THE RIEMANN HYPOTHESIS AND RELATED PROBLEMS

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need to write an introduction

For background information and basic definitions, see [[reference to be added]].

By a nontrivial zero we mean a zero of the completed L-function $\Lambda_L(s)$, equivalently, a zero of $L(s)$ which is not at the location of a pole of one of the Γ-factors that appears in the functional equation for $L(s)$.

1. The zeta function and other global L-functions

The Riemann Hypothesis for an L-function $L(s)$ is the assertion that the nontrivial zeros of $L(s)$ lie on the critical line.

For historical reasons there are names given to the Riemann hypothesis for various sets of L-functions. For example, the Generalized Riemann Hypothesis (GRH) is the Riemann Hypothesis for all Dirichlet L-functions. More examples collected below.

In certain applications there is a fundamental distinction between nontrivial zeros on the real axis and nontrivial zeros with a positive imaginary part. Here we use the adjective *modified* to indicate a Riemann Hypothesis except for the possibility of nontrivial zeros on the real axis. Thus, the Modified Generalized Riemann Hypothesis (MGRH) is the assertion that all nontrivial zeros of Dirichlet L-functions lie either on the critical line or on the real axis. A real zero which is very close to the point $s = 1$ is called a Landau-Siegel zero.

Precise statements of various Riemann Hypotheses are given below.

The Riemann Hypothesis

Problem 1.1. *The nontrivial zeros of the Riemann Zeta function, $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$.*

The Riemann Hypothesis is sometimes phrased in terms of the zeros in the critical strip, or the complex (meaning, non-real) zeros. For the Riemann zeta function these are equivalent statements. But for more general L-functions [[say more]].

The Generalized Riemann Hypothesis(GRH)

Problem 1.2. *Riemann Hypothesis is true, and in addition the nontrivial zeros of all Dirichlet L-functions lie on the critical line $\sigma = \frac{1}{2}$.*

Remark. GRH is occasionally called "Piltz conjecture," but the conjecture of a Riemann Hypothesis for Dirichlet L-functions is generally viewed as obvious generalization which should not be attributed to a particular person.

Remark. GRH is the conjecture of a Riemann Hypothesis for all degree 1 L-functions.

The Modified Generalized Riemann Hypothesis

Problem 1.3. *Riemann Hypothesis is true, and in addition the nontrivial zeros of all Dirichlet L-functions lie either on the real line or on the critical line* $\sigma = \frac{1}{2}$.

The Extended Riemann Hypothesis

Problem 1.4. *The nontrivial zeros of the Dedekind zeta function of any algebraic number field lie on the critical line.*

Note that ERH includes RH because the Riemann zeta function is the Dedekind zeta function of the rationals.

The Grand Riemann Hypothesis

Problem 1.5. *Nontrivial zeros of all L-functions lie on the critical line.*

The Grand Riemann Hypothesis is often phrased in terms of all *automorphic* L-functions.

The Modified Grand Riemann Hypothesis

Problem 1.6. *The nontrivial zeros of all L-functions lie either on the real line or on the critical line.*

2. Weaker statements about zeros of L-functions

The Riemann Hypothesis is the strongest possible statement about the horizontal distribution of the nontrivial zeros of an L-function. In this section we collect various weaker assertions.

Throughout this section let $L(s)$ be an L-function for which one might expect a Riemann Hypothesis to hold.

Quasi Riemann Hypothesis

Problem 2.1. $L(s)$ *has no zeros in a half-plane* $\sigma > \sigma_0$, *for some* $\sigma_0 < 1$.

The Density Hypothesis

Recall the standard notation:

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \ : \ \beta \geq \sigma, \ 0 < \gamma < T\}.$$

The Riemann Hypothesis is equivalent to $N(\sigma, T) = 0$ for $\sigma > \frac{1}{2}$.

The Density Hypothesis is the assertion:

Problem 2.2. *For all* $\epsilon > 0$*,*

$$N(\sigma, T) = O(T^{2(1-\sigma)+\varepsilon}).$$

This is nontrivial only when $\sigma > \frac{1}{2}$.

Remark. The importance of the Density Hypothesis is that, in terms of bounding the gaps between consecutive primes, the density hypothesis appears to be as strong as the Riemann Hypothesis.

Remark. Results on $N(\sigma, T)$ are generally obtained from mean values of the zeta-function. Further progress in this direction, particularly for σ close to $\frac{1}{2}$, appears to be hampered by the great difficulty in estimating the moments of the zeta-function on the critical line.

See Titchmarsh [88c:11049], Chapter 9, for an extensive discussion.

Remark. The Density Hypothesis follows from the Lindelöf Hypothesis.

Landau-Siegel zeros

This part needs to be written.

Problem 2.3. *Statement about Landau-Siegel zeros. [[to be added]]*

The 100% Hypothesis

Let $N(T)$ denote the counting function of the nontrivial zeros of $L(s)$, so $N(T) \sim \frac{d}{2\pi} T \log T$ where d is the degree of $L(s)$. And let $N_0(T)$ denote the counting function of the zeros of $L(s)$ on the critical line.

Problem 2.8. *The* 100% Hypothesis *for* $L(s)$ *asserts that* $N_0(T) \sim N(T)$ *as* $T \to \infty$*.*

An equivalent assertion is

$$N(T) - N_0(T) = o(T \log T),$$

which makes it clear that the 100% Hypothesis still allows quite a few zeros off the critical line.

Remark. The term "100% Hypothesis" is not standard.

Remark. In contrast to most of the other conjectures in this section, the 100% Hypothesis is not motivated by applications to the prime numbers. Indeed, at present there are no known consequences of this hypothesis.

3. Problems motivated by equivalences to the Riemann Hypothesis

The following problems could offer insight into a possible approach to the solving the Riemann Hypothesis. Most of these problems are motivated by items on the list of equivalences to the Riemann Hypothesis[1].

[1] http://aimpl.org/pl/rhequivalences/

Problem 3.1. *What is the constant in Baez-Duarte's equivalent*[2] *of RH?*

Redheffer's matrix

Problem 3.2. *Does the Riemann Hypothesis imply that the "nontrivial" eigenvalues of Redheffer's matrix*[3] *are inside the unit circle?*

[2]http://aimpl.org/rhequivalences/pl/7.3
[3]http://aimpl.org/rhequivalences/pl/2.3