Homogeneous Dynamics

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Welcome to the *Homogeneous Dynamics* course!

These lectures are intended to be an introduction to homogeneous dynamics, which nowadays is a very active subject of research. Homogeneous dynamics lies at the intersection of many areas in pure mathematics: of course, dynamics and ergodic theory, but also geometry, Lie group theory, representation theory, and more. There are also remarkable connections to several problems in number theory, some of which will be explored during the course.

The literature in the subject is vast and it would be impossible to cover it all. The choice I made to select the specific topics which will be discussed during these lectures was motivated mainly by two reasons. In part, of course, there are my personal preferences; more importantly, I wanted to focus on concrete examples (where computations can be carried out explicitly) which can help to build the intuition and provide insights on more general and abstract situations. It is my hope that this introduction can sparkle the curiosity in students to pursue this line of research.

One final disclaimer before starting: these lecture notes are a work-in-progress, and as such they need to be read with critical thinking. I tried to minimize the number of errors, but it would be widely optimistic of me to believe that there are none. If you spot mistakes, or have any comment in general, please let me know by sending me an email to `davide.ravotti@gmail.com`.

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Chapter 1

A quick recap: the case of linear flows on tori

In this first chapter, we will quickly review some basic notions in dynamics and ergodic theory, which the reader is assumed to be already familiar with. An exhaustive treatment of these topics can be found, for example, in [4, Chapters 2, 4.3].

In parallel, we will look at linear flows on tori. Very roughly speaking, the course consists in studying their non-Abelian analogues, as we will see later. Thus, focusing on this simple case can be a nice "warm-up" exercise.

1.1 Smooth flows on manifolds

The subject of this course is a special class of smooth flows. Let us recall the general definition.

Definition 1.1. *Let M be a smooth manifold, and let* $\mathrm{Diff}(M)$ *be the group of its diffeomorphisms. A* smooth flow $\varphi\colon \mathbb{R}\times M \to M$ *is a smooth map which satisfies*

$$\varphi_0 = \mathrm{Id}, \quad \textit{and} \quad \varphi_{t+s} = \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t, \quad \textit{for all } t,s\in\mathbb{R},$$

where $\varphi_t := \varphi(t,\cdot) \in \mathrm{Diff}(M)$.

In particular, Definition 1.1 implies that the continuous curve $t \mapsto \varphi_t$ is a group homomorphism between $\mathbb{R}$ and $\mathrm{Diff}(M)$, and $\{\varphi_t\}_{t\in\mathbb{R}}$ is said to be a *1-parameter group* of diffeomorphisms. We will often identify φ with $\{\varphi_t\}_{t\in\mathbb{R}}$.

Given a smooth flow φ, we can define a vector field X on M by

$$Xf(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\circ\varphi_t(p), \quad \text{for all } f\in\mathscr{C}^\infty(M) \text{ and } p\in M.$$

The vector field X is called the *infinitesimal generator* of φ. Vice-versa, one can prove that, at least when M is compact, for any given smooth vector field X, there exists a unique smooth flow φ with infinitesimal generator X.

From here onward, M always denotes a smooth manifold, not necessarily compact, and φ is a smooth flow on M.

Let us turn to a very concrete example. Let $\mathbb{T}^n$ be the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. We will denote points in $\mathbb{T}^n$ using the symbol $[\![\cdot]\!]$, namely $[\![\mathbf{x}]\!] := \mathbf{x}+\mathbb{Z}^n$. For any $\mathbf{v}\in\mathbb{R}^n\setminus\{0\}$, we define the *linear flow in direction* $\mathbf{v}$ to be the smooth flow $\varphi^{\mathbf{v}}$ on $\mathbb{T}^n$ given by

$$\varphi_t^{\mathbf{v}}([\![\mathbf{x}]\!]) = [\![\mathbf{x}+t\mathbf{v}]\!], \quad \text{for } t\in\mathbb{R}.$$

It is easy to check that indeed $\varphi^{\mathbf{v}}$ is a well-defined smooth flow according to Definition 1.1. The associated infinitesimal generator X is the derivative in direction $\mathbf{v}$: for any $p = [\![\mathbf{x}]\!] \in \mathbb{T}^n$,

$$Xf(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f([\![\mathbf{x}+t\mathbf{v}]\!]) = \mathbf{v}\cdot\nabla_p f.$$

In other words, under the usual identification of the tangent space $T_p\mathbb{T}^n$ at p with $\mathbb{R}^n$, we have $X = \mathbf{v}$. The associated 1-parameter subgroup consists of the translations $\varphi_t^{\mathbf{v}} \colon [\![\mathbf{x}]\!] \mapsto [\![\mathbf{x}+t\mathbf{v}]\!]$ in direction $\mathbf{v}$.

Let us rephrase the example above in more algebraic terms. Our setting was the following. We considered the Abelian group $(\mathbb{R}^n,+)$, and we fixed a 1-dimensional subgroup $V = \{t\mathbf{v} : t \in \mathbb{R}\} < \mathbb{R}^n$. This subgroup V is everywhere tangent to the constant vector field $\mathbf{v} \in \mathbb{R}^n$, where we identified $\mathbb{R}^n = T_{\mathbf{x}}\mathbb{R}^n$ for all $\mathbf{x} \in \mathbb{R}^n$. In turn, V is identified with the 1-parameter group of translations

$$\{(\mathbf{x} \mapsto \mathbf{x}+t\mathbf{v}) : t \in \mathbb{R}\} \subset \mathrm{Diff}(\mathbb{R}^n).$$

We then fixed the discrete subgroup $\mathbb{Z}^n < \mathbb{R}^n$ and we considered the quotient space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The key observation is that the 1-parameter group of translations $\mathbf{x} \mapsto \mathbf{x}+t\mathbf{v}$ associated to $\mathbf{v}$ *descends to the quotient*, which means that they commute with the canonical projection $\mathbf{x} \mapsto [\![\mathbf{x}]\!] = \mathbf{x}+\mathbb{Z}^n$. This tells us that, under the projection, we obtain a well-defined 1-parameter group of diffeomorphisms of $\mathbb{T}^n$, and hence a smooth flow $\varphi^{\mathbf{v}}$.

