Escola Tècnica Superior d'Enginyeria de Telecomunicació de Barcelona

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Departament de Teoria del Senyal i Comunicacions

FIBER-OPTIC COMMUNICATIONS

JOAN M. GENÉ BERNAUS

OPTICAL COMMUNICATIONS GROUP

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5. OPTICAL AMPLIFIERS

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- DOPED-FIBER AMPLIFIER
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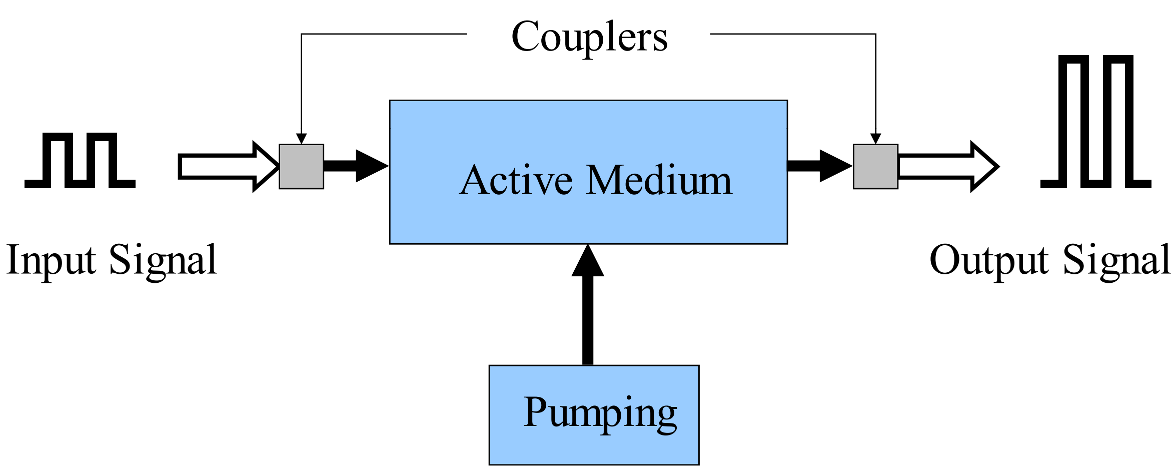
BASIC CONCEPTS

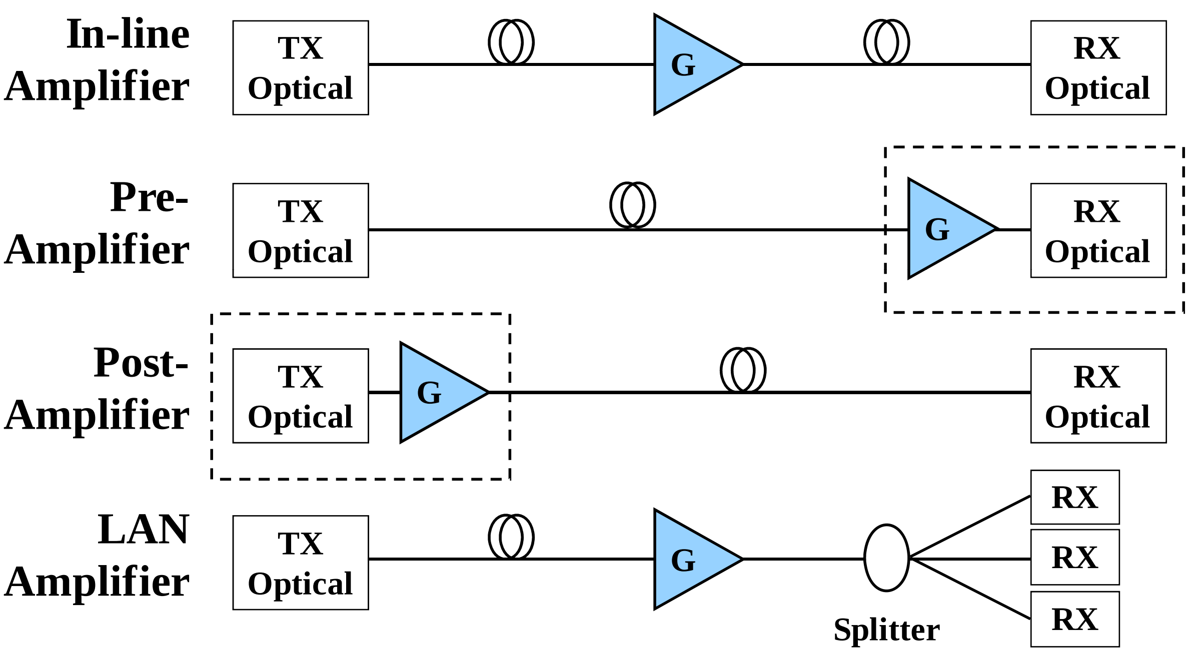
Generic Optical Amplifier



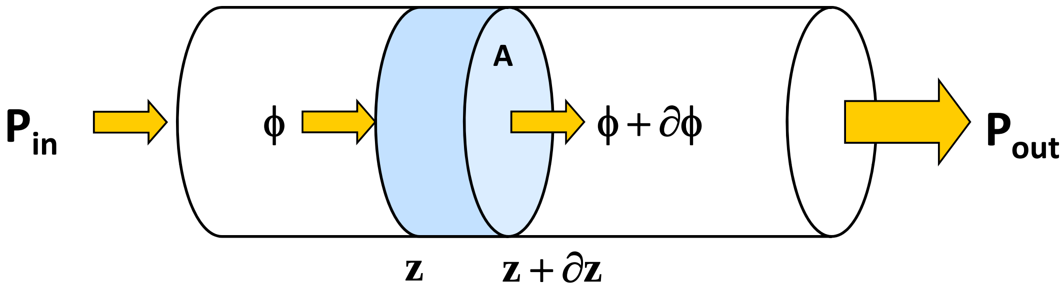
Characteristic Parameters

- ✓ Amplification Bandwidth
- ✓ Gain
- ✓ Noise Factor
- ✓ Operation Frequency
- ✓ Saturation Power
- ✓ Switching Speed
- ✓ Dimensions
- ✓ Cost

System Applications of Optical Amplifiers



Amplifier Gain



$$\phi = \frac{P}{hf \cdot A}$$ photon flux [$s^{-1}m^{-2}$]

$$\frac{\partial\phi}{\partial z} = g_0\,\phi \;\rightarrow\; \frac{\partial P}{\partial z} = g_0\,P$$ [$s^{-1}m^{-3}$]

small-signal gain

$$\boxed{P_{out} = P_{in}\,\underbrace{e^{g_0 L}}_{G_0}} \;\Rightarrow\; \boxed{G_0 \equiv \frac{P_{out}}{P_{in}} = e^{g_0 L}}$$

