Homogeneous Dynamics

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Welcome to the *Homogeneous Dynamics* course!

These lectures are intended to be an introduction to homogeneous dynamics, which nowadays is a very active subject of research. Homogeneous dynamics lies at the intersection of many areas in pure mathematics: of course, dynamics and ergodic theory, but also geometry, Lie group theory, representation theory, and more. There are also remarkable connections to several problems in number theory, some of which will be explored during the course.

The literature in the subject is vast and it would be impossible to cover it all. The choice I made to select the specific topics which will be discussed during these lectures was motivated mainly by two reasons. In part, of course, there are my personal preferences; more importantly, I wanted to focus on concrete examples (where computations can be carried out explicitly) which can help to build the intuition and provide insights on more general and abstract situations. It is my hope that this introduction can sparkle the curiosity in students to pursue this line of research.

One final disclaimer before starting: these lecture notes are a work-in-progress, and as such they need to be read with critical thinking. I tried to minimize the number of errors, but it would be widely optimistic of me to believe that there are none. If you spot mistakes, or have any comment in general, please let me know by sending me an email to `davide.ravotti@gmail.com`.

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Chapter 1

A quick recap: the case of linear flows on tori

In this first chapter, we will quickly review some basic notions in dynamics and ergodic theory, which the reader is assumed to be already familiar with. An exhaustive treatment of these topics can be found, for example, in [4, Chapters 2, 4.3].

In parallel, we will look at linear flows on tori. Very roughly speaking, the course consists in studying their non-Abelian analogues, as we will see later. Thus, focusing on this simple case can be a nice "warm-up" exercise.

1.1 Smooth flows on manifolds

The subject of this course is a special class of smooth flows. Let us recall the general definition.

Definition 1.1. *Let M be a smooth manifold, and let* $\mathrm{Diff}(M)$ *be the group of its diffeomorphisms. A* smooth flow $\varphi\colon \mathbb{R} \times M \to M$ *is a smooth map which satisfies*

$$\varphi_0 = \mathrm{Id}, \quad \textit{and} \quad \varphi_{t+s} = \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t, \quad \textit{for all } t,s \in \mathbb{R},$$

where $\varphi_t := \varphi(t,\cdot) \in \mathrm{Diff}(M)$.

In particular, Definition 1.1 implies that the continuous curve $t \mapsto \varphi_t$ is a group homomorphism between $\mathbb{R}$ and $\mathrm{Diff}(M)$, and $\{\varphi_t\}_{t\in\mathbb{R}}$ is said to be a *1-parameter group* of diffeomorphisms. We will often identify φ with $\{\varphi_t\}_{t\in\mathbb{R}}$.

Given a smooth flow φ, we can define a vector field X on M by

$$Xf(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \varphi_t(p), \quad \text{for all } f \in \mathscr{C}^\infty(M) \text{ and } p \in M.$$

The vector field X is called the *infinitesimal generator* of φ. Vice-versa, one can prove that, at least when M is compact, for any given smooth vector field X, there exists a unique smooth flow φ with infinitesimal generator X.

From here onward, M always denotes a smooth manifold, not necessarily compact, and φ is a smooth flow on M.

Let us turn to a very concrete example. Let $\mathbb{T}^n$ be the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. We will denote points in $\mathbb{T}^n$ using the symbol $[\![\cdot]\!]$, namely $[\![\mathbf{x}]\!] := \mathbf{x} + \mathbb{Z}^n$. For any $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$, we define the *linear flow in direction* $\mathbf{v}$ to be the smooth flow $\varphi^{\mathbf{v}}$ on $\mathbb{T}^n$ given by

$$\varphi^{\mathbf{v}}_t([\![\mathbf{x}]\!]) = [\![\mathbf{x} + t\mathbf{v}]\!], \quad \text{for } t \in \mathbb{R}.$$

It is easy to check that indeed $\varphi^{\mathbf{v}}$ is a well-defined smooth flow according to Definition 1.1. The associated infinitesimal generator X is the derivative in direction $\mathbf{v}$: for any $p = [\![\mathbf{x}]\!] \in \mathbb{T}^n$,

$$Xf(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f([\![\mathbf{x}+t\mathbf{v}]\!]) = \mathbf{v}\cdot\nabla_{[\![x]\!]} f = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}([\![x]\!]).$$

In other words, under the usual identification of the tangent space $T_p\mathbb{T}^n$ at p with $\mathbb{R}^n$, we have $X = \mathbf{v}$. The associated 1-parameter subgroup consists of the translations $\varphi_t^{\mathbf{v}} \colon [\![\mathbf{x}]\!] \mapsto [\![\mathbf{x}+t\mathbf{v}]\!]$ in direction $\mathbf{v}$.

Let us rephrase the example above in more algebraic terms. Our setting was the following. We considered the Abelian group $(\mathbb{R}^n, +)$, and we fixed a 1-dimensional subgroup $V = \{t\mathbf{v} : t \in \mathbb{R}\} < \mathbb{R}^n$. This subgroup V is everywhere tangent to the constant vector field $\mathbf{v} \in \mathbb{R}^n$, where we identified $\mathbb{R}^n = T_{\mathbf{x}}\mathbb{R}^n$ for all $\mathbf{x} \in \mathbb{R}^n$. In turn, V is identified with the 1-parameter group of translations

$$\{(\mathbf{x} \mapsto \mathbf{x}+t\mathbf{v}) : t \in \mathbb{R}\} \subset \mathrm{Diff}(\mathbb{R}^n).$$

We then fixed the discrete subgroup $\mathbb{Z}^n < \mathbb{R}^n$ and we considered the quotient space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The key observation is that the 1-parameter group of translations $\mathbf{x} \mapsto \mathbf{x}+t\mathbf{v}$ associated to $\mathbf{v}$ *descends to the quotient*, which means that they commute with the canonical projection $\mathbf{x} \mapsto [\![\mathbf{x}]\!] = \mathbf{x}+\mathbb{Z}^n$. This tells us that, under the projection, we obtain a well-defined 1-parameter group of diffeomorphisms of $\mathbb{T}^n$, and hence a smooth flow $\varphi^{\mathbf{v}}$.

Homogeneous flows, which are the subject of this course, are a "non-Abelian" generalization of this simple example. Namely, we will replace

- $\mathbb{R}^n$ with a Lie group G (the Heisenberg group in Chapter 3 and $\mathrm{SL}(2,\mathbb{R})$ in Chapters 4–7),
- $\mathbb{Z}^n$ with a *lattice* Γ (a discrete subgroup of G with some additional properties that we will discuss in §2.4),
- $\mathbb{T}^n$ with the *left*[1] quotient $\Gamma\backslash G = \{\Gamma g : g \in G\}$,
- $V = \{t\mathbf{v} : t \in \mathbb{R}\}$ with a 1-parameter subgroup $\{g_t : t \in \mathbb{R}\}$ of G (generated by a "constant" vector field, which in the case above was $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$),
- $\varphi_t^{\mathbf{v}} \colon [\![\mathbf{x}]\!] \mapsto [\![\mathbf{x}+t\mathbf{v}]\!]$ with the multiplication *on the right*[2] $\Gamma g \mapsto \Gamma g \cdot g_t$.

We will make this analogy precise in the next chapters.

1.2 The topology of orbits

Let $\varphi \colon \mathbb{R} \times M \to M$ be a smooth flow, and let $p \in M$. The *orbit* of p is the set

$$\mathcal{O}_\varphi(p) = \{\varphi_t(p) : t \in \mathbb{R}\} \subset M.$$

Note that the orbit of any point $p \in M$ is an *immersed* smooth curve in M.

In dynamics, one is interested in the behaviour of orbits: do they "close up"? Do they accumulate in some regions? Do they visit all parts of the space? From the topological point of view, it is particularly important to try to understand their accumulation points and closure $\overline{\mathcal{O}_\varphi(p)} \subseteq M$.

[1] Note that, in the Abelian case, left and right cosets coincide.

[2] Taking left quotients and multiplying on the right is the conventional choice, but of course one could do the opposite (taking right quotients and multiplying on the left). Note that, again, multiplying on the right and projecting on the left quotient $\Gamma\backslash G$ commute.

Definition 1.2. *A point p is a* fixed point *if* $\mathcal{O}_\varphi(p) = \{p\}$. *A point p is* periodic *if there exists $T > 0$ such that*

$$\varphi_T(p) = p. \tag{1.1}$$

If p is periodic but not a fixed point, its period *is the smallest $T > 0$ for which* (1.1) *holds.*

Exercise 1.3. (a) *Show that the set of $T \in \mathbb{R}$ for which* (1.1) *holds is a subgroup of $\mathbb{R}$, in particular if p is periodic but not a fixed point, its period is well-defined.*

(b) *Show that, if p is a periodic point of period T, then its orbit is an embedded closed curve and*

$$\mathcal{O}_\varphi(p) = \{\varphi_t(p) : t \in [0,T]\}.$$

Periodic and fixed points have the smallest possible orbit closures, since their orbits are themselves closed. On the opposite, we may have points with *dense* orbits, that is, points whose orbit closure is the largest possible.

Definition 1.4. *A smooth flow φ is* minimal *is all orbits are dense, namely if*

$$\overline{\mathcal{O}_\varphi(p)} = M, \quad \text{for all } p \in M.$$

Let us look at our motivating example. In the case of linear flows on the two dimensional torus, we have a pleasant dichotomy.

Theorem 1.5. *Let $\varphi^{\mathbf{v}} \colon \mathbb{T}^2 \to \mathbb{T}^2$ be a linear flow in direction $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 \setminus \{0\}$. If v_1 and v_2 are* rationally dependent*, then every orbit is periodic; otherwise, if v_1 and v_2 are* rationally independent*, the flow $\varphi^{\mathbf{v}}$ is minimal.*

We will say that $\varphi^{\mathbf{v}}$ is a *rational* linear flow if we are in the first case, and it is an *irrational* linear flow if we are in the second one.

Before diving into the proof of Theorem 1.5, let us make a couple of simple observations. First, note that a rescaling $a\mathbf{v}$ of $\mathbf{v}$ for some $a > 0$ does not change the behaviour of the orbits of the flow. If $v_2 = 0$, then $v_1 \neq 0$. It is clear that all orbits of $\varphi^{\mathbf{v}}$ are periodic of period $1/v_1$ and consist of horizontal cirlces of the form $\mathbb{T}^1 \times \{p_2\}$, with $p_2 \in \mathbb{T}^1$, hence the result is proved in this case. If $v_2 \neq 0$, then, without loss of generality, we can assume that $\mathbf{v} = (v, 1)$. We divide the proof of Theorem 1.5 into two cases: when $v \in \mathbb{Q}$ (the rational case) and when $v \notin \mathbb{Q}$ (the irrational case).

Proof of Theorem 1.5 - *Case $v \in \mathbb{Q}$.* Let us write $v = a/b$ in reduced terms. Then, we claim that all orbits are periodic of period b. Indeed, let $p = [\![x_1, x_2]\!] \in \mathbb{T}^2$. Then,

$$\varphi^{\mathbf{v}}_b(p) = [\![x_1 + b \cdot a/b, x_2 + b]\!] = [\![x_1, x_2]\!] + [\![a, b]\!] = p.$$

If $T > 0$ is such that $\varphi^{\mathbf{v}}_T(p) = p$, then, looking at its second coordinate, we see that $x_2 + T + \mathbb{Z} = x_2 + \mathbb{Z}$. Hence $T \in \mathbb{N}$, and, looking at the first coordinate, $x_1 + T \cdot a/b + \mathbb{Z} = x_1 + \mathbb{Z}$. This implies that $(Ta)/b \in \mathbb{Z}$. Since a and b are coprime by assumption, b divides T. This proves the claim and hence the theorem in the rational case. $\square$

Proof of Theorem 1.5 - *Case $v \notin \mathbb{Q}$.* We first claim that it is enough to prove the following statement.

($\star$) The *circle rotation* $R_v \colon \mathbb{T}^1 \to \mathbb{T}^1$ defined by $R_v([\![x]\!]) = [\![x + v]\!]$ is *minimal* (where, here, $[\![x]\!] = x + \mathbb{Z}$).

We leave as an exercise to the reader to check that indeed it is sufficient to prove $(\star)$. The idea is that the orbit of a point $p = [\![x_1, x_2]\!]$ under the flow $\varphi^{\mathbf{v}}$ is dense in $\mathbb{T}^2$ if and only if its intersection with the horizontal circle $\mathbb{T}^1 \times \{[\![x_2]\!]\}$ is dense in $\mathbb{T}^1 \times \{[\![x_2]\!]\}$. Indeed, the projection on the first coordinate of the intersection of the orbit of p with the circle $\mathbb{T}^1 \times \{[\![x_2]\!]\}$ is precisely the orbit of $[\![x_1]\!]$ under the rotation R_v.

We now focus on proving $(\star)$. Let $p = [\![x]\!] = x + \mathbb{Z} \in \mathbb{T}^1$ and $\varepsilon > 0$ be fixed; choose a natural number $N \geq \varepsilon^{-1}$ and partition $\mathbb{T}^1 \approx [0,1)$ into N intervals $I_k = [(k-1)N^{-1}, kN^{-1})$ for $k = 1, \dots, N$. We need to show that the orbit of p visits all intervals I_k.

