Movimiento Browniano *Geométrico* y la valorización de instrumentos financieros

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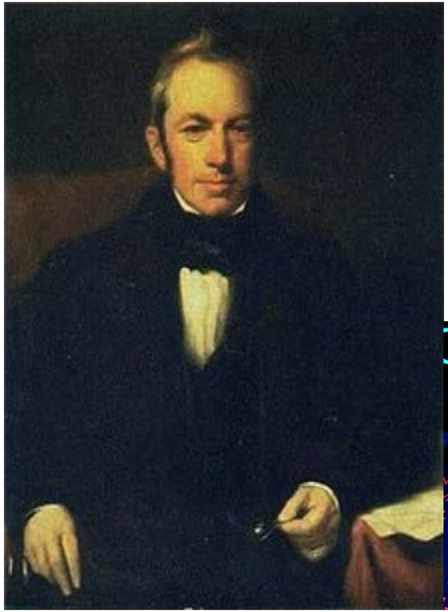


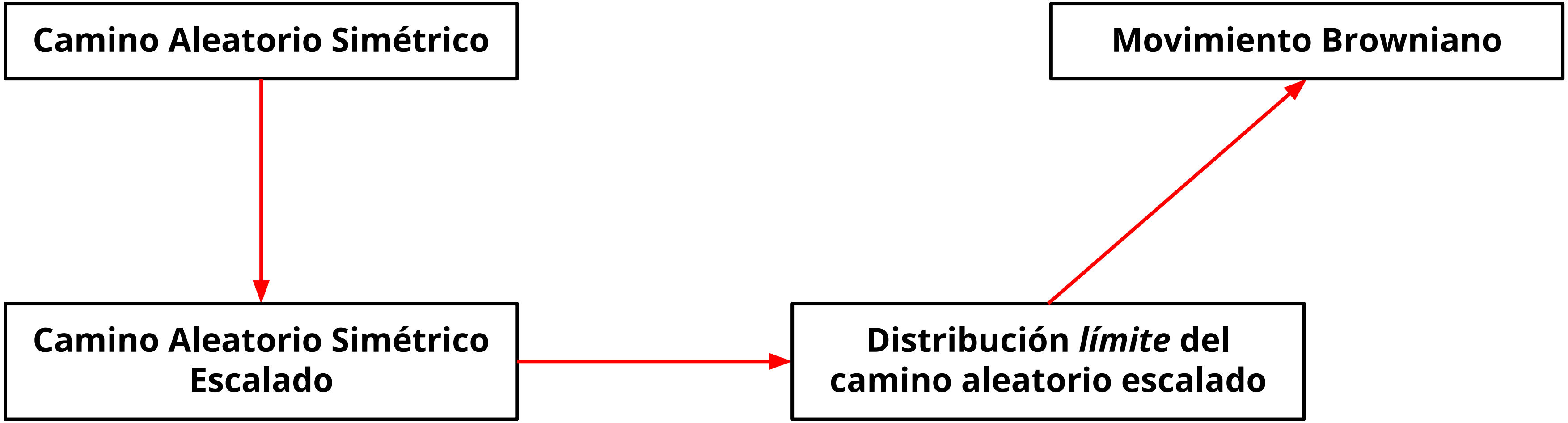
Introducción / Motivación

- Los modelos estocásticos han probado ser más útiles que los analíticos a la hora de describir el comportamiento de los mercados financieros.

- El movimiento Browniano y las ecuaciones diferenciales estocásticas, constituyen el pilar fundamental de las Finanzas Cuantitativas y en particular de los modelos para valorar derivados financieros.

- El Movimiento Browniano Geométrico (GBM) es quizás uno de los modelos más sencillos pero sirve para introducir los Procesos de Ito que son de suma importancia en el Calculo Estocástico.