Homogeneous flows, which are the subject of this course, are a “non-Abelian” generalization of this simple example. Namely, we will replace

- $\mathbb{R}^n$ with a Lie group G (the Heisenberg group in Chapter 3 and $\mathrm{SL}(2,\mathbb{R})$ in Chapters 4–7),
- $\mathbb{Z}^n$ with a *lattice* Γ (a discrete subgroup of G with some additional properties that we will discuss in §2.4),
- $\mathbb{T}^n$ with the *left*[1] quotient $\Gamma\backslash G = \{\Gamma g : g \in G\}$,
- $V = \{t\mathbf{v} : t \in \mathbb{R}\}$ with a 1-parameter subgroup $\{g_t : t \in \mathbb{R}\}$ of G (generated by a “constant” vector field, which in the case above was $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$),
- $\varphi_t^{\mathbf{v}} \colon [\![\mathbf{x}]\!] \mapsto [\![\mathbf{x}+t\mathbf{v}]\!]$ with the multiplication *on the right*[2] $\Gamma g \mapsto \Gamma g \cdot g_t$.

We will make this analogy precise in the next chapters.

1.2 The topology of orbits

Let $\varphi \colon \mathbb{R}\times M \to M$ be a smooth flow, and let $p \in M$. The *orbit* of p is the set

$$\mathcal{O}_\varphi(p) = \{\varphi_t(p) : t \in \mathbb{R}\} \subset M.$$

Note that the orbit of any point $p \in M$ is an *immersed* smooth curve in M.

In dynamics, one is interested in the behaviour of orbits: do they “close up”? Do they accumulate in some regions? Do they visit all parts of the space? From the topological point of view, it is particularly important to try to understand their accumulation points and closure $\overline{\mathcal{O}_\varphi(p)} \subseteq M$.

[1]Note that, in the Abelian case, left and right cosets coincide.

[2]Taking left quotients and multiplying on the right is the conventional choice, but of course one could do the opposite (taking right quotients and multiplying on the left). Note that, again, multiplying on the right and projecting on the left quotient $\Gamma\backslash G$ commute.

Definition 1.2. *A point p is a* fixed point *if* $\mathcal{O}_\varphi(p) = \{p\}$. *A point p is* periodic *if there exists $T > 0$ such that*

$$\varphi_T(p) = p. \tag{1.1}$$

If p is periodic but not a fixed point, its period *is the smallest $T > 0$ for which* (1.1) *holds.*

Exercise 1.3. (a) *Show that the set of $T \in \mathbb{R}$ for which* (1.1) *holds is a subgroup of $\mathbb{R}$, in particular if p is periodic but not a fixed point, its period is well-defined.*

(b) *Show that, if p is a periodic point of period T, then its orbit is an embedded closed curve and*

$$\mathcal{O}_\varphi(p) = \{\varphi_t(p) : t \in [0,T]\}.$$

Periodic and fixed points have the smallest possible orbit closures, since their orbits are themselves closed. On the opposite, we may have points with *dense* orbits, that is, points whose orbit closure is the largest possible.

Definition 1.4. *A smooth flow φ is* minimal *is all orbits are dense, namely if*

$$\overline{\mathcal{O}_\varphi(p)} = M, \quad \text{for all } p \in M.$$

Let us look at our motivating example. In the case of linear flows on the two dimensional torus, we have a pleasant dichotomy.

Theorem 1.5. *Let $\varphi^{\mathbf{v}} \colon \mathbb{T}^2 \to \mathbb{T}^2$ be a linear flow in direction $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 \setminus \{0\}$. If v_1 and v_2 are* rationally dependent*, then every orbit is periodic; otherwise, if v_1 and v_2 are* rationally independent*, the flow $\varphi^{\mathbf{v}}$ is minimal.*

We will say that $\varphi^{\mathbf{v}}$ is a *rational* linear flow if we are in the first case, and it is an *irrational* linear flow if we are in the second one.

Before diving into the proof of Theorem 1.5, let us make a couple of simple observations. First, note that a rescaling $a\mathbf{v}$ of $\mathbf{v}$ for some $a > 0$ does not change the behaviour of the orbits of the flow. If $v_2 = 0$, then $v_1 \neq 0$. It is clear that all orbits of $\varphi^{\mathbf{v}}$ are periodic of period $1/v_1$ and consist of horizontal cirlces of the form $\mathbb{T}^1 \times \{p_2\}$, with $p_2 \in \mathbb{T}^1$, hence the result is proved in this case. If $v_2 \neq 0$, then, without loss of generality, we can assume that $\mathbf{v} = (v, 1)$. We divide the proof of Theorem 1.5 into two cases: when $v \in \mathbb{Q}$ (the rational case) and when $v \notin \mathbb{Q}$ (the irrational case).

Proof of Theorem 1.5 - *Case $v \in \mathbb{Q}$.* Let us write $v = a/b$ in reduced terms. Then, we claim that all orbits are periodic of period b. Indeed, let $p = [\![x_1, x_2]\!] \in \mathbb{T}^2$. Then,

$$\varphi^{\mathbf{v}}_b(p) = [\![x_1 + b \cdot a/b, x_2 + b]\!] = [\![x_1, x_2]\!] + [\![a, b]\!] = p.$$

If $T > 0$ is such that $\varphi^{\mathbf{v}}_T(p) = p$, then, looking at its second coordinate, we see that $x_2 + T + \mathbb{Z} = x_2 + \mathbb{Z}$. Hence $T \in \mathbb{N}$, and, looking at the first coordinate, $x_1 + T \cdot a/b + \mathbb{Z} = x_1 + \mathbb{Z}$. This implies that $(Ta)/b \in \mathbb{Z}$. Since a and b are coprime by assumption, b divides T. This proves the claim and hence the theorem in the rational case. $\square$

Proof of Theorem 1.5 - *Case $v \notin \mathbb{Q}$.* We first claim that it is enough to prove the following statement.

($\star$) The *circle rotation* $R_v \colon \mathbb{T}^1 \to \mathbb{T}^1$ defined by $R_v([\![x]\!]) = [\![x + v]\!]$ is *minimal* (where, here, $[\![x]\!] = x + \mathbb{Z}$).

