APUNTS PROPIETAT DE:
MÀRIUS SERRA LÓPEZ

PER QUALSEVOL DUBTE O CONSULTA RESPECTE ELS APUNTS O EL QUE FACI FALTA ESCRIURE A:
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(l'assumpte ha de ser: Apunts ETSETB)

QUALSEVOL ERROR PRESENT S'ATRIBUEIX AL PROFE DE L'ASSIGNATURA QUE ME LA VA IMPARTIR!!

PROBLEMA

$$\int_0^{2\pi} \frac{dt}{1 + a \sin t} \qquad a \in \mathbb{R}, \; |a| < 1$$

$$z = e^{jt}$$

$$|z| = 1$$

$$dz = j e^{jt} dt \Rightarrow dt = \frac{1}{jz} dz$$

$$\sin t = \frac{e^{jt} - e^{-jt}}{2j} = \frac{z - \frac{1}{z}}{2j} = \frac{z^2 - 1}{2jz}$$

$$= \oint_C \frac{\frac{1}{jz}}{1 + a \frac{z^2-1}{2jz}} dz = 2 \oint_C \frac{dz}{2jz + az^2 - a} =$$

$$= \frac{2}{a} 2\pi j \, \mathrm{Res}\left(f, \frac{-j}{z} \right. \qquad = \frac{2\pi}{\sqrt{1-a^2}}$$

PROBLEMES

30-5-06

$$\oint_C \frac{z}{(9 - z^2)(z + j)} dz \qquad C \equiv |z| = 2$$

Recorregut en sentit positiu

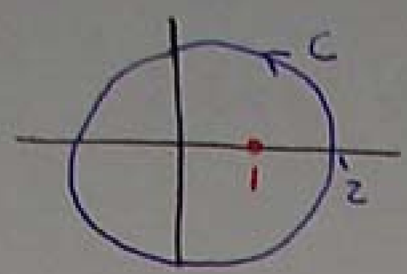
$\Rightarrow$ Pel. Teorema del Residu

$$= 2\pi j \cdot \mathrm{Res}(f, -j) = 2\pi j \lim_{z \to -j} (z + j) \cdot f(z) =$$

$$= 2\pi j \lim_{z \to -j} (z+j) \frac{z}{(9-z^2)(z+j)} = 2\pi j \left(\frac{-j}{10} \right) = \frac{\pi}{5}$$

* També es podia utilitzar la Formula Integral de Cauchy

PROBLEMA



$$\oint_C \frac{z^3+2z}{(z-1)^3}\,dz = 2\pi j \operatorname{Res}(f,1) =$$

$$= 2\pi j \lim_{z\to 1} \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z^3+2z}{(z-1)^3}\cdot(z-1)^3\right] = 2\pi j \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[z^3+2z\right] =$$

$$= 2\pi j \lim_{z\to 1}\frac{1}{2!}\left[6z\right] = 6\pi j$$

* $z=1$ és el pol d'ordre 3 de $f(z)$

PROBLEMA

$2y^2 = x+1$

$x = 2y^2-1$

Regió que no inclou el $z=0$

$$\int_{1+j}^{1-j} \frac{dz}{z^2}$$

a) 1

b) -1

c) j

d) -j

$$\int_{1+j}^{1-j}\frac{dz}{z^2} = \left.\frac{-1}{z}\right|_{1+j}^{1-j} = \frac{-1}{1-j}+\frac{1}{1+j} =$$

$$= \frac{-1-j+1-j}{1+1} = -j$$

* També podiem tancar la corba i fer servir el Teo. Integral de Cauchy

Però llavors li hem de restar la integral sobre la corba C_2.

PROBLEMA

$$\sum_{-\infty}^{+\infty} z^n$$

a) divergent $\forall z \in \mathbb{C}$

b) Defineix una f analítica en $|z|<1$

c) " " " " " $|z|>1$

d) " " " " " $\frac{1}{2}<|z|<2$

$$\sum_{-\infty}^{+\infty} z^n = \underbrace{\cdots \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}}_{\text{Part Principal}} + \underbrace{1+z+z^2+z^3+\cdots}_{\text{Part analítica}}$$

$$\frac{1}{z}\left.\frac{1}{1-\frac{1}{z}}\right|_{\left|\frac{1}{z}\right|<1 \to |z|>1}$$

Part Principal $\to \frac{1}{1-z}$ si $|z|>1$

Part analítica $\to \frac{1}{1-z}$ si $|z|<1$

$\leadsto$ Veiem que quan un convergeix, l'altre divergeix

PROBLEMA

$$f(z) = \frac{\sin z - z}{z^6} \underset{\text{Taylor}}{=} \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots - z}{z^6} = \frac{-\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots}{z^6} =$$

$$= \underbrace{\frac{-1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z}}_{\text{Part Principal}} \underbrace{- \frac{1}{7!} z + \frac{1}{9!} z^3 \ldots}_{\text{Part analítica}}$$

$z=0$ és un pol d'ordre 3

Residu de f en $z=0$ és $\frac{1}{5!}$

$$c_{-1} = \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[z^3 \frac{\sin z - z}{z^6}\right] = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}\left[\frac{\sin z - z}{z^3}\right] =$$

$$= \frac{1}{2!} \lim_{z \to 0} \left[\frac{(\cos z - 1) z^3 - 3z^2(\sin z - z)}{z^6}\right] =$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz}\Bigg[\underbrace{\frac{(\cos z - 1) z - 3(\sin z - z)}{z^4}}_{g'(x)}\Bigg]$$

$$g''(x) = \frac{z^4\left[z(-\sin z) - (\cos z - 1) - 3\cos z + 3\right] - 4z^3\left[(\cos z - 1)z - 3(\sin z - z)\right]}{z^8}$$

$$g''(x) = \frac{z\left[-z\sin z - \cos z + 4 - 3\cos z\right] - 4\left[z\cos z - z - 3\sin z + 3z\right]}{z^5}$$

$$g''(x) = \frac{-z^2\sin z - 4z\cos z + 4z - 4z\cos z + 4z + 12\sin z - 12z}{z^5}$$

$$g''(x) = \frac{-z^2\sin z - 8z\cos z - 4z + 12\sin z}{z^5}$$

$\rightsquigarrow$ Si anem aplicant hôpetal veurem que aconseguim la solució però tardem molt.

PROBLEMA

$$f(z) = \sin\left(\frac{z}{1-z}\right) = \sin\left(\frac{-z}{z-1}\right) = \sin\left(-1-\frac{1}{z-1}\right) = -\sin\left(1+\frac{1}{z-1}\right) =$$

$$= -\sin 1 \cdot \cos\left(\frac{1}{z-1}\right) - \sin\left(\frac{1}{z-1}\right)\cdot \cos 1 = ⊛$$

$$\cos z = \sum_{-\infty}^{+\infty} \frac{(-1)^n \cdot z^{2n}}{(2n)!}$$

$$\sin z = \sum_{-\infty}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

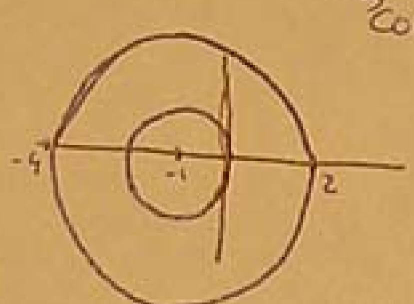
$$⊛ = -\sin(1) \sum_{-\infty}^{+\infty} (-1)^n \frac{1}{(2n)!\,(z-1)^{2n}} - \cos(1) \sum_{-\infty}^{+\infty} \frac{(-1)^n}{(2n+1)!\,(z-1)^{2n+1}}$$

$$= - \sum_{n\geq 0} \frac{\sin\left(1+\frac{\pi}{2}\right)}{n!\,(z-1)^n}$$

PROBLEMA

$$f(z) = \frac{1-2z}{z(z+1)}$$

$1 < |z+1| < 3 \rightarrow$ corona



$$f(z) = \frac{1}{z+1}\,\frac{1-2z}{z} = \frac{1}{z+1}\,\frac{3-2(z+1)}{-1+(z+1)} =$$

$$= \frac{2}{1-(z+1)} - \frac{3}{z+1}\,\frac{1}{1-(z+1)} = \frac{1}{1-(z+1)}\left[2+\frac{-3}{z+1}\right]$$

Si mirem a la corona: $\left|\frac{1}{z+1}\right| < 1$

$$= \frac{-1}{z+1}\,\frac{1}{-\frac{1}{z+1}+1}\left(2-\frac{3}{z+1}\right) =$$

$$= \frac{-1}{z+1}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}\right]\left(2-\frac{3}{z+1}\right)$$

$$\boxed{c_1 = 0 \qquad c_{-1} = -2}$$

PROBLEMA

$$\int_0^{2\pi} \frac{1}{2-\sin\theta}\, d\theta$$

Podem agafar:

Corba de Jordan $C \equiv |z| = 1$

$z = e^{j\theta} \quad 0 \leq \theta < 2\pi \qquad dz = j e^{j\theta} d\theta \rightarrow d\theta = \frac{dz}{j e^{j\theta}} = \frac{dz}{jz}$

$$\int_0^{2\pi} \frac{1}{2-\sin\theta}\, d\theta = \int_{|z|=1} \frac{1}{\frac{j2z}{jz} - \left[\frac{z^2-1}{2jz}\right]} \frac{1}{jz}\, dz = ⊛$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{e^{j\theta 2} - 1}{2e^{j\theta}\cdot j} = \frac{z^2-1}{2jz}$$

$$⊛ = \int_{|z|=1} \frac{1}{2jz - \frac{z^2}{2} + \frac{1}{2}}\, dz = \int_{|z|=1} \frac{2}{4jz - z^2 + 1}\, dz = -2\int_{|z|=1} \frac{1}{z^2 - 4jz - 1}\, dz =$$