Let us consider the set $O_N = \{p, R_v(p), \dots, R_v^N(p)\}$. Since $|O_N| = N+1$, by the Pigeonhole Principle, there exists a $\bar{k} \in \{1, \dots, N\}$ such that the interval $I_{\bar{k}}$ contains at least two distinct elements of O_N, say $R_v^n(p)$ and $R_v^m(p)$, with $n < m$. Let us call w the fractional part of $(m-n)v$. For any $y \in [0,1)$, we have

$$R_v^{m-n}([\![y]\!]) = [\![y + (m-n)v]\!] = [\![y + w]\!] = R_w([\![y]\!]),$$

namely, the map R_v^{m-n} is again a rotation of angle $w \in (0,1)$. Since we showed that the points $p' = R_v^n(p)$ and $R_w(p') = R_v^m(p)$ are both in the same interval $I_{\bar{k}}$, they are at distance less than N^{-1}. It follows that $0 < w < N^{-1} \leq \varepsilon$. Thus, the orbit of p under R_v contains the orbit of p under $R_v^{m-n} = R_w$, which is a rotation of angle less than ε. Since this latter set clearly intersects all intervals I_k, the proof is complete. □

In general, it is a hopeless task to try to understand all orbit closures. They can be quite complicated objects, with “fractal-like” structures and non-integer dimensions. However, in the particular case of linear flows on $\mathbb{T}^2$, orbit closures are well-behaved and we managed to classify all possibilities: we showed that all orbit closures are either the whole space $\mathbb{T}^2$ or circles isomorphic to $\mathbb{T}^1$. In higher dimensions, a similar phenomenon occurs: orbit closures of any linear flow on $\mathbb{T}^n$ are sub-tori isomorphic to $\mathbb{T}^k$, for some $k = 1, \dots, n$ (see Section 1.3.4 below).

1.3 Elements of Ergodic Theory

Ergodic theory is the study of dynamical systems from the point of view of measure theory. The measures on the phase space M that will be relevant for us are Borel invariant measures.

1.3.1 Invariant measures

Definition 1.6. *Let φ be a smooth flow on M. A Borel measure μ on M is an* invariant measure *for φ if for all Borel measurable sets $A \subset M$ and for all $t \in \mathbb{R}$,*

$$\mu(\varphi_t(A)) = \mu(A).$$

If $\mu(M) = 1$, then μ is a probability invariant measure. *The triple (M, φ, μ) is called a* probability preserving flow *(*ppf*, for short).*

The previous definition extends to all functions in $L^1(M) = L^1(M, \mu)$: if (M, φ, μ) is a ppf, then, for every function $f \in L^1(M)$ and for all $t \in \mathbb{R}$, the function $f \circ \varphi_t$ is in $L^1(M)$ and

$$\int_M f \circ \varphi_t \, \mathrm{d}\mu = \int_M f \, \mathrm{d}\mu.$$

Similarly, if $f \in L^2(M)$, then $f \circ \varphi_t \in L^2(M)$ for all $t \in \mathbb{R}$ and

$$\|f \circ \varphi_t\|_2 = \|f\|_2 . \tag{1.2}$$

Let us see some examples of invariant measures. Clearly, the Lebesgue measure on the torus $\mathbb{T}^2$ is an invariant measure for all linear flows $\varphi^{\mathbf{v}}$. If the flow is irrational, we will see in Section 1.3.3 that there are no other invariant probability measures. However, if $\varphi^{\mathbf{v}}$ is rational, then we have uncountably many invariant probability measures supported on periodic orbits. This is a general fact: for any periodic orbit, there is an invariant probability measure supported on such orbit.

Exercise 1.7. (a) *Let $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 \setminus \{0\}$, with v_1, v_2 rationally dependent. Let T be the period of all orbits of $\varphi^{\mathbf{v}}$. Show that*

$$T = \frac{\min\{\|\mathbf{w}\|_2 : \mathbf{w} \in \mathbb{R}\mathbf{v} \cap \mathbb{Z}^2,\ \mathbf{w} \neq 0\}}{\|\mathbf{v}\|_2}.$$

(b) *For any $p \in M$, let μ_p be the Borel measure defined by*

$$\mu_p(A) := \frac{1}{T}\,\mathrm{Leb}\{t \in [0,T] : \varphi^{\mathbf{v}}_t(p) \in A\}.$$

Show that μ_p is a probability invariant measure for $\varphi^{\mathbf{v}}$.

(c) *Prove that $\mu_p = \mu_q$ if and only if $\mathcal{O}_{\varphi^{\mathbf{v}}}(p) = \mathcal{O}_{\varphi^{\mathbf{v}}}(q)$. Deduce that $\varphi^{\mathbf{v}}$ has uncountably many probability invariant measures.*

It is actually easy to see that if there is more than one probability invariant measure, then there are uncountably many. Indeed, any convex combination of (probability) invariant measures is a (probability) invariant measure. In other words, probability invariant measures form a *simplex* in the space of probability measures on M.

The reader might wonder whether we are sure to find, in general, at least one probability invariant measure. When M is compact, the following result answers this question affirmatively.

Theorem 1.8 (Krylov-Bogolyubov)**.** *Let φ be a smooth flow on the compact manifold M. There exists one invariant probability measure.*

Proof. Recall that, when M is compact, the set of Borel (signed) measures coincides with $\mathscr{C}(M)^*$, the weak-$*$ dual of $\mathscr{C}(M)$. Recall also that, by Banach-Alaoglu's Theorem, the unit ball in $\mathscr{C}(M)^*$, which contains all (positive) probability measures, is weakly-$*$ compact. Fix any $p \in M$, and consider the family of (positive) probability measures $\{\mu_T\}_{T\in\mathbb{R}}$ given by

$$\mu_T(f) := \frac{1}{T}\int_0^T f \circ \varphi_t(p)\,\mathrm{d}t, \qquad \text{for } f \in \mathscr{C}(M).$$

By compactness, there exists an increasing sequence $T_n \to \infty$ such that μ_{T_n} weakly-$*$ converges to a (positive) probability measure μ. We claim that μ is invariant. Let $f \in \mathscr{C}(M)$ and $r \in \mathbb{R}$; then,

$$\begin{aligned}
|\mu_{T_n}(f\circ\varphi_r) - \mu_{T_n}(f)| &= \frac{1}{T_n}\left| \int_0^{T_n} f\circ\varphi_{t+r}(p)\,\mathrm{d}t - \int_0^{T_n} f\circ\varphi_t(p)\,\mathrm{d}t \right| \\
&= \frac{1}{T_n}\left| \int_{T_n}^{T_n+r} f\circ\varphi_t(p)\,\mathrm{d}t - \int_0^{r} f\circ\varphi_t(p)\,\mathrm{d}t \right| \\
&\le \frac{2r\,\|f\|_{\mathscr{C}(M)}}{T_n} \to 0.
\end{aligned}$$

Therefore,

$$0 = \lim_{n\to\infty} |\mu_{T_n}(f\circ\varphi_r) - \mu_{T_n}(f)| = |\mu(f\circ\varphi_r) - \mu(f)|,$$

which shows that μ is an invariant measure for φ. □

We will mostly be concerned with *smooth* invariant measures, namely measures given by integrating a volume form on M. In this case, we can check whether a smooth measure is invariant by computing its Lie derivative with respect to the infinitesimal generator of the flow.

Proposition 1.9. *Let φ be a smooth flow with infinitesimal generator X, and let μ be a smooth probability measure given by a volume form ω on M. Then μ is invariant if and only if $\mathscr{L}_X(\omega) = 0$, where $\mathscr{L}_X(\omega) = \mathrm{d}(i_X\omega)$ is the Lie derivative of ω with respect to X and i is the contraction operator.*

Proof. Let $(\varphi_t)^*$ denote the pull-back by φ_t. By definition of the Lie derivative,

$$\mathscr{L}_X(\omega) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_t)^*(\omega),$$

hence $(\varphi_t)^*(\omega) = \omega$ if and only if $\mathscr{L}_X(\omega) = 0$. By Cartan's formula,

$$\mathscr{L}_X(\omega) = \mathrm{d}(i_X\omega) + i_X(\mathrm{d}\omega) = \mathrm{d}(i_X\omega),$$

which follows from the fact that $\mathrm{d}\omega = 0$ since ω is a n-form, where $n = \dim(M)$. □

Exercise 1.10 (Invariant measures of time-changes)**.** *Let φ be a smooth flow on M with infinitesimal generator X, and let μ be a smooth probability invariant measure. Show that, for any smooth positive function $\alpha\colon M \to \mathbb{R}_{>0}$, the flow*[3] *generated by the vector field αX preserves the measure equivalent to μ with density $1/\alpha$.*

Once we have chosen a probability invariant measure, we can ask about the properties of *typical* points, in other words the properties that are satisfied up to exceptional sets of measure zero. A fundamental result is the recurrence theorem by Poincaré, which, roughly speaking, says that typical orbits will come back close to their initial point infinitely often.

Theorem 1.11 (Poincaré Recurrence Theorem)**.** *Let (M,φ,μ) be a ppf. If $A \subset M$ is a measurable (Borel) set, for* almost every *$p \in A$ there exists an increasing sequence $T_n \to \infty$ such that $\varphi_{T_n}(p) \in A$.*

1.3.2 Ergodicity and the Ergodic Theorems

Given a flow φ on M, we say that a measurable set $A \subset M$ is *invariant* if $\varphi_t(A) = A$ for all $t \in \mathbb{R}$. If (M,φ,μ) is a ppf and $A \subset M$ is an invariant set of positive measure, then we can consider the subsystem (A,φ,μ_A) given by the restriction of the flow φ to A with the conditional probability invariant measure defined by

$$\mu_A(B) := \mu(B \cap A)/\mu(A), \qquad \text{for any measurable set } B.$$

When we have an invariant set of positive measure, we can then reduce ourselves to study a "simpler" system. Intuitivley, the notion of ergodicity plays the role of "indecomposability" in the context of ppf's. That is to say, an ergodic ppf cannot be decomposed into non-trivial invariant subsystems.

Definition 1.12. *Let (M,φ,μ) be a ppf. We say that μ is* ergodic*, or that (M,φ,μ) is an ergodic flow*[4] *if for every invariant measurable set $A \subset M$ we have $\mu(A) = 0$ or $\mu(A) = 1$.*

We recall the following characterization of ergodicity.

[3] This flow is called the *time-change* generated by α.

[4] Sometimes, by a little abuse of notation, when the reference measure μ is clear from the context, we will say that φ is ergodic.

Proposition 1.13. *Let (M, φ, μ) be a ppf. The following are equivalent:*

1. *μ is ergodic,*
2. *for every measurable set $A \subset M$ such that $\mu(\varphi_t(A) \triangle A) = 0$ for all $t \in \mathbb{R}$, then $\mu(A) = 0$ or $\mu(A) = 1$,*
3. *if $f \colon M \to \mathbb{C}$ is a measurable function such that $f \circ \varphi_t = f$ almost everywhere for all $t \in \mathbb{R}$, then there exists $c \in \mathbb{C}$ such that $f = c$ almost everywhere,*
4. *if $f \in L^2(M)$ is an* invariant function*, namely if $f \circ \varphi_t = f$ in L^2 for all $t \in \mathbb{R}$, then there exists $c \in \mathbb{C}$ such that $f = c$ in L^2.*

Let us go back once more to the case of linear flows on tori and let us consider the ppf $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$. It is easy to see that, if the flow $\varphi^{\mathbf{v}}$ is rational, then it is *not* ergodic. Indeed, any set of the form

$$A_r = \bigcup \{ \mathcal{O}_{\varphi^{\mathbf{v}}}(p) : p = [\![x_1, 0]\!] \in \mathbb{T}^2 \text{ with } 0 \leq x_1 \leq r \}$$

is an invariant set of with $\mathrm{Leb}(A_r) = r$. Choosing $r \in (0,1)$ appropriately gives an example of a non-trivial invariant set, thus disproving ergodicity.

Exercise 1.14. (a) *Show that the measures μ_p of Exercise* 1.7 *are ergodic.*

(b) *Show that any non-trivial convex combination of μ_p and μ_q, for p and q on different orbits, is not ergodic.*

(c*) *Finally, show that if μ is an ergodic invariant probability measure, then $\mu = \mu_p$ for some $p \in \mathbb{T}^2$.*

On the other hand, the Lebesgue measure is ergodic when the flow $\varphi^{\mathbf{v}}$ is irrational. There are several ways of proving this fact, here we see a proof that uses Fourier analysis.

Theorem 1.15. *Let $\varphi^{\mathbf{v}}$ be an irrational linear flow on $\mathbb{T}^2$. Then, the Lebesgue measure* Leb *is ergodic.*

Proof. We denote by $\cdot$ the scalar product in $\mathbb{R}^2$. For any $f \in L^2(\mathbb{T}^2)$, we can write a Fourier expansion

$$f([\![\mathbf{x}]\!]) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}}, \quad \text{with} \sum_{\mathbf{n} \in \mathbb{Z}^2} |f_{\mathbf{n}}| = \|f\|_2^2.$$

Assume that f is an invariant function, that is assume that $f \circ \varphi_t^{\mathbf{v}} = f$ for all $t \in \mathbb{R}$, where the equality holds in $L^2(\mathbb{T}^2)$. We want to show it is constant in L^2. For all $\mathbf{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}$ we have

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}} = f([\![\mathbf{x}]\!]) = f([\![\mathbf{x} + t\mathbf{v}]\!]) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i \mathbf{n} \cdot (\mathbf{x} + t\mathbf{v})} = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}}.$$

By uniqueness of the coefficients, we must have

$$f_{\mathbf{n}} = f_{\mathbf{n}} e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} \quad \text{for all } \mathbf{n} \in \mathbb{Z}^2.$$

If $\mathbf{n} \neq 0$, then either $f_{\mathbf{n}} = 0$ or $e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} = 1$ for all $t \in \mathbb{R}$, and this latter condition is verified if and only if $\mathbf{n} \cdot \mathbf{v} = 0$. Since $\mathbf{v}$ has rationally independent coordinates, this second possibility cannot occur; hence we deduce $f_{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$. This proves that $f = f_0$ is equal to a constant in $L^2(\mathbb{T}^2)$, and thus completes the proof. □

Let (M, φ, μ) be an ergodic ppf. The ergodic theorems of Von Neumann and Birkhoff relate the *time averages* $\frac{1}{T} \int_0^T f \circ \varphi_t \, \mathrm{d}t$ of a measurable function $f \in L^2(M)$ (or $L^1(M)$) to the *space average* $\mu(f) = \int_M f \, \mathrm{d}\mu$.

