EQUIVALENCES TO THE RIEMANN HYPOTHESIS

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The Riemann Hypothesis(RH) is the assertion that all of the nontrivial zeros of the Riemann zeta function have real part equal to $\frac{1}{2}$.

The Riemann Hypothesis has been shown to be equivalent to an astounding variety of statements in several different areas of mathematics. Some of those equivalences are nearly trivial. For example, RH is equivalent to the nonvanishing of $\zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$. Other equivalences appear surprising and deep. Examples of both kinds are collected below.

For background on the Riemann zeta function and the Riemann Hypothesis, see ??? and also the Riemann Hypothesis and related problems[1] list.

1. Equivalences involving primes

The main point of Riemann's original paper is that the two sequences, of prime numbers on the one hand, and of zeros of ζ on the other hand, are in duality. A precise mathematical formulation of this fact is given by the so-called explicit formulas of prime number theory (Riemann, von Mangoldt, Guinand, Weil). Therefore, any statement about one of these two sequences must have a translation in terms of the other.

Some statements, such as the random matrix conjecture for the normalized neighbor spacing of the zeros, or the existence of infinitely many twin primes, do not seem to have a simple translation into a statement about the other sequence. But, as Riemann conjectured but did not prove, RH has a simple formulation in terms of the prime numbers.

Let Li be the "Logarithmic integral" function, defined by

$$Li(x) := \int_0^x \frac{dt}{\log t},$$

the integral being evaluated in principal value in the neighborhood of $x = 1$. And let

$$\begin{aligned} \pi(x) &= \sum_{p \le x} 1 && (1.1) \\ &= \text{the number of primes } p \le x. && (1.2) \end{aligned}$$

Here and throughout this document, p always stands for a prime number.

RH Equivalence 1.1. *The Riemann hypothesis is equivalent to*

$$\pi(x) = Li(x) + O(x^{1/2+\epsilon})$$

for any $\epsilon > 0$*.*

Roughly speaking, this means that the first half of the digits of the n-th prime are those of $Li^{-1}(n)$.

[1] http://aimpl.org/pl/rhrelated/

von Koch, Acta Mathematica 24 (1901), 159-182, showed:

RH Equivalence 1.15. *The Riemann hypothesis is equivalent to*

$$\pi(x) = Li(x) + O(\sqrt{x}\log x).$$

Remark. In 1976 L.Schoenfeld [56 15581b] gave a numerically explicit version of this equivalent form:

$$|\pi(x) - li(x)| \leq \frac{\sqrt{x}\log x}{8\pi} \text{ for } \geq 2657.$$

2. Averages of arithmetic functions

These equivalent statements have the following shape:

$$\sum_{n\leq x} f(n) = F(x) + O(x^{\alpha+\epsilon}), \quad x \to +\infty,$$

where f is an arithmetic function, $F(x)$ a smooth approximation to $\sum_{n\leq x} f(n)$, and α a real number.

The von Mangoldt function

The von Mangoldt function $\Lambda(n)$ is defined as $\log p$ if n is a power of a prime p, and 0 in the other cases. Define:

$$\psi(x) := \sum_{n\leq x} \Lambda(n).$$

RH Equivalence 2.1. *RH is equivalent to*

$$\psi(x) = x + O(x^{1/2+\epsilon}),$$

for every $\epsilon > 0$.

RH Equivalence 2.13. *RH is equivalent to*

$$\psi(x) = x + O(x^{1/2}\log^2)$$

L.Schoenfeld [56 15581b] refined Equivalence **??** to the numerically explicit form

RH Equivalence 2.17. *RH is equivalent to*

$$|\psi(x) - x| \leq \frac{x^{1/2}\log^2 x}{8\pi} \textit{ for } > 73.2.$$

The Möbius function

The Möbius function $\mu(n)$ is defined as $(-1)^r$ if n is a product of r distinct primes, and as 0 if the square of a prime divides n. Define:

$$M(x) := \sum_{n \le x} \mu(n).$$

Littlewood proved the following two equivalences.

