Abelian covers of horocycle flows

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joint work with Roberto Castorrini (SNS Pisa)

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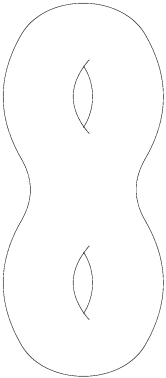
The setting



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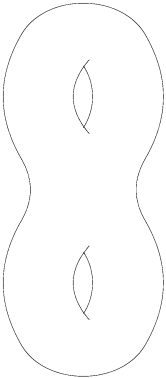

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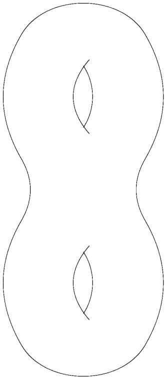

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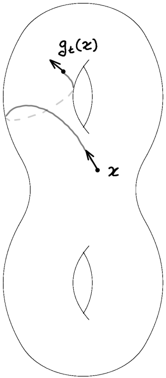

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The setting

The geodesic flow is an Anosov flow: there exist $C \geq 0$, $\lambda > 0$, and a continuous, $D\mathrm{g}_t$-invariant splitting

$$TM = E_- \oplus \langle X \rangle \oplus E_+,$$

where X is the vector field generating $(\mathrm{g}_t)_{t\in\mathbb{R}}$, such that

$$\| D\mathrm{g}_t|_{E_-} \| \leq C \cdot e^{-\lambda t} \qquad \text{and} \qquad \| D\mathrm{g}_{-t}|_{E_+} \| \leq C \cdot e^{-\lambda t},$$

for all $t \geq 0$.

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Theorem (Dolgopyat '98, Liverani '04)

The geodesic flow on (M, vol) is exponentially mixing.

For hyperbolic surfaces: Ratner '87.

Horocycle flows

Theorem (Hurder-Katok '90)

- The distributions $E_{\pm}$ are of class $\mathscr{C}^{2-\varepsilon}$, for all $\varepsilon > 0$.
- If E_+ or E_- is of class $\mathscr{C}^2$, then S has constant curvature.

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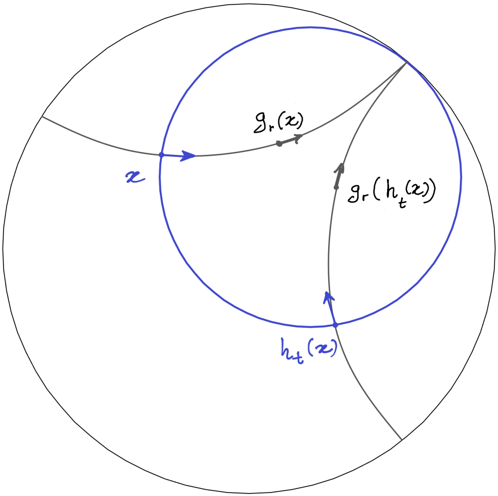
- The (stable) horocycle flow $(\mathrm{h}_t)_{t\in\mathbb{R}}$ is the unit speed motion along the leaves parallel to E_-.
- The generating vector field $U \in E_-$ is of class $\mathscr{C}^{2-\varepsilon}$, for all $\varepsilon > 0$.

Horocycle flows



Stable horocycle flow $(h_t)_{t\in\mathbb{R}}$ on $T^1\mathbb{D}$

Unique ergodicity

Theorem (Marcus '75, from Margulis '70)

There exists a unique, h_t-invariant probability measure μ.

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Theorem (Adam-Baladi '22)

Under a pinching condition, there exists $\delta > 0$ such that for all $f \in \mathscr{C}^2(M)$ we have

$$\int_0^T f \circ \mathrm{h}_t(x)\,\mathrm{d}x = T \cdot \mu(f) + O(T^{1-\delta}).$$

For hyperbolic surfaces: Burger '90, Flaminio-Forni '03, Bufetov-Forni '14, R. '23.