Gain Coefficient Model

small-signal gain

$$g(P) = \frac{g_0}{1 + (\omega - \omega_0)^2 T_2^2 + P/P_{sat}}$$

frequency profile
lorentzian

saturation

T_2: dipole relaxation time ⇨ **FWHM**

P_{sat}: saturation power ⇨ **Maximum Power**

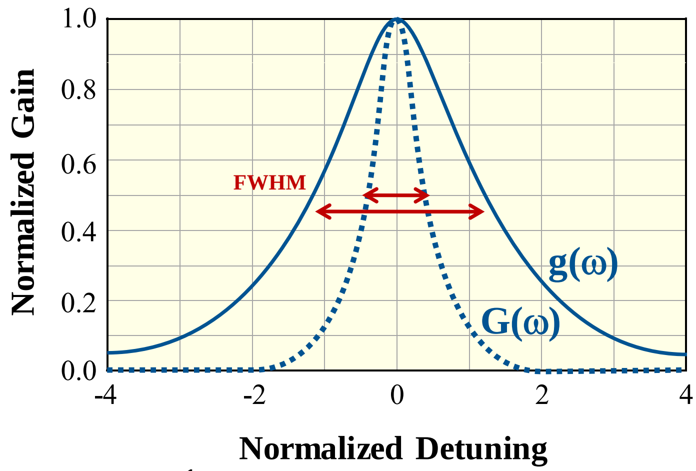
Gain Bandwidth

$$\boxed{g(P)=\frac{g_0}{1+(\omega-\omega_0)^2T_2^2}}$$

$g(\omega)$

$$\frac{\cancel{g_0}}{1+(\omega-\omega_0)^2T_2^2}=\frac{\cancel{g_0}}{2}$$

$$f_{FWHM-g}=\frac{1}{\pi T_2}$$



$$G(\omega)=e^{g(\omega)L}$$

$$e^{\frac{g_0L}{1+(\omega-\omega_0)^2T_2^2}}=\frac{1}{2}e^{g_0L}=G_0$$

$$\boxed{f_{FWHM-G}=\frac{1}{T_2}\left(\frac{\ln 2}{g_0L-\ln 2}\right)^{\frac{1}{2}}}=\left(\frac{\ln 2}{\ln(G_0/2)}\right)^{\frac{1}{2}}$$

Gain Saturation

$$\frac{\partial P}{\partial z}=g(P)\cdot P \qquad \boxed{g(P)=\frac{g_0}{1+P/P_{sat}}} \qquad \lim_{P\to\infty} g(P)=0$$

$$\partial P=g(P)\cdot P\cdot\partial z \quad =\frac{g_0}{1+\frac{P}{P_{sat}}}P\cdot\partial z \quad =\frac{1}{\frac{1}{P}+\frac{1}{P_{sat}}}g_0\partial z$$

$$g_0\partial z=\left(\frac{1}{P}+\frac{1}{P_{sat}}\right)\partial P \quad \rightarrow \quad \int_0^L g_0\partial z \;=\int_{P_{in}}^{P_{out}}\left(\frac{1}{P}+\frac{1}{P_{sat}}\right)\partial P$$

$$g_0L=\left(\ln[P]+\frac{P}{P_{sat}}\right)\Bigg|_{P_{in}}^{P_{out}}=\ln\left[\frac{P_{out}}{P_{in}}\right]+\frac{P_{out}-P_{in}}{P_{sat}}=\ln[G]+\frac{P_{in}}{P_{sat}}(G-1)$$

$$G=e^{g_0L-\frac{P_{in}}{P_{sat}}(G-1)}=\underbrace{e^{g_0L}}_{G_0}e^{-\frac{P_{in}}{P_{sat}}(G-1)} \Rightarrow \boxed{G=G_0e^{-\frac{P_{in}}{P_{sat}}(G-1)}} \qquad \lim_{P_{in}\to\infty} G(P_{in})=1$$

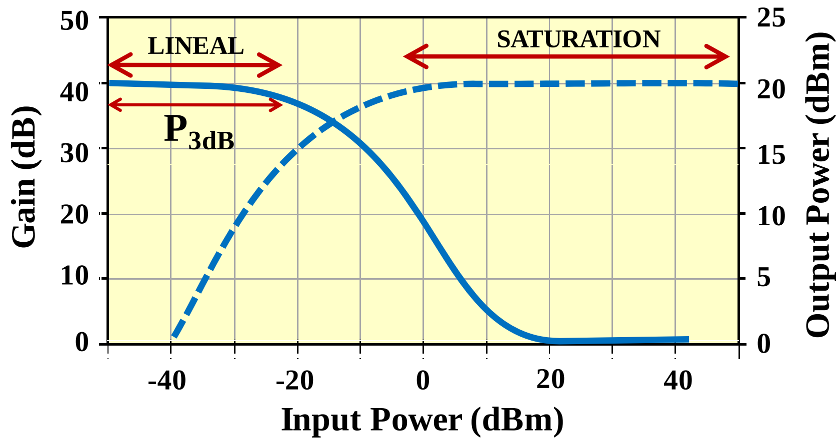
Gain Saturation

gain parameter

$$g(P)$$

$$\frac{g_0}{1+P_{3dB}/P_{sat}} = \frac{g_0}{2}$$

$$P_{3dB} = P_{sat}$$



absolute gain

$$G(P)$$

$$\frac{G_0}{2} = G_0 e^{-\frac{P_{3dB}}{P_{sat}}\left(\frac{G_0}{2}-1\right)}$$

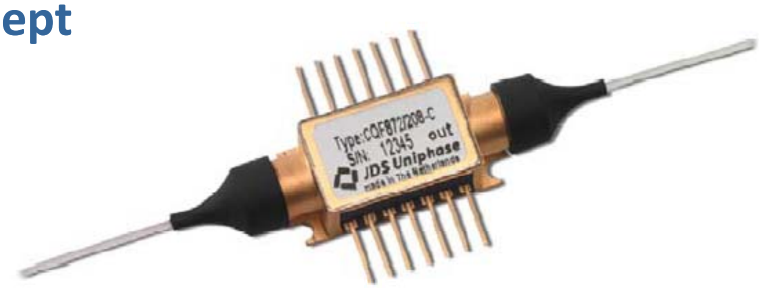
$$\ln 2 = \frac{P_{3dB}}{P_{sat}}\left(\frac{G_0}{2}-1\right) \approx \frac{P_{3dB}}{P_{sat}}\frac{G_0}{2} \Rightarrow \boxed{P_{3dB} = \frac{P_{sat}}{G_0} 2\ln 2}$$

