Chapter 1

ANALYTICAL TRIGONOMETRY

1.1 The Pythagorean Identities

In section Section **??**, we first encountered the concept of an **identity** when discussing Theorem **??**. Recall that an identity is an equation which is true regardless of the choice of variable. Identities are important in mathematics because they facilitate changing forms.[1]

We take a moment to generalize Theorem **??** below.

Theorem 1.1. Reciprocal and Quotient Identities: The following relationships hold for all angles θ provided each side of each equation is defined.

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$
- $\cos(\theta) = \dfrac{1}{\sec(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$
- $\sin(\theta) = \dfrac{1}{\csc(\theta)}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
- $\cot(\theta) = \dfrac{1}{\tan(\theta)}$
- $\tan(\theta) = \dfrac{1}{\cot(\theta)}$

It is important to remember that the equivalences stated in Theorem 1.1 are valid only when *all* quantities described therein are defined. As an example, $\tan(0) = 0$, but $\tan(0) \neq \frac{1}{\cot(0)}$ since $\cot(0)$ is undefined.

When it comes down to it, the Reciprocal and Quotient Identities amount to giving different ratios on the Unit Circle different names. The main focus of this section is on a more algebraic relationship between certain pairs of the circular functions: the **Pythagorean Identities**.

Recall in Definition **??**, the cosine and sine of an angle is defined as the x and y-coordinate, respectively, of a point on the Unit Circle. Since the coordinates of all points (x, y) on the Unit Circle satisfy the equation $x^2 + y^2 = 1$, we get for all angles θ, $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. An unfortunate[2] convention, which the authors are compelled to perpetuate, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. Rewriting the identity using this convention results in the following theorem, which is without a doubt one of the most important results in Trigonometry.

Theorem 1.2. The Pythagorean Identity: For any angle θ, $\cos^2(\theta) + \sin^2(\theta) = 1$.

The moniker 'Pythagorean' brings to mind the Pythagorean Theorem, from which both the Distance Formula and the equation for a circle are ultimately derived.[3] The word 'Identity' reminds us that, regardless of the angle θ, the equation in Theorem 1.2 is always true.

If one of $\cos(\theta)$ or $\sin(\theta)$ is known, Theorem 1.2 can be used to determine the other, up to a $(\pm)$ sign. If, in addition, we know where the terminal side of θ lies when in standard position, then we can remove the ambiguity of the $(\pm)$ and completely determine the missing value.[4] We illustrate this approach in the following example.

Example 1.1.1. Use Theorem 1.2 and the given information to find the indicated value.

[1] We've seen the utility of changing form throughout the text, most recently when we completed the square in Chapter **??** to put general quadratic equations into standard form in order to graph them.

[2] This is unfortunate from a 'function notation' perspective. See Section 1.3.

[3] See Sections **??** and **??** for details.

[4] See the illustration following Example **??** to refresh yourself which circular functions are positive in which quadrants.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, find $\cos(\theta)$.

2. If $\pi < t < \frac{3\pi}{2}$ with $\cos(t) = -\frac{\sqrt{5}}{5}$, find $\sin(t)$.

3. If $\sin(\theta) = 1$, find $\cos(\theta)$.

Solution.

1. When we substitute $\sin(\theta) = \frac{3}{5}$ into The Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\cos^2(\theta) + \frac{9}{25} = 1$. Solving, we find $\cos(\theta) = \pm\frac{4}{5}$. Since θ is a Quadrant II angle, we know $\cos(\theta) < 0$. Hence, we select $\cos(\theta) = -\frac{4}{5}$.

2. Here we're using the variable t instead θ which usually corresponds to a real number variable instead of an angle. As usual, we associate real numbers t with angles θ measuring t radians,[5] so the the Pythagorean Identity works equally well for all real numbers t as it does for all angles θ.

   Substituting $\cos(t) = -\frac{\sqrt{5}}{5}$ into $\cos^2(t) + \sin^2(t) = 1$ gives $\sin(t) = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. Since $\pi < t < \frac{3\pi}{2}$, we know t corresponds to a Quadrant III angle, so $\sin(t) < 0$. Hence, $\sin(t) = -\frac{2\sqrt{5}}{5}$.

3. When we substitute $\sin(\theta) = 1$ into $\cos^2(\theta) + \sin^2(\theta) = 1$, we find $\cos(\theta) = 0$. $\square$

The reader is encouraged to compare and contrast the solution strategies demonstrated in Example 1.1.1 with those showcases in Examples **??** and **??** in Section **??**.

As with many tools in mathematics, identities give us a different way to approach and solve problems.[6] As always, the key is to determine which approach makes the most sense (is more efficient, for instance) in the given scenario.

Our next task is to use use the Reciprocal and Quotient Identities found in Theorem 1.1 coupled with the Pythagorean Identity found in Theorem 1.2 to derive new Pythagorean-like identities for the remaining four circular functions.

Assuming $\cos(\theta) \neq 0$, we may start with $\cos^2(\theta) + \sin^2(\theta) = 1$ and divide both sides by $\cos^2(\theta)$ to obtain $1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$. Using properties of exponents along with the Reciprocal and Quotient Identities, this reduces to $1 + \tan^2(\theta) = \sec^2(\theta)$.

If $\sin(\theta) \neq 0$, we can divide both sides of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\sin^2(\theta)$, apply Theorem 1.1 once again, and obtain $\cot^2(\theta) + 1 = \csc^2(\theta)$.

These three Pythagorean Identities are worth memorizing and they, along with some of their other common forms, are summarized in the following theorem.

[5]See page **??** if you need a review of how we associate real numbers with angles in radian measure.

[6]For example, factoring, completing the square, and the quadratic formula are three different (yet equivalent) ways to solve a quadratic equation. See Section **??** for a refresher.

Theorem 1.3. The Pythagorean Identities:

1. $\cos^2(\theta) + \sin^2(\theta) = 1$.

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$

2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

   Common Alternate Forms:

   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$

3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

   Common Alternate Forms:

   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

As usual, the formulas states in Theorem 1.3 work equally well for (the applicable) angles as well as real numbers.

Example 1.1.2. Use Theorems 1.1 and 1.3 to find the indicated values.

1. If θ is a Quadrant IV angle with $\sec(\theta) = 3$, find $\tan(\theta)$.
2. Find $\csc(t)$ if $\pi < t < \frac{3\pi}{2}$ and $\cot(t) = 2$.
3. If θ is a Quadrant II angle with $\cos(\theta) = -\frac{3}{5}$, find the exact values of the remaining circular functions.

Solution.

1. Per Theorem 1.3, $\tan^2(\theta) = \sec^2(\theta) - 1$. Since $\sec(\theta) = 3$, we have $\tan^2(\theta) = (3)^2 - 1 = 8$, or $\tan(\theta) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since θ is a Quadrant IV angle, we know $\tan(\theta) < 0$ so $\tan(\theta) = -2\sqrt{2}$.

2. Again, using Theorem 1.3, we have $\csc^2(t) = 1 + \cot^2(t)$, so we have $\csc^2(t) = 1 + (2)^2 = 5$. This gives $\csc(t) = \pm\sqrt{5}$. Since $\pi < t < \frac{3\pi}{2}$, t corresponds to a Quadrant III angle, so $\csc(t) = -\sqrt{5}$.

3. With five function values to find, we have our work cut out for us. From Theorem 1.1, we know $\sec(\theta) = \frac{1}{\cos(\theta)}$, so we (quickly) get $\sec(\theta) = \frac{1}{-3/5} = -\frac{5}{3}$.

   Next, we go after $\sin(\theta)$ since between $\sin(\theta)$ and $\cos(\theta)$, we can get all of the remaining values courtesy of Theorem 1.1.

From Theorem 1.3, we have $\sin^2(\theta) = 1 - \cos^2(\theta)$, so $\sin^2(\theta) = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$. Hence, $\sin(\theta) = \pm\frac{4}{5}$ but since θ is a Quadrant II angle, we select $\sin(\theta) = \frac{4}{5}$.

Back to Theorem 1.1, we get $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{4/5} = \frac{5}{4}$, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{4/5}{-3/5} = -\frac{4}{3}$, and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{-3/5}{4/5} = -\frac{3}{4}$. □

Again, the reader is encouraged to study the solution methodology illustrated in Example 1.1.2 as compared with that employed in Example **??** in Section **??**.

Trigonometric identities play an important role in not just Trigonometry, but in Calculus as well. We'll use them in this book to find the values of the circular functions of an angle and solve equations and inequalities. In Calculus, they are needed to simplify otherwise complicated expressions. In the next example, we make good use of the Theorems 1.1 and 1.3.

Example 1.1.3. Verify the following identities. Assume that all quantities are defined.

1. $\tan(\theta) = \sin(\theta)\sec(\theta)$
2. $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = -1$
3. $\sin^2(x)\cos^3(x) = \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)$
4. $\dfrac{\sec(t)}{1 - \tan(t)} = \dfrac{1}{\cos(t) - \sin(t)}$
5. $6\sec(x)\tan(x) = \dfrac{3}{1 - \sin(x)} - \dfrac{3}{1 + \sin(x)}$
6. $\dfrac{\sin(\theta)}{1 - \cos(\theta)} = \dfrac{1 + \cos(\theta)}{\sin(\theta)}$

Solution. In verifying identities, we typically start with the more complicated side of the equation and use known identities to *transform* it into the other side of the equation.

1. Starting with the right hand side of $\tan(\theta) = \sin(\theta)\sec(\theta)$, we use $\sec(\theta) = \frac{1}{\cos(\theta)}$ and find:

$$\sin(\theta)\sec(\theta) = \sin(\theta)\frac{1}{\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta),$$

where the last equality is courtesy of Theorem 1.1.

2. Expanding the left hand side, we get: $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = \tan^2(t) - \sec^2(t)$. From Theorem 1.3, we know $\sec^2(t) - \tan^2(t) = 1$, which isn't *quite* what we have. We are off by a negative sign $(-)$, so we factor it out:

$$(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = \tan^2(t) - \sec^2(t) = (-1)(\sec^2(t) - \tan^2(t)) = (-1)(1) = -1.$$

3. Starting with the right hand side,[7] we notice we have a quantity we can immediately simplify per Theorem 1.3: $1 - \sin^2(x) = \cos^2(x)$. This increases the number of factors of cosine, (which is part of our goal in looking at the left hand side), so we proceed:

$$\sin^2(x)\left(1 - \sin^2(x)\right)\cos(x) = \sin^2(x)\cos^2(x)\cos(x) = \sin^2(x)\cos^3(x).$$

[7] We hope by this point a shift of variable to 'x' instead of 'θ' or 't' is a non-issue.

4. While both sides of our next identity contain fractions, the left side affords us more opportunities to use our identities.[8] Substituting $\sec(t) = \frac{1}{\cos(t)}$ and $\tan(t) = \frac{\sin(t)}{\cos(t)}$, we get:

$$\begin{aligned}
\frac{\sec(t)}{1-\tan(t)} &= \frac{\dfrac{1}{\cos(t)}}{1-\dfrac{\sin(t)}{\cos(t)}} = \frac{\dfrac{1}{\cos(t)}}{1-\dfrac{\sin(t)}{\cos(t)}} \cdot \frac{\cos(t)}{\cos(t)} \\
&= \frac{\left(\dfrac{1}{\cos(t)}\right)(\cos(t))}{\left(1-\dfrac{\sin(t)}{\cos(t)}\right)(\cos(t))} = \frac{1}{(1)(\cos(t)) - \left(\dfrac{\sin(t)}{\cos(t)}\right)(\cos(t))} \\
&= \frac{1}{\cos(t)-\sin(t)},
\end{aligned}$$

which is exactly what we had set out to show.

5. Starting with the right hand side, we can get started by obtaining common denominators to add:

$$\begin{aligned}
\frac{3}{1-\sin(x)} - \frac{3}{1+\sin(x)} &= \frac{3(1+\sin(x))}{(1-\sin(x))(1+\sin(x))} - \frac{3(1-\sin(x))}{(1+\sin(x))(1-\sin(x))} \\
&= \frac{3+3\sin(x)}{1-\sin^2(x)} - \frac{3-3\sin(x)}{1-\sin^2(x)} \\
&= \frac{(3+3\sin(x)) - (3-3\sin(x))}{1-\sin^2(x)} \\
&= \frac{6\sin(x)}{1-\sin^2(x)}
\end{aligned}$$

At this point, we have at least reduced the number of fractions from two to one, it may not be clear how to proceed. When this happens, it isn't a bad idea to start working with the other side of the identity to get some clues how to proceed.

Using a reciprocal and quotient identity, we find $6\sec(x)\tan(x) = 6\left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{6\sin(x)}{\cos^2(x)}$.

Theorem 1.3 tells us $1-\sin^2(x) = \cos^2(x)$, which means to our surprise and delight, we are much closer to our goal that we may have originally thought:

$$\begin{aligned}
\frac{3}{1-\sin(x)} - \frac{3}{1+\sin(x)} &= \frac{6\sin(x)}{1-\sin^2(x)} = \frac{6\sin(x)}{\cos^2(x)} \\
&= 6\left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) = 6\sec(x)\tan(x).
\end{aligned}$$

[8] Or, to put to another way, earn more partial credit if this were an exam question!

6. It is debatable which side of the identity is more complicated. One thing which stands out is that the denominator on the left hand side is $1 - \cos(\theta)$, while the numerator of the right hand side is $1+\cos(\theta)$. This suggests the strategy of starting with the left hand side and multiplying the numerator and denominator by the quantity $1 + \cos(\theta)$. Theorem 1.3 comes to our aid once more when we simplify $1 - \cos^2(\theta) = \sin^2(\theta)$:

$$\begin{aligned} \frac{\sin(\theta)}{1-\cos(\theta)} &= \frac{\sin(\theta)}{(1-\cos(\theta))} \cdot \frac{(1+\cos(\theta))}{(1+\cos(\theta))} = \frac{\sin(\theta)(1+\cos(\theta))}{(1-\cos(\theta))(1+\cos(\theta))} \\ &= \frac{\sin(\theta)(1+\cos(\theta))}{1-\cos^2(\theta)} = \frac{\sin(\theta)(1+\cos(\theta))}{\sin^2(\theta)} \\ &= \frac{\cancel{\sin(\theta)}(1+\cos(\theta))}{\cancel{\sin(\theta)}\sin(\theta)} = \frac{1+\cos(\theta)}{\sin(\theta)} \end{aligned}$$

□

In Example 1.1.3 number 6 above, we see that multiplying $1 - \cos(\theta)$ by $1 + \cos(\theta)$ produces a difference of squares that can be simplified to one term using Theorem 1.3.

This is exactly the same kind of phenomenon that occurs when we multiply expressions such as $1 - \sqrt{2}$ by $1 + \sqrt{2}$ or $3 - 4i$ by $3 + 4i$. In algebra, these sorts of expressions were called 'conjugates.'[9]

For this reason, the quantities $(1 - \cos(\theta))$ and $(1 + \cos(\theta))$ are called 'Pythagorean Conjugates.' Below is a list of other common Pythagorean Conjugates.

Pythagorean Conjugates

- $1 - \cos(\theta)$ and $1 + \cos(\theta)$: $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$
- $1 - \sin(\theta)$ and $1 + \sin(\theta)$: $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta) - 1$ and $\sec(\theta) + 1$: $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$
- $\sec(\theta) - \tan(\theta)$ and $\sec(\theta) + \tan(\theta)$: $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$
- $\csc(\theta) - 1$ and $\csc(\theta) + 1$: $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$
- $\csc(\theta) - \cot(\theta)$ and $\csc(\theta) + \cot(\theta)$: $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$

Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling with them just to become proficient in the basics.

Like many things in life, there is no short-cut here – there is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies which may be helpful (depending on the situation) is provided below and ample practice is provided for you in the Exercises.

[9] See Sections **??** and **??**.

Strategies for Verifying Identities

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in Theorem 1.1 to write functions on one side of the identity in terms of the functions on the other side of the identity.

  Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities in Theorem 1.3 to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities in Theorem 1.3.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.
- Try *something*. The more you work with identities, the better you'll get with identities.

1.1.1 Exercises

In Exercises 1 - 11, use the Reciprocal and Quotient Identities (Theorem 1.1) along with the Pythagorean Identities (Theorem 1.3), to find the value of the circular function requested below. (Find the exact value unless otherwise indicated.)

1. If $\sin(\theta) = \frac{\sqrt{5}}{5}$, find $\csc(\theta)$.
2. If $\sec(\theta) = -4$, find $\cos(\theta)$.
3. If $\tan(t) = 3$, find $\cot(t)$.
4. If θ is a Quadrant IV angle with $\cos(\theta) = \frac{5}{13}$, find $\sin(\theta)$.
5. If θ is a Quadrant III angle with $\tan(\theta) = 2$, find $\sec(\theta)$.
6. If $\frac{\pi}{2} < t < \pi$ with $\cot(t) = -2$, find $\csc(t)$.
7. If $\sec(\theta) = 3$ and $\sin(\theta) < 0$, find $\tan(\theta)$.
8. If $\sin(\theta) = -\frac{2}{3}$ but $\tan(\theta) > 0$, find $\cos(\theta)$.
9. If $0 < t < \frac{\pi}{2}$ and $\sin(t) = 0.42$, find $\cos(t)$, rounded to four decimal places.
10. If θ is Quadrant IV angle with $\sec(\theta) = 1.17$, find $\tan(\theta)$, rounded to four decimal places.
11. If $\pi < t < \frac{3\pi}{2}$ with $\cot(t) = 4.2$, find $\csc(t)$, rounded to four decimal places.

In Exercises 12 - 25, use the Reciprocal and Quotient Identities (Theorem 1.1) along with the Pythagorean Identities (Theorem 1.3), to find the exact values of the remaining circular functions. (Compare your methods with how you solved Exercises **??** - **??** in Section **??**.)

12. $\sin(\theta) = \dfrac{3}{5}$ with θ in Quadrant II
13. $\tan(\theta) = \dfrac{12}{5}$ with θ in Quadrant III
14. $\csc(\theta) = \dfrac{25}{24}$ with θ in Quadrant I
15. $\sec(\theta) = 7$ with θ in Quadrant IV
16. $\csc(\theta) = -\dfrac{10\sqrt{91}}{91}$ with θ in Quadrant III
17. $\cot(\theta) = -23$ with θ in Quadrant II
18. $\tan(\theta) = -2$ with θ in Quadrant IV.
19. $\sec(\theta) = -4$ with θ in Quadrant II.
20. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.
21. $\cos(\theta) = \dfrac{1}{3}$ with θ in Quadrant I.
22. $\cot(t) = 2$ with $0 < t < \dfrac{\pi}{2}$.
23. $\csc(t) = 5$ with $\dfrac{\pi}{2} < t < \pi$.
24. $\tan(t) = \sqrt{10}$ with $\pi < t < \dfrac{3\pi}{2}$.
25. $\sec(t) = 2\sqrt{5}$ with $\dfrac{3\pi}{2} < t < 2\pi$.
26. Skippy claims $\cos(\theta) + \sin(\theta) = 1$ is an identity because when $\theta = 0$, the equation is true. Is Skippy correct? Explain.

In Exercises 30 - 76, verify the identity. Assume that all quantities are defined.

27. $\cos(\theta)\sec(\theta) = 1$

28. $\tan(t)\cos(t) = \sin(t)$

29. $\sin(\theta)\csc(\theta) = 1$

30. $\tan(t)\cot(t) = 1$

31. $\csc(x)\cos(x) = \cot(x)$

32. $\dfrac{\sin(t)}{\cos^2(t)} = \sec(t)\tan(t)$

33. $\dfrac{\cos(\theta)}{\sin^2(\theta)} = \csc(\theta)\cot(\theta)$

34. $\dfrac{1+\sin(x)}{\cos(x)} = \sec(x) + \tan(x)$

35. $\dfrac{1-\cos(\theta)}{\sin(\theta)} = \csc(\theta) - \cot(\theta)$

36. $\dfrac{\cos(t)}{1-\sin^2(t)} = \sec(t)$

37. $\dfrac{\sin(x)}{1-\cos^2(x)} = \csc(x)$

38. $\dfrac{\sec(t)}{1+\tan^2(t)} = \cos(t)$

39. $\dfrac{\csc(\theta)}{1+\cot^2(\theta)} = \sin(\theta)$

40. $\dfrac{\tan(x)}{\sec^2(x)-1} = \cot(x)$

41. $\dfrac{\cot(t)}{\csc^2(t)-1} = \tan(t)$

42. $4\cos^2(\theta) + 4\sin^2(\theta) = 4$

43. $9 - \cos^2(t) - \sin^2(t) = 8$

44. $\tan^3(t) = \tan(t)\sec^2(t) - \tan(t)$

45. $\sin^5(x) = \left(1-\cos^2(x)\right)^2 \sin(x)$

46. $\sec^{10}(t) = \left(1+\tan^2(t)\right)^4 \sec^2(t)$

47. $\cos^2(x)\tan^3(x) = \tan(x) - \sin(x)\cos(x)$

48. $\sec^4(t) - \sec^2(t) = \tan^2(t) + \tan^4(t)$

49. $\dfrac{\cos(\theta)+1}{\cos(\theta)-1} = \dfrac{1+\sec(\theta)}{1-\sec(\theta)}$

50. $\dfrac{\sin(t)+1}{\sin(t)-1} = \dfrac{1+\csc(t)}{1-\csc(t)}$

51. $\dfrac{1-\cot(x)}{1+\cot(x)} = \dfrac{\tan(x)-1}{\tan(x)+1}$

52. $\dfrac{1-\tan(t)}{1+\tan(t)} = \dfrac{\cos(t)-\sin(t)}{\cos(t)+\sin(t)}$

53. $\tan(\theta) + \cot(\theta) = \sec(\theta)\csc(\theta)$

54. $\csc(t) - \sin(t) = \cot(t)\cos(t)$

55. $\cos(x) - \sec(x) = -\tan(x)\sin(x)$

56. $\cos(x)(\tan(x) + \cot(x)) = \csc(x)$

57. $\sin(t)(\tan(t) + \cot(t)) = \sec(t)$

58. $\dfrac{1}{1-\cos(\theta)} + \dfrac{1}{1+\cos(\theta)} = 2\csc^2(\theta)$

59. $\dfrac{1}{\sec(t)+1} + \dfrac{1}{\sec(t)-1} = 2\csc(t)\cot(t)$

60. $\dfrac{1}{\csc(x)+1} + \dfrac{1}{\csc(x)-1} = 2\sec(x)\tan(x)$

61. $\dfrac{1}{\csc(t) - \cot(t)} - \dfrac{1}{\csc(t) + \cot(t)} = 2\cot(t)$

62. $\dfrac{\cos(\theta)}{1 - \tan(\theta)} + \dfrac{\sin(\theta)}{1 - \cot(\theta)} = \sin(\theta) + \cos(\theta)$

63. $\dfrac{1}{\sec(t) + \tan(t)} = \sec(t) - \tan(t)$

64. $\dfrac{1}{\sec(x) - \tan(x)} = \sec(x) + \tan(x)$

65. $\dfrac{1}{\csc(t) - \cot(t)} = \csc(t) + \cot(t)$

66. $\dfrac{1}{\csc(\theta) + \cot(\theta)} = \csc(\theta) - \cot(\theta)$

67. $\dfrac{1}{1 - \sin(x)} = \sec^2(x) + \sec(x)\tan(x)$

68. $\dfrac{1}{1 + \sin(t)} = \sec^2(t) - \sec(t)\tan(t)$

69. $\dfrac{1}{1 - \cos(\theta)} = \csc^2(\theta) + \csc(\theta)\cot(\theta)$

70. $\dfrac{1}{1 + \cos(x)} = \csc^2(x) - \csc(x)\cot(x)$

71. $\dfrac{\cos(t)}{1 + \sin(t)} = \dfrac{1 - \sin(t)}{\cos(t)}$

72. $\csc(\theta) - \cot(\theta) = \dfrac{\sin(\theta)}{1 + \cos(\theta)}$

73. $\dfrac{1 - \sin(x)}{1 + \sin(x)} = (\sec(x) - \tan(x))^2$

In Exercises 77 - 80, verify the identity. You may need to consult Sections **??** and **??** for a review of the properties of absolute value and logarithms before proceeding.

74. $\ln|\sec(x)| = -\ln|\cos(x)|$

75. $-\ln|\csc(x)| = \ln|\sin(x)|$

76. $-\ln|\sec(x) - \tan(x)| = \ln|\sec(x) + \tan(x)|$

77. $-\ln|\csc(x) + \cot(x)| = \ln|\csc(x) - \cot(x)|$

1.1.2 Answers

1. $\csc(\theta) = \sqrt{5}$.
2. $\cos(\theta) = -\frac{1}{4}$.
3. $\cot(t) = \frac{1}{3}$.
4. $\sin(\theta) = -\frac{12}{13}$.
5. $\sec(\theta) = -\sqrt{5}$.
6. $\csc(t) = \sqrt{5}$.
7. $\tan(\theta) = -2\sqrt{2}$.
8. $\cos(\theta) = -\frac{\sqrt{5}}{3}$.
9. $\cos(t) \approx 0.9075$.
10. $\tan(\theta) \approx -0.6074$.
11. $\csc(t) \approx -4.079$.
12. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = -\frac{4}{5}, \tan(\theta) = -\frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = -\frac{5}{4}, \cot(\theta) = -\frac{4}{3}$
13. $\sin(\theta) = -\frac{12}{13}, \cos(\theta) = -\frac{5}{13}, \tan(\theta) = \frac{12}{5}, \csc(\theta) = -\frac{13}{12}, \sec(\theta) = -\frac{13}{5}, \cot(\theta) = \frac{5}{12}$
14. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$
15. $\sin(\theta) = \frac{-4\sqrt{3}}{7}, \cos(\theta) = \frac{1}{7}, \tan(\theta) = -4\sqrt{3}, \csc(\theta) = -\frac{7\sqrt{3}}{12}, \sec(\theta) = 7, \cot(\theta) = -\frac{\sqrt{3}}{12}$
16. $\sin(\theta) = -\frac{\sqrt{91}}{10}, \cos(\theta) = -\frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = -\frac{10\sqrt{91}}{91}, \sec(\theta) = -\frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$
17. $\sin(\theta) = \frac{\sqrt{530}}{530}, \cos(\theta) = -\frac{23\sqrt{530}}{530}, \tan(\theta) = -\frac{1}{23}, \csc(\theta) = \sqrt{530}, \sec(\theta) = -\frac{\sqrt{530}}{23}, \cot(\theta) = -23$
18. $\sin(\theta) = -\frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = -2, \csc(\theta) = -\frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = -\frac{1}{2}$
19. $\sin(\theta) = \frac{\sqrt{15}}{4}, \cos(\theta) = -\frac{1}{4}, \tan(\theta) = -\sqrt{15}, \csc(\theta) = \frac{4\sqrt{15}}{15}, \sec(\theta) = -4, \cot(\theta) = -\frac{\sqrt{15}}{15}$
20. $\sin(\theta) = -\frac{\sqrt{6}}{6}, \cos(\theta) = -\frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = -\sqrt{6}, \sec(\theta) = -\frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$
21. $\sin(\theta) = \frac{2\sqrt{2}}{3}, \cos(\theta) = \frac{1}{3}, \tan(\theta) = 2\sqrt{2}, \csc(\theta) = \frac{3\sqrt{2}}{4}, \sec(\theta) = 3, \cot(\theta) = \frac{\sqrt{2}}{4}$
22. $\sin(t) = \frac{\sqrt{5}}{5}, \cos(t) = \frac{2\sqrt{5}}{5}, \tan(t) = \frac{1}{2}, \csc(t) = \sqrt{5}, \sec(t) = \frac{\sqrt{5}}{2}, \cot(t) = 2$
23. $\sin(t) = \frac{1}{5}, \cos(t) = -\frac{2\sqrt{6}}{5}, \tan(t) = -\frac{\sqrt{6}}{12}, \csc(t) = 5, \sec(t) = -\frac{5\sqrt{6}}{12}, \cot(t) = -2\sqrt{6}$
24. $\sin(t) = -\frac{\sqrt{110}}{11}, \cos(t) = -\frac{\sqrt{11}}{11}, \tan(t) = \sqrt{10}, \csc(t) = -\frac{\sqrt{110}}{10}, \sec(t) = -\sqrt{11}, \cot(t) = \frac{\sqrt{10}}{10}$
25. $\sin(t) = -\frac{\sqrt{95}}{10}, \cos(t) = \frac{\sqrt{5}}{10}, \tan(t) = -\sqrt{19}, \csc(t) = -\frac{2\sqrt{95}}{19}, \sec(t) = 2\sqrt{5}, \cot(t) = -\frac{\sqrt{19}}{19}$
26. No, Skippy is not correct. In order to be an identity, an equation must hold for *all* applicable angles. For example, $\cos(\theta) + \sin(\theta) = 1$ does not hold when $\theta = \pi$.

1.2 More Identities

In Section 1.1, we saw the utility of identities in finding the values of the circular functions of a given angle as well as simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond.

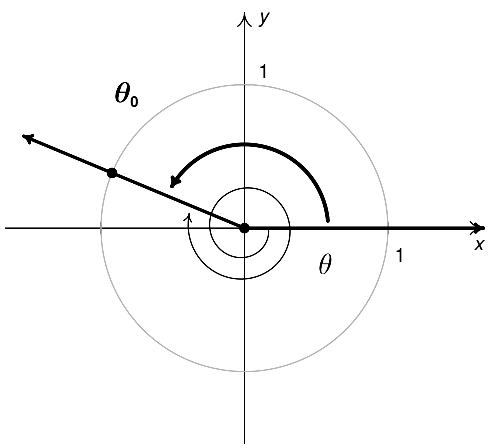
Our first set of identities is the 'Even / Odd' identities. We *observed* the even and odd properties of the circular functions graphically in Sections **??** and **??**. Here, we take the time to *prove* these properties from first principles. We state the theorem below for reference.

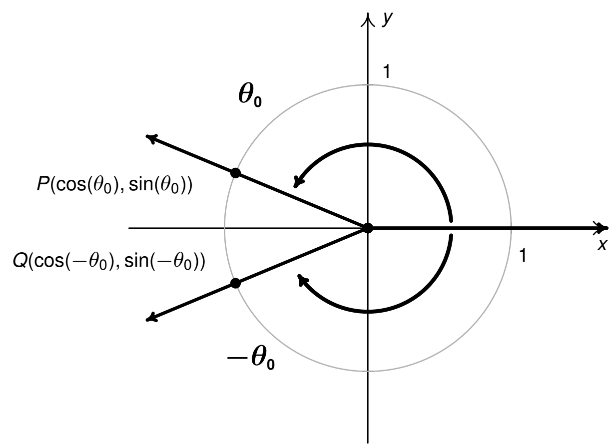
Theorem 1.4. Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

We start by proving $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Consider an angle θ plotted in standard position. Let θ_0 be the angle coterminal with θ with $0 \leq \theta_0 < 2\pi$. (We can construct the angle θ_0 by rotating counter-clockwise from the positive x-axis to the terminal side of θ as pictured below.) Since θ and θ_0 are coterminal, $\cos(\theta) = \cos(\theta_0)$ and $\sin(\theta) = \sin(\theta_0)$.





We now consider the angles $-\theta$ and $-\theta_0$. Since θ is coterminal with θ_0, there is some integer k so that $\theta = \theta_0 + 2\pi \cdot k$. Hence, $-\theta = -\theta_0 - 2\pi \cdot k = -\theta_0 + 2\pi \cdot (-k)$. Since k is an integer, so is $(-k)$, which means $-\theta$ is coterminal with $-\theta_0$. Therefore, $\cos(-\theta) = \cos(-\theta_0)$ and $\sin(-\theta) = \sin(-\theta_0)$.

Let P and Q denote the points on the terminal sides of θ_0 and $-\theta_0$, respectively, which lie on the Unit Circle. By definition, the coordinates of P are $(\cos(\theta_0), \sin(\theta_0))$ and the coordinates of Q are $(\cos(-\theta_0), \sin(-\theta_0))$.

Since θ_0 and $-\theta_0$ sweep out congruent central sectors of the Unit Circle, it follows that the points P and Q are symmetric about the x-axis. Thus, $\cos(-\theta_0) = \cos(\theta_0)$ and $\sin(-\theta_0) = -\sin(\theta_0)$.

Since the cosines and sines of θ_0 and $-\theta_0$ are the same as those for θ and $-\theta$, respectively, we get $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, as required.

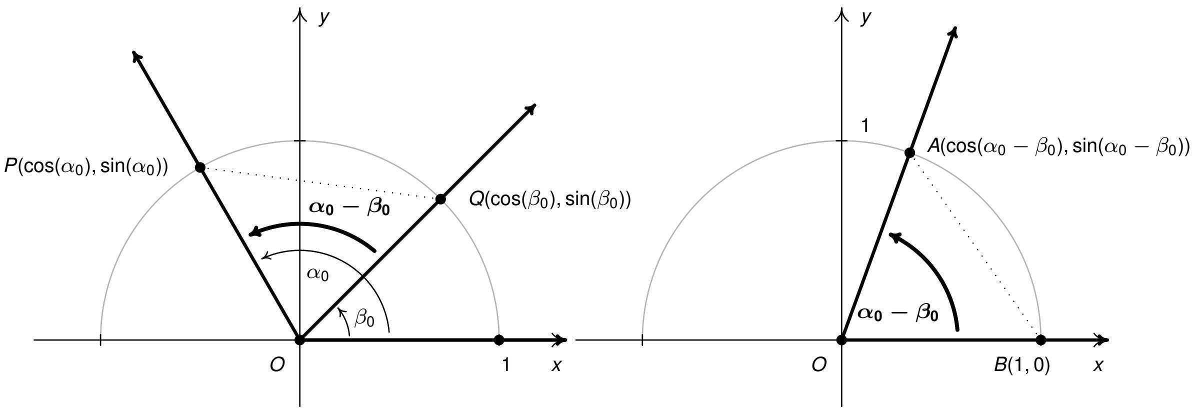
As we saw in Section **??**, the remaining four circular functions 'inherit' their even/odd nature from sine and cosine courtesy of the Reciprocal and Quotient Identities, Theorem 1.1.

Our next set of identities establish how the cosine function handles sums and differences of angles.

Theorem 1.5. Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

We first prove the result for differences. As in the proof of the Even / Odd Identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, each of which measure between 0 and 2π radians. Since α and α_0 are coterminal, as are β and β_0, it follows that $(\alpha-\beta)$ is coterminal with $(\alpha_0-\beta_0)$. Consider the case below where $\alpha_0 \geq \beta_0$.



Since the angles *POQ* and *AOB* are congruent, the distance between *P* and *Q* is equal to the distance between *A* and *B*.[1] The distance formula, Equation **??**, yields

$$\sqrt{(\cos(\alpha_0)-\cos(\beta_0))^2 + (\sin(\alpha_0)-\sin(\beta_0))^2} = \sqrt{(\cos(\alpha_0-\beta_0)-1)^2 + (\sin(\alpha_0-\beta_0)-0)^2}$$

Squaring both sides, we expand the left hand side of this equation as

$$\begin{aligned}
(\cos(\alpha_0)-\cos(\beta_0))^2 + (\sin(\alpha_0)-\sin(\beta_0))^2 &= \cos^2(\alpha_0) - 2\cos(\alpha_0)\cos(\beta_0) + \cos^2(\beta_0) \\
&\quad + \sin^2(\alpha_0) - 2\sin(\alpha_0)\sin(\beta_0) + \sin^2(\beta_0) \\
&= \cos^2(\alpha_0) + \sin^2(\alpha_0) + \cos^2(\beta_0) + \sin^2(\beta_0) \\
&\quad -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)
\end{aligned}$$

From the Pythagorean Identities, $\cos^2(\alpha_0)+\sin^2(\alpha_0) = 1$ and $\cos^2(\beta_0)+\sin^2(\beta_0) = 1$, so

[1] In the picture we've drawn, the triangles *POQ* and *AOB* are congruent, which is even better. However, $\alpha_0 - \beta_0$ could be 0 or it could be π, neither of which makes a triangle. It could also be larger than π, which makes a triangle, just not the one we've drawn. You should think about those three cases.

$$(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2 = 2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)$$

Turning our attention to the right hand side of our equation, we find

$$\begin{aligned}(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 &= \cos^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0) + 1 + \sin^2(\alpha_0 - \beta_0) \\ &= 1 + \cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0)\end{aligned}$$

Once again, we simplify $\cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) = 1$, so that

$$(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 = 2 - 2\cos(\alpha_0 - \beta_0)$$

Putting it all together, we get $2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) = 2 - 2\cos(\alpha_0 - \beta_0)$, which simplifies to: $\cos(\alpha_0 - \beta_0) = \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0)$.

Since α and α_0, β and β_0, and $(\alpha - \beta)$ and $(\alpha_0 - \beta_0)$ are all coterminal pairs of angles, we have established the identity: $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

For the case where $\alpha_0 \leq \beta_0$, we can apply the above argument to the angle $\beta_0 - \alpha_0$ to obtain the identity $\cos(\beta_0 - \alpha_0) = \cos(\beta_0)\cos(\alpha_0) + \sin(\beta_0)\sin(\alpha_0)$. Using this formula in conjunction with the Even Identity of cosine gives us the result in this case, too:

$$\begin{aligned}\cos(\alpha_0 - \beta_0) = \cos(-(\alpha_0 - \beta_0)) = \cos(\beta_0 - \alpha_0) &= \cos(\beta_0)\cos(\alpha_0) + \sin(\beta_0)\sin(\alpha_0) \\ &= \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0).\end{aligned}$$

To get the sum identity for cosine, we use the difference formula along with the Even/Odd Identities

$$\cos(\alpha + \beta) = \cos(\alpha - (-\beta)) = \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

We put these newfound identities to good use in the following example.

Example 1.2.1.

1. Find the exact value of $\cos(15^\circ)$.
2. Verify the identity: $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.
3. Suppose α is a Quadrant I angle with $\sin(\alpha) = \frac{3}{5}$ and β is a Quadrant IV angle with $\sec(\beta) = 4$. Find the exact value of $\cos(\alpha + \beta)$.

Solution.

1. In order to use Theorem 1.5 to find $\cos(15^\circ)$, we need to write 15° as a sum or difference of angles whose cosines and sines we know. One way to do so is to write $15^\circ = 45^\circ - 30^\circ$. We find:

$$\begin{aligned}
\cos(15^\circ) &= \cos(45^\circ - 30^\circ) \\
&= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}.
\end{aligned}$$

2. Using Theorem 1.5 gives:

$$\begin{aligned}
\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta) \\
&= (0)(\cos(\theta)) + (1)(\sin(\theta)) \\
&= \sin(\theta).
\end{aligned}$$

3. Per Theorem 1.5, we know $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. Hence, we need to find the sines and cosines of α and β to complete the problem.

   We are given $\sin(\alpha) = \frac{3}{5}$, so our first task is to find $\cos(\alpha)$. We can quickly get $\cos(\alpha)$ using the Pythagorean Identity $\cos^2(\alpha) = 1 - \sin^2(\alpha) = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$. We get $\cos(\alpha) = \frac{4}{5}$, choosing the positive root since α is a Quadrant I angle.

   Next, we need the $\sin(\beta)$ and $\cos(\beta)$. Since $\sec(\beta) = 4$, we immediately get $\cos(\beta) = \frac{1}{4}$ courtesy of the Reciprocal and Quotient Identities.

   To get $\sin(\beta)$, we employ the Pythagorean Identity: $\sin^2(\beta) = 1 - \cos^2(\beta) = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}$. Here, since β is a Quadrant IV angle, we get $\sin(\beta) = -\frac{\sqrt{15}}{4}$.

   Finally, we get: $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \left(\frac{4}{5}\right)\left(\frac{1}{4}\right) - \left(\frac{3}{5}\right)\left(-\frac{\sqrt{15}}{4}\right) = \frac{4+3\sqrt{15}}{20}$. □

The identity verified in Example 1.2.1, namely, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$, is the first of the celebrated 'cofunction' identities. These identities were first hinted at in Exercise **??** in Section **??**.

