FREE ANALYSIS

These problems arose during the AIM workshop “Free Analysis”, June 19 - 23, 2006, organized by Dimitri Shlyakhtenko and Dan Voiculescu.

1. X-constants and free Poincare inequality

[*Voiculescu*]

In a von Neumann algebra M with a faithful normal trace-state τ let $X = X^* \in M$ and let $1 \in B \subset M$ be an infinite-dimensional von Neumann subalgebra so that B and X are free in the algebraic sense and $M = W^*(X, B)$. start of problem block

Assume that $\partial_{X:B}$ is closable in $L^2(M, \tau)$ (this is the case for instantce if X is a free semicircular perturbation $X = X_0 + \varepsilon S$, with S a semicircular free from X_0 and B).

Problem 1.1. *Under what conditions are the L^2 solutions of*

$$\overline{\partial_{X:B}} u = 0$$

in $L^2(B, \tau)$?

end of problem block

A related question about a stronger condition:

start of problem block

Problem 1.2. *When does the free Poincare inequality*

$$C \|\partial_{X:B} \xi\|_2 \geq \|\xi - E_B \xi\|_2$$

hold for $\xi \in B\langle X \rangle$?

end of problem block

2. Large Deviations

[*Guionnet, Hiai, and Cabanal-Duvillar*]

Problem 2.1. *Given a tracial state τ corresponding to a free stochastic process, does there exist a sequence of tracial states $\tau_n \to \tau$ with $\chi_p^*(\tau_n) \to \chi_p^*(\tau)$ where τ_n corresponds to the process $dA_i(t) = dS_i(t) + k_t(A_1(s), \dots, A_m(s))_{s \leq t} dt$ with k_t stepwise constant in s, and χ_p^* denotes the quantity χ^* defined for processes in the paper of Guionnet and Cabanal-Duvillard.*

Problem 2.2. *In the one variable case, if $A(t)$ follows a process $dA(t) = dS(t) + k_t(A(s))_{s \leq t}$ then replacing $A(t)$ with $A(t) + C_\epsilon$ (with C having Cauchy distribution and free from $A(t)$) then k_t is replaced by $k_t^\epsilon = \tau\left(k_t | A(t) + C_\epsilon\right)$. Thus, k_t^ϵ is smooth. Is there an analog of this smoothing in the several-variable case?*

Problem 2.3. *We know that if $f : \mathbb{R} \to \mathbb{R}$ and A is an $n \times n$ Hermitian random matrix, then there exists a random matrix C_ϵ with Cauchy distribution such that $\mathbb{E} f(A + C_\epsilon) = P_\epsilon f(A)$ with $P_\epsilon f(x) = \int \frac{f(y)}{(y-x)^2 + i\epsilon^2} dy$ the usual Cauchy (Poisson) kernel. Can this be done for several variables?*

Problem 2.4. *Given $x_1, \ldots, x_m \in (\mathcal{A}, \tau)$ a tracial unital vN algebra, do the conjugate variables belong to the L^2 closure of cyclic gradient space? i.e. do there exist $H_k \in \mathbb{C}\langle \alpha_1, \ldots, \alpha_m \rangle$ such that $\mathcal{J}(x_i) = \lim_k D_i H_k$ where $\partial_{x_i} : L^2(\mathcal{A}, \tau) \to L^2(\mathcal{A}, \tau) \otimes L^2(\mathcal{A}, \tau)$ by $x_j \mapsto \delta_{ij} 1 \otimes 1$ as a densely defined operator, $\mathcal{J}(x_i) = \partial_{x_i}^*(1 \otimes 1)$, and $D_i = m \circ \partial_{x_i}$ (m is the flip-multiplication $x \otimes y \mapsto yx$).*

Problem 2.5. *Does the change of variables formula for χ also hold for χ^*?*

Problem 2.6. *Is there a change of variables formula for processes? i.e. suppose that we start with random variables $x_1, \ldots, x_m \in (\mathcal{A}, \tau)$ which can be reached by a process $dA_i(t) = dS_i(t) + k_t(A_1(s), \ldots, A_m(s))_{s \leq t}$, $\mu_{A_1(1), \ldots, A_m(1)} = \mu_{x_1, \ldots, x_m}$. We define new random variables via functional calculus $y_1 = f_1(x_1, \ldots, x_m), \ldots, y_m = f_m(x_1, \ldots, x_m)$. Can we apply a function P to k_t to get $dB_i(t) = dS_i(t) + P(k_t(B_1(s), \ldots, B_m(s))_{s \leq t})$ such that $\mu_{B_1(1), \ldots, B_m(1)} = \mu_{y_1, \ldots, y_m}$.*