RH Equivalence 2.2. *RH is equivalent to*

$$M(x) \ll x^{1/2+\epsilon},$$

for every positive ϵ

RH Equivalence 2.25. *RH is equivalent to*

$$M(x) \ll x^{1/2} \exp(A \log x / \log\log x),$$

for some $A > 0$.

Redheffer's matrix

The Redheffer matrix $A(n)$ is the $n \times n$ matrix of 0's and 1's defined by $A(i, j) = 1$ if $j = 1$ or if i divides j, and $A(i, j) = 0$ otherwise.

Redheffer proved that $A(n)$ has $n - [n \log 2] - 1$ eigenvalues equal to 1. Also, A has a real eigenvalue (the spectral radius) which is approximately $\sqrt{n}$, a negative eigenvalue which is approximately $-\sqrt{n}$ and the remaining eigenvalues are small.

The connection with the Riemann Hypothesis is that

$$\det A(n) = \sum_{1 \le j \le n} \mu(j).$$

Therefore by Equivalence **??**,

RH Equivalence 2.3. *The Riemann Hypothesis is equivalent to* $\det(A) = O(n^{1/2+\epsilon})$ *for every* $\epsilon > 0$.

Remark.

Barrett Forcade, Rodney, and Pollington [MR 89j:15029] give an easy proof of Redheffer's theorem. They also prove that the spectral radius of $A(n)$ is $= n^{1/2} + \frac{1}{2} \log n + O(1)$. See also the paper of Roesleren [MR 87i:11111].

Remark.

Vaughan [MR 94b:11086] and [MR 96m:11073] determines the dominant eigenvalues with an error term $O(n^{-2/3})$ and shows that the nontrivial eigenvalues are $\ll (\log n)^{2/5}$ (unconditionally), and $\ll \log\log(2 + n)$ on the Riemann Hypothesis.

Remark[Brian Conrey] It is possible that all the nontrivial eivenvalues lie in the unit disc.

3. Large values of arithmetic functions

RH is equivalent to several inequalities of the following type:

$$f(n) < F(n),$$

where f is an "arithmetic" or "irregular" function, and F an "analytic" or "regular" function.

The sum of divisors of n

Let

$$\sigma(n) = \sum_{d|n} d$$

denote the sum of the divisors of n.

Also let

$$H_n = \sum_{j=1}^{n} \frac{1}{j}$$

denote the nth harmonic number, and

$$\gamma = \lim_{n\to\infty} H_n - \log n \tag{3.3}$$
$$\approx 0.577215... \tag{3.4}$$

denote Euler's constant

G. Robin [86f:11069] showed that

RH Equivalence 3.1. *The Riemann Hypothesis is equivalent to*

$$\sigma(n) < e^{\gamma} n \log\log n$$

for all $n \geq 5041$.

That inequality does not leave much to spare, for Gronwall showed

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma},$$

and Robin showed unconditionally that

$$\sigma(n) < e^{\gamma} n \log\log n + 0.6482 \frac{n}{\log\log n},$$

for $n \geq 3$.

J. Lagarias [arXiv:math.NT/0008177] elaborated on Robin's work and showed that

RH Equivalence 3.15. *The Riemann Hypothesis is equivalent to*

$$\sigma(n) < H_n + \exp(H_n)\log(H_n)$$

for all $n \geq 2$.

Remark. By the definition (**??**) of γ, Lagarias' and Robin's inequalities are the same to leading order.

The Euler totient function

The Euler function $\phi(n)$ is defined as the number of positive integers not exceeding n and co-prime with n. That is, it is the multiplicative function which has $\phi(p^n) = p^n - p^{n-1}$.

Also, let N_k be the product of the first k prime numbers.

RH Equivalence 3.2. *The Riemann Hypothesis is equivalent to*

$$\frac{N_k}{\phi(N_k)} > e^\gamma \log\log N_k,$$

for all k.

RH Equivalence 3.25. *The Riemann Hypothesis is equivalent to*

$$\frac{N_k}{\phi(N_k)} > e^\gamma \log\log N_k,$$

for all but finitely many k.