Theorem 1.16 (Von Neumann Ergodic Theorem)**.** *Let (M,φ,μ) be a ppf. For every $f \in L^2(M)$, let $\mathrm{P}f \in L^2(M)$ be the projection of f onto the closed subspace of invariant functions. Then, the ergodic averages of f converge in $L^2(M)$ to $\mathrm{P}f$, namely*

$$\left\| \frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t - \mathrm{P}f(p) \right\|_2 \to 0.$$

In particular, if (M,φ,μ) is ergodic, $\mathrm{P}f = \mu(f)$ and hence

$$\frac{1}{T}\int_0^T f\circ\varphi_t\,\mathrm{d}t \to \mu(f) \quad \text{in } L^2(M).$$

Theorem 1.17 (Birkhoff Ergodic Theorem)**.** *Let (M,φ,μ) be a ppf. For every $f \in L^1(M)$, there exists $f^* \in L^1(M)$ with*

$$\mu(f) = \mu(f^*), \quad \text{and} \quad f^*\circ\varphi_t = f^* \quad \text{for all } t \in \mathbb{R},$$

where the latter equality holds in $L^1(M)$, such that

$$\frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t \to f^*(p),$$

for almost every *$p \in M$. If (M,φ,μ) is ergodic, then $f^*(p) = \mu(f)$ almost everywhere.*

1.3.3 Unique ergodicity

In Theorem 1.8, we saw that a smooth flow on a compact manifold M always has an invariant probability measure, and we also noticed that, if there is more than one, then there are uncountably many. The former case deserves a special name.

Definition 1.18. *Let φ be a smooth flow on a compact manifold M. If there exists only one invariant probability measure μ, the system (M,φ,μ) (or simply φ) is said to be* uniquely ergodic.

The reader might be wondering what the uniqueness of the invariant measure has to do with ergodicity. The following proposition shows that, in the case of a single invariant measure, ergodicity is automatically guaranteed.

Proposition 1.19. *Let φ be a smooth flow on a compact manifold M. The set of ergodic probability measures for φ coincides with the set of* extremal points[5] *of the simplex of invariant probability measures. In particular, if there exists a unique invariant probability measure μ, then it is ergodic.*

If (M,φ,μ) is uniquely ergodic, then, from the Ergodic Theorem, Theorem 1.17, we know that the ergodic averages of any L^1-function converge almost everywhere to its space average. On the other hand, one can show that, if the function is *continuous*, then the convergence is *uniform*.

Proposition 1.20. *Let φ be a smooth flow on a compact manifold M. The following are equivalent:*

1. *φ is uniquely ergodic,*
2. *there exists a unique ergodic invariant probability measure,*
3. *for every $f \in \mathscr{C}(M)$ there exists a constant C_f such that, for all $p \in M$,*

$$\frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t \to C_f, \tag{1.3}$$

[5] A point in a simplex is extremal if it cannot be expressed as a non-trivial convex combination of two other points.

4. *for every $f \in \mathscr{C}(M)$, the convergence in* (1.3) *is uniform over M.*

Under any of the assumptions above, the constant C_f in (1.3) *equals $\mu(f)$, where μ is the unique invariant probability measure.*

We have seen already that for rational linear flows $\varphi^{\mathbf{v}}$ on $\mathbb{T}^2$ there exist uncountably many invariant measures. Let us now see that in the other case, when the coordinates of $\mathbf{v}$ are rationally independent, the flow is uniquely ergodic.

Theorem 1.21. *Let $\varphi^{\mathbf{v}}$ be an irrational linear flow on $\mathbb{T}^2$. Then,* $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$ *is uniquely ergodic.*

Proof. Let $f \in \mathscr{C}(M)$ be fixed, and let us prove that the ergodic averages

$$A_T f(p) := \frac{1}{T} \int_0^T f \circ \varphi_t^{\mathbf{v}}(p) \,\mathrm{d}t$$

converge uniformly to $\mathrm{Leb}(f) = \int_{\mathbb{T}^2} f \,\mathrm{d}\mathrm{Leb}$. We claim that the family

$$\mathscr{A} := \{A_T f\}_{T>0} \subset \mathscr{C}(M).$$

is pre-compact in $\mathscr{C}(M)$, i.e., $\mathscr{A}$ has a compact closure. In order to do this, we check the assumptions of the Ascoli-Arzelà Theorem.

It is easy to see that $\mathscr{A}$ is equibounded: since $\|f \circ \varphi_t^{\mathbf{v}}\|_\infty = \|f\|_\infty$ for all $t \in \mathbb{R}$, it follows that, for any $T > 0$ and for all $p \in \mathbb{T}^2$, we have

$$|A_T f(p)| \leq \frac{1}{T} \int_0^T \|f\|_\infty \,\mathrm{d}t = \|f\|_\infty .$$

Let us verify that $\mathscr{A}$ is equicontinuous. We will use the fact that $\varphi_t^{\mathbf{v}}$ is an isometry for all $t \in \mathbb{R}$: if we denote by d the Euclidean distance on $\mathbb{T}^2$, we have that $d(\varphi_t^{\mathbf{v}}(p), \varphi_t^{\mathbf{v}}(q)) = d(p,q)$ for all $t \in \mathbb{R}$. With this in mind, let us fix $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(p) - f(q)| < \varepsilon$ whenever $d(p,q) < \delta$. Then, for any $T > 0$, if $p,q \in \mathbb{T}^2$ are such that $d(p,q) < \delta$, we get

$$|A_T(p) - A_T(q)| \leq \frac{1}{T} \int_0^T |f \circ \varphi_t^{\mathbf{v}}(p) - f \circ \varphi_t^{\mathbf{v}}(q)| \,\mathrm{d}t < \frac{1}{T} \int_0^T \varepsilon \,\mathrm{d}t = \varepsilon.$$

By the Ascoli-Arzelà Theorem, the closure of $\mathscr{A}$ is compact in $\mathscr{C}(M)$, in particular $\mathscr{A}$ has limit points. Let $T_n \to \infty$ and $g \in \mathscr{C}(M)$ be such that

$$A_{T_n} f \to g \quad \text{in } \mathscr{C}(M).$$

By Birkhoff Ergodic Theorem, Theorem 1.17, for almost every point p we have

$$A_{T_n} f(p) \to \int_{\mathbb{T}^2} f \,\mathrm{d}\mathrm{Leb},$$

therefore $g = \mathrm{Leb}(f)$ almost everywhere. Since g is continuous, the equality must hold everywhere.

We have showed that all limit points of $\mathscr{A}$ are the constant function $\mathrm{Leb}(f)$. Therefore, the limit point is unique and we conclude that the whole family converges in $\mathscr{C}(M)$, namely

$$A_T f = \frac{1}{T} \int_0^T f \circ \varphi_t^{\mathbf{v}} \,\mathrm{d}t \to \int_{\mathbb{T}^2} f \,\mathrm{d}\mathrm{Leb}$$

uniformly on $\mathbb{T}^2$, which concludes the proof. □

We remark that the proof of Theorem 1.21 works in a greater generality: any isometry of a compact space which has an ergodic measure with full support is uniquely ergodic.

Exercise 1.22. *Let* $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$ *be an irrational linear flow.*

(a) *Show that for any set* $A \subset \mathbb{T}^2$ with non-empty interior *there exists* $T_A > 0$ *such that* for all points $p \in M$ *there exists* $t \in [0, T_A]$ *such that* $\varphi_t^{\mathbf{v}}(p) \in A$ *(all points enter A before time* T_A*).*

(b*) *Provide a counterexample to* (a) *when we drop the assumption on A, namely give an example of a set* $A \subset \mathbb{T}^2$ with positive measure and empty interior *such that*

1. *almost every point enters A,*
2. *at least one point* $p \in M$ never *enters A,*
3. *for every* $T > 0$ *there exists a set* $B_T \subset \mathbb{T}^2$ of positive measure *such that all points in* B_T *do not enter A in the interval* $[0, T]$.

1.3.4 A glimpse at Ratner's Theorems

Let us summarize what we proved in the case of linear flows on the 2 dimensional torus:

- If the generator $\mathbf{v}$ has rationally independent coordinates, then
    1. the orbit closure of any point is the whole space $\mathbb{T}^2$ (Theorem 1.5),
    2. the orbit of any point equidistributes in $\mathbb{T}^2$ (Theorem 1.15),
    3. Leb is the only ergodic probability measure for $\varphi^{\mathbf{v}}$ (Theorem 1.21).
- If the generator $\mathbf{v}$ has rationally dependent coordinates, then
    1. all orbits are periodic, hence closed (Theorem 1.5),
    2. all orbits are not equidistributed in $\mathbb{T}^2$ (but, clearly, they equidistribute in their closure),
    3. any ergodic measure is the normalized Lebesgue measure on a periodic orbit (Exercise 1.14).

It is possible to generalize these results to linear flows on higher dimensional tori. Let us first recall some definitions.

A subspace $V < \mathbb{R}^n$ is called *rational* if the discrete Abelian group $V \cap \mathbb{Z}^n$ has rank precisely equal to $k := \dim(V)$. It is easy to see that this happens exactly when we can find a basis of V consisting of vectors in $\mathbb{Z}^n$. The subspace V carries a smooth measure, that we call Leb_V, given by the Lebesgue measure on V normalized so that the discrete subgroup $V \cap \mathbb{Z}^n$ has *covolume* 1 (see, e.g., Exercise 1.7). This measure descends to a measure on the k-dimensional torus $V/(V \cap \mathbb{Z}^n)$, as well as on its affine translates $\mathbf{x} + V/(V \cap \mathbb{Z}^n)$ for all $\mathbf{x} \in \mathbb{R}^n$. By a little abuse of notation, we will still call Leb_V any of these affine measures.

Theorem 1.23. *Let* $\varphi^{\mathbf{v}} \colon \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^n$ *be a linear flow on* $\mathbb{T}^n$*, with* $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$. *There exists a rational subspace* $V < \mathbb{R}^n$ *of dimension* $k \in \{1, \dots, n\}$ *which contains the line* $\mathbb{R}\mathbf{v}$ *for which the following holds.*

1. *(Orbit closure classification) The orbit closure of any point is an affine k-dimensional torus, namely for all* $p = [\![\mathbf{x}]\!] \in \mathbb{T}^n$ *we have*

$$\overline{\mathcal{O}_{\varphi^{\mathbf{v}}}(p)} = \mathbf{x} + V/(V \cap \mathbb{Z}^n).$$

2. *(Equidistribution) The orbit of any point $p \in \mathbb{T}^n$ equidistributes in its closure with respect to the affine measure* Leb_V.

3. *(Measure classification) Any ergodic measure for $\varphi^{\mathbf{v}}$ is an affine measure* Leb_V *on the affine torus* $\mathbf{x} + V/(V \cap \mathbb{Z}^n)$ *for some* $\mathbf{x} \in \mathbb{R}^n$.

Theorem 1.23 can be seen as a very simple case of a series of profound and general theorems by Marina Ratner which classify all possible orbit closures for unipotent actions, show that all orbits equidistribute in their closure, and prove that any ergodic measure is the affine translate of the Lebesgue (Haar) measure on a intermediate subgroup. The purpose of this complicated comment is only to whet you appetite for what will come in the rest of the course.

1.4 Further chaotic properties

1.4.1 Weak mixing

Let (M, φ, μ) be a ppf. As we have seen in (1.2), for every $t \in \mathbb{R}$ the *Koopman operator*

$$U_t \colon L^2(M) \to L^2(M), \quad U_t f = f \circ \varphi_t$$

is *unitary*. By Proposition 1.13, the flow φ is ergodic if and only if the eigenspace corresponding to the eigenvalue 1 has dimension 1, and consists of constant functions. Since U_t is unitary, if there are other eigenvalues, they must have modulus 1.

Definition 1.24. *We say that the ppf (M, φ, μ) is* weak mixing *if the only solutions to*

$$U_t f = e^{2\pi i t \alpha} f \quad \text{in } L^2(M) \text{ for all } t \in \mathbb{R}$$

are given by $\alpha = 0$ and $f = c$ for some $c \in \mathbb{C}$.

As usual, when the reference measure μ is clear from the context, we will often simply say that φ is weak mixing when the condition in Definition 1.24 is satisfied.

Clearly, a weak mixing ppf is also ergodic. The converse, however, is not true, and a family of counterexamples is given precisely by our irrational linear flows.

Lemma 1.25. *Any linear flow $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$ is not weak mixing.*

Proof. It is sufficient to consider the irrational case, since we already know that rational linear flows are not ergodic and hence cannot be weak mixing. We claim that for any $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$, the function

$$f_{\mathbf{n}}([\![\mathbf{x}]\!]) = e^{2\pi i \mathbf{n} \cdot \mathbf{x}} \in L^\infty(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$$

is a non-constant eigenfunction, and the α as in Definition 1.24 is $\alpha = \mathbf{n} \cdot \mathbf{v} \neq 0$. Indeed, for any $t \in \mathbb{R}$, we have

$$U_t f_{\mathbf{n}}([\![\mathbf{x}]\!]) = f_{\mathbf{n}}([\![\mathbf{x} + t\mathbf{v}]\!]) = e^{2\pi i \mathbf{n} \cdot (\mathbf{x} + t\mathbf{v})} = e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}} = e^{2\pi i t \alpha} f_{\mathbf{n}}([\![\mathbf{x}]\!]).$$

Thus, irrational linear flows are ergodic but not weak-mixing. □

Weak-mixing is a *spectral* property, in the sense that it concerns the spectrum of the Koopman operators U_t of the system. If they have no pure point component (no eigenvalues), the flow is weak mixing. There are other equivalent characterizations of weak-mixing, which have a more "dynamical flavour"; we summarize them in Proposition 1.26 below.