The noncompact setting

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- The associated cover $p\colon S_0 \to S$ has a Galois group isomorphic to $\mathbb{Z}^d$.
- Since $[\Gamma,\Gamma] \trianglelefteq \Gamma_0$ and $\Gamma/[\Gamma,\Gamma] \simeq H_1(S,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$, there are d linearly independent primitive cohomology classes $[\omega_1],\dots,[\omega_d] \in H^1_{\mathrm{dR}}(S,\mathbb{Z})$ so that $\int_\gamma \omega_i = 0$ for all $[\gamma] \in \Gamma_0/[\Gamma,\Gamma]$ and $p^*[\omega_i] = 0$.

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- We can take the forms $\omega_1, \dots, \omega_d$ to be harmonic.

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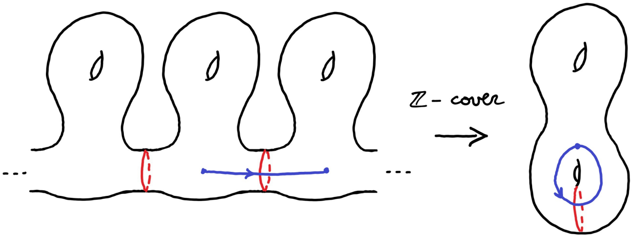

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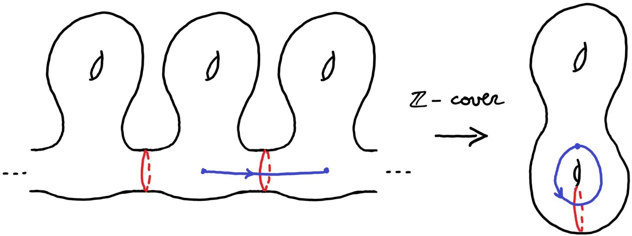

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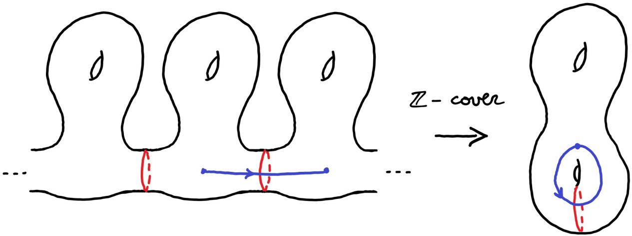

- We equip S_0 with the pullback Riemannian metric and measure μ.
- We consider the geodesic and horocycle flows $(\mathrm{g}_t)_{t\in\mathbb{R}}$ and $(\mathrm{h}_t)_{t\in\mathbb{R}}$ on $M_0 = T^1 S_0$.

Ergodic integrals

Some general facts about the ergodic integrals of integrable functions

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$$\limsup_{t\to\infty} \frac{\mathcal{I}_t f}{a(t)} = \infty \quad \mu\text{-a.e.,} \qquad \text{or} \qquad \liminf_{t\to\infty} \frac{\mathcal{I}_t f}{a(t)} = 0 \quad \mu\text{-a.e.,}$$

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- Hopf '37: for a positive integrable g,

$$\lim_{t \to \infty} \frac{\mathcal{I}_t f}{\mathcal{I}_t g} = \frac{\mu(f)}{\mu(g)} \quad \mu\text{-a.e.}$$

Ergodic integrals

Our goal: describe the ergodic integrals μ-a.e. as

$$\mathcal{I}_t f(x) = a(t) \cdot \mu(f) \cdot \Phi_t(x) \cdot (1 + o(1)).$$

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where

- $a(t)$ is the "correct" asymptotic rate and
- $\Phi_t(x)$ is an "oscillating" term, independent of f.