The maximal order of an element in the symmetric group

Let $g(n)$ be the maximal order of a permutation of n objects, $\omega(k)$ be the number of distinct prime divisors of the integer k and Li be the integral logarithm.

Massias, Nicolas and Robin [89i:11108] showed that

RH Equivalence 3.3. *The Riemann Hypothesis is equivalent to*

$$\log g(n) < \sqrt{Li^{-1}(n)} \quad \textit{for } n \textit{ large enough.}$$

RH Equivalence 3.35. *The Riemann Hypothesis is equivalent to*

$$\omega(g(n)) < Li(\sqrt{Li^{-1}(n)}) \quad \textit{for } n \textit{ large enough.}$$

4. Farey series

Let r_v be the elements of the Farey sequence of order N, $v = 1, 2, \ldots \Phi(N)$ where $\Phi(N) = \sum_{n=1}^{N} \phi(n)$. Let $\delta_v = r_v - v/\Phi(N)$.

A good (put possibly out-of-date) bibliography on Farey sequences and RH is available at `http://people.math.jussieu.fr/~miw/telecom/biblio-Amoroso.html`.

RH Equivalence 4.1. *The Riemann Hypothesis is equivalent to*

$$\sum_{v=1}^{\Phi(N)} \delta_v^2 \ll N^{-1+\epsilon}.$$

RH Equivalence 4.15. *The Riemann Hypothesis is equivalent to*

$$\sum_{v=1}^{\Phi(N)} |\delta_v| \ll N^{1/2+\epsilon}.$$

Amoroso's criterion

Amoroso [MR 98f:11113] has proven the following interesting equivalent to the Riemann Hypothesis. Let $\Phi_n(z)$ be the nth cyclotomic polynomial and let $F_N(z) = \prod_{n\le N} \Phi_n(z)$. Let

$$\tilde{h}(F_N) = (2\pi)^{-1} \int_{-\pi}^{\pi} \log^+ |F(e^{i\theta})|\, d\theta.$$

Then,

RH Equivalence 4.2. $\tilde{h}(F_n) \ll N^{\lambda+\epsilon}$ *is equivalent to the assertion that the Riemann zeta function does not vanish for* $\mathrm{Re} z \ge \lambda + \epsilon$.

5. Weil's positivity criterion

André Weil [MR 14,727e] proved the following explicit formula (see also A. P. Guinand [MR 10,104g] which specifically illustrates the dependence between primes and zeros. Let h be an even function which is holomorphic in the strip $|\Im t| \le 1/2 + \delta$ and satisfying $h(t) = O((1 + |t|)^{-2-\delta})$ for some $\delta > 0$, and let

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iur}\, dr.$$

Then we have the following duality between primes and zeros:

$$\sum_{\gamma} h(\gamma) = 2h(\tfrac{i}{2}) - g(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(\tfrac{1}{4} + \tfrac{1}{2}ir)\, dr - 2\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\log n).$$

In this formula, a zero is written as $\rho = 1/2 + i\gamma$ where $\gamma \in \mathbb{C}$; of course RH is the assertion that all of the γ are real. Using this duality Weil gave a criterion for RH.

Bombieri's refinement

Bombieri [1 841 692]has given the following version of Weil's criterion.
Let

$$\hat{g}(s) = \int_0^{\infty} g(x) x^{s-1}\, dx.$$

RH Equivalence 5.1. *The Riemann Hypothesis is equivalent to*

$$\sum_{\rho} \hat{g}(\rho)\hat{\bar{g}}(1-\rho) > 0$$

for every complex-valued $g(x) \in C_0^{\infty}(0, \infty)$ *which is not identically 0.*

Li's criterion

Xian-Jin Li [98d:11101] proved the following assertion:

RH Equivalence 5.2. *The Riemann Hypothesis is equivalent to* $\lambda_n \ge 0$ *for all n, where*

$$\lambda_n = \sum_{\rho} (1 - (1 - 1/\rho)^n),$$

where the sum is over the zeros of the zeta function.