$z^2 - 4jz - 1 = 0$

$$z = \frac{4j \pm \sqrt{-16+4}}{2} \begin{cases} z = 2j + j\sqrt{3} \\ z = 2j - j\sqrt{3} \end{cases}$$

$$= -2(2\pi j)\, \mathrm{Res}\left(f, j(2-\sqrt{3})\right) =$$

$$= -4\pi j \lim_{z \to j(2-\sqrt{3})} \left[(z - j(2-\sqrt{3})) \frac{1}{[z - (2j + j\sqrt{3})][z - j(2-\sqrt{3})]} \right]$$

$$= -4\pi j \lim_{z \to j(2-\sqrt{3})} \frac{1}{z - j(2+\sqrt{3})} =$$

$$= -4\pi j \frac{1}{j(2-\sqrt{3}) - j(2+\sqrt{3})} = \frac{-4\pi j}{-2j\sqrt{3}} =$$

$$= \frac{2\pi}{\sqrt{3}}$$

PROBLEMA

$$f(z) = \frac{z^4 + 3z^2}{1 - z^2} \qquad \left.\frac{d^8 f(z)}{dz^8}\right|_{z=0}$$

$$f(z) = (z^4 + 3z^2)\,\frac{1}{1-z^2} = (z^4 + 3z^2)\left[1 + z^2 + z^4 + z^6 + \dots\right]$$

$|z|<1$

El coeficient de z^8 és $\left.\begin{array}{l} z^4 \cdot z^4 = 1z^8 \\ 3z^2 \cdot z^6 = 3z^8 \end{array}\right\} 4z^8 \qquad \left.\frac{d^8 f(z)}{dz^8}\right|_{z=0} = \frac{4}{8!}$

PROBLEMA

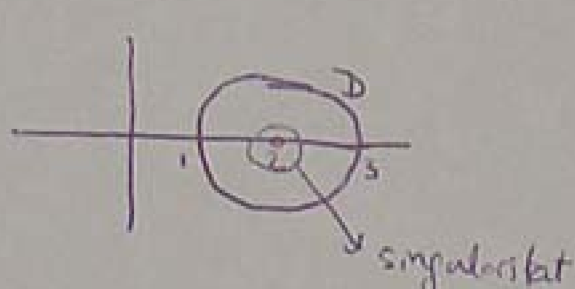
$D = \{ z \in \mathbb{C},\ |z-2| < 1 \} \qquad f(z) = \dfrac{(z^2-1)(z-2)^3}{\sin^3(\pi z)}$

a) f té singularitat essencial en D

b) f té pol d'ordre 1 en D

c) f és analítica en D amb independència del valor $f(2)$

d) " " " " si es defineix $f(2) = \frac{3}{\pi^3}$



Sing. essencial ≡ Pol d'ordre zero $\Rightarrow$ $0 \neq \lim\limits_{z\to 2} f(z) = \lim\limits_{z\to 2} \dfrac{(z^2-1)(z-2)^3}{\sin^3(\pi z)} \overset{\text{Hop.}}{=}$

$$= \lim_{z\to 2} \frac{\left[2z(z-2)^3 + (z^2-1)\,3(z-2)^2\right]\sin^3(\pi z) - 3\sin^2(\pi z)\cdot\cos(\pi z)\cdot\pi\,(z^2-1)(z-2)^3}{\sin^6(\pi z)}$$

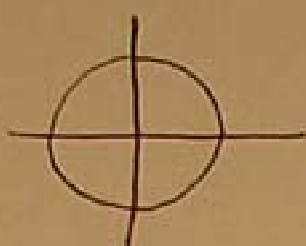
$$= \lim_{z\to 2} \frac{2z(z-2)^3 + (z^2+(-1))\,3(z-2)^2}{3\sin^2(\pi z)\cdot\pi\cos(\pi z)} = \lim_{z\to 2}$$ ⇝ Per aquest camí no acabariem

$\sin \pi z = \sin \pi(z-2) \simeq \pi(z-2)$

$$\lim_{z\to 2} \frac{(z^2-1)(z-2)^3}{\sin^3(\pi(z-2))} = \lim_{z\to 2} \frac{(z^2-1)(z-2)^3}{\pi^3 (z-2)^3} = \frac{3}{\pi^3} = f(z)$$

$\boxed{\text{PROBLEMA}}$

31-5-06



$$f(z) = \frac{1}{z^3+6z^2+9z} \quad . \; 0<|z|<3 \;\Rightarrow\; \left|\frac{z}{3}\right| < 1$$

$z=0$ és una singularitat aïllada de $f(z)$, a més es tracte de un pol simple, és a dir d'ordre 1.

En efecte:

$$\lim_{z\to 0} z \cdot \frac{1}{z^3+6z^2+9z} = \lim_{z\to 0} \frac{1}{z^2+6z+9} = \frac{1}{9} \neq 0$$

En aquest cas (pol d'ordre 1) ja sabem que $c_{-1} = \frac{1}{9}$

El desenvolupament en sèrie de Laurent, al voltant del zero, només presenta un terme a la part principal:

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \dots$$

En concret:

$$f(z) = \frac{1}{z}\frac{1}{(z+3)^2} = \frac{1}{z}\frac{1}{3^2}\left(\frac{1}{\frac{z}{3}+1}\right)^2 \overset{\left|\frac{z}{3}\right|<1}{=} \frac{1}{9z}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 \cdots\right)^2 =$$

$$= \frac{1}{9z}\left(1 - 2\frac{z}{3} + 3\left(\frac{z}{3}\right)^2 + (-4)\left(\frac{z}{3}\right)^3 \cdots\right) = \frac{1}{3^2 z} - \frac{2}{3^3} + \frac{3z}{3^4} - \frac{4z^2}{3^5} \cdots$$

$$\boxed{f(z) = \sum_{n\geq 0} (-1)^n \frac{n+1}{3^{n+2}} z^{n-1}}$$

PROBLEMA

a) $|e^{jz}| = |\cos z + j\sin z|$

b) $|e^{jz}| = \cos^2 z + \sin^2 z$

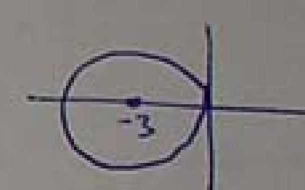
c) $\cos^2 z + \sin^2 z = 1$

d) $e^{jz} = \cos z + j \sin z$

$$\cos z = \cos(x+jy) = \cos x \cdot \text{ch}\, y$$
$$\sin z = \sin(x+jy) = \sin x \cdot \text{sh}\, y$$

PROBLEMA

$$f(z) = \frac{1+z}{z(z+3)} \quad , \quad 0 < |z+3| < 3 \Rightarrow \left|\frac{z+3}{3}\right| < 1$$



$$f(z) = \frac{1}{z+3}\,\frac{1+z}{z} = \frac{1}{z+3}\left(1+\frac{1}{z}\right) =$$

$$= \frac{1}{z+3}\left(1+\frac{1}{(z+3)-3}\right) = \frac{1}{z+3}\left(1 - \frac{1}{3}\,\frac{1}{1-\frac{z+3}{3}}\right) \underset{\left|\frac{z+3}{3}\right|<1}{=}$$

$$= \frac{1}{z+3}\left[1 - \frac{1}{3}\left(1 + \frac{z+3}{3} + \left(\frac{z+3}{3}\right)^2 + \cdots\right)\right] =$$

$$= \frac{2}{3}\,\frac{1}{z+3} - \frac{1}{9} - \frac{z+3}{9} + \cdots$$

PROBLEMA

$$f(z) = \frac{1}{\cos\left(\frac{1}{z-1}\right)}$$

té en $z=1$

a) singularitat essencial aïllada

b) pol simple

c) punt singular no aïllat

d) pol múltiple

$f(z)$ no és analítica...

$$\frac{1}{z-1} = \frac{\pi}{2} + k\pi = \pi\left(k+\frac{1}{2}\right)$$
$$\Downarrow$$
$$1 = \pi(z-1)\left(k+\frac{1}{2}\right)$$

$$z_k = \frac{1+\pi\left(k+\frac{1}{2}\right)}{\pi\left(k+\frac{1}{2}\right)} = 1 + \frac{1}{k\pi + \frac{\pi}{2}}$$

$$\lim_{n\to\infty} z_n = 1$$

PROBLEMA

$$f(z) = \frac{z}{z^3 + z^2(2-2j) - z(1+4j) - 2}$$ té en $z = j$ un residu que val?

$$\begin{array}{c|cccc} & 1 & 2-2j & -1-4j & -2 \\ j & & +j & 1+2j & 2 \\ \hline & 1 & 2-j & -2j & 0 \end{array}$$

$$f(z) = \frac{z}{(z+j)(z^2 + z(2-j) - 2j)}$$

$$z = \frac{(j-2) \pm \sqrt{-1+4-4j+8j}}{2} \nearrow z = \frac{j-2+\sqrt{3+4j}}{2}$$ ⇝ No es pot aplicar perquè són complexes

$$\begin{array}{c|ccc} & 1 & 2-j & -2j \\ j & & i & 2j \\ \hline & 1 & 2 & 0 \end{array}$$

$$f(z) = \frac{z}{(z+2)(z+j)(z+j)} = \frac{z}{(z+j)^2(z+2)}$$

$$\lim_{z\to j} (z+j)^2 \frac{z}{(z+j)^2(z+2)} = \frac{j}{j+2} \neq 0 \Rightarrow \text{pol doble}$$

$$c_{-1} = \lim_{z\to j} \frac{d}{dz}\left[(z+j)^2 \frac{z}{(z+j)^2(z+2)}\right] = \lim_{z\to j} \frac{d}{dz}\left[\frac{z}{z+2}\right] = \lim_{z\to j} \frac{2}{(z+2)^2} =$$

$$f(x) = \frac{x}{x+2} \qquad f'(x) = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

$$= \frac{2}{(j+2)^2}$$

PROBLEMA

$$f(z) = \frac{z+1}{z^2 + (j-2)z - 2j}$$

a) Taylor vàlid quan $|z-j| < 5$

b) " vàlid " $|z-j| < \sqrt{5}$

c) Laurent " " $0 < |z-j| < \sqrt{5}$

d) " " " $|z-j| > \sqrt{5}$

$$\begin{array}{c|ccc} & 1 & j-2 & -2j \\ -j & & -j & 2j \\ \hline & 1 & -2 & 0 \end{array}$$

$$f(z) = \frac{z+1}{(z+j)(z-2)} = \frac{1}{z+j}\left[1 + \frac{3}{z-2}\right] =$$

$$= \frac{1}{z+j}\left[1 + \frac{3}{-2-j+z+j}\right]$$

$\frac{1}{z+j} = \frac{1}{2j+z-j}$, $\frac{3}{z-2} = \frac{3}{-2+j+z-j}$

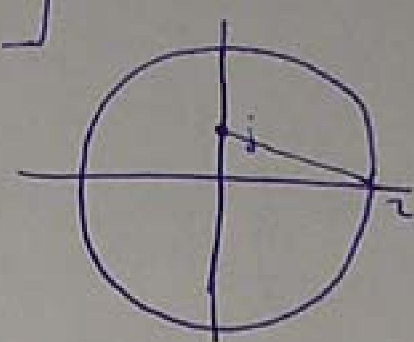
$$f(z) = \frac{1}{2j+z-j}\left[1+\frac{3}{-2+j+z-j}\right] =$$

$\rightsquigarrow$ ① ¿ $\left|\frac{2j}{z-j}\right|<1$, $\left|\frac{z-j}{2j}\right|<1$?

$$=\frac{1}{2j+z-j}\left[1+\frac{3}{j-2}\,\frac{1}{1+\frac{z-j}{j-2}}\right]=$$

$|j-2| = \sqrt{1+4} = \sqrt{5}$

② ¿ $\left|\frac{z-j}{j-2}\right|<1$, $\left|\frac{j-2}{z-j}\right|<1$?