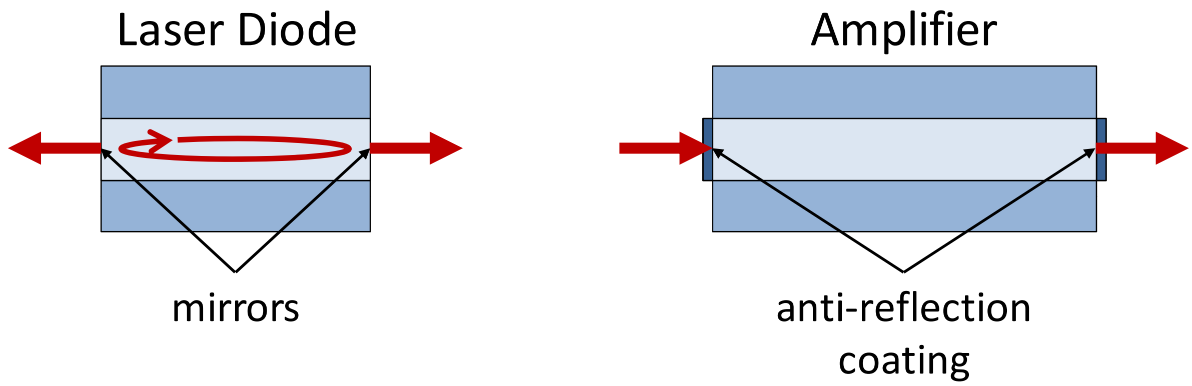
Types of Optical Amplifiers

- ❑ **Semiconductor Optical Amplifier (SOA)**
- ❑ **Fiber Optical Amplifier**
  - ❑ **Doped-Fiber Amplifier (EDFA)**
  - ❑ **Non-Linear Amplif. (Raman, Parametric)**

SEMICONDUCTOR OPTICAL AMPLIFIER

SOA Concept





Types of SOA

Fabry-Perot (FPOA)

- Reflectivity 30%
- BW moderate (10 nm)
- Gain moderate (15 dB)
- High Temp. Sensitivity

Under-threshold Laser

Traveling Wave (TWOA)

- Reflectivity 10^{-4}, 10^{-5}
- BW large (40 nm)
- Gain large (25 dB)
- Moderate Temp. Sensitivity

Rate Equation (single pass)

bandwidth not an issue

$$\boxed{\frac{\partial N}{\partial t} = \frac{I}{qV} - R_S - \frac{N}{\tau_r}}$$

$$R_S \equiv g\,v\,S = \Gamma a v\left(N - N_0\right)S$$

$$S = \frac{P}{v\,hf\,Wd}$$

incident photon density $\left[\frac{\text{fotons}}{m^3}\right]$

Stationary Regime

$$\frac{\partial N}{\partial t} = 0 \rightarrow \frac{I}{qV} = R_S + \frac{N}{\tau_r} = g\frac{P}{hf \cdot Wd} + \frac{1}{\tau_r}\left(\frac{g}{\Gamma a} + N_0\right)$$

$$\rightarrow g = \frac{\frac{I}{qV} - \frac{N_0}{\tau_r}}{\frac{P}{hf \cdot Wd} + \frac{1}{\Gamma a \tau_r}} = \frac{\Gamma a \tau_r\left(\frac{I}{qV} - \frac{N_0}{\tau_r}\right)}{1 + \Gamma a \tau_r \frac{P}{hf \cdot Wd}}$$

gain coefficient

$$\boxed{g = \frac{g_0}{1 + P/P_{sat}}}$$

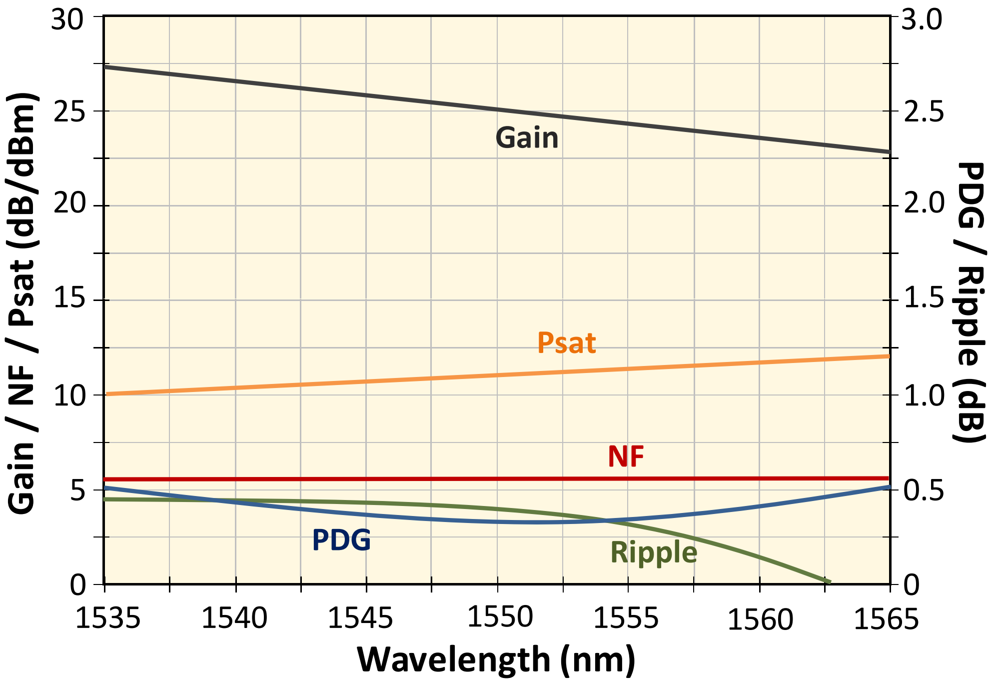
$$\boxed{g_0 \equiv \Gamma a \tau_r\left(\frac{I}{qV} - \frac{N_0}{\tau_r}\right)}$$