Proposition 1.26. *Let (M, φ, μ) be a ppf. The following are equivalent.*

1. (M, φ, μ) *is weak mixing.*
2. *For any* $f, g \in L^2(M)$,
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \int_M f \circ \varphi_t \cdot \bar{g} \, \mathrm{d}\mu - \mu(f)\, \mu(\bar{g}) \right| \mathrm{d}t = 0.$$
3. *For any* $f, g \in L^2(M)$,
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \int_M f \circ \varphi_t \cdot \bar{g} \, \mathrm{d}\mu - \mu(f)\, \mu(\bar{g}) \right|^2 \mathrm{d}t = 0.$$
4. *For any* $f, g \in L^2(M)$*, there exists a set* $J = J_{f,g} \subset \mathbb{R}$ *of zero density such that*
$$\lim_{T\to\infty, T\notin J} \int_M f \circ \varphi_T \cdot \bar{g} \, \mathrm{d}\mu = \mu(f)\, \mu(\bar{g}).$$
5. *The product measure* $\mu \times \mu$ *is ergodic for the flow* $\varphi \times \varphi$ *on* $M \times M$.
6. *The product measure* $\mu \times \mu$ *is weak mixing for the flow* $\varphi \times \varphi$ *on* $M \times M$.
7. *For any ergodic ppf* (N, ψ, ν)*, the system* $(M \times N, \varphi \times \psi, \mu \times \nu)$ *is ergodic.*

It might be worth for the reader to compare conditions 2–4 of Proposition 1.26 with the following equivalent definition of ergodicity.

Exercise 1.27. *Let* (M, φ, μ) *be a ppf. Show that it is ergodic if and only if for any* $f, g \in L^2(M)$ *we have*
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int_M f \circ \varphi_t \cdot \bar{g} \, \mathrm{d}\mu \right) \mathrm{d}t = \mu(f)\, \mu(\bar{g}).$$
Deduce in particular that a weak mixing ppf is ergodic.

1.4.2 Mixing

Mixing, sometimes called strong mixing, is an even stronger property that, roughly speaking, says that any two events become asymptotically independent.

Definition 1.28. *We say that the ppf* (M, φ, μ) *is* (strong) mixing *if for any two observables* $f, g \in L^2(M)$*, the* correlations decay*, namely if*
$$\langle f \circ \varphi_t, g \rangle = \int_M f \circ \varphi_t \cdot \bar{g} \, \mathrm{d}\mu \to \mu(f)\, \mu(\bar{g}),$$
as $t \to \infty$.

It is clear from Proposition 1.26-(4) that any mixing ppf is also weak mixing. The converse, however, is not true: there are weak mixing ppf's which are not strong mixing. The first examples of weak mixing but not mixing transformations were constructed by cutting-and-stacking methods. In the context of flows, typical translation flows on translation surfaces and typical minimal area-preserving flows on higher genus surfaces are also natural classes of examples of weak mixing flows that are not mixing. It is also interesting to notice that, by the Halmos-Rokhlin Theorem, weak mixing is a *generic* property, whereas mixing is *meager*. In this course, however, the flows we will encounter are either not weak mixing (the nilflows in Chapter 3) or mixing (the geodesic and horocycle flows in Chapters 4–7).

Returning to our case study, we already know from Lemma 1.25 that irrational linear flows are not weak mixing, hence they cannot possibly be mixing. We can actually prove a stronger result, namely they have the so-called *rigidity property*.

Exercise 1.29. *Let* $\mathbf{v} = (v, 1) \in \mathbb{R}^2$*, with* $v \notin \mathbb{Q}$*.*

(a) *Prove that the linear flow* $\varphi^{\mathbf{v}}$ *is* rigid*, namely there exists an increasing sequence* $t_n \to \infty$ *such that for any measurable set* $A \subset \mathbb{T}^2$ *we have*

$$\mathrm{Leb}(A \triangle \varphi^{\mathbf{v}}_{t_n}(A)) \to 0, \quad \textit{as } n \to \infty.$$

(b*) *Even more, find an explicit increasing sequence* $t_n \to \infty$ *such that for any set* Q *of the form* $Q = I_1 \times I_2 + \mathbb{Z}^2$*, where* $I_1, I_2 \subset [0,1)$ *are intervals, we have*

$$\mathrm{Leb}(Q \cap \varphi^{\mathbf{v}}_{t_n}(Q)) \geq \mathrm{Leb}(Q) - t_n^{-2},$$

for all $n \in \mathbb{N}$ *sufficiently large (Hint: it might be useful to consider the continued fraction expansion of* v*).*

(c) *Conclude in particular that* $\varphi^{\mathbf{v}}$ *is not mixing.*

One can also ask about the correlations of several events or observables, leading to the following definition.

Definition 1.30. *We say that the ppf* (M, φ, μ) *is* mixing of order k *or* k-mixing *if for any* k *(real-valued) observables* $f_1, \dots, f_k \in L^2(M)$ *we have*

$$\int_M f_1 \cdot f_2 \circ \varphi_{t_2} \cdots f_k \circ \varphi_{t_k} \, \mathrm{d}\mu \to \mu(f_1) \cdots \mu(f_k),$$

as $t_2, t_3 - t_2, \dots, t_k - t_{k-1} \to \infty$*.*

We say that the ppf (M, φ, μ) *is* mixing of all orders *if it is mixing of order* k *for all* $k \geq 2$*.*

It is currently unknown whether mixing implies mixing of all orders. This open question is known as the "Rokhlin Problem".

1.5 Outline of the course

In Chapter 2, we present all the relevant background material on matrix Lie groups. We will introduce their associated Lie algebras, which can be described as the space of all left-invariant vector fields. We will then study the induced flows using the exponential map. In Section 2.3, we introduce the Haar measure, which is the invariant measure we will be interested in, the Killing form and the Casimir operator. These last two objects will play a role in the final chapter of these notes. Finally, we define homogeneous spaces as the smooth manifolds obtained as quotients of Lie groups by lattices.

In Chapter 3, we focus on Heisenberg nilflows. We describe them using the so-called exponential coordinates and we classify all possible Heisenberg nilmanifolds. We then show that Heisenberg nilflows are never mixing, but typically relatively mixing and uniquely ergodic. In Section 3.3, we point out an interesting connection between Heisenberg nilflows and theta sums (or quadratic Weyl sums), which are classical objects in analytic number theory.

In Chapter 4, we study in detail the action of $\mathrm{PSL}(2,\mathbb{R})$ on the hyperbolic plane (namely, on its upper-half plane model), which is first introduced in §2.3. We define the geodesic and horocycle flow as particular cases of homogeneous flows on quotients of $\mathrm{PSL}(2,\mathbb{R})$. As an important example, we introduce the Modular Surface.

Chapter 5 is devoted to the study of the ergodic properties of geodesic and horocycle flow. We prove that they are ergodic and mixing.

In Chapter 6, we study the connection between the geodesic flow on the Modular Surface and continued fractions. This fascinating topic dates back to Artin [1], but we will follow an elegant presentation by Series [17].

The final part, Chapter 7, is devoted to the treatment of more advanced material. We prove unique ergodicity of the horocycle flow on compact manifolds, a result originally due to Furstenberg [8]. The proof we present in these notes is due to Coudène [3]. We then discuss the generalizations of this result to finite volume, noncompact spaces and we state Ratner's Theorem [15] on measure classification in the case of unipotent flows. We then study some quantitative properties. We present a special case of Ratner's quantitative mixing result [14] for geodesic and horocycle flow and a special case of Flaminio and Forni's result [6] on asymptotics of horocycle averages, but following the proof in [16].

Chapter 2

Lie Groups

In this section, we introduce and study Lie groups, in particular matrix Lie groups, and their Lie algebras. We introduce the objects and some fundamental tools we are going to study in this course: homogeneous flows, Haar measures, the Adjoint representation and the Casimir operator. Excellent references for these topics are the books [13] and [12]

The reader will benefit from some familiarity with basic notions in differential topology, such as tangent spaces, vector fields, and differential forms.

2.1 Matrix Lie groups

2.1.1 Definitions

Definition 2.1. *A* Lie group *G is a group $(G,\cdot)$ endowed with a differential structure such that both the multiplication map and the inverse map*

$$\begin{aligned} G\times G &\to G, & \text{and} \qquad G &\to G \\ (g,h) &\mapsto gh & g &\mapsto g^{-1} \end{aligned}$$

are smooth.

If G is a Lie group, it follows immediately from the definition that, for any $g \in G$, the *left multiplication map* $L_g \colon G \to G$ given by $L_g(h) = gh$ and the *right multiplication map* $R_g \colon G \to G$ given by $R_g(h) = hg$ are smooth maps.

The simplest example of a Lie group is $(\mathbb{R}^n, +)$ equipped with the trivial atlas. It is clear that the maps

$$\begin{aligned} \mathbb{R}^n\times\mathbb{R}^n &\to \mathbb{R}^n, & \text{and} \qquad \mathbb{R}^n &\to \mathbb{R}^n \\ (\mathbf{x},\mathbf{y}) &\mapsto \mathbf{x}+\mathbf{y} & \mathbf{x} &\mapsto -\mathbf{x} \end{aligned}$$

are smooth. The space $\mathrm{Mat}(n,\mathbb{R})$ of square matrices of size n with real coefficients is also a Lie group for the addition operation, since it is isomorphic to $\mathbb{R}^{n^2}$ (not only as Abelian groups, but also as vector spaces). Similarly, we can see that the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a Lie group. Notice that the projection map $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $\pi(\mathbf{x}) = [\![\mathbf{x}]\!] = \mathbf{x}+\mathbb{Z}^n$, is a local homeomorphism; that is, for every $p \in \mathbb{T}^n$ and any $r \in (0,1)$ there exists $\mathbf{x} \in \mathbb{R}^n$ such that the restriction $\pi|_{B(\mathbf{x},r)} \colon B(\mathbf{x},r) \to B(p,r)$ of π to the ball centered at $\mathbf{x}$ of radius r is a homeomorphism on its image. Then, one can construct an atlas on $\mathbb{T}^n$ by means of $(\pi|_{B(\mathbf{x},r)})^{-1} \colon B(p,r) \to \mathbb{R}^n$. The transition maps between charts are translations by elements of $\mathbb{Z}^n$. In this atlas, the group operations on $\mathbb{T}^n$, as a quotient group of $\mathbb{R}^n$ are smooth.

Exercise 2.2. (a) *Show that the circle* $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ *is a Lie group.*

(b) *Characterize for which* $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ *the subgroup* $\{[\![t\mathbf{v}]\!] : t \in \mathbb{R}\}$ *of* $\mathbb{T}^2$ *is a Lie group (with respect to the induced topology of* $\mathbb{T}^2$*).*

In all the simple examples we have seen above, the group is Abelian. This is not an interesting situation; our focus will be on non-Abelian groups.

Lemma 2.3. *The* general linear group

$$\mathrm{GL}(n,\mathbb{R}) = \{g \in \mathrm{Mat}(n,\mathbb{R}) : \det g \neq 0\}$$

with matrix multiplication is a (non-Abelian) Lie group.

Proof. It is clear that $\mathrm{GL}(n,\mathbb{R})$ is a smooth manifold, since it is an open subset of $\mathrm{Mat}(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$, and the restriction of the coordinate maps to this open subset defines an atlas of smooth charts. We only need to verify that multiplication and inversion are smooth with respect to this atlas. Matrix multiplication is smooth since, in these charts, it is a polynomial map; similarly, taking the inverse is also a polynomial map in coordinates by the Cramer's rule. □

As a consequence of the following lemma, whose proof can be found, for example, in [13, Proposition 7.11], we get many more examples of Lie groups.

Proposition 2.4. *Let H be a* closed *subgroup of a Lie group G. Then H is an embedded submanifold of G and hence a Lie group.*

Proposition 2.4 motivates the following definition.

Definition 2.5. *A* (real) matrix Lie group *is a closed subgroup G of* $\mathrm{GL}(n,\mathbb{R})$.

A matrix Lie group is thus a Lie group according to Definition 2.1. We remark that the converse is not true, namely there exists Lie groups that are not matrix Lie groups[1]. However, we will not deal with them in this course; actually, the examples of matrix Lie groups that we will mostly be interested in are the *special linear group of degree 2*

$$\mathrm{SL}(2,\mathbb{R}) = \{g \in \mathrm{GL}(2,\mathbb{R}) : \det g = 1\},$$

and the *Heisenberg group*

$$\mathrm{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

2.1.2 Tangent spaces and geometric tangent vectors

Proposition 2.4 states that a matrix Lie group G is an embedded submanifold in $\mathrm{Mat}(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$. In particular, we can look at the set of *geometric tangent vectors* at any point $g \in G$; that is, at the set of vectors in $\mathbb{R}^{n^2}$ which are parallel to the tangent space at g. We will denote by $\mathfrak{g}$ the set of geometric tangent vectors at the identity $e \in G$, more precisely we define

$$\mathfrak{g} := \left\{ \dot{\gamma}(0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \gamma(t) \ : \ \gamma\colon \mathbb{R} \to G \text{ is a smooth curve with } \gamma(0) = e \right\}.$$

Lemma 2.6. *If G is a matrix Lie group, the set* $\mathfrak{g}$ *is a subspace of* $\mathrm{Mat}(n,\mathbb{R})$.