Problem 2.7. *Open problem: Can we replace* lim sup *with* lim inf *in the microstates definition of the free entropy χ?*

Problem 2.8. *Hiai introduced the free pressure $\pi_R(h)$ for a self-adjoint element (regarded as a free hamiltonian) h of the universal free product C^*-algebra $\mathcal{A}^{(n)} = \bigstar_{i=1}^n C([-R, R])$, and defined a free entropy-like quantity $\eta_R(\tau)$ of a tracial state $\tau \in TS(\mathcal{A}^{(n)})$. The inequality $\eta_R(\tau) \geq \chi(\tau)$ holds. τ is called an equillibrium tracial state with respect to h if the variational equality $\eta_R(\tau) = \tau(h) + \pi_R(h)$ holds. Such a τ always exists for each h. For which h there is a unique equilibrium tracial state? A way to prove this is the free transportation inequality.*

f problem

It was recently shown by Guionnet and Maurel-Segala that for the vN algebra $(\mathcal{A}, \tau)$ generated by m free semicirculars,

$$\sup_{\tau \in \mathcal{TS}(\mathcal{A})} \left\{ \chi(\tau) - \tau\left(\sum t_i q_i\right) \right\} = \sum_{p_1, \ldots, p_m} \prod_{k_1, \ldots, k_m} \frac{(t_i)^{p_i}}{k_i!} C(q, k_1, \ldots, k_m)$$

where $C(q, k_1, \ldots, k_m)$ enumerated planar maps with colored edges and vertices of types $q, k_1, \ldots, k_m$.

Problem 2.9. *Is there a similar interpretation for the non-microstates analog*

$$\sup_{\tau \in \mathcal{TS}(\mathcal{A})} \left\{ \chi^*(\tau) - \tau\left(\sum t_i q_i\right) \right\} ?$$

f problem

3. Free von Neumann Algebra

[*Dykema, Ricard*]

Problem 3.1. *Given A, B free group factors with a common diffuse subalgebra $D \subset A, B$, what conditions on A, B, D guarantee that $A \star_D B$ is a free group factor?*

Problem 3.2. *For a regular weakly-rigid (in the sense of Popa) subalgebra of a von Neumann algebra, is the free entropy dimension ≤ 1?*

f problem

Problem 3.3. ***Open Problem**: For generators $\gamma_1, \ldots, \gamma_n \in \Gamma$ with the first L^2 -Betti number $\beta_1(\Gamma)$ large, is the microstates free entropy dimension of this family of generators large?*

Remark. This is known for the non-microstates free entropy dimension [work of Mineyev-Shlyakhtenko]

end of prob block

Problem 3.4. *Consider $\Delta = \sum_{i=1}^m \partial_{x_i}^* \partial_{x_i}$ and the corresponding completely positive map $\varphi_t = \exp(-t\Delta)$, where $(x_1, \ldots, x_m)$ have finite Free Fisher Information. Can φ_t converge uniformly to the identity map on the unit ball of $W^*(x_1, \ldots, x_m)$? If no, it follows that the von Neumann algebra generated by $(x_1, \ldots, x_m)$ is not weakly rigid if it is non-hyperfinite.*

start of prob block

Let $\Gamma_{q,n} = W^*(s_q(g) = l(g) + l(g)^* | g \in \mathcal{H}_{\mathbb{R}})$ with $n = \dim \mathcal{H}_{\mathbb{R}}$, $-1 < q < 1$ be the von Neumann algebra generated by fields operators acting on a q-deformed Fock space

Problem 3.5. *Does $\Gamma_{q,n}$ depend on q?*

A way to approach this question could come from the following observation. In the free case, $q = 0$, the natural orthormal basis of the Fock space consists of vectors $e_{\underline{i}} = e_{i_1}^{\otimes \alpha_1} \otimes \ldots \otimes e_{i_k}^{\otimes \alpha_k}$ with $i_1 \neq \ldots \neq i_k$ and $\alpha_1 > 0$. This basis can be recoved from the algebra as $e_{\underline{i}} = T_{\alpha_1}(s_0(e_1)) \ldots T_{\alpha_k}(s_0(e_q))\Omega$, where T_k are Chebytchev polynomials. It would be interesting to find an analogue for these formulas in the general case and to unterstand the underlying combinatorics.

end of prob block

start of prob block

The q-deformation leads to the commutation relations $l(e)^* l(f) = q l(f) l(e)^* + \langle f, e \rangle Id$. Instead consider themore general relations $l(e_i)^* l(e_j) = \sum_{s,t} t_{i,j}^{s,t} l(e_s) l(e_t)^* + \delta_{i,j} Id$.