Intuición sobre el Movimiento Browniano



Movimiento Browniano

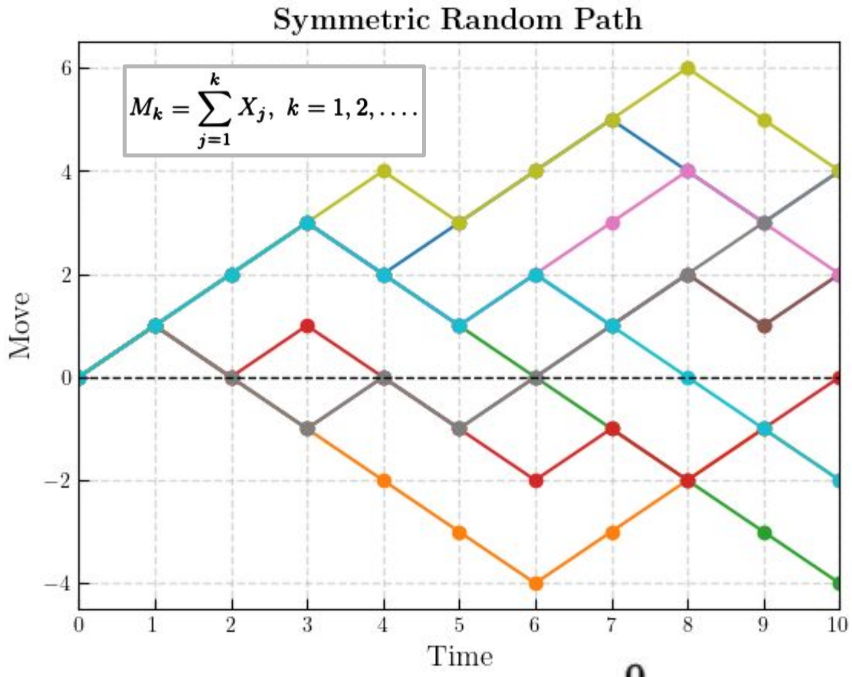
Camino Aleatorio Simétrico

$$X_j = \begin{cases} 1 & \text{if } \omega_j = H, \\ -1 & \text{if } \omega_j = T, \end{cases}$$

$$M_k = \sum_{j=1}^{k} X_j, \ k = 1, 2, \ldots .$$

$$M_0 = 0$$

```
Steps taken for each simulation
 [[ 1 -1]
 [ 1  1]
 [ 1 -1]
 [ 1 -1]
 [ 1 -1]
 [ 1  1]
 [-1 -1]
 [ 1  1]
 [ 1 -1]
 [-1 -1]]
Initial Positions for 2 simulations
 [[0. 0.]]
Final Paths for 2 simulations
 [[ 0.  0.]
 [ 1. -1.]
 [ 2.  0.]
 [ 3. -1.]
 [ 4. -2.]
 [ 5. -3.]
 [ 6. -2.]
 [ 5. -3.]
 [ 6. -2.]
 [ 7. -3.]
 [ 6. -4.]]
```



$$\begin{aligned} E[X] &= x_1 p_1 + \ldots + x_k p_k \\ &= (1)p + (-1)(1-p) \\ &= p - 1 + p = 2p - 1 \\ &= 2\left(\frac{1}{2}\right) - 1 \\ &= 0 \end{aligned}$$

0

$$\begin{aligned} \operatorname{Var}(X) &= E[X^2] - E^2[X] \\ &= E[X^2] \\ &= x_1^2 p_1 + \ldots + x_k^2 p_k \\ &= (1)^2 p + (-1)^2 (1-p) \\ &= p + 1 - p = 1 \end{aligned}$$