② $\begin{cases} |z-j|<\sqrt{5} \\ |z-j|>\sqrt{5} \end{cases}$

① $\begin{cases} |z-j|<2<\sqrt{5} \\ |z-j|>2 \end{cases}$

$$= \frac{1}{2j}\,\frac{1}{1+\frac{z-j}{2j}}\left(1+\frac{3}{j-2}\,\frac{1}{1+\frac{z-j}{j-2}}\right) =$$

$$= \left(\frac{1}{2j}\left(1-\frac{z-j}{2j}+\frac{(z-j)^2}{(2j)^2}+\cdots\right)\right)\left(1+\frac{3}{j-2}\left(1-\frac{z-j}{j-2}+\frac{(z-j)^2}{(j-2)^2}\cdots\right)\right)$$

Ha de complir $|z-j| < 2$

Ha de complir $|z-j| > \sqrt{5}$

$\Downarrow$

Com que no es poden complir les desigualtats alhora, ha de ser una serie de Laurent!!

$\boxed{\text{PROBLEMA}}$

i) $C \equiv |z|=1$

$$\oint_C \frac{dz}{\underbrace{z^3-3z^2}_{f(z)}} = \oint_C \frac{dz}{z^2(-3+z)}$$

$$= 2\pi j \,\mathrm{Res}(f,0) = 2\pi j \lim_{z\to 0} \frac{d}{dz}\left[z^2\,\frac{1}{z^2(z-3)}\right] = 2\pi j \lim_{z\to 0}\left[\frac{-1}{(z-3)^2}\right] = \frac{2\pi j}{-9}$$

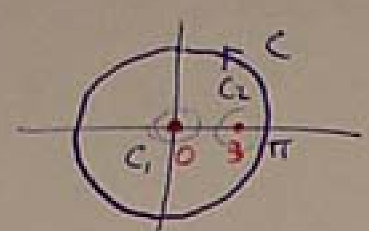
⇒ Una altra forma:

$$\oint_C \frac{\frac{1}{z-3}}{z^2}\,dz = \frac{2\pi j}{1!}\, g'(0) = \frac{-2\pi j}{9}$$

Formula Integral de Cauchy

$$g(x) = \frac{1}{x-3} \qquad g'(x) = \frac{-1}{(x-3)^2} \qquad g'(0) = \frac{-1}{9}$$

ii) $C: |z| = \pi$



$$\oint_C \frac{1}{z^2(z-3)}\,dz = \oint_{C_1} \frac{1}{z^2(z-3)}\,dz + \oint_{C_2} \frac{1}{z^2(z-3)}\,dz =$$

$$= \oint_{C_1} \frac{\overset{g(z)}{\frac{1}{(z-3)}}}{z^2}\,dz + \oint_{C_2} \frac{\overset{h(z)}{1/z^2}}{z-3}\,dz = 2\pi j\left[g'(0) + h(3) \right] =$$

$$= 2\pi j\left[\frac{-1}{9} + \frac{1}{9}\right] = \boxed{\underline{\underline{0}}!!}$$

$g'(0) = \frac{-1}{9}$

$h(z) = \frac{1}{z^2}$

$h(3) = \frac{1}{9}$

PROBLEMA

$$\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{2e}$$

$$I = \int_C \frac{e^{jz}}{1+z^2}\,dz = 2\pi j\ \mathrm{Res}(f, j) = 2\pi j \lim_{z\to j} (z-j)\frac{e^{jz}}{(z-j)(z+j)} =$$

$$= 2\pi j \lim_{z\to j} \frac{e^{jz}}{z+j} = 2\pi j\,\frac{e^{-1}}{2j} = \pi e^{-1} = \boxed{\frac{\pi}{e}}$$

$$\left|\int_{CR} f\right| \le \int_{CR} \frac{|e^{jz}|}{|1+z^2|}\,|dz| \le \frac{1}{R^2-1}\,\pi R \xrightarrow[R\to\infty]{} 0 \Rightarrow I = \int_{CR} f + \int_{-R}^{R} f \qquad \frac{\pi}{e} = \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$$

$$|e^{jz}| = |e^{jx}\cdot e^{-y}| = |e^{jx}|\cdot|e^{-y}| = e^{-y} \le 1$$

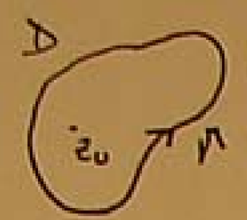
$$|1+z^2| \ge \left|1-|z|^2\right| = \left||z|^2-1\right| = R^2-1$$

$$= \int_{-\infty}^{+\infty} \frac{e^{jx}}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx + j\int_{-\infty}^{+\infty} \frac{\sin x}{1+x^2}\,dx =$$

$$= 2\int_0^\infty \frac{\cos x}{1+x^2}\,dx$$

MÉS PROBLEMES DE VARIABLE COMPLEXA

Fórmula integral de Cauchy:



f és analítica dins i sobre Γ

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0}\, dz$$

Fórmula generalitzada:

En les mateixes condicions

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad , \forall n \geq 0$$

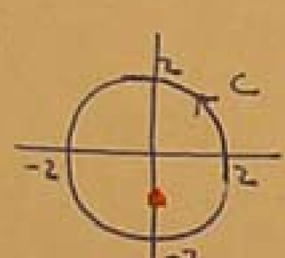
→ Si és derivable 1 cop, ho és infinitament.

Problemes:

$\boxed{\#18}$ Si $C = \{ z \in \mathbb{C} : |z| = 2 \}$, calculeu:

a) $\displaystyle\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$ b) $\displaystyle\oint \frac{z^3+2z}{(z-1)^3}\, dz$

a)



$$g(z) = \frac{z}{(9-z^2)} \rightsquigarrow$$ Arrels: $z = -i$ $z = \pm 3$

$$\oint_C \frac{z/(9-z^2)}{z+i}\, dz = \oint_C \frac{g(z)}{z+i}\, dz \overset{z_0=-i}{=} 2\pi i\, g(-i) = 2\pi i \frac{-i}{9+1} = \frac{2\pi}{10} = \frac{\pi}{5}$$

Teoria:

$g(z) \Rightarrow$ no és analítica en z_0 i z_1

$$\oint_{\Gamma} g(z)\,dz = \oint_{\Gamma_0} g(z)\,dz + \int_{\Gamma_1} g(z)\,dz$$

b)

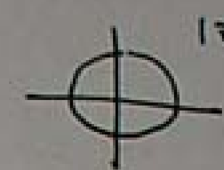
$$\oint_C \frac{z^3+2z}{(z-1)^3}\,dz = \int \frac{f(z)}{(z-1)^3}\,dz = 2\pi i\,\frac{f^{(2)}(+1)}{2!} = 6\pi i$$

$z_0 = 1$, $n=2$

$f'(z) = 3z^2+2$. $f^{(2)}(z) = 6z$

#17

$$\oint \frac{\cos z}{z^n}\,dz = I_n$$



$|z|=1$

$f(z) = \frac{\cos z}{z^n}$

$I_n = 0 \Leftarrow n \leq 0$

$n>0$?

$$I_n = 2\pi i\,\frac{\cos^{(n-1)}(0)}{(n-1)!}$$

$\cos^{(2m)}(0) = (-1)^m \rightarrow$ derivades parells

$n-1 = 2m-1$ (senar) $\quad I_n = 0$

$n-1 = \text{parell} = 2m \quad I_n = \frac{2\pi i}{(n-1)!}(-1)^{\frac{n-1}{2}}$

#21 Calculeu la integral $\frac{1}{2\pi i}\oint_C \frac{e^z}{z(1-z)^3}\,dz$, si:

a) El punt 0 es troba dins i l'1 fora de C.

b) 1 dins, 0 fora.

c) 1 i 0 dins.

a) $f(z) = \frac{e^z}{(1-z)^3}$ $\quad I = \frac{1}{2\pi i}\oint \frac{f(z)}{z}dz = f(0) = \frac{1}{1} = 1$

b) $f(z) = \frac{-e^z}{z}$ $\quad I = \frac{1}{2\pi i}\oint \frac{f(z)}{(1-z)^3}\,dz = \frac{f^{(2)}(1)}{2!} = \frac{-e}{2}$

$f'(z) = -\frac{ze^z - e^z}{z^2} = +e^z(z^{-2} - z^{-1})$

$f''(z) = +e^z(z^{-2}-z^{-1}) + e^z(-2z^{-3}+z^{-2}) = e^z\left(\frac{-1}{z} + \frac{2}{z^2} - \frac{2}{z^3}\right)$

c)

$$I = \frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\left(\oint_{\gamma_0} f + \int_{\gamma_1} f\right) = 1 - \frac{e}{2}$$

↪ A partir dels resultats anteriors

$\boxed{\#22}$ Estudieu si la integral:

$$\int_{-\pi i}^{2\pi i} \underbrace{e^z \cdot \cos(e^z)}_{f(z)}\,dz = F(2\pi i) - F(-\pi i) = \sin(e^{2\pi i}) - \sin(e^{-\pi i}) =$$

$$= \sin(1) - \sin(-1) = 2\sin(1)$$

és independent del camí escollit. Calculeu-la.

$f(z)$ és entera $\Rightarrow$ No depèn del camí escollit en tot $\mathbb{C}$.