small-signal gain coefficient

$$\boxed{P_{sat} \equiv \frac{hf \cdot Wd}{\Gamma a \tau_r}}$$

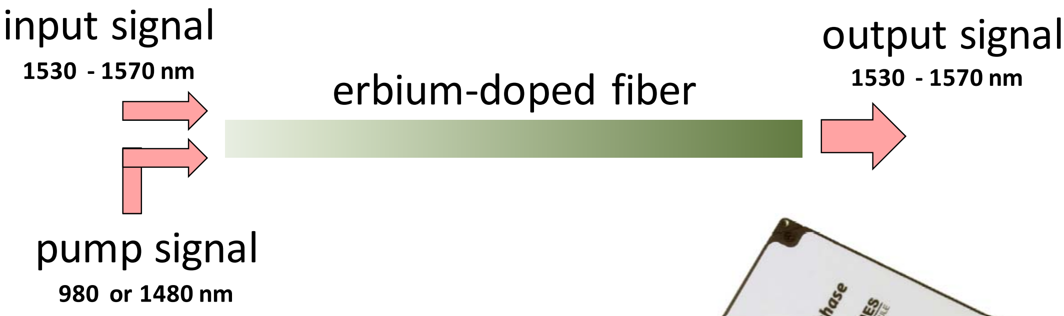
saturation power

SOA performance



DOPED-FIBER AMPLIFIER

DFA Concept



Types of Dopants

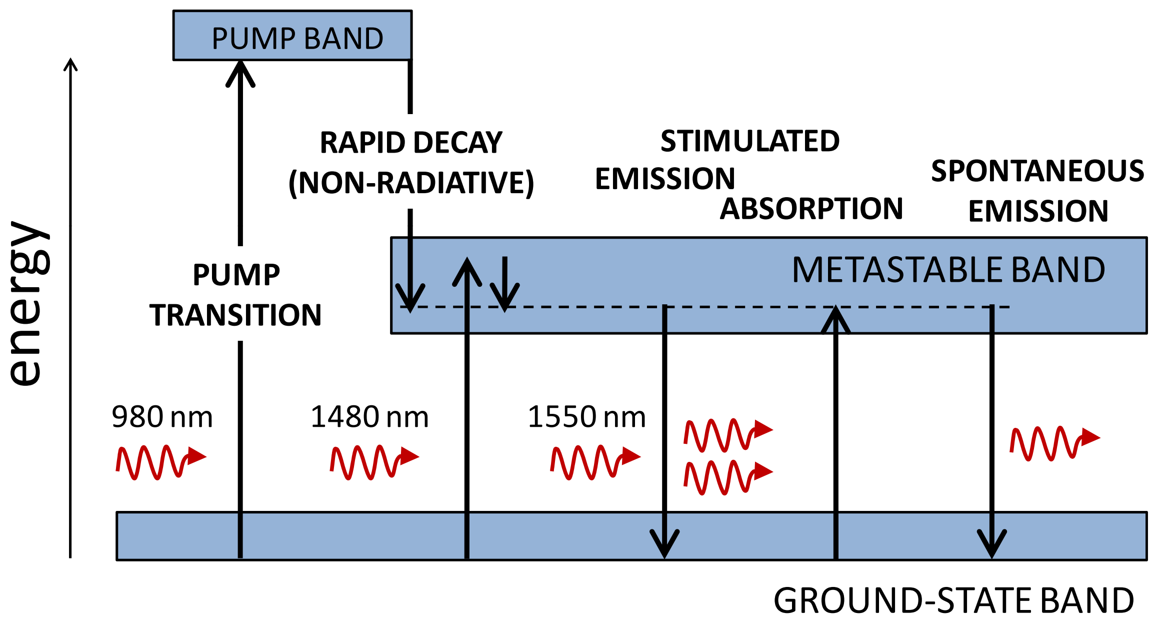
Erbium → 3rd Window

Neodimium → 2nd Window

Praseodimium → 2nd Window

Amplification Mechanism

Erbium energy-level diagram

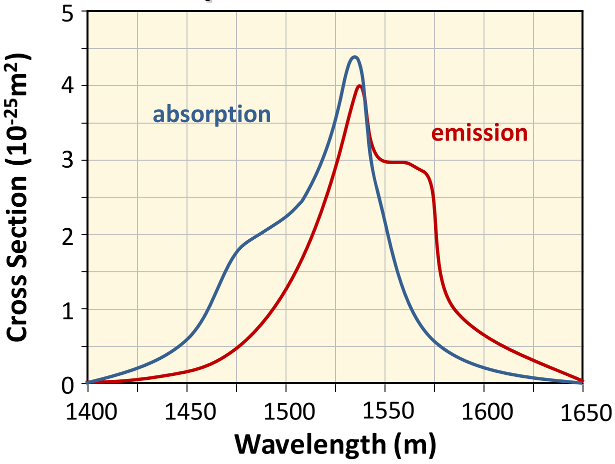


Conversion Efficiency & Gain

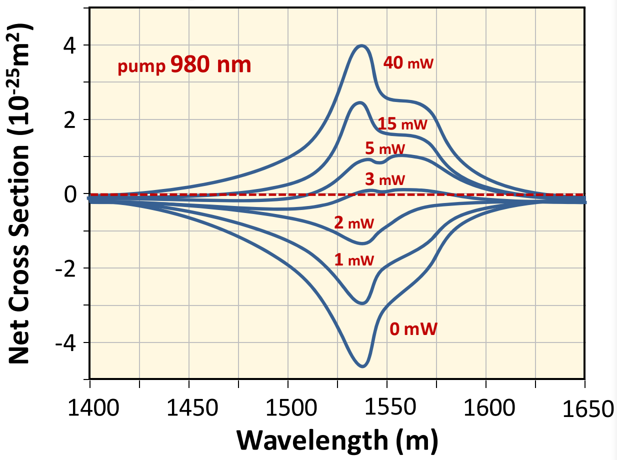
$$g_0 = \varepsilon \sigma_e \rightarrow G_0 = \exp\left[\varepsilon \sigma_e L\right]$$

σ_e: Erbium Emission Parameter
ε: Erbium Concentration

Er-doped fiber cross-sections



Er-doped fiber net cross-section

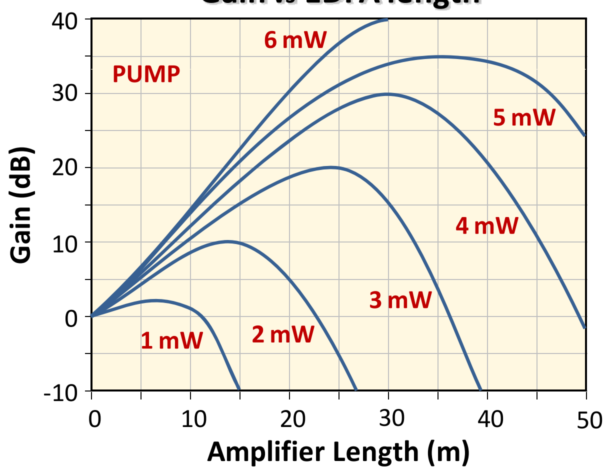


Conversion Efficiency & Gain

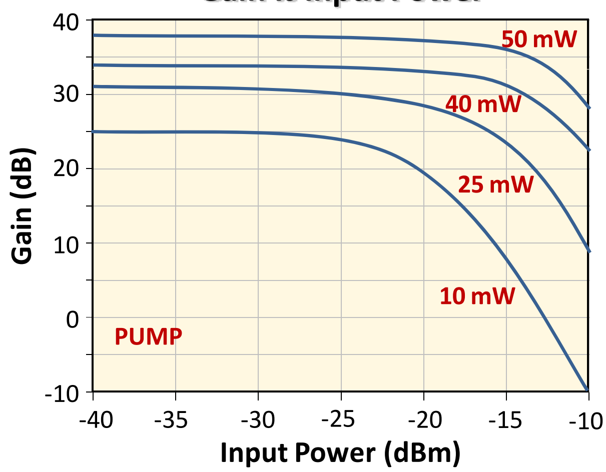
$$g_0 = \varepsilon \sigma_e \rightarrow G_0 = \exp\left[\varepsilon \sigma_e L\right]$$