Movimiento Browniano

Camino Aleatorio Simétrico

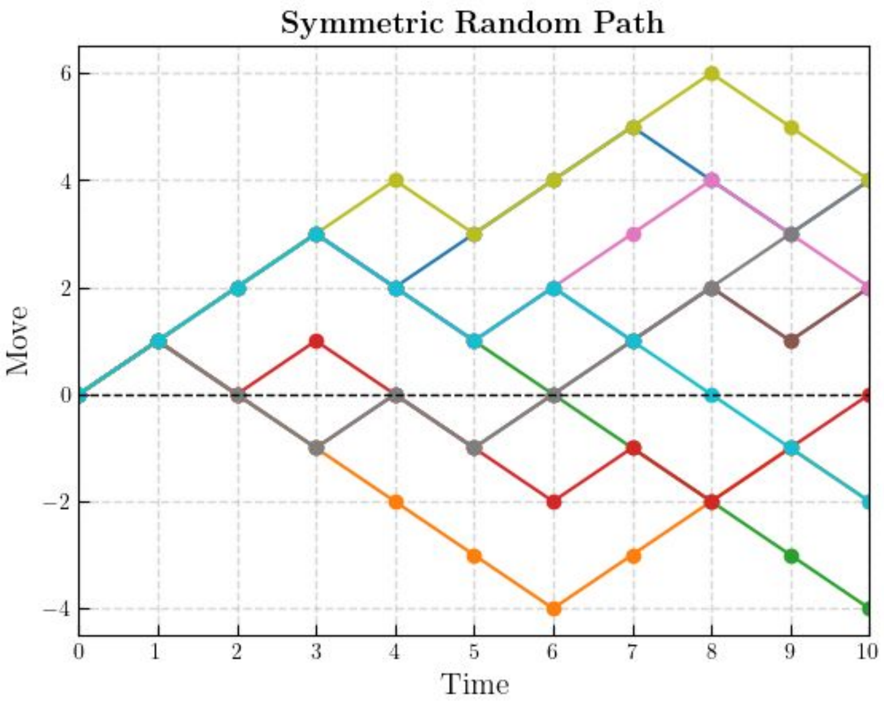
- Incrementos Estadísticamente independientes:

$$0 = k_0 < k_1 < \cdots < k_m$$

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

- La varianza se acumula a una tasa de 1 por unidad de tiempo

$$\mathrm{Var}\left(M_{k_{i+1}} - M_{k_i}\right) = \sum_{j=k_i+1}^{k_{i+1}} \mathrm{Var}(X_j) = \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i$$



- La variación cuadrática es diferente de cero:

$$[M, M]_k = \sum_{j=1}^{k} \underbrace{\left(M_j - M_{j-1}\right)^2}_{\pm 1} = k$$

