

FYS3150  
Project 3 -

Hugounet, Antoine & Villeneuve, Ethel

September 2017  
University of Oslo  
<https://github.com/kryzar/Perseids.git>

## **Abstract**

# Contents

# Introduction

# Chapter 1

## Theory

In a first place, we will begin with an Earth-Sun system with the Earth orbiting around the Sun. We will test our simulation on this simple case and add the other planets afterwards.

### 1.1 Earth-Sun system

#### 1.1.1 Physical conditions of the system

The only force applied to this system is the gravity. According to the Newton's law, we have

$$F_G = \frac{GM_\odot M_\oplus}{r^2} \quad (1.1)$$

with  $F_G$  the norm of the gravitational force,  $G$  the gravitational constant ( $G = 6.674 \times 10^{-11} \text{N.m}^2.\text{kg}^{-2}$ ),  $M_\odot$  the mass of the Sun,  $M_\oplus$  the mass of the Earth and  $r$  the distance between the Earth and the Sun.

We will neglect the motion of the Sun here as the mass of the Sun is much larger than the mass of the Earth ( $M_\odot = 2 \times 10^{30} \text{kg}$  against  $M_\oplus = 6 \times 10^{24} \text{kg}$ ). We want to establish the motion of the Earth around the Sun. Moreover, we will assume that the orbit of the Earth around the Sun is coplanar in the  $xy$ -plane.

The Newton's second law of motion states that the total force applied to any system in a Galilean referential is  $\vec{F} = M\vec{a}$  with  $M$  the mass of the body concerned and  $\vec{a}$  the acceleration. Applied to our case, we have  $\vec{F}_G = M_\oplus \vec{a}$ . Since  $a$  is the second derivative of the position :

$$F_{G,x} = M_\oplus \frac{d^2x}{dt^2}$$

$$F_{G,y} = M_\oplus \frac{d^2y}{dt^2}$$

or

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_\oplus} \quad (1)$$

$$\frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_\oplus}. \quad (2)$$

with  $F_{G,x}$  and  $F_{G,y}$  the components of the gravitational force.

**Remark 1.1.** *We will not use the SI units but the Astronomical units (AU) for the distances (with  $1\text{AU}=\text{average distance Earth-Sun} = 1.5 \times 10^{11}\text{m}$ ), kg for masses and years for time units. Thus the initial position of the Earth will be  $x_{\oplus} = 1$ ,  $y_{\oplus} = 0$  with the Sun the origin ( $x_{\odot} = 0$ ,  $y_{\odot} = 0$ ).*

To simplify a little bit, we will assume that the Earth's orbit is circular around the Sun. So we can write :

$$F_G = \frac{M_{\oplus}v^2}{r} = \frac{GM_{\odot}M_{\oplus}}{r^2} \quad (3)$$

with  $v$  the velocity of Earth. From here we have

$$v^2r = GM_{\odot} = 4\pi^2\text{AU}^3/\text{yr}^2 \quad (4)$$

### 1.1.2 Escape velocity

In the case where only the gravitational force is applied, the mechanical energy  $E_m = E_c + E_p$  with  $E_c$  the kinetic energy and  $E_p$  the potential energy is conserved (the gravitational force is conservative). For a planet at a distance of 1 AU from the Sun (let's take the Earth as an example), the kinetic energy is given by  $E_c = \frac{1}{2}M_{\oplus} \times v^2$  and the potential energy by  $E_p = -\frac{GM_{\odot}M_{\oplus}}{r}$  with  $r = 1$  AU.

We want this planet to escape from the gravitational attraction of the Sun on it. The escape velocity will be the minimal speed  $v_e$  at which the planet escapes from the gravitational influence of the Sun. We consider two positions of the planet :  $p_i$  the initial position when  $r_i = 1\text{AU}$  and  $p_f$  the final position when the planet has escaped and is at an infinite distance from the Sun  $r_f = \infty$ .

Let's consider the mechanical energy at the final position. We have  $E_{m_f} = E_{c_f} + E_{p_f}$ ,  $E_{c_f} = 0$  as the velocity of the planet is zero at the final state,  $E_{p_f} = 0$  since  $r_f = \infty$  and  $E_{m_f} = 0$ .