Problem 3.6. *When does the C^*-algebra generated by these operators is an extension of a Cuntz algebra by compacts ? When does the fields operators associated to them produce a type II_1 factor ?*

end of prob block

start of prob block

Consider the projection P_k from $\Gamma_{q,n}$ to its subspace consisting of x such that $x.\Omega$ has length at most k in the Fock space.

Problem 3.7. *Is $\|P_k\|_{cb}$ polynomially bounded in k ?*

This would prove the CBAP for the associated L_p spaces $(1 < p < \infty)$ and the exactness of the C^*-algebra generated by q-gaussians.

end of prob block

Problem 3.8. *To prove the existence of an embedding $\Gamma_{q,n} \to \mathcal{R}^\omega$, one uses Speicher's central limit theorem. In this procedure, is it possible to find explictely uniformly bounded matrix whose mixed moments approach those of q-gaussians ? More precisely, let $c_{i,j}$ be unitary generators of the CAR-algebra (or -1-gaussians), are the matrices $\frac{1}{\sqrt{n}}[c_{i,j}]_{i,j \leq n}$ uniformly bounded ?*

start of prob block

Problem 3.9. *For the random matrix model $\exp(-nTr(p(A_1, A_1^*, \ldots, A_m, A_m^*))$ we know that the conjugate variables satisfy $\mathcal{J}_i = \mathcal{D}_i P$. Is the operator $\exp(-t \sum \partial_j^* \partial_j)$ compact in the limit $n \to \infty$ (where ∂_j is Voiculescu's partial difference quotient on the limit algebra with respect to the limit of A_j)?*

Remark. As a starting point, consider $P = \sum A_i^2 + \sum t_i q_i(A_1, \ldots, A_m)$ where Guionnet and Maurel-Segala have shown convergence of the model.

end of prob block

4. Focus Group on Free Entropy

The problems in this section arose during a discussion group on Free Entropy during the 3rd day of the AIM workshop.

f problem

Let

$$\delta^* = n - \limsup_{t\downarrow 0} \frac{\chi^*(x_1+\sqrt{t}s_1,\ldots,x_n+\sqrt{t}s_m)}{\log t^{1/2}}$$

and

$$\delta^\star = n - \limsup_{t\to 0} \sum_{i=1}^{n} t\Phi^*(x_1+\sqrt{t}s_1,\ldots x_m+\sqrt{t}s_m).$$

Problem 4.1. *Does $\delta^* = \delta^\star$?*

problem

Problem 4.2. *What is the non-microstates analogue of free entropy in the presence, $\chi(x_1,\ldots,x_n : y_1,\ldots,y_n)$?*

5. Focus Group on Operator Theory

The problems in this section arose during a discussion group on Operator Theory during the 3rd day of the AIM workshop.

Problem 5.1. *What is the boundary behavior of the subordination functions which appear in free convolution of operator-valued random variables?*

f problem

Problem 5.2. *What are examples/conditions for freely strongly unimodal variables, i.e. unimodal random variables that when freely convolved with a unimodal variables remain unimodal?*

Remark. Unimodal means that the law of the random variable has a smooth density with a unique maximum; example: Gaussian law or the semicircle law.

problem

Problem 5.3. *More specifically, if μ,ν are symmetric unimodal distribution, is $\mu\boxplus\nu$ unimodal?*

6. Invariant Subspaces for an Operator

[*Haagerup*]

Problem 6.1. *Let x, y be two free circular elements, and let S, T be two operators in a II_1 factor, which is free from x, y. In the Haagerup-Schultz estimate*

$$(\star\star) \qquad \left\|(S+xy^{-1})^{-1}-(T+xy^{-1})^{-1}\right\|_p \le c(p)\,\|S-T\|_p < \infty$$

with $0<p<\frac{2}{3}$, can one use x instead of xy^{-1}?

f problem

(Brown measure of unbounded operators): As defined by (Haagerup and Schultz), $\Delta(T)$ makes sense for $T\in M^\Delta$ where $M^\Delta = \left\{T\in\tilde{M} \mid \int_0^\infty \log t\, d\mu_T(t)<\infty\right\}$. Then $\Delta(T) = \exp(\int_0^\infty \log t\, d\mu_T(t)) \in [0,\infty]$.