From $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$, we get: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - \theta\right]\right) = \cos(\theta)$, which says, in words, that the 'co'sine of an angle is the sine of its 'co'mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

Theorem 1.6. Cofunction Identities: For all applicable angles θ,

- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$

The Cofunction Identities enable us to derive the sum and difference formulas for sine. We first convert to sine to cosine and expand:

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) \\
&= \cos\left(\left[\frac{\pi}{2}-\alpha\right]-\beta\right) \\
&= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta) \\
&= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)
\end{aligned}$$

We can derive the difference formula for sine by rewriting $\sin(\alpha-\beta)$ as $\sin(\alpha+(-\beta))$ and using the sum formula and the Even / Odd Identities. Again, we leave the details to the reader.

Theorem 1.7. Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$

We try out these new identities in the next example.

Example 1.2.2.

1. Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$
2. Suppose α is a Quadrant II angle with $\sin(\alpha)=\frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta)=2$. Find the exact value of $\sin(\alpha-\beta)$.
3. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution.

1. As in Example 1.2.1, we need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination[2] is $\frac{19\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$. Applying Theorem 1.7, we get

$$\begin{aligned}
\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right) \\
&= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{-\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

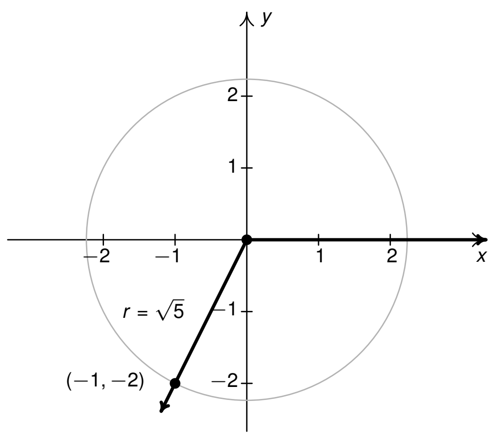
[2] It takes some trial and error to find this combination. One alternative is to convert to degrees . . .

2. In order to find $\sin(\alpha - \beta)$ using Theorem 1.7, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$.

   To find $\cos(\alpha)$, we use the Pythagorean Identity $\cos^2(\alpha) = 1 - \sin^2(\alpha) = 1 - \left(\frac{5}{13}\right)^2 = \frac{144}{169}$. We get $\cos(\alpha) = -\frac{12}{13}$, the negative, here, owing to the fact that α is a Quadrant II angle.

   We now set about finding $\sin(\beta)$ and $\cos(\beta)$. We have several ways to proceed at this point, but since there isn't a direct way to get from $\tan(\beta) = 2$ to either $\sin(\beta)$ or $\cos(\beta)$, we opt for a more geometric approach as presented in Section **??**.

   Since β is a Quadrant III angle with $\tan(\beta) = 2 = \frac{-2}{-1}$, we know the point $Q(x, y) = (-1, -2)$ is on the terminal side of β as illustrated below.[3]



   the terminal side of β contains $Q(-1, -2)$

   We find $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$, so per Theorem **??**, $\sin(\beta) = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$ and $\cos(\beta) = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$.

   At last, we have $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) = \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right) = -\frac{29\sqrt{5}}{65}$.

3. We can start expanding $\tan(\alpha + \beta)$ using a quotient identity and our sum formulas

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \end{aligned}$$

   Since $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$, it looks as though if we divide both numerator and denominator by $\cos(\alpha)\cos(\beta)$ we will have what we want

[3] Note that even though $\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$, we *cannot* take $\sin(\beta) = -2$ and $\cos(\beta) = -1$. Recall that $\sin(\beta)$ and $\cos(\beta)$ are the *y* and *x* coordinates on a *specific* circle, the Unit Circle. As we'll see shortly, $(-1, -2)$ lies on a circle of $\sqrt{5}$, so *not* the Unit Circle.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\dfrac{1}{\cos(\alpha)\cos(\beta)}}{\dfrac{1}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\dfrac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\dfrac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}}+\dfrac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\dfrac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}-\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}
\end{aligned}$$

Naturally, this formula is limited to those cases where all of the tangents are defined. □

The formula developed in Exercise 1.2.2 for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan(\alpha+(-\beta))$ and using the odd property of tangent. (The reader is encouraged to fill in the details.) Below we summarize all of the sum and difference formulas.

Theorem 1.8. Sum and Difference Identities: For all applicable angles α and β,

- $\cos(\alpha\pm\beta)=\cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$
- $\tan(\alpha\pm\beta)=\dfrac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$

In the statement of Theorem 1.8, we have combined the cases for the sum '+' and difference '−' of angles into one formula. The convention here is that if you want the formula for the sum '+' of two angles, you use the top sign in the formula; for the difference, '−', use the bottom sign. For example,

$$\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

If we set $\alpha=\beta$ in the sum formulas in Theorem 1.8, we obtain the following 'Double Angle' Identities:

Theorem 1.9. Double Angle Identities: For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

The three different forms for $\cos(2\theta)$ can be explained by our ability to 'exchange' squares of cosine and sine via the Pythagorean Identity. For instance, if we substitute $\sin^2(\theta) = 1 - \cos^2(\theta)$ into the first formula for $\cos(2\theta)$, we get $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1$.

It is interesting to note that to determine the value of $\cos(2\theta)$, only *one* piece of information is required: either $\cos(\theta)$ or $\sin(\theta)$. To determine $\sin(2\theta)$, however, it appears that we must know both $\sin(\theta)$ and $\cos(\theta)$. In the next example, we show how we can find $\sin(2\theta)$ knowing just one piece of information, namely $\tan(\theta)$.

Example 1.2.3.

1. Suppose $P(-3, 4)$ lies on the terminal side of θ when θ is plotted in standard position.

   Find $\cos(2\theta)$ and $\sin(2\theta)$ and determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.
2. If $\sin(\theta) = x$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.
3. Verify the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.
4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

Solution.

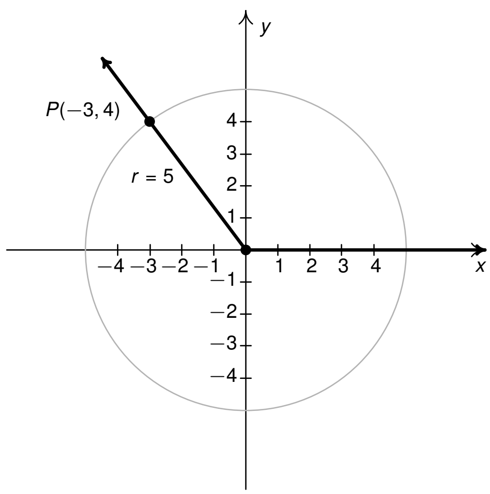
1. We sketch the terminal side of θ below on the left. Using Theorem **??** from Section **??** with $x = -3$ and $y = 4$, we find $r = \sqrt{x^2 + y^2} = 5$. Hence, $\cos(\theta) = -\frac{3}{5}$ and $\sin(\theta) = \frac{4}{5}$.

   Theorem 1.9 gives us three different formulas to choose from to find $\cos(2\theta)$. Using the first formula,[4] we get: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\frac{7}{25}$. For $\sin(2\theta)$, we get $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$.

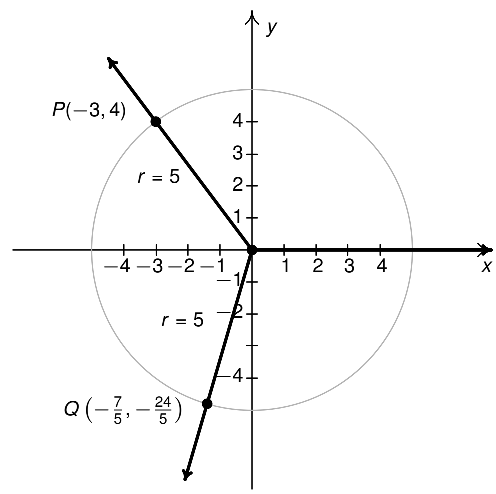
   Since both cosine and sine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III. To see this more clearly, we plot the terminal side of 2θ, along with the terminal side of θ below on the right.

[4] We invite the reader to check this answer using the other two formulas.

Note that in order to find the point $Q(x, y)$ on the terminal side of 2θ of a circle of radius 5, we use Theorem **??** again and find $x = r\cos(2\theta) = 5\left(-\frac{7}{25}\right) = -\frac{7}{5}$ and $y = r\sin(2\theta) = 5\left(-\frac{24}{25}\right) = -\frac{24}{5}$.



$P(-3, 4)$ is on the terminal side of θ.



$Q\left(-\frac{7}{5}, -\frac{24}{5}\right)$ is on the terminal side of 2θ.

2. If your first reaction to ‘$\sin(\theta) = x$’ is ‘No it’s not, $\cos(\theta) = x$!’ then you have indeed learned something, and we take comfort in that.

   While we have mostly used ‘x’ to represent the x-coordinate of the point the terminal side of an angle θ, here, ‘x’ represents the quantity $\sin(\theta)$ and our task is to express $\sin(2\theta)$ in terms of x.

   Since $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\cos(\theta)$, what remains is to express $\cos(\theta)$ in terms of x.

   Substituting $\sin(\theta) = x$ into the Pythagorean Identity, we get $\cos^2(\theta) = 1 - \sin^2(\theta) = 1 - x^2$, or $\cos(\theta) = \pm\sqrt{1 - x^2}$. Since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$, and thus $\cos(\theta) = \sqrt{1 - x^2}$.

   Our final answer is $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\sqrt{1 - x^2}$.

3. We start with the right hand side of the identity and note that $1 + \tan^2(\theta) = \sec^2(\theta)$. Next, we use the Reciprocal and Quotient Identities to rewrite $\tan(\theta)$ and $\sec(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$:

$$\begin{aligned}
\frac{2\tan(\theta)}{1 + \tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} = \frac{2\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}{\dfrac{1}{\cos^2(\theta)}} = 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\
&= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta).
\end{aligned}$$

4. In Theorem 1.9, one of the formulas for $\cos(2\theta)$, namely $\cos(2\theta) = 2\cos^2(\theta) - 1$, expresses $\cos(2\theta)$ as a polynomial in terms of $\cos(\theta)$. We are now asked to find such an identity for $\cos(3\theta)$.

   Using the sum formula for cosine, we begin with

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta+\theta)\\ &= \cos(2\theta)\cos(\theta)-\sin(2\theta)\sin(\theta).\end{aligned}$$

Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ only. To that end, we substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ which yields:

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta)\cos(\theta)-\sin(2\theta)\sin(\theta)\\ &= \left(2\cos^2(\theta)-1\right)\cos(\theta)-(2\sin(\theta)\cos(\theta))\sin(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\sin^2(\theta)\cos(\theta)\end{aligned}$$

Finally, we exchange $\sin^2(\theta) = 1 - \cos^2(\theta)$ courtesy of the Pythagorean Identity, and get

$$\begin{aligned}\cos(3\theta) &= 2\cos^3(\theta)-\cos(\theta)-2\sin^2(\theta)\cos(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\left(1-\cos^2(\theta)\right)\cos(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\cos(\theta)+2\cos^3(\theta)\\ &= 4\cos^3(\theta)-3\cos(\theta).\end{aligned}$$

Hence, $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. □

In the last problem in Example 1.2.3, we saw how we could rewrite $\cos(3\theta)$ as sums of powers of $\cos(\theta)$. In Calculus, we have occasion to do the reverse; that is, reduce the power of cosine and sine.

Solving the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ for $\cos^2(\theta)$ and the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ for $\sin^2(\theta)$ results in the aptly-named 'Power Reduction' formulas below.

Theorem 1.10. Power Reduction Formulas: For all angles θ,

- $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

Our next example is a typical application of Theorem 1.10 that you'll likely see in Calculus.

Example 1.2.4. Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and difference of cosines to the first power.

Solution. We begin with a straightforward application of Theorem 1.10

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)\\ &= \frac{1}{4}\left(1-\cos^2(2\theta)\right)\\ &= \frac{1}{4}-\frac{1}{4}\cos^2(2\theta)\end{aligned}$$

Next, we apply the power reduction formula to $\cos^2(2\theta)$ to finish the reduction

$$\begin{aligned}
\sin^2(\theta)\cos^2(\theta) &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta) \\
&= \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos(2(2\theta))}{2}\right) \\
&= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta) \\
&= \frac{1}{8} - \frac{1}{8}\cos(4\theta)
\end{aligned}$$

□

Another application of the Power Reduction Formulas is the Half Angle Formulas. To start, we apply the Power Reduction Formula to $\cos^2\left(\frac{\theta}{2}\right)$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2} = \frac{1+\cos(\theta)}{2}.$$

We can obtain a formula for $\cos\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine, and by using a quotient formula, obtain a half angle formula for tangent.

We summarize these formulas below.

Theorem 1.11. Half Angle Formulas: For all applicable angles θ,

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Example 1.2.5.

1. Use a half angle formula to find the exact value of $\cos(15^\circ)$.
2. Suppose $-\pi \leq t \leq 0$ with $\cos(t) = -\frac{3}{5}$. Find $\sin\left(\frac{t}{2}\right)$.
3. Use the identity given in number 3 of Example 1.2.3 to derive the identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$

Solution.

1. To use the half angle formula, we note that $15^\circ = \frac{30^\circ}{2}$ and since 15° is a Quadrant I angle, its cosine is positive. Thus we have

$$\begin{aligned} \cos(15^\circ) &= +\sqrt{\frac{1+\cos(30^\circ)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2} \end{aligned}$$

Back in Example 1.2.1, we found $\cos(15^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$ by using the difference formula for cosine. The reader is encouraged to prove that these two expressions are equal algebraically.

2. If $-\pi \leq t \leq 0$, then $-\frac{\pi}{2} \leq \frac{t}{2} \leq 0$, which means $\frac{t}{2}$ corresponds to a Quadrant IV angle. Hence, $\sin\left(\frac{t}{2}\right) < 0$, so we choose the negative root formula from Theorem 1.11:

$$\begin{aligned} \sin\left(\frac{t}{2}\right) &= -\sqrt{\frac{1-\cos(t)}{2}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} \\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}\cdot\frac{5}{5}} = -\sqrt{\frac{8}{10}} = -\frac{2\sqrt{5}}{5} \end{aligned}$$

3. Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity we proved in number 3 of Example 1.2.3 and manipulate it into the identity we are asked to prove.

   The identity we are asked to start with is $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$. If we are to use this to derive an identity for $\tan\left(\frac{\theta}{2}\right)$, it seems reasonable to proceed by replacing each occurrence of θ with $\frac{\theta}{2}$

$$\begin{aligned} \sin\left(2\left(\tfrac{\theta}{2}\right)\right) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \end{aligned}$$

   We now have the $\sin(\theta)$ we need, but we somehow need to get a factor of $1+\cos(\theta)$ involved. We substitute $1+\tan^2\left(\frac{\theta}{2}\right) = \sec^2\left(\frac{\theta}{2}\right)$, and continue to manipulate our given identity by converting secants to cosines.

$$\begin{aligned} \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right) \end{aligned}$$

Finally, we apply a power reduction formula, and then solve for $\tan\left(\frac{\theta}{2}\right)$

$$\begin{aligned}
\sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right) \\
\sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\left(\frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\right) \\
\sin(\theta) &= \tan\left(\tfrac{\theta}{2}\right)(1+\cos(\theta)) \\
\tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\theta)}{1+\cos(\theta)}
\end{aligned}$$

□

Our next batch of identities, the Product to Sum Formulas,[5] are easily verified by expanding each of the right hand sides in accordance with Theorem 1.8 and as you should expect by now we leave the details as exercises. They are of particular use in Calculus, and we list them here for reference.

Theorem 1.12. Product to Sum Formulas: For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Related to the Product to Sum Formulas are the Sum to Product Formulas, which we will have need of in Section 1.4. These are essentially restatements of the Product to Sum Formulas (by re-labeling the arguments of the sine and cosine functions) and as such, their proofs are left as exercises.

Theorem 1.13. Sum to Product Formulas: For all angles α and β,

- $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
- $\sin(\alpha)\pm\sin(\beta) = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$

Example 1.2.6.

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
2. Write $\sin(\theta)-\sin(3\theta)$ as a product.

[5] These are also known as the Prosthaphaeresis Formulas and have a rich history. The authors recommend that you conduct some research on them as your schedule allows.

Solution.

1. Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$\begin{aligned} \cos(2\theta)\cos(6\theta) &= \tfrac{1}{2}\left[\cos(2\theta - 6\theta) + \cos(2\theta + 6\theta)\right] \\ &= \tfrac{1}{2}\cos(-4\theta) + \tfrac{1}{2}\cos(8\theta) \\ &= \tfrac{1}{2}\cos(4\theta) + \tfrac{1}{2}\cos(8\theta), \end{aligned}$$

   where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta) = \cos(4\theta)$.

2. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta), \end{aligned}$$

   where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$. □

The reader is reminded that all of the identities presented in this section which regard the circular functions as functions of angles (in radian measure) apply equally well to the circular (trigonometric) functions regarded as functions of real numbers.

1.2.1 Sinusoids, Revisited

We first studied sinusoids in Section **??**. Using the sum formulas for sine and cosine, we can expand the forms given to us in Theorem **??**:

$$S(t) = A\sin(\omega t + \phi) + B = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B,$$

and

$$C(t) = A\cos(\omega t + \phi) + B = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B.$$

As we'll see in the next example, recognizing these 'expanded' forms of sinusoids allows us to graph functions as sinusoids which, at first glance, don't appear to fit the forms of either $C(t)$ or $S(t)$.

Example 1.2.7. Consider the function $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$. Find a formula for $f(t)$:

1. in the form $C(t) = A\cos(\omega t + \phi) + B$ for $\omega > 0$
2. in the form $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$

Check your answers analytically using identities and using a graphing utility.

Solution.

1. The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. We start by equating $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of $C(t) = A\cos(\omega t + \phi) + B$: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B$.

   If we take $\omega = 2$ and $B = 0$, we get: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(2t)\cos(\phi) - A\sin(2t)\sin(\phi)$.

   To determine A and ϕ, a bit more work is involved. We get started by equating the coefficients of the trigonometric functions on either side of the equation.

   On the left hand side, the coefficient of $\cos(2t)$ is 1, while on the right hand side, it is $A\cos(\phi)$. Since this equation is to hold for all real numbers, we must have[6] that $A\cos(\phi) = 1$.

   Similarly, we find by equating the coefficients of $\sin(2t)$ that $A\sin(\phi) = \sqrt{3}$. In conjunction with $A\cos(\phi) = 1$, we have a system of two (nonlinear) equations and two unknowns.

   As usual, our first task is to reduce this system of two equations and two unknowns to one equation and one unknown. We can temporarily eliminate the dependence on ϕ by using a Pythagorean Identity. From $\cos^2(\phi) + \sin^2(\phi) = 1$, we multiply through by A^2 to get $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$.

   In our case, $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, hence $A^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) = 1^2 + (\sqrt{3})^2 = 4$ so $A = \pm 2$. In much the same way we fit a sinusoid to a graph in Example **??**, we *choose* $A = 2$, and then find the phase angle ϕ associated with this choice.

   Substituting $A = 2$ into our two equations, $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, we get $2\cos(\phi) = 1$ and $2\sin(\phi) = \sqrt{3}$. After some rearrangement, $\cos(\phi) = \frac{1}{2}$ and $\sin(\phi) = \frac{\sqrt{3}}{2}$. One such angle ϕ which satisfies this criteria is $\phi = \frac{\pi}{3}$.

   Hence, one way to write $f(t)$ as a sinusoid is $f(t) = 2\cos\left(2t + \frac{\pi}{3}\right)$. We can check our answer using the sum formula for cosine :

$$\begin{aligned} f(t) &= 2\cos\left(2t + \tfrac{\pi}{3}\right) \\ &= 2\left[\cos(2t)\cos\left(\tfrac{\pi}{3}\right) - \sin(2t)\sin\left(\tfrac{\pi}{3}\right)\right] \\ &= 2\left[\cos(2t)\left(\tfrac{1}{2}\right) - \sin(2t)\left(\tfrac{\sqrt{3}}{2}\right)\right] \\ &= \cos(2t) - \sqrt{3}\sin(2t). \end{aligned}$$

2. Proceeding as before, we equate $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of of the sinusoid $S(t) = A\sin(\omega t + \phi) + B$ to get: $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B$.

   Taking $\omega = 2$ and $B = 0$, we get $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(2t)\cos(\phi) + A\cos(2t)\sin(\phi)$. We equate[7] the coefficients of $\cos(2t)$ on either side and get $A\sin(\phi) = 1$ and $A\cos(\phi) = -\sqrt{3}$.

   Using $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$ as before, we get $A = \pm 2$, and again we choose $A = 2$.

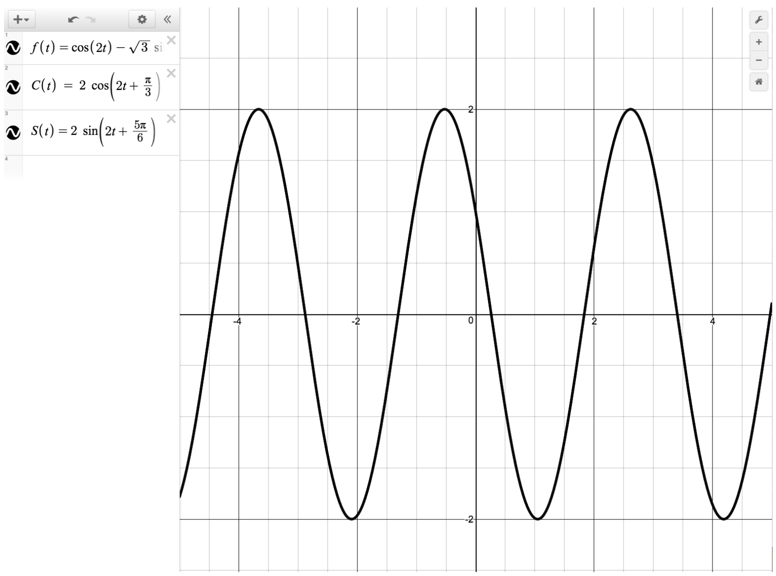
[6]This should remind you of equation coefficients of like powers of x in Section **??**.
[7]Be careful here!

This means $2\sin(\phi) = 1$, or $\sin(\phi) = \frac{1}{2}$, and $2\cos(\phi) = -\sqrt{3}$, so $\cos(\phi) = -\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $\phi = \frac{5\pi}{6}$.

Hence, we have $f(t) = 2\sin\left(2t + \frac{5\pi}{6}\right)$. Checking our work analytically, we have

$$\begin{aligned} f(t) &= 2\sin\left(2t + \frac{5\pi}{6}\right) \\ &= 2\left[\sin(2t)\cos\left(\frac{5\pi}{6}\right) + \cos(2t)\sin\left(\frac{5\pi}{6}\right)\right] \\ &= 2\left[\sin(2t)\left(-\frac{\sqrt{3}}{2}\right) + \cos(2t)\left(\frac{1}{2}\right)\right] \\ &= \cos(2t) - \sqrt{3}\sin(2t) \end{aligned}$$

Graphing the three formulas for $f(t)$ result in the identical curve, verifying the work done analytically.



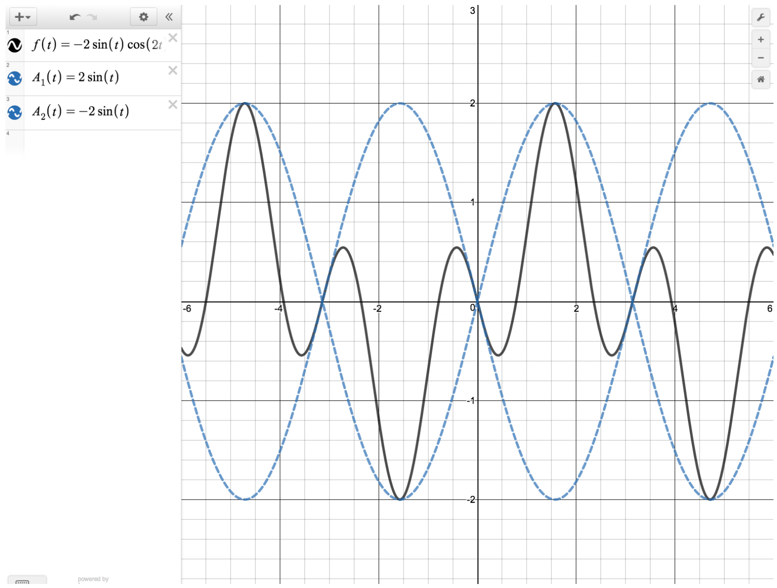
□

A couple of remarks about Example 1.2.7 are in order. First, had we chosen $A = -2$ instead of $A = 2$ as we worked through Example 1.2.7, our final answers would have *looked* different. The reader is encouraged to rework Example 1.2.7 using $A = -2$ to see what these differences are, and then for a challenging exercise, use identities to show that the formulas are all equivalent.[8]

It is important to note that in order for the technique presented in Example 1.2.7 to fit a function into one of the forms in Theorem **??**, the *frequencies* of the sine and cosine terms much match. For example, in the Exercises, you'll be asked to write $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t)$ in the form of $S(t)$ and $C(t)$ above, and since both the sine and cosine terms have frequency 3, this is possible.

[8]The general equations to fit a function of the form $f(x) = a\cos(\omega x) + b\sin(\omega x) + B$ into one of the forms in Theorem **??** are explored in Exercise 36.

However, a function such as $f(t) = \sin(t) - \sin(3t)$ cannot be written in the form of $S(t)$ or $C(t)$. The quickest way to see this is to examine its graph below which is decidedly not a sinusoid. That being said, we can still analyze this curve using identities.



Using our result from number 2 Example 1.2.6, we may rewrite $f(t) = \sin(t) - \sin(3t) = -2\sin(t)\cos(2t)$. Grouping factors, we can view $f(t) = [-2\sin(t)]\cos(2t) = A(t)\cos(2t)$ as the curve $y = \cos(2t)$ with a *variable* amplitude, $A(t) = -2\sin(t)$.

Overlaying the graphs of $f(t)$ with the (dashed) graphs of $A_1(t) = 2\sin(t)$ and $A_2(t) = -2\sin(t)$, we can see the role these two curves play in the graph of $y = f(t)$. They create a kind of 'wave envelope' for the graph of $y = f(t)$. This is an example of the beats phenomenon. Note that when written as a product of sinusoids, it is always the *lower* frequency factor which creates the 'wave-envelope' of the curve.

Note that in order to rewrite a sum or difference of sine and cosine functions with different frequencies into a product using the sum to product identities, Theorem 1.13, we need the *amplitudes* of each term to be the same. We explore more examples of these functions and this behavior in the Exercises.

1.2.2 Exercises

In Exercises 1 - 6, use the Even / Odd Identities to verify the identity. Assume all quantities are defined.

1. $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$
2. $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$
3. $\tan(-x^2 + 1) = -\tan(x^2 - 1)$
4. $\csc(-\theta - 5) = -\csc(\theta + 5)$
5. $\sec(-6x) = \sec(6x)$
6. $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

In Exercises 7 - 21, use the Sum and Difference Identities to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

7. $\cos(75^\circ)$
8. $\sec(165^\circ)$
9. $\sin(105^\circ)$
10. $\csc(195^\circ)$
11. $\cot(255^\circ)$
12. $\tan(375^\circ)$
13. $\cos\left(\frac{13\pi}{12}\right)$
14. $\sin\left(\frac{11\pi}{12}\right)$
15. $\tan\left(\frac{13\pi}{12}\right)$
16. $\cos\left(\frac{7\pi}{12}\right)$
17. $\tan\left(\frac{17\pi}{12}\right)$
18. $\sin\left(\frac{\pi}{12}\right)$
19. $\cot\left(\frac{11\pi}{12}\right)$
20. $\csc\left(\frac{5\pi}{12}\right)$
21. $\sec\left(-\frac{\pi}{12}\right)$
22. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$
23. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$
24. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find
    (a) $\sin(\alpha + \beta)$
    (b) $\cos(\alpha - \beta)$
    (c) $\tan(\alpha - \beta)$
25. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find
    (a) $\csc(\alpha - \beta)$
    (b) $\sec(\alpha + \beta)$
    (c) $\cot(\alpha + \beta)$

In Exercises 26 - 35, use Example 1.2.7 as a guide to show that the function is a sinusoid by rewriting it in the forms $C(t) = A\cos(\omega t + \phi) + B$ and $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$ and $0 \le \phi < 2\pi$.

26. $f(t) = \sqrt{2}\sin(t) + \sqrt{2}\cos(t) + 1$

27. $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t)$

28. $f(t) = -\sin(t) + \cos(t) - 2$

29. $f(t) = -\frac{1}{2}\sin(2t) - \frac{\sqrt{3}}{2}\cos(2t)$

30. $f(t) = 2\sqrt{3}\cos(t) - 2\sin(t)$

31. $f(t) = \frac{3}{2}\cos(2t) - \frac{3\sqrt{3}}{2}\sin(2t) + 6$

32. $f(t) = -\frac{1}{2}\cos(5t) - \frac{\sqrt{3}}{2}\sin(5t)$

33. $f(t) = -6\sqrt{3}\cos(3t) - 6\sin(3t) - 3$

34. $f(t) = \frac{5\sqrt{2}}{2}\sin(t) - \frac{5\sqrt{2}}{2}\cos(t)$

35. $f(t) = 3\sin\left(\frac{t}{6}\right) - 3\sqrt{3}\cos\left(\frac{t}{6}\right)$

36. In Exercises 26 - 35, you should have noticed a relationship between the phases ϕ for the $S(t)$ and $C(t)$. Show that if $f(t) = A\sin(\omega t + \alpha) + B$, then $f(t) = A\cos(\omega t + \beta) + B$ where $\beta = \alpha - \frac{\pi}{2}$.

37. Let ϕ be an angle measured in radians and let $P(a, b)$ be a point on the terminal side of ϕ when it is drawn in standard position. Use Theorem **??** and the sum identity for sine in Theorem 1.7 to show that $f(t) = a\sin(\omega t) + b\cos(\omega t) + B$ (with $\omega > 0$) can be rewritten as $f(t) = \sqrt{a^2 + b^2}\sin(\omega t + \phi) + B$.

38. In Example **??** in Section **??**, we developed two (seemingly) different formulas to model the hours of daylight, $H(t)$: $H_1(t) = 9.25\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 12.55$ and $H_2(t) = -8.13\sin\left(\frac{\pi}{6}t - 4.70\right) + 12.5$. Use the difference identities for sine to expand $H_1(t)$ and $H_2(t)$. How different are they?

In Exercises 39 - 53, verify the identity.[9]

39. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(t)$

40. $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(t)$

41. $\cos(\theta - \pi) = -\cos(\theta)$

42. $\sin(\pi - \theta) = \sin(\theta)$

43. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

44. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

45. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

46. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

47. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$

48. $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$

49. $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

50. $\dfrac{\tan(\alpha + \beta)}{\tan(\alpha - \beta)} = \dfrac{\sin(\alpha)\cos(\alpha) + \sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha) - \sin(\beta)\cos(\beta)}$

51. $\dfrac{\sin(t + h) - \sin(t)}{h} = \cos(t)\left(\dfrac{\sin(h)}{h}\right) + \sin(t)\left(\dfrac{\cos(h) - 1}{h}\right)$

[9] Note: numbers 39 and 40 are the conversion formulas stated in Theorem **??** in Section **??**.

52. $\dfrac{\cos(t+h)-\cos(t)}{h} = \cos(t)\left(\dfrac{\cos(h)-1}{h}\right) - \sin(t)\left(\dfrac{\sin(h)}{h}\right)$

53. $\dfrac{\tan(t+h)-\tan(t)}{h} = \left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

In Exercises 54 - 63, use the Half Angle Formulas to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

54. $\cos(75^\circ)$ (compare with Exercise 7)

55. $\sin(105^\circ)$ (compare with Exercise 9)

56. $\cos(67.5^\circ)$

57. $\sin(157.5^\circ)$

58. $\tan(112.5^\circ)$

59. $\cos\left(\frac{7\pi}{12}\right)$ (compare with Exercise 16)

60. $\sin\left(\frac{\pi}{12}\right)$ (compare with Exercise 18)

61. $\cos\left(\frac{\pi}{8}\right)$

62. $\sin\left(\frac{5\pi}{8}\right)$

63. $\tan\left(\frac{7\pi}{8}\right)$

In Exercises 64 - 73, use the given information about θ to find the exact values of

- $\sin(2\theta)$
- $\cos(2\theta)$
- $\tan(2\theta)$
- $\sin\left(\frac{\theta}{2}\right)$
- $\cos\left(\frac{\theta}{2}\right)$
- $\tan\left(\frac{\theta}{2}\right)$

64. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$

65. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$

66. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

67. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$

68. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$

69. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

70. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$

71. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$

72. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$

73. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

In Exercises 74 - 88, verify the identity. Assume all quantities are defined.

74. $(\cos(\theta)+\sin(\theta))^2 = 1+\sin(2\theta)$

75. $(\cos(\theta)-\sin(\theta))^2 = 1-\sin(2\theta)$

76. $\tan(2t) = \frac{1}{1-\tan(t)} - \frac{1}{1+\tan(t)}$

77. $\csc(2\theta) = \frac{\cot(\theta)+\tan(\theta)}{2}$

78. $8\sin^4(x) = \cos(4x) - 4\cos(2x) + 3$

79. $8\cos^4(x) = \cos(4x) + 4\cos(2x) + 3$

80. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

81. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$

82. $32\sin^2(t)\cos^4(t) = 2 + \cos(2t) - 2\cos(4t) - \cos(6t)$

83. $32\sin^4(t)\cos^2(t) = 2 - \cos(2t) - 2\cos(4t) + \cos(6t)$

84. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

85. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$ (HINT: Use the result to 84.)

86. $\sec(2x) = \dfrac{\cos(x)}{\cos(x)+\sin(x)} + \dfrac{\sin(x)}{\cos(x)-\sin(x)}$

87. $\dfrac{1}{\cos(\theta)-\sin(\theta)} + \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$

88. $\dfrac{1}{\cos(\theta)-\sin(\theta)} - \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

89. Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas

    (a) $\cos(\theta) = \sqrt{1-x^2}$
    (b) $\sin(2\theta) = 2x\sqrt{1-x^2}$
    (c) $\cos(2\theta) = 1 - 2x^2$

90. Discuss with your classmates how each of the formulas, if any, in Exercise 89 change if we change assume θ is a Quadrant II, III, or IV angle.

91. Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas

    (a) $\cos(\theta) = \dfrac{1}{\sqrt{x^2+1}}$
    (b) $\sin(\theta) = \dfrac{x}{\sqrt{x^2+1}}$
    (c) $\sin(2\theta) = \dfrac{2x}{x^2+1}$
    (d) $\cos(2\theta) = \dfrac{1-x^2}{x^2+1}$

92. Discuss with your classmates how each of the formulas, if any, in Exercise 91 change if we change assume θ is a Quadrant II, III, or IV angle.

93. If $\sin(t) = x$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\tan(t)$ in terms of x.

94. If $\tan(\theta) = x$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sec(\theta)$ in terms of x.

95. If $\sec(\theta) = x$ where θ is a Quadrant II angle, find an expression for $\tan(\theta)$ in terns of x.

96. If $\sin(t) = \frac{x}{2}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\cos(2t)$ in terms of x.

97. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

98. If $\sec(t) = \frac{x}{4}$ for $0 < t < \frac{\pi}{2}$, find an expression for $\ln|\sec(t) + \tan(t)|$ in terms of x.

99. Show that $\cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$ for all θ.

100. Let θ be a Quadrant III angle with $\cos(\theta) = -\frac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\frac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \frac{7\pi}{2}$ and then assuming $\pi < \theta < \frac{3\pi}{2}$ and computing $\sin\left(\frac{\theta}{2}\right)$ in both cases.

101. Without using your calculator, show that $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

102. In part 4 of Example 1.2.3, we wrote $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$. In Exercise 84, we had you verify an identity which expresses $\cos(4\theta)$ as a polynomial in terms of $\cos(\theta)$. Can you find a polynomial in terms of $\cos(\theta)$ for $\cos(5\theta)$? $\cos(6\theta)$? Can you find a pattern so that $\cos(n\theta)$ could be written as a polynomial in cosine for any natural number n?

103. In Exercise 80, we has you verify an identity which expresses $\sin(3\theta)$ as a polynomial in terms of $\sin(\theta)$. Can you do the same for $\sin(5\theta)$? What about for $\sin(4\theta)$? If not, what goes wrong?

In Exercises 101 - 106, verify the identity by graphing the right and left hand using a graphing utility.

104. $\sin^2(t) + \cos^2(t) = 1$

105. $\sec^2(x) - \tan^2(x) = 1$

106. $\cos(t) = \sin\left(\frac{\pi}{2} - t\right)$

107. $\tan(x + \pi) = \tan(x)$

108. $\sin(2t) = 2\sin(t)\cos(t)$

109. $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$

In Exercises 107 - 112, write the given product as a sum. Note: you may need to use an Even/Odd Identity to match the answer provided.

110. $\cos(3\theta)\cos(5\theta)$

111. $\sin(2t)\sin(7t)$

112. $\sin(9x)\cos(x)$

113. $\cos(2\theta)\cos(6\theta)$

114. $\sin(3t)\sin(2t)$

115. $\cos(x)\sin(3x)$

In Exercises 113 - 118, write the given sum as a product. Note: you may need to use an Even/Odd or Cofunction Identity to match the answer provided.

116. $\cos(3\theta) + \cos(5\theta)$

117. $\sin(2t) - \sin(7t)$

118. $\cos(5x) - \cos(6x)$

119. $\sin(9\theta) - \sin(-\theta)$

120. $\sin(t) + \cos(t)$

121. $\cos(x) - \sin(x)$

In Exercises 119 - 122, using the remarks following Example 1.2.7 on page 29 as a guide, rewrite the given function $f(t)$ as a product of sinusoids. Identify the functions which create the 'wave envelope.' Check your answer by graphing the function along with the 'wave-envelope' using a graphing utility.

122. $f(t) = \cos(3t) + \cos(5t)$

123. $f(t) = 3\cos(5t) - 3\cos(6t)$

124. $f(t) = \frac{1}{2}\sin(9t) + \frac{1}{2}\sin(t)$

125. $f(t) = \frac{2}{3}\sin(2t) - \frac{2}{3}\sin(7t)$

126. Verify the Even / Odd Identities for tangent, secant, cosecant and cotangent.

127. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.
128. Verify the Difference Identities for sine and tangent.
129. Verify the Product to Sum Identities.
130. Verify the Sum to Product Identities.