Another expression for λ_n is given by

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}(s^{n-1}\log\xi(s))|_{s=1}$$

where

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma(s/2)\zeta(s)$$

RH Equivalence 5.3. *The Riemann Hypothesis is equivalent to the equality*

$$\sum_{\rho}\frac{1}{|\rho|^2} = 2 + \gamma - \log 4\pi$$

where the sum is over all complex zeros $\rho = \beta + i\gamma$ *of* $\zeta(s)$ *in the critical strip* $0 < \beta < 1$.

6. Function-theoretic properties of ζ

Speiser's criterion

A. Speiser (Math Annahlen 110 (1934) 514-521) prove

RH Equivalence 6.1. *The Riemann Hypothesis is equivalent to the nonvanishing of the derivative* $\zeta'(s)$ *in the left-half of the critical strip* $0 < \sigma < 1/2$.

Remark. Levinson and Montgomery [MR 54 5135] gave an alternative, quantitative version of this result. This led to Levinson's [MR 58 27837] discovery of his method for counting zeros on the critical line, which he used to prove that at least 1/3 of the zeros of $\zeta(s)$ are on the critical line.

RH Equivalence 6.2. *The Riemann Hypothesis is equivalent to*

$$\Re\frac{\xi'(s)}{\xi(s)} > 0$$

for $\Re s > 1/2$.

V. V. Volchkov [MR 96g:11111] proved

RH Equivalence 6.3. *The Riemann Hypothesis is equivalent to*

$$\int_0^\infty\int_{1/2}^\infty \frac{1-12y^2}{(1+4y^2)^3}\log(|\zeta(x+iy)|)\,dx\,dy = \pi\frac{3-\gamma}{32}$$

7. Function spaces

Beginning with Wiener's paper, "Tauberian Theorems" in the Annals of Math (1932) a number of functional analytic equivalences of RH have been proven. These involve the completeness of various spaces. M. Balazard has written a survey on these developments (See Surveys in Number Theory, Papers from the Millennial Conference on Number Theory, A. K. Peters, 2003.)

The Beurling-Nyman Criterion

Let $\mathcal{N}_{(0,1)}$ be the space of functions

$$f(t) = \sum_{k=1}^{n} c_k \rho(\theta_k/t)$$

for which $\theta_k \in (0,1)$ and such that $\sum_{k=1}^{n} c_k = 0$.

RH Equivalence 7.1. *The Riemann Hypothesis is equivalent to the assertion that $\mathcal{N}_{(0,1)}$ is dense in $L^2(0,1)$.*

Beurling [MR 17,15a] proved that the following equivalences of a quasi-Riemann Hypothesis.

RH Equivalence 7.2. *Suppose $1 < p < \infty$. The following are equivalent:*

(1) *$\zeta(s)$ has no zeros in $\sigma > 1/p$,*
(2) *$\mathcal{N}_{(0,1)}$ is dense in $L^p(0,1)$,*
(3) *The characteristic function $\chi_{(0,1)}$ is in the closure of $\mathcal{N}_{(0,1)}$ in $L^p(0,1)$.*

Mollifiers

Baez-Duarte [arXiv:math.NT/0202141] proved

RH Equivalence 7.3. *The Riemann Hypothesis is equivalent to*

$$\inf_{A_N(s)} \int_{-\infty}^{\infty} |1 - A_N(1/2+it)\zeta(1/2+it|^2)\, \frac{dt}{\frac{1}{4}+t^2} \to 0$$

as $N \to \infty$, where the infimum is over all Dirichlet polynomials of length N:

$$A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

Salem's criterion

Let

$$k_\sigma(x) = \frac{x^{\sigma-1}}{e^x+1}.$$

R. Salem [MR 14,727a] proved an equivalence for the nonvanishing of the zeta function on a vertical line:

RH Equivalence 7.4. *The non-vanishing of $\zeta(s)$ on the σ –line is equivalent to the completeness in $L^1(0,\infty)$ of $\{k_\sigma(\lambda x),\ \lambda > 0\}$.*

8. The zeta function at the positive integers

These equivalences involve $\zeta(n)$ at integers $n \geq 2$, or the Bernoulli numbers via the identity

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}$$

Riesz series

M. Riesz (Sur l'hypothèse de Riemann, Acta Math. 40 (1916), 185-190) proved

RH Equivalence 8.1. *The Riemann Hypothesis is equivalent to*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} \ll x^{1/2+\epsilon}$$

for any $\epsilon > 0$.

