FYS3150
Project 3 -

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https://github.com/kryzar/Perseids.git

Abstract

Contents

Introduction

Chapter 1

Theory

In a first place, we will begin with an Earth-Sun system with the Earth orbiting around the Sun to test a simple algorithm and then we will add the other planets to have a simulation of the complete Solar System.

1.1 Earth-Sun system

1.1.1 Physical conditions of the system

The only force applied to this system is the gravity. According to the Newton's law, we have

$$F_G = \frac{GM_\odot M_\oplus}{r^2}$$

with F_G the gravitational force, G the gravitational constant ($G = 6.674 \times 10^{-11}\mathrm{N.m^2.kg^{-2}}$), $M_\odot$ the mass of the Sun, $M_\oplus$ the mass of the Earth and r the distance between the Earth and the Sun.
We will neglect the motion of the Sun here as the mass of the Sun is much larger than the mass of the Earth ($M_\odot = 2 \times 10^{30}$kg against $M_\oplus = 6 \times 10^{24}$kg). We want to establish the motion of the Earth around the Sun. Moreover, we will assume that the orbit of the Earth around the Sun is coplanar in the xy-plane.

The Newton's second law of motion is given by $F = M \times a$ with F the force applied to the system, M the mass of the body concerned and a the acceleration. Applied to our case, we have $F_G = M_\oplus \times a$, a being the second derivative of the position, which can be written in two equations :

$$F_{G,x} = M_\oplus \frac{d^2x}{dt^2}$$
$$F_{G,y} = M_\oplus \frac{d^2y}{dt^2}$$

or

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_\oplus} \quad (1)$$

$$\frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_\oplus}. \quad (2)$$

with $F_{G,x}$ and $F_{G,y}$ the components of the gravitational force.
We will not use the SI units but the Astronomical units (AU) for the distances (with 1AU=average distance Earth-Sun $= 1.5 \times 10^{11}$m), kg for masses and years for time units.
In this case, the initial position of the Earth will be $x_\oplus = 1,\ y_\oplus = 0$ with the Sun the origin ($x_\odot = 0$, $y_\odot = 0$).
To simplify a little bit, we will assume that the Earth's orbit is circular around the Sun. So we can write :

$$F_G = \frac{M_\oplus v^2}{r} = \frac{GM_\odot M_\oplus}{r^2} \quad (3)$$

with v the velocity of Earth. From here we have

$$v^2 r = GM_\odot = 4\pi^2 \mathrm{AU}^3/\mathrm{yr}^2 \quad (4)$$

1.1.2 Escape velocity

In the case where only the gravitational force is applied, the mechanical energy $E_m = E_c + E_p$ with E_c the kinetic energy and E_p the potential energy is conserved as the gravitational force is conservative. For a planet at a distance of 1 AU from the Sun (let's take the Earth as an example), the kinetic energy is given by $E_c = \frac{1}{2}M_\oplus \times v^2$ and the potential energy by $E_p = -\frac{GM_\odot M_\oplus}{r}$ with $r = 1$ AU.
We want this planet to escape from the gravitational attraction of the Sun on it. The escape velocity will be the minimal speed v_e at which the planet escapes from the gravitational influence of the Sun. We consider two positions of the planet : p_i the initial position when $r_i = 1$AU and p_f the final position when the planet has escaped and is at an infinite distance from the Sun $r_f = \infty$. Let's consider the mechanical energy at the final position. We have $E_{m_f} = E_{c_f} + E_{p_f}$, $E_{c_f} = 0$ as the velocity of the planet is zero at the final state, $E_{p_f} = 0$ as the planet has escaped from the gravitational attraction of the Sun. $E_{m_f} = 0$.
So, with the planet at 1 AU from the Sun at its inital position, of which velocity is the escape velocity v_e, in the case where there is only the gravitational force applied to the system planet-Sun, we have :

$$E_{c_i} + E_{p_i} = \frac{1}{2}M_\oplus \times v_e^2 - \frac{GM_\odot M_\oplus}{r_i} = E_{c_f} + E_{p_f} = 0$$

according to the law of conservation of energy. From this we can write

$$\frac{1}{2}M_\oplus \times v_e^2 - \frac{GM_\odot M_\oplus}{r_i} = 0$$
$$\Rightarrow v_e = \sqrt{\frac{2GM_\odot}{r_i}}$$
$$\Rightarrow v_e = \sqrt{2GM_\odot}$$

which only depends on the mass of the Sun.

$$v_e = \sqrt{2 \times 4\pi^2}, \quad \text{from}(4)$$
$$v_e = 2\sqrt{2}\pi \ \text{AU/yr}$$

Then, for a velocity $v \geq 2\sqrt{2}\pi$, the Earth would escape from the Sun's gravitational attraction.