We leave as an exercise to the reader to check that indeed it is sufficient to prove $(\star)$. The idea is that the orbit of a point $p = [\![x_1, x_2]\!]$ under the flow $\varphi^{\mathbf{v}}$ is dense in $\mathbb{T}^2$ if and only if its intersection with the horizontal circle $\mathbb{T}^1 \times \{[\![x_2]\!]\}$ is dense in $\mathbb{T}^1 \times \{[\![x_2]\!]\}$. Indeed, the projection on the first coordinate of the intersection of the orbit of p with the circle $\mathbb{T}^1 \times \{[\![x_2]\!]\}$ is precisely the orbit of $[\![x_1]\!]$ under the rotation R_v.

We now focus on proving $(\star)$. Let $p = [\![x]\!] = x + \mathbb{Z} \in \mathbb{T}^1$ and $\varepsilon > 0$ be fixed; choose a natural number $N \geq \varepsilon^{-1}$ and partition $\mathbb{T}^1 \approx [0,1)$ into N intervals $I_k = [(k-1)N^{-1}, kN^{-1})$ for $k = 1, \dots, N$. We need to show that the orbit of p visits all intervals I_k.

Let us consider the set $O_N = \{p, R_v(p), \dots, R_v^N(p)\}$. Since $|O_N| = N+1$, by the Pigeonhole Principle, there exists a $\bar{k} \in \{1, \dots, N\}$ such that the interval $I_{\bar{k}}$ contains at least two distinct elements of O_N, say $R_v^n(p)$ and $R_v^m(p)$, with $n < m$. Let us call w the fractional part of $(m-n)v$. For any $y \in [0,1)$, we have

$$R_v^{m-n}([\![y]\!]) = [\![y + (m-n)v]\!] = [\![y + w]\!] = R_w([\![y]\!]),$$

namely, the map R_v^{m-n} is again a rotation of angle $w \in (0,1)$. Since we showed that the points $p' = R_v^n(p)$ and $R_w(p') = R_v^m(p)$ are both in the same interval $I_{\bar{k}}$, they are at distance less than N^{-1}. It follows that $0 < w < N^{-1} \leq \varepsilon$. Thus, the orbit of p under R_v contains the orbit of p under $R_v^{m-n} = R_w$, which is a rotation of angle less than ε. Since this latter set clearly intersects all intervals I_k, the proof is complete. □

In general, it is a hopeless task to try to understand all orbit closures. They can be quite complicated objects, with "fractal-like" structures and non-integer dimensions. However, in the particular case of linear flows on $\mathbb{T}^2$, orbit closures are well-behaved and we managed to classify all possibilities: we showed that all orbit closures are either the whole space $\mathbb{T}^2$ or circles isomorphic to $\mathbb{T}^1$. In higher dimensions, a similar phenomenon occurs: orbit closures of any linear flow on $\mathbb{T}^n$ are sub-tori isomorphic to $\mathbb{T}^k$, for some $k = 1, \dots, n$ (see Section 1.3.4 below).

1.3 Elements of Ergodic Theory

Ergodic theory is the study of dynamical systems from the point of view of measure theory. The measures on the phase space M that will be relevant for us are Borel invariant measures.

1.3.1 Invariant measures

Definition 1.6. *Let φ be a smooth flow on M. A Borel measure μ on M is an* invariant measure *for φ if for all Borel measurable sets $A \subset M$ and for all $t \in \mathbb{R}$,*

$$\mu(\varphi_t(A)) = \mu(A).$$

If $\mu(M) = 1$, then μ is a probability invariant measure. *The triple (M, φ, μ) is called a* probability preserving flow *(*ppf*, for short).*

The previous definition extends to all functions in $L^1(M) = L^1(M,\mu)$: if (M, φ, μ) is a ppf, then, for every function $f \in L^1(M)$ and for all $t \in \mathbb{R}$, the function $f \circ \varphi_t$ is in $L^1(M)$ and

$$\int_M f \circ \varphi_t \, \mathrm{d}\mu = \int_M f \, \mathrm{d}\mu.$$

Similarly, if $f \in L^2(M)$, then $f \circ \varphi_t \in L^2(M)$ for all $t \in \mathbb{R}$ and

$$\|f \circ \varphi_t\|_2 = \|f\|_2 . \tag{1.2}$$

Let us see some examples of invariant measures. Clearly, the Lebesgue measure on the torus $\mathbb{T}^2$ is an invariant measure for all linear flows $\varphi^{\mathbf{v}}$. If the flow is irrational, we will see in Section 1.3.3 that there are no other invariant probability measures. However, if $\varphi^{\mathbf{v}}$ is rational, then we have uncountably many invariant probability measures supported on periodic orbits. This is a general fact: for any periodic orbit, there is an invariant probability measure supported on such orbit.

Exercise 1.7. (a) *Let $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 \setminus \{0\}$, with v_1, v_2 rationally dependent. Let T be the period of all orbits of $\varphi^{\mathbf{v}}$. Show that T is the ratio between the* covolume *of the discrete subgroup $\mathbb{R}\mathbf{v} \cap \mathbb{Z}^2$ of the line $\mathbb{R}\mathbf{v} \subset \mathbb{R}^2$ and $\|\mathbf{v}\|_2$.*

(b) *For any $p \in M$, let μ_p be the Borel measure defined by*

$$\mu_p(A) := \frac{1}{T} \operatorname{Leb}\{t \in [0,T] : \varphi_t^{\mathbf{v}}(p) \in A\}.$$

Show that μ_p is a probability invariant measure for $\varphi^{\mathbf{v}}$.

(c) *Prove that $\mu_p = \mu_q$ if and only if $\mathcal{O}_{\varphi^{\mathbf{v}}}(p) = \mathcal{O}_{\varphi^{\mathbf{v}}}(q)$. Deduce that $\varphi^{\mathbf{v}}$ has uncountably many probability invariant measures.*

It is actually easy to see that if there is more than one probability invariant measure, then there are uncountably many. Indeed, any convex combination of (probability) invariant measures is a (probability) invariant measure. In other words, probability invariant measures form a *simplex* in the space of probability measures on M.

The reader might wonder whether we are sure to find, in general, at least one probability invariant measure. When M is compact, the following result answers this question affirmatively.