$\Downarrow$

Tenim una regla de Barrow en tot $\mathbb{C}$

Una primitiva és $\sin(e^z) = F(z)$

$\boxed{\#23}$ Calculeu:

$$\int_0^{2\pi} e^{e^{it}}\,dt$$

Indicació: Transformeu-la en una integral sobre $|z|=1$

$$z = e^{it} \qquad dz = ie^{it}\,dt \qquad dt = \frac{dz}{iz}$$

$$\int_0^{2\pi} e^{e^{it}}\,dt = \oint_{|z|=1} e^z \frac{dz}{iz} = -i\oint \frac{\overset{f(z)}{e^z}}{z}\,dz = -i \cdot 2\pi i \cdot f(0) = 2\pi$$

Sèries de potències:

Una sèrie de potències <u>centrada en z_0</u> és una serie (funcional) del tipus:

$$\sum_{k\geq 0} a_k (z-z_0)^k \qquad a_k \in \mathbb{C}, \ \forall k$$

Cada sèrie de potències té un <u>radi de convergència</u> de la sèrie R. Llavors la sèrie convergeix:

* Uniformament en $|z-z_0| \leq r < R$
* Absolutament en $|z-z_0| < R$
* ? en $|z-z_0| = R$ $\rightsquigarrow$ En aquest cas no es sap si CONV. o si DIVERXEIX

+ Teoria:

Donada una funció analítica f en D i z_0 dins D

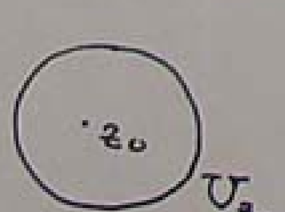
podem desenvolupar $f(z)$ en sèrie de potències centrada en z_0:

$$f(z) = \sum_{k\geq 0} a_k (z-z_0)^k$$

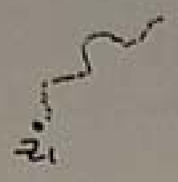
$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz$$

Def.

Un <u>punt singular aïllat</u> de f, z_0, és un punt on f no és analítica, però és analítica en tot un entorn. $U \setminus \{z_0\}$



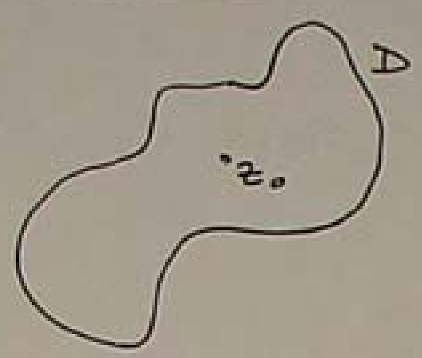
Un punt singular de f, z_1, és un punt on f no és analítica i



tampoc ho és en una successió.

$$\{ Y_n \} \longrightarrow z_1$$

Teorema (Desenvolupament en un punt regular):



f analítica en un entorn de z_0

Llavors, podem desenvolupar

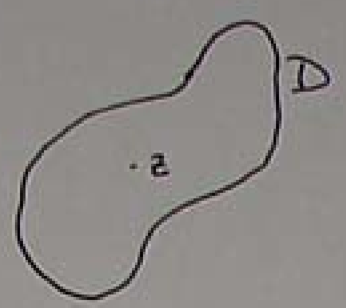
$$f(z) = \sum_{k \geq 0} a_k \cdot (z - z_0)^k$$

$$a_k = \frac{f^{(k)}(z_0)}{k!}, \quad k \geq 0 \qquad a_k \text{ tb } = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{k+1}} dz$$

El radi de convergència d'aquest desenvolupament és la distància al punt singular (aïllat) més proper a z_0.

Def.

- En les condicions de la fórmula integral de Cauchy:



$f^{(n)}$ derivable en tot $z \in D$ $\Rightarrow$ Les funcions coordenades $u(z)$ i $v(z)$ de f són $C^\infty(D)$.

$\Downarrow$

$f^{(n)}$ analítica en D.

Teorema de Morera:

$$\left.\begin{array}{l} f \text{ és contínua en } D \\ \oint_\Gamma f = 0 \quad \forall \Gamma \subset D \text{ determinada} \end{array}\right\} \Rightarrow f \text{ és analítica en } D.$$

$F' = f \Rightarrow F$ analítica en $D \Rightarrow F^{(n)}$ analítica en D $\forall n \geq 0$

Resultats genèrics sobre sèries de potències:

$\sum_{n\geq 0} a_n (z-z_0)^n$ amb radi de convergència, R.

Quan tenim convergència uniforme:

$$f'(z) = \sum_{n\geq 1} n \cdot a_n (z-z_0)^{n-1} \text{ amb el mateix } R.$$

$$\int_{z_0}^{z} f(w)\,dw = \sum_{n\geq 0} \frac{a_n}{n+1} (z-z_0)^{n+1} \text{ amb el mateix } R.$$

OPERACIONS:

1)
$$\left(\sum_{n\geq 0} a_n (z-z_0)^n\right)\left(\sum_{m\geq 0} b_m (z-z_0)^m\right) = \sum_{k\geq 0} p_k (z-z_0)^k$$

$$p_k = \sum_{l=0}^{k} a_l \cdot b_{k-l}$$

2)

$$\frac{\sum\limits_{n\geq 0} a_n (z-z_0)^n}{\sum\limits_{m\geq 0} b_n (z-z_0)^m} = \sum_{k\geq 0} d_k (z-z_0)^k$$

$$\Rightarrow \sum_{n\geq 0} a_n (z-z_0)^n = \left[\sum d_k (z-z_0)^k\right]\left[\sum b_n (z-z_0)^n\right]$$

$$a_0 = d_0 \cdot b_0 \rightarrow d_0 = \frac{a_0}{b_0}$$

$$a_1 = d_0 b_1 + d_1 b_0 \rightsquigarrow d_1$$

Def:

f analítica en D (i en z_0)

$$f(z) = \sum_{n\geq 0} a_n \cdot (z-z_0)^n$$

SERIES DE TAYLOR

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

R = distància de z_0 al punt més singular proper.

Suposem que z_0 és un punt singular aïllat de f.

f és analítica dins de la corona circular $s \leq |z-z_0| \leq t$

$$f(z) = \sum_{n=-\infty} a_n (z-z_0)$$

conv. uniforme

Serie de Laurent

$$a_n = \frac{1}{2\pi i} \oint \frac{}{(z-z_1)^{n+1}}\, dz$$

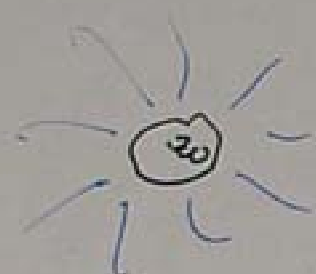
$$\sum_{n=1}^{+\infty} a_n (z-z_0)^{-n} = \sum_{n=1}^{+\infty} \frac{a_{-n}}{(z-z_0)^n}$$ Part principal ~~solució~~ S. L.

$$\sum_{n\geq 0} a_n (z-z_0)^2$$ Part Analítica de la S. L.

f és analítica en z_0, no té part principal.

Part analítica

Part. Anal + Part. Principal



Part principal

Problemes:

#24 Desenvolupeu en series de potencies de z:

(a) e^z (b) $\sin z$ (c) $\cos z$

En tots els casos el radi de convergència serà $R=\infty$

a/
$$\left.\frac{d^{(n)}}{dz^n}(e^z)\right|_{z=0} ? = \left. e^z \right|_{z=0} = 1 \qquad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} \quad \forall n\geq 0$$

$$e^z = \sum_{n\geq 0} \frac{1}{n!} z^n$$

b)

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = \frac{1}{2i}\left(\underset{R_1=\infty}{\sum_{n\geq 0}\frac{(iz)^n}{n!}} - \underset{R_2=\infty}{\sum_{n\geq 0}\frac{(-iz)^n}{n!}}\right) \underset{R=\min\{R_1,R_2\}=\infty}{=}$$

$$= \frac{1}{2i}\sum_{n\geq 0}\frac{i^n-(-i)^n}{n!}z^n = \frac{1}{2i}\sum_{n\geq 0}\frac{i^n\left[1-(-1)^n\right]}{n!}z^n = \frac{1}{2i}\sum_{n\geq 1}\frac{i^{2n-1}\cdot 2}{(2n-1)!}z^{2n-1} =$$

$$= \frac{1}{i^2}\sum_{n\geq 1}\frac{(-1)^n}{(2n-1)!}z^{2n-1} = \sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)!}z^{2n-1}$$

c)

$$\cos z = \frac{d}{dz}\sin z = \frac{d}{dz}\left[\sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-1)!}z^{2n-1}\right] = \sum_{n\geq 1}\frac{(-1)^{n+1}}{(2n-2)!}z^{\underset{2(n-1)}{2n-2}} \underset{m=n-1}{=}$$

$$= \sum_{n\geq 0}\frac{(-1)^m}{(2m)!}z^{2m}$$

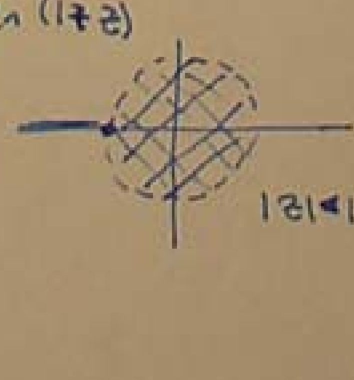
$\boxed{\#25}$ Desenvolupeu en potències de z, $f(z)=2^z$.

$$2^z = e^{z\ln 2} \underset{R=\infty}{=} \sum_{n\geq 0}\frac{(z\ln 2)^n}{n!} = \sum_{n\geq 0}\frac{\ln^n 2}{n!}z^n \qquad \left.\frac{d^n}{dz^n}(2^z)\right|_{z=0} = \boxed{\ln^n 2}$$

$\boxed{\#26}$ Amb $z=re^{i\theta}$, $-\pi<\theta\leq\pi$, definim $\ln z = \ln r + i\theta$. Comproveu que $\ln(1+z)$ és analítica a l'origen. Calculeu la seva sèrie de potències.