1.2 Three-body problem

We add Jupiter to the previous system.

1.2.1 The Sun as the center-of-mass

In the previous case, the Earth's motion had a stable circular orbit in time. Adding Jupiter, the Earth's motion will inevitably be disturbed. We can write the gravitational between the Earth and Jupiter as

$$F_{\oplus-♃} = \frac{GM_{♃}M_{\oplus}}{r^2_{\oplus-♃}}$$

with $M_{♃}$ the mass of Jupiter and $r_{\oplus-♃}$ the distance between the Earth and Jupiter.
The algorithm used for this case will be exactly the same as the previous case. We will see in section 1.4 how to discretize and implement the algorithm.

1.2.2 Real center-of-mass

To get closer to the reality, we now do not consider the Sun as a static body anymore. Each of three bodies are in motion. Moreover, the center-of-mass will not be the Sun but the real one. Let's compute the center-of-mass coordinates.

$$x_c = \frac{1}{M_{tot}} \sum_i M_i x_i$$
$$y_c = \frac{1}{M_{tot}} \sum_i M_i y_i$$

We consider the system Earth-Jupiter-Sun.

$$M_{tot} = M_{\oplus} + M_{♃} + M_{\odot}$$
$$M_{tot} = 6 \times 10^{24} + 1.9 \times 10^{27} + 2 \times 10^{30}$$
$$M_{tot} = 2.00191 \times 10^{30}\text{kg}$$

$$\begin{cases} x_c = \dfrac{1}{M_{tot}}[M_{\oplus}x_{\oplus_i} + M_{♃}x_{♃_i} + M_{\odot}x_{\odot_i}] \\ y_c = \dfrac{1}{M_{tot}}[M_{\oplus}y_{\oplus_i} + M_{♃}y_{♃_i} + M_{\odot}y_{\odot_i}] \end{cases}$$

$$\begin{cases} x_c = \dfrac{1}{2.00191 \times 10^{30}}[(2 \times 10^{24})(8.307575140688143 \times 10^{-1}) + (1.9 \times 10^{27})(-4.544631370382659) \\ \qquad\qquad\qquad\qquad + (2 \times 10^{30})(2.171123049041335 \times 10^{-3})] \\ y_c = \dfrac{1}{2.00191 \times 10^{30}}[(2 \times 10^{24})(5.546449644310195 \times 10^{-1}) + (1.9 \times 10^{27})(-2.980887268959417) \\ \qquad\qquad\qquad\qquad + (2 \times 10^{30})(5.784524554879186 \times 10^{-3})] \end{cases}$$

The initial positions were taken from the NASA website (`http://ssd.jpl.nasa.gov/horizons.cgi#top`) on 27th of October 2017, 00:00:00. The coordinates are expressed with respect to the "Solar System Barycenter".

$$\begin{cases} x_c = \dfrac{1}{2.00191 \times 10^{30}}[-4.29089 \times 10^{27}] \\ y_c = \dfrac{1}{2.00191 \times 10^{30}}[5.90647 \times 10^{27}] \end{cases}$$

$$\begin{cases} x_c = 3.11898 \times 10^{-3} \\ y_c = 2.95042 \times 10^{-3} \end{cases} \tag{5}$$

We take those coordinates as origin. Let's now compute the Sun's initial velocity so that the total momentum of the three-body system would be equal to zero.

$$\vec{p_{tot}} = \sum_k M_k \vec{k}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = M_\oplus \begin{pmatrix} v_{\oplus_x} \\ v_{\oplus_y} \end{pmatrix} + M_{♃} \begin{pmatrix} v_{♃_x} \\ v_{♃_y} \end{pmatrix} + M_\odot \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\begin{cases} M_\oplus v_{\oplus_x} + M_{♃} v_{♃_x} + M_\odot v_x = 0 \\ M_\oplus v_{\oplus_y} + M_{♃} v_{♃_y} + M_\odot v_y = 0 \end{cases}$$

Again, we take the initial velocities of Jupiter and the Earth from the NASA website.

$$\begin{cases} (6 \times 10^{24})(-9.791937393985607 \times 10^{-3}) + (1.9 \times 10^{27})(4.050196424242171 \times 10^{-3}) \\ \qquad\qquad + (2 \times 10^{30}) v_x = 0 \\ (6 \times 10^{24})(1.428201625774134 \times 10^{-2}) + (1.9 \times 10^{27})(-5.951356984409009 \times 10^{-3}) \\ \qquad\qquad + (2 \times 10^{30}) v_y = 0 \end{cases}$$

$$\begin{cases} v_x = -3.81831 \times 10^{-6} \\ v_y = 5.61094 \times 10^{-6} \end{cases} \tag{6}$$

To have a fixed center-of-mass, the Sun will need a velocity $v = \begin{pmatrix} -3.81831 \times 10^{-6} \\ 5.61094 \times 10^{-6} \end{pmatrix}$.

