Up and down an infinite staircase

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Measurable Dynamics

Let $(X, \mathscr{A}, \mu)$ be a measure space. A measure preserving flow ϕ on X is a measurable $\mathbb{R}$-action

$$\phi : \mathbb{R} \times X \to X, \qquad \phi(t, \cdot) = \phi_t(\cdot),$$

such that

$$\mu(\phi_t(A)) = \mu(A)$$

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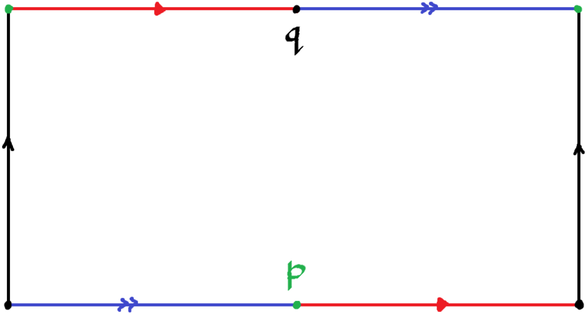
The orbit of $x \in X$ is the measurable curve $t \mapsto \phi_t(x)$ in X.

Question

What is the behaviour of typical orbits?

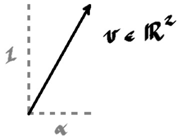
Example

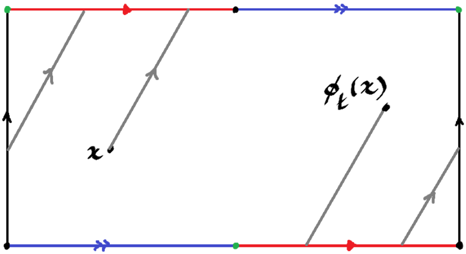


The surface X is a flat torus with two marked points $\Sigma = \{p, q\}$.

Example

Given $\mathbf{v} = (\alpha, 1) \in \mathbb{R}^2$, we consider the straight line flow ϕ in direction $\mathbf{v}$.





Typical orbits

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Ergodicity

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then for every $A \in \mathscr{B}$,

$$\frac{1}{t} \int_0^t \mathbb{1}_A \circ \phi_r \, \mathrm{d}r \to \frac{m(A)}{m(X)}, \qquad m\text{-a.e., as } t \to \infty.$$

Equivalently, for any $f \in L^1(X)$,

$$\frac{1}{t} \int_0^t f \circ \phi_r \, \mathrm{d}r \to \frac{1}{m(X)} \int_X f \, \mathrm{d}m, \qquad m\text{-a.e., as } t \to \infty.$$

Speed of convergence

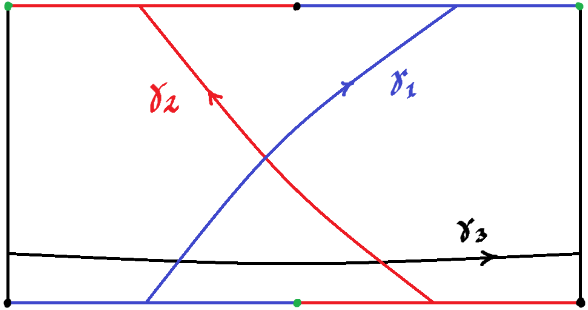
How fast the convergence happens depends on the Diophantine properties of α.

Consider, e.g., $\alpha = 1 - \sqrt{2}$. Then, for every $f \in \mathscr{C}(X)$ and for every $x \in X$,

$$\left| \int_0^t f \circ \phi_r(x) \, \mathrm{d}r - \frac{t}{m(X)} \int_X f \, \mathrm{d}m \right| = O(\log t).$$

An infinite staircase

Fix the loops γ_1, γ_2, and γ_3 in $X \setminus \Sigma$ as follows.