Theorem 1.8 (Krylov-Bogolyubov)**.** *Let φ be a smooth flow on the compact manifold M. There exists one invariant probability measure.*

Proof. Recall that, when M is compact, the set of Borel (signed) measures coincides with $\mathscr{C}(M)^*$, the weak-$*$ dual of $\mathscr{C}(M)$. Recall also that, by Banach-Alaoglu's Theorem, the unit ball in $\mathscr{C}(M)^*$, which contains all (positive) probability measures, is weakly-$*$ compact. Fix any $p \in M$, and consider the family of (positive) probability measures $\{\mu_T\}_{T\in\mathbb{R}}$ given by

$$\mu_T(f) := \frac{1}{T} \int_0^T f \circ \varphi_t(p) \,\mathrm{d}t, \qquad \text{for } f \in \mathscr{C}(M).$$

By compactness, there exists an increasing sequence $T_n \to \infty$ such that μ_{T_n} weakly-$*$ converges to a (positive) probability measure μ. We claim that μ is invariant. Let $f \in \mathscr{C}(M)$ and $r \in \mathbb{R}$; then,

$$\begin{aligned} |\mu_{T_n}(f\circ\varphi_r) - \mu_{T_n}(f)| &= \frac{1}{T_n} \left| \int_0^{T_n} f \circ \varphi_{t+r}(p)\,\mathrm{d}t - \int_0^{T_n} f\circ\varphi_t(p)\,\mathrm{d}t \right| \\ &= \frac{1}{T_n} \left| \int_{T_n}^{T_n+r} f \circ \varphi_t(p)\,\mathrm{d}t - \int_0^{r} f\circ\varphi_t(p)\,\mathrm{d}t \right| \\ &\leq \frac{2r\,\|f\|_{\mathscr{C}(M)}}{T_n} \to 0. \end{aligned}$$

Therefore,

$$0 = \lim_{n\to\infty} |\mu_{T_n}(f\circ\varphi_r) - \mu_{T_n}(f)| = |\mu(f\circ\varphi_r) - \mu(f)|,$$

which shows that μ is an invariant measure for φ. □

We will mostly be concerned with *smooth* invariant measures, namely measures given by integrating a volume form on M. In this case, we can check whether a smooth measure is invariant by computing its Lie derivative with respect to the infinitesimal generator of the flow.

Proposition 1.9. *Let φ be a smooth flow with infinitesimal generator X, and let μ be a smooth probability measure given by a volume form ω on M. Then μ is invariant if and only if $\mathscr{L}_X(\omega) = 0$, where $\mathscr{L}_X(\omega) = \mathrm{d}(i_X\omega)$ is the Lie derivative of ω with respect to X and i is the contraction operator.*

Proof. Let $(\varphi_t)^*$ denote the pull-back by φ_t. By definition of the Lie derivative,

$$\mathscr{L}_X(\omega) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_t)^*(\omega),$$

hence $(\varphi_t)^*(\omega) = \omega$ if and only if $\mathscr{L}_X(\omega) = 0$. By Cartan's formula,

$$\mathscr{L}_X(\omega) = \mathrm{d}(i_X\omega) + i_X(\mathrm{d}\omega) = \mathrm{d}(i_X\omega),$$

which follows from the fact that $\mathrm{d}\omega = 0$ since ω is a n-form, where $n = \dim(M)$. □

Exercise 1.10 (Invariant measures of time-changes)**.** *Let φ be a smooth flow on M with infinitesimal generator X, and let μ be a smooth probability invariant measure. Show that, for any smooth positive function $\alpha \colon M \to \mathbb{R}_{>0}$, the flow*[3] *generated by the vector field αX preserves the measure equivalent to μ with density $1/\alpha$.*

Once we have chosen a probability invariant measure, we can ask about the properties of *typical* points, in other words the properties that are satisfied up to exceptional sets of measure zero. A fundamental result is the recurrence theorem by Poincaré, which, roughly speaking, says that typical orbits will come back close to their initial point infinitely often.

Theorem 1.11 (Poincaré Recurrence Theorem)**.** *Let (M,φ,μ) be a ppf. If $A \subset M$ is a measurable (Borel) set, for* almost every *$p \in A$ there exists an increasing sequence $T_n \to \infty$ such that $\varphi_{T_n}(p) \in A$.*

1.3.2 Ergodicity and the Ergodic Theorems

Given a flow φ on M, we say that a measurable set $A \subset M$ is *invariant* if $\varphi_t(A) = A$ for all $t \in \mathbb{R}$. If (M,φ,μ) is a ppf and $A \subset M$ is an invariant set of positive measure, then we can consider the subsystem (A,φ,μ_A) given by the restriction of the flow φ to A with the conditional probability invariant measure defined by

$$\mu_A(B) := \mu(B\cap A)/\mu(A), \qquad \text{for any measurable set } B.$$

When we have an invariant set of positive measure, we can then reduce ourselves to study a "simpler" system. Intuitivley, the notion of ergodicity plays the role of "indecomposability" in the context of ppf's. That is to say, an ergodic ppf cannot be decomposed into non-trivial invariant subsystems.

Definition 1.12. *Let (M,φ,μ) be a ppf. We say that μ is* ergodic*, or that (M,φ,μ) is an ergodic flow*[4] *if for every invariant measurable set $A \subset M$ we have $\mu(A) = 0$ or $\mu(A) = 1$.*

We recall the following characterization of ergodicity.

Proposition 1.13. *Let (M,φ,μ) be a ppf. The following are equivalent:*

1. *μ is ergodic,*

[3] This flow is called the *time-change* generated by α.

[4] Sometimes, by a little abuse of notation, when the reference measure μ is clear from the context, we will say that φ is ergodic.

2. *for every measurable set $A \subset M$ such that $\mu(\varphi_t(A) \triangle A) = 0$ for all $t \in \mathbb{R}$, then $\mu(A) = 0$ or $\mu(A) = 1$,*

3. *if $f \colon M \to \mathbb{C}$ is a measurable function such that $f \circ \varphi_t = f$ almost everywhere for all $t \in \mathbb{R}$, then there exists $c \in \mathbb{C}$ such that $f = c$ almost everywhere,*

4. *if $f \in L^2(M)$ is an* invariant function*, namely if $f \circ \varphi_t = f$ in L^2 for all $t \in \mathbb{R}$, then there exists $c \in \mathbb{C}$ such that $f = c$ in L^2.*

Let us go back once more to the case of linear flows on tori and let us consider the ppf $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$. It is easy to see that, if the flow $\varphi^{\mathbf{v}}$ is rational, then it is *not* ergodic. Indeed, any set of the form

$$A_r = \bigcup \{\mathcal{O}_{\varphi^{\mathbf{v}}}(p) : p = [\![x_1, 0]\!] \in \mathbb{T}^2 \text{ with } 0 \leq x_1 \leq r\}$$

is an invariant set of with $\mathrm{Leb}(A_r) = r$. Choosing $r \in (0,1)$ appropriately gives an example of a non-trivial invariant set, thus disproving ergodicity.