1.3 The complete Solar system

1.4 Discretization

1.4.1 Euler's method

Acceleration. To implement the accelerations, we normalize the masses with respect to the mass of the Sun. Thus, $M_\odot = 1$, $M_{k_N} = \frac{M_k}{M_\odot}$ is the normalized mass for each body k.

From the Newton's law of motion, the acceleration of a given body can be written as $a = \frac{F_G}{M}$, with M the mass of the body.

$$a_i = \frac{\sum_k \frac{GMM_{k_N}}{r_k^2}}{M}$$

with k the number of bodies involved in the system.

$$a_i = \sum_k \frac{GM_{k_N}}{r_k^2}$$

From (4), we have $GM_\odot = 4\pi^2 \mathrm{AU}^3/\mathrm{yr}^2$ and as $M_\odot = 1$, $G = 4\pi^2 \mathrm{AU}^3/\mathrm{yr}^2$. So,

$$a_i = \sum_k \frac{4\pi^2 M_{k_N}}{r_k^2}$$

$$a_i = \sum_k \frac{4\pi^2 M_{k_N}(x_i - x_{i_k})}{r_k^3} \tag{7}$$

With Euler's method, we can write $a_i = \dfrac{v_{i+1} - v_i}{h}$ with $h = \dfrac{b-a}{N}$ the step length and N the number of mesh points. Therefore,

$$\frac{v_{i+1} - v_i}{h} = \sum_k \frac{4\pi^2 M_k(x_i - x_{i_k})}{r_k^3}$$

$$\Rightarrow v_{i+1} = h\sum_k \frac{4\pi^2 M_k(x_i - x_{i_k})}{r_k^3} + v_i \tag{8}$$

Velocity. The velocity can easily be discretized as

$$v_i = \frac{x_{i+1} - x_i}{h}$$

Therefore,

$$x_{i+1} = hv_i + x_i \tag{9}$$

The initial position and velocity of the body are obtained with the data from the NASA website.

1.4.2 Verlet's method

The Verlets method is almost the same as the Euler's one but is more precise. It is based on a Taylor expansion of the position. We consider two instants $t_i + h$ as $t_i - h$:

$$\begin{cases} x_{i+1} = x(t_i + h) = x(t_i) + h\dfrac{dx}{dt}(t_i) + \dfrac{h^2}{2!}\dfrac{d^2x}{dt^2}(t_i) + \dfrac{h^3}{3!}\dfrac{d^3x}{dt^3}(t_i) + \mathcal{O}(h^4) \\ x_{i-1} = x(t_i - h) = x(t_i) - h\dfrac{dx}{dt}(t_i) + \dfrac{h^2}{2!}\dfrac{d^2x}{dt^2}(t_i) - \dfrac{h^3}{3!}\dfrac{d^3x}{dt^3}(t_i) + \mathcal{O}(h^4) \end{cases}$$

We sum up the two lines to have

$$x_{i+1} + x_{i-1} = 2x_i + h^2\frac{d^2x_i}{dt^2} + \mathcal{O}(h^4)$$
$$\Rightarrow x_{i+1} = 2x_i - x_{i-1} + h^2\frac{d^2x_i}{dt^2} + \mathcal{O}(h^4)$$

Chapter 2

Implementation

2.1 Earth-Sun system

2.1.1

2.1.2 Tests

The mechanical energy $E_m = E_c + E_p$ should be conserved because the only force taken into account is the gravitational force, which is conservative. The potential energy is constant in time ($E_p = -\frac{GM_\oplus M_\odot}{r} \forall t$), so is conserved. The kinetic energy depends on the velocity of the Earth ($E_c = \frac{1}{2}M_\oplus v^2$, which is constant in time as we have a uniform circular motion of the Earth around the Sun. So as both of the kinetic and potential energies are constant in time, for any t_i and t_f we have

$$\begin{cases} E_{c_i} = E_{p_f} \\ E_{p_i} = E_{p_f} \end{cases} \Rightarrow E_{c_i} + E_{p_i} = E_{c_f} + E_{p_f}$$

$$\Rightarrow E_{m_i} = E_{m_f}$$

2.2 Adding one planet

2.3 The complete Solar system

Chapter 3

Results

3.1 Earth-Sun system

3.1.1

3.2 Three-body problem

3.2.1 The Sun as the center-of-mass

3.2.2 The real center-of-mass

3.3 Complete Solar System

3.3.1

3.3.2

Conclusion

Bibliography

- NASA website http://ssd.jpl.nasa.gov/horizons.cgi#top
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