[1] For example, the universal cover of $\mathrm{SL}(2,\mathbb{R})$.

Proof. The zero matrix $\mathbf{0} \in \mathrm{Mat}(n,\mathbb{R})$ belongs to $\mathfrak{g}$, since it is the derivative of the constant curve $t \mapsto e \in G$. Let $\dot\gamma(0)$ and $\dot\eta(0)$ be two geometric tangent vectors at e, and let us check that their sum is a geometric tangent vector as well. Define the curve $(\gamma\cdot\eta)(t) := \gamma(t)\eta(t)$. Then, $(\gamma\cdot\eta)(0) = e$, and, using the product rule, its geometric tangent vector is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\gamma\cdot\eta)(t) = \dot\gamma(0)\eta(0) + \gamma(0)\dot\eta(0) = \dot\gamma(0) + \dot\eta(0),$$

hence $\dot\gamma(0)+\dot\eta(0) \in \mathfrak{g}$. Finally, let us check that $\mathfrak{g}$ is closed under scalar multiplication. Let $\dot\gamma(0) \in \mathfrak{g}$, and let $a \in \mathbb{R}$. Then, the curve $(a\gamma)(t) := \gamma(at)$ is a smooth curve such that $(a\gamma)(0) = e$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (a\gamma)(t) = a\dot\gamma(0) \in \mathfrak{g},$$

which completes the proof. □

Let us see a concrete example: let us find the space $\mathfrak{sl}(2,\mathbb{R})$ of geometric tangent vectors of $\mathrm{SL}(2,\mathbb{R})$ at the identity $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$. If we define $u(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1\end{smallmatrix}\right)$, then we obtain a smooth function $u \colon \mathbb{R} \to \mathrm{SL}(2,\mathbb{R})$ and

$$\mathbf{u} := \dot u(0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}).$$

In a similar way, by looking at the smooth curves $a(t) := \left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{smallmatrix}\right)$ and $v(t) := \left(\begin{smallmatrix} 1 & 0 \\ t & 1\end{smallmatrix}\right)$, we find that $\mathbf{u},\mathbf{a},\mathbf{v} \in \mathfrak{sl}(2,\mathbb{R})$, where

$$\mathbf{a} := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \text{and} \quad \mathbf{v} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In particular, if we denote by $\mathrm{tr}(\mathbf{x})$ the trace of the matrix $\mathbf{x}$, by Lemma 2.6 we get

$$\mathrm{span}\{\mathbf{u},\mathbf{a},\mathbf{v}\} = \{\mathbf{x} \in \mathrm{Mat}(2,\mathbb{R}) : \mathrm{tr}(\mathbf{x}) = 0\} \subseteq \mathfrak{sl}(2,\mathbb{R}).$$

Let us show that equality holds. If $\gamma(t) = \left(\begin{smallmatrix} a(t) & b(t) \\ c(t) & d(t)\end{smallmatrix}\right)$ is a smooth curve in $\mathrm{SL}(2,\mathbb{R})$ with $\gamma(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$, then we have $a(t)d(t) - b(t)c(t) = 1$ for all $t \in \mathbb{R}$. Differentiating at $t = 0$, we get

$$0 = \dot a(0)d(0) + a(0)\dot d(0) - \dot b(0)c(0) - b(0)\dot c(0) = \dot a(0) + \dot d(0).$$

This shows that $\dot\gamma(0) \in \{\mathbf{x} \in \mathrm{Mat}(2,\mathbb{R}) : \mathrm{tr}(\mathbf{x}) = 0\}$, and hence proves the equality

$$\mathfrak{sl}(2,\mathbb{R}) = \{\mathbf{x} \in \mathrm{Mat}(2,\mathbb{R}) : \mathrm{tr}(\mathbf{x}) = 0\}.$$

Exercise 2.7. *Find the space $\mathfrak{h}$ of geometric tangent vectors of* Heis *at the identity.*

If $\mathbf{x},\mathbf{y} \in \mathfrak{g}$ are two geometric tangent vectors of a matrix Lie group G, by Lemma 2.6, their sum $\mathbf{x}+\mathbf{y}$ is still in $\mathfrak{g}$, as well as any of their scalar multiples. We define now a bilinear, antisymmetric operation on geometric tangent vectors, which we call the *bracket*, and will turn the space $\mathfrak{g}$ into an *algebra*. The geometric interpretation of this operation will become clear in a little while.

If $G \subset \mathrm{GL}(n,\mathbb{R})$ is a matrix Lie group, define the *bracket operation*

$$[\cdot,\cdot]_{\mathfrak{g}} \colon \mathfrak{g}\times\mathfrak{g} \to \mathrm{Mat}(n,\mathbb{R}) \quad \text{by} \quad [\mathbf{x},\mathbf{y}]_{\mathfrak{g}} := \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x},$$

where $\cdot$ denotes the matrix multiplication in $\mathrm{Mat}(n,\mathbb{R})$. From now on, we will often suppress the symbol $\cdot$, which should be clear from the context. The definition immediately implies that the bracket of a vector with itself is zero.

Exercise 2.8. *Show that* $[\cdot,\cdot]_{\mathfrak{g}}$ *is bilinear, antisymmetric and satisfies the* Jacobi identity: *for all* $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathfrak{g}$*, we have*

$$[[\mathbf{x},\mathbf{y}]_{\mathfrak{g}},\mathbf{z}]_{\mathfrak{g}} + [[\mathbf{y},\mathbf{z}]_{\mathfrak{g}},\mathbf{x}]_{\mathfrak{g}} + [[\mathbf{z},\mathbf{x}]_{\mathfrak{g}},\mathbf{y}]_{\mathfrak{g}} = 0.$$

In $\mathfrak{sl}(2,\mathbb{R})$, one can check that the nontrivial possible brackets of the basis elements $\mathbf{u},\mathbf{a},\mathbf{v}$ are

$$[\mathbf{a},\mathbf{u}]_{\mathfrak{sl}(2,\mathbb{R})} = -[\mathbf{u},\mathbf{a}]_{\mathfrak{sl}(2,\mathbb{R})} = \mathbf{u}, \qquad [\mathbf{a},\mathbf{v}]_{\mathfrak{sl}(2,\mathbb{R})} = -[\mathbf{v},\mathbf{a}]_{\mathfrak{sl}(2,\mathbb{R})} = -\mathbf{v}, \tag{2.1}$$

$$[\mathbf{u},\mathbf{v}]_{\mathfrak{sl}(2,\mathbb{R})} = -[\mathbf{v},\mathbf{u}]_{\mathfrak{sl}(2,\mathbb{R})} = 2\mathbf{a}. \tag{2.2}$$

Notice in particular that the bracket of any two vectors in $\mathfrak{sl}(2,\mathbb{R})$ is again an element of $\mathfrak{sl}(2,\mathbb{R})$. Indeed, this is no coincidence, as the next lemma shows.

Lemma 2.9. *(a) For all* $g \in G$ *and all* $\mathbf{x} \in \mathfrak{g}$*, we have*

$$g^{-1}\mathbf{x}g \in \mathfrak{g}.$$

(b) The space $\mathfrak{g}$ *is closed under bracket* $[\cdot,\cdot]_{\mathfrak{g}}$.

Before proving the lemma, let us note that, by part (a), the map

$$\mathrm{Ad}(g)\colon \mathbf{x} \mapsto g^{-1}\mathbf{x}g$$

is an invertible linear transformation of $\mathfrak{g}$ for any $g \in G$, the inverse being $(\mathrm{Ad}(g))^{-1} = \mathrm{Ad}(g^{-1})$. We call $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$ the *Adjoint of* g. As we mentioned, a geometric interpretation of these facts will come later on.

Proof of Lemma 2.9. Let γ be a smooth curve such that $\dot{\gamma}(0) = \mathbf{x}$. Then, for all $g \in G$, the map $\gamma_g(t) := g^{-1}\gamma(t)g$ is a smooth curve in G with $\gamma_g(0) = g^{-1}g = e$. Differentiating at $t = 0$, we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma_g(t) = g^{-1}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\gamma(t)\right) g = g^{-1}\mathbf{x}g \in \mathfrak{g},$$

which proves (a).

In order to prove (b), let $\mathbf{x},\mathbf{y} \in \mathfrak{g}$; we need to show that $[\mathbf{x},\mathbf{y}]_{\mathfrak{g}} \in \mathfrak{g}$. Let $\gamma(t)$ be such that $\dot{\gamma}(0) = \mathbf{x}$. By (a), we know that

$$\eta(t) := \gamma(t)\mathbf{y}\gamma(t)^{-1} \in \mathfrak{g} \quad \text{ for all } t \in \mathbb{R}.$$

Moreover, since γ is smooth and multiplying matrices is a smooth map as well, the function η defines a smooth curve in $\mathfrak{g}$. In particular, the derivative $\dot{\eta}(0)$ of η at $t = 0$ exists. By Lemma 2.6, $\mathfrak{g}$ is a closed subset, hence the limit

$$\dot{\eta}(0) = \lim_{t\to 0}\frac{\eta(t)-\eta(0)}{t}$$

belongs to $\mathfrak{g}$. Let us compute it. First of all, since $\gamma(t)^{-1}\gamma(t) = e$ for all $t \in \mathbb{R}$ and $\gamma(0) = e$, by differentiating, we get

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\gamma(t)^{-1}\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\gamma(t)^{-1}) + \dot{\gamma}(0).$$

From this, we conclude

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\eta(t) = \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\gamma(t)\right)\mathbf{y}\gamma(0)^{-1} + \gamma(0)\mathbf{y}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\gamma(t)^{-1}\right) = \mathbf{x}\mathbf{y} - \mathbf{y}\mathbf{x},$$

which shows that $[\mathbf{x},\mathbf{y}]_{\mathfrak{g}} \in \mathfrak{g}$. $\square$

Finally, let us recall that the definition of tangent space in the general context of differentiable manifolds is given in terms of *derivations* as follows. For any $g \in G$, the *tangent space* T_gG at g is the vector space of all possible derivations at g; that is, the space of all linear maps $X_g \colon \mathscr{C}^\infty(G) \to \mathbb{R}$ on smooth functions on G which satisfy the Leibniz rule

$$X_g(f_1 f_2) = X_g(f_1)\, f_2(g) + f_1(g)\, X_g(f_2).$$

It is not hard to see that the tangent space at any point can be identified with the space of its geometric tangent vectors; in particular we have the following identification.

Lemma 2.10. *If G is a matrix Lie group, then $T_eG \simeq \mathfrak{g}$.*

Proof. The isomorphism between $\mathfrak{g}$ and T_eG is defined as follows: if $\dot\gamma(0) \in \mathfrak{g}$, then we associate the derivation

$$X_e \colon f \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \gamma(t).$$

In coordinates, if $\dot\gamma(0) = \mathbf{x}$, where $\mathbf{x} = (x_{i,j})_{i,j=1}^n \in \mathrm{Mat}(n,\mathbb{R})$, then the associated derivation is $X_e(f) = \sum_{i,j=1}^n x_{i,j} \partial_{i,j} f(e)$, where $\partial_{i,j}$ is the partial derivative with respect to the (i,j)-coordinate. It is an easy exercise to verify that this map is indeed a linear isomoprhism, for details see, e.g., [13, Proposition 3.2]. □

2.2 The Lie algebra of a Lie group

2.2.1 Left-invariant vector fields

Let G be a Lie group, and let $F \colon G \to G$ be a smooth map. The *differential* DF of F is a smooth map on the tangent bundle TG of G defined as follows: the differential $DF(g)$ at $g \in G$ is a linear map from the tangent space T_gG at g to the tangent space $T_{F(g)}G$ at $F(g)$ which sends a derivation $X_g \in T_gG$ to the derivation $DF(g)X_g \in T_{F(g)}G$ given by

$$[DF(g)X_g](f) = X_g(f \circ F).$$

The reader can check that $DF(g)X_g$ is indeed a derivation (namely, it is a linear map on $\mathscr{C}^\infty(G)$ which satisfies the Leibniz rule).

If G is a matrix Lie group, then we can define the differential DF of F in terms of geometric tangent vectors as well, following Lemma 2.10. Let $\mathbf{x} = \dot\gamma(0) \in \mathfrak{g}$ be a geometric tangent vector at the identity. We define

$$DF(e)\mathbf{x} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F \circ \gamma(t),$$

which is a geometric tangent vector at $F(e) \in G$.

Using smooth maps, we can therefore "move" tangent vectors around G. Recall that, for any $g \in G$, the left-multiplication map $L_g \colon h \mapsto gh$ is smooth. Then, for any element $\mathbf{x} \in \mathfrak{g}$ we can associate a tangent vector at any other point g by the aid of the differential of L_g at e. In other words, we can define a *vector field* X on G by setting

$$X_g := DL_g(e)X_e, \qquad \text{for all } g \in G, \tag{2.3}$$

where, again, $X_e \in T_eG$ is the derivation associated to $\mathbf{x}$ according to Lemma 2.10. Let us express this in coordinates. If $\mathbf{x} = \dot\gamma(0) \in \mathfrak{g}$, by Taylor's Theorem we can write

$$\gamma(t) = e + t\mathbf{x} + tR(t),$$

where $t \mapsto R(t) \in \mathrm{Mat}(n,\mathbb{R})$ is a smooth map and $R(t) \to \mathbf{0}$ as $t \to 0$. Then

$$DL_g(e)\mathbf{x} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_g \circ \gamma(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g + tg\mathbf{x} + tgR(t)) = g\mathbf{x} \in \mathrm{Mat}(n,\mathbb{R}). \tag{2.4}$$

Proposition 2.11. *For any $\mathbf{x} = X_e \in T_eG$, the associated vector field X defined by* (2.3) *is smooth and* left-invariant*, namely for all $g \in G$ we have $DL(g)X = X$.*

Proof. In coordinates, the fact that X is smooth comes from expression (2.4). More formally, in order to check that X is smooth it is enough to show that $Xf \colon G \to \mathbb{R}$ is a smooth function for any $f \in \mathscr{C}^\infty(G)$. Let $\gamma(t)$ be a smooth curve such that $\dot\gamma(0) = X_e$, and fix any such $f \in \mathscr{C}^\infty(G)$. Define $F(g,t) := f(g\gamma(t))$; then it is clear that $F \colon G \times \mathbb{R} \to \mathbb{R}$ is a smooth function, and so is its derivative $\frac{\partial}{\partial t}F(g,0)$ at $t = 0$. Thus,

$$X_g f = (DL_g(e)X_e)(f) = X_e(f \circ L_g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ L_g(\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g\gamma(t)) = \frac{\partial}{\partial t}F(g,0)$$

is a smooth function in g, which proves that X is smooth.