The main result

Theorem (Castorrini-R. '24)

There exist $C_M \geq 1$, $\delta > 0$, and

- a $d \times d$ positive definite symmetric matrix Σ,
- a function $t_* \colon M \times [C_M, \infty) \to \mathbb{R}_{\geq 0}$ satisfying

$$\| T - e^{h_{\mathrm{top}} \cdot t_*(\cdot, T)} \|_\infty \leq C_M T^{1-\delta},$$

- a vector-valued function $F_* \colon M \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ satisfying

$$\frac{F_*(x,t)}{\sqrt{t}} \to \mathcal{N}(0, \Sigma) \quad \text{in distribution, as } x \sim \mathrm{vol}$$

for which the following holds. (cont'd)

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Theorem (Castorrini-R. '24)

For every $f \in \mathscr{C}^2_c(M_0)$ and every $x \in M_0$, there exists a constant $C(f,x)$

depending (explicitly) of the $\mathscr{C}^2$-norm of f, its support $\mathrm{supp}(f)$, and the distance between x and $\mathrm{supp}(f)$,

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$$\Bigg| \mathcal{I}_T f(x) - \frac{h_{\mathrm{top}}^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} \sqrt{\det \Sigma}} \frac{T}{(\log T)^{\frac{d}{2}}} \mu(f) \times \times \exp\left(-\frac{1}{2} \frac{F_*(p(x), t_*) \cdot \Sigma F_*(p(x), t_*)}{t_*} \right) \Bigg| \leq C_M \, C(f,x) \frac{T \log\log T}{(\log T)^{\frac{d+1}{2}}}.$$

Remarks

- For hyperbolic surfaces, the result (without error rates) is due to Ledrappier-Sarig '06.

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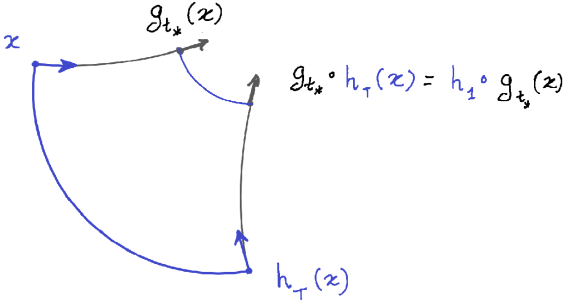
- Results of this type have been obtained for some (very special) classes of $\mathbb{Z}^d$-covers of translation flows, including Avila-Doglopyat-Duryev-Sarig '15 and Bruin-Fougeron-R.-Terhesiu '24.

Renormalization time

The function t_* is a renormalization time, namely is defined by the equality

$$\mathrm{g}_{t_*(x,T)} \circ \mathrm{h}_T(x) = \mathrm{h}_1 \circ \mathrm{g}_{t_*(x,T)}(x).$$



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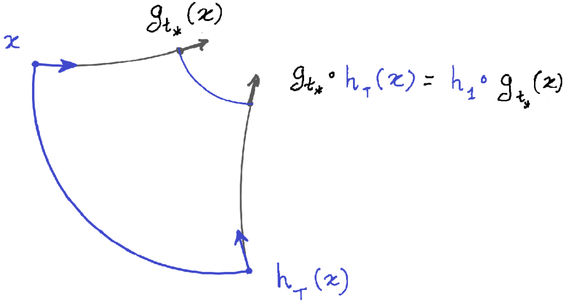

In particular, for hyperbolic surfaces, $t_*(x, T) = \log(T)$.

The Frobenius function

- Recall that there are d harmonic 1-forms $\omega_1, \dots, \omega_d$ on S so that their pullback $p^*\omega_1, \dots, p^*\omega_d$ are exact on S_0.

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- The Frobenius function/geodesic winding cycle $F_*(x,t)$ is the d-dimensional vector whose i-th component is

$$\int_x^{\mathrm{g}_t(x)} p^*\omega_i = \int_0^t \langle \omega_i, X \rangle \circ \mathrm{g}_r(x) \, \mathrm{d}r.$$

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- It is invariant by deck transformations, hence it is a well-defined function on M.