So, with the planet at 1 AU from the Sun at its initial position, of which velocity is the escape velocity  $v_e$ , in the case where there is only the gravitational force applied to the system planet-Sun, we have :

$$E_{c_i} + E_{p_i} = \frac{1}{2}M_{\oplus} \times v_e^2 - \frac{GM_{\odot}M_{\oplus}}{r_i} = E_{c_f} + E_{p_f} = 0$$

according to the law of conservation of energy. From this we can write

$$\begin{aligned} \frac{1}{2}M_{\oplus} \times v_e^2 - \frac{GM_{\odot}M_{\oplus}}{r_i} &= 0 \\ \Rightarrow v_e &= \sqrt{\frac{2GM_{\odot}}{r_i}} \\ \Rightarrow v_e &= \sqrt{2GM_{\odot}} \end{aligned}$$

which only depends on the mass of the Sun.

$$\begin{aligned} v_e &= \sqrt{2 \times 4\pi^2}, \quad \text{from (4)} \\ v_e &= 2\sqrt{2}\pi \text{ AU/yr} \end{aligned}$$

Then, for a velocity  $v \geq 2\sqrt{2}\pi$ , the Earth would escape from the Sun's gravitational attraction.

## 1.2 Three-body problem

We add Jupiter to the previous system and place it on the same  $x$ -axis that the Earth is on.

### 1.2.1 The Sun as the center-of-mass

In the previous case, the Earth's motion had a stable circular orbit in time. Adding Jupiter, the Earth's motion will inevitably be disturbed. We can write the gravitational between the Earth and Jupiter as

$$F_{\oplus-\jmath} = \frac{GM_{\jmath}M_{\oplus}}{r_{\oplus-\jmath}^2}$$

with  $M_{\jmath}$  the mass of Jupiter and  $r_{\oplus-\jmath}$  the distance between the Earth and Jupiter. The algorithm used for this case will be exactly the same as the previous case. We will see in section 1.4 how to discretize and implement the algorithm.

### 1.2.2 Real center-of-mass

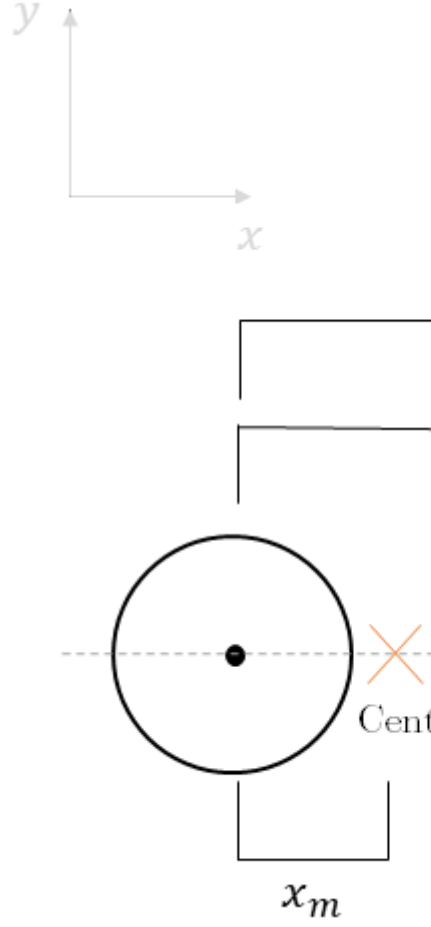
To get closer to the reality, we now do not consider the Sun as a static body anymore. Each of three bodies are in motion. Moreover, the center-of-mass will not be the Sun but the real one. Let's compute the center-of-mass coordinates.

$$\begin{cases} x_c = \frac{1}{M_{tot}} \sum_i M_i x_i \\ y_c = \frac{1}{M_{tot}} \sum_i M_i y_i \end{cases}$$

We consider the system Earth-Jupiter-Sun.

$$\begin{aligned} M_{tot} &= M_{\oplus} + M_{\jmath} + M_{\odot} \\ &= 6 \times 10^{24} + 1.9 \times 10^{27} + 2 \times 10^{30} \\ &= 2.00191 \times 10^{30} \text{ kg} \end{aligned}$$

$$\begin{cases} x_c = \frac{1}{M_{tot}} [M_{\oplus} x_{\oplus} + M_{\jmath} x_{\jmath} + M_{\odot} x_{\odot}] \\ y_c = \frac{1}{M_{tot}} [M_{\oplus} y_{\oplus} + M_{\jmath} y_{\jmath} + M_{\odot} y_{\odot}] \end{cases}$$