1.2.3 Answers

7. $\cos(75^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$

8. $\sec(165^\circ) = -\frac{4}{\sqrt{2}+\sqrt{6}} = \sqrt{2}-\sqrt{6}$

9. $\sin(105^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$

10. $\csc(195^\circ) = \frac{4}{\sqrt{2}-\sqrt{6}} = -(\sqrt{2}+\sqrt{6})$

11. $\cot(255^\circ) = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

12. $\tan(375^\circ) = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2-\sqrt{3}$

13. $\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{6}+\sqrt{2}}{4}$

14. $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

15. $\tan\left(\frac{13\pi}{12}\right) = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2-\sqrt{3}$

16. $\cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$

17. $\tan\left(\frac{17\pi}{12}\right) = 2+\sqrt{3}$

18. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

19. $\cot\left(\frac{11\pi}{12}\right) = -(2+\sqrt{3})$

20. $\csc\left(\frac{5\pi}{12}\right) = \sqrt{6}-\sqrt{2}$

21. $\sec\left(-\frac{\pi}{12}\right) = \sqrt{6}-\sqrt{2}$

22. (a) $\cos(\alpha+\beta) = -\frac{\sqrt{2}}{10}$
    (b) $\sin(\alpha+\beta) = \frac{7\sqrt{2}}{10}$
    (c) $\tan(\alpha+\beta) = -7$
    (d) $\cos(\alpha-\beta) = -\frac{\sqrt{2}}{2}$
    (e) $\sin(\alpha-\beta) = \frac{\sqrt{2}}{2}$
    (f) $\tan(\alpha-\beta) = -1$

23. (a) $\cos(\alpha+\beta) = -\frac{4+7\sqrt{2}}{30}$
    (b) $\sin(\alpha+\beta) = \frac{28-\sqrt{2}}{30}$
    (c) $\tan(\alpha+\beta) = \frac{-28+\sqrt{2}}{4+7\sqrt{2}} = \frac{63-100\sqrt{2}}{41}$
    (d) $\cos(\alpha-\beta) = \frac{-4+7\sqrt{2}}{30}$
    (e) $\sin(\alpha-\beta) = -\frac{28+\sqrt{2}}{30}$
    (f) $\tan(\alpha-\beta) = \frac{28+\sqrt{2}}{4-7\sqrt{2}} = -\frac{63+100\sqrt{2}}{41}$

24. (a) $\sin(\alpha+\beta) = \frac{16}{65}$
    (b) $\cos(\alpha-\beta) = \frac{33}{65}$
    (c) $\tan(\alpha-\beta) = \frac{56}{33}$

25. (a) $\csc(\alpha-\beta) = -\frac{5}{4}$
    (b) $\sec(\alpha+\beta) = \frac{125}{117}$
    (c) $\cot(\alpha+\beta) = \frac{117}{44}$

26. $f(t) = \sqrt{2}\sin(t) + \sqrt{2}\cos(t) + 1 = 2\sin\left(t+\frac{\pi}{4}\right) + 1 = 2\cos\left(t+\frac{7\pi}{4}\right) + 1$

27. $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t) = 6\sin\left(3t+\frac{11\pi}{6}\right) = 6\cos\left(3t+\frac{4\pi}{3}\right)$

28. $f(t) = -\sin(t) + \cos(t) - 2 = \sqrt{2}\sin\left(t+\frac{3\pi}{4}\right) - 2 = \sqrt{2}\cos\left(t+\frac{\pi}{4}\right) - 2$

29. $f(t) = -\frac{1}{2}\sin(2t) - \frac{\sqrt{3}}{2}\cos(2t) = \sin\left(2t+\frac{4\pi}{3}\right) = \cos\left(2t+\frac{5\pi}{6}\right)$

30. $f(t) = 2\sqrt{3}\cos(t) - 2\sin(t) = 4\sin\left(t+\frac{2\pi}{3}\right) = 4\cos\left(t+\frac{\pi}{6}\right)$

31. $f(t) = \frac{3}{2}\cos(2t) - \frac{3\sqrt{3}}{2}\sin(2t) + 6 = 3\sin\left(2t+\frac{5\pi}{6}\right) + 6 = 3\cos\left(2t+\frac{\pi}{3}\right) + 6$

32. $f(t) = -\frac{1}{2}\cos(5t) - \frac{\sqrt{3}}{2}\sin(5t) = \sin\left(5t + \frac{7\pi}{6}\right) = \cos\left(5t + \frac{2\pi}{3}\right)$

33. $f(t) = -6\sqrt{3}\cos(3t) - 6\sin(3t) - 3 = 12\sin\left(3t + \frac{4\pi}{3}\right) - 3 = 12\cos\left(3t + \frac{5\pi}{6}\right) - 3$

34. $f(t) = \frac{5\sqrt{2}}{2}\sin(t) - \frac{5\sqrt{2}}{2}\cos(t) = 5\sin\left(t + \frac{7\pi}{4}\right) = 5\cos\left(t + \frac{5\pi}{4}\right)$

35. $f(t) = 3\sin\left(\frac{t}{6}\right) - 3\sqrt{3}\cos\left(\frac{t}{6}\right) = 6\sin\left(\frac{t}{6} + \frac{5\pi}{3}\right) = 6\cos\left(\frac{t}{6} + \frac{7\pi}{6}\right)$

54. $\cos(75^\circ) = \frac{\sqrt{2-\sqrt{3}}}{2}$

55. $\sin(105^\circ) = \frac{\sqrt{2+\sqrt{3}}}{2}$

56. $\cos(67.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2}$

57. $\sin(157.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2}$

58. $\tan(112.5^\circ) = -\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = -1 - \sqrt{2}$

59. $\cos\left(\frac{7\pi}{12}\right) = -\frac{\sqrt{2-\sqrt{3}}}{2}$

60. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$

61. $\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$

62. $\sin\left(\frac{5\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$

63. $\tan\left(\frac{7\pi}{8}\right) = -\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = 1 - \sqrt{2}$

64.
- $\sin(2\theta) = -\frac{336}{625}$
- $\cos(2\theta) = \frac{527}{625}$
- $\tan(2\theta) = -\frac{336}{527}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{10}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{7\sqrt{2}}{10}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{7}$

65.
- $\sin(2\theta) = \frac{2520}{2809}$
- $\cos(2\theta) = -\frac{1241}{2809}$
- $\tan(2\theta) = -\frac{2520}{1241}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{106}}{106}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{9\sqrt{106}}{106}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{5}{9}$

66.
- $\sin(2\theta) = \frac{120}{169}$
- $\cos(2\theta) = -\frac{119}{169}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{3\sqrt{13}}{13}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{2\sqrt{13}}{13}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{3}{2}$

67.
- $\sin(2\theta) = -\frac{\sqrt{15}}{8}$
- $\cos(2\theta) = \frac{7}{8}$
- $\tan(2\theta) = -\frac{\sqrt{15}}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{8+2\sqrt{15}}}{4}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{8-2\sqrt{15}}}{4}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}}}$
  $\tan\left(\frac{\theta}{2}\right) = 4 + \sqrt{15}$

68.
- $\sin(2\theta) = \frac{24}{25}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{5}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{1}{2}$

69.
- $\sin(2\theta) = \frac{24}{25}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{5}}{5}$
- $\tan\left(\frac{\theta}{2}\right) = -2$

70.
- $\sin(2\theta) = -\frac{120}{169}$
- $\cos(2\theta) = \frac{119}{169}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{5\sqrt{26}}{26}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{5}$

71.
- $\sin(2\theta) = -\frac{120}{169}$
- $\cos(2\theta) = \frac{119}{169}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{26}}{26}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\tan\left(\frac{\theta}{2}\right) = 5$

72.
- $\sin(2\theta) = -\frac{4}{5}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\tan(2\theta) = \frac{4}{3}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50-10\sqrt{5}}}{10}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{50+10\sqrt{5}}}{10}$
- $\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{5-\sqrt{5}}{5+\sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5-5\sqrt{5}}{10}$

73.
- $\sin(2\theta) = -\frac{4}{5}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\tan(2\theta) = \frac{4}{3}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50+10\sqrt{5}}}{10}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{50-10\sqrt{5}}}{10}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5+5\sqrt{5}}{10}$

93. $\tan(t) = \frac{x}{\sqrt{1-x^2}}$

94. $\sec(\theta) = \sqrt{1+x^2}$

95. $\tan(\theta) = -\sqrt{x^2-1}$

96. $\cos(2t) = 1 - \dfrac{x^2}{2}$

97. $\sin(2\theta) = \dfrac{14x}{x^2+49}$

98. $\ln|\sec(t)+\tan(t)| = \ln\left|x+\sqrt{x^2+16}\right| - \ln(4)$

110. $\dfrac{\cos(2\theta)+\cos(8\theta)}{2}$

111. $\dfrac{\cos(5t)-\cos(9t)}{2}$

112. $\dfrac{\sin(8x)+\sin(10x)}{2}$

113. $\dfrac{\cos(4\theta)+\cos(8\theta)}{2}$

114. $\dfrac{\cos(t)-\cos(5t)}{2}$

115. $\dfrac{\sin(2x)+\sin(4x)}{2}$

116. $2\cos(4\theta)\cos(\theta)$

117. $-2\cos\left(\frac{9}{2}t\right)\sin\left(\frac{5}{2}t\right)$

118. $2\sin\left(\frac{11}{2}x\right)\sin\left(\frac{1}{2}x\right)$

119. $2\cos(4\theta)\sin(5\theta)$

120. $\sqrt{2}\cos\left(t-\frac{\pi}{4}\right)$

121. $-\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

122. $f(t) = [2\cos(t)]\cos(4t)$, $A(t) = 2\cos(t)$, wave-envelope: $y = \pm 2\cos(t)$.

123. $f(t) = \left[6\sin\left(\frac{1}{2}t\right)\right]\sin\left(\frac{11}{2}t\right)$, $A(t) = 6\sin\left(\frac{1}{2}t\right)$, wave-envelope: $y = \pm 6\sin\left(\frac{1}{2}t\right)$.

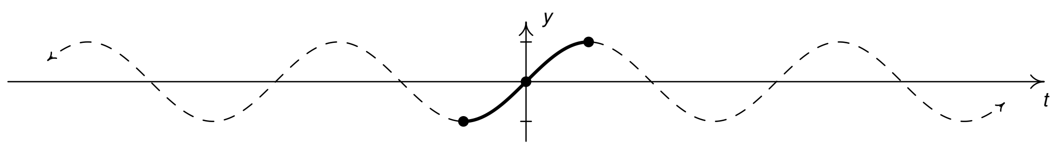
124. $f(t) = [\cos(4t)]\sin(5t)$, $A(t) = \cos(4t)$, wave-envelope: $y = \pm\cos(4t)$.

125. $f(t) = \left[-\frac{4}{3}\sin\left(\frac{5}{2}t\right)\right]\cos\left(\frac{9}{2}t\right)$, $A(t) = -\frac{4}{3}\sin\left(\frac{5}{2}t\right)$, wave-envelope: $y = \pm\frac{4}{3}\sin\left(\frac{5}{2}t\right)$.

1.3 The Inverse Circular Functions

In this section we concern ourselves with finding inverses of the circular (trigonometric) functions.[1] Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domains of the circular functions in the same way we restricted the domain of the quadratic function in Example **??** in Section **??** to obtain a one-to-one function.

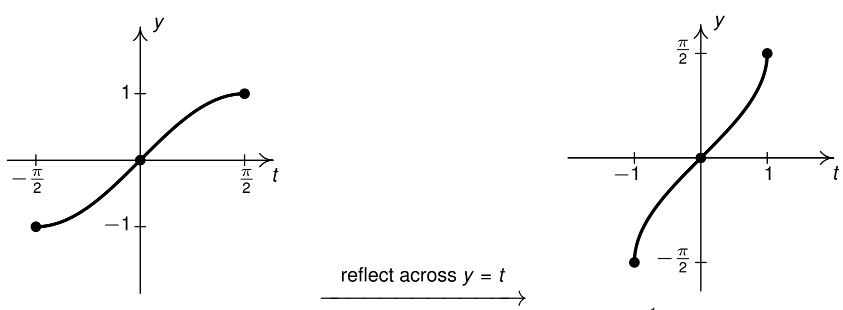
We start with $f(t) = \sin(t)$ and restrict our domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in order to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.



Restricting the domain of $f(t) = \sin(t)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

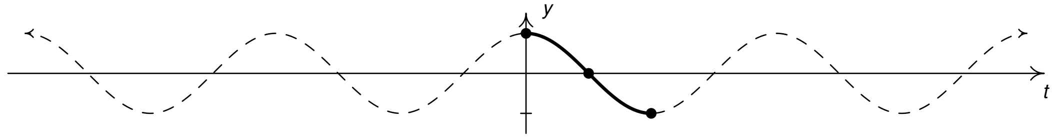
Recall from Section **??** that the inverse of a function f is typically denoted f^{-1}. For this reason, some textbooks use the notation $f^{-1}(t) = \sin^{-1}(t)$ for the inverse of $f(t) = \sin(t)$. The obvious pitfall here is our convention of writing $(\sin(t))^2$ as $\sin^2(t)$, $(\sin(t))^3$ as $\sin^3(t)$ and so on. It is far too easy to confuse $\sin^{-1}(t)$ with $\frac{1}{\sin(t)} = \csc(t)$ so we will not use this notation in our text.[2]

Instead, we use the notation $f^{-1}(t) = \arcsin(t)$, read 'arc-sine of t'. We'll explain the 'arc' in 'arcsine' shortly. For now, we graph $f(t) = \sin(t)$ and $f^{-1}(t) = \arcsin(t)$, where we obtain the latter from the former by reflecting it across the line $y = t$, in accordance with Theorem **??**.



$f(t) = \sin(t)$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. $f^{-1}(t) = \arcsin(t)$.

Next, we consider $g(t) = \cos(t)$. Here, we select the interval $[0, \pi]$ for our restriction.

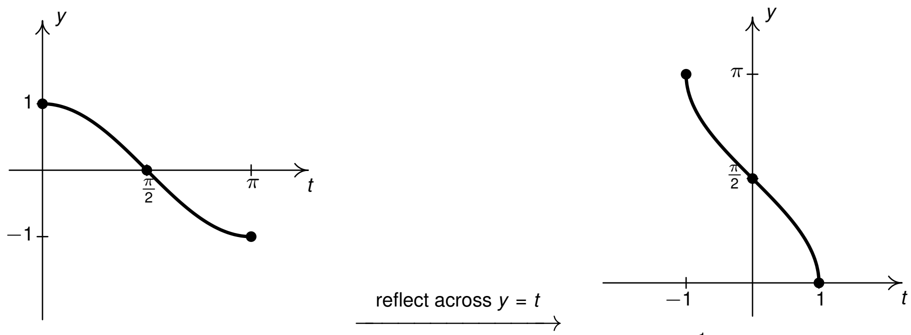


Restricting the domain of $f(t) = \cos(t)$ to $[0, \pi]$.

[1] We have already discussed this concept in Section **??** as the 'angle finder' in the context of acute angles in right triangles.

[2] But be aware that many books do! As always, be sure to check the context!

Reflecting the across the line $y = t$ produces the graph $y = g^{-1}(t) = \arccos(t)$.



$f(t) = \cos(t)$, $0 \leq t \leq \pi$ $\quad$ $f^{-1}(t) = \arccos(t)$.

We list some important facts about the arcsine and arccosine functions in the following theorem.[3] Everything in Theorem 1.14 is a direct consequence of Theorem **??** as applied to the (restricted) sine and cosine functions, and as such, its proof is left to the reader.

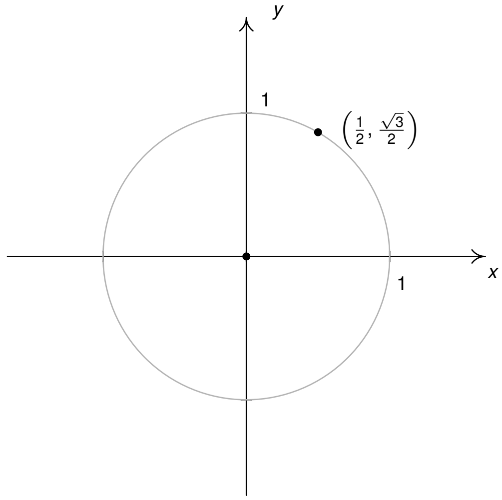
Theorem 1.14. Properties of the Arccosine and Arcsine Functions

- Properties of $F(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $\sin(t) = x$ and $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $\sin(\arcsin(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arcsin(\sin(t)) = t$ provided $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $F(x) = \arcsin(x)$ is odd
- Properties of $G(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $\cos(t) = x$ and $0 \leq t \leq \pi$
  - $\cos(\arccos(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arccos(\cos(t)) = t$ provided $0 \leq t \leq \pi$

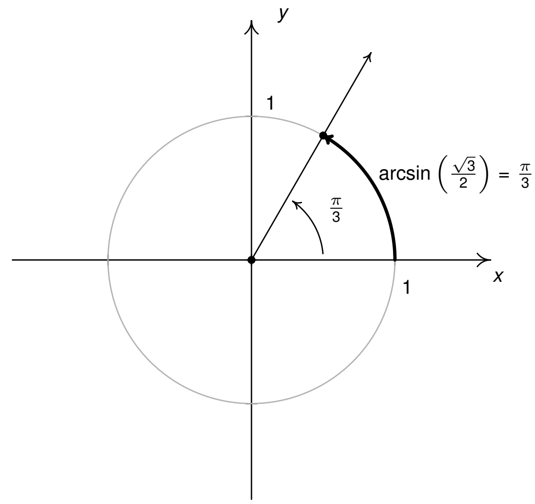
Before moving to an example, we take a moment to understand the 'arc' in 'arcsine.' Consider the figure below which illustrates the specific case of $\arcsin\left(\frac{\sqrt{3}}{2}\right)$.

[3]We switch the input variable to the arcsine and arccosine functions to 'x' to avoid confusion with the outputs we label 't.'

By definition, the real number $t = \arcsin\left(\frac{\sqrt{3}}{2}\right)$ satisfies $\sin(t) = \frac{\sqrt{3}}{2}$ with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. In other words, we are looking for angle measuring t radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a sine of $\frac{\sqrt{3}}{2}$. Hence, $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. In terms of oriented arcs[4], if we start at $(1, 0)$ and travel along the Unit Circle in the positive (counterclockwise) direction for $\frac{\pi}{3}$ units, we will arrive at the point whose y-coordinate is $\frac{\sqrt{3}}{2}$. Hence, the real number $\frac{\pi}{3}$ also corresponds to 'arc' corresponding to the 'sine' that is $\frac{\sqrt{3}}{2}$.



y-value on Unit Circle is $\frac{\sqrt{3}}{2}$.



corresponding oriented arc: $t = \frac{\pi}{3}$

In general, the function $f(t) = \sin(t)$ takes a real number input t, associates it with the angle $\theta = t$ radians, and returns the value $\sin(\theta)$. The value $\sin(\theta) = \sin(t)$ is the y-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1, 0)$.

Hence, we may view the inputs to $f(t) = \sin(t)$ as oriented arcs and the outputs as y-coordinates on the Unit Circle. Therefore, the function f^{-1} reverses this process and takes y-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arcsine.

It is high time for an example.

Example 1.3.1.

1. Find the exact values of the following.

   (a) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

   (b) $\arccos\left(\frac{1}{2}\right)$

   (c) $\arcsin\left(-\frac{1}{2}\right)$

   (d) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

[4]See page **??** if you need a review of how we associate real numbers with angles in radian measure.

(e) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$

(f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$

(g) $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$

(h) $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

(a) $f(x) = \tan(\arccos(x))$

(b) $g(x) = \cos(2\arcsin(x))$

Solution.
The best way to approach these problems is to remember that $\arcsin(x)$ and $\arccos(x)$ are real numbers which correspond to the radian measure of angles that fall within a certain prescribed range.

1. (a) To find $\arcsin\left(\frac{\sqrt{2}}{2}\right)$, we need the angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = \frac{\sqrt{2}}{2}$. Hence, $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

   (b) To find $\arccos\left(\frac{1}{2}\right)$, we are looking for the angle measuring t radians which lies between 0 and π that has $\cos(t) = \frac{1}{2}$. Our answer is $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (c) For $\arcsin\left(-\frac{1}{2}\right)$, we are looking for an angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = -\frac{1}{2}$. Hence, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.
   Alternatively, we could use the fact that the arcsine function is odd, so $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right)$. We find $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$, so $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right) = -\frac{\pi}{6}$.

   (d) For $\arccos\left(-\frac{\sqrt{2}}{2}\right)$, we need the angle measuring t radians which lies between 0 and π with $\cos(t) = -\frac{\sqrt{2}}{2}$. Hence, $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

   (e) Since $0 \leq \frac{\pi}{6} \leq \pi$, we could simply invoke Theorem 1.14 to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$.
   However, in order to make sure we understand *why* this is the case, we choose to work the example through using the definition of arccosine.
   Working from the inside out, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. To find $\arccos\left(\frac{\sqrt{3}}{2}\right)$, we need an angle measuring t radians which lies between 0 and π that has $\cos(t) = \frac{\sqrt{3}}{2}$. We get $t = \frac{\pi}{6}$, so that $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

   (f) Since $\frac{11\pi}{6}$ does not fall between 0 and π, Theorem 1.14 does not apply. We are forced to work through from the inside out starting with $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. From the previous problem, we know $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. Hence, $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$.

   (g) One way to simplify $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$ is to use Theorem 1.14 directly. Since $-\frac{3}{5}$ is between -1 and 1, we have that $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = -\frac{3}{5}$ and we are done.
   However, as before, to really understand *why* this cancellation occurs, we let $t = \arccos\left(-\frac{3}{5}\right)$. By definition, $\cos(t) = -\frac{3}{5}$. Hence, $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = \cos(t) = -\frac{3}{5}$, and we are finished in (nearly) the same amount of time.

(h) As in the previous example, we let $t = \arccos\left(-\frac{3}{5}\right)$ so that $\cos(t) = -\frac{3}{5}$ for some angle measuring t radians between 0 and π.

Since $\cos(t) < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < t < \pi$, so that t corresponds to an angle in Quadrant II.

In terms of t, then, we need to find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(t)$, and since we know $\cos(t)$, the fastest route is through the Pythagorean Identity.

We get $\sin^2(t) = 1 - \cos^2(t) = 1 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25}$. Since t corresponds to a Quadrant II angle, we choose the positive root, $\sin(t) = \frac{4}{5}$, so $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \frac{4}{5}$.

2. (a) We begin this problem in the same manner we began the previous two problems. We let $t = \arccos(x)$, so our goal is to find a way to express $\tan(\arccos(x)) = \tan(t)$ in terms of x.

Since $t = \arccos(x)$, we know $\cos(t) = x$ where $0 \leq t \leq \pi$. One approach[5] to finding $\tan(t)$ is to use the quotient identity $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Since we know $\cos(t)$, we just need to find $\sin(t)$.

Using the Pythagorean Identity, we get $\sin^2(t) = 1 - \cos^2(t) = 1 - x^2$ so that $\sin(t) = \pm\sqrt{1 - x^2}$. Since $0 \leq t \leq \pi$, $\sin(t) \geq 0$, so we choose $\sin(t) = \sqrt{1 - x^2}$.

Thus, $\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{\sqrt{1-x^2}}{x}$, so $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$.

To determine the domain, we harken back to Section **??**. The function $f(x) = \tan(\arccos(x))$ can be thought of as a two step process: first, take the arccosine of a number, and second, take the tangent of whatever comes out of the arccosine.

Since the domain of $\arccos(x)$ is $-1 \leq x \leq 1$, the domain of f will be some subset of $[-1, 1]$. The range of $\arccos(x)$ is $[0, \pi]$, and of these values, only $\frac{\pi}{2}$ will cause a problem for the tangent function. Since $\arccos(x) = \frac{\pi}{2}$ happens when $x = \cos\left(\frac{\pi}{2}\right) = 0$, we exclude $x = 0$ from our domain. Hence, the domain of $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$ is $[-1, 0) \cup (0, 1]$.

Note that *in this particular case*, we could have obtained the correct domain of f using its algebraic description: $f(x) = \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$. This is not always true, however, as we'll see in the next problem.

(b) We proceed as in the previous problem by writing $t = \arcsin(x)$ so that t lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = x$. We aim to express $\cos(2\arcsin(x)) = \cos(2t)$ in terms of x.

Thanks to Theorem 1.9, we have three choices for rewriting $\cos(2t)$: $\cos(2t) = \cos^2(t) - \sin^2(t)$, $\cos(2t) = 2\cos^2(t) - 1$ and $\cos(2t) = 1 - 2\sin^2(t)$.

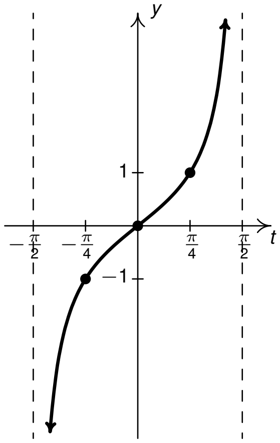
Since we know $x = \sin(t)$, we choose: $\cos(2\arcsin(x)) = \cos(2t) = 1 - 2\sin^2(t) = 1 - 2x^2$. Hence, $g(x) = \cos(2\arcsin(x)) = 1 - 2x^2$.

To find the domain of $g(x) = \cos(2\arcsin(x))$, we once again appeal to what we learned in Section **??**. The domain of $\arcsin(x)$ is $[-1, 1]$, and since there are no domain restrictions on cosine, the domain of g is $[-1, 1]$.

[5]Alternatively, we could use the identity: $1 + \tan^2(t) = \sec^2(t)$. Since $x = \cos(t)$, $\sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$. The reader is invited to work through this approach to see what, if any, difficulties arise.

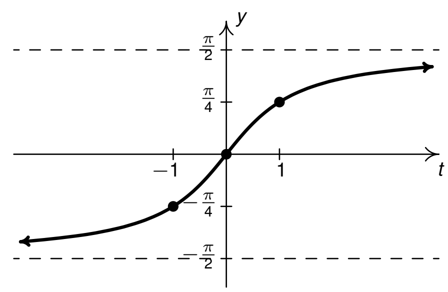
It is important to note that in this case, even though the algebraic expression $1 - 2x^2$ is defined for all real numbers, the domain of g is limited to that of $\arcsin(x)$, namely $[-1, 1]$. The adage 'find the domain *before* you simplify' rings as true here as it did in Chapter **??**. $\square$

The next pair of functions we wish to discuss are the inverses of tangent and cotangent. First, we restrict $f(t) = \tan(t)$ to its fundamental cycle on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain the arctangent function, $f^{-1}(t) = \arctan(t)$. Among other things, note that the *vertical* asymptotes $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$ of the graph of $f(t) = \tan(t)$ become the *horizontal* asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(t) = \arctan(t)$.



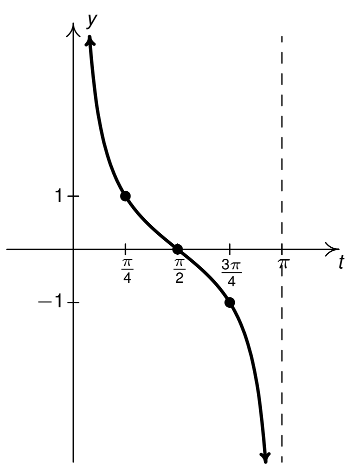
$f(t) = \tan(t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

reflect across $y = t$ $\longrightarrow$ switch t and y coordinates



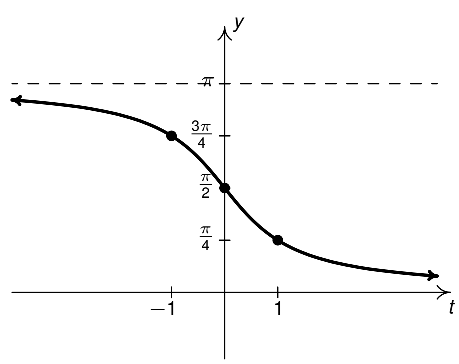
$f^{-1}(t) = \arctan(t)$.

Next, we restrict $g(t) = \cot(t)$ to its fundamental cycle on $(0, \pi)$ to obtain $g^{-1}(t) = \operatorname{arccot}(t)$, the arccotangent function. Once again, the vertical asymptotes $t = 0$ and $t = \pi$ of the graph of $g(t) = \cot(t)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ of the graph of $g^{-1}(t) = \operatorname{arccot}(t)$.



$g(t) = \cot(t)$, $0 < t < \pi$.

reflect across $y = t$ $\longrightarrow$ switch t and y coordinates



$g^{-1}(t) = \operatorname{arccot}(t)$.

Below we summarize the important properties of the arctangent and arccotangent functions.

Theorem 1.15. Properties of the Arctangent and Arccotangent Functions

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}^{+}$; as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}^{-}$
  - $\arctan(x) = t$ if and only if $\tan(t) = x$ and $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $\arctan(x) = \operatorname{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(t)) = t$ provided $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $F(x) = \arctan(x)$ is odd
- Properties of $G(x) = \operatorname{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - as $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^{-}$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^{+}$
  - $\operatorname{arccot}(x) = t$ if and only if $\cot(t) = x$ and $0 < t < \pi$
  - $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(t)) = t$ provided $0 < t < \pi$

The properties listed in Theorem 1.15 are consequences of the definitions of the arctangent and arccotangent functions along with Theorem **??**, and its proof is left to the reader.

Example 1.3.2.

1. Find the exact values of the following.

   (a) $\arctan(\sqrt{3})$

   (b) $\operatorname{arccot}(-\sqrt{3})$

   (c) $\cot(\operatorname{arccot}(-5))$

   (d) $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $\tan(2\arctan(x))$

   (b) $\cos(\operatorname{arccot}(2x))$

Solution.

1. (a) To find $\arctan(\sqrt{3})$, we need the angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t) = \sqrt{3}$. We find $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

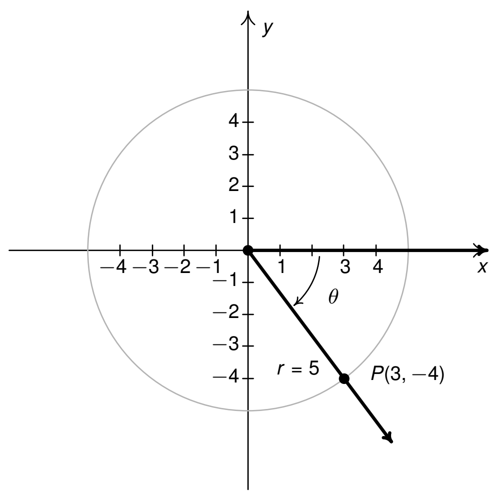
   (b) To find $\operatorname{arccot}(-\sqrt{3})$, we need the angle measuring t radians which lies between 0 and π with $\cot(t) = -\sqrt{3}$. Hence, $\operatorname{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$.

   (c) We can apply Theorem 1.15 directly and obtain $\cot(\operatorname{arccot}(-5)) = -5$. However, working it through provides us with yet another opportunity to understand why this is the case.

   Letting $t = \operatorname{arccot}(-5)$, by definition, $\cot(t) = -5$. Hence, $\cot(\operatorname{arccot}(-5)) = \cot(t) = -5$.

   (d) We start simplifying $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$ by letting $t = \arctan\left(-\frac{4}{3}\right)$. By definition, $\tan(t) = -\frac{4}{3}$ for some angle measuring t radians which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\tan(t) < 0$, we know, in fact, t corresponds to a Quadrant IV angle.

   We are given $\tan(t)$ but wish to know $\sin(t)$. Since there is no direct identity to marry the two, we make a quick sketch of the situation below. Since $\tan(t) = -\frac{4}{3} = \frac{-4}{3}$, we take $P(3, -4)$ as a point on the terminal side of $\theta = t = \arctan\left(-\frac{4}{3}\right)$ radians.



$P(3, -4)$ is on the terminal side of θ.

   We find $r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = 5$, so $\sin(t) = -\frac{4}{5}$. Hence, $\sin\left(\arctan\left(-\frac{4}{3}\right)\right) = -\frac{4}{5}$.

2. (a) We proceed as above and let $t = \arctan(x)$. We have $\tan(t) = x$ where $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Our goal is to express $\tan(2\arctan(x)) = \tan(2t)$ in terms of x.

   From Theorem 1.9, we know $\tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)} = \frac{2x}{1-x^2}$. Hence $f(x) = \tan(2\arctan(x)) = \frac{2x}{1-x^2}$.

   To find the domain, we once again think of $f(x) = \tan(2\arctan(x))$ as a sequence of steps and work from the inside out.

   The first step is to find the arctangent of a real number. Since the domain of $\arctan(x)$ is all real numbers, we have no restrictions here and we get out all values $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

   The next step is to multiply $\arctan(x)$ by 2. There are no restrictions here, either. Since the range of $\arctan(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the range of $2\arctan(x)$ is $(-\pi, \pi)$.

The last step is to take the tangent of $2\arctan(x)$. Since we are taking the tangent of values in the interval $(-\pi, \pi)$, we will run into trouble if $2\arctan(x) = \pm\frac{\pi}{2}$, that is, if $\arctan(x) = \pm\frac{\pi}{4}$. Since this happens exactly when $x = \tan\left(\pm\frac{\pi}{4}\right) = \pm 1$, we must exclude $x = \pm 1$ from the domain of f.

Hence, the domain of $f(x) = \tan(2\arctan(x))$ is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. In this example, we could have obtained the correct answer by looking at the algebraic equivalence, $f(x) = \frac{2x}{1-x^2}$. However, as we saw in Example 1.3.1, number 2b, this is not always the case.

(b) To get started, we let $t = \operatorname{arccot}(2x)$ so that $\cot(t) = 2x$ where $0 < t < \pi$. In terms of t, $\cos(\operatorname{arccot}(2x)) = \cos(t)$, and our goal is to express the latter in terms of x.

One way to proceed is to rewrite the identity $\cot(t) = \frac{\cos(t)}{\sin(t)}$ as $\cos(t) = \cot(t)\sin(t)$ and use the fact that $\cot(t) = 2x$ to find $\sin(t)$ in terms of x. This isn't as hopeless as it might seem, since the Pythagorean Identity $\csc^2(t) = 1 + \cot^2(t)$ relates cotangent to cosecant, and $\sin(t) = \frac{1}{\csc(t)}$.

Following this strategy, we get $\csc^2(t) = 1 + \cot^2(t) = 1 + (2x)^2 = 1 + 4x^2$ so $\csc(t) = \pm\sqrt{4x^2 + 1}$. Since t is between 0 and π, $\csc(t) > 0$. Hence, $\csc(t) = \sqrt{4x^2 + 1}$, so $\sin(t) = \frac{1}{\csc(t)} = \frac{1}{\sqrt{4x^2+1}}$.

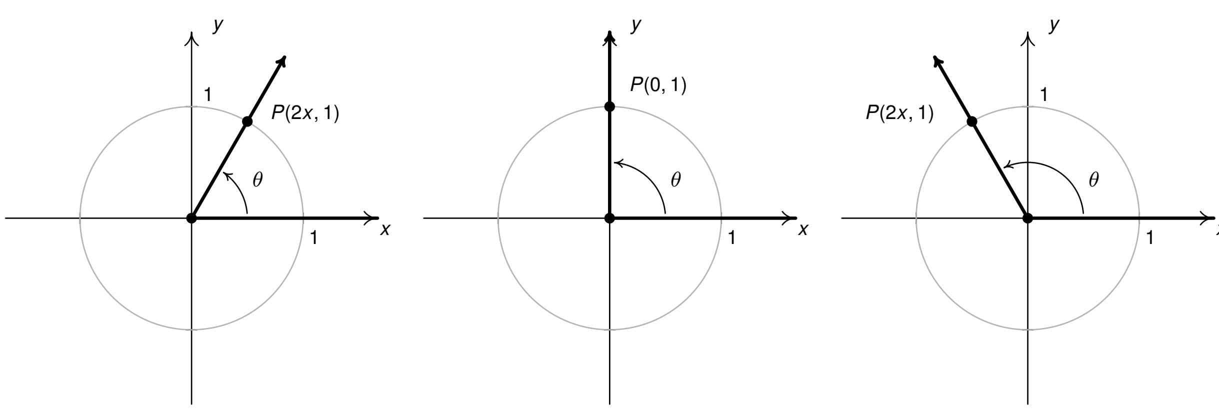
We find $\cos(t) = \cot(t)\sin(t) = \frac{2x}{\sqrt{4x^2+1}}$. Hence, $g(x) = \cos(\operatorname{arccot}(2x)) = \frac{2x}{\sqrt{4x^2+1}}$.

Viewing $g(x) = \cos(\operatorname{arccot}(2x))$ as a sequence of steps, we see we first double the input x, then take the arccotangent, and, finally, take the cosine. Since each of these processes are valid for all real numbers, the domain of g is $(-\infty, \infty)$. □

The reader may well wonder if there isn't a more direct way to handle Example 1.3.2 number 2b. Indeed, we can take some inspiration from Section **??** and imagine an angle θ measuring t radians so that $\cot(\theta) = \cot(t) = 2x$ where $0 < \theta < \pi$.

Thinking of $\cot(\theta)$ as a ratio of coordinates on a circle, we may rewrite $\cot(\theta) = 2x = \frac{2x}{1}$ and we would like to identify a point $P(2x, 1)$ on the terminal side of θ.

We need to be careful here. Since $\cot(\theta) = 2x$, $x = \frac{1}{2}\cot(\theta)$, so as θ ranges between 0 and π, x can take on positive, negative or 0 values. We need to argue that the point $P(2x, 1)$ lies in the quadrant we expect (as depicted below) in all cases before we delve too far into our analysis.



$0 < \theta < \frac{\pi}{2}$, $x > 0$. $\theta = \frac{\pi}{2}$, $x = 0$. $\frac{\pi}{2} < \theta < \pi$, $x < 0$.

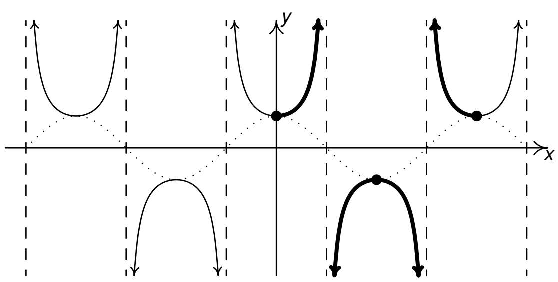
If $0 < \theta < \frac{\pi}{2}$, then $\cot(\theta) > 0$. Hence, $x > 0$ so the point $P(2x, 1)$ is in Quadrant I, as required. If $\theta = \frac{\pi}{2}$, then $x = 0$, and our point $P(2x, 1) = (0, 1)$, as required. If $\frac{\pi}{2} < \theta < \pi$, then $\cot(\theta) < 0$. Hence, $x < 0$, so $P(2x, 1)$ is in Quadrant II, as required.

Hence, in all three cases, our formula for the point $P(2x, 1)$ determines a point in the same quadrant as the terminal side of θ, as illustrated above.

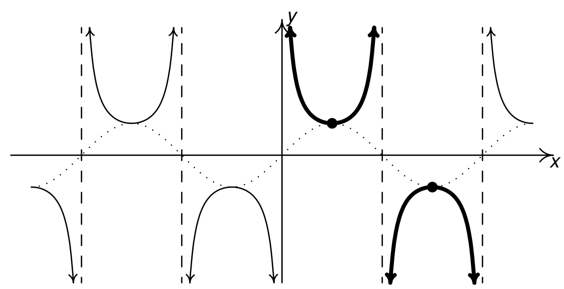
This allows us to use Theorem **??** from Section **??**. We find $r = \sqrt{(2x)^2 + 1^2} = \sqrt{4x^2 + 1}$, and hence, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$, which agrees with our answer from Example 1.3.2.

It shouldn't surprise the reader that there are some cases where the approach outlined above doesn't go as smoothly (as we'll see in the discussion following Example 1.3.3.)

The last two functions to invert are secant and cosecant. A portion of each of their graphs, which were first discussed in Subsection **??**, are given below with the fundamental cycles highlighted.



The graph of $y = \sec(x)$.



The graph of $y = \csc(x)$.

It is clear from the graph of secant that we cannot find one single continuous piece of its graph which covers its entire range of $(-\infty, -1] \cup [1, \infty)$ and restricts the domain of the function so that it is one-to-one. The same is true for cosecant.

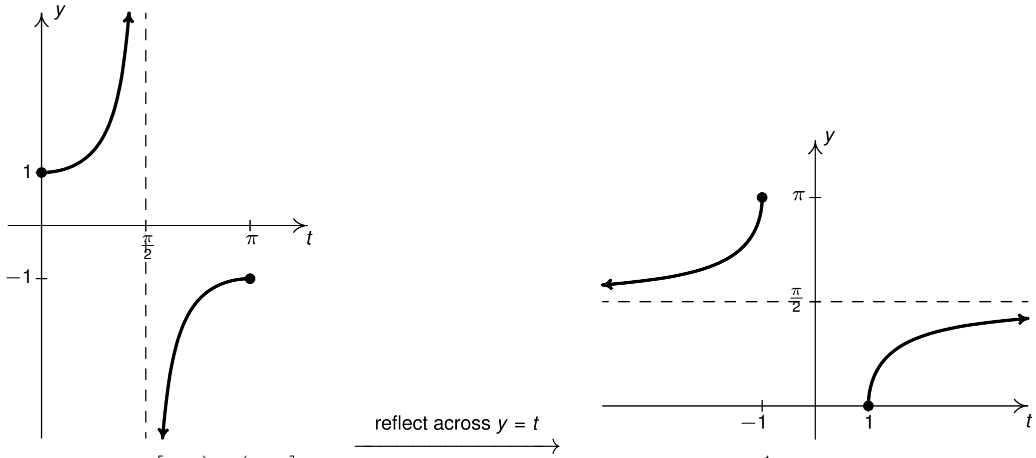
Thus in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$.

There are two generally accepted ways make these choices which restrict the domains of these functions so that they are one-to-one. One approach simplifies the Trigonometry associated with the inverse functions, but complicates the Calculus; the other makes the Calculus easier, but the Trigonometry less so.

For completeness, we present both points of view, each in its own subsection.