Problem 6.2. *Can one make sense of μ_T for such unbounded T?*

problem

7. Free Group Factor

[Ozawa] start of prob block

Conjecture 7.1. *If $\mathcal{H}$ an M-M bimodule $M = L\mathbb{F}_n$, and ${}_M\mathcal{H}_M \preceq L^2M \otimes L^2M$, (weak containment) then*

$$Hom({}_M\mathcal{H} \underset{M}{\otimes} \mathcal{H} \underset{M}{\otimes} \mathcal{H}_M, L^2M \otimes L^2M) \neq 0.$$

Note that the assumption of weak containment is equivalent that the map

$$x \otimes y \mapsto (\lambda(x)\rho(y) : \mathcal{H}_M \ni h \mapsto xhy) \in B({}_M\mathcal{H}_M)$$

is continuous for the min-tensor product on $M \otimes M$. Examples of bimodules with this property come from the basic construction

$${}_M\mathcal{H}_M = M \otimes_A M$$

over a hyperfinite subalgebra $A \subset M$. *end of prob block*

8. Combinatorics of Random Matrix Models

The material in this section arose during a discussion group on the 4th day of the AIM workshop.

Given random matrices A_n and B_n with corresponding measures μ_{A_n} and μ_{B_n} on $M_n(\mathbb{C})$, we define their Itzykson-Zuber integral as

$$IZ(A_n, B_n) = \int \exp(-nTr(AU^*BU))d\mu_{A_n}(A)d\mu_{B_n}(B).$$

Theorem 8.1. *[*Guionnet and Zeitouni*] If $\|A_n\| < c$, $\|B_n\| < c$ then $IZ(A_n, B_n) \sim \exp(-n\psi)$.*

Problem 8.2. *There is another result that states that*

$$\frac{\partial^n}{\partial t^n} \log IZ(tA_n, B_n)|_{t=0} \text{ converges.}$$

Does this expression match ψ above? Can we extend Guionnet and Zeitouni's result to complex parameters?

Problem 8.3. *Extend the model $\exp(-nTr(P(A_1, \dots, A_m) + \frac{1}{2}\sum_{i=1}^m A_i^2))dA_1 \dots dA_m$ of Guionnet and Maurel-Segala to non-selfadjoint P (i.e.polynomials with complex coefficients).*

Problem 8.4. *Is there a combinatorial interpretation of free cumulants in terms of enumeration of maps and operations on maps?*

start of prob block

Consider the spherical integrals

$$I_n(z, E_n) := \int \exp\{n\mathrm{tr}(UD_nU^*E_n)\}d_{m_n}(U),$$

where $D_n = \mathrm{diag}(\mathrm{z}, 0, 0, \dots, 0)$, $z \in \mathbb{C}$, and E_n is a sequence of $n \times n$ selfadjoint (diagonal) matrices, with spectrum uniformly bounded in n, and converging in distribution to μ_E

The sequence of functions of z

$$f_n(z) = \partial_z \frac{1}{n} \log I_n(z, E_n),$$

has been shown by Guionnet and Maida to converge to $R_{\mu_E}(z)$ for $|z|$ small enough.

Problem 8.5. *What is the largest domain in the complex plane on which this convergence takes place? If μ_E is $\boxplus$-infinitely divisible, is the convergence happening on all the upper half-plane? Is there any possible generalization to measures with noncompact support?*

Remark. One could probably approach this problem by trying to study the normality of the family/sequence f_n.

f *problem*

9. Invariant Subspaces

The questions in this section arose during the focus group on Invariant Subspaces during the 4th day of the workshop.