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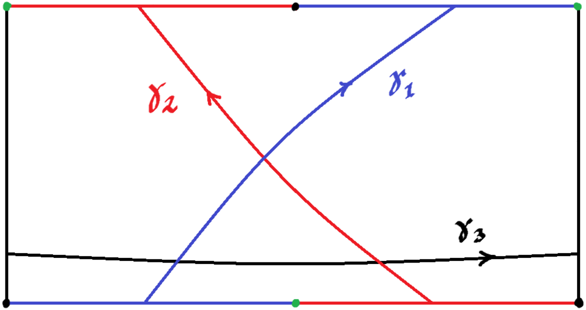
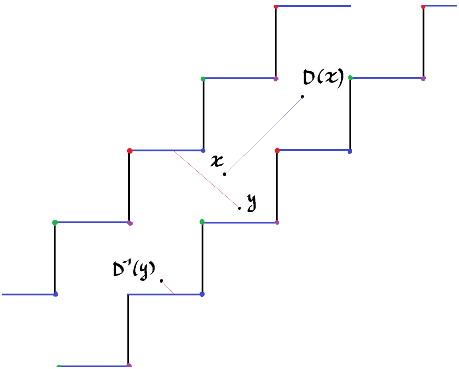


Let

$$\Gamma = \langle [\gamma_1] + [\gamma_2], [\gamma_3] \rangle \leq H_1(X \setminus \Sigma, \mathbb{Z}).$$

We consider the cover $p \colon \widetilde{X} \to X$ associated to Γ.

An infinite staircase



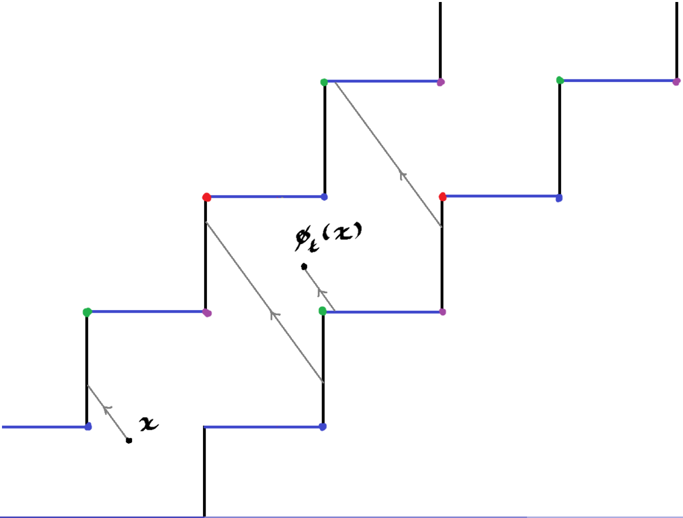
The group of Deck transformations of $\widetilde{X}$ is isomorphic to $\mathbb{Z}$ via

$$\text{Deck} = \langle D \rangle, \qquad D = [\gamma_1] + \Gamma.$$

Note that $D^{-1} = -[\gamma_1] + \Gamma = [\gamma_2] + \Gamma$.

Translation flow

We consider the analogous translation flow ϕ in direction $\mathbf{v} = (\alpha, 1)$ as before, with $\alpha = 1 - \sqrt{2}$.



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Aaronson's Theorem

Let $a(t)$ be any positive function. Then, for every $f \in L^1(\widetilde{X})$, $f \geq 0$,

$$\textit{either} \qquad \limsup_{t\to\infty} \frac{1}{a(t)} \int_0^t f \circ \phi_r \, \mathrm{d}r = \infty, \qquad m\text{-a.e.,}$$

$$\textit{or} \qquad \liminf_{t\to\infty} \frac{1}{a(t)} \int_0^t f \circ \phi_r \, \mathrm{d}r = 0, \qquad m\text{-a.e.}$$

Typical orbits

One good news:

Hopf's Theorem

For every $f, g \in L^1(\widetilde{X})$, $g > 0$,

$$\frac{\int_0^t f \circ \phi_r \, \mathrm{d}r}{\int_0^t g \circ \phi_r \, \mathrm{d}r} \to \frac{\int_{\widetilde{X}} f \, \mathrm{d}m}{\int_{\widetilde{X}} g \, \mathrm{d}m} \qquad m\text{-a.e., as } t \to \infty.$$

Typical orbits

Idea: describe the ergodic integrals as

$$\int_0^t f \circ \phi_r \, \mathrm{d}r = a(t) \left(\int_{\widetilde{X}} f \, \mathrm{d}m \right) \mathrm{Osc}_t(x) \, (1 + o(1)),$$