The fact that X is left-invariant is an easy exercise that we leave to the reader. □

So far, we have seen that in a matrix Lie group G we can identify the tangent space at the identity with the space of geometric tangent vectors $\mathfrak{g} \subset \mathrm{Mat}(n,\mathbb{R})$. Moreover, given any element $X_e \in T_eG$, we can define a smooth vector field X on G which is left-invariant.

Definition 2.12. *Let G be a Lie group. The vector space of all smooth, left-invariant vector fields*

$$\mathrm{Lie}(G) := \{X \colon G \to TG : X \textit{ is smooth and } DL_gX = X \textit{ for all } g \in G\}$$

is called the Lie algebra *of G.*

The use of the word "algebra" will become clear later on. The following result should come as no surprise.

Lemma 2.13. *The evaluation at the identity map* $\mathrm{ev} \colon \mathrm{Lie}(G) \to T_eG$ *defined by* $\mathrm{ev}(X) = X_e$ *is a linear isomorphism.*

Proof. It immediately follows from the definition that ev is linear. If $\mathrm{ev}(X) = X_e = 0$, then for every $g \in G$ we have $X_g = DL_g(e)0 = 0$, which implies $X = 0$. This shows that ev is injective.

Let now $X_e \in T_eG$, and define a vector field X as in (2.3). By Proposition 2.11, X is smooth and left-invariant, hence $X \in \mathrm{Lie}(G)$. By construction, $\mathrm{ev}(X) = X_e$, which proves surjectivity and completes the proof. □

In the following, we will often identify $\mathbf{x} \in \mathfrak{g}$, $X_e \in T_eG$ and $X \in \mathrm{Lie}(G)$. We now know that (geometric) tangent vectors at the identity are in 1-to-1 correspondence with smooth left-invariant vector fields. Each of the latters generate a smooth flow on G; more precisely, if $X \in \mathrm{Lie}(G)$, there exists a unique $\varphi^X = \{\varphi^X_t\}_{t \in I}$ (which is defined at least on small intervals I containing 0) whose infinitesimal generator is X, namely such that

$$X_g f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \varphi^X_t(g), \qquad \text{for all } g \in G,$$

where f is any smooth function. The goal now is to find an expression for φ^X and show that it is defined for all $t \in \mathbb{R}$.

2.2.2 The exponential map

On the space of matrices $\mathrm{Mat}(n,\mathbb{R})$, we introduce the following map

$$\begin{aligned} \exp\colon\ \mathrm{Mat}(n,\mathbb{R}) &\to \mathrm{Mat}(n,\mathbb{R}) \\ \mathbf{x} &\mapsto \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{x}^k, \end{aligned} \tag{2.5}$$

called the *matrix exponential*.

Proposition 2.14. *The matrix exponential is well-defined and satisfies the following properties:*

1. $\exp(\mathbf{x})\exp(\mathbf{y}) = \exp(\mathbf{x}+\mathbf{y})$ *if* $\mathbf{x}$ *and* $\mathbf{y}$ *commute,*
2. $\exp(\mathbf{x}) \in \mathrm{GL}(n,\mathbb{R})$ *and* $\exp(\mathbf{x})^{-1} = \exp(-\mathbf{x})$*,*
3. $\gamma_{\mathbf{x}}(t) := \exp(t\mathbf{x})$ *is a smooth curve in* $\mathrm{GL}(n,\mathbb{R})$ *whose geometric tangent vector at e is* $\mathbf{x}$*,*
4. $\exp(\mathrm{Ad}(g)\mathbf{x}) = \exp(g^{-1}\mathbf{x}g) = g^{-1}\exp(\mathbf{x})g$ *for all* $g \in \mathrm{GL}(n,\mathbb{R})$*,*
5. $\det(\exp(\mathbf{x})) = e^{\mathrm{tr}(\mathbf{x})}$*,*
6. $\exp\colon\ \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ *is a smooth map.*

Proof. In oder to show that exp is well-defined, we prove that for any $\mathbf{x} \in \mathrm{Mat}(n,\mathbb{R})$, the series $\sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{x}^k$ converges. For any submultiplicative norm $\|\cdot\|$, we have

$$\sum_{k=0}^{\infty} \left\| \frac{1}{k!}\mathbf{x}^k \right\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|\mathbf{x}\|^k < \infty,$$

that is, $\sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{x}^k$ is absolutely convergent. Since $\mathrm{Mat}(n,\mathbb{R})$ is complete, this shows that the series is convergent. We now verify the other claims.

1. if $\mathbf{x}$ and $\mathbf{y}$ commute, we have

$$\begin{aligned} \exp(\mathbf{x})\exp(\mathbf{y}) &= \left(\sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{x}^k\right)\left(\sum_{l=0}^{\infty} \frac{1}{l!}\mathbf{y}^l\right) = \sum_{k,l=0}^{\infty} \frac{1}{k!l!}\mathbf{x}^k\mathbf{y}^l = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{1}{l!(k-l)!}\mathbf{x}^l\mathbf{y}^{k-l} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\sum_{l=0}^{k} \binom{k}{l}\mathbf{x}^l\mathbf{y}^{k-l} = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{x}+\mathbf{y})^k = \exp(\mathbf{x}+\mathbf{y}). \end{aligned}$$

2. Take $\mathbf{y} = -\mathbf{x}$ in part 1, and notice that $\exp(\mathbf{0}) = e$.

3. We proved that $\sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{x}^k$ is absolutely convergent, so the following equalities hold

$$\dot{\gamma}_{\mathbf{x}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\exp(t\mathbf{x}) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\mathrm{d}}{\mathrm{d}t}(t\mathbf{x})^k = \sum_{k=0}^{\infty} \frac{1}{(k-1)!}\mathbf{x}^k t^{k-1} = \mathbf{x}\sum_{k=0}^{\infty} \frac{1}{k!}(t\mathbf{x})^k = \mathbf{x}\gamma_{\mathbf{x}}(t).$$

   Hence $\dot{\gamma}_{\mathbf{x}}(0) = \mathbf{x}$.

4. For any $g \in \mathrm{GL}(n,\mathbb{R})$,

$$\exp(g^{-1}\mathbf{x}g) = \sum_{k=0}^{\infty} \frac{1}{k!}(g^{-1}\mathbf{x}g)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(g^{-1}\mathbf{x}^k g) = g^{-1}\left(\sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{x}^k\right)g = g^{-1}\exp(\mathbf{x})g.$$

5. If $\mathbf{x} \in \mathrm{Mat}(n,\mathbb{R})$ is an upper triangular matrix, it is easy to see that the diagonal entries of $\exp(\mathbf{x})$ are the exponentials of the diagonal entries of $\mathbf{x}$; in particular the equality $\det(\exp(\mathbf{x})) = e^{\mathrm{tr}(\mathbf{x})}$ holds for these matrices. If $\mathbf{x}$ is arbitrary, write $\mathbf{x} = g^{-1}\mathbf{y}g$ in Jordan normal form and apply part 4.

6. The fact that exp is a smooth map follows from the rules of differentiations for series of functions.

□

For any $\mathbf{x} \in \mathfrak{g}$ and $t \in \mathbb{R}$ let us define

$$\varphi_t^{\mathbf{x}}(g) = g\exp(t\mathbf{x}). \tag{2.6}$$

Notice that, a priori, $\varphi_t^{\mathbf{x}} \colon G \to \mathrm{GL}(n,\mathbb{R})$. We shall now see that $\varphi_t^{\mathbf{x}}$ has values in G and defined the integral curves of the vector field $X \in \mathrm{Lie}(G)$.

Proposition 2.15. *The map $\varphi^{\mathbf{x}} \colon \mathbb{R} \times G \to \mathrm{GL}(n,\mathbb{R})$ defined by $\varphi^{\mathbf{x}}(t,g) = \varphi_t^{\mathbf{x}}(g)$ as in* (2.6) *is a smooth flow with infinitesimal generator $X \in \mathrm{Lie}(G)$, where $X_e = \mathbf{x}$.*

Proof. The fact that $\varphi^{\mathbf{x}}$ is a smooth map follows from Proposition 2.14-(6). For $t = 0$, we have $\varphi_0^{\mathbf{x}}(g) = g\exp(\mathbf{0}) = g$ and for any $t,s \in \mathbb{R}$ we have

$$\varphi_{t+s}^{\mathbf{x}}(g) = g\exp((t+s)\mathbf{x}) = g\exp(t\mathbf{x}+s\mathbf{x}) = g\exp(t\mathbf{x})\exp(s\mathbf{x}) = \varphi_s^{\mathbf{x}} \circ \varphi_t^{\mathbf{x}}(g),$$

where we have used Proposition 2.14-(1). Hence, $\varphi^{\mathbf{x}}$ is a smooth flow.

We now check that its infinitesimal generator is $X \in \mathrm{Lie}(G)$. By Proposition 2.14-(3), we know that the tangent vector at the identity associated to $\varphi^{\mathbf{x}}$ is $\mathbf{x} = X_e$. Then, for any smooth function f and any $g \in G$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \varphi_t^{\mathbf{x}}(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g\exp(t\mathbf{x})) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ L_g)(\exp(t\mathbf{x})) = [DL_g(e)X_e](f) = X_g f,$$

which proves that the infinitesimal generator is X. □

Corollary 2.16. *We have* $\exp\colon \mathfrak{g} \to G$*. Moreover,* exp *is a smooth diffeomorphism between a neighbourhood of $\mathbf{0} \in \mathfrak{g}$ and a neighbourhood of $e \in G$.*

Proof. Let $X \in \mathfrak{g} = \mathrm{Lie}(G)$. By the standard theory of ODEs, there exists a unique smooth solution $\gamma\colon I \to G$ to the equation $\dot\gamma(t) = X(\gamma(t))$ with the initial condition $\gamma(0) = e$. The solution γ is a smooth curve in G defined on an interval $I = (-\varepsilon,\varepsilon)$ for some $\varepsilon > 0$. By Proposition 2.15, the smooth curve $t \mapsto \varphi_t^{\mathbf{x}}(e) = \exp(t\mathbf{x})$ satisfies the ODE $\dot\gamma = X(\gamma)$, hence $\exp(t\mathbf{x}) \in G$ for all $t \in (-\varepsilon,\varepsilon)$. Let $N \in \mathbb{Z}$ be such that $|N|^{-1} < \varepsilon$. By Proposition 2.14-(1), we conclude

$$\exp(\mathbf{x}) = \exp(N^{-1}\mathbf{x})^N \in G,$$

since G is a group. This shows that exp maps $\mathfrak{g}$ into G.

Let us show it is a local diffeomorphism. By the Inverse Function Theorem, it is enough to show that the differential $D\exp(\mathbf{0})$ from $T_{\mathbf{0}}\mathfrak{g} \simeq \mathfrak{g}$ to $T_eG = \mathfrak{g}$ is invertible. Indeed, for any $\mathbf{x} \in \mathfrak{g}$ we have $D\exp(\mathbf{0})\mathbf{x} = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\exp(t\mathbf{x}) = \mathbf{x}$; that is, $D\exp(\mathbf{0})$ is the identity. This completes the proof. □

We have then shown that for any $\mathbf{x} \in \mathfrak{g}$ there exists a smooth flow $\varphi^{\mathbf{x}}$ on G defined for all times $t \in \mathbb{R}$ given by the action by multiplication on the right by the 1-parameter subgroup $\{\exp(t\mathbf{x}) : t \in \mathbb{R}\}$ generated by the left-invariant vector field X associated to $\mathbf{x}$.

Definition 2.17. *For any $\mathbf{x} \in \mathfrak{g} \setminus \{\mathbf{0}\}$, the flow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ given by $\varphi_t^{\mathbf{x}}(g) = g\exp(t\mathbf{x})$ is called the* homogeneous flow generated by $\mathbf{x}$. *We will write interchangeably $\varphi_t^{\mathbf{x}}$ and φ_t^X.*

For example, we can check that

$$\exp(t\mathbf{u}) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \exp(t\mathbf{a}) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \qquad \exp(t\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}.$$

As we saw, exp is a diffeomorphism between a neighbourhood of $\mathbf{0} \in \mathfrak{g}$ and $e \in G$, but in general exp is neither injective nor surjective.