If M is a II_1 factor, $T_1, \ldots, T_n \in M$, $[T_i, T_j] = 0$, then we have the "Brown Measure" defined as the unique measure on $\mathbb{C}^n$ such that

$$(\star) \qquad \log \Delta(1 - \sum \alpha_i T_i) = \int \log(1 - \sum \alpha_i \zeta_i) d\mu_{T_!,\ldots,T_n}(\zeta_1, \ldots, \zeta_n).$$

Problem 9.1. *Is $\mathrm{supp}\mu_{T_1,\ldots,T_n} \subset \sigma(T_1, \ldots, T_n)$, the Taylor spectrum of $T_1, \ldots, T_n$?*

Problem 9.2. *Which functions on $\mathbb{C}^n$ have an integral representation as in $(\star)$o*

f problem

M a II_1 factor and $T \in M$. Define

$$K(T, r) = \left\{\xi \in \mathcal{H} | \exists \xi_n \in \mathcal{H} \text{ s.t. } \|\xi_n - \xi\|_2 \to 0 \text{ and } \limsup \|T^n \xi_n\|^{1/n} \to 0\right\},$$

$$\text{and } E(T, r) = \left\{\xi \in \mathcal{H} | \limsup \|T^n \xi_n\|^{1/n} \to 0\right\}.$$

Problem 9.3. *Does $K(T, r) = E(T, r)$?*

Remark. The DT quasinilpotent operator may be a counterexample.

f *problem*

Problem 9.4. *Let c be a circular element ($\sigma(c) = \bar{\mathbb{D}}$), and let $f \in C^\infty(\mathbb{C})$. Can we make sense of $f(c)$ as an (unbounded) operator affiliated with $\{c\}''$?*

Problem 9.5. *Let (Γ, τ) be a II_1 factor, $T \in \Gamma$, $\mu_T = \delta_0$. Does T have a non-trivial invariant subspace affiliated with Γ?*

Problem 9.6. *Let $B_c$be a band limited operator obtained from c a circular element, and let D be the band limited operator obtained from the identity. Then D is uniformly distributed on $[0, 1]$ and $\star$-free from $\{B_c, B_c^*\}$.*

Is $D \in W^*(B_c)$? Or is $W^*(B_c) = L\mathbb{F}_t$ with $t = 1 + 2c(1 - \frac{c}{2})$?

10. Infinite Divisibility

f problem

[*Nica*]

Given $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ in a vNa such that $\{x_1, \ldots, x_k\}$ is tensor-independent of $\{y_1, \ldots, y_k\}$ and such that $\mu_{x_1,\ldots,x_k}, \nu_{y_1,\ldots,y_k}$ are freely infinitely divisible, we can apply the Fourier transform to get the power-series of the classical convolution of $\mu_{x_1,\ldots,x_k}$ and $\nu_{y_1,\ldots,y_k}$.

Problem 10.1. *How do such power-series relate to the noncommutative power series obtained from free convolution?*

In other words how does the set of classically obtainable power-series relate to the set of freely obtainable power-series?

problem

f problem

Problem 10.2. *Can we make sense of the R-transform for x_1, x_2 unbounded (power-series are insufficient to encode all the information)?*

Easier question is for infinitely divisible unbounded operators.

end of prob block

Problem 10.3. *If c is unbounded R-diagonal, what is the R-transform of c, c^*?*

11. Dirichlet Forms, from Classical to Quantum

This section comes from a focus group on the 5th day of the AIM workshop.

start of prob block

Problem 11.1. *For the q-deformed semicircular, the analogue of $\partial^*\partial$ exists (it is the number operator). Describe explicitly the associated ∂.*

It exists by the work of Sauvageot.

end of prob block

Problem 11.2. *More generally, given a negative definite function on a group Γ (i.e. a Dirichlet form), we know it gives a representation by affine actions on $L^2\Gamma$. When is it a multiple of the left regular representation? What conditions on the negative definite function guarantee this?*

Problem 11.3. *What conditions on a Dirichlet form $\delta^*\delta$ guarantee that the bimodule associated to δ embeds into $\bigoplus L^2N \otimes L^2N$?*

Problem 11.4. *What is the analogue of the Bakry-Emery criterion in the noncommutative case? i.e. what is Γ_2 for noncommutative Dirichlet forms?*

Problem 11.5. *Let $\partial : M \to L^2(M)\bar{\otimes}L^2(M^o)$ be a closable derivation, and let $\Delta = \partial^*\partial$, $S_t = \exp(-t\Delta)$. If the semigroup S_t converges uniformly to the identity in $\|\cdot\|_2$ on the unit ball, is the derivation inner when considered with values in the algebra of unbounded operators affiliated to $M\bar{\otimes}M^o$?*