$\ln z$

$\ln(1+z)$



$R=1$
$z_0=0$

$f(z) = \ln(1+z)$

$$f'(z) = \frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n\geq 0} (-z)^n = \sum_{n\geq 0} (-1)^n z^n$$

$$\ln(1+z) = \int_0^z f'(\omega)\,d\omega = f(z) - f(0) = \sum_{n\geq 0} \frac{(-1)^n}{n+1} z^{n+1} \underset{m=n+1}{=} \sum \frac{(-1)^{m-1}}{m} z^m$$

$\boxed{\#29}$ Determinen els 4 primers termes de $f(z) = e^z \cdot \ln(1+z)$ desenvolupada en $z=0$.

$$f(z) = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z \ldots\right)\left(z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^6 + \ldots\right) =$$

$$= z + z^2\left(\frac{1}{2}+1\right) + z^3\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{2}\right) + z^4\left(-\frac{1}{4} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6}\right) + z^5\left(\frac{1}{5} - \frac{1}{4} + \frac{1}{6} - \frac{1}{\sqrt{2}} + \frac{1}{24}\right) =$$

$$= z + \frac{1}{2}z^2 + \frac{1}{3}z^3$$

$\boxed{\#27}$ D

un punt de z les funcions:

a) $\displaystyle\int_0^z e^{s^2}\,ds$ b) $\displaystyle\int_0^z \frac{\sin s}{s}\,ds$

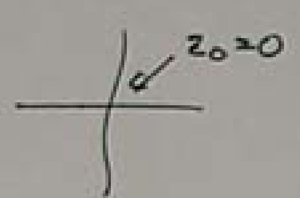
a) $$\int_0^z \left(\underset{R=\infty}{\sum_{n\geq 0} \frac{s^{2n}}{n!}}\right) ds \underset{R=\infty}{=} \sum_{n\geq 0} \frac{z^{n+1}}{(2n+1)\,n!}$$

b)
$$\int_0^z \left(\frac{1}{s} \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)!} s^{2n-1} \right) ds = \int_0^z \sum \frac{(-1)^{n+1}}{(2n-1)!} s^{2n-2} =$$

$R = \infty$

$$= \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)!} \cdot \frac{z^{2n-1}}{2n-1} = \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)!\,(2n-1)} z^{2n-1}$$

#30



$b \in \mathbb{R}$

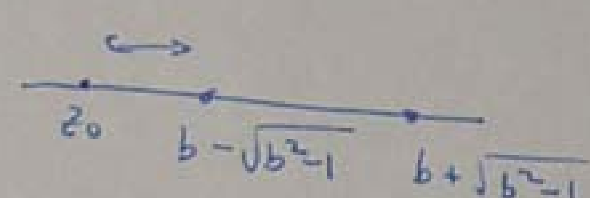
$$f(z) = \frac{1}{z^2 - 2bz + 1} = \sum_{n\geq 0} a_n z^n$$

$$\Delta = 4b^2 - 4 = 4(b^2-1)$$

$|b| > 1$

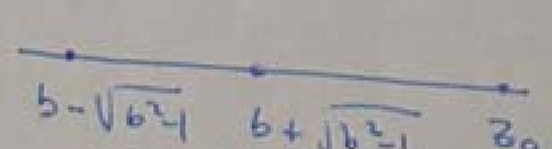
$$x_\pm = \frac{2b \pm 2\sqrt{b^2-1}}{2} = b \pm \sqrt{b^2-1}$$

$b > 1$



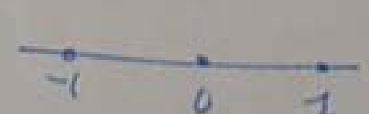
$R = b - \sqrt{b^2-1}$

$b < -1$



$R = b + \sqrt{b^2-1}$

$|b| = 1$ $\quad \Delta = 0$



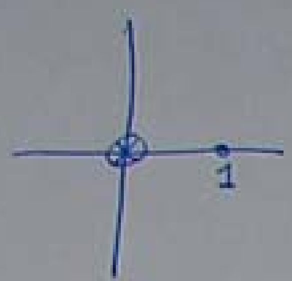
$R = 1$

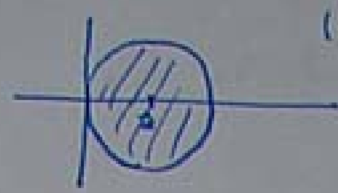
$|b| < 1$ $\quad \Delta < 0 \quad w_\pm = b \pm i\sqrt{1-b^2}$

$$|w| = \sqrt{b^2 + \left(\sqrt{1-b^2}\right)^2} = \sqrt{b^2 + 1 - b^2} = 1 \qquad R = 1$$

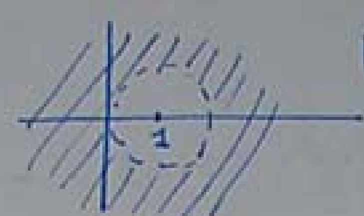
#31, Desenvolupem $\frac{1}{z}$ en potències de $z-1$

$z_0 = 1$





$|z-1|<1$



$|z-1|>1$

$$\frac{1}{z} = \frac{1}{1+(z-1)}$$

$|z-1|<1$

$$\frac{1}{1+(z-1)} = \sum_{n\geq 0}\left[-(z-1)\right]^n = \sum_{n\geq 0}(-1)^n (z-1)^n$$

$|z-1|>1 \iff 1 > \frac{1}{|z-1|}$

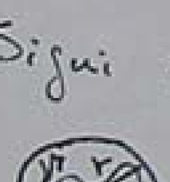
$$\frac{1}{(z-1)+1} = \frac{1}{z-1}\cdot\frac{1}{1+\frac{1}{z-1}} = \frac{1}{z-1}\sum_{n\geq 0}\frac{1}{(z-1)^n} =$$

$$= \sum_{n\geq 1}\frac{1}{(z-1)^n}$$

9-12-04

Def:

Sigui z_0 un punt singular aïllat. de f. Aleshores existeix $r>0$ tal que



en la regió $0<|z-z_0|<r$ $f(z)$ és analítica.

Llavors, podem desenvolupar f en aquesta regió:

$$f(z) = \underbrace{\dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0}}_{\substack{\text{p. s.} \\ \hookrightarrow \text{part singular}}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}_{\substack{\text{p. a.} \\ \hookrightarrow \text{part analítica}}}$$

$$a_n = \frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Def: $\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \oint_{\Gamma} f(z)\,dz$

Notes:

1) Si f és analítica en z_0, tenim $\operatorname{Res}(f, z_0) = 0$

2) Si z_0 és singular no aïllat, no ho estudiem.

3) La corba tancada Γ ha d'estar continguda dins una regió del tipus $0 < |z - z_0| < r$

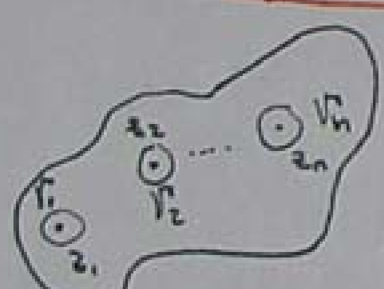
Exemple:

1) $f(z) = \frac{1}{z-1} \Rightarrow \operatorname{Res}(f, 1) = 1$

$g(z) = e^{1/z} \Rightarrow \operatorname{Res}(g, 0) = 1$

$$\underset{|z|>0}{e^{1/z}} = \sum_{k \geq 0} \frac{(1/z)^n}{n!} = 1 + \frac{1}{z} + \dots$$

Teorema dels residus:



Sigui $f(z)$ analítica dins i sobre Γ, excepte en un nombre finit de punts singulars aïllats

$z_1, z_2, \dots, z_n$

Aleshores:

$$\oint_{\Gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}(f, z_k)$$

Comprovació:

$$\oint_{\Gamma} f(z)\,dz \underset{\text{T. Cauchy 2}}{=} \sum_{k=1}^{m} \oint_{\Gamma_k} f(z)\,dz = \sum_{k=1}^{m} 2\pi i \operatorname{Res}(f, z_k)$$

Def:

Un pol d'ordre $n \geq 1$, z_0, de $f(z)$ és un punt singular aïllat tal que:

$$\lim_{z \to z_0} (z - z_0)^n \cdot f(z_0) = C \neq 0, \infty$$

Proposició:

Si z_0 és un pol d'ordre n de f, el desenvolupament de Laurent de f en $0 < |z - z_0| < r$ és del tipus:

$a_{-n} \neq 0$ $$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1 (z-z_0) + \cdots$$

Càlcul de $Res(f, z_0)$ on z_0 és un pol d'ordre n de f

Del desenvolupament de Laurent de $f(z)$ en $0 < |z - z_0| < r$, la funció auxiliar

$$\varphi(z) = \begin{cases} (z-z_0)^n \cdot f(z) & z \neq z_0 \\ a_{-n} & z = z_0 \end{cases}$$

és analítica en z_0:

Si $z \neq z_0$ (en un entorn de z_0), φ és analítica

Si $z = z_0$, anem a veure que és derivable:

$$\lim_{z \to z_0} \frac{\varphi(z) - \varphi(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(z-z_0)^n \cdot f(z) - a_{-n}}{z - z_0} = (*)$$

$$(z-z_0)^n f(z) = (z-z_0)^n \left[\frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + a_1 (z-z_0) + \dots \right] =$$

$$= a_{-n} + a_{-n+1}(z-z_0) + \underset{\uparrow \text{ infinitesim}}{O(z-z_0)} \qquad (z \to z_0)$$

$$(*) = \lim_{z \to z_0} \frac{\cancel{a_{-n}} + a_{-n+1}(z-z_0) + O(z-z_0) - \cancel{a_{-n}}}{z-z_0} = \lim_{z \to z_0} a_{-n+1} + \frac{O(z-z_0)}{z-z_0}$$

$$\boxed{a_{-n-1}}$$

φ analítica en z_0. Podem fer operacions analítiques sobre φ per aconseguir a_{-1}:

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n \cdot f(z) \right] = \text{Res}(f, z_0) \rightarrow \text{sempre i quan } z_0 \equiv \text{pol}$$

Exemples:

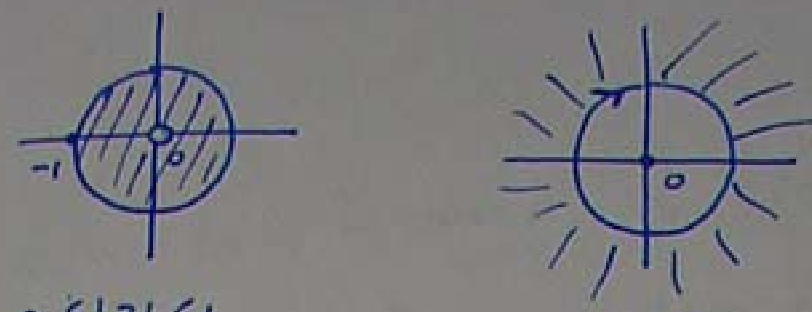
$\boxed{\#35}$ Calculen el residu en $z_0=0$ de $f(z) = \dfrac{1+2z}{z^2(1+z)}$

a) Considerant $z_0=0$ com a pol múltiple

b) A partir del desenvolupament de potències de z

b) $z_0=0$ és un pol d'ordre 2:

$$\lim_{z \to 0} z^2 \cdot f(z) = \lim_{z \to 0} \frac{1+2z}{1+z} = 1 \neq 0, \infty$$



$0 < |z| < 1$ $\qquad$ $|z| > 1$

$$f(z) = \frac{1}{z^2}(1+2z)\frac{1}{1+z} = \frac{1}{z^2}\boxed{(1+2z)\sum_{k\geq 0}(-1)^n z^n} =$$

$$= \frac{1}{z^2}\left[\sum_{k\geq 0}(-1)^n z^n + 2\sum_{m\geq 0}(-1)^n z^{m+1}\right] = \frac{1}{z^2}\left[\sum_{k\geq 0}(-1)^n z^n + 2\sum_{m\geq 1}(-1)^{m+1}\cdot z^m\right] =$$

$$= \frac{1}{z^2}\left(1 + \sum_{k\geq 1}\left[(-1)^k + 2(-1)^{k+1}\right]z^k\right) = \frac{1}{z^2}\left(1 + \sum_{k\geq 1}(-1)^{k+1}\cdot z^k\right) = \frac{1}{z^2} + \sum_{k\geq 1}(-1)^{k+1}\cdot z^{k-2}$$

$(-1)^k(1-2) = (-1)^{k+1}$

$$= \frac{1}{z^2} + \frac{1}{z} + \dots \longrightarrow \operatorname{Res}(f, 0) = 1$$

a) Hem comprovat que $z_0 = 0$ és un pol doble, per tant posem $n = 2$ a la fórmula.

$$\operatorname{Res}(f, 0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0}\frac{d}{dt}\left[\frac{1+2z}{1+z}\right] =$$

$$= \lim_{z\to 0}\frac{2(1+z)-(1+2z)}{(1+z)^2} = \frac{2-1}{1} = 1$$

Comentaris:

(1) Quan el punt singular sigui 1 pol, en general, és més ràpid calcular-li el residu mitjançant la fórmula.

(2) Quan <u>no sigui un pol</u>, no ens estalviarem el càlcul de la sèrie de Laurent corresponent.

Ex:

$f(z) = e^{1/z}$ $\quad z_0 = 0$

$\lim_{z \to 0} z^n \cdot f(z) = \infty \quad \forall n \geq 1 \quad \Rightarrow \quad |z|>0 \quad \sum_{k \geq 0} \frac{1}{k!} \frac{1}{z^k} = \dots + \frac{1}{2z^2} + \frac{1}{z} + 1$

↪ z_0 no és pol

Residu

#36 Calculeu els residus de:

a) $f(z) = e^{1/z}$ $\qquad$ b) $f(z) = \dfrac{\sin^2 z}{z^4 - 1}$.

en els punts singulars.

b) $z^4 - 1 = (z^2-1)(z^2+1) \qquad \pm 1, \pm i \rightarrow$ pols simples

$$f(z) = \frac{\sin^2 z}{(z+1)(z-1)(z+i)(z-i)}$$

$$\mathrm{Res}(f, +1) = \lim_{z \to +1} (z-1) f(z) = \lim_{z \to +1} \frac{\sin^2 z}{(z+1)(z^2+1)} = \frac{\sin^2 1}{4} \; \frac{1}{4}\sin^2 1$$

$$\mathrm{Res}(f, -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{\sin^2 z}{(z-1)(z^2+1)} = \frac{\sin^2 -1}{-4} = \frac{1}{4}\sin^2 1$$

$$\sin^2(i) = \left(\frac{e^{-1}-e^{1}}{2i}\right)^2 = -\left(\frac{e^{1}-e^{-1}}{2}\right)^2 = -\mathrm{sh}^2 1$$

$$\mathrm{Res}(f,-i) = \lim_{z\to -i} (z+i)f(z) = \lim_{z\to -i} \frac{\sin^2(z)}{(z-i)(z^2-1)} = \frac{\sin^2(-i)}{4i} = \frac{-\mathrm{sh}^2 1}{4i}$$

$\boxed{\#37}$ Comproveu que les singularitats següents són pols i determineu el seu ordre:

a) $\dfrac{1-\mathrm{ch}\, z}{z^3}$ b) $\dfrac{1-e^{2z}}{z^4}$ c) $\dfrac{e^{2z}}{(z-1)^2}$

a) $$\mathrm{ch}\, z = \frac{e^{z}+e^{-z}}{2} = \sum_{k\geq 0} \frac{1}{(2k)!} z^{2k} = 1 + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$

$$1-\mathrm{ch}\, z = -\frac{z^2}{2} + \frac{z^4}{24} + \dots$$

$$f(z) = \frac{-1/2}{z} + \underbrace{\tfrac{1}{6} + \dots}_{p.a.}$$

b) $$f(z) = \frac{1-e^{2z}}{z^4} = -\frac{1}{z^4}\left(2z + 2z^2 + \dots\right) \underset{\uparrow}{=} \frac{2}{z^3} + \frac{2}{z^2} + \dots$$

pol d'ordre 3

$$e^{2z} = 1 + 2z + \frac{4}{2}z^2 + \dots$$

Classificació de les singularitats aïllades:

Sigui z_0 una singularitat aïllada de $f(z)$.

Sigui:

$$\underbrace{\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}}_{\text{part principal}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{\text{part. aillada}}$$

El desenvolupament de Laurent de $f(z)$, centrat en z_0, en la regió $0<|z-z_0|<r$.

1) La part principal té un nombre finit de sumands: z_0 és un pol.

2) La part principal no té cap sumand $\neq 0$, en z_0 tenim una singularitat evitable.

Exemple: $f(z) = \frac{\sin z}{z}$, $z_0 = 0$

$$f(z) = \frac{1}{z}\sum_{k\geq 1} \frac{(-1)^{k+1}}{(2k-1)!} z^{2k-1} = \sum_{k\geq 1} \frac{(-1)^{k+1}}{(2k-1)!} z^{2k-2} = 1 + \cdots$$

3) La part principal té infinits sumands, en z_0 tenim una singularitat essencial.

Exemple:

$e^{1/z}$, $z_0 = 0$

$$\underset{|z|>0}{e^{1/z}} = \sum_{k\geq 0} \frac{1}{k!}\frac{1}{z^k} = \underbrace{1}_{p.a.} + \underbrace{\frac{1}{z} + \frac{1}{2z^2} + \cdots}_{p.p.}$$

Aplicació del T^a dels residus al càlcul d'integrals:

$$x(t) = \frac{1}{1+t^2} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

INTENTEM FER TRANSFORMADA DE FOURIER:

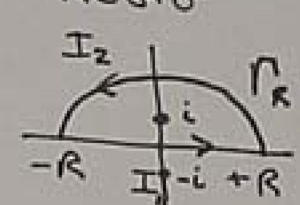
$$X(f) = \int_{-\infty}^{\infty} \frac{e^{-i2\pi f t}}{1+t^2}\,dt = \int_{-\infty}^{+\infty} \frac{\cos(2\pi f t)}{1+t^2}\,dt + i\cdot 0$$

Generalitzant:

$$\int_{-\infty}^{+\infty} \frac{\cos(mt)}{a^2+t^2}\,dt \qquad m, a > 0$$

$$f(z) = \frac{e^{imz}}{a^2+z^2},$$

REGIÓ



$\Gamma_R = I_1 + I_2$

TEO. RESIDUS DE LA $f(z)$ sobre Γ_R:

$$\oint_{\Gamma_R} \frac{e^{imz}}{a^2+z^2}\,dz \underset{(*)}{=} 2\pi i \cdot \operatorname{Res}(+i)$$

$$\oint_{\Gamma_R} f(z)\,dz = \int_{I_2} f(z)\,dz + \int_{I_1} f(z)\,dz$$

$$\int_{I_1} \frac{e^{imz}}{a^2+z^2}\,dz = \int_{-R}^{R} \frac{e^{imt}}{a^2+t^2}\,dt = \int_{-R}^{R} \frac{\cos(mt)}{a^2+t^2}\,dt$$

La igualtat (*) val $\forall R>1$. Per tant, si fem $R\to+\infty$, tb és valida:

$$\lim_{R\to+\infty} J_R + \int_{-\infty}^{+\infty} \frac{\cos(mt)}{a^2+t^2}\,dt = 2\pi i \cdot \mathrm{Res}(+ai)$$

$$|J_R| = \left|\int_0^\pi \frac{e^{imz(\theta)}}{a^2+z(\theta)^2}\, z'(\theta)\, d\theta\right| \le \int_0^\pi \left|\frac{e^{imRe^{i\theta}}}{a^2+R^2e^{2i\theta}}\, iRe^{i\theta}\right| d\theta \le \int_0^\pi \frac{R}{|a^2+R^2e^{i2\theta}|}\,d\theta \quad (**)$$

$$z(\theta) = Re^{i\theta}$$

$$\left|e^{imRe^{i\theta}}\right| = \left|e^{imR(\cos\theta + i\sin\theta)}\right| = \left|e^{imR\cos\theta}\right| \cdot e^{-mR\sin\theta} \le 1$$

$$|a^2+R^2e^{i2\theta}| \ge \Big|\,|R^2e^{i2\theta}| - |a^2|\,\Big| = R^2-a^2$$

$$(**) \leadsto \le \frac{R}{R^2-a^2}\pi \qquad |J_R| \le \frac{\pi R}{R^2-a^2} \xrightarrow[R\to\infty]{} 0$$

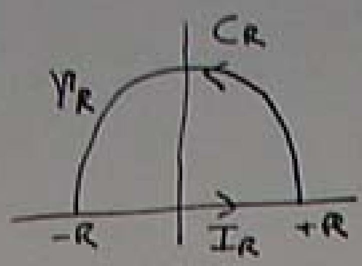
$$\int_{-\infty}^{+\infty} \frac{\cos(mt)}{1+t^2}\,dt = 2\pi i \cdot \mathrm{Res}(+ai) = \frac{\pi}{ae^{ma}}$$

$$R(+ai) = \lim_{z\to ai} (z-ai)\cdot f(z) = \lim_{z\to ai} \frac{e^{imz}}{z+ai} = \frac{e^{-ma}}{2ai}$$

$$X(f) = \frac{\pi}{e^{2\pi f}}$$

Truc:

$\Gamma_R = C_R + I_R$



En general quan volem evaluar

$$\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\,dx \quad \text{on } p, q \text{ són polinomis}$$

tals que:

- $q(x)$ no té arrels reals
- $gr(q) \geq gr(p) + 2$

podem elegir la mateixa regió i $f(z) = \frac{p(z)}{q(z)}$. També quan tenim una integral del tipus $\int_0^{2\pi} R(\sin t, \cos t)\,dt$, el canvi $z = e^{it}$ la porta a una del tipus $\oint_{|z|=1} \frac{p(z)}{q(z)}\,dz$.