Exercise 2.18. *(a) Show that* $\exp\colon \mathfrak{h} \to \mathrm{Heis}$ *is a global diffeomorphism.*

(b) Show that* $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ *is neither injective nor surjective: find countably many $\mathbf{x}_n \in \mathfrak{sl}(2,\mathbb{R})$ such that $\exp(\mathbf{x}_n) = e$ and find a matrix $g \in \mathrm{SL}(2,\mathbb{R})$ which cannot be written as $\exp(\mathbf{x})$ for $\mathbf{x} \in \mathfrak{sl}(2,\mathbb{R})$.*

2.2.3 Adjoints, Lie derivatives and Lie brackets

Let G be a matrix Lie group and $\mathfrak{g}$ its Lie algebra; call $k = \dim \mathfrak{g}$. Let us recall, from Lemma 2.9, that, for all $g \in G$, we can define a linear map $\mathrm{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$ called the Adjoint of g by $\mathrm{Ad}(g)\mathbf{x} = g^{-1}\mathbf{x}g$. The map

$$\begin{aligned} \mathrm{Ad}\colon G &\to \mathrm{GL}(\mathfrak{g}) \simeq \mathrm{GL}(k,\mathbb{R}) \\ g &\mapsto \mathrm{Ad}(g) \end{aligned} \tag{2.7}$$

is a group anti-homomorphism, since

$$\mathrm{Ad}(gh)\mathbf{x} = (gh)^{-1}\mathbf{x}(gh) = h^{-1}(g^{-1}\mathbf{x}g)h = (\mathrm{Ad}(h) \circ \mathrm{Ad}(g))\mathbf{x},$$

for all $\mathbf{x} \in \mathfrak{g}$. We call Ad the *Adjoint representation of G*.

Let us comment again on its dynamical significance. Let $\mathbf{x} \in \mathfrak{g} \setminus \{\mathbf{0}\}$, and let $\varphi^{\mathbf{x}}$ be the associated flow. We want to study the divergence of nearby points under $\varphi^{\mathbf{x}}$. By Proposition 2.14-(6), the exponential map is a smooth diffeomorphism when restricted to a sufficiently small neighbourhood $\mathcal{U}$ of $\mathbf{0} \in \mathfrak{g}$. Let $g \in \exp(\mathcal{U}) \subset G$ be a point sufficiently close to the identity e, so that we can write $g = \exp(\mathbf{z})$ for some $\mathbf{z} \in \mathcal{U}$. If we want to move between the points $\varphi_t^{\mathbf{x}}(e) = \exp(t\mathbf{x})$ and $\varphi_t^{\mathbf{x}}(g) = g\exp(t\mathbf{x})$, we need to multiply by

$$\exp(-t\mathbf{x})g\exp(tx) = \exp(-t\mathbf{x})\exp(\mathbf{z})\exp(tx) = \exp(\exp(-t\mathbf{x})\mathbf{z}\exp(tx)) = \exp(\mathrm{Ad}(\exp(t\mathbf{x}))\mathbf{z}),$$

where we used Proposition 2.14-(4). In other words, the exponential of the Adjoint tells us how nearby points diverge.

Let us be more precise. Let us fix $t \in \mathbb{R}$ and consider the time-t map $\varphi_t^{\mathbf{x}}\colon G \to G$. We compute its differential $D\varphi_t^{\mathbf{x}}$ acting on tangent vectors. Fix $g \in G$ and $Z \in \mathrm{Lie}(G)$, which we identify with $\mathbf{z} \in \mathfrak{g}$ as usual. In order to compute the image $[D\varphi_t^{\mathbf{x}}(Z)]_g$ of the vector field Z at the point g, we fix an arbitrary smooth function f on G so that

$$\begin{aligned} [D\varphi_t^{\mathbf{x}}(Z)]_g(f) &= Z_{\varphi_{-t}^{\mathbf{x}}(g)}(f \circ \varphi_t^{\mathbf{x}}) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} (f \circ \varphi_t^{\mathbf{x}})(\varphi_{-t}^{\mathbf{x}}(g)\exp(s\mathbf{z})) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} f(g\exp(-t\mathbf{x})\exp(s\mathbf{z})\exp(t\mathbf{x})) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} f\big(g\exp[s\,\mathrm{Ad}(\exp(t\mathbf{x}))\mathbf{z}]\big) \\ &= [\mathrm{Ad}(\exp(t\mathbf{x}))\mathbf{z}]_g(f), \end{aligned}$$

where, in the last equality, we have used Proposition 2.14-(4). Therefore, we conclude

$$D\varphi_t^{\mathbf{x}}(Z) = \mathrm{Ad}(\exp(t\mathbf{x}))\mathbf{z}, \tag{2.8}$$

that is, the action of $\mathrm{Ad}(\exp(t\mathbf{x}))$ on $\mathfrak{g}$ describes how tangent vectors evolve under $\varphi^{\mathbf{x}}$.

Exercise 2.19. *Let $\mathscr{B} := \{\mathbf{u},\mathbf{a},\mathbf{v}\}$ the basis of $\mathfrak{sl}(2,\mathbb{R})$ we introduced in §2.1.2. For all $\mathbf{x} \in \mathscr{B}$ and any given $t \in \mathbb{R}$, compute explicitly the matrix associated to $D\varphi_t^{\mathbf{x}}$ with respect to $\mathscr{B}$. What is the difference between* $\mathbf{a}$ *and the other two elements of $\mathscr{B}$?*

For all $\mathbf{x} \in \mathfrak{g}$, let us call $\mathfrak{ad}_{\mathbf{x}} := [\mathbf{x},\cdot]_{\mathfrak{g}}$ the linear map $\mathfrak{ad}_{\mathbf{x}} \colon \mathfrak{g} \to \mathfrak{g}$. It is called the *adjoint endomorphism of* $\mathbf{x}$, and it can be expressed by a matrix $\mathfrak{ad} \in \mathrm{Mat}(k,\mathbb{R})$. Clearly, this matrix is not invertible. From the proof of Lemma 2.9, it follows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(-t\mathbf{x})) = \mathfrak{ad}_{\mathbf{x}}, \tag{2.9}$$

from which one deduces

$$\mathrm{Ad}(\exp(t\mathbf{x})) = \exp(\mathfrak{ad}_{\mathbf{x}}) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathfrak{ad}_{\mathbf{x}}^k.$$

While $\mathrm{Ad}(\exp(t\mathbf{x}))$ describes the divergence of left-invariant vector fields (and nearby points) under the flow $\varphi^{\mathbf{x}}$, the map $\mathfrak{ad}_{\mathbf{x}} = \mathscr{L}_X$ describes its infinitesimal version, that is how vector fields diverge "infinitesimally".

Again, let us be more precise. Let us consider two flows φ_t^X and φ_t^Y generated by the vector fields $X,Y \in \mathrm{Lie}(G)$. If φ_t^X and φ_t^Y commute, that is if $\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X$ for all $t,s \in \mathbb{R}$, then for any fixed $t \in \mathbb{R}$, the differential $D\varphi_t^X$ of the smooth map φ_t^X maps the vector field Y into itself. If the two flows do not commute, then $D\varphi_t^X$ maps Y smoothly into another smooth vector field $Z = Z(t)$. The Lie derivative describes the "infinitesimal change" of Y when moved by $D\varphi_t^X$. More precisely, the *Lie derivative $\mathscr{L}_X(Y)$ of Y with respect to X at $g \in G$* is defined by

$$\mathscr{L}_X(Y) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} D\varphi_{-t}^X(Y) = \lim_{t\to 0} \frac{D\varphi_{-t}^X(Y) - Y}{t}.$$

The reader might be familiar with the formula

$$\mathscr{L}_X(Y) = XY - YX,$$

we now show that the Lie derivative coincides with the bracket operation we defined in §2.1.2.

Proposition 2.20. *Let $\mathbf{x},\mathbf{y} \in \mathfrak{g}$, identified with $X,Y \in \mathrm{Lie}(G)$. Then,*

$$\mathscr{L}_X(Y) = [\mathbf{x},\mathbf{y}]_{\mathfrak{g}},$$

which is called the Lie bracket *of* $\mathbf{x}$ *and* $\mathbf{y}$.

Proof. The claim follows immediately from (2.8) and (2.9), since

$$\mathscr{L}_X(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} D\varphi_{-t}^X(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(-t\mathbf{x}))\mathbf{y} = \mathfrak{ad}_{\mathbf{x}}(\mathbf{y}) = [\mathbf{x},\mathbf{y}]_{\mathfrak{g}}.$$

□

Therefore, the geometric interpretation of the Lie bracket $[\mathbf{x},\mathbf{y}]_{\mathfrak{g}}$ is to describe the infinitesimal distortion of the left-invariant vector field $Y = \mathbf{y}$ under the action of the flow generated by $X = \mathbf{x}$. The Lie derivative (the Lie bracket) makes the space $\mathrm{Lie}(G) = \mathfrak{g}$ in an algebra.

Recall that, by definition, an *ideal* $\mathfrak{k}$ of $\mathfrak{g}$ is a vector subspace with the property that $[\mathfrak{k},\mathfrak{g}]_{\mathfrak{g}} \subset \mathfrak{k}$.

Definition 2.21. *A Lie algebra $\mathfrak{g}$ is* simple *if has no non-trivial ideals, namely if $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{k} = \mathfrak{g}$ or $\mathfrak{k} = \{\mathbf{0}\}$. A matrix Lie group G is* simple *if its lie algebra $\mathfrak{g}$ is simple.*

Exercise 2.22. *Show that $\mathrm{SL}(2,\mathbb{R})$ is simple.*

2.3 Haar, Killing, Casimir

2.3.1 The Haar measure

Using the differential of the left-multiplication maps we can not only define vector fields starting from a single vector at the identity, but we can also construct a measure on G starting from "a determinant" on $\mathfrak{g}$. This measure will be one of the fundamental objects of this course. Let us see how to do this.

Let V be a k-dimensional real vector space. It is a standard fact from linear algebra that there exists a k-multilinear alternating form on V which is unique up to scalar multiplication; that is, the space $\wedge^k V^*$ is one dimensional. In order to explicilty write one of such multilinear alternating forms ω, one can do the following: choose a basis $\{v_1, \dots, v_k\}$ of V and identify V with $\mathbb{R}^k$ by means of this basis (i.e., identify $w = a_1 v_1 + \cdots + a_k v_k$ with $(a_1, \dots, a_k) \in \mathbb{R}^k$). Then, for any k vectors $w^{(1)}, \dots, w^{(k)}$, with $w^{(j)} = \sum_i a_i^{(j)} v_i$, consider the matrix W whose j-th row is $(a_1^{(j)}, \dots, a_k^{(j)})$. Associated to this choice of basis, we can define ω by

$$\omega(w^{(1)}, \dots, w^{(k)}) = \det W.$$

A different choice of basis would have the effect of multiplying ω by the determinant of the matrix expressing the change of basis; in particular all possible multilinear alternating forms are multiples of each other.

Let us now turn to matrix Lie groups. Let k be the dimension of $\mathfrak{g}$, and fix a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ of $\mathfrak{g}$. Let ω_e be the associated multilinear alternating form on $\mathfrak{g} \simeq T_e G$. Using the left-multiplication maps L_g, we can define a multilinear alternating form ω_g on the tangent space of any other point $g \in G$ by pull-back, namely

$$\omega_g(X_g^{(1)}, \dots, X_g^{(k)}) := \omega_e(DL_{g^{-1}}(g)X_g^{(1)}, \dots, DL_{g^{-1}}(g)X_g^{(k)}) \quad \text{for any } X_g^{(1)}, \dots, X_g^{(k)} \in T_g G.$$

Notice that, indeed, since $L_{g^{-1}}(g) = e$, its differential $DL_{g^{-1}}(g)$ maps $T_g G$ to $T_e G$.

In other words, from any choice of basis on $\mathfrak{g}$, we have defined a *k-differential form*; in formal terms, a section of the vector bundle $\wedge^k T^* G \to G$. From a k-differential form ω, we obtain a positive measure μ by taking its absolute value, $\mathrm{d}\mu = |\omega|$.

Concretely, we fix a basis $X_e^{(1)}, \dots, X_e^{(k)}$ of $T_e G$. Using the differentials of L_g, we obtain vector fields $X^{(1)}, \dots, X^{(k)} \in \mathrm{Lie}(G)$ by $X_g^{(j)} = DL_g(e)X_e^{(j)}$. Then, we take their dual $\mathrm{d}X^{(j)}$: these are differential 1-forms defined by saying that for all $g \in G$, we have $\mathrm{d}X_g^{(j)}(X_g^{(i)}) = \delta_{i,j}$ (i.e., 1 if $i = j$ and 0 otherwise). We define the measure μ by saying that for all continuous functions $f \colon G \to \mathbb{R}$,

$$\int_G f(g)\, \mathrm{d}\mu(g) = \int_G f(g)\, \mathrm{d}X_g^{(1)} \cdots \mathrm{d}X_g^{(k)}.$$

With a little extra effort, we can complete the proof of the following important result.

Theorem 2.23. *Let G be a matrix Lie group. There exists a smooth measure μ on G which is invariant by all left-multiplication maps, namely for all continuous functions $f \colon G \to \mathbb{R}$ and for all $h \in G$ we have*

$$\int_G f(hg)\, \mathrm{d}\mu(g) = \int_G f(g)\, \mathrm{d}\mu(g).$$

This measure μ is unique up to scalar and is called the (left) Haar measure *on G.*

Proof. Let us consider a measure μ constructed as above. The fact that μ is a smooth measure is a consequence of Proposition 2.11, since the cotangent vector fields $\mathrm{d}X^{(j)}$ are smooth. Let us verify that $\mu = L_g^* \mu$. First of all, we claim that the pullback $(L_g)^*(\mathrm{d}X^{(i)})$ of $\mathrm{d}X^{(i)}$ is again $\mathrm{d}X^{(i)}$

for all $i = 1,\dots,k$. In order to show this, it is enough to show that for all fixed $h \in G$ we have $[(L_g)^*(\mathrm{d}X^{(i)})]_h(X_h^{(j)}) = \delta_{i,j}$, by the definition of $\mathrm{d}X^{(i)}$. Indeed, we have that

$$[(L_g)^*(\mathrm{d}X^{(i)})]_h(X_h^{(j)}) = \mathrm{d}X^{(i)}_{L_g(h)}(DL_g(X_h^{(j)})) = \mathrm{d}X^{(i)}_{gh}(X^{(j)}_{gh}) = \delta_{i,j},$$

hence our claim is proved.