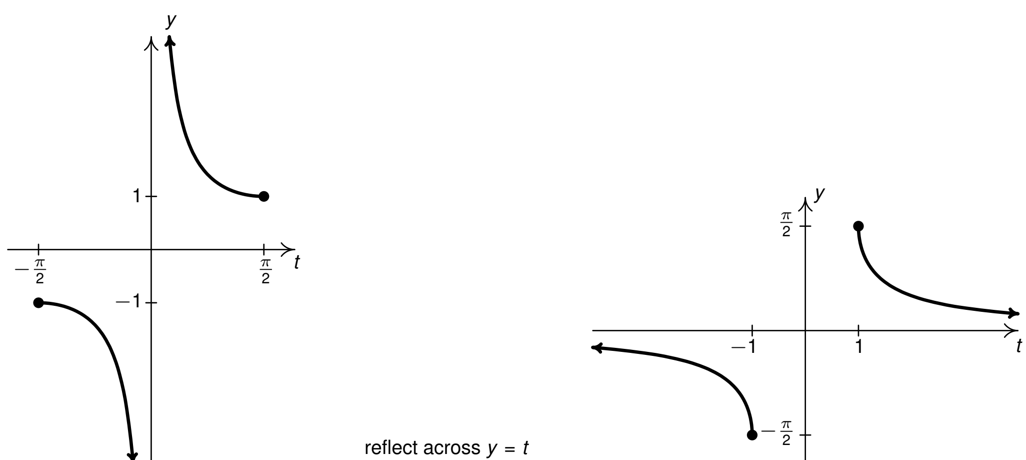
1.3.1 Inverses of Secant and Cosecant: Trigonometry Friendly Approach

In this subsection, we restrict the secant and cosecant functions to coincide with the restrictions on cosine and sine, respectively. For $f(t) = \sec(t)$, we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



$f(t) = \sec(t)$ on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ switch t and y coordinates $f^{-1}(t) = \operatorname{arcsec}(t)$

and we restrict $g(t) = \csc(t)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.



$g(t) = \csc(t)$ on $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ switch t and y coordinates $g^{-1}(t) = \operatorname{arccsc}(t)$

Note that for both arcsecant and arccosecant, the domain is $(-\infty, -1] \cup [1, \infty)$. Taking a page from Section **??**, we can rewrite this as $\{x \mid |x| \geq 1\}$. (This is often done in Calculus textbooks, so we include it here for completeness.)

Using these definitions along with Theorem **??**, we get the following properties of the arcsecant and arccosecant functions.

Theorem 1.16. Properties of the Arcsecant and Arccosecant Functions[a]

- Properties of $F(x) = \text{arcsec}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
  - as $x \to -\infty$, $\text{arcsec}(x) \to \frac{\pi}{2}^+$; as $x \to \infty$, $\text{arcsec}(x) \to \frac{\pi}{2}^-$
  - $\text{arcsec}(x) = t$ if and only if $\sec(t) = x$ and $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
  - $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\sec(\text{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\text{arcsec}(\sec(t)) = t$ provided $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
- Properties of $G(x) = \text{arccsc}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
  - as $x \to -\infty$, $\text{arccsc}(x) \to 0^-$; as $x \to \infty$, $\text{arccsc}(x) \to 0^+$
  - $\text{arccsc}(x) = t$ if and only if $\csc(t) = x$ and $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\csc(\text{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\text{arccsc}(\csc(t)) = t$ provided $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $G(x) = \text{arccsc}(x)$ is odd

[a] . . . assuming the "Trigonometry Friendly" ranges are used.

The reason the ranges here are called 'Trigonometry Friendly' is specifically because of two properties listed in Theorem 1.16: $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

These formulas essentially allow us to always convert arcsecants and arccosecants back to arccosines and arcsines, respectively. We see this play out in our next example.

Example 1.3.3.

1. Find the exact values of the following.

   (a) $\text{arcsec}(2)$ (b) $\text{arccsc}(-2)$ (c) $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$ (d) $\cot(\text{arccsc}(-3))$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $f(x) = \tan(\text{arcsec}(x))$

   (b) $g(x) = \cos(\text{arccsc}(4x))$

Solution.

1. (a) Using Theorem 1.16, we have $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) Once again, Theorem 1.16 comes to our aid giving $\text{arccsc}(-2) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

   (c) Since $\frac{5\pi}{4}$ doesn't fall between 0 and $\frac{\pi}{2}$ or $\frac{\pi}{2}$ and π, we cannot use the inverse property stated in Theorem 1.16. Hence, we work from the 'inside out.'

   We get: $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \text{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

   (d) We begin simplifying $\cot(\text{arccsc}(-3))$ by letting $t = \text{arccsc}(-3)$. Then, $\csc(t) = -3$. Since $\csc(t) < 0$, t lies in the interval $\left[-\frac{\pi}{2}, 0\right)$, so t corresponds to a Quadrant IV angle.

   To find $\cot(\text{arccsc}(-3)) = \cot(t)$, we use the Pythagorean Identity: $\cot^2(t) = \csc^2(t) - 1$. We get $\csc^2(t) = (-3)^2 - 1 = 8$, or $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$.

   Since t corresponds to a Quadrant IV angle, $\cot(t) < 0$. Hence, $\cot(\text{arccsc}(-3)) = -2\sqrt{2}$.

2. (a) Proceeding as above, we let $t = \text{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. We seek a formula for $\tan(\text{arcsec}(x)) = \tan(t)$ in terms of x.

   To relate $\sec(t)$ to $\tan(t)$, we use the Pythagorean Identity: $\tan^2(t) = \sec^2(t) - 1$. Substituting $\sec(t) = x$, we get $\tan^2(t) = \sec^2(t) - 1 = x^2 - 1$, so $\tan(t) = \pm\sqrt{x^2 - 1}$.

   If t belongs to $\left[0, \frac{\pi}{2}\right)$ then $\tan(t) \geq 0$. On the the other hand, if t belongs to $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(t) \leq 0$. As a result, we get a *piecewise defined* function for $\tan(t)$:

$$\tan(t) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } 0 \leq t < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

   Now we need to determine what these conditions on t mean for x. Since $x = \sec(t)$, when $0 \leq t < \frac{\pi}{2}$, $x \geq 1$, and when $\frac{\pi}{2} < t \leq \pi$, $x \leq -1$. Hence,

$$f(x) = \tan(\text{arcsec}(x)) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -\sqrt{x^2 - 1}, & \text{if } x \leq -1 \end{cases}$$

   To find the domain of f, we consider $f(x) = \tan(\text{arcsec}(x))$ as a two step process. First, we have the arcsecant function, whose domain is $(\infty, -1] \cup [1, \infty)$.

   Since the range of $\text{arcsec}(x)$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, taking the tangent of any output from $\text{arcsec}(x)$ is defined. Hence, the domain of f is $(-\infty, -1] \cup [1, \infty)$.

(b) Taking a cue from the previous problem, we start by letting $t = \text{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Our goal is to rewrite $\cos(\text{arccsc}(4x)) = \cos(t)$ in terms of x.

From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use of the Pythagorean Identity: $\cos^2(t) = 1 - \sin^2(t)$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) = 1 - \left(\frac{1}{4x}\right)^2 = 1 - \frac{1}{16x^2}$. Getting a common denominator and extracting square roots, we obtain:

$$\cos(t) = \pm\sqrt{\frac{16x^2 - 1}{16x^2}} = \pm\frac{\sqrt{16x^2 - 1}}{4|x|}.$$

Since t belongs to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we know $\cos(t) \geq 0$, so we choose $\cos(t) = \frac{\sqrt{16-x^2}}{4|x|}$. (The absolute values here are necessary, since x could be negative.) Hence,

$$g(x) = \cos(\text{arccsc}(4x)) = \frac{\sqrt{16 - x^2}}{4|x|}.$$

To find the domain of $g(x) = \cos(\text{arccsc}(4x))$, as usual, we think of g as a series of processes. First, we take the input, x, and multiply it by 4. Since this can be done to any real number, we have no restrictions here.

Next, we take the arccosecant of $4x$. Using interval notation, the domain of the arccosecant function is written as: $(-\infty, -1] \cup [1, \infty)$. Hence to take the arccosecant of $4x$, the quantity $4x$ must lie in one of these two intervals.[6] That is, $4x \leq -1$ or $4x \geq 1$, so $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$.

The third and final process coded in $g(x) = \cos(\text{arccsc}(4x))$ is to take the cosine of $\text{arccsc}(4x)$. Since the cosine accepts any real number, we have no additional restrictions. Hence, the domain of g is $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$. □

As promised in the discussion following Example 1.3.2, in which we used the methods from Section **??** to circumvent some onerous identity work, we take some time here to revisit number 2a to see what issues arise when we take a Section **??** approach here.

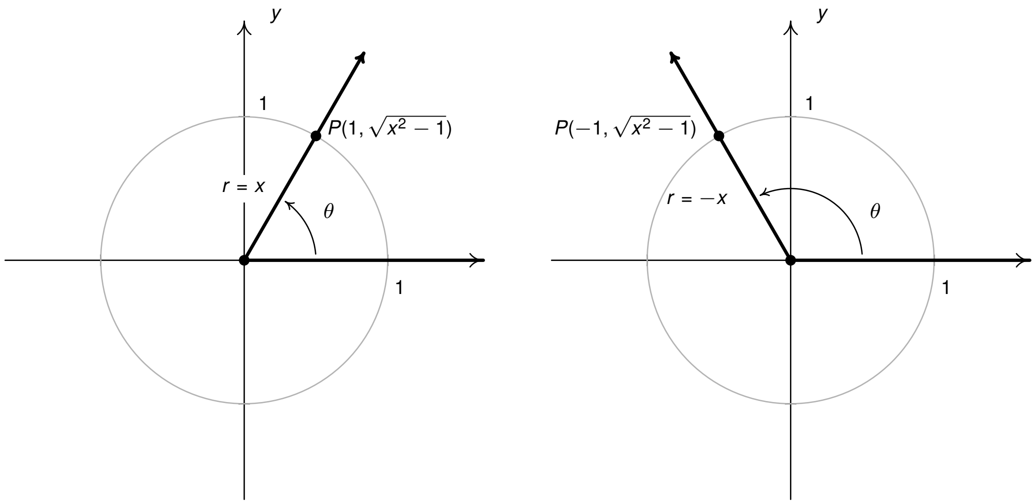
As above, we start rewriting $f(x) = \tan(\text{arcsec}(x))$ by letting $t = \text{arcsec}(x)$ so that $\sec(t) = x$ where $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$. We let $\theta = t$ radians and wish to view $\sec(\theta) = \sec(t) = x$ as described in Theorem **??**: the ratio of the radius of a circle, r centered at the origin, divided by the abscissa[7] of a point on the terminal side of θ which intersects said circle.

If we make the usual identification $\sec(\theta) = x = \frac{x}{1}$, we see that if $0 \leq \theta < \frac{\pi}{2}$, then $x = \sec\theta \geq 1$, so it makes sense to identify the quantity x as the radius of the circle with 1 as the abscissa of the point where the terminal side of θ intersects said circle. To find the associated ordinate (y-coordinate), we have $1^2 + y^2 = x^2$ so $y = \sqrt{x^2 - 1}$, where we have chosen the positive root since we are in Quadrant I. We sketch out this scenario below on the left.

If, however, $\frac{\pi}{2} < t \leq \pi$, then $x = \sec(t) \leq -1$, so we need to rewrite $\sec(\theta) = x = \frac{x}{1} = \frac{-x}{-1}$ in order to keep the radius of the circle, $r = -x > 0$ and the abscissa, $-1 < 0$. From $(-1)^2 + y^2 = (-x)^2$, we still get $y = \sqrt{x^2 - 1}$, as shown below on the right.

[6] Alternatively, we can write the domain of $\text{arccsc}(x)$ as $|x| \geq 1$, so the domain of $\text{arccsc}(4x)$ is $|4x| \geq 1$.

[7] We'll avoid the label 'x-coordinate' here since as we'll see, the quantity x in this problem is tied to the radius as opposed to the coordinates of points on the terminal side of θ.

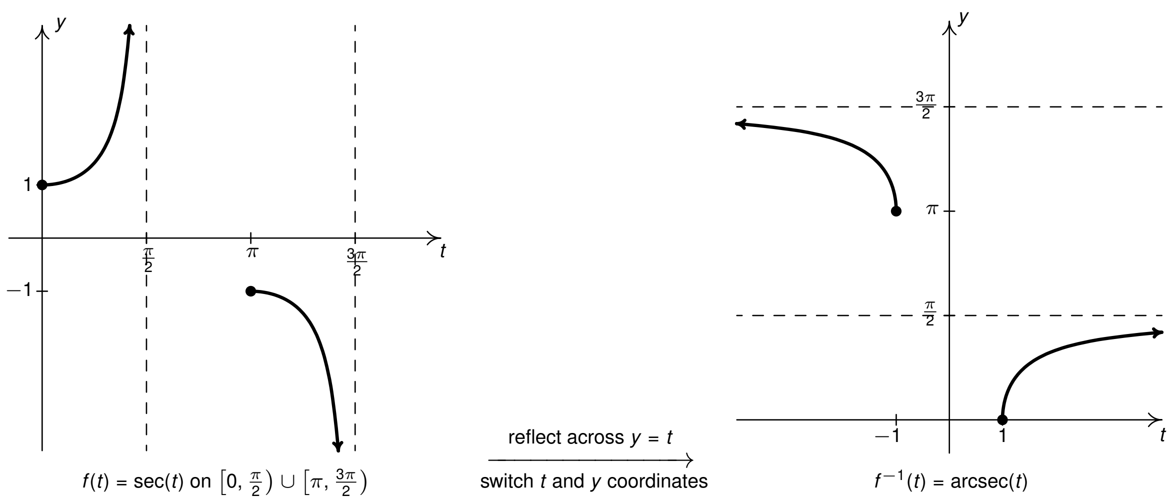


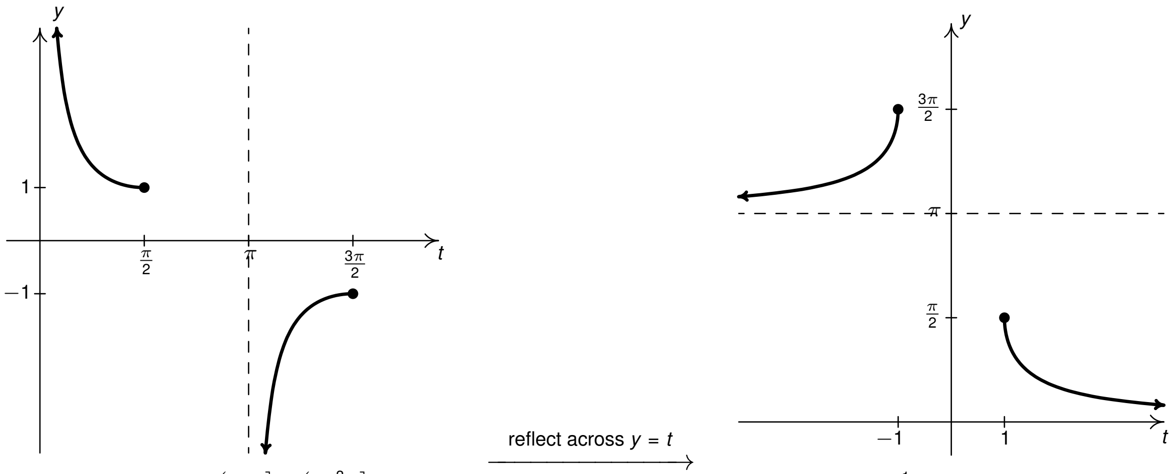
$0 < \theta < \frac{\pi}{2}$, $x \geq 1$, $r = x$. $\frac{\pi}{2} < \theta < \pi$, $x \leq -1$, $r = -x$.

In the Quadrant I case, when $x \geq 1$, we get $\tan(\theta) = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$. In Quadrant II, when $x \leq -1$, we obtain $\tan(\theta) = \frac{\sqrt{x^2-1}}{-1} = -\sqrt{x^2-1}$. Hence, we get the piecewise definition for $f(x)$ as we did in number 2a above: $f(x) = \tan(\operatorname{arcsec}(x)) = \sqrt{x^2-1}$ if $x \geq 1$ and $f(x) = \tan(\operatorname{arcsec}(x)) = -\sqrt{x^2-1}$ if $x \leq -1$.
The moral of the story here is that you are free to choose whichever route you like to simplify expressions like those found in Example 1.3.3 number 2a. Whether you choose identities or a more geometric route, just be careful to keep in mind which quadrants are in play, which variables represent which quantities, and what signs ($\pm$) each should have.

1.3.2 Inverses of Secant and Cosecant: Calculus Friendly Approach

In this subsection, we restrict $f(t) = \sec(t)$ to $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and $g(t) = \csc(t)$ to $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.





$g(t) = \csc(t)$ on $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ $\qquad$ $g^{-1}(t) = \operatorname{arccsc}(t)$

Using these definitions, we get the following.

Theorem 1.17. Properties of the Arcsecant and Arccosecant Functions[a]

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$
  - as $x \to -\infty$, $\operatorname{arcsec}(x) \to \frac{3\pi}{2}^-$; as $x \to \infty$, $\operatorname{arcsec}(x) \to \frac{\pi}{2}^-$
  - $\operatorname{arcsec}(x) = t$ if and only if $\sec(t) = x$ and $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$.
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $x \geq 1$ only
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(t)) = t$ provided $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x \mid |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$
  - as $x \to -\infty$, $\operatorname{arccsc}(x) \to \pi^+$; as $x \to \infty$, $\operatorname{arccsc}(x) \to 0^+$
  - $\operatorname{arccsc}(x) = t$ if and only if $\csc(t) = x$ and $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $x \geq 1$ only
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(t)) = t$ provided $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$

[a] . . . assuming the "Calculus Friendly" ranges are used.

While it is difficult to explain why the choices here for the ranges for the arcsecant an arccosecant are, indeed, 'Calculus Friendly,' we can demonstrate how they are slightly less 'Trigonometry Friendly.' Note the equivalences $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ hold for $x \geq 1$ only, and not for all x in the domain. We will need to remember this as we work through the problems in the next example.

Speaking of which, our next example is a duplicate of Example 1.3.3. The interested reader is invited to see what differences are to be had as a consequence of the change in ranges.

Example 1.3.4.

1. Find the exact values of the following.

   (a) $\operatorname{arcsec}(2)$ (b) $\operatorname{arccsc}(-2)$ (c) $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$ (d) $\cot\left(\operatorname{arccsc}(-3)\right)$

2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $\tan(\operatorname{arcsec}(x))$ (b) $\cos(\operatorname{arccsc}(4x))$

Solution.

1. (a) Since $2 \geq 1$, we may invoke Theorem 1.17 to get $\operatorname{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) Unfortunately, -2 is not greater to or equal to 1, so we cannot apply Theorem 1.17 to $\operatorname{arccsc}(-2)$ and convert this into an arcsine problem. Instead, we appeal to the definition.

   To find $t = \operatorname{arccsc}(-2)$, we need the angle measuring t radians with $\csc(t) = -2$ and is either between 0 and $\frac{\pi}{2}$ or between π and $\frac{3\pi}{2}$.

   Since $\csc(t) < 0$, we know t corresponds to an angle between π and $\frac{3\pi}{2}$, and since $\csc(t) = -2$, we know $\sin(t) = -\frac{1}{2}$. Hence, $t = \operatorname{arccsc}(-2) = \frac{7\pi}{6}$.

   (c) Since $\frac{5\pi}{4}$ lies between π and $\frac{3\pi}{2}$, Theorem 1.17 applies: $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$.

   We encourage the reader to work this through using the definition as we have done in the previous examples to see how it goes.

   (d) To simplify $\cot(\operatorname{arccsc}(-3))$ we let $t = \operatorname{arccsc}(-3)$ so that $\cot(\operatorname{arccsc}(-3)) = \cot(t)$.

   We know $\csc(t) = -3$, and since this is negative, t lies in $\left(\pi, \frac{3\pi}{2}\right]$. To get from $\csc(t)$ to $\cot(t)$, we use the Pythagorean Identity: $\cot^2(t) = \csc^2(t) - 1$. We find $\cot^2(t) = (-3)^2 - 1 = 8$ so that $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$.

   Since t is in the interval $\left(\pi, \frac{3\pi}{2}\right]$, t corresponds to a Quadrant III angle, so we know $\cot(t) > 0$. Hence, our answer is $\cot(\operatorname{arccsc}(-3)) = 2\sqrt{2}$.

2. (a) We begin rewriting $f(x) = \tan(\operatorname{arcsec}(x))$ by letting $t = \operatorname{arcsec}(x)$. Hence, $\sec(t) = x$ where either $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$. Our goal is to find an expression for $\tan(t)$ in terms of x.

   To relate $\sec(t)$ to $\tan(t)$, we use the Pythagorean Identity: $\tan^2(t) = \sec^2(t) - 1 = x^2 - 1$ so that $\tan(t) = \pm\sqrt{x^2-1}$. Since t lies in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, $\tan(t) \geq 0$, so we choose $\tan(t) = \sqrt{x^2-1}$. Hence, $f(x) = \tan(\operatorname{arcsec}(x)) = \sqrt{x^2-1}$.

For the domain of f, we note that the domain of $\text{arcsec}(x)$ is $(-\infty, -1] \cup [1, \infty)$. Since all values in the range of arcsecant, $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, are in the domain of the tangent function, we find the domain of f is $(-\infty, -1] \cup [1, \infty)$.

(b) To rewrite $g(x) = \cos(\text{arccsc}(4x))$, we start by letting $t = \text{arccsc}(4x)$. Then $\csc(t) = 4x$ where either $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$. Our goal is to find an expression for $\cos(t)$ in terms of x.

From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use of the Pythagorean Identity: $\cos^2(t) = 1 - \sin^2(t)$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) = 1 - \left(\frac{1}{4x}\right)^2 = 1 - \frac{1}{16x^2}$. Getting a common denominator and extracting square roots, we obtain:

$$\cos(t) = \pm\sqrt{\frac{16x^2 - 1}{16x^2}} = \pm\frac{\sqrt{16x^2 - 1}}{4|x|}.$$

If t lies in $\left(0, \frac{\pi}{2}\right]$, then $\cos(t) \geq 0$, and we choose $\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|}$. Here, $x = \frac{1}{4}\csc(t) > 0$ as well, so we can disregard the absolute values here and write $\cos(t) = \frac{\sqrt{16x^2-1}}{4x}$.

If t belongs to $\left(\pi, \frac{3\pi}{2}\right]$, then $\cos(t) \leq 0$, so, we choose $\cos(t) = -\frac{\sqrt{16x^2-1}}{4|x|}$. In this case, $x = \frac{1}{4}\csc(t) < 0$, so $|x| = -x$ (see Section **??** for a refresher, if needs be!) and so,

$$\cos(t) = -\frac{\sqrt{16x^2 - 1}}{4|x|} = -\frac{\sqrt{16x^2 - 1}}{4(-x)} = \frac{\sqrt{16x^2 - 1}}{4x}.$$

Hence, in both cases, we get

$$g(x) = \cos(\text{arccsc}(4x)) = \frac{\sqrt{16x^2 - 1}}{4x}.$$

To find the domain of $g(x) = \cos(\text{arccsc}(4x))$, as usual, we think of g as a series of processes. First, we take the input, x, and multiply it by 4. Since this can be done to any real number, we have no restrictions here.

Next, we take the arccosecant of $4x$. Using interval notation, the domain of the arccosecant function is written as: $(-\infty, -1] \cup [1, \infty)$. Hence to take the arccosecant of $4x$, the quantity $4x$ must lie in one of these two intervals.[8] That is, $4x \leq -1$ or $4x \geq 1$, so $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$.

The third and final process coded in $g(x) = \cos(\text{arccsc}(4x))$ is to take the cosine of $\text{arccsc}(4x)$. Since the cosine accepts any real number, we have no additional restrictions. Hence, the domain of g is $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$. $\square$

For completeness, we embark here on a discussion of how the techniques from Section **??**, in particular Theorem **??** can be used to circumvent some of the identity work in number 2a above.[9]

As above, we start rewriting $f(x) = \tan(\text{arcsec}(x))$ by letting $t = \text{arcsec}(x)$ so that $\sec(t) = x$ where $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$. We let $\theta = t$ radians and wish to view $\sec(\theta) = \sec(t) = x$ as described in

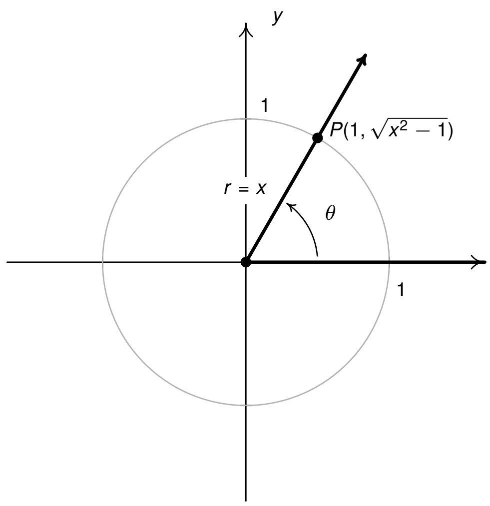
[8] Alternatively, we can write the domain of $\text{arccsc}(x)$ as $|x| \geq 1$, so the domain of $\text{arccsc}(4x)$ is $|4x| \geq 1$.

[9] See also the remarks following Examples 1.3.2 and 1.3.3.

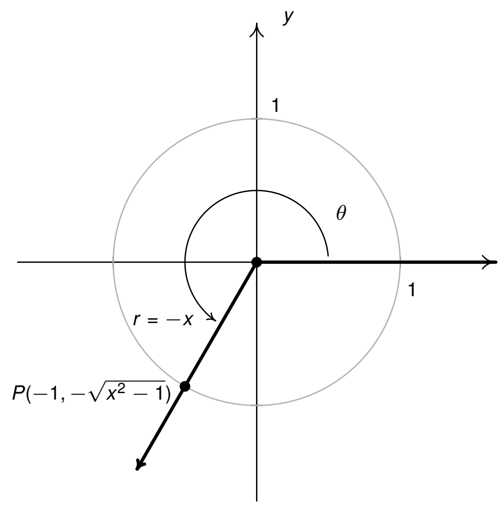
Theorem **??**: the ratio of the radius of a circle, r centered at the origin, divided by the abscissa[10] of a point on the terminal side of θ which intersects said circle.

If we make the usual identification $\sec(\theta) = x = \frac{x}{1}$, we see that if $0 \leq \theta < \frac{\pi}{2}$, then $x = \sec\theta \geq 1$, so it makes sense to identify the quantity x as the radius of the circle with 1 as the abscissa of the point where the terminal side of θ intersects said circle. To find the associated ordinate (y-coordinate), we have $1^2 + y^2 = x^2$ so $y = \sqrt{x^2 - 1}$, where we have chosen the positive root since we are in Quadrant I. We sketch out this scenario below on the left.

If, however, $\pi \leq t < \frac{3\pi}{2}$, then $x = \sec(t) \leq -1$, so we need to rewrite $\sec(\theta) = x = \frac{x}{1} = \frac{-x}{-1}$ in order to keep the radius of the circle, $r = -x > 0$ and the abscissa, $-1 < 0$. From $(-1)^2 + y^2 = (-x)^2$, we get $y = -\sqrt{x^2 - 1}$, in this case choosing the negative root since we are in Quadrant III.



$0 < \theta < \frac{\pi}{2}$, $x \geq 1$, $r = x$.



$\pi < \theta < \frac{3\pi}{2}$, $x \leq -1$, $r = -x$.

In the Quadrant I case, when $x \geq 1$, we get $\tan(\theta) = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2 - 1}$. In Quadrant III, when $x \leq -1$, we obtain $\tan(\theta) = \frac{-\sqrt{x^2-1}}{-1} = \sqrt{x^2 - 1}$. Hence, in both cases, we obtain the same answer as we did in number 2a above: $f(x) = \tan(\text{arcsec}(x)) = \sqrt{x^2 - 1}$ for x in $(-\infty, -1] \cup [1, \infty)$.

1.3.3 Calculators and the Inverse Circular Functions.

In the sections to come, we will have need to approximate the values of the inverse circular functions. On most calculators, only the arcsine, arccosine and arctangent functions are available and they are usually labeled as $\sin^{-1}, \cos^{-1}$ and $\tan^{-1}$, respectively. If we are asked to approximate these values, it is a simple matter to punch up the appropriate decimal on the calculator.

If we are asked for an arccotangent, arcsecant or arccosecant, however, we often need to employ some ingenuity, as our next example illustrates.

[10] We'll avoid the label 'x-coordinate' here since as we'll see, the quantity x in this problem is tied to the radius as opposed to the coordinates of points on the terminal side of θ.

Example 1.3.5.

1. Use a calculator to approximate the following values to four decimal places.

   (a) arccot(2) (b) arcsec(5) (c) arccot(-2) (d) arccsc $\left(-\dfrac{3}{2}\right)$

2. Find the domain and range of the following functions. Check your answers using a graphing utility.

   (a) $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$ (b) $g(x) = 3\arctan(4x)$.

Solution.

1. (a) Since $2 > 0$, we can use the property listed in Theorem 1.15 to get: arccot(2) = arctan $\left(\frac{1}{2}\right)$. In 'radian' mode, we find arccot(2) = arctan $\left(\frac{1}{2}\right) \approx 0.4636$.

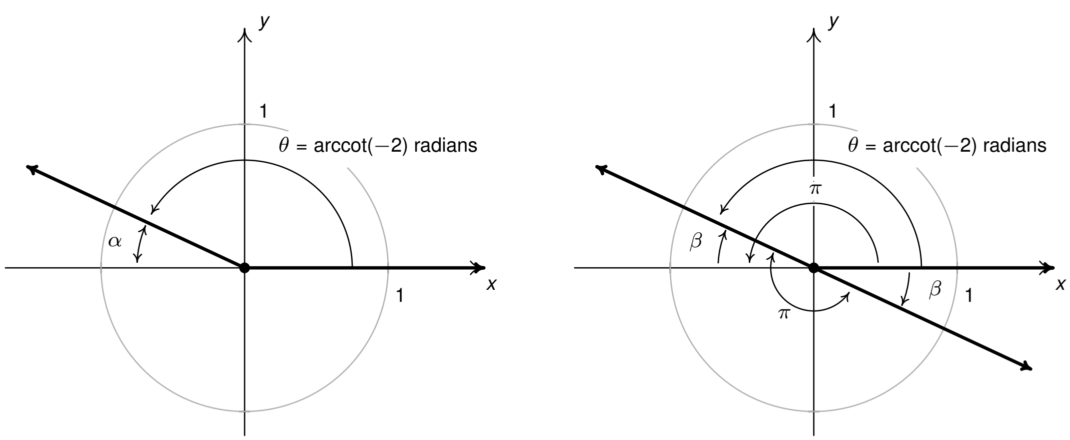
   (b) Since $5 \geq 1$, we can use the property from either Theorem 1.16 or Theorem 1.17 to write arcsec(5) = arccos $\left(\frac{1}{5}\right) \approx 1.3694$.

   (c) Since the argument -2 is negative, we cannot directly apply Theorem 1.15 to help us find arccot(-2), so we appeal to the definition.

   The number t = arccot(-2) corresponds to an angle $\theta = t$ radians with $\cot(\theta) = -2$ which lies between 0 and π. Moreover, since $\cot(\theta) < 0$, we know θ is a Quadrant II angle.

   Let α be the reference angle for θ, as pictured below on the left. By definition, α is an acute angle which means $0 < \alpha < \frac{\pi}{2}$. By The Reference Angle Theorem, Theorem **??**, we also know $\cot(\alpha) = 2$. Hence, by definition, α = arccot(2) radians.

   Since the argument of arccotangent is now a *positive* 2, we can use Theorem 1.15 to get α = arccot(2) = arctan $\left(\frac{1}{2}\right)$ radians. Since $\theta = \pi - \alpha = \pi -$ arctan $\left(\frac{1}{2}\right) \approx 2.6779$ radians, we get arccot(-2) ≈ 2.6779.



α is a reference angle of θ β = arctan $\left(-\frac{1}{2}\right)$ radians

Another way to attack the problem is to use $\arctan\left(-\frac{1}{2}\right)$. By definition, we have the real number $t = \arctan\left(-\frac{1}{2}\right)$ satisfies $\tan(t) = -\frac{1}{2}$ with $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $\tan(t) < 0$, we know more specifically that $-\frac{\pi}{2} < t < 0$, so t corresponds to an angle β in Quadrant IV. We sketch β along with $\theta = \operatorname{arccot}(-2)$ radians above on the right.

To find the value of $\operatorname{arccot}(-2)$, we once again visualize the angle $\theta = \operatorname{arccot}(-2)$ radians and note that it is a Quadrant II angle with $\tan(\theta) = -\frac{1}{2}$. This means it is exactly π units away from β, and we get $\theta = \pi + \beta = \pi + \arctan\left(-\frac{1}{2}\right) \approx 2.6779$ radians. Hence, as before, $\operatorname{arccot}(-2) \approx 2.6779$.

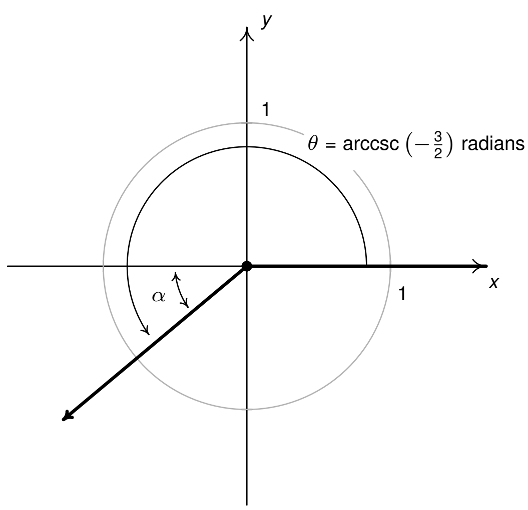
(d) If the range of arccosecant is taken to be $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we can use Theorem 1.16 to get $\operatorname{arccsc}\left(-\frac{3}{2}\right) = \arcsin\left(-\frac{2}{3}\right) \approx -0.7297$.

If, on the other hand, the range of arccosecant is taken to be $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, then we proceed as in the previous problem by letting $t = \operatorname{arccsc}\left(-\frac{3}{2}\right)$.

Then t is a real number with $\csc(t) = -\frac{3}{2}$ and since $\csc(t) < 0$, we know $\pi < \theta \leq \frac{3\pi}{2}$, Hence, t corresponds to a Quadrant III angle, θ, as depicted below.

As above, we let α be the reference angle for θ. Then $0 < \alpha < \frac{\pi}{2}$ and $\csc(\alpha) = \frac{3}{2}$, which means $\alpha = \operatorname{arccsc}\left(\frac{3}{2}\right)$ radians.

Since the argument of arccosecant is now positive, Theorem 1.17 applies so we can rewrite $\alpha = \operatorname{arccsc}\left(\frac{3}{2}\right) = \arcsin\left(\frac{2}{3}\right)$ radians. Since $\theta = \pi + \alpha = \pi + \arcsin\left(\frac{2}{3}\right) \approx 3.8713$ radians, we have that in this case, $\operatorname{arccsc}\left(-\frac{3}{2}\right) \approx 3.8713$.



2. (a) To find the domain of f, we can think of the function as a sequence of steps and track our inputs through each step and track the restrictions that arise. To that end, we rewrite $f(x)$ as we did in Section **??**: $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right) = -\arccos\left(\frac{x}{5}\right) + \frac{\pi}{2}$.

Starting with a real number x, we divide by 5 to obtain $\frac{x}{5}$. So far, we have no restrictions.

Next, we take the arccosine of $\frac{x}{5}$. Since the arccosine function only admits inputs between -1 and 1 inclusive, we require that $-1 \leq \frac{x}{5} \leq 1$. Solving, we get $-5 \leq x \leq 5$.

Moving outside the arccosine, we multiply the outputs from the arccosine by -1 and then add $\frac{\pi}{2}$. Since these are defined for all real numbers, we have our domain restricted only by the arccosine itself. Hence, the domain of f is $[-5, 5]$.

To determine the range of f, we work through the steps above, this time paying attention to the outputs from each step. We know our domain is restricted to $[-5, 5]$ due to the arccosine, so we start with the range of arccosine: $[0, \pi]$.

Let y represent a typical output from $\arccos(x)$. Then $0 \leq y \leq \pi$. From our work above, we know the arccosine values are first multiplied by -1 and then added to $\frac{\pi}{2}$, so we apply these same operations to the inequality $0 \leq y \leq \pi$.

Multiplying this inequality through by -1 gives $-\pi \leq -y \leq 0$. Adding through by $\frac{\pi}{2}$ gives $-\frac{\pi}{2} \leq -y + \frac{\pi}{2} \leq \frac{\pi}{2}$. Hence the range of $f(x) = -\arccos\left(\frac{x}{5}\right) + \frac{\pi}{2}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Our graph below on the left confirms our results.[11]

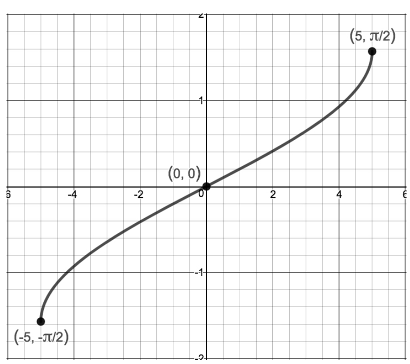
(b) As with the previous example, we think of $g(x) = 3\arctan(4x)$ as a series of steps in order to find the domain and track the range.

Starting with an input x, we multiply it by 4, take the arctangent, and then multiply that result by 3. Since all of these operations are defined for all real numbers, we conclude the domain of g is also all real numbers, or $(-\infty, \infty)$.

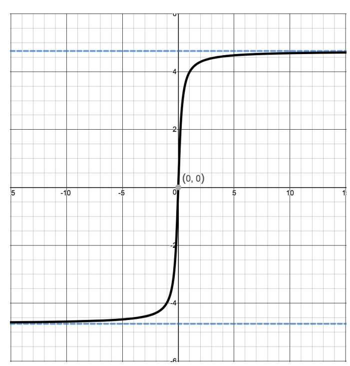
Looking at outputs, we find that the range first becomes limited when taking the arctangent. If y represent a typical output from $\arctan(x)$, then $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

In our formula for $g(x)$, the outputs from the arctangent are multiplied by 3. Multiplying the inequality $-\frac{\pi}{2} < y < \frac{\pi}{2}$ through by 3 gives $-\frac{3\pi}{2} < 3y < \frac{3\pi}{2}$.

Hence the range of g is $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$. Our answers are confirmed by examining the graph of g below on the right.



$y = f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$



$y = g(x) = 3\arctan(4x)$

□

[11] If this sort of analysis sounds familiar, it should. We are really just tracking the effect of transformations as in Section **??**.

1.3.4 Solving Equations Using the Inverse Trigonometric Functions.

In Sections **??** and **??**, we learned how to solve equations like $\sin(\theta) = \frac{1}{2}$ and $\tan(t) = -1$. In each case, we ultimately appealed to the Unit Circle and relied on the fact that the answers corresponded to a set of 'common angles' listed on page **??**.

If, on the other hand, we had been asked to find all angles with $\sin(\theta) = \frac{1}{3}$ or solve $\tan(t) = -2$ for real numbers t, we would have been hard-pressed to do so. With the introduction of the inverse trigonometric functions, however, we are now in a position to solve these equations.

A good parallel to keep in mind is how the square root function can be used to solve certain quadratic equations. The equation $x^2 = 4$ is a lot like $\sin(\theta) = \frac{1}{2}$ in that it has friendly, 'common value' answers $x = \pm 2$. The equation $x^2 = 7$, on the other hand, is a lot like $\sin(\theta) = \frac{1}{3}$. We know there are answers, but we can't express them using 'friendly' numbers.

To solve $x^2 = 7$, we make use of the square root function (which is an inverse to $f(x) = x^2$ on a restricted domain) and write our answer as $x = \pm\sqrt{7}$. We need the $\pm$ to adjust for the fact that $\sqrt{7}$ is defined to be positive only, but we know we have two solutions, one positive and one negative. Using a calculator, we can certainly *approximate* the values $\pm\sqrt{7}$, but as far as exact answers go, we leave them as $x = \pm\sqrt{7}$.

In the same way, we will use the arcsine function (the inverse to the sine function on a restricted domain) to solve $\sin(\theta) = \frac{1}{3}$. However, we will need to adjust for the fact that there is more than one answer to this equation (infinitely many, in fact!) As it turns out, we will be able to express every solution in terms of $\arcsin\left(\frac{1}{3}\right)$, as our next example illustrates.