Hardy-Littlewood series

Hardy and Littlewood (Acta Mathematica 41 (1918), 119 - 196) showed

RH Equivalence 8.2. *The Riemann Hypothesis is equivalent to*

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!\zeta(2k+1)} \ll x^{-1/4}$$

as $x \to \infty$.

Carey's series

RH Equivalence 8.3. *The Riemann Hypothesis is equivalent to the convergence of the series*

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \left| \sum_{k=0}^{n} \frac{c_{2n+1,2k+1}}{2k+2} \log\left(\frac{2k+1}{2k+2} \frac{(-1)^k B_{2k+2}(2\pi)^{2k+2}}{2(2k+2)!} \right) \right|^2 < \infty$$

where $c_{m,r}$ denotes the coefficient of x^r in the Legendre polynomial of degree m. Specifically,

$$c_{2n+1,2k+1} = \frac{(-1)^{n-k}(2n+2k+2)!}{2^{2n+1}(n-k)!(n+k+1)!(2k+1)!}.$$

9. Analytic estimates

Polya's integral criterion

Polya (see Collected Works, Volume 2, Paper 102, section 7) gave a number of integral criteria for Fourier transforms to have only real zeros.

Let

$$\Phi(u) = 2\sum_{n=1}^{\infty}(2n^4\pi^2 e^{\frac{9}{2}u} - 3n^2\pi e^{\frac{5}{2}u})e^{-n^2\pi e^{2u}}$$

so that

$$\xi(\tfrac{1}{2} + iz) = \int_{-\infty}^{\infty} \Phi(t)e^{iz}\, dt.$$

One of Polya's criteria gives

RH Equivalence 9.1. *The Riemann Hypothesis is equivalent to*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\alpha)\Phi(\beta)e^{i(\alpha+\beta)x}e^{(\alpha-\beta)y}(\alpha-\beta)^2\, d\alpha\, d\beta \geq 0.$$

Newman's criterion

Charles Newman [MR 55 #7944], building on work of deBruijn [MR 12,250] defined

$$\Xi_\lambda(z) = \int_{-\infty}^{\infty} \Phi(t)e^{-\lambda t^2}e^{iz}\, dt.$$

Note that $\Xi_0(z) = \Xi(z) := \xi(\frac{1}{2} + iz)$.

Newman proved that there exists a constant Λ (with $-1/8 \leq \Lambda < \infty$) such that $\Xi_\lambda(z)$ has only real zeros if and only if $\lambda \geq \Lambda$.

RH Equivalence 9.2. *The Riemann Hypothesis is equivalent to* $\Lambda \leq 0$.

The constant Λ (which Newman conjectured is equal to 0) is now called the deBruijn-Newman constant.

Remark. A. Odlyzko [MR 2002a:30046] has recently proven that $-2.710^{-9} < \Lambda$.

Remark. Conjectures for the distribution of gaps between zeros, based on random matrix theory, imply that $\Lambda \geq 0$.

Grommer inequalities

Let

$$-\frac{\Xi'}{\Xi}(t) = s_1 + s_2 t + s_3 t^2 + \dots.$$

Let M_n be the matrix whose i, j entry is s_{i+j}. J. Grommer (J. Reine Angew. Math. 144 (1914), 114–165) proved

RH Equivalence 9.3. *The Riemann Hypothesis is equivalent to* $\det M_n > 0$ *for all* $n \geq 1$.

Remark. See also the paper of R. Alter [MR 36 1399].