σ_e: Erbium Emission Parameter
ε: Erbium Concentration

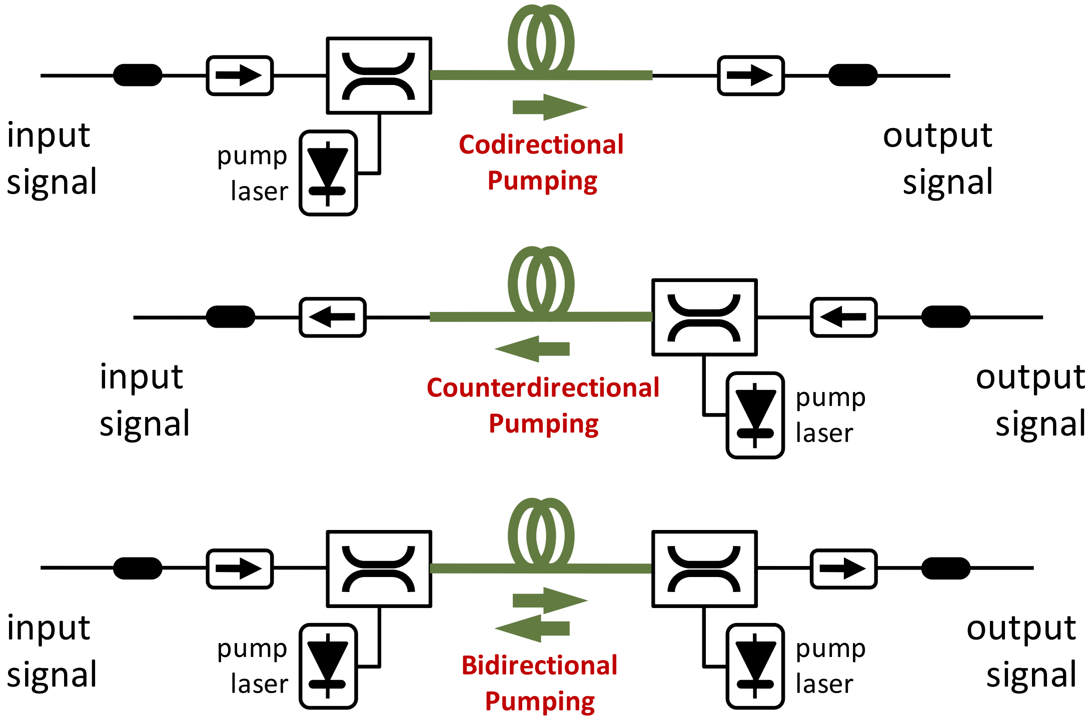
Gain vs EDFA length



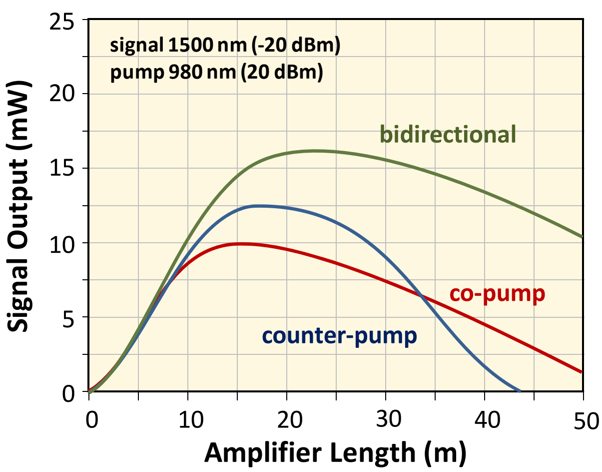
Gain vs Input Power

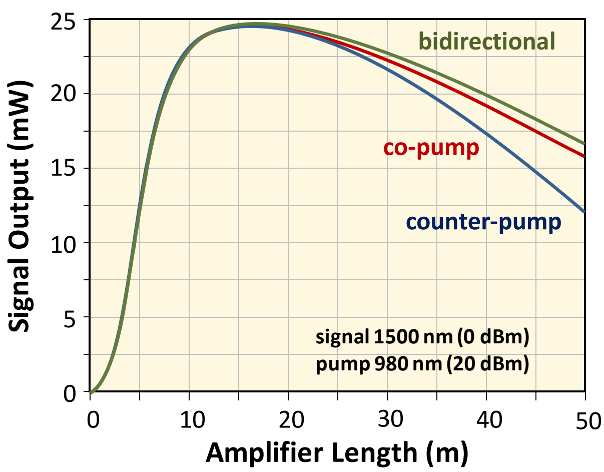


EDFA Architectures – pumping direction

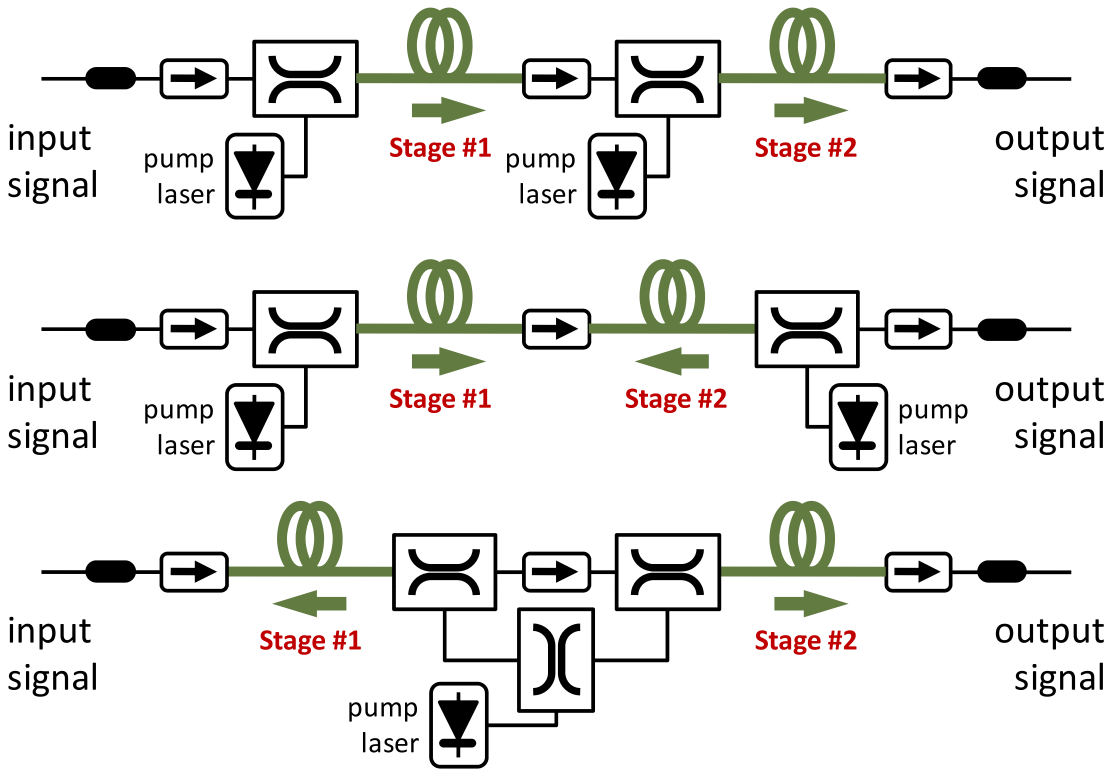


EDFA Architectures – pumping direction





EDFA Architectures – multistage

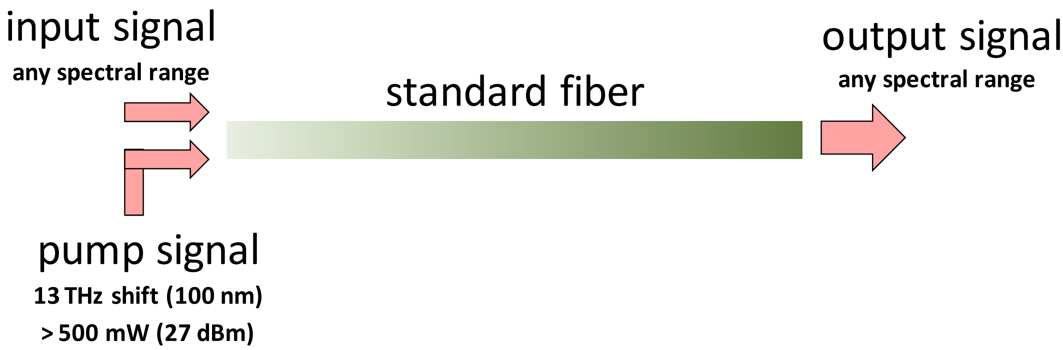


EDFA vs SOA

SOA	EDFA
Gain → 15-20 dB	Gain → 30-40 dB
Saturation → 8-10 dBm	Saturation → 20 dBm