#43

a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$ b) $\displaystyle\int_0^{2\pi} \sin^{2n}(\theta)\,d\theta$, $n \geq 0$

b)

$z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{1}{2i}\cdot\frac{z^2-1}{z}$$

$$\int_0^{2\pi} \sin^{2n}(\theta)\,d\theta \underset{z=e^{i\theta}}{=} \oint_{|z|=1} \left(\frac{z^2-1}{z}\right)^{2n} \cdot \frac{1}{(2i)^{2n}} \cdot \frac{dz}{iz} = \frac{1}{i\,4^n(-1)^n} \oint_{|z|=1} \frac{(z^2-1)^{2n}}{z^{2n+1}}\,dz =$$

$$= \frac{2\pi i}{i\,4^n(-1)^n}\,\mathrm{Res}(0) = (*)$$

$\mathrm{Res}\left(\frac{(z^2-1)^{2n}}{z^{2n+1}}, 0\right) =$ Calculem-lo amb la sèrie de Laurent

$$\frac{1}{z^{2n+1}}\left(z^2+1\right)^{2n} = \frac{1}{z^{2n+1}} \sum_{k=0}^{2n} \binom{2n}{k} z^{2k} (-1)^{2n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} z^{2(k-n)-1}$$

$$2(k-n)-1 = -1 \Rightarrow k = n$$

$$\Downarrow$$

$$\mathrm{Res}(0) = (-1)^n \binom{2n}{n}$$

$$(*) = \frac{2\pi}{4^n(-1)^n}(-1)^n \binom{2n}{n} = \frac{2\pi}{4^n}\binom{2n}{n}$$

a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$

$z = e^{i\theta}$ $\quad \sin\theta = \frac{1}{2i}\cdot\frac{z^2-1}{z}$ $\quad d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_{|z|=1} \frac{1}{iz\left[2+\frac{1}{2i}\cdot\frac{z^2-1}{z}\right]}dz = 2\oint_{|z|=1}\frac{dz}{4iz+z^2-1} =$$

$$= 4\pi i \, \mathrm{Res}\left[(-2+\sqrt{3})i\right] = \frac{2\pi}{\sqrt{3}}$$

$z^2 + 4iz - 1 = 0$ $\qquad \Delta = -16-4(-1) = -12$

$$z_\pm = \frac{-4i \pm 2\sqrt{3}i}{2} = -2i \pm \sqrt{3}i = (-2\pm\sqrt{3})i$$

$$\mathrm{Res}\left[(-2\pm\sqrt{3})i\right] = \lim_{z\to(-2+\sqrt{3})i} \frac{z-(-2+\sqrt{3})i}{z^2+4iz-1} = \lim_{z\to(-2+\sqrt{3})i}\frac{1}{z+(2+\sqrt{3})i} = \frac{1}{2\sqrt{3}i}$$

$\boxed{\#41}$

c) $\displaystyle\oint_{|z|=r} \sin\left(\frac{1}{z}\right)\cdot dz$ $\qquad$ d) $\displaystyle\oint_{|z|=r} \sin^2\left(\frac{1}{z}\right)dz$ $\qquad$ e) $\displaystyle\oint_{|z|=r} z^n\cdot e^{2/z}\cdot dz$

c) $$\oint_{|z|=r}\sin\left(\frac{1}{z}\right)dz = 2\pi i\,\mathrm{Res}\left(\sin\frac{1}{z},\ z=0\right) = 2\pi i$$

$$\sin\frac{1}{z} = \sum_{k\geq 1}\frac{(-1)^{k+1}}{(2k-1)!}\frac{1}{z^{2k-1}}$$

d) $\sin^2\left(\frac{1}{z}\right) = \left(\frac{1}{z} - \frac{1}{6z^3} + \cdots\right)\left(\frac{1}{z} - \frac{1}{6z^3} + \cdots\right) = \frac{1}{z^2} + \cdots$

$$\oint_{|z|=r} \sin^2\left(\tfrac{1}{z}\right) dz = 0$$

e) $$\oint_{|z|=r} z^n e^{2/z} dz = 2\pi i \cdot \mathrm{Res}(0) = (*)$$

$$e^{2/z} = 1 + \frac{2}{z} + \cdots$$

$\mathrm{Res}(0) = 0 \Rightarrow n < -1$

$\underline{n \geq -1}$:

$$z^n \sum_{k\geq 0} \frac{2^k}{k!}\frac{1}{z^k} = \sum_{k\geq 0} \frac{2^k}{k!}\frac{1}{z^{k-n}}$$

$k - n = 1 \Rightarrow k = n+1$

$$\mathrm{Res}(0) = \frac{2^{n+1}}{(n+1)!}$$

$$(*) = \begin{cases} 0 & n < -1 \\ \pi \dfrac{2^{n+2}}{(n+1)!} i & n \geq -1 \end{cases}$$

$\boxed{\# 32}$

$f(z) = \dfrac{1}{z^2 - 4z + 5}$, en potencies de $z-1$ en

a) $|z-1| < \sqrt{2}$ b) $|z-1| > \sqrt{2}$

$z^2 - 4z + 5 = 0$ $\Delta = 16 - 4\cdot 5 = -4$

$$z_\pm = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$\frac{1}{z^2-4z+5} = \left(\frac{1}{z-(2-i)} - \frac{1}{(z-(2+i))}\right)\left(\frac{-1}{2i}\right) = \frac{i}{2}\left[\frac{1}{(z-1)-(1-i)} - \frac{1}{(z-1)-(1+i)}\right] =$$

$$= \frac{i}{2}\left[\frac{1}{1-i}\,\frac{1}{\frac{z-1}{1-i}-1} - \frac{1}{1+i}\,\frac{1}{\frac{z-1}{1+i}-1}\right] = \frac{i}{2}\left[\frac{-1}{1-i}\,\frac{1}{1-\frac{z-1}{1-i}} + \frac{1}{1-\frac{z-1}{1+i}}\right] =$$

$$\left|\frac{z-1}{1-i}\right| = \frac{|z-1|}{\sqrt{2}} = \left|\frac{z-1}{1+i}\right|$$

$$= \frac{-i}{2}\,\frac{1}{1-i}\sum_{k\geq 0}\left(\frac{z-1}{1-i}\right)^k + \frac{i}{2}\,\frac{1}{1+i}\sum_{k\geq 0}\left(\frac{z-1}{1+i}\right)^k = \frac{i}{2}\sum_{k\geq 0}\left[\frac{1}{(1+i)^{k+1}} - \frac{1}{(1-i)^{k+1}}\right]\cdot$$

$$\cdot (z-1)^k$$

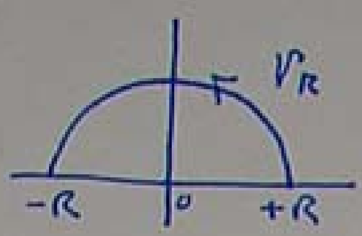
$$1+i = \sqrt{2}\,e^{i\pi/4} \qquad 1-i = \sqrt{2}\,e^{-i\frac{\pi}{4}}$$

$$\frac{1}{(1+i)^{k+1}} - \frac{1}{(1-i)^{k+1}} = \left(\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}\right)^{k+1} - \left(\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\right)^{k+1} = \frac{e^{-i(k+1)\frac{\pi}{4}} - e^{i(k+1)\frac{\pi}{4}}}{2^{\frac{k+1}{2}}}$$

$$(*) = \sum_{k\geq 0}\frac{1}{2^{\frac{k+1}{2}}}\cdot\frac{e^{i(k+1)\frac{\pi}{4}} - e^{-i(k+1)\frac{\pi}{4}}}{2i}\,(z-1)^k = \sum_{k\geq 0}\frac{\sin\left[(k+1)\frac{\pi}{4}\right]}{2^{\frac{k+1}{2}}}(z-1)^k$$

Sigui Γ_R el semicercle superior de radi R centrat en 0, que va de $+R$ a $-R$. Sigui $I_R = \int_{\Gamma_R} \frac{dz}{z}$. Llavors el límit: $\lim_{R\to\infty} I_R$ val:

a) 0 b) $i\pi$ c) $2\pi i$ d) ∞



$$I_R = \int_{\Gamma_R} \frac{dz}{z} = \int_0^{\pi} \frac{\cancel{R} i \cancel{e^{i\theta}} d\theta}{\cancel{R e^{i\theta}}} = i\pi$$

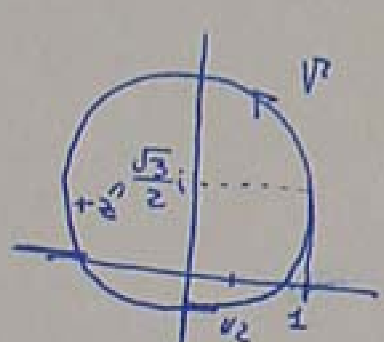
$z(\theta) = Re^{i\theta}$

$\theta \in [0, \pi]$

Sigui Γ el cercle de radi 1, centrat en $\frac{\sqrt{3}}{2}i$. Llavors, la integral

$$I = \oint_{\Gamma} \frac{dz}{z^2 - z + 1} \quad \text{val:}$$

a) $\sqrt{3}\pi i$ b) $-\frac{2\pi}{\sqrt{3}}$ c) $\frac{2\pi}{\sqrt{3}}$ d) $\frac{\sqrt{3}}{2}\pi i$