The proof: renormalization

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Commutation

There exists a function $\tau\colon M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{g}_t \circ \mathrm{h}_s(x) = \mathrm{h}_{\tau(s,t,p(x))} \circ \mathrm{g}_t(x),$$

for all $x \in M_0$ and $t, s \in \mathbb{R}$.

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For $t \geq 0$, it satisfies

$$C^{-1} e^{h_{\mathrm{top}} t} s \leq \tau(s, -t, x) \leq C e^{h_{\mathrm{top}} t} s,$$

for some constant $C \geq 1$.

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Weighted transfer operators

We are led to consider the operators $\mathcal{L}_t \colon f \mapsto f \circ \mathrm{g}_{-t} \cdot J_{-t}$ on $\mathscr{C}_c^{2-\varepsilon}(M_0)$.

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Then, every homomorphism $\chi \colon \mathrm{Deck} \to U(1)$ is of the form

$$\chi(D) = \chi([\gamma] + \Gamma_0) = \exp\left(2\pi i \int_{[\gamma]} \omega \right),$$

for a unique $\omega \in \mathbb{T}^d = \mathcal{H}/\mathcal{H}(\mathbb{Z})$.

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For $r < 2$, define

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We have

$$\pi_\omega \colon \mathscr{C}^r_c(M_0) \to \mathscr{C}^r(M,\omega), \qquad \pi_\omega(f) = \sum_{D \in \mathrm{Deck}} \chi_\omega(D) f \circ D$$

and

$$f = \int_{\mathbb{T}^d} \pi_\omega(f) \, \mathrm{d}\omega.$$

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Define

$$\Xi_\omega \colon \mathscr{C}^r(M, \eta+\omega) \to \mathscr{C}^r(M, \eta), \qquad \Xi_\omega(f)(x) = f(x) \exp\left(2\pi i \int_{x_0}^{x} p^* \omega\right)$$

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$$\Xi_\omega \colon \mathscr{C}^r(M, \eta + \omega) \to \mathscr{C}^r(M, \eta), \qquad \Xi_\omega(f)(x) = f(x) \exp\left(2\pi i \int_{x_0}^{x} p^*\omega\right)$$

We have

$$\begin{array}{ccc} \mathscr{C}^r(M,\omega) & \xrightarrow{\mathcal{L}_t} & \mathscr{C}^r(M,\omega) \\ \Big\uparrow \Xi_{-\omega} & & \Xi_\omega \Big\downarrow \\ \mathscr{C}^r(M,0) & & \mathscr{C}^r(M,0) \end{array}$$

The proof: a Fourier decomposition

Define

$$\Xi_\omega \colon \mathscr{C}^r(M,\eta+\omega) \to \mathscr{C}^r(M,\eta), \qquad \Xi_\omega(f)(x) = f(x)\exp\left(2\pi i \int_{x_0}^{x} p^*\omega\right)$$

We have

$$\begin{tikzcd} \mathscr{C}^r(M,\omega) \arrow[r, "\mathcal{L}_t"] & \mathscr{C}^r(M,\omega) \arrow[d, "\Xi_\omega"'] \\ \mathscr{C}^r(M,0) \arrow[u, "\Xi_{-\omega}"'] & \mathscr{C}^r(M,0) \end{tikzcd}$$

Twisted (and weighted) transfer operators

We study the family of operators $\mathcal{L}_t^{(\omega)} \colon \mathscr{C}^r(M) \to \mathscr{C}^r(M)$ given by

$$\mathcal{L}_t^{(\omega)} f = f \circ \mathrm{g}_{-t} \cdot J_{-t} \cdot \exp\left(2\pi i \int_0^t \langle \omega, X \rangle \circ \mathrm{g}_{-s}\, \mathrm{d}s\right)$$