Now, for any continuous function $f\colon G \to \mathbb{R}$, by the change of variable formula, we have

$$\begin{aligned}\int_G f \circ L_h \,\mathrm{d}\mu &= \int_{L_h(G)} f\,(L_h)^*(\mathrm{d}X^{(1)}\cdots \mathrm{d}X^{(k)}) = \int_{L_h(G)} f\,[(L_h)^*(\mathrm{d}X^{(1)})]\cdots[(L_h)^*(\mathrm{d}X^{(k)})] \\ &= \int_{hG} f\,\mathrm{d}X^{(1)}\cdots \mathrm{d}X^{(k)} = \int_G f\,\mathrm{d}\mu.\end{aligned}$$

This completes the proof of the existence of a measure as in the statement of the theorem.

Let us verify the uniqueness claim. The idea is that a left-invariant differential k-form is uniquely determined by its restriction to T_eG, and, by the previous discussion, all multilinear alternating forms on T_eG are multiples of each other. Formally, let ν be another smooth measure as in the statement of the theorem. Then, ν is defined by integrating a smooth k-differential form. In particular, there exists a smooth function $f\colon G \to \mathbb{R}_{\geq 0}$ such that for all $g \in G$ we have $\mathrm{d}\nu(g) = f(g)\,\mathrm{d}X^{(1)}\cdots\mathrm{d}X^{(k)}$. By invariance under L_h for all $h \in G$, we deduce that f must be constant. This proves the uniqueness claim and hence completes the proof. □

The important point to remember is that on any matrix Lie group, up to a normalization factor, there is a unique smooth measure that is invariant by all left translations $g \mapsto hg$. We will come back to the case of $\mathrm{SL}(2,\mathbb{R})$ in Chapter 4. In the case of the Heisenberg group, the Haar measure is actually the Lebesgue measure on $\mathbb{R}^3$, as the next exercise shows.

Exercise 2.24. *Let μ denote the Haar measure on* Heis*, normalized so that*

$$\mu\left(\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} : x,y,z \in [0,1]\right\}\right) = 1.$$

For any function $f\colon$ Heis $\to \mathbb{R}$ and any $g = \left(\begin{smallmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{smallmatrix}\right) \in$ Heis, write $f(g) = f(x,y,z)$. Show that for any continuous function $f\colon$ Heis $\to \mathbb{R}$ we have

$$\int_{\mathrm{Heis}} f(g)\,\mathrm{d}\mu(g) = \int_{\mathbb{R}^3} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

In oder words, μ coincides with the Lebesgue measure on $\mathbb{R}^3$.

2.3.2 The Killing form

We can define a symmetric bilinear form on $\mathfrak{g}$ as follows. Recall that, for all $\mathbf{x} \in \mathfrak{g}$, its adjoint is given by $\mathfrak{ad}_{\mathbf{x}} = [\mathbf{x},\cdot]_{\mathfrak{g}} \in \mathrm{Mat}(k,\mathbb{R})$, where $k = \dim \mathfrak{g}$.

Definition 2.25. *The Killing form* B *is a bilinear symmetric form on $\mathfrak{g}$ defined by*

$$\mathrm{B}(\mathbf{x},\mathbf{y}) := \mathrm{tr}(\mathfrak{ad}_{\mathbf{x}} \circ \mathfrak{ad}_{\mathbf{y}}), \qquad \textit{for all } \mathbf{x},\mathbf{y} \in \mathfrak{g}.$$

The fact that B is bilinear follows from the linearity of the Lie bracket $\mathfrak{ad}_{a\mathbf{x}+b\mathbf{y}} = [a\mathbf{x}+b\mathbf{y},\cdot]_{\mathfrak{g}} = a[\mathbf{x},\cdot]_{\mathfrak{g}} + b[\mathbf{y},\cdot]_{\mathfrak{g}} = a\mathfrak{ad}_{\mathbf{x}} + b\mathfrak{ad}_{\mathbf{y}}$, as the reader can easily check. The symmetry of B follows from the properties of the trace: for any matrices A,B we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

The Killing form is Ad-invariant, as the next lemma shows.

Lemma 2.26. *For all $g \in G$, we have*

$$\mathrm{B}(\mathrm{Ad}(g)\mathbf{x}, \mathrm{Ad}(g)\mathbf{y}) = \mathrm{B}(\mathbf{x},\mathbf{y}), \qquad \text{for all } \mathbf{x},\mathbf{y} \in \mathfrak{g}.$$

Proof. We first claim that, for all $g \in G$ and $\mathbf{x} \in \mathfrak{g}$, we have

$$\mathfrak{ad}_{\mathrm{Ad}(g)\mathbf{x}} = \mathrm{Ad}(g) \circ \mathfrak{ad}_{\mathbf{x}} \circ \mathrm{Ad}(g)^{-1}.$$

Indeed, let $\mathbf{y} \in \mathfrak{g}$. Straightforward computations give us

$$\begin{aligned}\mathfrak{ad}_{\mathrm{Ad}(g)\mathbf{x}}(\mathbf{y}) &= [\mathrm{Ad}(g)\mathbf{x},\mathbf{y}]_{\mathfrak{g}} = g^{-1}\mathbf{x}g\mathbf{y} - \mathbf{y}g^{-1}\mathbf{x}g = g^{-1}(\mathbf{x}g\mathbf{y}g^{-1} - g\mathbf{y}g^{-1}\mathbf{x})g \\ &= \mathrm{Ad}(g)(\mathbf{x}\,\mathrm{Ad}(g^{-1})\mathbf{y} - \mathrm{Ad}(g^{-1})\mathbf{y}\mathbf{x}) = (\mathrm{Ad}(g) \circ \mathfrak{ad}_{\mathbf{x}} \circ \mathrm{Ad}(g)^{-1})(\mathbf{y}),\end{aligned}$$

which proves our claim. Then,

$$\begin{aligned}\mathrm{B}(\mathrm{Ad}(g)\mathbf{x}, \mathrm{Ad}(g)\mathbf{y}) &= \mathrm{tr}(\mathfrak{ad}_{\mathrm{Ad}(g)\mathbf{x}} \circ \mathfrak{ad}_{\mathrm{Ad}(g)\mathbf{y}}) \\ &= \mathrm{tr}\left((\mathrm{Ad}(g) \circ \mathfrak{ad}_{\mathbf{x}} \circ \mathrm{Ad}(g)^{-1}) \circ (\mathrm{Ad}(g) \circ \mathfrak{ad}_{\mathbf{y}} \circ \mathrm{Ad}(g)^{-1})\right) \\ &= \mathrm{tr}(\mathrm{Ad}(g) \circ \mathfrak{ad}_{\mathbf{x}} \circ \mathfrak{ad}_{\mathbf{y}} \circ \mathrm{Ad}(g)^{-1}) \\ &= \mathrm{tr}(\mathfrak{ad}_{\mathbf{x}} \circ \mathfrak{ad}_{\mathbf{y}}) = \mathrm{B}(\mathbf{x},\mathbf{y}),\end{aligned}$$

where we used the fact that the trace is invariant under conjugation. □

Let us compute the Killing form in the case of $\mathfrak{sl}(2,\mathbb{R})$. Let us fix the usual basis $\{\mathbf{u},\mathbf{a},\mathbf{v}\}$ as in §2.1.2. Then, using the computations (2.1), we can write

$$\mathfrak{ad}_{\mathbf{u}} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{ad}_{\mathbf{a}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mathfrak{ad}_{\mathbf{v}} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In order to compute the Killing form, it is enough to compute six matrices, for example $\mathfrak{ad}_{\mathbf{u}}^2, \mathfrak{ad}_{\mathbf{a}}^2, \mathfrak{ad}_{\mathbf{v}}^2$, and $\mathfrak{ad}_{\mathbf{u}} \circ \mathfrak{ad}_{\mathbf{a}}, \mathfrak{ad}_{\mathbf{a}} \circ \mathfrak{ad}_{\mathbf{v}}, \mathfrak{ad}_{\mathbf{u}} \circ \mathfrak{ad}_{\mathbf{v}}$, and look at their traces. In matrix form, we get

$$\mathrm{B}(\mathbf{x},\mathbf{y}) = \mathbf{x}^T \begin{pmatrix} 0 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix} \mathbf{y} \tag{2.10}$$

where ^{T} denotes the transpose. We conclude that the Killing form B on $\mathfrak{sl}(2,\mathbb{R})$ is non-degenerate and has signature $(2,1)$. From this, we can prove an important geometrical fact that links the algebraic properties of $\mathrm{SL}(2,\mathbb{R})$ to hyperbolic geometry.

Proposition 2.27. *Let $\mathscr{H}$ be the hyperboloid model of the hyperbolic plane, that is the set*

$$\mathscr{H} := \{\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 > 0 \text{ and } x_1^2 - x_2^2 - x_3^2 = 1\},$$

equipped with the hyperbolic distance

$$d_{\mathscr{H}}(\mathbf{x},\mathbf{y}) := \operatorname{arcosh}(x_1y_1 - x_2y_2 - x_3y_3).$$

The group

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm e\}$$

acts on $\mathscr{H}$ by hyperbolic isometries.

Proof. We start by diagonalizing the Killing form, namely we can find positive constants a_1, a_2, a_3 such that the Killing form with respect to the basis $\{a_1(\mathbf{u}-\mathbf{v}), a_2\mathbf{a}, a_3(\mathbf{u}+\mathbf{v})\}$ can be expressed as

$$\mathrm{B}(\mathbf{x},\mathbf{y}) = -x_1y_1 + x_2y_2 + x_3y_3.$$

By Lemma 2.26, since $\mathrm{Ad}(g)$ preserves B, we have that $\mathrm{Ad}(g)$ maps the set of vectors $\mathbf{x} \in \mathbb{R}^3$ which satisfy $\mathrm{B}(\mathbf{x},\mathbf{x}) = -1$ into itself. Moreover, one can verify by hand that if $x_1 > 0$, then the first coordinate of $\mathrm{Ad}(g)\mathbf{x}$ also is positive. Therefore, $\mathrm{Ad}(g)$ maps $\mathscr{H} = \{\mathbf{x} \in \mathbb{R}^3 : \mathrm{B}(\mathbf{x},\mathbf{x}) = -1 \text{ and } x_1 > 0\}$ into itself. Again by Lemma 2.26, $\mathrm{Ad}(g)$ is an isometry with respect to $d_{\mathscr{H}}$.

We have shown that

$$\mathrm{Ad}\colon \mathrm{SL}(2,\mathbb{R}) \to O(2,1)$$

is a smooth homomorphism into the orthogonal group of signature $(2,1)$. It remains to show that its kernel is $\{\pm e\}$. Clearly, $-e \in \ker(\mathrm{Ad})$, so we need to verify the other inclusion. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \ker(\mathrm{Ad})$, then $\mathrm{Ad}(g)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ implies $d = \pm 1$ and $c = 0$, so that $a = \mp 1$. Repeating the same argument with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ shows that $b = 0$ and hence $g = \pm e$, which concludes the proof. $\square$

As we just saw, in the case of $\mathrm{SL}(2,\mathbb{R})$, the Killing form is non-degenerate. This is not always the case.

Exercise 2.28. *Show that the Killing form on* Heis *is identically zero, that is* $\mathrm{B}(\mathbf{x},\mathbf{y}) = 0$ *for all* $\mathbf{x},\mathbf{y} \in \mathfrak{h}$.

Lie groups for which the Killing form is non-degenerate have a special name. Proposition 2.30 below, which we will not prove, gives some equivalent conditions.

Definition 2.29. *A matrix Lie group G for which the Killing form on its Lie algebra $\mathfrak{g}$ is non-degenerate is called* semisimple.

Proposition 2.30 (Cartan's Criterion for semisimplicity)**.** *Let G be a matrix Lie group and $\mathfrak{g}$ its Lie algebra. The following are equivalent:*

1. *G is semisimple,*
2. *$\mathfrak{g}$ is a direct sum of simple algebras,*
3. *$\mathfrak{g}$ has no non-zero abelian ideals.*

2.3.3 The Casimir operator

We conclude this section by introducing the Casimir operator, which will play a key role when we discuss the quantitative ergodic properties of homogeneous flows on $\mathrm{SL}(2,\mathbb{R})$ in Chapter 7.

Recall that any element $\mathbf{x} \in \mathfrak{g}$ can be seen as a (left-invariant) vector field $X = \mathbf{x} \in \mathrm{Lie}(G)$ on G, and hence as a first order differential operator. The Casimir operator is a second order differential operator on G, which, roughly speaking, plays the same role in the harmonic analysis on G that the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $\mathbb{R}$ does in the Fourier analysis in one variable.

Let G be a semisimple matrix Lie group, and let B be the Killing form. Let $\mathscr{B} = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis of $\mathfrak{g}$. Since B is non-degenerate, we can construct the *dual basis* $\widehat{\mathscr{B}} = \{\widehat{\mathbf{x}}_1, \dots, \widehat{\mathbf{x}}_k\}$ given by the condition $\mathrm{B}(\mathbf{x}_i, \widehat{\mathbf{x}}_j) = \delta_{i,j}$.

Definition 2.31. *The* Casimir operator $\square = \square_{\mathscr{B}}$ *associated to the basis $\mathscr{B}$ is the second order differential operator on G given by*

$$\square = \sum_{i=1}^{k} \mathbf{x}_i \widehat{\mathbf{x}}_i.$$