$I = 2\pi i \quad \text{Res}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{2\pi \cancel{i}}{\sqrt{3}\cancel{i}}$

$z^2 - z + 1 = 0 \qquad \Delta = 1 - 4 = 3 \qquad z_+ = \frac{1 + \sqrt{3}i}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$$\text{Res}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \lim_{z \to \frac{1}{2} + \frac{\sqrt{3}}{2}i} \frac{1}{z - \frac{1}{2} + \frac{\sqrt{3}}{2}i} = \frac{1}{\sqrt{3}i}$$

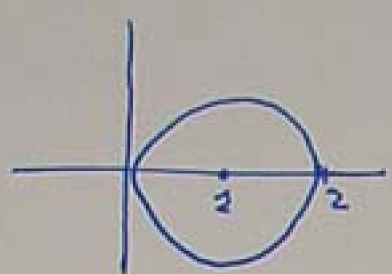
Donada $f(z) = \frac{1}{z-1} + \frac{1}{z-2}$, el seu desenvolupament de Laurent en potències de $z-1$, en $0 < |z-1| < 1$ és:

a) $\frac{1}{z-1} + \sum_{n\geq 0} \frac{1}{2^n} (z-1)^n$

c) $\sum_{n\geq -1} \frac{(-1)^n}{2^n} (z-1)^n$

b) $\frac{1}{z-1} - \sum_{n\geq 0} (z-1)^n$

d) $\sum_{n\geq -1} 2^n (z-1)^n$



$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)}$$

Si la part real de una funció entera, $f = u + vi$, és $u(x,y) = (2x-1)y$ amb $f(0) = 0$, llavors $v(x,y)$ val:

a) $y^2 - x^2 + x$

b) $\frac{2}{3}y^3 - x^2 + x - i$

c) $\frac{2}{3}y^3 - x^2 + x + 1$

d) $x^2 - y^2 + x + 1$

$$u_x = v_y = 2y \Rightarrow v = y^2 + C(x)$$

$$-u_y = v_x = -(2x-1) = -2x+1 = C'(x) \Rightarrow C(x) = -x^2 + x + C$$

$$\left.\begin{array}{l} v(x,y) = y^2 - x^2 + x + C \\ v(0,0) = 0 \end{array}\right\} \Rightarrow C = 0$$

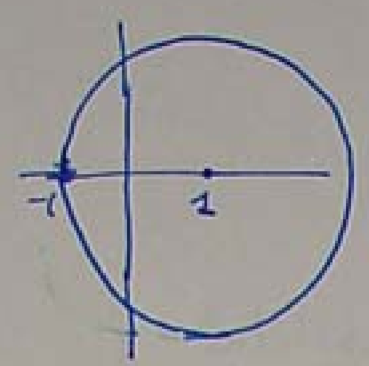
El desenvolupament de Laurent de $\frac{1}{z^2-1}$, en $0<|z-1|<2$, en potències de $z-1$, és:

a) $\sum_{n\geq 0} (z-1)^{n-1}$

b) $\sum_{n\geq 0} \frac{1}{2^{n+1}} (z-1)^{n-1}$

c) $\sum_{n\geq 0} 2^n (z-1)^{n-1}$

d) $\sum_{n\geq 0} \frac{(-1)^n}{2^{n+1}} (z-1)^{n-1}$



$z^2-1=0$; $z=\pm 1$

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{(z-1)} \cdot \frac{1}{2+(z-1)} =$$

$$= \frac{1}{2(z-1)} \sum_{n\geq 0} \frac{(-1)^n}{2^n} (z-1)^n = \sum_{n\geq 0} \frac{(-1)^n}{2^{n+1}} (z-1)^{n-1}$$

Sigui $f(z) = \ln(z+1)$, amb $\ln w = \ln r + i\theta$ $\theta \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$. Llavors f és analítica en A i no és analítica en B amb:

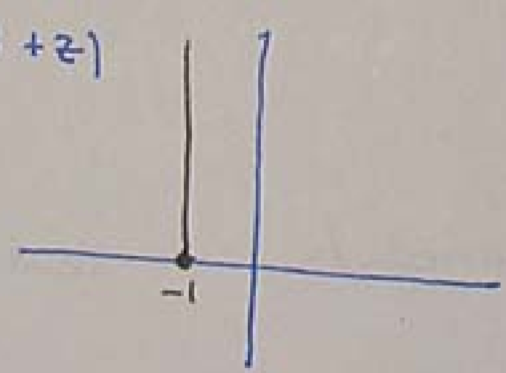
a) $A=0$, $B=-1+\frac{1}{2}i$

b) $A=-1+\frac{1}{2}i$, $B=-1-\frac{1}{2}i$

c) $A=e^{i\pi/2}$, $B=i$

d) $A=1+i$, $B=1-i$

$\ln(1+z)$



Sigui $f = u + iv$ entera. Si $u_x(x,y) = v_y(x,y) = 2x$ i $v_x = -u_y = 2y$.

llavors $f'(1)$ val:

a) i b) 3 c) 2 d) $2i$

$$f'(z) = u_x(z) + i v_x(z)$$

$$f'(1) = 2 + i0 = 2$$

Juny 2003

Siguin C, D i A els conjunts de continuitat, derivabilitat i analicitat de la funció $\bar{z}^2$, llavors:

a) $C = \mathbb{C}$, $D = A = \emptyset$

b) $C = D = A = \mathbb{C}$

c) $C = \mathbb{C}$, $D = \{0\}$, $A = \emptyset$

d) Cap de les altres

$$\frac{\partial}{\partial \bar{z}} f = 0; \qquad 2\bar{z} = 0 \; ; \; z = 0$$

$$\lim_{z\to 0} \frac{\bar{z}^2 - 0}{z - 0} = \lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$$

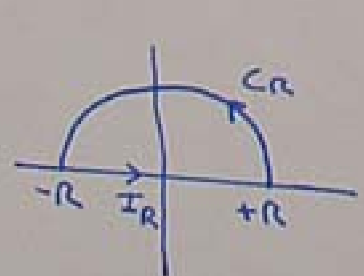
$$\left| \frac{\bar{z}^2}{z} \right| = \frac{|z|^2}{|z|} = |z| \xrightarrow[z\to 0]{} 0$$

Sigui $C_R = \{ z \in \mathbb{C} \mid |z| = R, \ \mathrm{Im}(z) \geq 0 \}$, recorreguda en sentit positiu.

Calculeu:

$$\int_{C_R} e^z \, dz$$

a) $3e^R$ b) $\sin R$ c) $\cos R$ d) $-2\,\mathrm{sh}\, R$



$$\int_{C_R} e^z \, dz = \int_{+R}^{-R} e^z \, dz = e^{-R} - e^{+R} = 2\, \frac{e^{-R} - e^{+R}}{2} = -2\,\mathrm{sh}\, R$$

Altre forma: $\gamma_R = C_R + I_R$ $\quad \oint_{\gamma_R} e^z dz = 0 = \left(\int_{C_R} + \int_{I_R} \right) f \Rightarrow \int_{C_R} f = -\int_{I_R} f = -\int_{-R}^{R} e^x dx$

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El desenvolupament de $f(z) = \frac{1}{z}$ en potències de $z-i$, en $|z-i|>1$, és:

a) $\sum_{m\geq 1} (z-i)^m$ b) $\sum_{m\geq 1} (-1)^{m-1} \cdot i^{m-1} \cdot (z-i)^{-m}$

c) $\sum_{m\geq 1} \frac{(-1)^{m-1}}{i^{m-1}} (z-i)^m$ d) $\sum_{m\geq 1} i^{m-1} \cdot (z-i)^{-m}$

$$\frac{1}{z} = \frac{1}{z+i-i} = \frac{1}{z-i} \frac{1}{1+\frac{i}{z-i}} =$$

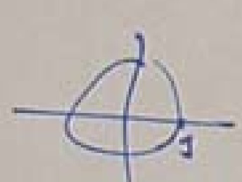
$|z-i|>1$
$$= \frac{1}{z-i} \sum_{m\geq 0} (-1)^m \cdot i^m \cdot (z-i)^{-m} =$$

$$= \sum_{m\geq 0} (-1)^m \cdot i^m \cdot (z-i)^{-m-1} \underset{m+1=m}{=} \sum_{m\geq 1} (-1)^{m-1} \cdot i^{m-1} \cdot (z-i)^{-m}$$

El valor de la integral $\oint z^2 \cdot \sin\frac{1}{z}\, dz$ és:

a) $-\frac{\pi}{3} i$ b) $\frac{\pi}{60} i$ c) 1 d) $\frac{2}{3}\pi$

$$\oint z^2 \cdot \sin\frac{1}{z}\, dz = 2\pi i \operatorname{Res}(0) = 2\pi i \cdot \frac{-1}{6} = -\frac{\pi}{3} i$$

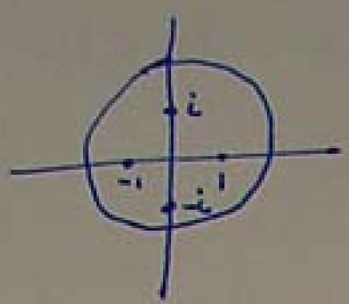


$$z^2 \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n-1)!} = \sum_{n\geq 0} \frac{(-1)^{n+1}}{(2n-1)!} z^{-2n+3}$$

$-2n+3=-1$
$-2n=-4$
$\boxed{n=2}$

El valor de:

$$\oint_{|z|=2} \frac{dz}{z^4-1}$$ és: a) $2\pi i$ b) 1 c) 0 d) $\frac{\pi}{3}+i$



$$2\pi i \; \left(Res(+1) + Res(-1) + Res(i) + Res(-i) \right) = 0$$

$$z^4 - 1 = 0 \qquad (z^2+1)(z^2-1) = 0$$

$$Res(+1) = \lim_{z \to +1} \frac{1}{(z^2+1)(z+1)} = \frac{1}{4}$$

$$Res(-1) = \lim_{z \to -1} \frac{1}{(z^2+1)(z-1)} = -\frac{1}{4}$$

$$Res(i) = \lim_{z \to i} \frac{1}{(z+i)(z^2-1)} = -\frac{1}{4i}$$

$$Res(-i) = \lim_{z \to -i} \frac{1}{(z \; \cdots} = \frac{1}{4i}$$