The proof: spectral theory

The proof: spectral theory

We consider a pair of Banach spaces $\mathcal{B}_w, \mathcal{B}$ with dense inclusions

$$\mathscr{C}^{2-\varepsilon}(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow \mathscr{C}^{2-\varepsilon}(M)^*,$$

The proof: spectral theory

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$$\mathscr{C}^{2-\varepsilon}(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow \mathscr{C}^{2-\varepsilon}(M)^*,$$

on which the linear functionals

$$L_{x,\varphi} \colon f \mapsto \int_0^1 f \circ \mathrm{h}_s(x) \cdot \varphi(s) \, \mathrm{d}s, \qquad \text{for } x \in M \text{ and } \varphi \in \mathscr{C}_c^{2-\varepsilon}((0,1)),$$

defined for $f \in \mathscr{C}^{2-\varepsilon}(M)$, extend by continuity.

The proof: spectral theory

We prove that there exists $\delta > 0$ such that, for all $v \in \mathscr{C}^{2-\varepsilon}(M)$, we have

$$e^{-h_{\mathrm{top}} t} \mathcal{L}_t^{(\omega)} v = e^{z(\omega)t} \Pi_\omega v + Q_{\omega,t} v,$$

where

The proof: spectral theory

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- $z(\omega) \leq 0$ and $z(\omega) = 0$ if and only if $\omega = 0$,

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where

- Π_ω is a projection,
- $\| Q_{\omega,t} v \|_{\mathcal{B}_w} \leq e^{-\delta t} \| v \|_{\mathscr{C}^{2-\varepsilon}}$,

and

- $z(\omega) \leq 0$ and $z(\omega) = 0$ if and only if $\omega = 0$,
- for all $\omega \in B(0,\delta) \subseteq \mathbb{T}^d$, we have

$$z(\omega) = -4\pi^2 \omega \cdot \Sigma \omega + O(|\omega|^3).$$

The proof: combining the pieces

The proof: combining the pieces

- Step 1: Fourier decomposition

$$\mathcal{I}_T f(x) = \int_{\mathbb{T}^d} \int_0^T \pi_\omega(f) \circ \mathrm{h}_s(x) \, \mathrm{d}s \, \mathrm{d}\omega$$
$$= \int_{\mathbb{T}^d} \int_0^T \Xi_{-\omega} \circ \pi_0(f) \circ \Xi_\omega \circ \mathrm{h}_s(x) \, \mathrm{d}s \, \mathrm{d}\omega.$$

The proof: combining the pieces

- Step 1: Fourier decomposition

$$\mathcal{I}_T f(x) = \int_{\mathbb{T}^d} \int_0^T \Xi_{-\omega}(f_\omega) \circ \mathrm{h}_s(x) \, \mathrm{d}s \, \mathrm{d}\omega.$$

- Step 2: renormalization

The proof: combining the pieces

- Step 1: Fourier decomposition

$$\mathcal{I}_T f(x) = \int_{\mathbb{T}^d} \int_0^T \Xi_{-\omega}(f_\omega) \circ \mathrm{h}_s(x) \,\mathrm{d}s\,\mathrm{d}\omega.$$

- Step 2: renormalization

$$\begin{aligned}\mathcal{I}_T f(x) &= \int_{\mathbb{T}^d} \int_0^1 \mathcal{L}_{t_*}[\Xi_{-\omega}(f_\omega)] \circ \mathrm{h}_s(x) \,\mathrm{d}s\,\mathrm{d}\omega \\ &= \int_{\mathbb{T}^d} \int_0^1 \Xi_{-\omega} \circ \mathcal{L}_{t_*}^{(\omega)}(f_\omega) \circ \mathrm{h}_s(\mathrm{g}_{t_*}(x)) \,\mathrm{d}s\,\mathrm{d}\omega \\ &\approx \int_{\mathbb{T}^d} e^{-2\pi i \int_x^{\mathrm{g}_{t_*}(x)} p^*\omega} L_{\mathrm{g}_{t_*}(x),\varphi}\left[\mathcal{L}_{t_*}^{(\omega)}(f_\omega)\right] \mathrm{d}\omega\end{aligned}$$