Exercise 1.14. (a) *Show that the measures μ_p of Exercise* 1.7 *are ergodic.*

(b) *Show that any non-trivial convex combination of μ_p and μ_q, for p and q on different orbits, is not ergodic.*

(c*) *Finally, show that if μ is an ergodic invariant probability measure, then $\mu = \mu_p$ for some $p \in \mathbb{T}^2$.*

On the other hand, the Lebesgue measure is ergodic when the flow $\varphi^{\mathbf{v}}$ is irrational. There are several ways of proving this fact, here we see a proof that uses Fourier analysis.

Theorem 1.15. *Let $\varphi^{\mathbf{v}}$ be an irrational linear flow on $\mathbb{T}^2$. Then, the Lebesgue measure* Leb *is ergodic.*

Proof. We denote by $\cdot$ the scalar product in $\mathbb{R}^2$. For any $f \in L^2(\mathbb{T}^2)$, we can write a Fourier expansion

$$f([\![\mathbf{x}]\!]) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}}, \quad \text{with } \sum_{\mathbf{n} \in \mathbb{Z}^2} |f_{\mathbf{n}}| = \|f\|_2^2.$$

Assume that f is an invariant function, that is assume that $f \circ \varphi_t^{\mathbf{v}} = f$ for all $t \in \mathbb{R}$, where the equality holds in $L^2(\mathbb{T}^2)$. We want to show it is constant in L^2. For all $\mathbf{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}$ we have

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}} = f([\![\mathbf{x}]\!]) = f([\![\mathbf{x} + t\mathbf{v}]\!]) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i \mathbf{n} \cdot (\mathbf{x} + t\mathbf{v})} = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}}.$$

By uniqueness of the coefficients, we must have

$$f_{\mathbf{n}} = f_{\mathbf{n}} e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} \quad \text{for all } \mathbf{n} \in \mathbb{Z}^2.$$

If $\mathbf{n} \neq 0$, then either $f_{\mathbf{n}} = 0$ or $e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} = 1$ for all $t \in \mathbb{R}$, and this latter condition is verified if and only if $\mathbf{n} \cdot \mathbf{v} = 0$. Since $\mathbf{v}$ has rationally independent coordinates, this second possibility cannot occur; hence we deduce $f_{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$. This proves that $f = f_0$ is equal to a constant in $L^2(\mathbb{T}^2)$, and thus completes the proof. □

Let (M, φ, μ) be an ergodic ppf. The ergodic theorems of Von Neumann and Birkhoff relate the *time averages* $\frac{1}{T} \int_0^T f \circ \varphi_t \, \mathrm{d}t$ of a measurable function $f \in L^2(M)$ (or $L^1(M)$) to the *space average* $\mu(f) = \int_M f \, \mathrm{d}\mu$.

Theorem 1.16 (Von Neumann Ergodic Theorem)**.** *Let (M,φ,μ) be a ppf. For every $f \in L^2(M)$, let $\mathrm{P}f \in L^2(M)$ be the projection of f onto the closed subspace of invariant functions. Then, the ergodic averages of f converge in $L^2(M)$ to $\mathrm{P}f$, namely*

$$\left\| \frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t - \mathrm{P}f(p) \right\|_2 \to 0.$$

In particular, if (M,φ,μ) is ergodic, $\mathrm{P}f = \mu(f)$ and hence

$$\frac{1}{T}\int_0^T f\circ\varphi_t\,\mathrm{d}t \to \mu(f) \quad \text{in } L^2(M).$$

Theorem 1.17 (Birkhoff Ergodic Theorem)**.** *Let (M,φ,μ) be a ppf. For every $f \in L^1(M)$, there exists $f^* \in L^1(M)$ with*

$$\mu(f) = \mu(f^*), \quad \text{and} \quad f^*\circ\varphi_t = f^* \quad \text{for all } t \in \mathbb{R},$$

where the latter equality holds in $L^1(M)$, such that

$$\frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t \to f^*(p),$$

for almost every *$p \in M$. If (M,φ,μ) is ergodic, then $f^*(p) = \mu(f)$ almost everywhere.*

1.3.3 Unique ergodicity

In Theorem 1.8, we saw that a smooth flow on a compact manifold M always has an invariant probability measure, and we also noticed that, if there is more than one, then there are uncountably many. The former case deserves a special name.

Definition 1.18. *Let φ be a smooth flow on a compact manifold M. If there exists only one invariant probability measure μ, the system (M,φ,μ) (or simply φ) is said to be* uniquely ergodic.

The reader might be wondering what the uniqueness of the invariant measure has to do with ergodicity. The following proposition shows that, in the case of a single invariant measure, ergodicity is automatically guaranteed.

Proposition 1.19. *Let φ be a smooth flow on a compact manifold M. The set of ergodic probability measures for φ coincides with the set of* extremal points[5] *of the simplex of invariant probability measures. In particular, if there exists a unique invariant probability measure μ, then it is ergodic.*

If (M,φ,μ) is uniquely ergodic, then, from the Ergodic Theorem, Theorem 1.17, we know that the ergodic averages of any L^1-function converge almost everywhere to its space average. On the other hand, one can show that, if the function is *continuous*, then the convergence is *uniform*.

Proposition 1.20. *Let φ be a smooth flow on a compact manifold M. The following are equivalent:*

1. *φ is uniquely ergodic,*
2. *there exists a unique ergodic invariant probability measure,*
3. *for every $f \in \mathscr{C}(M)$ there exists a constant C_f such that, for all $p \in M$,*

$$\frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t \to C_f, \tag{1.3}$$

[5] A point in a simplex is extremal if it cannot be expressed as a non-trivial convex combination of two other points.