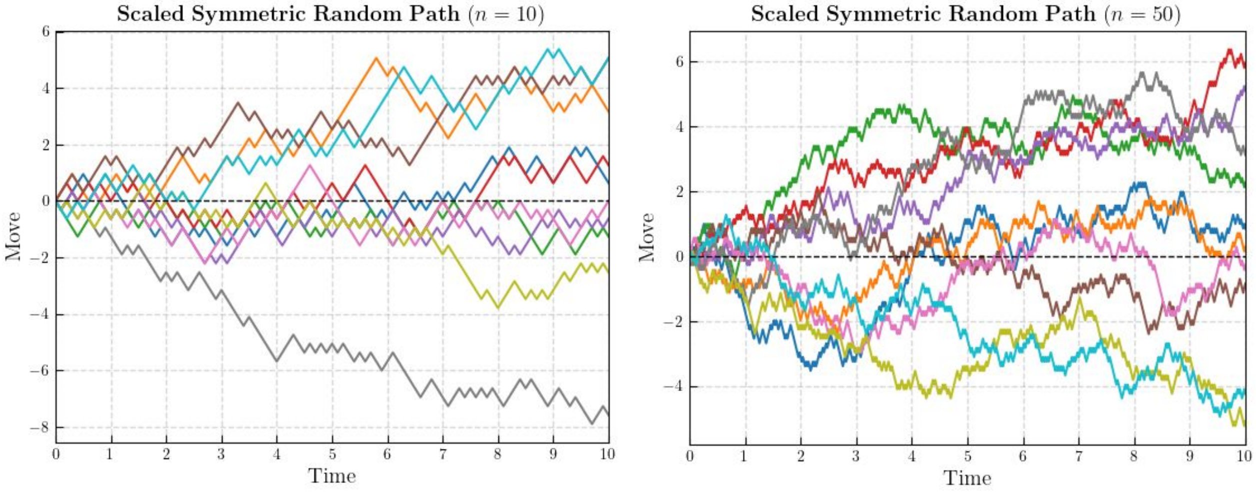
Movimiento Browniano

Camino Aleatorio Simétrico Escalado

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

- Se reescala el paso (ya no es de 1 en 1), ahora tenemos *nt* pasos.

- Ganamos granularidad y nos acercamos más al movimiento Browniano real

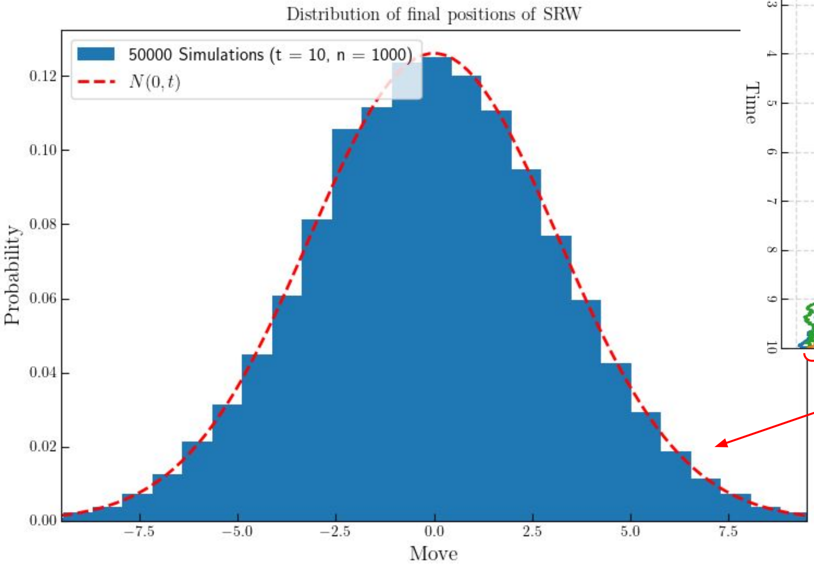


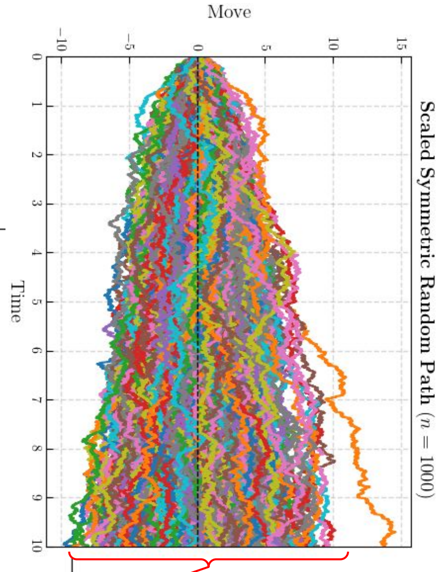
Hereda todas las propiedades del camino aleatorio simétrico regular

Movimiento Browniano

Distribución límite del camino aleatorio escalado

A medida que *n* crece, la distribución Binomial (que es la que describe el camino aleatorio escalado) tiende a una distribución normal con varianza *t*.