The proof: combining the pieces

- Step 1: Fourier decomposition

$$\mathcal{I}_T f(x) = \int_{\mathbb{T}^d} \int_0^T \Xi_{-\omega}(f_\omega) \circ \mathrm{h}_s(x) \, \mathrm{d}s \, \mathrm{d}\omega.$$

- Step 2: renormalization

$$\mathcal{I}_T f(x) \approx \int_{\mathbb{T}^d} e^{-2\pi i \int_x^{\mathrm{g}_{t_*}(x)} p^* \omega} L_{\mathrm{g}_{t_*}(x), \varphi} \left[\mathcal{L}_{t_*}^{(\omega)}(f_\omega) \right] \mathrm{d}\omega$$

- Step 3: spectral theory

The proof: combining the pieces

- Step 1: Fourier decomposition

$$\mathcal{I}_T f(x) = \int_{\mathbb{T}^d} \int_0^T \Xi_{-\omega}(f_\omega) \circ \mathrm{h}_s(x) \, \mathrm{d}s \, \mathrm{d}\omega.$$

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$$\mathcal{I}_T f(x) \approx \int_{\mathbb{T}^d} e^{-2\pi i \int_x^{\mathrm{g}_{t_*}(x)} p^* \omega} L_{\mathrm{g}_{t_*}(x),\varphi} \left[\mathcal{L}_{t_*}^{(\omega)}(f_\omega) \right] \mathrm{d}\omega$$

- Step 3: spectral theory

$$\begin{aligned} \mathcal{I}_T f(x) &\approx \int_{\mathbb{T}^d} e^{-2\pi i \int_x^{\mathrm{g}_{t_*}(x)} p^* \omega} e^{h_{\mathrm{top}} t_*} L_{\mathrm{g}_{t_*}(x),\varphi} \left[e^{z(\omega) t} \Pi_\omega f_\omega + Q_{\omega,t} f_\omega \right] \mathrm{d}\omega \\ &\approx \int_{B(0,\delta)} e^{-2\pi i \int_x^{\mathrm{g}_{t_*}(x)} p^* \omega} e^{h_{\mathrm{top}} t_*} e^{-4\pi^2 \omega \cdot \Sigma \omega t_*} L_{\mathrm{g}_{t_*}(x),\varphi} [\Pi_\omega f_\omega] \, \mathrm{d}\omega \end{aligned}$$

The proof: combining the pieces

- Step 1: Fourier decomposition

$$\mathcal{I}_T f(x) = \int_{\mathbb{T}^d} \int_0^T \Xi_{-\omega}(f_\omega) \circ \mathrm{h}_s(x) \,\mathrm{d}s \,\mathrm{d}\omega.$$

- Step 2: renormalization

$$\mathcal{I}_T f(x) \approx \int_{\mathbb{T}^d} e^{-2\pi i \int_x^{\mathrm{g}_{t_*}(x)} p^*\omega} L_{\mathrm{g}_{t_*}(x),\varphi} \left[\mathcal{L}_{t_*}^{(\omega)}(f_\omega) \right] \mathrm{d}\omega$$

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$$\mathcal{I}_T f(x) \approx \int_{B(0,\delta)} e^{-2\pi i \int_x^{\mathrm{g}_{t_*}(x)} p^*\omega} e^{h_{\mathrm{top}} t_*} e^{-4\pi^2 \omega \cdot \Sigma \omega t_*} L_{\mathrm{g}_{t_*}(x),\varphi} [\Pi_\omega f_\omega] \,\mathrm{d}\omega$$

- Step 4: computations of exponential integrals ("stationary phase"-type estimates)

Thank you for your attention.

Happy birthday, Giovanni!