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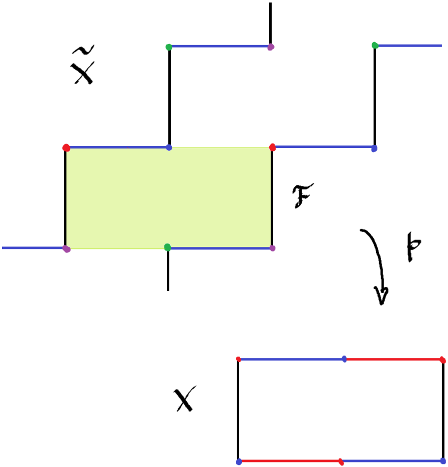
where

- $a(t)$ is the "correct" growth rate,
- $\mathrm{Osc}_t(x)$ is an oscillating term that
  - does not depend on f,
  - does not converge pointwise almost everywhere,
  - but maybe converges in some weaker sense.

The setting



We normalize m so that $m(\mathcal{F}) = 1$.

The result

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Theorem

There exists $\sigma > 0$ such that for every $f \in \mathscr{C}_c^1(\widetilde{X})$, for every $t \geq 1$ and for m-almost every $x \in \mathcal{F}$ we have

$$\int_0^t f \circ \phi_r \, \mathrm{d}r = \frac{1}{\sigma\sqrt{2\pi}} \frac{t}{\sqrt{N}} \left(\int_{\widetilde{X}} f \, \mathrm{d}m \right) \mathrm{Osc}_t(x) + O\left(\frac{t}{N} \right),$$

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and

$$\sqrt{\log \mathrm{Osc}_t} \xrightarrow{\text{dist}} \mathcal{N}(0, \sigma^2).$$

A higher order ergodic theorem

Corollary

Let
$$a(t) = \frac{1}{\sigma\sqrt{2\pi}} \frac{t}{\sqrt{N}}.$$

For every $f \in \mathscr{C}_c(\widetilde{X})$ and for m-almost every $x \in \widetilde{X}$ we have
$$\frac{1}{\log\log T} \int_e^T \frac{1}{t\log t} \left(\frac{1}{a(t)} \int_0^t f \circ \phi_r \,\mathrm{d}r \right) \mathrm{d}t \to \int_{\widetilde{X}} f \,\mathrm{d}m,$$

as $T \to \infty$.

Linear (pseudo-)Anosov

We chose the vector $\mathbf{v} = (1-\sqrt{2}, 1)$ as direction because it is a stable eigenvector of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix},$$

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The matrix A induces a linear hyperbolic map $\psi \colon X \to X$.

Self-similarity

For every $x \in X$ and $t \in \mathbb{R}$,

$$\psi \circ \phi_t(x) = \phi_{\lambda t} \circ \psi(x).$$

Lifting the automorphism

The matrix A satisfies the following properties:

- $A_*(\Gamma) \subseteq \Gamma$,
- $A_*([\gamma_1] + \Gamma) = [\gamma_1] + \Gamma$.

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- $A_*(\Gamma) \subseteq \Gamma$,
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Thus,

- ψ can be lifted to a map $\widetilde{\psi} \colon \widetilde{X} \to \widetilde{X}$,
- $\widetilde{\psi} \circ D = D \circ \widetilde{\psi}$ for every $D \in \mathrm{Deck}$.

The oscillating term

- The factor
$$N = N(t) = \left\lfloor -\frac{\log t}{\log(3-2\sqrt{2})} \right\rfloor + 1$$
is the number of iterates of $\widetilde{\psi}$ we need to apply to "normalize" an orbit of length t to one of size $O(1)$.

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- Let $\xi : \widetilde{X} \to \mathbb{Z}$ be "$\mathbb{Z}$-coordinate". Then, the function
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is well-defined on X.
- The oscillating term can be expressed in terms of
$$\frac{F_N(x)}{\sqrt{N}},$$
which converges in distribution to a (nontrivial) Gaussian random variable, as $N \to \infty$.

Thank you for your attention!