4. *for every $f \in \mathscr{C}(M)$, the convergence in* (1.3) *is uniform over M.*

Under any of the assumptions above, the constant C_f in (1.3) *equals $\mu(f)$, where μ is the unique invariant probability measure.*

We have seen already that for rational linear flows $\varphi^{\mathbf{v}}$ on $\mathbb{T}^2$ there exist uncountably many invariant measures. Let us now see that in the other case, when the coordinates of $\mathbf{v}$ are rationally independent, the flow is uniquely ergodic.

Theorem 1.21. *Let $\varphi^{\mathbf{v}}$ be an irrational linear flow on $\mathbb{T}^2$. Then,* $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$ *is uniquely ergodic.*

Proof. Let $f \in \mathscr{C}(M)$ be fixed, and let us prove that the ergodic averages

$$A_T f(p) := \frac{1}{T} \int_0^T f \circ \varphi_t^{\mathbf{v}}(p) \, \mathrm{d}t$$

converge uniformly to $\mathrm{Leb}(f) = \int_{\mathbb{T}^2} f \, \mathrm{d}\mathrm{Leb}$. We claim that the family

$$\mathscr{A} := \{A_T f\}_{T>0} \subset \mathscr{C}(M).$$

is pre-compact in $\mathscr{C}(M)$, i.e., $\mathscr{A}$ has a compact closure. In order to do this, we check the assumptions of the Ascoli-Arzelà Theorem.

It is easy to see that $\mathscr{A}$ is equibounded: since $\|f \circ \varphi_t^{\mathbf{v}}\|_\infty = \|f\|_\infty$ for all $t \in \mathbb{R}$, it follows that, for any $T > 0$ and for all $p \in \mathbb{T}^2$, we have

$$|A_T f(p)| \leq \frac{1}{T} \int_0^T \|f\|_\infty \, \mathrm{d}t = \|f\|_\infty .$$

Let us verify that $\mathscr{A}$ is equicontinuous. We will use the fact that $\varphi_t^{\mathbf{v}}$ is an isometry for all $t \in \mathbb{R}$: if we denote by d the Euclidean distance on $\mathbb{T}^2$, we have that $d(\varphi_t^{\mathbf{v}}(p), \varphi_t^{\mathbf{v}}(q)) = d(p,q)$ for all $t \in \mathbb{R}$. With this in mind, let us fix $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(p) - f(q)| < \varepsilon$ whenever $d(p,q) < \delta$. Then, for any $T > 0$, if $p, q \in \mathbb{T}^2$ are such that $d(p,q) < \delta$, we get

$$|A_T(p) - A_T(q)| \leq \frac{1}{T} \int_0^T |f \circ \varphi_t^{\mathbf{v}}(p) - f \circ \varphi_t^{\mathbf{v}}(q)| \, \mathrm{d}t < \frac{1}{T} \int_0^T \varepsilon \, \mathrm{d}t = \varepsilon.$$

By the Ascoli-Arzelà Theorem, the closure of $\mathscr{A}$ is compact in $\mathscr{C}(M)$, in particular $\mathscr{A}$ has limit points. Let $T_n \to \infty$ and $g \in \mathscr{C}(M)$ be such that

$$A_{T_n} f \to g \quad \text{ in } \mathscr{C}(M).$$

By Birkhoff Ergodic Theorem, Theorem 1.17, for almost every point p we have

$$A_{T_n} f(p) \to \int_{\mathbb{T}^2} f \, \mathrm{d}\mathrm{Leb},$$

therefore $g = \mathrm{Leb}(f)$ almost everywhere. Since g is continuous, the equality must hold everywhere.

We have showed that all limit points of $\mathscr{A}$ are the constant function $\mathrm{Leb}(f)$. Therefore, the limit point is unique and we conclude that the whole family converges in $\mathscr{C}(M)$, namely

$$A_T f = \frac{1}{T} \int_0^T f \circ \varphi_t^{\mathbf{v}} \, \mathrm{d}t \to \int_{\mathbb{T}^2} f \, \mathrm{d}\mathrm{Leb}$$

uniformly on $\mathbb{T}^2$, which concludes the proof. $\square$

We remark that the proof of Theorem 1.21 works in a greater generality: any isometry of a compact space which has an ergodic measure with full support is uniquely ergodic.

Exercise 1.22. *Let* $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$ *be an irrational linear flow.*

(a) *Show that for any set* $A \subset \mathbb{T}^2$ with non-empty interior *there exists* $T_A > 0$ *such that* for all points $p \in M$ *there exists* $t \in [0, T_A]$ *such that* $\varphi^{\mathbf{v}}_t(p) \in A$ *(all points enter A before time* T_A*).*

(b*) *Provide a counterexample to* (a) *when we drop the assumption on A, namely give an example of a set* $A \subset \mathbb{T}^2$ with positive measure and empty interior *such that*

1. *almost every point enters A,*
2. *at least one point* $p \in M$ never *enters A,*
3. *for every* $T > 0$ *there exists a set* $B_T \subset \mathbb{T}^2$ of positive measure *such that all points in* B_T *do not enter A in the interval* $[0, T]$.

1.3.4 A glimpse at Ratner's Theorems

Let us summarize what we proved in the case of linear flows on the 2 dimensional torus:

- If the generator $\mathbf{v}$ has rationally independent coordinates, then
  1. the orbit closure of any point is the whole space $\mathbb{T}^2$ (Theorem 1.5),
  2. the orbit of any point equidistributes in $\mathbb{T}^2$ (Theorem 1.15),
  3. Leb is the only ergodic probability measure for $\varphi^{\mathbf{v}}$ (Theorem 1.21).
- If the generator $\mathbf{v}$ has rationally dependent coordinates, then
  1. all orbits are periodic, hence closed (Theorem 1.5),
  2. all orbits are not equidistributed in $\mathbb{T}^2$ (but, clearly, they equidistribute in their closure),
  3. any ergodic measure is the normalized Lebesgue measure on a periodic orbit (Exercise 1.14).

It is possible to generalize these results to linear flows on higher dimensional tori. Let us first recall some definitions.