Noise → Moderate	Noise → Low
Polarization → Sensitive	Polarization → Indep.
Crosstalk → High	Crosstalk → Low
Switching → Fast	Switching → Slow
Integrable → Yes	Integrable → No
Cost → Moderate	Cost → High

RAMAN AMPLIFIER

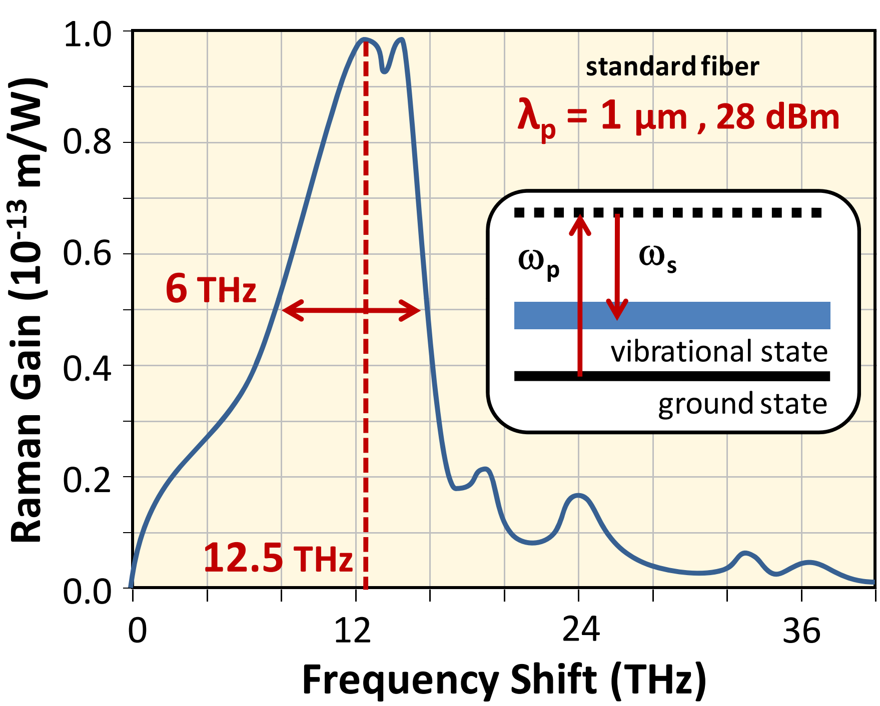
Raman Amplifier Concept



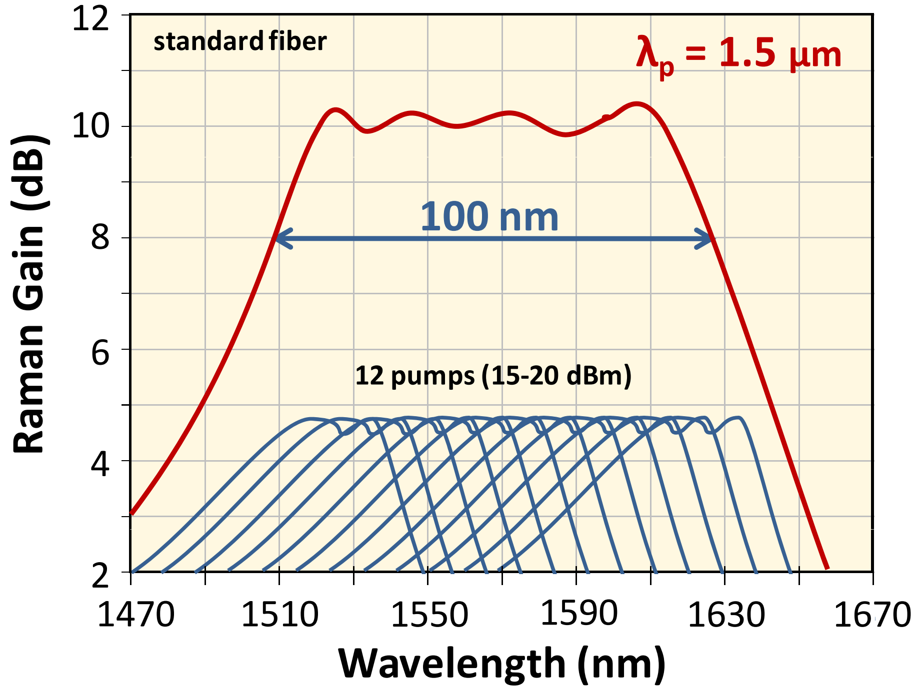
Main Characteristics

1. forward/backward capability
2. 13 THz frequency shift
3. 40 THz bandwidth
4. any frequency window
5. longer fiber lengths (>1Km)