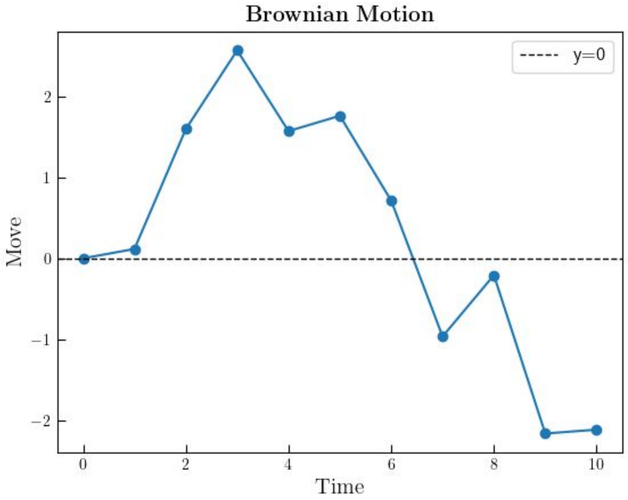
Theorem 3.2.1 (Central limit). *Fix $t \geq 0$. As $n \to \infty$, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t.*

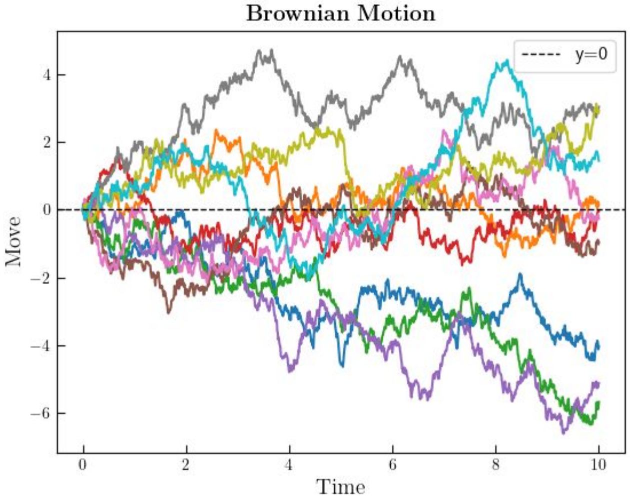
Movimiento Browniano

```
Movements =
 [[ 0.11573802]
 [ 1.48121204]
 [ 0.97292647]
 [-0.99778914]
 [ 0.18933288]
 [-1.04832045]
 [-1.67895699]
 [ 0.75788098]
 [-1.95692253]
 [ 0.04815497]]
Final Path by summing =
 [[ 0.        ]
 [ 0.11573802]
 [ 1.59695006]
 [ 2.56987653]
 [ 1.57208739]
 [ 1.76142026]
 [ 0.71309982]
 [-0.96585717]
 [-0.2079762 ]
 [-2.16489872]
 [-2.11674376]]
```

```python
# Parameters
M = 1     # Number of simulations
t = 10   # Time
n = 10 # Steps
dt = t/n # Time step

# Sample from a normal distribution with variance t
movements = np.random.normal(0, np.sqrt(dt), size=(M, n)).T
```





Procesos de Ito

- Está dividido en 2 partes:

1. Integral de un proceso en el tiempo
2. Integral de un proceso sobre un Movimiento Browniano.

- Podemos verlo a partir de la ecuación de Langevin con ruido multiplicativo:

$$\boxed{\frac{dX}{dt} = a(x, \langle \mu \rangle) + b(x)\eta(x)}$$

$$\frac{du(t)}{dt} = f(u,t) + g(u,t)\eta(t)$$

$$du(t) = f(u,t)dt + g(u,t)\eta(t)dt$$

$$du(t) = f(u,t)dt + g(u,t)\eta(t)dW_t$$

$$W_t = \int_0^t \eta(s)ds$$

Procesos de Ito

$$\frac{du(t)}{dt} = f(u,t) + g(u,t)\eta(t)$$

$$du(t) = f(u,t)dt + g(u,t)\eta(t)dt$$

$$du(t) = f(u,t)dt + g(u,t)\eta(t)dW_t$$

$$W_t = \int_0^t \eta(s)ds$$

$$u(t) = u(0) + \underbrace{\overbrace{\boxed{\int_0^t f(u,s)ds}}^{\textit{Drift}}}_{\textit{Integral de Riemann}} + \underbrace{\overbrace{\boxed{\int_0^t g(u,t)dW_t}}^{\textit{Diffusion}}}_{\textit{Integral de Ito}}$$