We fix the initial positions with respect to the new center of mass.  
So we look for the  $x_m$  so that  $x_c$  and  $y_c$  are equal to zero.

$$\begin{cases} x_c = \frac{1}{M_{tot}}[(2 \times 10^{24}) \times (1 + x_m) + (1.9 \times 10^{27}) \times (5.20 + x_m) + (2 \times 10^{30}) \times x_m] = 0 \\ y_c = 0 \end{cases}$$

$$\begin{cases} \frac{1}{2.00191 \times 10^{30}}[x_m(2 \times 10^{24} + 1.9 \times 10^{27} + 2 \times 10^{30}) + (2 \times 10^{24} + 5.20 \times 1.9 \times 10^{27})] = 0 \\ y_m = 0 \end{cases}$$

$$\begin{cases} x_m = -\frac{2 \times 10^{24} + 5.20 \times 1.9 \times 10^{27}}{M_{tot}} \\ y_m = 0 \end{cases}$$

$$\begin{cases} x_m = -4.936 \times 10^{-3} \\ y_m = 0 \end{cases} \quad (5)$$

These are the coordinates of the Sun when the center-of-mass of the system is set as the origin. Let's now compute the Sun's initial velocity so that the total momentum of the



three-body system would be equal to zero.

$$\begin{aligned}
\vec{p}_{tot} &= \sum_k M_k \vec{k} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= M_{\oplus} \begin{pmatrix} v_{\oplus x} \\ v_{\oplus y} \end{pmatrix} + M_{\text{J}} \begin{pmatrix} v_{\text{J}x} \\ v_{\text{J}y} \end{pmatrix} + M_{\odot} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\
\begin{cases} M_{\oplus} v_{\oplus x} + M_{\text{J}} v_{\text{J}x} + M_{\odot} v_x = 0 \\ M_{\oplus} v_{\oplus y} + M_{\text{J}} v_{\text{J}y} + M_{\odot} v_y = 0 \end{cases}
\end{aligned}$$

We take the initial velocities of the Earth and Jupiter so that their orbit will be circular. If the orbit of the Earth is circular,  $v_{\oplus} = 2\pi$  AU/yr, because the distance travelled in one Earth-year is  $2\pi \times r$  with  $r = 1$  AU. The velocity is constant so the initial velocity will be  $v_{\oplus i} = 2\pi$  AU/yr. In the initial configuration represented above, the initial velocity is

oriented in the direction of  $y$ . So,  $v_{\oplus i} = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix}$

Jupiter has an orbital period of 11.862yr. The distance travelled by Jupiter in this period is  $2\pi \times r = 2\pi \times 5.20 = 32.673 \text{ AU}$ . So  $v_{\text{J}i} = \frac{2\pi \times r}{T} = \frac{2\pi \times 5.20}{11.862} = 2.7544$  AU/yr. Similarly, the initial velocity can be expressed as  $v_{\text{J}i} = \begin{pmatrix} 0 \\ 2.7544 \end{pmatrix}$

$$\begin{aligned}
\begin{cases} (6 \times 10^{24}) \times 0 + (1.9 \times 10^{27}) \times 0 + (2 \times 10^{30}) \times v_x = 0 \\ (6 \times 10^{24}) \times 2\pi + (1.9 \times 10^{27}) \times 2.7544 + (2 \times 10^{30}) \times v_y = 0 \end{cases} \\
\begin{cases} v_x = 0 \\ v_y = -\frac{(6 \times 10^{24}) \times 2\pi + (1.9 \times 10^{27}) \times 2.7544}{2 \times 10^{30}} \end{cases} \\
\begin{cases} v_x = 0 \\ v_y = 2.6355 \times 10^{-3} \end{cases} \tag{6}
\end{aligned}$$

To have a fixed center-of-mass, the Sun will need a velocity  $v = \begin{pmatrix} 0 \\ 2.6355 \times 10^{-3} \end{pmatrix}$ .

### 1.3 The complete Solar system

### 1.4 Discretization

#### 1.4.1 Euler's method

**Acceleration.** To implement the accelerations, we normalize the masses with respect to the mass of the Sun. Thus,  $M_{\odot} = 1$ ,  $M_{k_N} = \frac{M_k}{M_{\odot}}$  is the normalized mass for each body  $k$ .