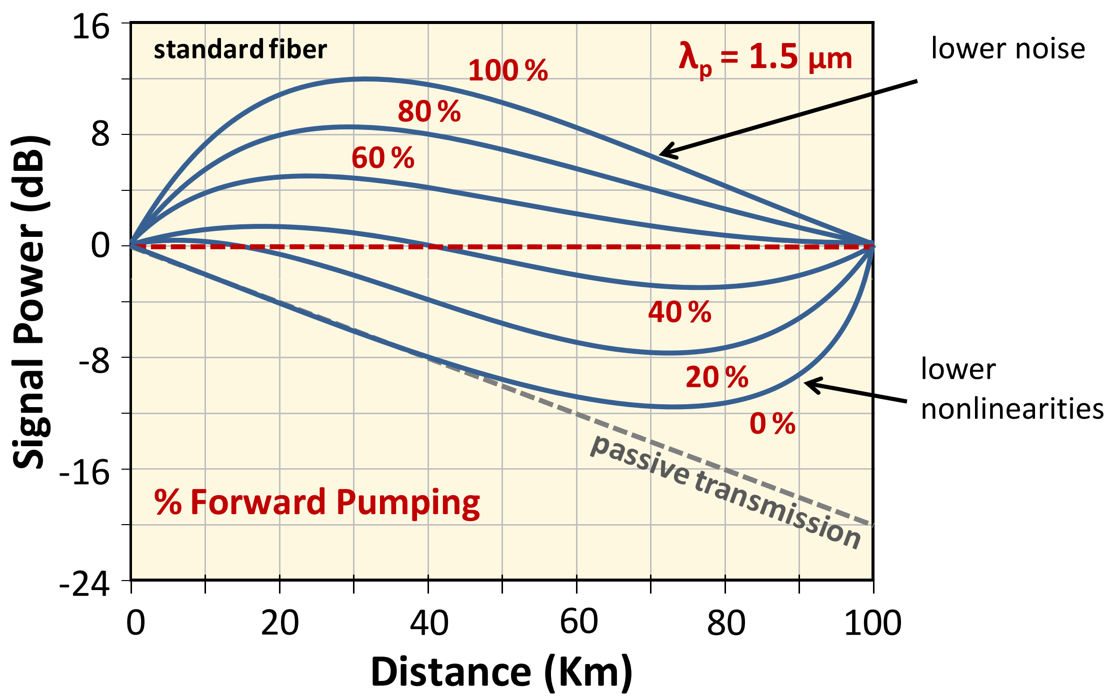
Stimulated Raman Scattering



Multiple-Pump Raman Amplifiers



Distributed Raman Amplification



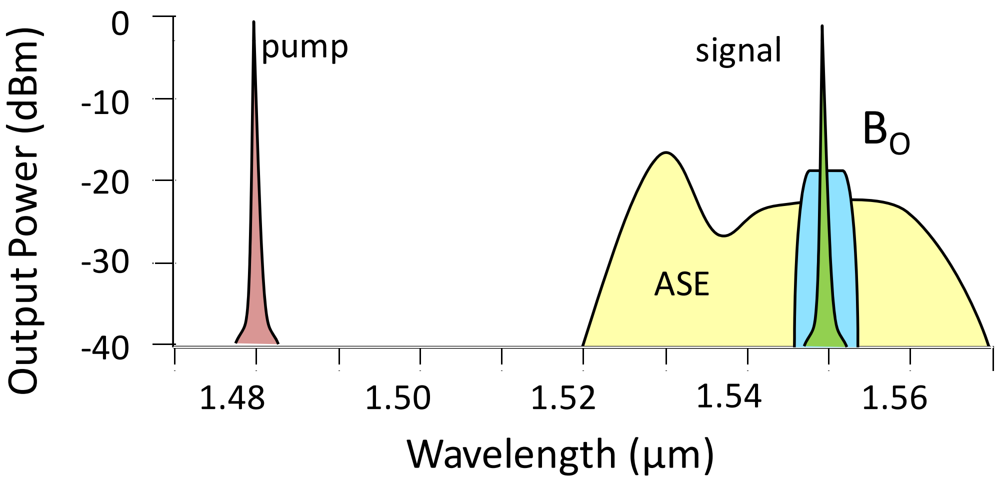
Performance Limiting Factors

1. Spontaneous Raman Scattering (noise)
2. Rayleigh Backscattering (multipath interference → most limiting factor)
3. Polarization Dependent Gain (copolarized pumping → high PMD penalty)
4. Pump-Noise Transfer (exponential pump power dependence)

NOISE IN OPTICAL AMPLIFIERS

Amplified Spontaneous Emission (ASE)

Spontaneous recombination of electron-hole pairs in the active region is the origin of ASE noise



ASE spectrum

$$S_{ASE}(f) = hf\left[G(f) - 1\right]\rho$$

$$\rho = \frac{N_2}{N_2 - N_1} \geq 1$$

spontaneous emission factor

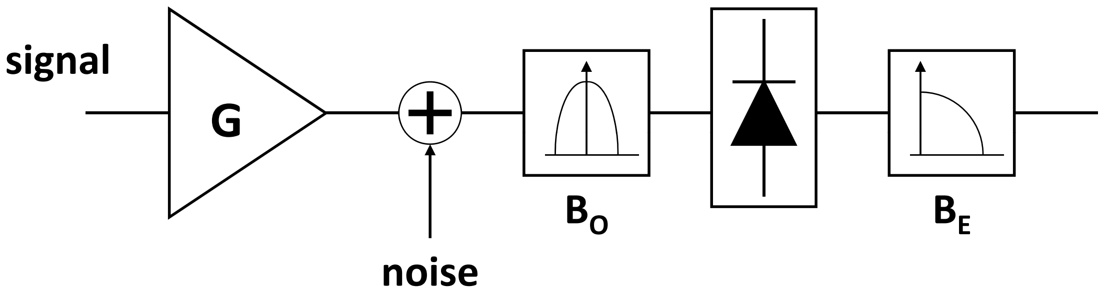
operating bandwidth (B_O)

