Big O notation: a proof example

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1 Valid proof

We want to proof that $8n^2 + 5n + 2$ is $O(n^2)$. From theory, we know that this means to proof that:

$$\exists c > 0 : 8n^2 + 5n + 2 \leq c \cdot n^2 \tag{1}$$

If the equation is factorised by the higher n-element n^2, it became:

$$n^2 \cdot \left(8 + \frac{5}{n} + \frac{2}{n^2}\right) \leq c \cdot n^2 \tag{2}$$

Now divide both terms for the common factor n^2:

$$8 + \frac{5}{n} + \frac{2}{n^2} \leq c \tag{3}$$

The big O notation describes the behaviour of a function for greater values of the variable. In other words, it is possible to replace the original function with a big O notation if a proof of its limiting behaviour is given.

For this reason, it is possible to reduce the analysis to behaviour with high values of the variable. In this case, consider very high value of the variable n, say $n \to \infty$.

$$8 + \lim_{n\to\infty} \frac{5}{n} + \lim_{n\to\infty} \frac{2}{n^2} \leq c \tag{4}$$

The two terms depending from n tend to 0: they have the same behaviour of the functions $y = \frac{1}{n}$ and $y = \frac{1}{n^2}$ respectively. The equation above could be simplified in the following.

$$8 \leq c \tag{5}$$

This simply ends the proof because every value of $c \geq 8$ make the equation 1 true.

2 Invalid proof

A very bad students stated that $8n^2 + 5n + 2$ is $O(n)$. We can try to proof this and see that we reach a contradiction.

From theory, we know that this means to proof that:

$$\exists c > 0 : 8n^2 + 5n + 2 \leq c \cdot n \tag{6}$$

If the equation is factorised by the higher n-element n, it became:

$$n^2 \cdot \left(8 + \frac{5}{n} + \frac{2}{n^2}\right) \leq c \cdot n \tag{7}$$

Now divide both terms for the common factor n^2:

$$8 + \frac{5}{n} + \frac{2}{n^2} \leq \frac{c}{n} \tag{8}$$

The big O notation describes the behaviour of a function for greater values of the variable. In other words, it is possible to replace the original function with a big O notation if a proof of its limiting behaviour is given.

For this reason, it is possible to reduce the analysis to behaviour with high values of the variable. In this case, consider very high value of the variable n, say $n \to \infty$.

$$8 + \lim_{n\to\infty} \frac{5}{n} + \lim_{n\to\infty} \frac{2}{n^2} \leq \lim_{n\to\infty} \frac{c}{n} \tag{9}$$

The two terms depending from n tend to 0: they have the same behaviour of the functions $y = \frac{1}{n}$ and $y = \frac{1}{n^2}$ respectively. The right term has also the same behaviour of $y = \frac{1}{n}$, because c is a constant.

The equation above could be simplified in the following.

$$8 \leq 0 \tag{10}$$

This is a contradiction. In fact, it is not possible to find a positive value of c that make this disequation valid because it is not even dependant from c.

This ends the proof: the original function is *not* a $O(n)$.