Example 1.3.6. Solve the following equations.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.
2. Find all real numbers t for which $\tan(t) = -2$
3. Solve $\sec(x) = -\frac{5}{3}$ for x.

Solution.

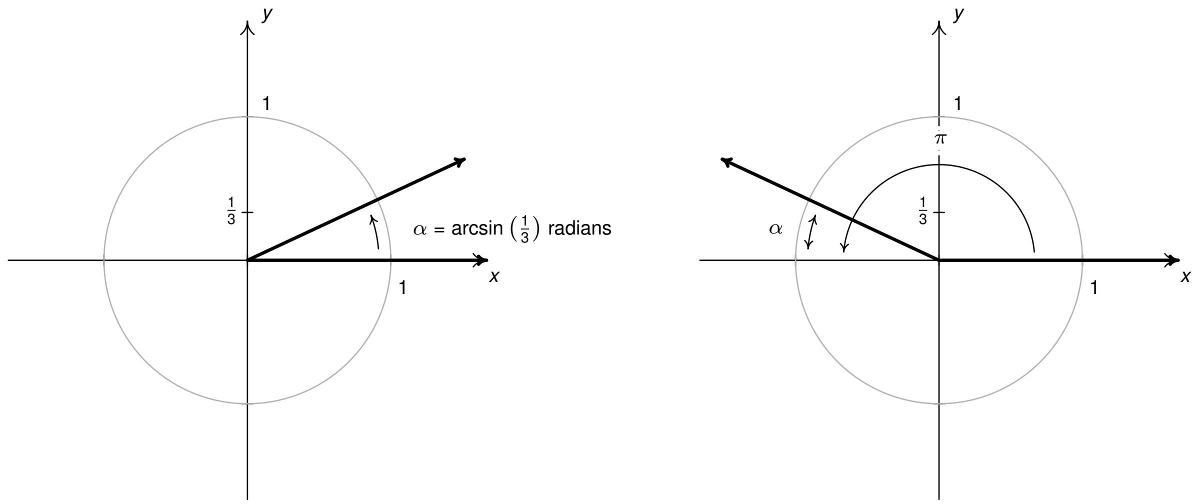
1. If $\sin(\theta) = \frac{1}{3}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $y = \frac{1}{3}$. Geometrically, we see that this happens at two places: in Quadrant I and Quadrant II.

   If we let α denote the acute solution to the equation, then all the solutions to this equation in Quadrant I are coterminal with α, and α serves as the reference angle for all of the solutions to this equation in Quadrant II as seen below.

   Since $\frac{1}{3}$ isn't the sine of any of the 'common angles' we've encountered, we use the arcsine functions to express our answers. By definition, real number $t = \arcsin\left(\frac{1}{3}\right)$ $\sin(t) = \frac{1}{3}$ with $0 < t < \frac{\pi}{2}$.

   Hence, $\alpha = \arcsin\left(\frac{1}{3}\right)$ radians is an acute angle with $\sin(\alpha) = \frac{1}{3}$. Since all of the Quadrant I solutions θ are all coterminal with α, we get $\theta = \alpha + 2\pi k = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k.

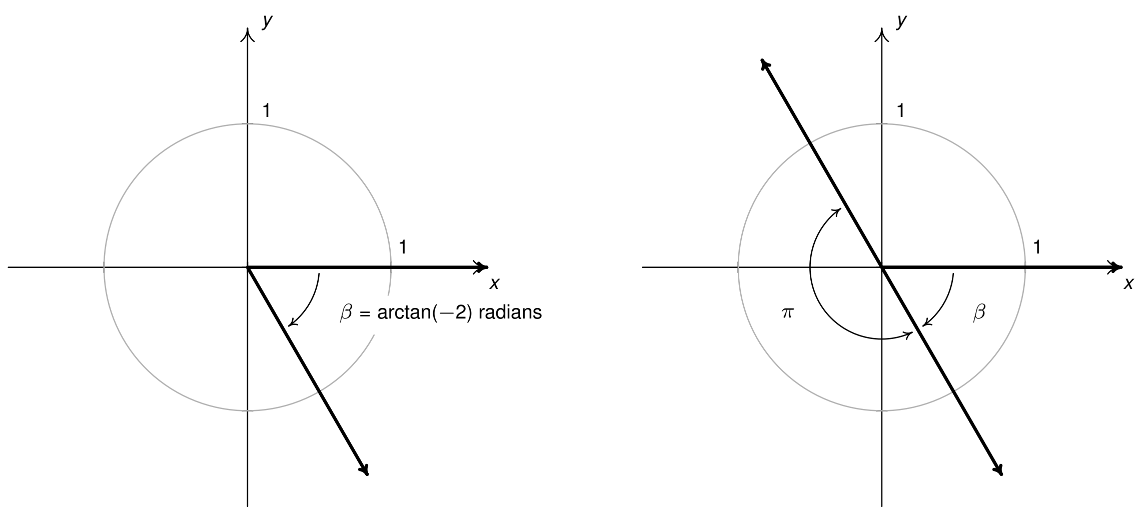
   Turning our attention to Quadrant II, we get one solution to be $\pi - \alpha$. Hence, the Quadrant II solutions are $\theta = \pi - \alpha + 2\pi k = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$, for integers k.



2. The real number solutions to $\tan(t) = -2$ correspond to angles θ with $\tan(\theta) = -2$. Since tangent is negative only in Quadrants II and IV, we focus our efforts there.

   The real number $t = \arctan(-2)$ satisfies $\tan(t) = -2$ and $-\frac{\pi}{2} < t < 0$. If we let $\beta = \arctan(-2)$ radians, then all of the Quadrant IV solutions to $\tan(\theta) = -2$ are coterminal with β.

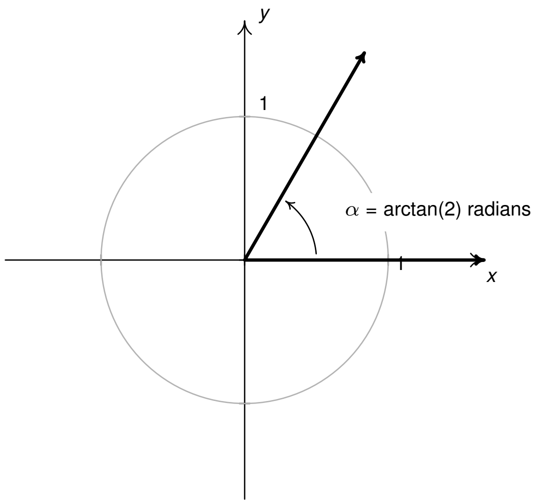
   Moreover, as seen below on the right, the solutions from Quadrant II differ by exactly π units from the solutions in Quadrant IV (recall, the period of the tangent function is π.) Hence, all of the solutions to $\tan(\theta) = -2$ are of the form $\theta = \beta + \pi k = \arctan(-2) + \pi k$ for some integer k. Switching back to the variable t, we record our final answer to $\tan(t) = -2$ as $t = \arctan(-2) + \pi k$ for integers k.

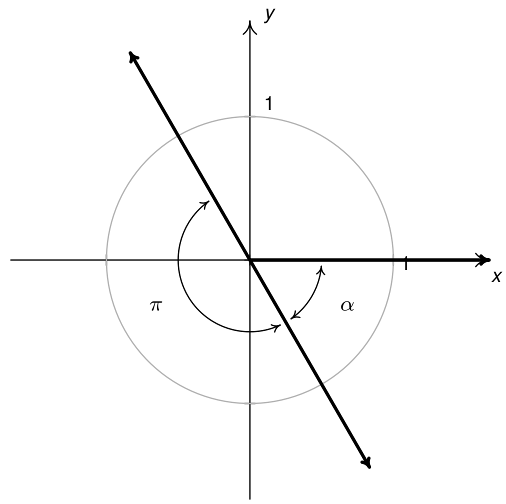


   Another tact we could have taken to solve this problem is to use reference angles. Consider the (angle) equation: $\tan(\theta) = -2$. If we let α be the reference angle for the solutions θ, we know α is an acute angle with $\tan(\alpha) = 2$.

By definition, the real number $t = \arctan(2)$ satisfies $0 < t < \frac{\pi}{2}$ with $\tan(t) = 2$. Hence, the angle $\alpha = \arctan(2)$ radians is the reference angle for our solutions to $\tan(\theta) = -2$.

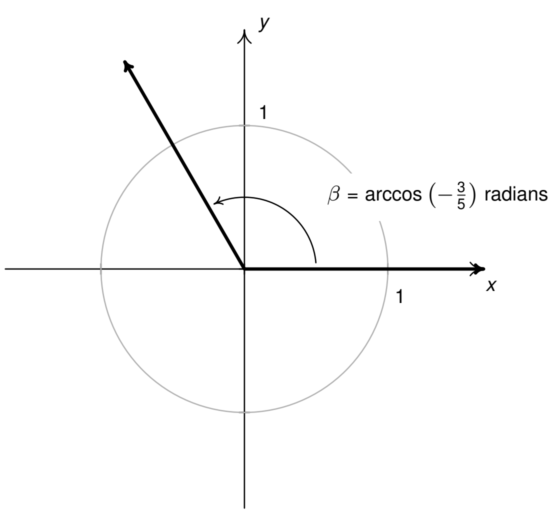
Adjusting for quadrants, we get our answers to $\tan(\theta) = -2$ are $\theta = -\alpha + \pi k = -\arctan(2) + \pi k$ for integers k. Again, we cosmetically change the variable from θ back to t so our answer to $\tan(t) = -2$ is $t = -\arctan(2) + \pi k$. Thanks to the odd property of arctangent, $\arctan(-2) = -\arctan(2)$ and we see this family of solutions is identical to what we obtained earlier.

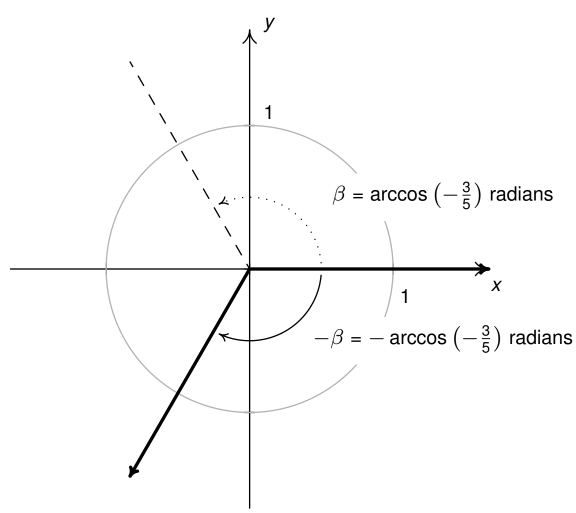




3. In the last equation, $\sec(x) = -\frac{5}{3}$, we are not told whether or not x represents an angle or a real number. This isn't really much of an issue, since we attack both problems the same way.

   Taking a cue from our work in Section **??** and use a Reciprocal Identity to convert the equation $\sec(x) = -\frac{5}{3}$ to $\cos(x) = -\frac{3}{5}$. Thinking geometrically, we are looking for angles θ with $\cos(\theta) = -\frac{3}{5}$. Since $\cos(\theta) < 0$, we are looking for solutions in Quadrants II and III. Since $-\frac{3}{5}$ isn't the cosine of any of the 'common angles', we'll need to express our solutions in terms of the arccosine function.





□

The real number $t = \arccos\left(-\frac{3}{5}\right)$ is defined so that $\frac{\pi}{2} < t < \pi$ with $\cos(t) = -\frac{3}{5}$. Hence, the angle $\beta = \arccos\left(-\frac{3}{5}\right)$ radians is a Quadrant II angle which satisfies $\cos(\beta) = -\frac{3}{5}$. To obtain a Quadrant III angle solution, we may simply use $-\beta = -\arccos\left(-\frac{3}{5}\right)$ as seen above on the right.

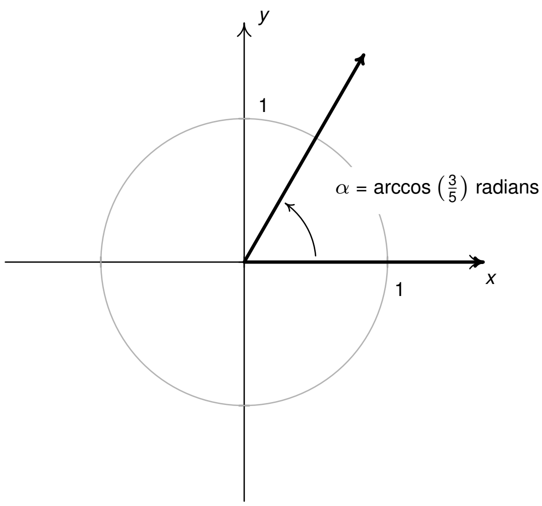
Since all angle solutions are coterminal with β or $-\beta$, we get our solutions to $\cos(\theta) = -\frac{3}{5}$ to be $\theta = \beta + 2\pi k = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $\theta = -\beta + 2\pi k = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k.

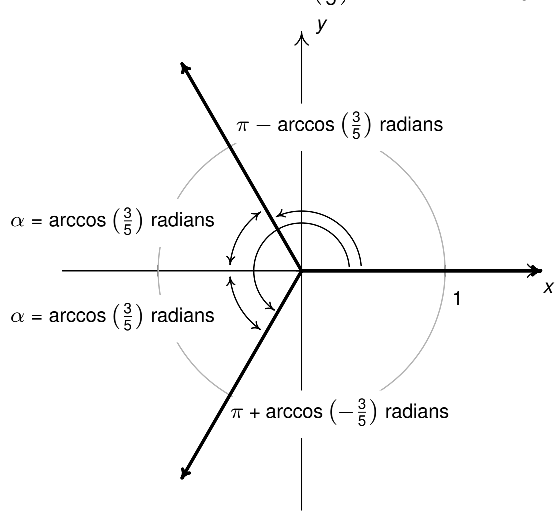
Switching back to the variable x, we record our final answer to $\sec(x) = -\frac{5}{3}$ as $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k.

As with the previous problem, we can approach solving $\cos(\theta) = -\frac{3}{5}$ using reference angles. Letting α represent the reference angle for the solutions θ, we know α is an acute angle with $\cos(\alpha) = \frac{3}{5}$.

We know the real number $t = \arccos\left(\frac{3}{5}\right)$ satisfies $\cos(t) = \frac{3}{5}$ and $0 < t < \frac{\pi}{2}$, hence $\alpha = \arccos\left(\frac{3}{5}\right)$ radians is the reference angle for the solutions to $\cos(\theta) = -\frac{3}{5}$.

Hence, the Quadrant II solutions to $\cos(\theta) = -\frac{3}{5}$ are $\theta = \pi - \alpha + 2\pi k = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ while the Quadrant IV solutions to $\cos(\theta) = -\frac{3}{5}$ are $\theta = \pi + \alpha + 2\pi k = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k.





□

Shifting back to the variable x, we get our solution to $\sec(x) = -\frac{5}{3}$ are $x = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ or $x = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k. While these certainly look quite different than what we obtained before, $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k, they are, in fact, equivalent. To show this, we start with $\arccos\left(-\frac{3}{5}\right) = \pi - \arccos\left(\frac{3}{5}\right)$ and begin writing out specific solutions from each family by choosing specific values of k. We leave these details to the reader. □

We close this section with one last sinusoid example.

Example 1.3.7. Consider the function $f(t) = 3\cos(6t) - 4\sin(6t)$. Find a formula for $f(t)$:

1. in the form $C(t) = A\cos(\omega t + \phi) + B$ for $\omega > 0$
2. in the form $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$

Solution.

1. As in Example 1.2.7, we compare the expanded form of $C(t) = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B$ with $f(t) = 3\cos(6t) - 4\sin(6t)$. We identify $\omega = 6$ and $B = 0$ and by equating coefficients of $\cos(6t)$ and $\sin(6t)$ get the two equations: $A\cos(\phi) = 3$ and $A\sin(\phi) = 4$.

   Using the Pythagorean Identity to eliminate ϕ, we get $A^2 = (A\cos(\phi))^2 + (A\sin(\phi))^2 = 3^2 + 4^2 = 25$. We choose $A = 5$ and work to find the phase angle ϕ.

   Substituting $A = 5$ into our two equations relating A and ϕ, we get $5\cos(\phi) = 3$, or $\cos(\phi) = \frac{3}{5}$ and $5\sin(\phi) = 4$, so $\sin(\phi) = \frac{4}{5}$. Since both $\sin(\phi)$ and $\cos(\phi)$ are positive, we know ϕ is a Quadrant I angle. However, since neither the sine nor cosine value of ϕ corresponds to a common angle, we need to express ϕ in terms of either an arcsine or arccosine.

   Since the real number $t = \arccos\left(\frac{3}{5}\right)$ satisfies $\cos(t) = \frac{3}{5}$ and $0 < t < \frac{\pi}{2}$, we know the angle $\phi = \arccos\left(\frac{3}{5}\right)$ radians is an acute (Quadrant I) angle which satisfies $\cos(\phi) = \frac{3}{5}$. Hence, we can take $\phi = \arccos\left(\frac{3}{5}\right)$ and write $f(t) = 5\cos\left(6t + \arccos\left(\frac{3}{5}\right)\right)$.

   In addition, the real number $t = \arcsin\left(\frac{4}{5}\right)$ satisfies $\sin(t) = \frac{4}{5}$ and $0 < t < \frac{\pi}{2}$. Hence $\phi = \arcsin\left(\frac{4}{5}\right)$ radians is Quadrant I angle with $\sin(\phi) = \frac{4}{5}$. This means we could also take $\phi = \arcsin\left(\frac{4}{5}\right)$ and write $f(t) = 5\cos\left(6t + \arcsin\left(\frac{4}{5}\right)\right)$. (We could also express ϕ in terms of arctangents, if we wanted!)

   We leave it to the reader to verify (both) solutions analytically and graphically.

2. Once again, we equate the expanded form of $S(t) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B$ with $f(t) = 3\cos(6t) - 4\sin(6t)$. Once again, we get $\omega = 6$ and $B = 0$. Here, our two equations for A and ϕ are $A\cos(\phi) = -4$ and $A\sin(\phi) = 3$.

   As before, we get $A^2 = (A\cos(\phi))^2 + (A\sin(\phi))^2 = (-4)^2 + 3^2 = 25$, and we choose $A = 5$. Our equations for ϕ become: $\cos(\phi) = -\frac{4}{5}$ and $\sin(\phi) = \frac{3}{5}$. Since $\cos(\phi) < 0$ but $\sin(\phi) > 0$, we know ϕ is a Quadrant II angle. As before, since neither the sine nor cosine value of ϕ corresponds to a common angle, we need to express ϕ in terms of either an arcsine or arccosine.

   Here, we opt to use the arccosine function, since the range of arccosine, $[0, \pi]$ covers Quadrant II. From $\cos(\phi) = -\frac{4}{5}$, we get $\phi = \arccos\left(-\frac{4}{5}\right)$, so $f(t) = 5\sin\left(6t + \arccos\left(-\frac{4}{5}\right)\right)$.

   Had we chosen to work with arcsines, we would need a Quadrant II solution to $\sin(\phi) = \frac{3}{5}$. Going through the usual machinations, we arrive at $\phi = \pi - \arcsin\left(\frac{3}{5}\right)$. Hence, an alternative form of our answer is $f(t) = 5\sin\left(6t + \pi - \arcsin\left(\frac{3}{5}\right)\right)$. We leave the checks to the reader. $\square$

1.3.5 Exercises

In Exercises 1 - 40, find the exact value.

1. $\arcsin(-1)$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
4. $\arcsin\left(-\frac{1}{2}\right)$
5. $\arcsin(0)$
6. $\arcsin\left(\frac{1}{2}\right)$
7. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
9. $\arcsin(1)$
10. $\arccos(-1)$
11. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
13. $\arccos\left(-\frac{1}{2}\right)$
14. $\arccos(0)$
15. $\arccos\left(\frac{1}{2}\right)$
16. $\arccos\left(\frac{\sqrt{2}}{2}\right)$
17. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
18. $\arccos(1)$
19. $\arctan\left(-\sqrt{3}\right)$
20. $\arctan(-1)$
21. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$
22. $\arctan(0)$
23. $\arctan\left(\frac{\sqrt{3}}{3}\right)$
24. $\arctan(1)$
25. $\arctan\left(\sqrt{3}\right)$
26. $\operatorname{arccot}\left(-\sqrt{3}\right)$
27. $\operatorname{arccot}(-1)$
28. $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$
29. $\operatorname{arccot}(0)$
30. $\operatorname{arccot}\left(\frac{\sqrt{3}}{3}\right)$
31. $\operatorname{arccot}(1)$
32. $\operatorname{arccot}\left(\sqrt{3}\right)$
33. $\operatorname{arcsec}(2)$
34. $\operatorname{arccsc}(2)$
35. $\operatorname{arcsec}\left(\sqrt{2}\right)$
36. $\operatorname{arccsc}\left(\sqrt{2}\right)$
37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
38. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
39. $\operatorname{arcsec}(1)$
40. $\operatorname{arccsc}(1)$

In Exercises 41 - 48, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value. (See Section 1.3.1.)

41. $\operatorname{arcsec}(-2)$
42. $\operatorname{arcsec}\left(-\sqrt{2}\right)$
43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
44. $\operatorname{arcsec}(-1)$
45. $\operatorname{arccsc}(-2)$
46. $\operatorname{arccsc}\left(-\sqrt{2}\right)$
47. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
48. $\operatorname{arccsc}(-1)$

In Exercises 49 - 56, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value. (See Section 1.3.2.)

49. $\text{arcsec}\,(-2)$
50. $\text{arcsec}\left(-\sqrt{2}\right)$
51. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
52. $\text{arcsec}\,(-1)$
53. $\text{arccsc}\,(-2)$
54. $\text{arccsc}\left(-\sqrt{2}\right)$
55. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
56. $\text{arccsc}\,(-1)$

In Exercises 57 - 86, find the exact value or state that it is undefined.

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$
58. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$
59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$
60. $\sin\left(\arcsin\left(-0.42\right)\right)$
61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$
62. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$
63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$
64. $\cos\left(\arccos\left(\frac{5}{13}\right)\right)$
65. $\cos\left(\arccos\left(-0.998\right)\right)$
66. $\cos\left(\arccos\left(\pi\right)\right)$
67. $\tan\left(\arctan\left(-1\right)\right)$
68. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$
69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$
70. $\tan\left(\arctan\left(0.965\right)\right)$
71. $\tan\left(\arctan\left(3\pi\right)\right)$
72. $\cot\left(\text{arccot}\left(1\right)\right)$
73. $\cot\left(\text{arccot}\left(-\sqrt{3}\right)\right)$
74. $\cot\left(\text{arccot}\left(-\frac{7}{24}\right)\right)$
75. $\cot\left(\text{arccot}\left(-0.001\right)\right)$
76. $\cot\left(\text{arccot}\left(\frac{17\pi}{4}\right)\right)$
77. $\sec\left(\text{arcsec}\left(2\right)\right)$
78. $\sec\left(\text{arcsec}\left(-1\right)\right)$
79. $\sec\left(\text{arcsec}\left(\frac{1}{2}\right)\right)$
80. $\sec\left(\text{arcsec}\left(0.75\right)\right)$
81. $\sec\left(\text{arcsec}\left(117\pi\right)\right)$
82. $\csc\left(\text{arccsc}\left(\sqrt{2}\right)\right)$
83. $\csc\left(\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$
84. $\csc\left(\text{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$
85. $\csc\left(\text{arccsc}\left(1.0001\right)\right)$
86. $\csc\left(\text{arccsc}\left(\frac{\pi}{4}\right)\right)$

In Exercises 87 - 106, find the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$
88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$
89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$
91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$
92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$
93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$
94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$
95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$
96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$
97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$
98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$
99. $\arctan\left(\tan\left(\pi\right)\right)$
100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$
102. $\text{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$
103. $\text{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$
104. $\text{arccot}\left(\cot\left(\pi\right)\right)$
105. $\text{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$
106. $\text{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

In Exercises 107 - 118, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value. (See Section 1.3.1.)

107. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$
108. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$
109. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$
110. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$
111. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$
112. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$
113. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$
114. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$
115. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$
116. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$
117. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$
118. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 119 - 130, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value. (See Section 1.3.2.)

119. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$
120. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$
121. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$
122. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$
123. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$
124. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$
125. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$
126. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$
127. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$
128. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$
129. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$
130. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 131 - 154, find the exact value or state that it is undefined.

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$

132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$

133. $\sin\left(\arctan\left(-2\right)\right)$

134. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right)$

135. $\sin\left(\operatorname{arccsc}\left(-3\right)\right)$

136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$

138. $\cos\left(\operatorname{arccot}\left(3\right)\right)$

139. $\cos\left(\operatorname{arcsec}\left(5\right)\right)$

140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$

142. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$

143. $\tan\left(\operatorname{arccot}\left(12\right)\right)$

144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

146. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right)$

147. $\cot\left(\arctan\left(0.25\right)\right)$

148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$

150. $\sec\left(\arctan\left(10\right)\right)$

151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$

152. $\csc\left(\operatorname{arccot}\left(9\right)\right)$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$

154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

In Exercises 155 - 164, find the exact value or state that it is undefined.

155. $\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$

156. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$

157. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$

158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$

159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right)$

160. $\sin\left(2\arctan\left(2\right)\right)$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

162. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right)$

163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

164. $\sin\left(\frac{\arctan(2)}{2}\right)$

In Exercises 165 - 184, rewrite each of the following composite functions as algebraic functions of x and state the domain.

165. $f(x) = \sin(\arccos(x))$
166. $f(x) = \cos(\arctan(x))$
167. $f(x) = \tan(\arcsin(x))$
168. $f(x) = \sec(\arctan(x))$
169. $f(x) = \csc(\arccos(x))$
170. $f(x) = \sin(2\arctan(x))$
171. $f(x) = \sin(2\arccos(x))$
172. $f(x) = \cos(2\arctan(x))$
173. $f(x) = \sin(\arccos(2x))$
174. $f(x) = \sin\left(\arccos\left(\frac{x}{5}\right)\right)$
175. $f(x) = \cos\left(\arcsin\left(\frac{x}{2}\right)\right)$
176. $f(x) = \cos(\arctan(3x))$
177. $f(x) = \sin(2\arcsin(7x))$
178. $f(x) = \sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right)$
179. $f(x) = \cos(2\arcsin(4x))$
180. $f(x) = \sec(\arctan(2x))\tan(\arctan(2x))$
181. $f(x) = \sin(\arcsin(x) + \arccos(x))$
182. $f(x) = \cos(\arcsin(x) + \arctan(x))$
183. $f(x) = \tan(2\arcsin(x))$
184. $f(x) = \sin\left(\frac{1}{2}\arctan(x)\right)$
185. If $\theta = \arcsin\left(\frac{x}{2}\right)$, find an expression for $\theta + \sin(2\theta)$ in terms of x.
186. If $\theta = \arctan\left(\frac{x}{7}\right)$, find an expression for $\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta)$ in terms of x.
187. If $\theta = \operatorname{arcsec}\left(\frac{x}{4}\right)$, find an expression for $4\tan(\theta) - 4\theta$ in terms of x assuming $x \geq 4$.

In Exercises 188 - 207, solve the equation using the techniques discussed in Example 1.3.6 then approximate the solutions which lie in the interval $[0, 2\pi)$ to four decimal places.

188. $\sin(\theta) = \frac{7}{11}$
189. $\cos(\theta) = -\frac{2}{9}$
190. $\sin(\theta) = -0.569$
191. $\cos(\theta) = 0.117$
192. $\sin(\theta) = 0.008$
193. $\cos(\theta) = \frac{359}{360}$
194. $\tan(t) = 117$
195. $\cot(t) = -12$
196. $\sec(t) = \frac{3}{2}$
197. $\csc(t) = -\frac{90}{17}$
198. $\tan(t) = -\sqrt{10}$
199. $\sin(t) = \frac{3}{8}$
200. $\cos(x) = -\frac{7}{16}$
201. $\tan(x) = 0.03$
202. $\sin(x) = 0.3502$

203. $\sin(x) = -0.721$

204. $\cos(x) = 0.9824$

205. $\cos(x) = -0.5637$

206. $\cot(x) = \dfrac{1}{117}$

207. $\tan(x) = -0.6109$

In Exercises 208 - 213, rewrite the given function as a sinusoid of the form $C(t) = A\cos(\omega t + \phi)$ and $S(t) = A\sin(\omega t + \phi)$ (See Example 1.3.7.) Approximate the value of ϕ (which is in radians, of course) to four decimal places.

208. $f(t) = 5\sin(3t) + 12\cos(3t)$

209. $f(t) = 3\cos(2t) + 4\sin(2t)$

210. $f(t) = \cos(t) - 3\sin(t)$

211. $f(t) = 7\sin(10t) - 24\cos(10t)$

212. $f(t) = -\cos(t) - 2\sqrt{2}\sin(t)$

213. $f(t) = 2\sin(t) - \cos(t)$

In Exercises 214 - 225, find the domain of the given function. Write your answers in interval notation.

214. $f(x) = \arcsin(5x)$

215. $f(x) = \arccos\left(\dfrac{3x-1}{2}\right)$

216. $f(x) = \arcsin\left(2x^2\right)$

217. $f(x) = \arccos\left(\dfrac{1}{x^2-4}\right)$

218. $f(x) = \arctan(4x)$

219. $f(x) = \operatorname{arccot}\left(\dfrac{2x}{x^2-9}\right)$

220. $f(x) = \arctan(\ln(2x-1))$

221. $f(x) = \operatorname{arccot}(\sqrt{2x-1})$

222. $f(x) = \operatorname{arcsec}(12x)$

223. $f(x) = \operatorname{arccsc}(x+5)$

224. $f(x) = \operatorname{arcsec}\left(\dfrac{x^3}{8}\right)$

225. $f(x) = \operatorname{arccsc}\left(e^{2x}\right)$

226. Find a nonzero number x where $\operatorname{arccot}(x) \neq \arctan\left(\dfrac{1}{x}\right)$.

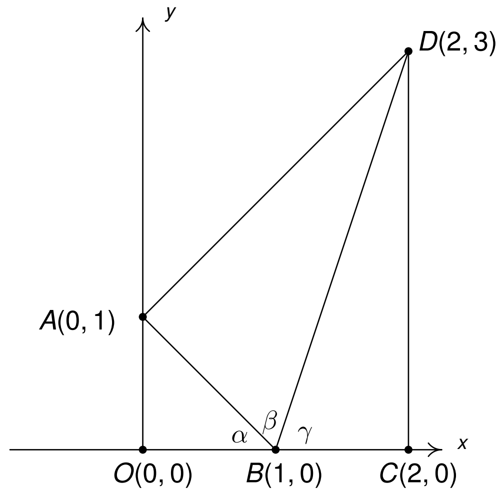
227. Find an example where $\operatorname{arcsec}(x) \neq \arccos\left(\dfrac{1}{x}\right)$ if we use $\left[0, \dfrac{\pi}{2}\right) \cup \left[\pi, \dfrac{3\pi}{2}\right)$ as the range of $f(x) = \operatorname{arcsec}(x)$.

228. Show that $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$ for $-1 \leq x \leq 1$.

229. Discuss with your classmates why $\arcsin\left(\dfrac{1}{2}\right) \neq 30$.

230. Use the diagram below along with the accompanying questions to show:

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$



(a) Clearly $\triangle AOB$ and $\triangle BCD$ are right triangles because the line through O and A and the line through C and D are perpendicular to the x-axis. Use the distance formula to show that $\triangle BAD$ is also a right triangle (with $\angle BAD$ being the right angle) by showing that the sides of the triangle satisfy the Pythagorean Theorem.

(b) Use $\triangle AOB$ to show that $\alpha = \arctan(1)$

(c) Use $\triangle BAD$ to show that $\beta = \arctan(2)$

(d) Use $\triangle BCD$ to show that $\gamma = \arctan(3)$

(e) Use the fact that O, B and C all lie on the x-axis to conclude that $\alpha + \beta + \gamma = \pi$. Thus $\arctan(1) + \arctan(2) + \arctan(3) = \pi$.

1.3.6 Answers

1. $\arcsin(-1) = -\frac{\pi}{2}$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$
3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$
4. $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$
5. $\arcsin(0) = 0$
6. $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$
7. $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
9. $\arcsin(1) = \frac{\pi}{2}$
10. $\arccos(-1) = \pi$
11. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$
13. $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$
14. $\arccos(0) = \frac{\pi}{2}$
15. $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$
16. $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$
17. $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$
18. $\arccos(1) = 0$
19. $\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$
20. $\arctan(-1) = -\frac{\pi}{4}$
21. $\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$
22. $\arctan(0) = 0$
23. $\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$
24. $\arctan(1) = \frac{\pi}{4}$
25. $\arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$
26. $\operatorname{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$
27. $\operatorname{arccot}(-1) = \frac{3\pi}{4}$
28. $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right) = \frac{2\pi}{3}$
29. $\operatorname{arccot}(0) = \frac{\pi}{2}$
30. $\operatorname{arccot}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{3}$
31. $\operatorname{arccot}(1) = \frac{\pi}{4}$
32. $\operatorname{arccot}\left(\sqrt{3}\right) = \frac{\pi}{6}$
33. $\operatorname{arcsec}(2) = \frac{\pi}{3}$
34. $\operatorname{arccsc}(2) = \frac{\pi}{6}$
35. $\operatorname{arcsec}\left(\sqrt{2}\right) = \frac{\pi}{4}$
36. $\operatorname{arccsc}\left(\sqrt{2}\right) = \frac{\pi}{4}$
37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{6}$
38. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{3}$
39. $\operatorname{arcsec}(1) = 0$
40. $\operatorname{arccsc}(1) = \frac{\pi}{2}$
41. $\operatorname{arcsec}(-2) = \frac{2\pi}{3}$
42. $\operatorname{arcsec}\left(-\sqrt{2}\right) = \frac{3\pi}{4}$
43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$
44. $\operatorname{arcsec}(-1) = \pi$
45. $\operatorname{arccsc}(-2) = -\frac{\pi}{6}$

46. $\text{arccsc}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$

47. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$

48. $\text{arccsc}(-1) = -\frac{\pi}{2}$

49. $\text{arcsec}(-2) = \frac{4\pi}{3}$

50. $\text{arcsec}\left(-\sqrt{2}\right) = \frac{5\pi}{4}$

51. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{7\pi}{6}$

52. $\text{arcsec}(-1) = \pi$

53. $\text{arccsc}(-2) = \frac{7\pi}{6}$

54. $\text{arccsc}\left(-\sqrt{2}\right) = \frac{5\pi}{4}$

55. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{4\pi}{3}$

56. $\text{arccsc}(-1) = \frac{3\pi}{2}$

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{1}{2}$

58. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right) = -\frac{\sqrt{2}}{2}$

59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{3}{5}$

60. $\sin(\arcsin(-0.42)) = -0.42$

61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$ is undefined.

62. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right) = \frac{\sqrt{2}}{2}$

63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

64. $\cos\left(\arccos\left(\frac{5}{13}\right)\right) = \frac{5}{13}$

65. $\cos(\arccos(-0.998)) = -0.998$

66. $\cos(\arccos(\pi))$ is undefined.

67. $\tan(\arctan(-1)) = -1$

68. $\tan\left(\arctan\left(\sqrt{3}\right)\right) = \sqrt{3}$

69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{5}{12}$

70. $\tan(\arctan(0.965)) = 0.965$

71. $\tan(\arctan(3\pi)) = 3\pi$

72. $\cot(\text{arccot}(1)) = 1$

73. $\cot\left(\text{arccot}\left(-\sqrt{3}\right)\right) = -\sqrt{3}$

74. $\cot\left(\text{arccot}\left(-\frac{7}{24}\right)\right) = -\frac{7}{24}$

75. $\cot(\text{arccot}(-0.001)) = -0.001$

76. $\cot\left(\text{arccot}\left(\frac{17\pi}{4}\right)\right) = \frac{17\pi}{4}$

77. $\sec(\text{arcsec}(2)) = 2$

78. $\sec(\text{arcsec}(-1)) = -1$

79. $\sec\left(\text{arcsec}\left(\frac{1}{2}\right)\right)$ is undefined.

80. $\sec(\text{arcsec}(0.75))$ is undefined.

81. $\sec(\text{arcsec}(117\pi)) = 117\pi$

82. $\csc\left(\text{arccsc}\left(\sqrt{2}\right)\right) = \sqrt{2}$

83. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right) = -\frac{2\sqrt{3}}{3}$

84. $\csc\left(\operatorname{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$ is undefined.

85. $\csc(\operatorname{arccsc}(1.0001)) = 1.0001$

86. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$ is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right) = -\frac{\pi}{6}$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{\pi}{3}$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right) = \frac{3\pi}{4}$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

99. $\arctan(\tan(\pi)) = 0$

100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$ is undefined

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right) = -\frac{\pi}{3}$

102. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

103. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right) = \frac{3\pi}{4}$

104. $\operatorname{arccot}(\cot(\pi))$ is undefined

105. $\operatorname{arccot}\left(\cot\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

106. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

107. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

108. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right) = \frac{2\pi}{3}$

109. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6}$

110. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$ is undefined.

111. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

112. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

113. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4}$

114. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$

115. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{2}$

116. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right) = -\frac{\pi}{6}$

117. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{11\pi}{12}$

118. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right) = -\frac{\pi}{8}$

119. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

120. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right) = \frac{4\pi}{3}$

121. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{7\pi}{6}$

122. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$ is undefined.

123. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

124. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

125. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$

126. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$

127. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = \frac{3\pi}{2}$

128. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right) = \frac{7\pi}{6}$

129. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{13\pi}{12}$

130. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right) = \frac{9\pi}{8}$

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right) = \frac{4}{5}$

133. $\sin(\arctan(-2)) = -\frac{2\sqrt{5}}{5}$

134. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right) = \frac{\sqrt{6}}{6}$

135. $\sin(\operatorname{arccsc}(-3)) = -\frac{1}{3}$

136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right) = \frac{12}{13}$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right) = \frac{\sqrt{2}}{4}$

138. $\cos(\operatorname{arccot}(3)) = \frac{3\sqrt{10}}{10}$

139. $\cos(\operatorname{arcsec}(5)) = \frac{1}{5}$

140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right) = -2$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right) = -\sqrt{3}$

142. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right) = \frac{4}{3}$

143. $\tan(\operatorname{arccot}(12)) = \frac{1}{12}$

144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right) = \frac{5}{12}$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \sqrt{3}$

146. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right) = 2$

147. $\cot(\arctan(0.25)) = 4$

148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{2\sqrt{3}}{3}$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right) = \frac{13}{5}$

150. $\sec(\arctan(10)) = \sqrt{101}$

151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right) = -\sqrt{11}$

152. $\csc(\operatorname{arccot}(9)) = \sqrt{82}$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{5}{3}$

154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right) = -\frac{\sqrt{13}}{2}$

155. $\sin\left(\arcsin\left(\frac{5}{13}\right) + \frac{\pi}{4}\right) = \frac{17\sqrt{2}}{26}$

156. $\cos(\operatorname{arcsec}(3) + \arctan(2)) = \frac{\sqrt{5} - 4\sqrt{10}}{15}$

157. $\tan\left(\arctan(3) + \arccos\left(-\frac{3}{5}\right)\right) = \frac{1}{3}$

158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right) = -\frac{24}{25}$

159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right) = \frac{120}{169}$

160. $\sin(2\arctan(2)) = \frac{4}{5}$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right) = \frac{7}{25}$

162. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right) = -\frac{527}{625}$

163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right) = \frac{2}{3}$

164. $\sin\left(\frac{\arctan(2)}{2}\right) = \sqrt{\frac{5-\sqrt{5}}{10}}$

165. $f(x) = \sin(\arccos(x)) = \sqrt{1-x^2}$ for $-1 \leq x \leq 1$

166. $f(x) = \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$ for all x

167. $f(x) = \tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

168. $f(x) = \sec(\arctan(x)) = \sqrt{1+x^2}$ for all x

169. $f(x) = \csc(\arccos(x)) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

170. $f(x) = \sin(2\arctan(x)) = \frac{2x}{x^2+1}$ for all x

171. $f(x) = \sin(2\arccos(x)) = 2x\sqrt{1-x^2}$ for $-1 \leq x \leq 1$

172. $f(x) = \cos(2\arctan(x)) = \frac{1-x^2}{1+x^2}$ for all x

173. $f(x) = \sin(\arccos(2x)) = \sqrt{1-4x^2}$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$

174. $f(x) = \sin\left(\arccos\left(\frac{x}{5}\right)\right) = \frac{\sqrt{25-x^2}}{5}$ for $-5 \leq x \leq 5$

175. $f(x) = \cos\left(\arcsin\left(\frac{x}{2}\right)\right) = \frac{\sqrt{4-x^2}}{2}$ for $-2 \leq x \leq 2$

176. $f(x) = \cos(\arctan(3x)) = \frac{1}{\sqrt{1+9x^2}}$ for all x

177. $f(x) = \sin(2\arcsin(7x)) = 14x\sqrt{1-49x^2}$ for $-\frac{1}{7} \leq x \leq \frac{1}{7}$

178. $f(x) = \sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right) = \frac{2x\sqrt{3-x^2}}{3}$ for $-\sqrt{3} \leq x \leq \sqrt{3}$

179. $f(x) = \cos(2\arcsin(4x)) = 1 - 32x^2$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$

180. $f(x) = \sec(\arctan(2x))\tan(\arctan(2x)) = 2x\sqrt{1+4x^2}$ for all x

181. $f(x) = \sin(\arcsin(x) + \arccos(x)) = 1$ for $-1 \leq x \leq 1$

182. $f(x) = \cos(\arcsin(x) + \arctan(x)) = \frac{\sqrt{1-x^2} - x^2}{\sqrt{1+x^2}}$ for $-1 \leq x \leq 1$

183. [12] $f(x) = \tan(2\arcsin(x)) = \frac{2x\sqrt{1-x^2}}{1-2x^2}$ for x in $\left(-1, -\frac{\sqrt{2}}{2}\right) \cup \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 1\right)$

184. $f(x) = \sin\left(\frac{1}{2}\arctan(x)\right) = \begin{cases} \sqrt{\frac{\sqrt{x^2+1}-1}{2\sqrt{x^2+1}}} & \text{for } x \geq 0 \\ -\sqrt{\frac{\sqrt{x^2+1}-1}{2\sqrt{x^2+1}}} & \text{for } x < 0 \end{cases}$

185. $\theta + \sin(2\theta) = \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2}$

186. $\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta) = \frac{1}{2}\arctan\left(\frac{x}{7}\right) - \frac{7x}{x^2+49}$

187. $4\tan(\theta) - 4\theta = \sqrt{x^2-16} - 4\operatorname{arcsec}\left(\frac{x}{4}\right)$

188. $\theta = \arcsin\left(\frac{7}{11}\right) + 2\pi k$ or $\theta = \pi - \arcsin\left(\frac{7}{11}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.6898, 2.4518$

[12] The equivalence for $x = \pm 1$ can be verified independently of the derivation of the formula, but Calculus is required to fully understand what is happening at those x values. You'll see what we mean when you work through the details of the identity for $\tan(2t)$. For now, we exclude $x = \pm 1$ from our answer.

189. $\theta = \arccos\left(-\frac{2}{9}\right) + 2\pi k$ or $\theta = -\arccos\left(-\frac{2}{9}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 1.7949, 4.4883$

190. $\theta = \pi + \arcsin(0.569) + 2\pi k$ or $\theta = 2\pi - \arcsin(0.569) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 3.7469, 5.6779$

191. $\theta = \arccos(0.117) + 2\pi k$ or $\theta = 2\pi - \arccos(0.117) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 1.4535, 4.8297$

192. $\theta = \arcsin(0.008) + 2\pi k$ or $\theta = \pi - \arcsin(0.008) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.0080, 3.1336$

193. $\theta = \arccos\left(\frac{359}{360}\right) + 2\pi k$ or $\theta = 2\pi - \arccos\left(\frac{359}{360}\right) + 2\pi k$, in $[0, 2\pi)$, $\theta \approx 0.0746, 6.2086$

194. $t = \arctan(117) + \pi k$, in $[0, 2\pi)$, $t \approx 1.56225, 4.70384$

195. $t = \arctan\left(-\frac{1}{12}\right) + \pi k$, in $[0, 2\pi)$, $t \approx 3.0585, 6.2000$

196. $t = \arccos\left(\frac{2}{3}\right) + 2\pi k$ or $t = 2\pi - \arccos\left(\frac{2}{3}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 0.8411, 5.4422$

197. $t = \pi + \arcsin\left(\frac{17}{90}\right) + 2\pi k$ or $t = 2\pi - \arcsin\left(\frac{17}{90}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 3.3316, 6.0932$

198. $t = \arctan\left(-\sqrt{10}\right) + \pi k$, in $[0, 2\pi)$, $t \approx 1.8771, 5.0187$

199. $t = \arcsin\left(\frac{3}{8}\right) + 2\pi k$ or $t = \pi - \arcsin\left(\frac{3}{8}\right) + 2\pi k$, in $[0, 2\pi)$, $t \approx 0.3844, 2.7572$

200. $x = \arccos\left(-\frac{7}{16}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{7}{16}\right) + 2\pi k$, in $[0, 2\pi)$, $x \approx 2.0236, 4.2596$

201. $x = \arctan(0.03) + \pi k$, in $[0, 2\pi)$, $x \approx 0.0300, 3.1716$

202. $x = \arcsin(0.3502) + 2\pi k$ or $x = \pi - \arcsin(0.3502) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.3578, 2.784$

203. $x = \pi + \arcsin(0.721) + 2\pi k$ or $x = 2\pi - \arcsin(0.721) + 2\pi k$, in $[0, 2\pi)$, $x \approx 3.9468, 5.4780$

204. $x = \arccos(0.9824) + 2\pi k$ or $x = 2\pi - \arccos(0.9824) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.1879, 6.0953$

205. $x = \arccos(-0.5637) + 2\pi k$ or $x = -\arccos(-0.5637) + 2\pi k$, in $[0, 2\pi)$, $x \approx 2.1697, 4.1135$

206. $x = \arctan(117) + \pi k$, in $[0, 2\pi)$, $x \approx 1.5622, 4.7038$

207. $x = \arctan(-0.6109) + \pi k$, in $[0, 2\pi)$, $x \approx 2.5932, 5.7348$

208. $f(t) = 5\sin(3t) + 12\cos(3t) = 13\sin\left(3t + \arcsin\left(\frac{12}{13}\right)\right) \approx 13\sin(3t + 1.1760)$

$f(t) = 5\sin(3t) + 12\cos(3t) = 13\cos\left(3t + \arcsin\left(-\frac{5}{13}\right)\right) \approx 13\cos(3t - 0.3948)$

209. $f(t) = 3\cos(2t) + 4\sin(2t) = 5\sin\left(2t + \arcsin\left(\frac{3}{5}\right)\right) \approx 5\sin(2t + 0.6435)$

$f(t) = 3\cos(2t) + 4\sin(2t) = 5\cos\left(2t + \arcsin\left(-\frac{4}{5}\right)\right) \approx 5\cos(2t - 0.9273)$

210. $f(t) = \cos(t) - 3\sin(t) = \sqrt{10}\sin\left(t + \arccos\left(-\frac{3\sqrt{10}}{10}\right)\right) \approx \sqrt{10}\sin(t + 2.8198)$

$f(t) = \cos(t) - 3\sin(t) = \sqrt{10}\cos\left(t + \arcsin\left(\frac{3\sqrt{10}}{10}\right)\right) \approx \sqrt{10}\cos(t + 1.2490)$

211. $f(t) = 7\sin(10t) - 24\cos(10t) = 25\sin\left(10t + \arcsin\left(-\frac{24}{25}\right)\right) \approx 25\sin(10t - 1.2870)$

$f(t) = 7\sin(10t) - 24\cos(10t) = 25\cos\left(10t + \pi + \arcsin\left(\frac{7}{25}\right)\right) \approx 25\cos(10t + 3.4254)$

212. $f(t) = -\cos(t) - 2\sqrt{2}\sin(t) = 3\sin\left(t + \pi + \arcsin\left(\frac{1}{3}\right)\right) \approx 3\sin(t + 3.4814)$

$f(t) = -\cos(t) - 2\sqrt{2}\sin(t) = 3\cos\left(t + \arccos\left(-\frac{1}{3}\right)\right) \approx 3\sin(t + 1.9106)$

213. $f(t) = 2\sin(t) - \cos(t) = \sqrt{5}\sin\left(t + \arcsin\left(-\frac{\sqrt{5}}{5}\right)\right) \approx \sqrt{5}\sin(t - 0.4636)$

$f(t) = 2\sin(t) - \cos(t) = \sqrt{5}\cos\left(t + \pi + \arcsin\left(\frac{2\sqrt{5}}{5}\right)\right) \approx \sqrt{5}\cos(t + 4.2487)$

214. $\left[-\frac{1}{5}, \frac{1}{5}\right]$

215. $\left[-\frac{1}{3}, 1\right]$

216. $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$

217. $(-\infty, -\sqrt{5}] \cup [-\sqrt{3}, \sqrt{3}] \cup [\sqrt{5}, \infty)$

218. $(-\infty, \infty)$

219. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

220. $\left(\frac{1}{2}, \infty\right)$

221. $\left[\frac{1}{2}, \infty\right)$

222. $\left(-\infty, -\frac{1}{12}\right] \cup \left[\frac{1}{12}, \infty\right)$

223. $(-\infty, -6] \cup [-4, \infty)$

224. $(-\infty, -2] \cup [2, \infty)$

225. $[0, \infty)$

1.4 Equations and Inequalities Involving the Circular Functions

In Sections **??**, **??** and most recently 1.3, we solved some basic equations involving the trigonometric functions. Below we summarize the techniques we've employed thus far. Note that we use the neutral letter 'u' as the argument of each circular function for generality.

Strategies for Solving Basic Equations Involving the Circular Functions

- To solve $\cos(u) = c$ or $\sin(u) = c$ for $-1 \leq c \leq 1$, first solve for u in the interval $[0, 2\pi)$ and add integer multiples of the period 2π. If $c < -1$ or of $c > 1$, there are no real solutions.
- To solve $\sec(u) = c$ or $\csc(u) = c$ for $c \leq -1$ or $c \geq 1$, convert to cosine or sine, respectively, and solve as above. If $-1 < c < 1$, there are no real solutions.
- To solve $\tan(u) = c$ for any real number c, first solve for u in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and add integer multiples of the period π.
- To solve $\cot(u) = c$ for $c \neq 0$, convert to tangent and solve as above. If $c = 0$, the solution to $\cot(u) = 0$ is $u = \frac{\pi}{2} + \pi k$ for integers k.

Using the above guidelines, we can comfortably solve $\sin(x) = \frac{1}{2}$ and find the solution $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for integers k. But how do we solve the related equation $\sin(3x) = \frac{1}{2}$?

Since this equation has the *form* $\sin(u) = \frac{1}{2}$, we know the solutions take the form $u = \frac{\pi}{6} + 2\pi k$ or $u = \frac{5\pi}{6} + 2\pi k$ for integers k. Since the argument of sine here is $3x$, we have $3x = \frac{\pi}{6} + 2\pi k$ or $3x = \frac{5\pi}{6} + 2\pi k$.

To solve for x, we divide both sides[1] of these equations by 3, and obtain $x = \frac{\pi}{18} + \frac{2\pi}{3}k$ or $x = \frac{5\pi}{18} + \frac{2\pi}{3}k$ for integers k. This is the technique employed in the example below.

Example 1.4.1. Solve the following equations and check your answers analytically. List the solutions which lie in the interval $[0, 2\pi)$ and verify them using a graphing utility.

1. $\cos(2\theta) = -\frac{\sqrt{3}}{2}$
2. $\csc\left(\frac{1}{3}\theta - \pi\right) = \sqrt{2}$
3. $\cot(3t) = 0$
4. $\sec^2(t) = 4$
5. $\tan\left(\frac{x}{2}\right) = -3$
6. $\sin(2x) = 0.87$

Solution.

1. The solutions to $\cos(u) = -\frac{\sqrt{3}}{2}$ are $u = \frac{5\pi}{6} + 2\pi k$ or $u = \frac{7\pi}{6} + 2\pi k$ for integers k.

   Since the argument of cosine here is 2θ, this means $2\theta = \frac{5\pi}{6} + 2\pi k$ or $2\theta = \frac{7\pi}{6} + 2\pi k$ for integers k. Solving for θ gives $\theta = \frac{5\pi}{12} + \pi k$ or $\theta = \frac{7\pi}{12} + \pi k$ for integers k.

   To check these answers analytically, we substitute them into the original equation. For any integer k:

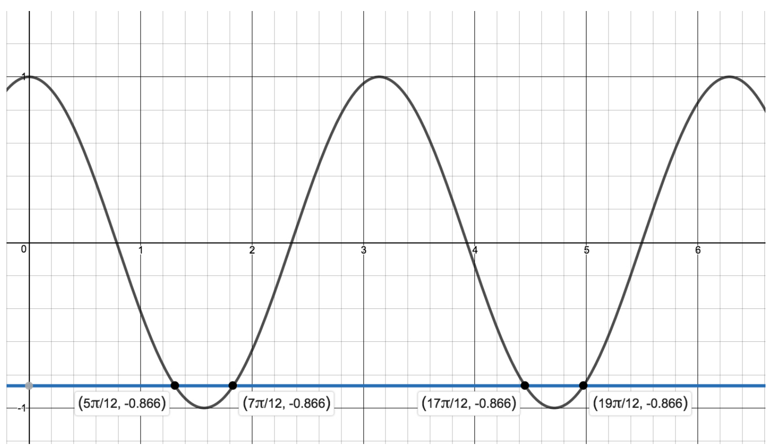
[1] Don't forget to divide the $2\pi k$ by 3 as well!

$$\begin{aligned}
\cos\left(2\left[\tfrac{5\pi}{12}+\pi k\right]\right) &= \cos\left(\tfrac{5\pi}{6}+2\pi k\right) \\
&= \cos\left(\tfrac{5\pi}{6}\right) \qquad \text{(the period of cosine is } 2\pi\text{)} \\
&= -\tfrac{\sqrt{3}}{2}
\end{aligned}$$

Similarly, we find $\cos\left(2\left[\frac{7\pi}{12}+\pi k\right]\right) = \cos\left(\frac{7\pi}{6}+2\pi k\right) = \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

To determine which of our solutions lie in $[0, 2\pi)$, we substitute integer values for k. The solutions we keep come from the values of $k = 0$ and $k = 1$ and are $\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}$ and $\frac{19\pi}{12}$.

Using a calculator, we graph $y = \cos(2\theta)$ and $y = -\frac{\sqrt{3}}{2}$ over $[0, 2\pi)$ and examine where these two graphs intersect to verify our answers.



$\boldsymbol{y = \cos(2\theta)}$ and $y = -\frac{\sqrt{3}}{2}$

2. Since this equation has the form $\csc(u) = \sqrt{2}$, we rewrite this as $\sin(u) = \frac{\sqrt{2}}{2}$ and find $u = \frac{\pi}{4} + 2\pi k$ or $u = \frac{3\pi}{4} + 2\pi k$ for integers k.

   Since the argument of cosecant here is $\left(\frac{1}{3}\theta - \pi\right)$, $\frac{1}{3}\theta - \pi = \frac{\pi}{4} + 2\pi k$ or $\frac{1}{3}\theta - \pi = \frac{3\pi}{4} + 2\pi k$.

   To solve $\frac{1}{3}\theta - \pi = \frac{\pi}{4} + 2\pi k$, we first add π to both sides to get $\frac{1}{3}\theta = \frac{\pi}{4} + 2\pi k + \pi$. A common error is to treat the '$2\pi k$' and 'π' terms as 'like' terms and try to combine them when they are not.

   We can, however, combine the 'π' and '$\frac{\pi}{4}$' terms to get $\frac{1}{3}\theta = \frac{5\pi}{4} + 2\pi k$.

   We now finish by multiplying both sides by 3 to get $\theta = 3\left(\frac{5\pi}{4} + 2\pi k\right) = \frac{15\pi}{4} + 6\pi k$, where k, as always, runs through the integers.

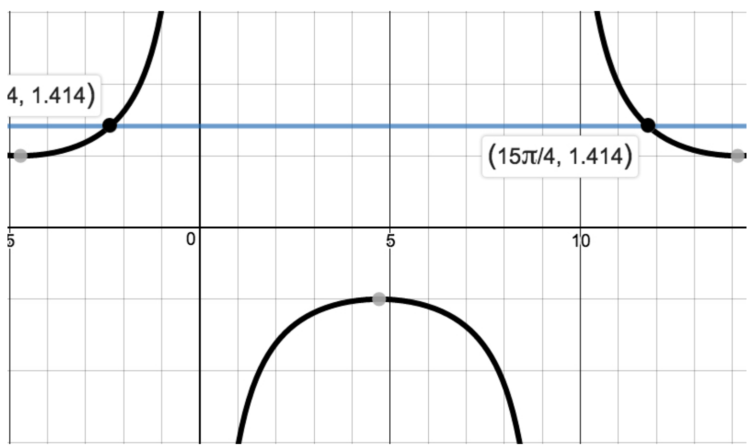
   Solving the other equation, $\frac{1}{3}\theta - \pi = \frac{3\pi}{4} + 2\pi k$ produces $\theta = \frac{21\pi}{4} + 6\pi k$ for integers k. To check the first family of answers, we substitute, combine line terms, and simplify.

$$\begin{aligned}
\csc\left(\tfrac{1}{3}\left[\tfrac{15\pi}{4}+6\pi k\right]-\pi\right) &= \csc\left(\tfrac{5\pi}{4}+2\pi k-\pi\right) \\
&= \csc\left(\tfrac{\pi}{4}+2\pi k\right) \\
&= \csc\left(\tfrac{\pi}{4}\right) && \text{(the period of cosecant is } 2\pi\text{)} \\
&= \sqrt{2}
\end{aligned}$$

The family $\theta = \frac{21\pi}{4} + 6\pi k$ checks similarly.

Despite having infinitely many solutions, we find that *none* of them lie in $[0, 2\pi)$.

To verify this graphically, we check that $y = \csc\left(\frac{1}{3}\theta - \pi\right)$ and $y = \sqrt{2}$ do not intersect at all over the interval $[0, 2\pi)$.



$\boldsymbol{y = \csc\left(\frac{1}{3}\theta - \pi\right)}$ and $y = \sqrt{2}$

3. Since $\cot(3t) = 0$ has the form $\cot(u) = 0$, we know $u = \frac{\pi}{2} + \pi k$, so, in this case, $3t = \frac{\pi}{2} + \pi k$ for integers k.

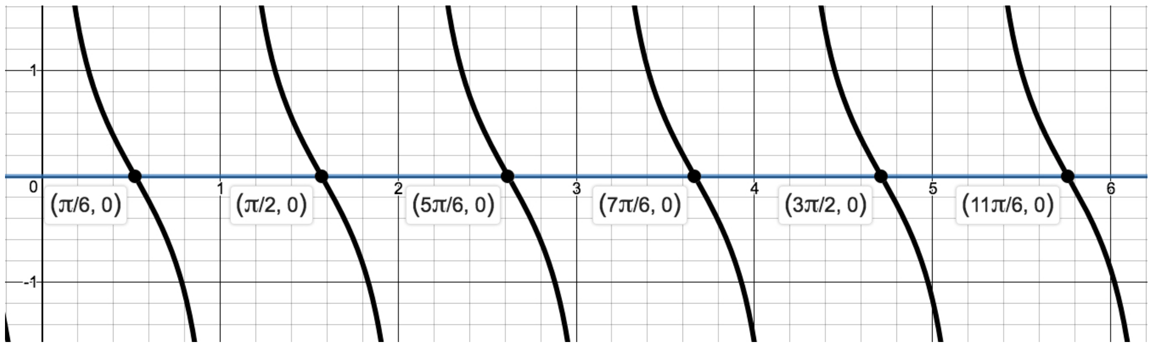
   Solving for t yields $t = \frac{\pi}{6} + \frac{\pi}{3}k$. Checking our answers, we get

$$\begin{aligned}
\cot\left(3\left[\tfrac{\pi}{6}+\tfrac{\pi}{3}k\right]\right) &= \cot\left(\tfrac{\pi}{2}+\pi k\right) \\
&= \cot\left(\tfrac{\pi}{2}\right) && \text{(the period of cotangent is } \pi\text{)} \\
&= 0
\end{aligned}$$

   As k runs through the integers, we obtain six answers, corresponding to $k = 0$ through $k = 5$, which lie in $[0, 2\pi)$: $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$ and $\frac{11\pi}{6}$.

   Graphing $y = \cot(3t)$ and $y = 0$ (the t-axis), we confirm our result.[2]

[2] On many calculators, there is no function button for cotangent. In that case, we would use the quotient identity and graph $y = \frac{\cos(3t)}{\sin(3t)}$ instead. The reader is invited to see what happens if we would graph $y = \frac{1}{\tan(3t)}$ instead.



$\boldsymbol{y = \cot(3t)}$ and $y = 0$

4. The complication in solving an equation like $\sec^2(t) = 4$ comes not from the argument of secant, which is just t, but rather, the fact the secant is being squared: $\sec^2(t) = (\sec(t))^2 = 4$.

   To get this equation to look like one of the forms listed on page 80, we extract square roots to get $\sec(t) = \pm 2$. Converting to cosines, we have $\cos(t) = \pm\frac{1}{2}$.

   For $\cos(t) = \frac{1}{2}$, we get $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for integers k. For $\cos(t) = -\frac{1}{2}$, we get $t = \frac{2\pi}{3} + 2\pi k$ or $t = \frac{4\pi}{3} + 2\pi k$ for integers k.

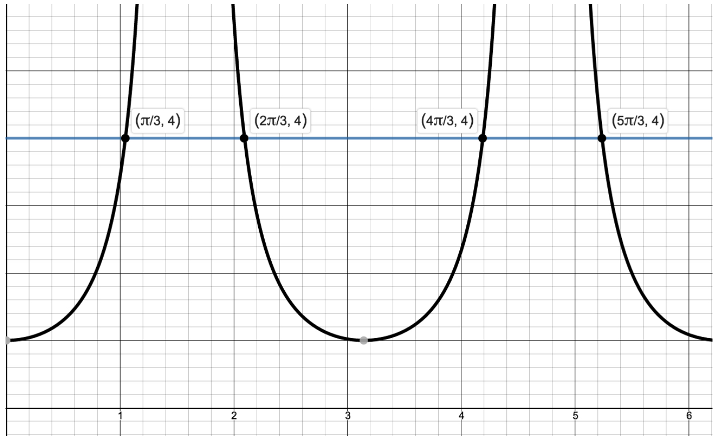
   If we take a step back and think of these families of solutions geometrically, we see we are finding the measures of all angles with a reference angle of $\frac{\pi}{3}$. As a result, these solutions can be combined and we may write our solutions as $t = \frac{\pi}{3} + \pi k$ and $t = \frac{2\pi}{3} + \pi k$ for integers k.

   To check the first family of solutions, we note that, depending on the integer k, $\sec\left(\frac{\pi}{3} + \pi k\right)$ doesn't always equal $\sec\left(\frac{\pi}{3}\right)$. It is true, though, that for all integers k, $\sec\left(\frac{\pi}{3} + \pi k\right) = \pm\sec\left(\frac{\pi}{3}\right) = \pm 2$. (Can you show this?) Hence, checking our first family of solutions gives:

$$\begin{aligned}
\sec^2\left(\frac{\pi}{3} + \pi k\right) &= \left(\pm\sec\left(\frac{\pi}{3}\right)\right)^2 \\
&= (\pm 2)^2 \\
&= 4
\end{aligned}$$

   The check for the family of solutions $t = \frac{2\pi}{3} + \pi k$ is similar.

   The solutions which lie in $[0, 2\pi)$ come from the values $k = 0$ and $k = 1$, namely $t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$. Graphing $y = (\sec(t))^2$ and $y = 4$ confirms our results.



$\boldsymbol{y = (\sec(t))^2}$ and $y = 4$

5. The equation $\tan\left(\frac{x}{2}\right) = -3$ has the form $\tan(u) = -3$, whose solution is $u = \arctan(-3) + \pi k$.

   Hence, $\frac{x}{2} = \arctan(-3) + \pi k$, so $x = 2\arctan(-3) + 2\pi k$ for integers k. To check, we note

$$\begin{array}{rcll} \tan\left(\frac{2\arctan(-3)+2\pi k}{2}\right) & = & \tan\left(\arctan(-3) + \pi k\right) & \\ & = & \tan\left(\arctan(-3)\right) & \text{(the period of tangent is } \pi\text{)} \\ & = & -3 & \text{(See Theorem 1.15)} \end{array}$$

   To determine which of our answers lie in the interval $[0, 2\pi)$, we first need to get an idea of the value of $2\arctan(-3)$. While we could easily find an approximation using a calculator,[3] we proceed analytically, as is our custom.

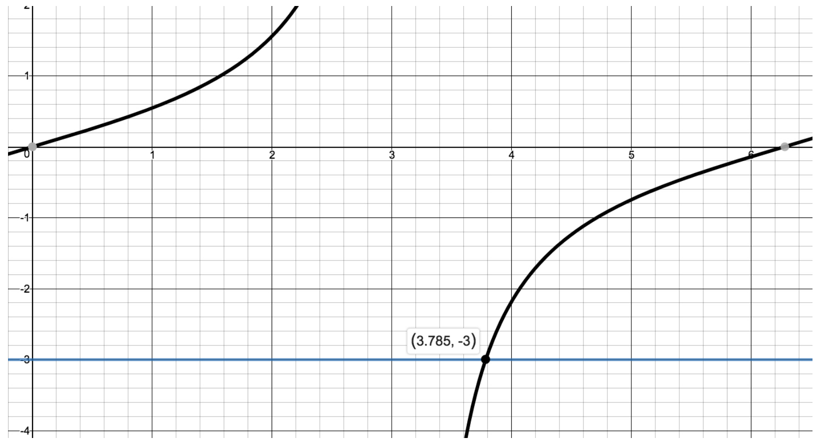
   To get started, we note that since $-3 < 0$, it $-\frac{\pi}{2} < \arctan(-3) < 0$. Hence, $-\pi < 2\arctan(-3) < 0$. With regard to our solutions, $x = 2\arctan(-3) + 2\pi k$, we see for $k = 0$, we get $x = 2\arctan(-3) < 0$, so we discard this answer and all answers $x = 2\arctan(-3) + 2\pi k$ where $k < 0$.

   Next, we turn our attention to $k = 1$ and get $x = 2\arctan(-3) + 2\pi$. Starting with the inequality $-\pi < 2\arctan(-3) < 0$, we add through 2π and get $\pi < 2\arctan(-3) + 2\pi < 2\pi$. This means $x = 2\arctan(-3) + 2\pi$ lies in $[0, 2\pi)$.

   Advancing k to 2 produces $x = 2\arctan(-3) + 4\pi$. Once again, we get from $-\pi < 2\arctan(-3) < 0$ that $3\pi < 2\arctan(-3) + 4\pi < 4\pi$. Since this is outside the interval of interest, $[0, 2\pi)$, we discard $x = 2\arctan(-3) + 4\pi$ and all solutions of the form $x = 2\arctan(-3) + 2\pi k$ for $k > 2$.

   Graphically, $y = \tan\left(\frac{x}{2}\right)$ and $y = -3$ intersect only once on $[0, 2\pi)$ at $x = 2\arctan(-3) + 2\pi \approx 3.785$.

[3] Your instructor will let you know if you should abandon the analytic route at this point and use your calculator.



$\boldsymbol{y = \tan\left(\frac{x}{2}\right)}$ and $y = -3$

6. To solve $\sin(2x) = 0.87$, we first note that it has the form $\sin(u) = 0.87$, which has the family of solutions $u = \arcsin(0.87) + 2\pi k$ or $u = \pi - \arcsin(0.87) + 2\pi k$ for integers k.

   Since the argument of sine here is $2x$, we get $2x = \arcsin(0.87) + 2\pi k$ or $2x = \pi - \arcsin(0.87) + 2\pi k$ which gives $x = \frac{1}{2}\arcsin(0.87) + \pi k$ or $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$ for integers k. To check,

$$\begin{aligned} \sin\left(2\left[\tfrac{1}{2}\arcsin(0.87) + \pi k\right]\right) &= \sin\left(\arcsin(0.87) + 2\pi k\right) \\ &= \sin\left(\arcsin(0.87)\right) && \text{(the period of sine is } 2\pi\text{)} \\ &= 0.87 && \text{(See Theorem 1.14)} \end{aligned}$$

   For the family $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$, we get

$$\begin{aligned} \sin\left(2\left[\tfrac{\pi}{2} - \tfrac{1}{2}\arcsin(0.87) + \pi k\right]\right) &= \sin\left(\pi - \arcsin(0.87) + 2\pi k\right) \\ &= \sin\left(\pi - \arcsin(0.87)\right) && \text{(the period of sine is } 2\pi\text{)} \\ &= \sin\left(\arcsin(0.87)\right) && (\sin(\pi - t) = \sin(t)) \\ &= 0.87 && \text{(See Theorem 1.14)} \end{aligned}$$

   To determine which of these solutions lie in $[0, 2\pi)$, we first need to get an idea of the value of $x = \frac{1}{2}\arcsin(0.87)$. Once again, we could use the calculator, but we adopt an analytic route here.

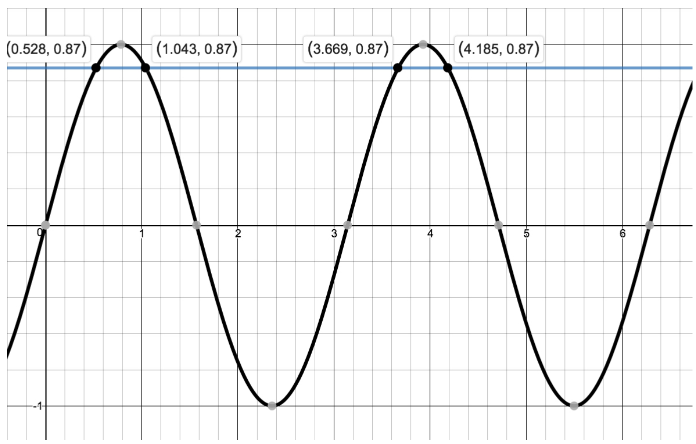
   By definition, $0 < \arcsin(0.87) < \frac{\pi}{2}$ so that multiplying through by $\frac{1}{2}$ gives us $0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4}$.

   Starting with the family of solutions $x = \frac{1}{2}\arcsin(0.87) + \pi k$, we use the same kind of arguments as in our solution to number 5 above and find only the solutions corresponding to $k = 0$ and $k = 1$ lie in $[0, 2\pi)$: $x = \frac{1}{2}\arcsin(0.87)$ and $x = \frac{1}{2}\arcsin(0.87) + \pi$.

Next, we move to the family $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$ for integers k. Here, we need to get a better estimate of $\frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$. From the inequality $0 < \frac{1}{2}\arcsin(0.87) < \frac{\pi}{4}$, we first multiply through by -1 and then add $\frac{\pi}{2}$ to get $\frac{\pi}{2} > \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) > \frac{\pi}{4}$, or $\frac{\pi}{4} < \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) < \frac{\pi}{2}$.

Proceeding with the usual arguments, we find the only solutions which lie in $[0, 2\pi)$ correspond to $k = 0$ and $k = 1$, namely $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87)$ and $x = \frac{3\pi}{2} - \frac{1}{2}\arcsin(0.87)$.

All told, we have found four solutions to $\sin(2x) = 0.87$ in $[0, 2\pi)$: $x = \frac{1}{2}\arcsin(0.87) \approx 0.528$, $x = \frac{1}{2}\arcsin(0.87) + \pi \approx 3.669$, $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) \approx 1.043$ and $x = \frac{3\pi}{2} - \frac{1}{2}\arcsin(0.87) \approx 4.185$. By graphing $y = \sin(2x)$ and $y = 0.87$, we confirm our results.



$\boldsymbol{y = \sin(2x)}$ and $y = 0.87$

□

If one looks closely at the equations and solutions in Example 1.4.1, an interesting relationship evolves between the frequency of the circular function involved in the equation and how many solutions one can expect in the interval $[0, 2\pi)$. This relationship is explored in Exercise 108.

Each of the problems in Example 1.4.1 featured one circular function. If an equation involves two different circular functions or if the equation contains the same circular function but with different arguments, we will need to employ identities and Algebra to reduce the equation to the same form as those given on page 80. We demonstrate these techniques in the following example.

Example 1.4.2. Solve the following equations and list the solutions which lie in the interval $[0, 2\pi)$. Verify your solutions on $[0, 2\pi)$ graphically.

1. $3\sin^3(\theta) = \sin^2(\theta)$
2. $\sec^2(\theta) = \tan(\theta) + 3$
3. $\cos(2t) = 3\cos(t) - 2$
4. $\cos(3t) = 2 - \cos(t)$
5. $\cos(3x) = \cos(5x)$
6. $\sin(2x) = \sqrt{3}\cos(x)$

7. $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$

8. $\cos(x) - \sqrt{3}\sin(x) = 2$

Solution.

1. One approach to solving $3\sin^3(\theta) = \sin^2(\theta)$ begins with dividing both sides by $\sin^2(\theta)$. Doing so, however, assumes that $\sin^2(\theta) \neq 0$ which means we risk losing solutions.

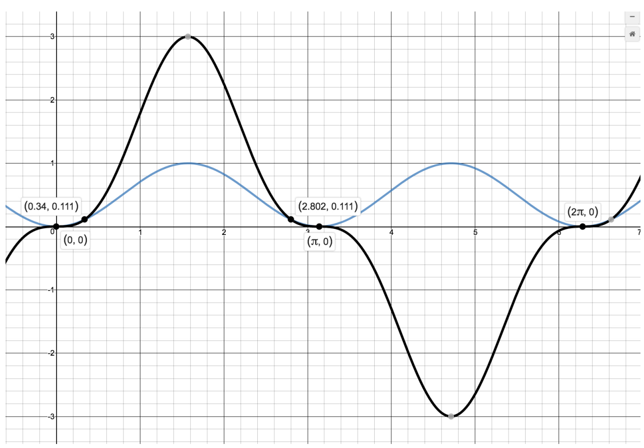
   Instead, we take a cue from Chapter **??** (since what we have here is a polynomial equation in terms of $\sin(\theta)$) and gather all the nonzero terms on one side and factor:

$$\begin{array}{rcll} 3\sin^3(\theta) & = & \sin^2(\theta) & \\ 3\sin^3(\theta) - \sin^2(\theta) & = & 0 & \\ \sin^2(\theta)(3\sin(\theta) - 1) & = & 0 & \text{Factor out } \sin^2(\theta) \text{ from both terms.} \end{array}$$

   We get $\sin^2(\theta) = 0$ or $3\sin(\theta) - 1 = 0$, so $\sin(\theta) = 0$ or $\sin(\theta) = \frac{1}{3}$. The solution to $\sin(\theta) = 0$ is $\theta = \pi k$, with $\theta = 0$ and $\theta = \pi$ being the two solutions which lie in $[0, 2\pi)$.

   To solve $\sin(\theta) = \frac{1}{3}$, we use the arcsine function to get $\theta = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ or $\theta = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k. We find the two solutions here which lie in $[0, 2\pi)$ to be $\theta = \arcsin\left(\frac{1}{3}\right) \approx 0.34$ and $\theta = \pi - \arcsin\left(\frac{1}{3}\right) \approx 2.80$.

   To check graphically, we plot $y = 3(\sin(\theta))^3$ and $y = (\sin(\theta))^2$ and find the θ-coordinates of the intersection points of these two curves.[4] (Some extra zooming may be required near $\theta = 0$ and $\theta = \pi$ to verify that these two curves do in fact intersect four times.)



$y = 3(\sin(\theta))^3$ and $y = (\sin(\theta))^2$

2. We see immediately in the equation $\sec^2(\theta) = \tan(\theta) + 3$ that there are two different circular functions present, so we look for an identity to express both sides in terms of the same function.

   We use the Pythagorean Identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ to exchange $\sec^2(\theta)$ for tangents. What results is a 'quadratic in disguise:'[5]

[4]Note that we do *not* list $\theta = 2\pi$ as part of the solution over the interval $[0, 2\pi)$ since 2π is not in $[0, 2\pi)$.

[5]See Section **??** for a review of this concept.

$$\begin{aligned}
\sec^2(\theta) &= \tan(\theta) + 3 \\
1 + \tan^2(\theta) &= \tan(\theta) + 3 \quad \text{(Since } \sec^2(\theta) = 1 + \tan^2(\theta).\text{)} \\
\tan^2(\theta) - \tan(\theta) - 2 &= 0 \\
u^2 - u - 2 &= 0 \qquad \text{Let } u = \tan(\theta). \\
(u+1)(u-2) &= 0
\end{aligned}$$

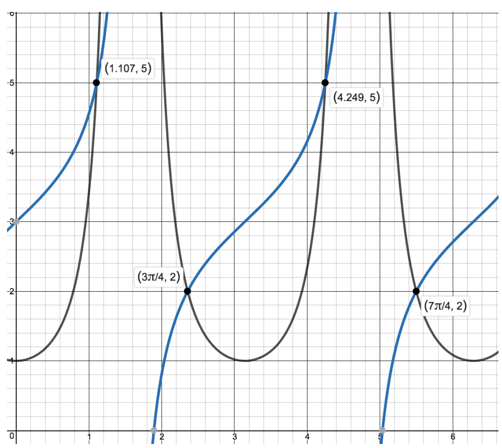
This gives $u = -1$ or $u = 2$. Since $u = \tan(\theta)$, we have $\tan(\theta) = -1$ or $\tan(\theta) = 2$.

From $\tan(\theta) = -1$, we get $\theta = -\frac{\pi}{4} + \pi k$ for integers k. To solve $\tan(\theta) = 2$, we employ the arctangent function and get $\theta = \arctan(2) + \pi k$ for integers k.

From the first set of solutions, we get $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$ as our answers which lie in $[0, 2\pi)$.

Using the same sort of argument we saw in Example 1.4.1, we get $\theta = \arctan(2) \approx 1.107$ and $\theta = \pi + \arctan(2) \approx 4.249$ as answers from our second set of solutions which lie in $[0, 2\pi)$.

We verify our solutions below graphically.



$y = (\sec(\theta))^2$ and $y = \tan(\theta) + 3$

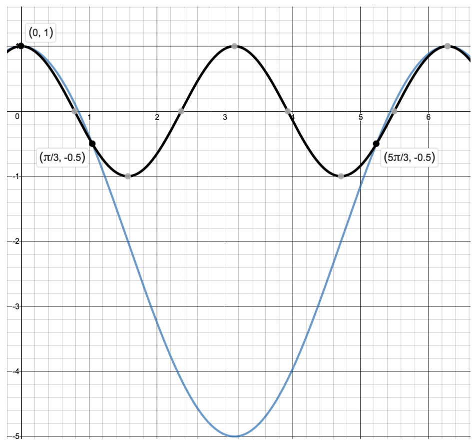
3. The good news is that in the equation $\cos(2t) = 3\cos(t) - 2$, we have the same circular function, cosine, throughout. The bad news is that we have different arguments, $2t$ and t.

   Using the double angle identity $\cos(2t) = 2\cos^2(t) - 1$ results in another quadratic in disguise:'

$$\begin{aligned}
\cos(2t) &= 3\cos(t) - 2 \\
2\cos^2(t) - 1 &= 3\cos(t) - 2 \quad \text{(Since } \cos(2t) = 2\cos^2(t) - 1.\text{)} \\
2\cos^2(t) - 3\cos(t) + 1 &= 0 \\
2u^2 - 3u + 1 &= 0 \qquad \text{Let } u = \cos(t). \\
(2u-1)(u-1) &= 0
\end{aligned}$$

We get $u = \frac{1}{2}$ or $u = 1$, so $\cos(t) = \frac{1}{2}$ or $\cos(t) = 1$. Solving $\cos(t) = \frac{1}{2}$, we get $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for integers k. From $\cos(t) = 1$, we get $t = 2\pi k$ for integers k.

The answers which lie in $[0, 2\pi)$ are $t = 0$, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$. Graphing $y = \cos(2t)$ and $y = 3\cos(t) - 2$, we find that the curves intersect in three places on $[0, 2\pi)$ and confirm our results.