Movimiento Browniano Geométrico como un proceso de Ito

Se quiere evaluar un contrato financiero que depende del precio subyacente S_t:

$$C_t = f(S_t)$$

El precio subyacente está descrito por un proceso de Ito:

$$\boxed{dS_t = \mu S_t dt + \sigma S_t dW_t}$$

Drift **Diffusion**

Nos interesa estudiar la dinámica de esta función en el tiempo. Consideremos intervalo de tiempo *dt* infinitesimal y hagamos una expansión en series de Taylor:

$$df(S_t) = f(S_t + dt) - f(S_t)$$

$$df\,(S_t) = \frac{\partial f}{\partial S}(dS_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2 + \frac{1}{6}\frac{\partial^3 f}{\partial S^3}(dS_t)^3 + \cdots$$

?

Fórmula de Ito-Doeblin (Lema de Ito)

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$df(t,W(t)) = f_t(t,W(t))\,dt + f_x(t,W(t))\,dW(t) + \frac{1}{2}f_{xx}(t,W(t))\,dt.$$

$$df\,(S_t) = \frac{\partial f}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2 \longrightarrow df(S_t) = \frac{\partial f}{\partial S}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2 dt$$

$\boxed{dS_t = \mu S_t dt + \sigma S_t dW_t}$

$\boxed{(dS_t)^2 = (\mu S_t dt + \sigma S_t dW_t)^2 = (\sigma S_t dW_t)^2 = \sigma^2 S_t^2 dt}$

Fórmula de Ito-Doeblin (Lema de Ito)

$$df(S_t) = \frac{\partial f}{\partial S}\left(\mu S_t dt + \sigma S_t dW_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2 dt$$

Hagamos la expansión para ln(s):

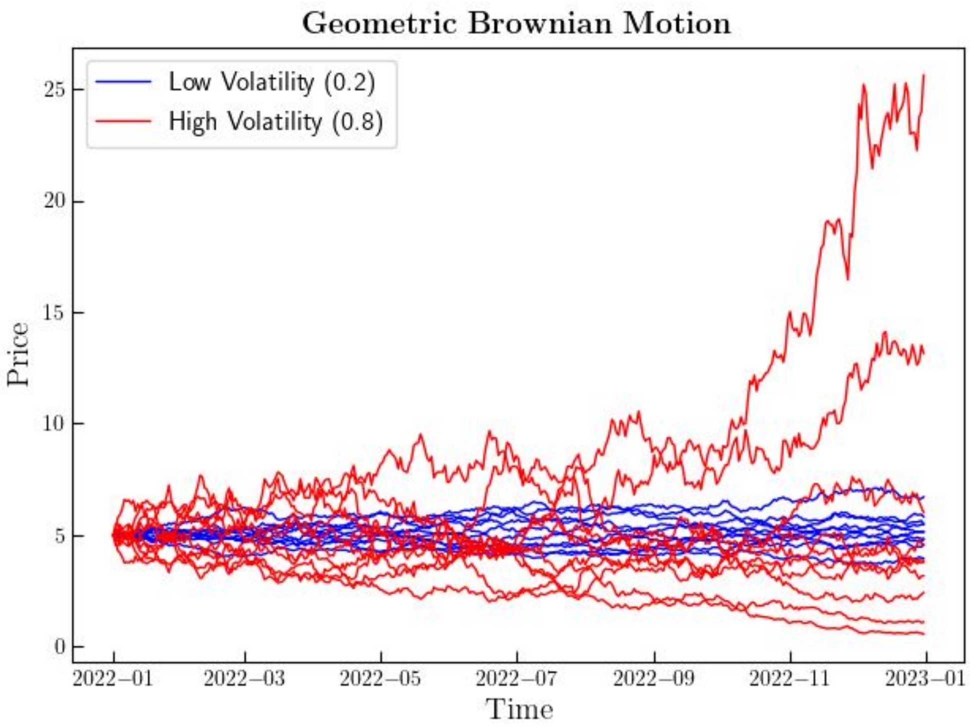
$$f(S_t) = \ln(S_t), f'(S_t) = \frac{1}{S_t}, f''(S_t) = -\frac{1}{S_t^2}$$