A subspace $V < \mathbb{R}^n$ is called *rational* if the discrete Abelian group $V \cap \mathbb{Z}^n$ has rank precisely equal to $k := \dim(V)$. It is easy to see that this happens exactly when we can find a basis of V consisting of vectors in $\mathbb{Z}^n$. The subspace V carries a smooth measure, that we call Leb_V, given by the Lebesgue measure on V normalized so that the discrete subgroup $V \cap \mathbb{Z}^n$ has *covolume* 1 (see, e.g., Exercise 1.7). This measure descends to a measure on the k-dimensional torus $V/(V \cap \mathbb{Z}^n)$, as well as on its affine translates $\mathbf{x} + V/(V \cap \mathbb{Z}^n)$ for all $\mathbf{x} \in \mathbb{R}^n$. By a little abuse of notation, we will still call Leb_V any of these affine measures.

Theorem 1.23. *Let* $\varphi^{\mathbf{v}} \colon \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^n$ *be a linear flow on* $\mathbb{T}^n$*, with* $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$*. There exists a rational subspace* $V < \mathbb{R}^n$ *of dimension* $k \in \{1, \dots, n\}$ *which contains the line* $\mathbb{R}\mathbf{v}$ *for which the following holds.*

1. *(Orbit closure classification) The orbit closure of any point is an affine k-dimensional torus, namely for all* $p = [\![\mathbf{x}]\!] \in \mathbb{T}^n$ *we have*
$$\overline{\mathcal{O}_{\varphi^{\mathbf{v}}}(p)} = \mathbf{x} + V/(V \cap \mathbb{Z}^n).$$

2. *(Equidistribution) The orbit of any point $p \in \mathbb{T}^n$ equidistributes in its closure with respect to the affine measure* Leb_V.

3. *(Measure classification) Any ergodic measure for $\varphi^{\mathbf{v}}$ is an affine measure* Leb_V *on the affine torus $\mathbf{x} + V/(V \cap \mathbb{Z}^n)$ for some $\mathbf{x} \in \mathbb{R}^n$.*

Theorem 1.23 can be seen as a very simple case of a series of profound and general theorems by Marina Ratner which classify all possible orbit closures for unipotent actions, show that all orbits equidistribute in their closure, and prove that any ergodic measure is the affine translate of the Lebesgue (Haar) measure on a intermediate subgroup. The purpose of this complicated comment is only to whet you appetite for what will come in the rest of the course.

1.4 Further chaotic properties

1.4.1 Weak mixing

Let (M, φ, μ) be a ppf. As we have seen in (1.2), for every $t \in \mathbb{R}$ the *Koopman operator*

$$U_t \colon L^2(M) \to L^2(M), \quad U_t f = f \circ \varphi_t$$

is *unitary*. By Proposition 1.13, the flow φ is ergodic if and only if the eigenspace corresponding to the eigenvalue 1 has dimension 1, and consists of constant functions. Since U_t is unitary, if there are other eigenvalues, they must have modulus 1.

Definition 1.24. *We say that the ppf (M, φ, μ) is* weak mixing *if the only solutions to*

$$U_t f = e^{2\pi i t \alpha} f \quad \text{in } L^2(M) \text{ for all } t \in \mathbb{R}$$

are given by $\alpha = 0$ and $f = c$ for some $c \in \mathbb{C}$.

As usual, when the reference measure μ is clear from the context, we will often simply say that φ is weak mixing when the condition in Definition 1.24 is satisfied.

Clearly, a weak mixing ppf is also ergodic. The converse, however, is not true, and a family of counterexamples is given precisely by our irrational linear flows.

Lemma 1.25. *Any linear flow $(\mathbb{T}^2, \varphi^{\mathbf{v}}, \mathrm{Leb})$ is not weak mixing.*

Proof. It is sufficient to consider the irrational case, since we already know that rational linear flows are not ergodic and hence cannot be weak mixing. We claim that for any $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$, the function

$$f_{\mathbf{n}}([\![\mathbf{x}]\!]) = e^{2\pi i \mathbf{n} \cdot \mathbf{x}} \in L^\infty(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$$

is a non-constant eigenfunction, and the α as in Definition 1.24 is $\alpha = \mathbf{n} \cdot \mathbf{v} \neq 0$. Indeed, for any $t \in \mathbb{R}$, we have

$$U_t f_{\mathbf{n}}([\![\mathbf{x}]\!]) = f_{\mathbf{n}}([\![\mathbf{x} + t\mathbf{v}]\!]) = e^{2\pi i \mathbf{n} \cdot (\mathbf{x} + t\mathbf{v})} = e^{2\pi i t \mathbf{n} \cdot \mathbf{v}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}} = e^{2\pi i t \alpha} f_{\mathbf{n}}([\![\mathbf{x}]\!]).$$

Thus, irrational linear flows are ergodic but not weak-mixing. □

Weak-mixing is a *spectral* property, in the sense that it concerns the spectrum of the Koopman operators U_t of the system. If they have no pure point component (no eigenvalues), the flow is weak mixing. There are other equivalent characterizations of weak-mixing, which have a more "dynamical flavour"; we summarize them in Proposition 1.26 below.

Proposition 1.26. *Let (M, φ, μ) be a ppf. The following are equivalent.*

1. *(M,φ,μ) is weak mixing.*
2. *For any $f,g \in L^2(M)$,*
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left| \int_M f\circ\varphi_t \cdot \bar{g}\,\mathrm{d}\mu - \mu(f)\,\mu(\bar{g}) \right| \mathrm{d}t = 0.$$
3. *For any $f,g \in L^2(M)$,*
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left| \int_M f\circ\varphi_t \cdot \bar{g}\,\mathrm{d}\mu - \mu(f)\,\mu(\bar{g}) \right|^2 \mathrm{d}t = 0.$$
4. *For any $f,g \in L^2(M)$, there exists a set $J = J_{f,g} \subset \mathbb{R}$ of measure zero such that*
$$\lim_{T\to\infty, T\notin J} \int_M f\circ\varphi_T \cdot \bar{g}\,\mathrm{d}\mu = \mu(f)\,\mu(\bar{g}).$$
5. *The product measure $\mu\times\mu$ is ergodic for the flow $\varphi\times\varphi$ on $M\times M$.*
6. *The product measure $\mu\times\mu$ is weak mixing for the flow $\varphi\times\varphi$ on $M\times M$.*
7. *For any ergodic ppf (N,ψ,ν), the system $(M\times N, \varphi\times\psi, \mu\times\nu)$ is ergodic.*

It might be worth for the reader to compare conditions 2–4 of Proposition 1.26 with the following equivalent definition of ergodicity.

