 2.2.1.

22 Mar: The exponential matrix (definition and properties). Definition of homogeneous flows. Geometric interpretation of the Adjoint and of the Lie brackets.
Lecture notes: §§2.2.2, 2.2.3.

05 Apr: Construction of the Haar measure. Definition of the Killing form and proof of the invariance under Adjoint. Action of $\mathrm{PSL}(2,\mathbb{R})$ on the hyperbolic plane by isometries. Definition of the Casimir operator.
Lecture notes: §§2.3.1, 2.3.2, 2.3.3 (up to Proposition 2.32).

26 Apr: The Casimir operator commutes with all elements of the Lie algebra. Existence of left-invariant metrics. The quotient of a Lie group by a discrete subgroup is a smooth manifold.
Lecture notes: §§2.3.3, 2.4.

03 May: Generalities on the Heisenberg group (exponential coordinates, Baker-Campbell-Hausdorff Formula, computation of the Haar measure). Classification of Heisenberg nilmanifolds up to group automorphisms (to be finished).
Lecture notes: §3.1.

10 May: Classification of Heisenberg nilmanifolds up to group automorphisms (conclusion). Heisenberg nilflows (formula, invariance of Haar measure). Heisenberg niflows are never weak mixing and are either totally periodic or relatively mixing.
Lecture notes: §3.2.1.

17 May: Heisenberg niflows as special flows over skew-translations on the 2-torus. Equivalence of unique ergodicity of Heisenberg niflows with minimality of their projected linear toral factor.
Lecture notes: §3.2.2.

24 May: The sequence $\{an^2+bn+c : n \in \mathbb{N}\}$ is uniformly distributed mod 1 when $a \notin \mathbb{Q}$. The group $\mathrm{PSL}(2,\mathbb{R})$ acts simply transitively on $T^1\mathbb{H}$ and preserves the hyperbolic metric.
Lecture notes: §§3.3, 4.1.1, 4.1.2.

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