$\boldsymbol{y = \cos(2t)}$ and $y = 3\cos(t) - 2$

4. To solve $\cos(3t) = 2 - \cos(t)$, we take a cue from the previous problem and look for an identity to rewrite $\cos(3t)$ in terms of $\cos(t)$.

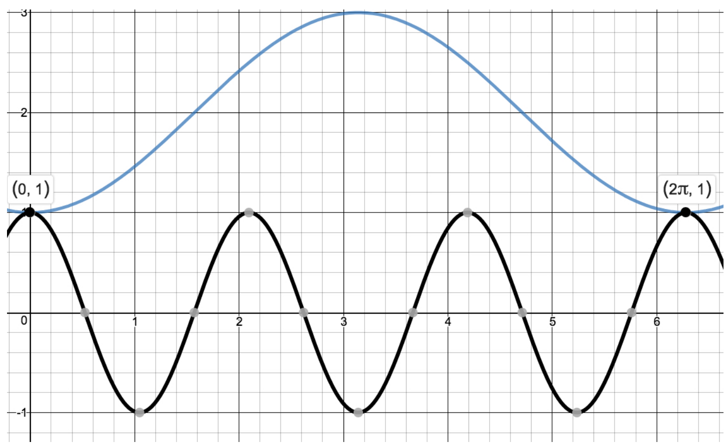
   From Example 1.2.3, number 4, we know that $\cos(3t) = 4\cos^3(t) - 3\cos(t)$. This transforms the equation into a polynomial in terms of $\cos(t)$.

$$\begin{array}{rcll} \cos(3t) & = & 2 - \cos(t) & \\ 4\cos^3(t) - 3\cos(t) & = & 2 - \cos(t) & \\ 2\cos^3(t) - 2\cos(t) - 2 & = & 0 & \\ 4u^3 - 2u - 2 & = & 0 & \text{Let } u = \cos(t). \end{array}$$

   Using what we know from Chapter **??**, we factor $4u^3 - 2u - 2$ as $(u - 1)\left(4u^2 + 4u + 2\right)$ and set each factor equal to 0.

   We get either $u - 1 = 0$ or $4u^2 + 2u + 2 = 0$, and since the discriminant of the latter is negative, the only real solution to $4u^3 - 2u - 2 = 0$ is $u = 1$.

   Since $u = \cos(t)$, we get $\cos(t) = 1$, so $t = 2\pi k$ for integers k. The only solution which lies in $[0, 2\pi)$ is $t = 0$. Our graph below confirms this.



$\boldsymbol{y = \cos(3t)}$ and $y = 2 - \cos(t)$

5. While we could approach solving the equation $\cos(3x) = \cos(5x)$ in the same manner as we did the previous two problems, we choose instead to showcase the utility of the Sum to Product Identities.[6]

   From $\cos(3x) = \cos(5x)$, we get $\cos(5x) - \cos(3x) = 0$, and it is the presence of 0 on the right hand side that indicates a switch to a product would be a good move.[7]

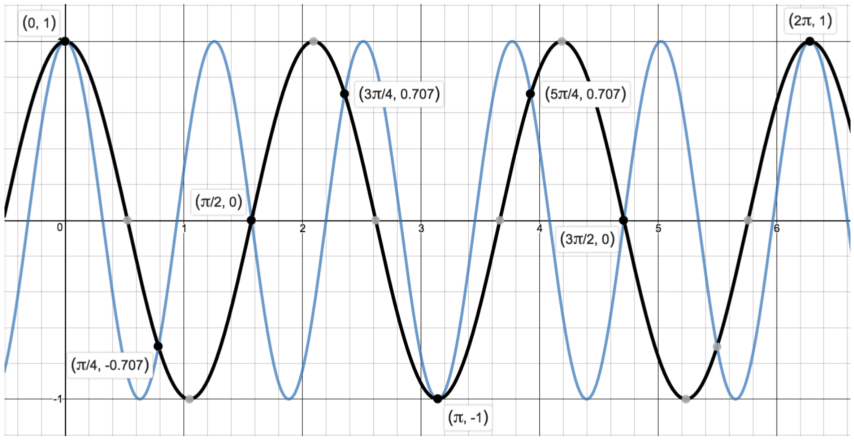
   Using Theorem 1.13, we rewrite $\cos(5x) - \cos(3x)$ as $-2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) = -2\sin(4x)\sin(x)$. Hence, our original equation $\cos(3x) = \cos(5x)$ is equivalent to $-2\sin(4x)\sin(x) = 0$.

   From $-2\sin(4x)\sin(x) = 0$, we get either $\sin(4x) = 0$ or $\sin(x) = 0$. Solving $\sin(4x) = 0$ gives $x = \frac{\pi}{4}k$ for integers k, and the solution to $\sin(x) = 0$ is $x = \pi k$ for integers k.

   The second set of solutions is contained in the first set of solutions,[8] so our final solution to $\cos(5x) = \cos(3x)$ is $x = \frac{\pi}{4}k$ for integers k.

   There are eight of these answers which lie in $[0, 2\pi)$: $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}$ and $\frac{7\pi}{4}$. Our plot of the graphs of $y = \cos(3x)$ and $y = \cos(5x)$ below (after some careful zooming) bears this out.

[6] We invite the reader to try the 'polynomial approach' used in the previous problem to see what difficulties are encountered.
[7] Since a *product* equalling zero means, necessarily, one or both *factors* is 0. See page **??**.
[8] As always, when in doubt, write it out!



***y* = cos(3*x*)** and *y* = *cos*(5*x*)

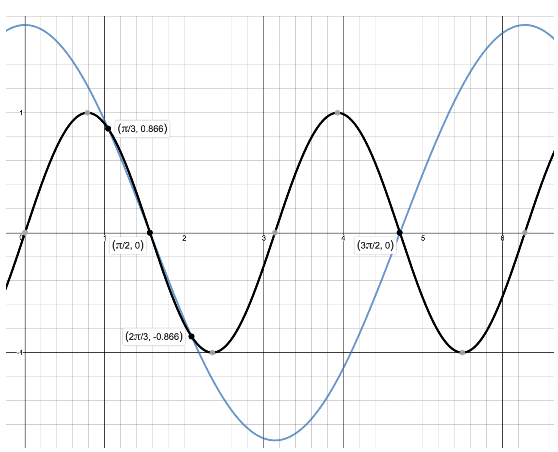
6. In the equation $\sin(2x) = \sqrt{3}\cos(x)$, we not only have different circular functions involved, but we also have different arguments to contend with.

   Using the double angle identity $\sin(2x) = 2\sin(x)\cos(x)$ makes all of the arguments the same and we proceed to gather all of the nonzero terms on one side of the equation and factor.

$$\begin{array}{rcll} \sin(2x) & = & \sqrt{3}\cos(x) & \\ 2\sin(x)\cos(x) & = & \sqrt{3}\cos(x) & (\text{Since } \sin(2x) = 2\sin(x)\cos(x).) \\ 2\sin(x)\cos(x) - \sqrt{3}\cos(x) & = & 0 & \\ \cos(x)(2\sin(x) - \sqrt{3}) & = & 0 & \end{array}$$

   We get $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$. From $\cos(x) = 0$, we obtain $x = \frac{\pi}{2} + \pi k$ for integers k. From $\sin(x) = \frac{\sqrt{3}}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for integers k.

   The answers which lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$, as verified graphically below.

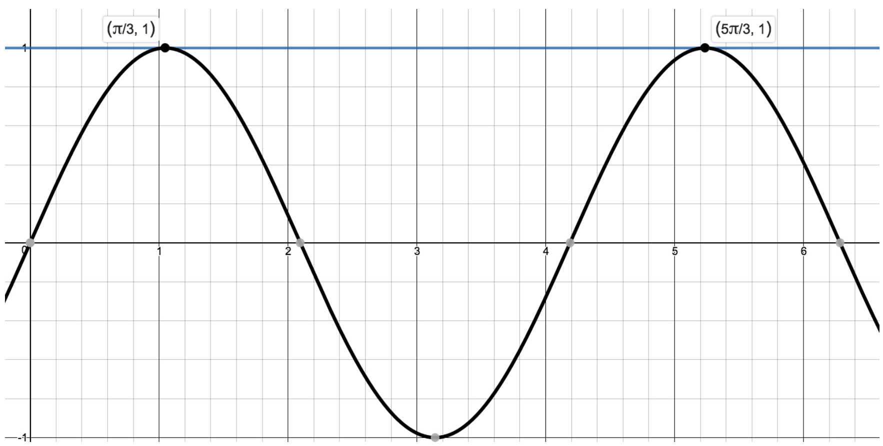


$y = \sin(2x)$ and $y = \sqrt{3}\cos(x)$

7. Unlike the previous problem, there seems to be no quick way to get the circular functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$.

   If we stare at it long enough, however, we realize that the left hand side is the expanded form of the sum formula for $\sin\left(x + \frac{x}{2}\right)$. Hence, our original equation is equivalent to $\sin\left(\frac{3}{2}x\right) = 1$.

   Solving, we find $x = \frac{\pi}{3} + \frac{4\pi}{3}k$ for integers k. Two of these solutions lie in $[0, 2\pi)$: $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$. Graphing $y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)$ and $y = 1$ validates our solutions.



$y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)$ and $y = 1$

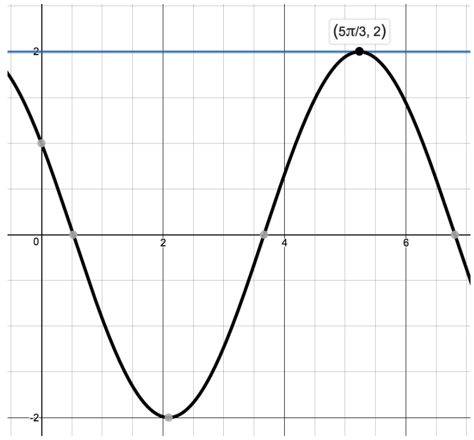
8. With the absence of double angles or squares, there doesn't seem to be much we can do with the equation $\cos(x) - \sqrt{3}\sin(x) = 2$.

   However, since the frequencies of the sine and cosine terms are the same, we can rewrite the left hand side of this equation as a sinusoid.

   To fit $f(x) = \cos(x) - \sqrt{3}\sin(x)$ to the form $A\sin(\omega t + \phi) + B$, we use what we learned in Example 1.2.7 and find $A = 2$, $B = 0$, $\omega = 1$ and $\phi = \frac{5\pi}{6}$.

   Hence, we can rewrite the equation $\cos(x) - \sqrt{3}\sin(x) = 2$ as $2\sin\left(x + \frac{5\pi}{6}\right) = 2$, or $\sin\left(x + \frac{5\pi}{6}\right) = 1$. Solving, we get $x = -\frac{\pi}{3} + 2\pi k$ for integers k.

   Only one of our solutions, $x = \frac{5\pi}{3}$, which corresponds to $k = 1$, lies in $[0, 2\pi)$. Geometrically, we see that $y = \cos(x) - \sqrt{3}\sin(x)$ and $y = 2$ intersect just once, supporting our answer.



$y = \cos(x) - \sqrt{3}\sin(x)$ and $y = 2$

An alternative way to solve this problem is to *introduce* squares in order to exchange sines and cosines using a Pythagorean Identity.

From $\cos(x) - \sqrt{3}\sin(x) = 2$ we get $\sqrt{3}\sin(x) = \cos(x) - 2$ so that $\left(\sqrt{3}\sin(x)\right)^2 = (\cos(x) - 2)^2$. Simplifying, we get: $3\sin^2(x) = \cos^2(x) - 4\cos(x) + 4$.

Substituting $\sin^2(x) = 1 - \cos^2(x)$, we get $3(1 - \cos^2(x)) = \cos^2(x) - 4\cos(x) + 4$ which results in the quadratic equation: $4\cos^2(x) - 4\cos(x) + 1 = 0$.

Letting $u = \cos(x)$, we get $4u^2 - 4u + 1 = 0$ or $(2u - 1)^2 = 0$. We get $u = \cos(x) = \frac{1}{2}$. Solving $\cos(x) = \frac{1}{2}$ gives $x = \frac{\pi}{3} + 2\pi k$ as well as $x = \frac{5\pi}{3} + 2\pi k$ for integers, k.

Of these two families, only solutions of the form $x = \frac{5\pi}{3} + 2\pi k$ checks in our original equation.[9] We leave it the reader to verify this representation of solutions to $\cos(x) - \sqrt{3}\sin(x) = 2$ is equivalent to the one we found previously. □

We repeat here the advice given when solving systems of nonlinear equations in section **??** – when it comes to solving equations involving the circular functions, it helps to just try something.

Next, we focus on solving inequalities involving the circular functions. Since these functions are continuous on their domains, we may use the sign diagram technique we've used in the past to solve the inequalities.[10]

Example 1.4.3. Solve the following inequalities on $[0, 2\pi)$. Express your answers using interval notation and verify your answers graphically.

1. $2\sin(t) \leq 1$
2. $\sin(2x) > \cos(x)$
3. $\tan(x) \geq 3$

[9] We've seen how squaring both sides can lead to extraneous solutions in Section **??** and Chapter **??**. Here, squaring both sides admits an entire *family* of extraneous solutions.

[10] See pages **??**, **??**, **??**, as well as Examples **??** and **??** for a review of this technique, as needed.

Solution.

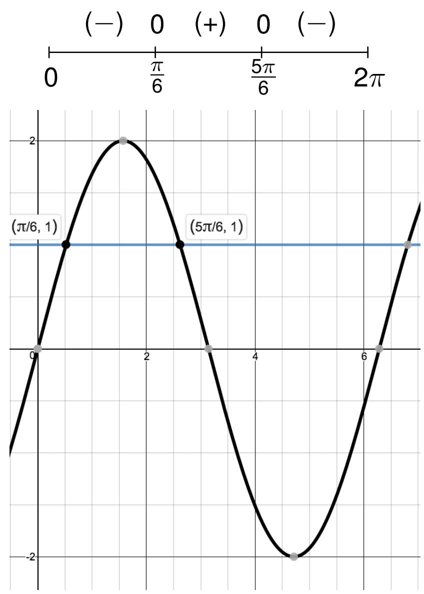
1. We begin solving $2\sin(t) \leq 1$ by collecting all of the terms on one side of the equation and zero on the other to get $2\sin(t) - 1 \leq 0$.

   Next, we let $f(t) = 2\sin(t) - 1$ and note that our original inequality is equivalent to solving $f(t) \leq 0$. We now look to see where, if ever, f is undefined and where $f(t) = 0$.

   Since the domain of f is all real numbers, we can immediately set about finding the zeros of f. Solving $f(t) = 0$, we have $2\sin(t) - 1 = 0$ or $\sin(t) = \frac{1}{2}$. The solutions here are $t = \frac{\pi}{6} + 2\pi k$ and $t = \frac{5\pi}{6} + 2\pi k$ for integers k. Since we are restricting our attention to $[0, 2\pi)$, only $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$ are of concern.

   Next, we choose test values in $[0, 2\pi)$ other than the zeros and determine if f is positive or negative there. For $t = 0$ we have $f(0) = -1$, for $t = \frac{\pi}{2}$ we get $f\left(\frac{\pi}{2}\right) = 1$ and for $t = \pi$ we get $f(\pi) = -1$.

   Since our original inequality is equivalent to $f(t) \leq 0$, we are looking for where the function is negative $(-)$ or 0, and we get the intervals $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right)$. We can confirm our answer graphically by seeing where the graph of $y = 2\sin(t)$ crosses or is below the graph of $y = 1$.



***y* = 2 sin(*t*)** and $y = 1$

2. We first rewrite $\sin(2x) > \cos(x)$ as $\sin(2x) - \cos(x) > 0$ and let $f(x) = \sin(2x) - \cos(x)$.

   Our original inequality is thus equivalent to $f(x) > 0$. The domain of f is all real numbers, so we can advance to finding the zeros of f.

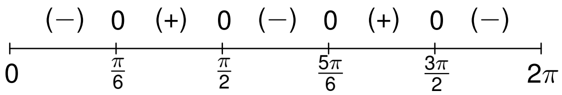
   Setting $f(x) = 0$ yields $\sin(2x) - \cos(x) = 0$, which, by way of the double angle identity for sine, becomes $2\sin(x)\cos(x) - \cos(x) = 0$ or $\cos(x)(2\sin(x) - 1) = 0$.

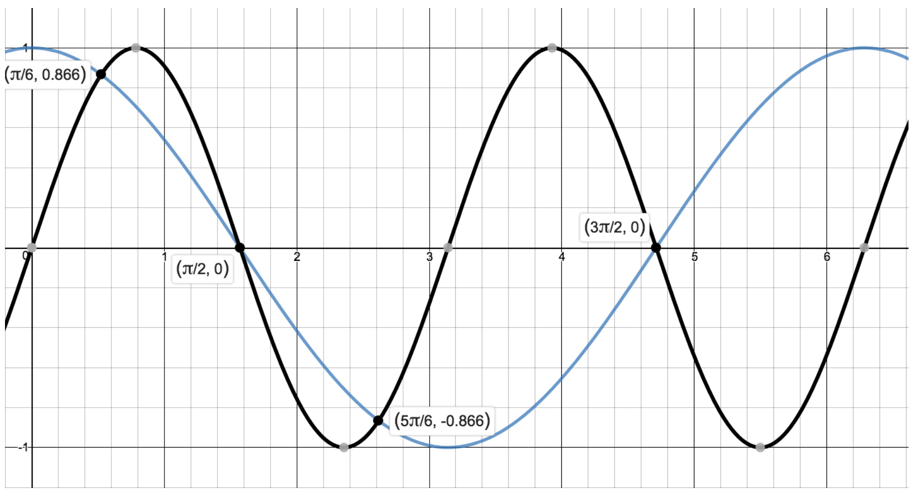
From $\cos(x) = 0$, we get $x = \frac{\pi}{2} + \pi k$ for integers k of which only $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ lie in $[0, 2\pi)$.

For $2\sin(x) - 1 = 0$, we get $\sin(x) = \frac{1}{2}$ which gives $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for integers k. Of those, only $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ lie in $[0, 2\pi)$.

Choosing test values, we get: for $x = 0$ we find $f(0) = -1$; when $x = \frac{\pi}{4}$ we get $f\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$; for $x = \frac{3\pi}{4}$ we get $f\left(\frac{3\pi}{4}\right) = -1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-2}{2}$; when $x = \pi$ we have $f(\pi) = 1$, and lastly, for $x = \frac{7\pi}{4}$ we get $f\left(\frac{7\pi}{4}\right) = -1 - \frac{\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2}$.

We see $f(x) > 0$ on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, so this is our answer. Geometrically, we see the graph of $y = \sin(2x)$ is indeed above the graph of $y = \cos(x)$ on those intervals.





$\boldsymbol{y = \sin(2x)}$ and $y = \cos(x)$

3. Proceeding as above, we rewrite $\tan(x) \geq 3$ as $\tan(x) - 3 \geq 0$ and let $f(x) = \tan(x) - 3$.

   We note that on $[0, 2\pi)$, f is undefined at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, so those values will need the usual disclaimer on the sign diagram.[11]

   Moving along to zeros, solving $f(x) = \tan(x) - 3 = 0$ requires the arctangent function. We find $x = \arctan(3) + \pi k$ for integers k and of these, only $x = \arctan(3)$ and $x = \arctan(3) + \pi$ lie in $[0, 2\pi)$. Since $3 > 0$, we know $0 < \arctan(3) < \frac{\pi}{2}$ which allows us to position these zeros correctly on the sign diagram.

   To choose test values, we begin with $x = 0$ and find $f(0) = -3$. Finding a convenient test value in the interval $\left(\arctan(3), \frac{\pi}{2}\right)$ is a bit more challenging. Since the arctangent function is increasing and

[11]See page **??** for a discussion of the non-standard character known as the interrobang.

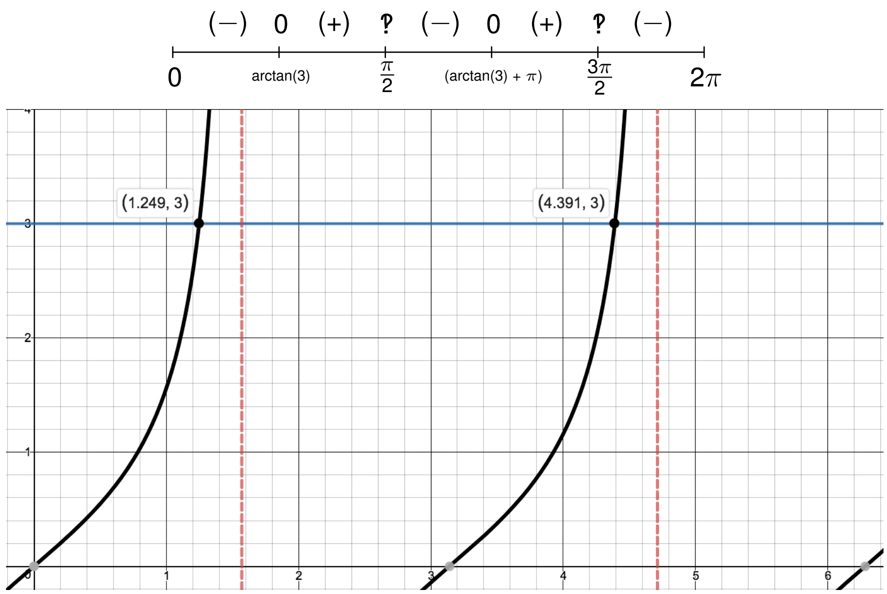
is bounded above by $\frac{\pi}{2}$, the number $x = \arctan(117)$ is guaranteed[12] to lie between $\arctan(3)$ and $\frac{\pi}{2}$. We see that $f(\arctan(117)) = \tan(\arctan(117)) - 3 = 114$.

For our next test value, we take $x = \pi$ and find $f(\pi) = -3$, which brings us to finding a test value in the interval $\left(\arctan(3) + \pi, \frac{3\pi}{2}\right)$.

From $\arctan(3) < \arctan(117) < \frac{\pi}{2}$ we get $\arctan(3) + \pi < \arctan(117) + \pi < \frac{3\pi}{2}$ by adding π through the inequality. We find $f(\arctan(117) + \pi) = \tan(\arctan(117) + \pi) - 3 = \tan(\arctan(117)) - 3 = 114$.

For our last test value, we choose $x = \frac{7\pi}{4}$ and find $f\left(\frac{7\pi}{4}\right) = -4$.

Since we want $f(x) \geq 0$, we see that our answer is $\left[\arctan(3), \frac{\pi}{2}\right) \cup \left[\arctan(3) + \pi, \frac{3\pi}{2}\right)$. Using the graphs of $y = \tan(x)$ and $y = 3$, we see when the graph of the former is above (or meets) the graph of the latter. (Note, $\arctan(3) \approx 1.249$ and $\arctan(3) + \pi \approx 4.391$.)



***y* = tan(*x*)** and $y = 3$

□

Our next example puts solving equations and inequalities to good use – finding domains of functions.

Example 1.4.4. Express the domain of the following functions using extended interval notation.[13]

1. $f(x) = \csc\left(2x + \frac{\pi}{3}\right)$
2. $f(t) = \dfrac{\sin(t)}{2\cos(t) - 1}$
3. $f(x) = \sqrt{1 - \cot(x)}$

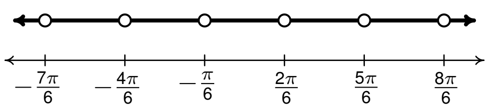
[12] We could have chosen any value $\arctan(t)$ where $t > 3$.

[13] See Section **??** for details about this notation.

Solution.

1. To find the domain of $f(x) = \csc\left(2x + \frac{\pi}{3}\right)$, we rewrite f in terms of sine as $f(x) = \frac{1}{\sin\left(2x+\frac{\pi}{3}\right)}$. Since the sine function is defined everywhere, our only concern comes from zeros in the denominator.

   Solving $\sin\left(2x + \frac{\pi}{3}\right) = 0$, we get $x = -\frac{\pi}{6} + \frac{\pi}{2}k$ for integers k. In set-builder notation, our domain is $\left\{x \mid x \neq -\frac{\pi}{6} + \frac{\pi}{2}k \text{ for integers } k\right\}$. To help visualize the domain, we follow the old mantra 'When in doubt, write it out!' We get $\left\{x \mid x \neq -\frac{\pi}{6}, \frac{2\pi}{6}, -\frac{4\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}, \frac{8\pi}{6}, \ldots\right\}$, where we have kept the denominators 6 throughout to help see the pattern. Graphing the situation on a number line, we have

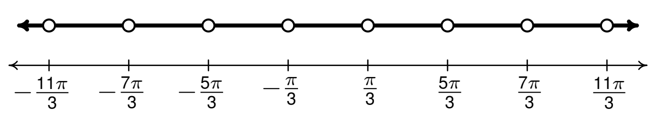
   Proceeding as in Section **??**, we let x_k denote the kth number excluded from the domain and we have $x_k = -\frac{\pi}{6} + \frac{\pi}{2}k = \frac{(3k-1)\pi}{6}$ for integers k. The intervals which comprise the domain are of the form $(x_k, x_{k+1}) = \left(\frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6}\right)$ as k runs through the integers. Using extended interval notation, we have that the domain is

   $$\bigcup_{k=-\infty}^{\infty} \left(\frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6}\right)$$

   We can check our answer by substituting in values of k to see that it matches our diagram.

2. Since the domains of $\sin(t)$ and $\cos(t)$ are all real numbers, the only concern when finding the domain of $f(t) = \frac{\sin(t)}{2\cos(t)-1}$ is division by zero so we set the denominator equal to zero and solve.

   From $2\cos(t) - 1 = 0$ we get $\cos(t) = \frac{1}{2}$ so $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{5\pi}{3} + 2\pi k$ for integers k. Using set-builder notation, the domain is $\left\{t \mid t \neq \frac{\pi}{3} + 2\pi k \text{ and } t \neq \frac{5\pi}{3} + 2\pi k \text{ for integers } k\right\}$. Writing this out, we find the domain is $\left\{t \mid t \neq \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \pm\frac{11\pi}{3}, \ldots\right\}$, so we have

   Unlike the previous example, we have *two* different families of points to consider, and we present two ways of dealing with this kind of situation. One way is to generalize what we did in the previous example and use the formulas we found in our domain work to describe the intervals.

   To that end, we let $a_k = \frac{\pi}{3} + 2\pi k = \frac{(6k+1)\pi}{3}$ and $b_k = \frac{5\pi}{3} + 2\pi k = \frac{(6k+5)\pi}{3}$ for integers k. The goal now is to write the domain in terms of the a's an b's. We find $a_0 = \frac{\pi}{3}$, $a_1 = \frac{7\pi}{3}$, $a_{-1} = -\frac{5\pi}{3}$, $a_2 = \frac{13\pi}{3}$, $a_{-2} = -\frac{11\pi}{3}$, $b_0 = \frac{5\pi}{3}$, $b_1 = \frac{11\pi}{3}$, $b_{-1} = -\frac{\pi}{3}$, $b_2 = \frac{17\pi}{3}$ and $b_{-2} = -\frac{7\pi}{3}$.

   Hence, in terms of the a's and b's, our domain is

$$\ldots (a_{-2}, b_{-2}) \cup (b_{-2}, a_{-1}) \cup (a_{-1}, b_{-1}) \cup (b_{-1}, a_0) \cup (a_0, b_0) \cup (b_0, a_1) \cup (a_1, b_1) \cup \ldots$$

If we group these intervals in pairs, $(a_{-2}, b_{-2}) \cup (b_{-2}, a_{-1})$, $(a_{-1}, b_{-1}) \cup (b_{-1}, a_0)$, $(a_0, b_0) \cup (b_0, a_1)$ and so forth, we see a pattern emerge of the form $(a_k, b_k) \cup (b_k, a_{k+1})$ for integers k so that our domain can be written as

$$\bigcup_{k=-\infty}^{\infty} (a_k, b_k) \cup (b_k, a_{k+1}) = \bigcup_{k=-\infty}^{\infty} \left(\frac{(6k+1)\pi}{3}, \frac{(6k+5)\pi}{3} \right) \cup \left(\frac{(6k+5)\pi}{3}, \frac{(6k+7)\pi}{3} \right)$$

A second approach to the problem exploits the periodic nature of f. Since $\cos(t)$ and $\sin(t)$ have period 2π, it's not too difficult to show the function f repeats itself every 2π units.[14] This means if we can find a formula for the domain on an interval of length 2π, we can express the entire domain by translating our answer left and right on the t-axis by adding integer multiples of 2π.

One such interval that arises naturally from our domain work is $\left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$. The portion of the domain here is $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right) \cup \left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$. Adding integer multiples of 2π, we obtain the family of intervals: $\left(\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k\right) \cup \left(\frac{5\pi}{3} + 2\pi k, \frac{7\pi}{3} + 2\pi k\right)$ for integers k. We leave it to the reader to show that getting common denominators leads to our previous answer.

3. To find the domain of $f(x) = \sqrt{1 - \cot(x)}$, we first note that, due to the presence of the $\cot(x)$ term, $x \neq \pi k$ for integers k.

   Next, we recall that for the square root to be defined, we need $1 - \cot(x) \geq 0$. Unlike the inequalities we solved in Example 1.4.3, we are not restricted here to a given interval. Our strategy is to solve this inequality over $(0, \pi)$ (the same interval which generates a fundamental cycle of cotangent) and then add integer multiples of the period, in this case, π.

   We let $g(x) = 1 - \cot(x)$ and set about making a sign diagram for g over the interval $(0, \pi)$ to find where $g(x) \geq 0$. We note that g is undefined for $x = \pi k$ for integers k, in particular, at the endpoints of our interval $x = 0$ and $x = \pi$.

   Next, we look for the zeros of g. Solving $g(x) = 0$, we get $\cot(x) = 1$ or $x = \frac{\pi}{4} + \pi k$ for integers k and only one of these, $x = \frac{\pi}{4}$, lies in $(0, \pi)$. Choosing the test values $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$, we get $g\left(\frac{\pi}{6}\right) = 1 - \sqrt{3}$, and $g\left(\frac{\pi}{2}\right) = 1$. We construct the sign diagram for g over the interval $(0, \pi)$ below:

‽	$(-)$	0	$(+)$	‽
0		$\frac{\pi}{4}$		π

   We find $g(x) \geq 0$ on $\left[\frac{\pi}{4}, \pi\right)$. Adding multiples of the period we get our solution to consist of the intervals $\left[\frac{\pi}{4} + \pi k, \pi + \pi k\right) = \left[\frac{(4k+1)\pi}{4}, (k+1)\pi\right)$.

[14] This doesn't necessarily mean the period of f is 2π. The tangent function is comprised of sine and cosine, but its period is half theirs. The reader is invited to investigate the period of f.

Using extended interval notation, we have our final answer:

$$\bigcup_{k=-\infty}^{\infty} \left[\frac{(4k+1)\pi}{4}, (k+1)\pi \right)$$

□

In our next example, we solve equations and inequalities involving the *inverse* circular functions.

Example 1.4.5. Solve the following equations and inequalities analytically. Check your answers using a graphing utility.

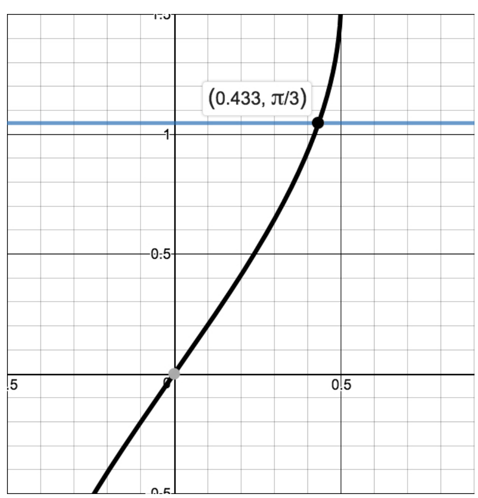
1. $\arcsin(2x) = \frac{\pi}{3}$
2. $4\arccos(t) - 3\pi = 0$
3. $3\,\mathrm{arcsec}(2x-1) + \pi = 2\pi$
4. $4\arctan^2(t) - 3\pi\arctan(t) - \pi^2 = 0$
5. $\pi^2 - 4\arccos^2(x) < 0$
6. $4\,\mathrm{arccot}(3t) > \pi$

Solution.

1. To solve $\arcsin(2x) = \frac{\pi}{3}$, we first note that $\frac{\pi}{3}$ is in the range of the arcsine function (so a solution exists!) Next, we exploit the inverse property of sine and arcsine from Theorem 1.14

$$\begin{aligned} \arcsin(2x) &= \tfrac{\pi}{3} \\ \sin(\arcsin(2x)) &= \sin\left(\tfrac{\pi}{3}\right) \\ 2x &= \tfrac{\sqrt{3}}{2} \qquad \text{Since } \sin(\arcsin(u)) = u \\ x &= \tfrac{\sqrt{3}}{4} \end{aligned}$$

Below we see the graphs of $y = \arcsin(2x)$ and $y = \frac{\pi}{3}$, intersect at $x = \frac{\sqrt{3}}{4} \approx 0.4430$.

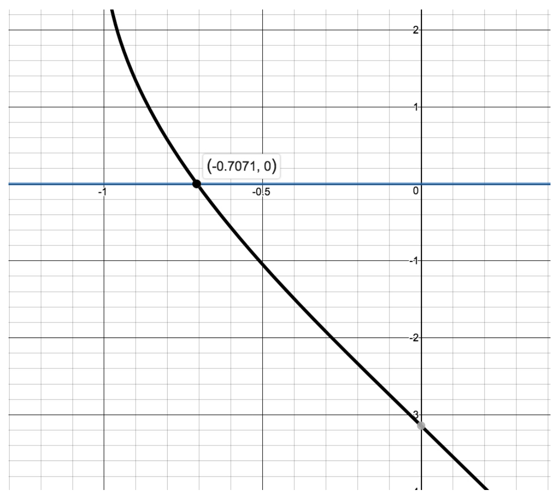


***y* = arcsin(2*x*)** and $y = \frac{\pi}{3}$

2. Our first step in solving $4\arccos(t) - 3\pi = 0$ is to isolate the arccosine. We get $\arccos(t) = \frac{3\pi}{4}$. Since $\frac{3\pi}{4}$ is in the range of arccosine, we may apply Theorem 1.14

$$
\begin{aligned}
\arccos(t) &= \frac{3\pi}{4} \\
\cos(\arccos(t)) &= \cos\left(\frac{3\pi}{4}\right) \\
t &= -\frac{\sqrt{2}}{2} \qquad \text{Since } \cos(\arccos(u)) = u
\end{aligned}
$$

Below we see the graph of $y = 4\arccos(t) - 3\pi$ crosses $y = 0$ at $t = -\frac{\sqrt{2}}{2} \approx -0.7071$.

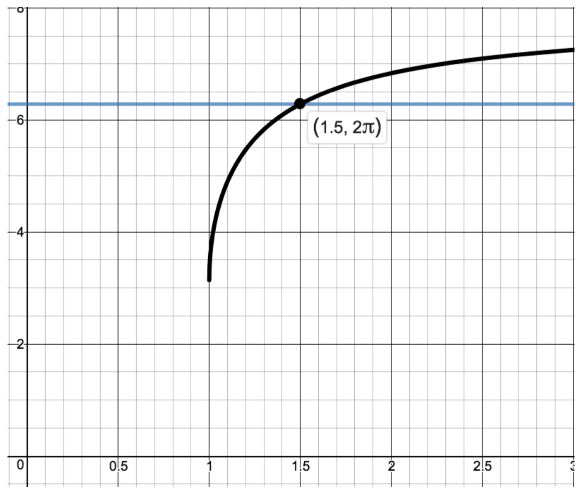


$y = 4\arccos(t) - 3\pi$ and $y = 0$

3. From $3\,\mathrm{arcsec}(2x - 1) + \pi = 2\pi$, we get $\mathrm{arcsec}(2x - 1) = \frac{\pi}{3}$. Regardless of how the range of arcsecant is chosen, since $0 \leq \frac{\pi}{3} < \frac{\pi}{2}$, both Theorems 1.16 1.17, apply:

$$
\begin{aligned}
\mathrm{arcsec}(2x-1) &= \frac{\pi}{3} \\
\sec(\mathrm{arcsec}(2x-1)) &= \sec\left(\frac{\pi}{3}\right) \\
2x - 1 &= 2 \qquad \text{Since } \sec(\mathrm{arcsec}(u)) = u \\
x &= \frac{3}{2}
\end{aligned}
$$

Below we see the graphs of $y = 3\,\mathrm{arcsec}(2x - 1) + \pi$ and $y = 2\pi$ intersect at $x = \frac{3}{2} = 1.5$.



$y = 3\,\mathrm{arcsec}(2x - 1) + \pi$ and $y = 2\pi$

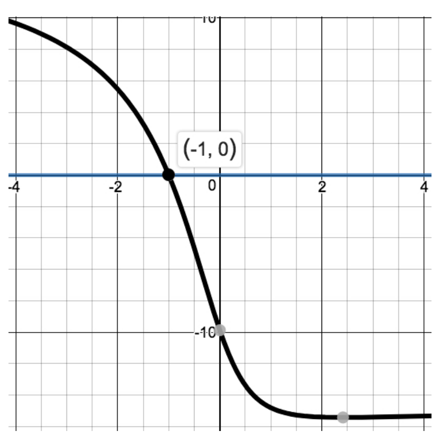
4. With the presence of both $\arctan^2(t)$ ($= (\arctan(t))^2$) and $\arctan(t)$, we substitute $u = \arctan(t)$ to reveal a quadratic in disguise: $4u^2 - 3\pi u - \pi^2 = 0$.

   Factoring, (don't let the π throw you!) we get $(4u + \pi)(u - \pi) = 0$, so $u = \arctan(t) = -\frac{\pi}{4}$ or $u = \arctan(t) = \pi$.

   Since $-\frac{\pi}{4}$ is in the range of arctangent, but π is not, we only get solutions from the first equation. Using Theorem 1.15, we get

$$\begin{array}{rcll} \arctan(t) & = & -\frac{\pi}{4} & \\ \tan(\arctan(t)) & = & \tan\left(-\frac{\pi}{4}\right) & \\ t & = & -1 & \text{Since } \tan(\arctan(u)) = u. \end{array}$$

   We verify this result graphically below.



$$\boldsymbol{y = 4\arctan^2(t) - 3\pi\arctan(t) - \pi^2} \text{ and } y = 0.$$

5. Since the inverse circular functions are continuous on their domains, we can solve inequalities featuring these functions using sign diagrams.

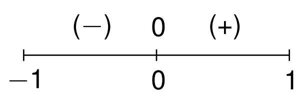
   Since all of the nonzero terms of $\pi^2 - 4\arccos^2(x) < 0$ are on one side of the inequality, we let $f(x) = \pi^2 - 4\arccos^2(x)$ and note the domain of f is limited by the $\arccos(x)$ to $[-1, 1]$.

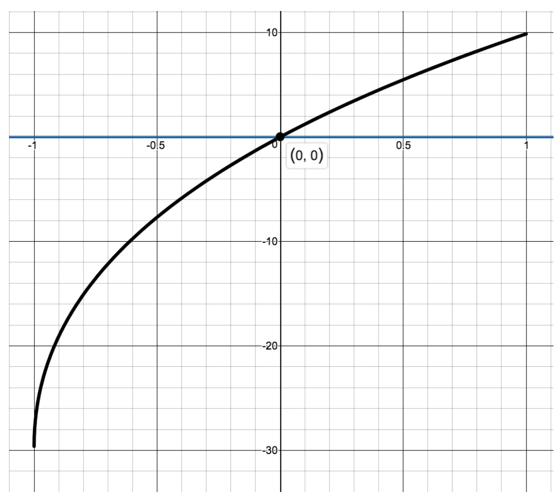
   Next, we find the zeros of f by setting $f(x) = \pi^2 - 4\arccos^2(x) = 0$. We get $\arccos(x) = \pm\frac{\pi}{2}$, and since the range of arccosine is $[0, \pi]$, we focus our attention on $\arccos(x) = \frac{\pi}{2}$.

   Using Theorem 1.14, we get $x = \cos\left(\frac{\pi}{2}\right) = 0$ as our only zero which breaks our domain $[-1, 1]$ into two test intervals: $[-1, 0)$ and $(0, 1]$.

   Choosing test values $x = \pm 1$, we get $f(-1) = -3\pi^2 < 0$ and $f(1) = \pi^2 > 0$. Since we are looking for where $f(x) = \pi^2 - 4\arccos^2(x) < 0$, our answer is $[-1, 0)$.