From the Newton's law of motion, the acceleration of a given body can be written as  $a = \frac{F_G}{M}$ , with  $M$  the mass of the body.

$$a_i = \frac{\sum_k \frac{G M M_{k_N}}{r_k^2}}{M}$$

with  $k$  the number of bodies involved in the system.

$$a_i = \sum_k \frac{G M_{k_N}}{r_k^2}$$

From (4), we have  $GM_\odot = 4\pi^2 \text{AU}^3/\text{yr}^2$  and as  $M_\odot = 1$ ,  $G = 4\pi^2 \text{AU}^3/\text{yr}^2$ . So,

$$\begin{aligned} a_i &= \sum_k \frac{4\pi^2 M_{k_N}}{r_k^2} \\ a_i &= \sum_k \frac{4\pi^2 M_{k_N} (x_i - x_{i_k})}{r_k^3} \end{aligned} \quad (7)$$

With Euler's method, we can write  $a_i = \frac{v_{i+1} - v_i}{h}$  with  $h = \frac{b-a}{N}$  the step length and  $N$  the number of mesh points. Therefore,

$$\begin{aligned} \frac{v_{i+1} - v_i}{h} &= \sum_k \frac{4\pi^2 M_k (x_i - x_{i_k})}{r_k^3} \\ \Rightarrow v_{i+1} &= h \sum_k \frac{4\pi^2 M_k (x_i - x_{i_k})}{r_k^3} + v_i \end{aligned} \quad (8)$$

**Velocity.** The velocity can easily be discretized as

$$v_i = \frac{x_{i+1} - x_i}{h}$$

Therefore,

$$x_{i+1} = h v_i + x_i \quad (9)$$

The initial position and velocity of the body are obtained with the data from the NASA website.

### 1.4.2 Verlet's method

The Verlets method is almost the same as the Euler's one but is more precise. It is based on a Taylor expansion of the position. We consider two instants  $t_i + h$  as  $t_i - h$  :

$$\begin{cases} x_{i+1} = x(t_i + h) = x(t_i) + h \frac{dx}{dt}(t_i) + \frac{h^2}{2!} \frac{d^2x}{dt^2}(t_i) + \frac{h^3}{3!} \frac{d^3x}{dt^3}(t_i) + \mathcal{O}(h^4) \\ x_{i-1} = x(t_i - h) = x(t_i) - h \frac{dx}{dt}(t_i) + \frac{h^2}{2!} \frac{d^2x}{dt^2}(t_i) - \frac{h^3}{3!} \frac{d^3x}{dt^3}(t_i) + \mathcal{O}(h^4) \end{cases}$$

We sum up the two lines to have

$$\begin{aligned}x_{i+1} + x_{i-1} &= 2x_i + h^2 \frac{d^2 x_i}{dt^2} + \mathcal{O}(h^4) \\ \Rightarrow x_{i+1} &= 2x_i - x_{i-1} + h^2 \frac{d^2 x_i}{dt^2} + \mathcal{O}(h^4)\end{aligned}$$

## Chapter 2

# Implementation

### 2.1 Earth-Sun system

#### 2.1.1

#### 2.1.2 Tests

The mechanical energy  $E_m = E_c + E_p$  should be conserved because the only force taken into account is the gravitational force, which is conservative. The potential energy is constant in time ( $E_p = -\frac{GM_\oplus M_\odot}{r} \forall t$ ), so is conserved. The kinetic energy depends on the velocity of the Earth ( $E_c = \frac{1}{2}M_\oplus v^2$ , which is constant in time as we have a uniform circular motion of the Earth around the Sun. So as both of the kinetic and potential energies are constant in time, for any  $t_i$  and  $t_f$  we have

$$\begin{aligned} \begin{cases} E_{c_i} = E_{p_f} \\ E_{p_i} = E_{p_f} \end{cases} &\Rightarrow E_{c_i} + E_{p_i} = E_{c_f} + E_{p_f} \\ &\Rightarrow E_{m_i} = E_{m_f} \end{aligned}$$

### 2.2 Adding one planet

### 2.3 The complete Solar system

## Chapter 3

# Results

### 3.1 Earth-Sun system

#### 3.1.1

### 3.2 Three-body problem

#### 3.2.1 The Sun as the center-of-mass

#### 3.2.2 The real center-of-mass

### 3.3 Complete Solar System

#### 3.3.1

#### 3.3.2

# Conclusion

# Bibliography

- NASA website <http://ssd.jpl.nasa.gov/horizons.cgi#top>
-