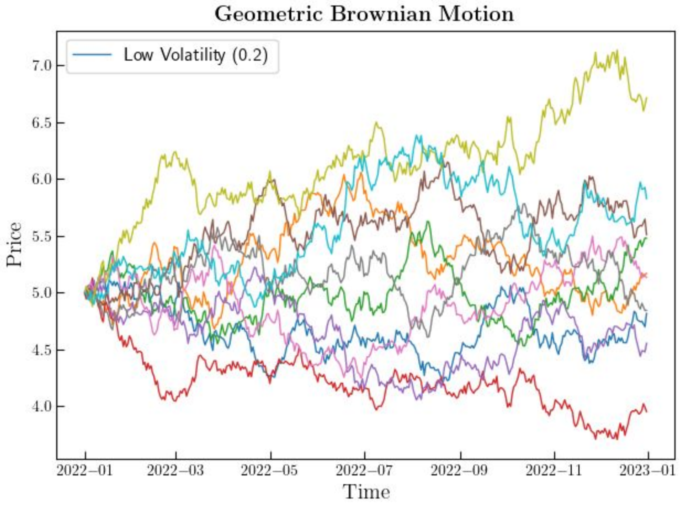
$$\begin{aligned} d\ln\left(S_t\right) &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}\left(dS_t\right)^2 \\ &= \frac{1}{S_t}\left(\mu S_t dt + \sigma S_t dW_t\right) - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 dt \end{aligned}$$

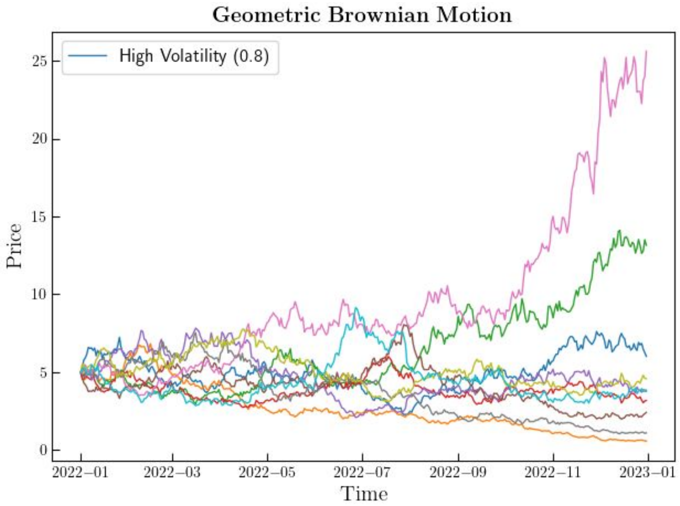
$$d\ \ln\left(S_t\right) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t$$

$$\ln S_t = \ln S_0 + \underbrace{\boxed{\int_0^t \left(\mu - \frac{\sigma^2}{2}\right) du}}_{\textit{Drift}} + \underbrace{\boxed{\int_0^t \sigma dW_u}}_{\textit{Diffusion}}$$

$$\boxed{S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]}$$


Geometric Brownian Motion
Low Volatility (0.2)
High Volatility (0.8)
Price
0
5
10
15
20
25
Time
2022−01
2022−03
2022−05
2022−07
2022−09
2022−11
2023−01


Geometric Brownian Motion
Low Volatility (0.2)
Price
4.0
4.5
5.0
5.5
6.0
6.5
7.0
Time
2022−01
2022−03
2022−05
2022−07
2022−09
2022−11
2023−01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2022−03
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2022−07
2022−09
2022−11
2023−01

Referencias

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