   Geometrically, we find the graph of $y = \pi^2 - 4\arccos^2(x)$ is below $y = 0$ (the x-axis) on $[-1, 0)$.





$\boldsymbol{y = \pi^2 - 4\arccos^2(x)}$ and $y = 0$

6. As in the previous problem, we will use a sign diagram to solve $4\,\text{arccot}(3t) > \pi$. Our first step is to rewrite the inequality as $4\,\text{arccot}(3t) - \pi > 0$.

   We let $f(t) = 4\,\text{arccot}(3t) - \pi$, and find the domain of f is all real numbers, $(-\infty, \infty)$.

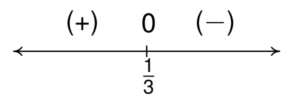
   To find the zeros of f, we set $f(t) = 4\,\text{arccot}(3t) - \pi = 0$ and solve. We get $\text{arccot}(3t) = \frac{\pi}{4}$, and since $\frac{\pi}{4}$ is in the range of arccotangent, we may apply Theorem 1.15 and solve

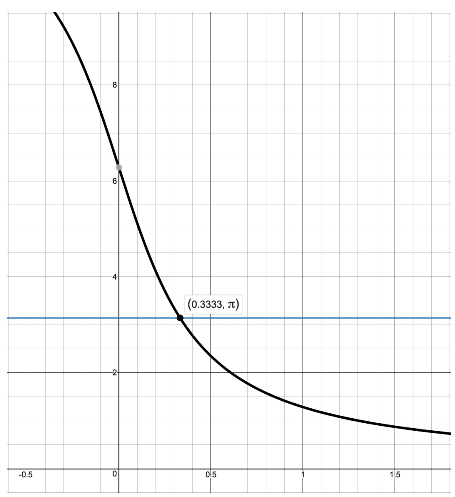
$$\begin{aligned} \operatorname{arccot}(3t) &= \tfrac{\pi}{4} \\ \cot(\operatorname{arccot}(3t)) &= \cot\left(\tfrac{\pi}{4}\right) \\ 3t &= 1 && \text{Since } \cot(\operatorname{arccot}(u)) = u. \\ t &= \tfrac{1}{3} \end{aligned}$$

Next, we make a sign diagram for f. Since the domain of f is all real numbers, and there is only one zero of f, $t = \frac{1}{3}$, we have two test intervals, $\left(-\infty, \frac{1}{3}\right)$ and $\left(\frac{1}{3}, \infty\right)$.

Ideally, we wish to find test values t in these intervals so that $\operatorname{arccot}(3t)$ corresponds to one of our oft-used 'common' angles. After a bit of computation,[15] we choose $t = 0$ for the interval $\left(-\infty, \frac{1}{3}\right)$ and $t = \frac{\sqrt{3}}{3}$ for the interval $\left(\frac{1}{3}, \infty\right)$.

We find $f(0) = \pi > 0$ and $f\left(\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{3} < 0$. Since we are looking for where $f(t) > 0$, we get our answer $\left(-\infty, \frac{1}{3}\right)$. Graphically, we see the graph of $y = 4\operatorname{arccot}(3t)$ is above the horizontal line $y = \pi$ on $\left(-\infty, \frac{1}{3}\right) = \left(-\infty, 0.\overline{3}\right)$.





$y = 4 \operatorname{arccot}(3t)$ and $y = \pi$

□

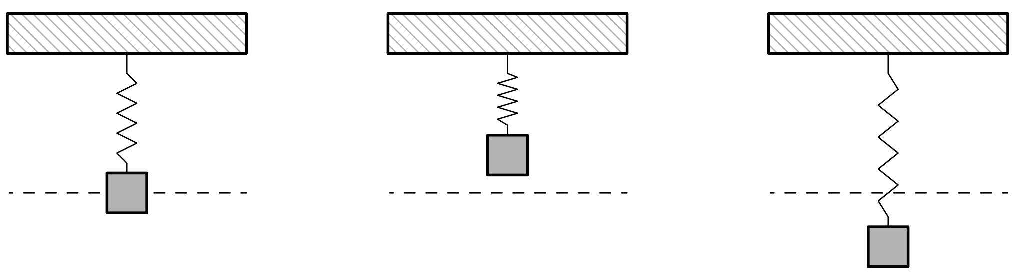
[15] Set $3t$ equal to the cotangents of the 'common angles' and choose accordingly.

1.4.1 Harmonic Motion

One of the major applications of the circular functions (sinusoids in particular!) in Science and Engineering is the study of **harmonic motion**, We close this chapter with a brief foray into this topic since it pulls together many important concepts from both Chapters **??** and **??**. The equations for harmonic motion can be used to describe a wide range of phenomena, from the motion of an object on a spring, to the response of an electronic circuit. In this subsection, we restrict our attention to modeling a simple spring system. Before we jump into the Mathematics, there are some Physics terms and concepts we need to discuss.

In Physics, 'mass' is defined as a measure of an object's resistance to straight-line motion whereas 'weight' is the amount of force (pull) gravity exerts on an object. An object's mass cannot change,[16] while its weight could change. An object which weighs 6 pounds on the surface of the Earth would weigh 1 pound on the surface of the Moon, but its mass is the same in both places. In the English system of units, 'pounds' (lbs.) is a measure of force (weight), and the corresponding unit of mass is the 'slug'. In the SI system, the unit of force is 'Newtons' (N) and the associated unit of mass is the 'kilogram' (kg).

We convert between mass and weight using the formula[17] $w = mg$. Here, w is the weight of the object, m is the mass and g is the acceleration due to gravity. In the English system, $g = 32 \frac{\text{feet}}{\text{second}^2}$, and in the SI system, $g = 9.8 \frac{\text{meters}}{\text{second}^2}$. Hence, on Earth a *mass* of 1 slug *weighs* 32 lbs. and a *mass* of 1 kg *weighs* 9.8 N.[18] Suppose we attach an object with mass m to a spring as depicted below.



$x(t) = 0$ at the equilibrium position

$x(t) < 0$ above the equilibrium position

$x(t) > 0$ below the equilibrium position

The weight of the object will stretch the spring. The system is said to be in 'equilibrium' when the weight of the object is perfectly balanced with the restorative force of the spring. How far the spring stretches to reach equilibrium depends on the spring's 'spring constant'. Usually denoted by the letter k, the spring constant relates the force F applied to the spring to the amount d the spring stretches in accordance with Hooke's Law[19] $F = kd$.

If the object is released above or below the equilibrium position, or if the object is released with an upward or downward velocity, the object will bounce up and down on the end of the spring until some external force

[16] Well, assuming the object isn't subjected to relativistic speeds . . .

[17] This is a consequence of Newton's Second Law of Motion $F = ma$ where F is force, m is mass and a is acceleration. In our present setting, the force involved is weight which is caused by the acceleration due to gravity.

[18] Note that 1 pound = 1 $\frac{\text{slug foot}}{\text{second}^2}$ and 1 Newton = 1 $\frac{\text{kg meter}}{\text{second}^2}$.

[19] Look familiar? We saw Hooke's Law in Section **??**.

stops it. If we let $x(t)$ denote the object's displacement from the equilibrium position at time t, then $x(t) = 0$ means the object is at the equilibrium position, $x(t) < 0$ means the object is *above* the equilibrium position, and $x(t) > 0$ means the object is *below* the equilibrium position. The function $x(t)$ is called the 'equation of motion' of the object.[20]

If we ignore all other influences on the system except gravity and the spring force, then Physics tells us that gravity and the spring force will battle each other forever and the object will oscillate indefinitely. In this case, we describe the motion as 'free' (meaning there is no external force causing the motion) and 'undamped' (meaning we ignore friction caused by surrounding medium, which in our case is air).

The following theorem, which comes from Differential Equations, gives $x(t)$ as a function of the mass m of the object, the spring constant k, the initial displacement x_0 of the object and initial velocity v_0 of the object. As with $x(t)$, $x_0 = 0$ means the object is released from the equilibrium position, $x_0 < 0$ means the object is released *above* the equilibrium position and $x_0 > 0$ means the object is released *below* the equilibrium position. As far as the initial velocity v_0 is concerned, $v_0 = 0$ means the object is released 'from rest,' $v_0 < 0$ means the object is heading *upwards* and $v_0 > 0$ means the object is heading *downwards*.[21]

Theorem 1.18. Equation for Free Undamped Harmonic Motion: Suppose an object of mass m is suspended from a spring with spring constant k. If the initial displacement from the equilibrium position is x_0 and the initial velocity of the object is v_0, then the displacement x from the equilibrium position at time t is given by $x(t) = A\sin(\omega t + \phi)$ where

- $\omega = \sqrt{\dfrac{k}{m}}$ and $A = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega}\right)^2}$
- $A\sin(\phi) = x_0$ and $A\omega\cos(\phi) = v_0$.

It is a great exercise in 'dimensional analysis' to verify that the formulas given in Theorem 1.18 work out so that ω has units $\frac{1}{s}$ and A has units ft. or m, depending on which system we choose.

Example 1.4.6. Suppose an object weighing 64 pounds stretches a spring 8 feet.

1. If the object is attached to the spring and released 3 feet below the equilibrium position from rest, find the equation of motion of the object, $x(t)$. When does the object first pass through the equilibrium position? Is the object heading upwards or downwards at this instant?

2. If the object is attached to the spring and released 3 feet below the equilibrium position with an upward velocity of 8 feet per second, find the equation of motion of the object, $x(t)$. What is the longest distance the object travels *above* the equilibrium position? When does this first happen? Confirm your result using a graphing utility.

[20] To keep units compatible, if we are using the English system, we use feet (ft.) to measure displacement. If we are in the SI system, we measure displacement in meters (m). Time is always measured in seconds (s).

[21] The sign conventions here are carried over from Physics. If not for the spring, the object would fall towards the ground, which is the 'natural' or 'positive' direction. Since the spring force acts in direct opposition to gravity, any movement upwards is considered 'negative'.

Solution. In order to use the formulas in Theorem 1.18, we first need to determine the spring constant k and the mass of the object m.

To find k, we use Hooke's Law $F = kd$. We know the object weighs 64 lbs. and stretches the spring 8 ft.. Using $F = 64$ and $d = 8$, we get $64 = k \cdot 8$, or $k = 8 \frac{\text{lbs.}}{\text{ft.}}$.

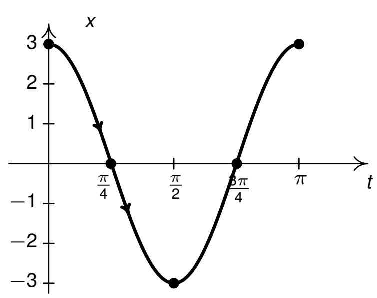
To find m, we use $w = mg$ with $w = 64$ lbs. and $g = 32 \frac{\text{ft.}}{s^2}$. We get $m = 2$ slugs. We can now proceed to apply Theorem 1.18.

1. With $k = 8$ and $m = 2$, we get $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$. Since the object is released 3 feet *below* the equilibrium position 'from rest,' $x_0 = 3$ and $v_0 = 0$. Therefore, $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + 0^2} = 3$.

   To determine the phase ϕ, we have $A\sin(\phi) = x_0$, which in this case gives $3\sin(\phi) = 3$ so $\sin(\phi) = 1$. Only $\phi = \frac{\pi}{2}$ and angles coterminal to it satisfy this condition, so we pick[22] the phase to be $\phi = \frac{\pi}{2}$. Hence, the equation of motion is $x(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$.

   To find when the object passes through the equilibrium position we solve $x(t) = 3\sin\left(2t + \frac{\pi}{2}\right) = 0$. Going through the usual analysis we find $t = -\frac{\pi}{4} + \frac{\pi}{2}k$ for integers k. Since we are interested in the *first* time the object passes through the equilibrium position, we look for the smallest positive t value which in this case is $t = \frac{\pi}{4} \approx 0.78$ seconds after the start of the motion.

   Common sense suggests that if we release the object *below* the equilibrium position, the object should be traveling *upwards* when it first passes through it. To check this answer, we graph one cycle of $x(t)$. Since our applied domain in this situation is $t \geq 0$, and the period of $x(t)$ is $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$, we graph $x(t)$ over the interval $[0, \pi]$. Remembering that $x(t) > 0$ means the object is below the equilibrium position and $x(t) < 0$ means the object is above the equilibrium position, the fact our graph is crossing through the t-axis from positive x to negative x at $t = \frac{\pi}{4}$ confirms our answer.



$x(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$

2. The only difference between this problem and the previous problem is that we now release the object with an upward velocity of $8\,\frac{\text{ft}}{s}$. We still have $\omega = 2$ and $x_0 = 3$, but now we have $v_0 = -8$, the negative indicating the velocity is directed upwards.

[22] For confirmation, we note that $A\omega\cos(\phi) = v_0$, which in this case reduces to $6\cos(\phi) = 0$.

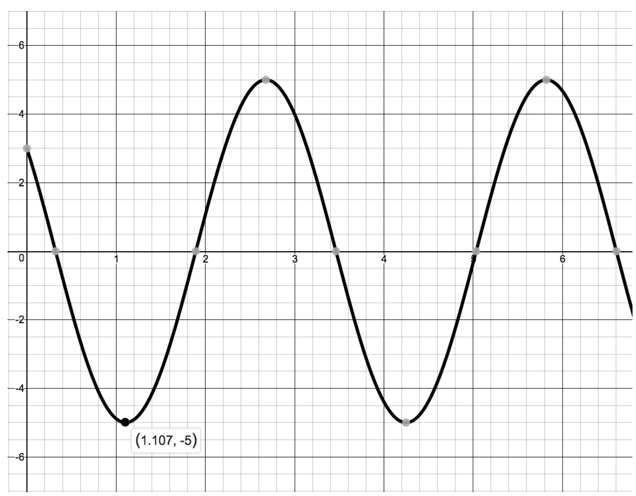
Here, we get $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + (-4)^2} = 5$. From $A\sin(\phi) = x_0$, we get $5\sin(\phi) = 3$ which gives $\sin(\phi) = \frac{3}{5}$. From $A\omega\cos(\phi) = v_0$, we get $10\cos(\phi) = -8$, or $\cos(\phi) = -\frac{4}{5}$.

Hence, ϕ is a Quadrant II angle which we can describe in terms of either arcsine or arccosine. Since the range of arccosine covers Quadrant II, we choose to express ϕ in terms of the arccosine: $\phi = \arccos\left(-\frac{4}{5}\right)$. Hence, $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$.

Since the amplitude of $x(t)$ is 5, the object will travel at most 5 feet above the equilibrium position. To find when this happens, we solve the equation $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right) = -5$, the negative once again signifying that the object is *above* the equilibrium position.

Going through the usual machinations, we get $t = -\frac{1}{2}\arccos\left(-\frac{4}{5}\right) - \frac{\pi}{4} + \pi k$ for integers k. The smallest (positive) of these values occurs when $k = 1$, that is, $t = -\frac{1}{2}\arccos\left(-\frac{4}{5}\right) + \frac{3\pi}{4} \approx 1.107$ seconds after the start of the motion.

Graphing $x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$, we find the coordinates of the first relative minimum of to be approximately $(1.107, -5)$.



$x(t) = 5\sin\left(2t + \arccos\left(-\frac{4}{5}\right)\right)$

□

Though beyond the scope of this course, it is possible to model the effects of friction and other external forces acting on the system.[23]

While we may not have the Physics and Calculus background to *derive* equations of motion for these scenarios, we can certainly analyze them. We examine three cases in the following example.

[23]Take a good Differential Equations class to see this!

Example 1.4.7.

1. Write $x(t) = 5e^{-t/5}\cos(t) + 5e^{-t/5}\sqrt{3}\sin(t)$ in the form $x(t) = A(t)\sin(\omega t + \phi)$. Graph $x(t)$ using a graphing utility.

2. Write $x(t) = (t+3)\sqrt{2}\cos(2t) + (t+3)\sqrt{2}\sin(2t)$ in the form $x(t) = A(t)\sin(\omega t + \phi)$. Graph $x(t)$ using a graphing utility.

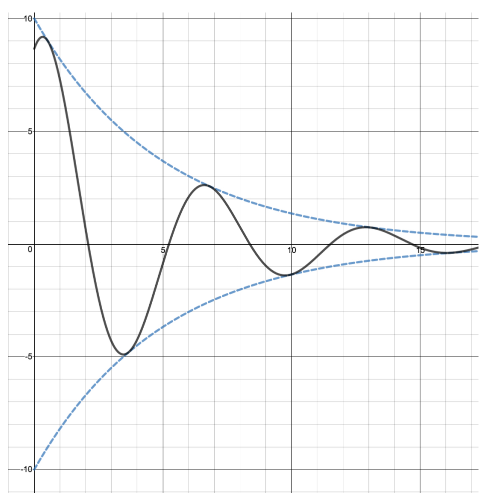
3. Find the period of $x(t) = 5\sin(6t) - 5\sin(8t)$. Graph $x(t)$ using a graphing utility.

Solution.

1. We start rewriting $x(t) = 5e^{-t/5}\cos(t) + 5e^{-t/5}\sqrt{3}\sin(t)$ by factoring out $5e^{-t/5}$ from both terms to get $x(t) = 5e^{-t/5}\left(\cos(t) + \sqrt{3}\sin(t)\right)$. We convert what's left in parentheses to the required form using the technique introduced in Example 1.2.1 from Section 1.2. We find $\left(\cos(t) + \sqrt{3}\sin(t)\right) = 2\sin\left(t + \frac{\pi}{3}\right)$ so that $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$.

   Graphing $x(t)$ reveals some interesting behavior. The sinusoidal nature continues indefinitely, but it is being attenuated. In the sinusoid $A\sin(\omega t + \phi)$, the coefficient A of the sine function is the amplitude. In the case of $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$, we can think of the *function* $A(t) = 10e^{-t/5}$ as the amplitude.[24] As $t \to \infty$, $10e^{-t/5} \to 0$ which means the amplitude shrinks towards zero.

   Indeed, if we graph $x = \pm 10e^{-t/5}$ along with $x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$, we see this attenuation taking place with the exponentials acting as a 'wave envelope.'



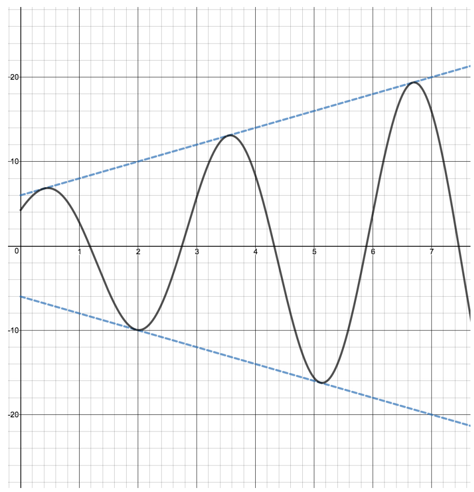
$x(t) = 10e^{-t/5}\sin\left(t + \frac{\pi}{3}\right)$ and $x = \pm 10e^{-t/5}$

   In this case, the function $x(t)$ corresponds to the motion of an object on a spring where there is a slight force which acts to 'damp', or slow the motion. An example of this kind of force would be the friction of the object against the air. According to this model, the object oscillates forever, but with increasingly smaller and smaller amplitude.

[24] This is the same sort of phenomenon we saw on page 29 in Section 1.2.1.

2. Proceeding as in the first example, we factor out $(t+3)\sqrt{2}$ from each term in the function $x(t)$ to get $x(t) = (t+3)\sqrt{2}(\cos(2t) + \sin(2t))$. We find $(\cos(2t) + \sin(2t)) = \sqrt{2}\sin\left(2t + \frac{\pi}{4}\right)$, so an equivalent form of $x(t)$ is $x(t) = 2(t+3)\sin\left(2t + \frac{\pi}{4}\right)$.

   Graphing $x(t)$, we find the sinusoid's amplitude growing. This isn't too surprising since our amplitude function here is $A(t) = 2(t+3) = 2t+6$, grows without bound as $t \to \infty$.



$\boldsymbol{x(t) = 2(t+3)\sin\left(2t + \frac{\pi}{4}\right)}$ and $x = \pm 2(t+3)$

   The phenomenon illustrated here is 'forced' motion. That is, we imagine that the entire apparatus on which the spring is attached is oscillating as well.

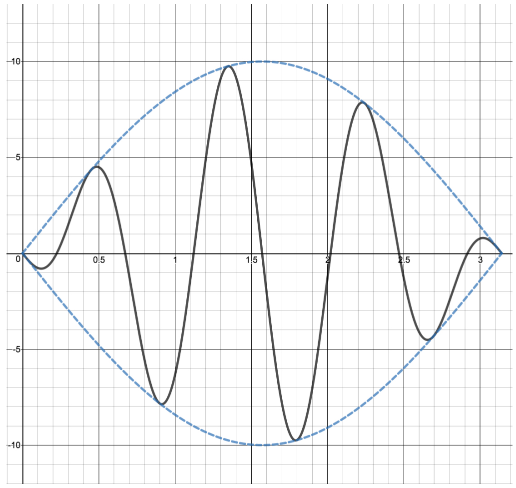
   In this particular case, we are witnessing a 'resonance' effect – the frequency of the external oscillation matches the frequency of the motion of the object on the spring. In a mechanical system, this will result in some sort of structural failure.[25]

3. Last, but not least, we come to $x(t) = 5\sin(6t) - 5\sin(8t)$. To find the period of this function, we need to determine the length of the smallest interval on which both $f(t) = 5\sin(6t)$ and $g(t) = 5\sin(8t)$ complete a whole number of cycles.

   To do this, we take the ratio of their frequencies and reduce to lowest terms: $\frac{6}{8} = \frac{3}{4}$. This tells us that for every 3 cycles f makes, g makes 4. Hence, the period of $x(t)$ is three times the period of $f(t)$ (which is four times the period of $g(t)$), or π. We check our work by graphing $x(t)$ over $[0, \pi]$

   The reader may recognize $x(t)$ an example of the 'beats' phenomenon we first saw on 29 in Section 1.2.1. Indeed, using a sum to product identity, we may rewrite $x(t)$ as $x(t) = -10\sin(t)\cos(7t)$. As we saw on 29 (and Exercises 119 - 122 in Section 1.2), the lower frequency factor, $-10\sin(t)$ determines the 'wave-envelope,' $x = \pm 10\sin(t)$.

[25]The reader is invited to investigate the destructive implications of resonance.



$x(t) = 5\sin(6t) - 5\sin(8t)$ and $x = \pm 10\sin(t)$ over $[0, \pi]$

This equation of motion also results from 'forced' motion, but here the frequency of the external oscillation is different than that of the object on the spring. Since the sinusoids here have different frequencies, they are 'out of sync' and do not amplify each other as in the previous example. Instead, through a combination of constructive and destructive interference, the mass continues to oscillate no more than 10 units from its equilibrium position indefinitely. □

1.4.2 Exercises

In Exercises 1 - 18, find all of the exact solutions of the equation and then list those solutions which are in the interval $[0, 2\pi)$.

1. $\sin(5\theta) = 0$
2. $\cos(3t) = \frac{1}{2}$
3. $\sin(-2x) = \frac{\sqrt{3}}{2}$
4. $\tan(6\theta) = 1$
5. $\csc(4t) = -1$
6. $\sec(3x) = \sqrt{2}$
7. $\cot(2\theta) = -\frac{\sqrt{3}}{3}$
8. $\cos(9t) = 9$
9. $\sin\left(\frac{x}{3}\right) = \frac{\sqrt{2}}{2}$
10. $\cos\left(\theta + \frac{5\pi}{6}\right) = 0$
11. $\sin\left(2t - \frac{\pi}{3}\right) = -\frac{1}{2}$
12. $2\cos\left(x + \frac{7\pi}{4}\right) = \sqrt{3}$
13. $\csc(\theta) = 0$
14. $\tan(2t - \pi) = 1$
15. $\tan^2(x) = 3$
16. $\sec^2(\theta) = \frac{4}{3}$
17. $\cos^2(t) = \frac{1}{2}$
18. $\sin^2(x) = \frac{3}{4}$

In Exercises 19 - 42, solve the equation, giving the exact solutions which lie in $[0, 2\pi)$

19. $\sin(\theta) = \cos(\theta)$
20. $\sin(2t) = \sin(t)$
21. $\sin(2x) = \cos(x)$
22. $\cos(2\theta) = \sin(\theta)$
23. $\cos(2t) = \cos(t)$
24. $\cos(2x) = 2 - 5\cos(x)$
25. $3\cos(2\theta) + \cos(\theta) + 2 = 0$
26. $\cos(2t) = 5\sin(t) - 2$
27. $3\cos(2x) = \sin(x) + 2$
28. $2\sec^2(\theta) = 3 - \tan(\theta)$
29. $\tan^2(t) = 1 - \sec(t)$
30. $\cot^2(x) = 3\csc(x) - 3$
31. $\sec(\theta) = 2\csc(\theta)$
32. $\cos(t)\csc(t)\cot(t) = 6 - \cot^2(t)$
33. $\sin(2x) = \tan(x)$
34. $\cot^4(\theta) = 4\csc^2(\theta) - 7$
35. $\cos(2t) + \csc^2(t) = 0$
36. $\tan^3(x) = 3\tan(x)$
37. $\tan^2(\theta) = \frac{3}{2}\sec(\theta)$
38. $\cos^3(t) = -\cos(t)$
39. $\tan(2x) - 2\cos(x) = 0$
40. $\csc^3(\theta) + \csc^2(\theta) = 4\csc(\theta) + 4$
41. $2\tan(t) = 1 - \tan^2(t)$
42. $\tan(x) = \sec(x)$

In Exercises 43 - 58, solve the equation, giving the exact solutions which lie in $[0, 2\pi)$

43. $\sin(6\theta)\cos(\theta) = -\cos(6\theta)\sin(\theta)$
44. $\sin(3t)\cos(t) = \cos(3t)\sin(t)$
45. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$
46. $\cos(5\theta)\cos(3\theta) - \sin(5\theta)\sin(3\theta) = \dfrac{\sqrt{3}}{2}$
47. $\sin(t) + \cos(t) = 1$
48. $\sin(x) + \sqrt{3}\cos(x) = 1$
49. $\sqrt{2}\cos(\theta) - \sqrt{2}\sin(\theta) = 1$
50. $\sqrt{3}\sin(2t) + \cos(2t) = 1$
51. $\cos(2x) - \sqrt{3}\sin(2x) = \sqrt{2}$
52. $3\sqrt{3}\sin(3\theta) - 3\cos(3\theta) = 3\sqrt{3}$
53. $\cos(3t) = \cos(5t)$
54. $\cos(4x) = \cos(2x)$
55. $\sin(5\theta) = \sin(3\theta)$
56. $\cos(5t) = -\cos(2t)$
57. $\sin(6x) + \sin(x) = 0$
58. $\tan(x) = \cos(x)$

In Exercises 59 - 68, solve the equation.

59. $\arccos(2x) = \pi$
60. $\pi - 2\arcsin(t) = 2\pi$
61. $4\arctan(3x - 1) - \pi = 0$
62. $6\,\mathrm{arccot}(2t) - 5\pi = 0$
63. $4\,\mathrm{arcsec}\left(\frac{x}{2}\right) = \pi$
64. $12\,\mathrm{arccsc}\left(\frac{t}{3}\right) = 2\pi$
65. $9\arcsin^2(x) - \pi^2 = 0$
66. $9\arccos^2(t) - \pi^2 = 0$
67. $8\,\mathrm{arccot}^2(x) + 3\pi^2 = 10\pi\,\mathrm{arccot}(x)$
68. $6\arctan(t)^2 = \pi\arctan(x) + \pi^2$

In Exercises 69 - 80, solve the inequality. Express the exact answer in interval notation, restricting your attention to $0 \leq x \leq 2\pi$.

69. $\sin(x) \leq 0$
70. $\tan(t) \geq \sqrt{3}$
71. $\sec^2(x) \leq 4$
72. $\cos^2(t) > \dfrac{1}{2}$
73. $\cos(2x) \leq 0$
74. $\sin\left(t + \dfrac{\pi}{3}\right) > \dfrac{1}{2}$
75. $\cot^2(x) \geq \dfrac{1}{3}$
76. $2\cos(t) \geq 1$
77. $\sin(5x) \geq 5$
78. $\cos(3t) \leq 1$
79. $\sec(x) \leq \sqrt{2}$
80. $\cot(t) \leq 4$

In Exercises 81 - 86, solve the inequality. Express the exact answer in interval notation, restricting your attention to $-\pi \leq x \leq \pi$.

81. $\cos(x) > \dfrac{\sqrt{3}}{2}$

82. $\sin(t) > \dfrac{1}{3}$

83. $\sec(x) \leq 2$

84. $\sin^2(t) < \dfrac{3}{4}$

85. $\cot(x) \geq -1$

86. $\cos(t) \geq \sin(t)$

In Exercises 87 - 92, solve the inequality. Express the exact answer in interval notation, restricting your attention to $-2\pi \leq x \leq 2\pi$.

87. $\csc(x) > 1$

88. $\cos(t) \leq \dfrac{5}{3}$

89. $\cot(x) \geq 5$

90. $\tan^2(t) \geq 1$

91. $\sin(2x) \geq \sin(x)$

92. $\cos(2t) \leq \sin(x)$

In Exercises 93 - 98, solve the given inequality.

93. $\arcsin(2x) > 0$

94. $3\arccos(t) \leq \pi$

95. $6\,\text{arccot}(7x) \geq \pi$

96. $\pi > 2\arctan(t)$

97. $2\arcsin(x)^2 > \pi\arcsin(x)$

98. $12\arccos(t)^2 + 2\pi^2 > 11\pi\arccos(t)$

In Exercises 99 - 107, express the domain of the function using the extended interval notation. (See Example 1.4.4 and Section **??** for details.)

99. $f(x) = \dfrac{1}{\cos(x) - 1}$

100. $f(t) = \dfrac{\cos(t)}{\sin(t) + 1}$

101. $f(x) = \sqrt{\tan^2(x) - 1}$

102. $f(t) = \sqrt{2 - \sec(t)}$

103. $f(x) = \csc(2x)$

104. $f(t) = \dfrac{\sin(t)}{2 + \cos(t)}$

105. $f(x) = 3\csc(x) + 4\sec(x)$

106. $f(t) = \ln\left(|\cos(t)|\right)$

107. $f(x) = \arcsin(\tan(x))$

108. (a) With the help of your classmates, determine the number of solutions to $\sin(x) = \frac{1}{2}$ in $[0, 2\pi)$. Then find the number of solutions to $\sin(2x) = \frac{1}{2}$, $\sin(3x) = \frac{1}{2}$ and $\sin(4x) = \frac{1}{2}$ in $[0, 2\pi)$. What pattern emerges? Explain how this pattern would help you solve equations like $\sin(11x) = \frac{1}{2}$.

(b) Repeat the above exercise focusing on $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$, $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ and $\sin\left(\frac{5x}{2}\right) = \frac{1}{2}$. What pattern emerges here?

(c) Replace sine with tangent and $\frac{1}{2}$ with 1 and repeat the whole exploration.

109. Suppose an object weighing 10 pounds is suspended from the ceiling by a spring which stretches 2 feet to its equilibrium position when the object is attached.

    (a) Find the spring constant k in $\frac{\text{lbs.}}{\text{ft.}}$ and the mass of the object in slugs.
    (b) Find the equation of motion of the object if it is released from 1 foot *below* the equilibrium position from rest. When is the first time the object passes through the equilibrium position? In which direction is it heading?
    (c) Find the equation of motion of the object if it is released from 6 inches *above* the equilibrium position with a *downward* velocity of 2 feet per second. Find when the object passes through the equilibrium position heading downwards for the third time.

1.4.3 Answers

1. $\theta = \frac{\pi k}{5}$; $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}$
2. $t = \frac{\pi}{9} + \frac{2\pi k}{3}$ or $t = \frac{5\pi}{9} + \frac{2\pi k}{3}$; $t = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$
3. $x = \frac{2\pi}{3} + \pi k$ or $x = \frac{5\pi}{6} + \pi k$; $x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$
4. $\theta = \frac{\pi}{24} + \frac{\pi k}{6}$; $\theta = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{41\pi}{24}, \frac{15\pi}{8}$
5. $t = \frac{3\pi}{8} + \frac{\pi k}{2}$; $t = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$
6. $x = \frac{\pi}{12} + \frac{2\pi k}{3}$ or $x = \frac{7\pi}{12} + \frac{2\pi k}{3}$; $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$
7. $\theta = \frac{\pi}{3} + \frac{\pi k}{2}$; $\theta = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$
8. No solution
9. $x = \frac{3\pi}{4} + 6\pi k$ or $x = \frac{9\pi}{4} + 6\pi k$; $x = \frac{3\pi}{4}$
10. $\theta = -\frac{\pi}{3} + \pi k$; $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$
11. $t = \frac{3\pi}{4} + \pi k$ or $t = \frac{13\pi}{12} + \pi k$; $t = \frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$
12. $x = -\frac{19\pi}{12} + 2\pi k$ or $x = \frac{\pi}{12} + 2\pi k$; $x = \frac{\pi}{12}, \frac{5\pi}{12}$
13. No solution
14. $t = \frac{5\pi}{8} + \frac{\pi k}{2}$; $t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
15. $x = \frac{\pi}{3} + \pi k$ or $x = \frac{2\pi}{3} + \pi k$; $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
16. $\theta = \frac{\pi}{6} + \pi k$ or $\theta = \frac{5\pi}{6} + \pi k$; $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
17. $t = \frac{\pi}{4} + \frac{\pi k}{2}$; $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
18. $x = \frac{\pi}{3} + \pi k$ or $x = \frac{2\pi}{3} + \pi k$; $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

19. $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

20. $t = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

21. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

22. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

23. $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

24. $x = \frac{\pi}{3}, \frac{5\pi}{3}$

25. $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \arccos\left(\frac{1}{3}\right), 2\pi - \arccos\left(\frac{1}{3}\right)$

26. $t = \frac{\pi}{6}, \frac{5\pi}{6}$

27. $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \arcsin\left(\frac{1}{3}\right), \pi - \arcsin\left(\frac{1}{3}\right)$

28. $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \arctan\left(\frac{1}{2}\right), \pi + \arctan\left(\frac{1}{2}\right)$

29. $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

30. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

31. $\theta = \arctan(2), \pi + \arctan(2)$

32. $t = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

33. $x = 0, \pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

34. $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$

35. $t = \frac{\pi}{2}, \frac{3\pi}{2}$

36. $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

37. $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

38. $t = \frac{\pi}{2}, \frac{3\pi}{2}$

39. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

40. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

41. $t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

42. No solution

43. $\theta = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{8\pi}{7}, \frac{9\pi}{7}, \frac{10\pi}{7}, \frac{11\pi}{7}, \frac{12\pi}{7}, \frac{13\pi}{7}$

44. $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

45. $x = 0$

46. $\theta = \frac{\pi}{48}, \frac{11\pi}{48}, \frac{13\pi}{48}, \frac{23\pi}{48}, \frac{25\pi}{48}, \frac{35\pi}{48}, \frac{37\pi}{48}, \frac{47\pi}{48}, \frac{49\pi}{48}, \frac{59\pi}{48}, \frac{61\pi}{48}, \frac{71\pi}{48}, \frac{73\pi}{48}, \frac{83\pi}{48}, \frac{85\pi}{48}, \frac{95\pi}{48}$

47. $t = 0, \frac{\pi}{2}$

48. $x = \frac{\pi}{2}, \frac{11\pi}{6}$

49. $\theta = \frac{\pi}{12}, \frac{17\pi}{12}$

50. $t = 0, \pi, \frac{\pi}{3}, \frac{4\pi}{3}$

51. $x = \frac{17\pi}{24}, \frac{41\pi}{24}, \frac{23\pi}{24}, \frac{47\pi}{24}$

52. $\theta = \frac{\pi}{6}, \frac{5\pi}{18}, \frac{5\pi}{6}, \frac{17\pi}{18}, \frac{3\pi}{2}, \frac{29\pi}{18}$

53. $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

54. $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

55. $\theta = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

56. $t = \frac{\pi}{7}, \frac{\pi}{3}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{5\pi}{3}, \frac{13\pi}{7}$

57. $x = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

58. $x = \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 0.6662, \pi - \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 2.4754$

59. $x = -\frac{1}{2}$

60. $t = -1$

61. $x = \frac{2}{3}$

62. $t = -\frac{\sqrt{3}}{2}$

63. $x = 2\sqrt{2}$

64. $t = 6$

65. $x = \pm\frac{\sqrt{3}}{2}$

66. $t = \frac{1}{2}$

67. $x = -1, 0$

68. $t = -\sqrt{3}$

69. $[\pi, 2\pi]$

70. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left[\frac{4\pi}{3}, \frac{3\pi}{2}\right)$

71. $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$

72. $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right]$

73. $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$

74. $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right]$

75. $\left(0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \pi\right) \cup \left(\pi, \frac{4\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right)$

76. $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$

77. No solution

78. $[0, 2\pi]$

79. $\left[0, \frac{\pi}{4}\right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left[\frac{7\pi}{4}, 2\pi\right]$

80. $[\operatorname{arccot}(4), \pi) \cup [\pi + \operatorname{arccot}(4), 2\pi)$

81. $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

82. $\left(\arcsin\left(\frac{1}{3}\right), \pi - \arcsin\left(\frac{1}{3}\right)\right)$

83. $\left[-\pi, -\frac{\pi}{2}\right) \cup \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \cup \left(\frac{\pi}{2}, \pi\right]$

84. $\left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$

85. $\left(-\pi, -\frac{\pi}{4}\right] \cup \left(0, \frac{3\pi}{4}\right]$

86. $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

87. $\left(-2\pi, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\pi\right) \cup \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

88. $[-2\pi, 2\pi]$

89. $(-2\pi, \operatorname{arccot}(5) - 2\pi] \cup (-\pi, \operatorname{arccot}(5) - \pi] \cup (0, \operatorname{arccot}(5)] \cup (\pi, \pi + \operatorname{arccot}(5)]$

90. $\left[-\frac{7\pi}{4}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{5\pi}{4}\right] \cup \left[-\frac{3\pi}{4}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right]$

91. $\left[-2\pi, -\frac{5\pi}{3}\right] \cup \left[-\pi, -\frac{\pi}{3}\right] \cup \left[0, \frac{\pi}{3}\right] \cup \left[\pi, \frac{5\pi}{3}\right]$

92. $\left[-\frac{11\pi}{6}, -\frac{7\pi}{6}\right] \cup \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \cup, \left\{-\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

93. $\left(0, \frac{1}{2}\right]$

94. $\left[\frac{1}{2}, 1\right]$

95. $\left(-\infty, \frac{\sqrt{3}}{7}\right]$

96. $(-\infty, \infty)$

97. $[-1, 0)$

98. $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 1\right]$

99. $\displaystyle\bigcup_{k=-\infty}^{\infty} (2k\pi, (2k+2)\pi)$

100. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(4k-1)\pi}{2}, \frac{(4k+3)\pi}{2}\right)$

101. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left\{\left[\frac{(4k+1)\pi}{4}, \frac{(2k+1)\pi}{2}\right) \cup \left(\frac{(2k+1)\pi}{2}, \frac{(4k+3)\pi}{4}\right]\right\}$

102. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left\{\left[\frac{(6k-1)\pi}{3}, \frac{(6k+1)\pi}{3}\right] \cup \left(\frac{(4k+1)\pi}{2}, \frac{(4k+3)\pi}{2}\right)\right\}$

103. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$

104. $(-\infty, \infty)$

105. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$

106. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$

107. $\displaystyle\bigcup_{k=-\infty}^{\infty} \left[\frac{(4k-1)\pi}{4}, \frac{(4k+1)\pi}{4}\right]$

109. (a) $k = 5\ \frac{\text{lbs.}}{\text{ft.}}$ and $m = \frac{5}{16}$ slugs

(b) $x(t) = \sin\left(4t + \frac{\pi}{2}\right)$. The object first passes through the equilibrium point when $t = \frac{\pi}{8} \approx 0.39$ seconds after the motion starts. At this time, the object is heading upwards.

(c) $x(t) = \frac{\sqrt{2}}{2}\sin\left(4t + \frac{7\pi}{4}\right)$. The object passes through the equilibrium point heading downwards for the third time when $t = \frac{17\pi}{16} \approx 3.34$ seconds.