Exercise 1.27. *Let (M,φ,μ) be a ppf. Show that it is ergodic if and only if for any $f,g \in L^2(M)$ we have*
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_M f\circ\varphi_t \cdot \bar{g}\,\mathrm{d}\mu \right) \mathrm{d}t = \mu(f)\,\mu(\bar{g}).$$
Deduce in particular that a weak mixing ppf is ergodic.

1.4.2 Mixing

Mixing, sometimes called strong mixing, is an even stronger property that, roughly speaking, says that any two events become asymptotically independent.

Definition 1.28. *We say that the ppf (M,φ,μ) is* (strong) mixing *if for any two observables $f,g \in L^2(M)$, the* correlations decay*, namely if*
$$\langle f\circ\varphi_t, g\rangle = \int_M f\circ\varphi_t \cdot \bar{g}\,\mathrm{d}\mu \to \mu(f)\,\mu(\bar{g}),$$
as $t\to\infty$.

It is clear from Proposition 1.26-(4) that any mixing ppf is also weak mixing. The converse, however, is not true: there are weak mixing ppf's which are not strong mixing. The first examples of weak mixing but not mixing transformations were constructed by cutting-and-stacking methods. In the context of flows, typical translation flows on translation surfaces and typical minimal area-preserving flows on higher genus surfaces are also natural classes of examples of weak mixing flows that are not mixing. It is also interesting to notice that, by the Halmos-Rokhlin Theorem, weak mixing is a *generic* property, whereas mixing is *meager*. In this course, however, the flows we will encounter are either not weak mixing (the nilflows in Chapter 3) or mixing (the geodesic and horocycle flows in Chapters 4–7).

Returning to our case study, we already know from Lemma 1.25 that irrational linear flows are not weak mixing, hence they cannot possibly be mixing. We can actually prove a stronger result, namely they have the so-called *rigidity property*.

Exercise 1.29. *Let* $\mathbf{v} = (v, 1) \in \mathbb{R}^2$*, with* $v \notin \mathbb{Q}$*.*

(a*) *Find an explicit increasing sequence* $t_n \to \infty$ *such that for any set* Q *of the form* $Q = I_1 \times I_2 + \mathbb{Z}^2$*, where* $I_1, I_2 \subset [0,1)$ *are intervals, we have*

$$\mathrm{Leb}(Q \cap \varphi^{\mathbf{v}}_{t_n}(Q)) \geq \mathrm{Leb}(Q) - t_n^{-2},$$

for all $n \in \mathbb{N}$ *sufficiently large (Hint: it might be useful to consider the continued fraction expansion of* v*).*

(b) *Deduce that the linear flow* $\varphi^{\mathbf{v}}$ *is* rigid*, namely there exists an increasing sequence* $t_n \to \infty$ *such that for any measurable set* $A \subset \mathbb{T}^2$ *we have*

$$\mathrm{Leb}(A \triangle \varphi^{\mathbf{v}}_{t_n}(A)) \to 0, \quad \text{as } n \to \infty.$$

(c) *Conclude in particular that* $\varphi^{\mathbf{v}}$ *is not mixing.*

One can also ask about the correlations of several events or observables, leading to the following definition.

Definition 1.30. *We say that the ppf* (M, φ, μ) *is* mixing of order k *or* k-mixing *if for any* k *(real-valued) observables* $f_1, \dots, f_k \in L^2(M)$ *we have*

$$\int_M f_1 \cdot f_2 \circ \varphi_{t_2} \cdots f_k \circ \varphi_{t_k} \, \mathrm{d}\mu \to \mu(f_1) \cdots \mu(f_k),$$

as $t_2, t_3 - t_2, \dots, t_k - t_{k-1} \to \infty$*.*

We say that the ppf (M, φ, μ) *is* mixing of all orders *if it is mixing of order* k *for all* $k \geq 2$*.*

It is currently unknown whether mixing implies mixing of all orders. This open question is known as the “Rokhlin Problem”.

1.5 Outline of the course

In Chapter 2, we present all the relevant background material on matrix Lie groups. We will introduce their associated Lie algebras, which can be described as the space of all left-invariant vector fields. We will then study the induced flows using the exponential map. In Section 2.3, we introduce the Haar measure, which is the invariant measure we will be interested in, the Killing form and the Casimir operator. These last two objects will play a role in the final chapter of these notes. Finally, we define homogeneous spaces as the smooth manifolds obtained as quotients of Lie groups by lattices.

In Chapter 3, we focus on Heisenberg nilflows. We describe them using the so-called exponential coordinates and we classify all possible Heisenberg nilmanifolds. We then show that Heisenberg nilflows are never mixing, but typically relatively mixing and uniquely ergodic. In Section 3.3, we point out an interesting connection between Heisenberg nilflows and theta sums (or quadratic Weyl sums), which are classical objects in analytic number theory.

In Chapter 4, we study in detail the action of $\mathrm{PSL}(2,\mathbb{R})$ on the hyperbolic plane (namely, on its upper-half plane model), which is first introduced in §2.3. We define the geodesic and horocycle flow as particular cases of homogeneous flows on quotients of $\mathrm{PSL}(2,\mathbb{R})$. As an important example, we introduce the Modular Surface.

Chapter 5 is devoted to the study of the ergodic properties of geodesic and horocycle flow. We prove that they are ergodic and mixing.

In Chapter 6, we study the connection between the geodesic flow on the Modular Surface and continued fractions. This fascinating topic dates back to Artin [1], but we will follow an elegant presentation by Series [17].

The final part, Chapter 7, is devoted to the treatment of more advanced material. We prove unique ergodicity of the horocycle flow on compact manifolds, a result originally due to Furstenberg [8]. The proof we present in these notes is due to Coudène [3]. We then discuss the generalizations of this result to finite volume, noncompact spaces and we state Ratner's Theorem [15] on measure classification in the case of unipotent flows. We then study some quantitative properties. We present a special case of Ratner's quantitative mixing result [14] for geodesic and horocycle flow and a special case of Flaminio and Forni's result [6] on asymptotics of horocycle averages, but following the proof in [16].