$$S_{ASE}(f) \approx hf_p\left[G(f_p) - 1\right]\rho = ct$$

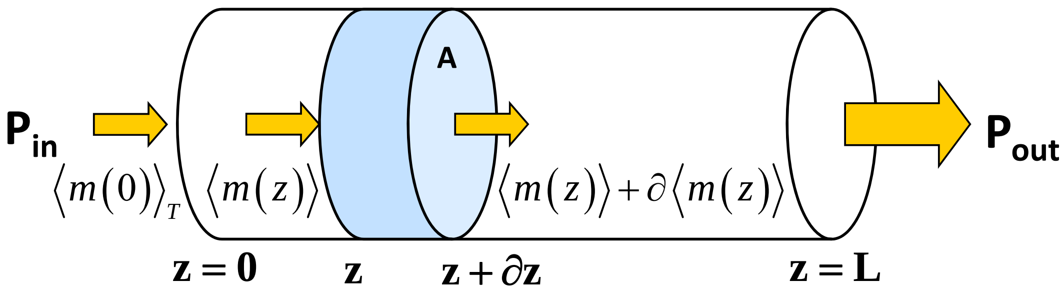
white noise

$$P_{out} = GP_{in} + S_{ASE}B_O$$

OSNR



Amplifier Statistical Model



A ≡ Probability per photon, per second of generating a new photon by stimulated emission

B ≡ Probability per photon, per second of absorbing /scatter a photon by stimulated absorption/scatt.

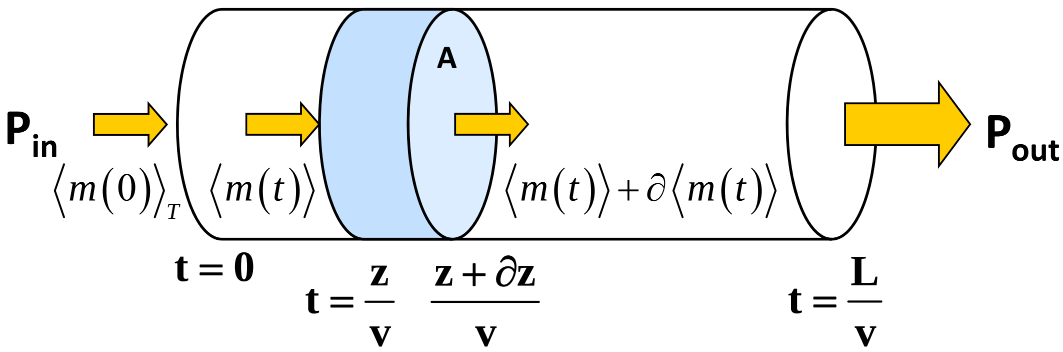
C ≡ Probability per second of generating a new photon by spontaneous emission

laser theory parameter identification

$$A = v \cdot \Gamma a N$$

$$B = \underbrace{v \cdot \Gamma a N_0}_{B_R} + \underbrace{v \cdot \alpha_s}_{B_{NR}}$$

Amplifier Statistical Model



first order moment

$$\frac{\partial\langle m(t)\rangle}{\partial t} = (A-B)\langle m(t)\rangle + C$$

$$\langle m(t)\rangle = \langle m(0)\rangle e^{(A-B)t} + \frac{C}{A-B}\left\{e^{(A-B)t} - 1\right\}$$

$$\langle m(z)\rangle = \langle m(0)\rangle e^{\frac{A-B}{v}z} + \frac{C}{A-B}\left\{e^{\frac{A-B}{v}z} - 1\right\}$$

$$\langle m(L)\rangle = \langle m(0)\rangle e^{\frac{A-B}{v}L} + \frac{C}{A-B}\left\{e^{\frac{A-B}{v}L} - 1\right\} = \langle m(0)\rangle G + \frac{C}{A-B}(G-1)$$

$$G \equiv e^{\frac{A-B}{v}L}$$ single-passing gain

$$\langle m(L)\rangle = \langle m(0)\rangle G + \frac{C}{A-B}(G-1) = \langle m(0)\rangle G + \frac{C}{A}\rho(G-1)$$

$$\rho \equiv \frac{A}{A-B}$$ spontaneous emission parameter

$0 \le B \le A \rightarrow \rho \ge 1$

single-mode cavity → C=A

$$\boxed{\langle m(L)\rangle = \langle m(0)\rangle G + \rho(G-1)}$$

second order moment (variance)

$$v\frac{\partial\langle m^2\rangle}{\partial z}=2(A-B)\langle m^2\rangle+(A+B+2C)\langle m\rangle+C \qquad G\equiv e^{\frac{A-B}{v}L}$$

demo
APPENDIX 1

$$\sigma_m^2=\langle m^2\rangle-\langle m\rangle^2 \qquad \rho\equiv\frac{A}{A-B}$$

$$\sigma_m^2(L)=G^2\left\{\sigma_{m(0)}^2-\langle m(0)\rangle\right\}+G\langle m(0)\rangle+\frac{C}{A-B}(G-1)+$$
$$+2\frac{A}{A-B}G(G-1)\langle m(0)\rangle+\frac{C}{A}\left(\frac{A}{A-B}\right)^2(G-1)^2$$

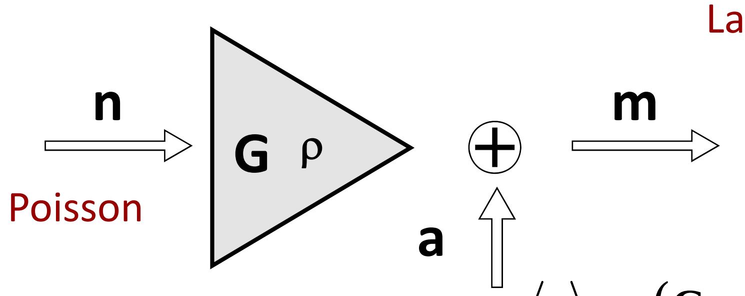
single-mode cavity → C=A

$$\boxed{\sigma_m^2(L)=G^2\left\{\sigma_{m(0)}^2-\langle m(0)\rangle\right\}+G\langle m(0)\rangle+\rho(G-1)+2\rho G(G-1)\langle m(0)\rangle+\rho^2(G-1)^2}$$

Optical amplification statistics

Laguerre

$$\langle \mathbf{n}\rangle \qquad \sigma_n^2=\langle \mathbf{n}\rangle$$



$$\langle \mathbf{m}\rangle \qquad \sigma_m^2$$

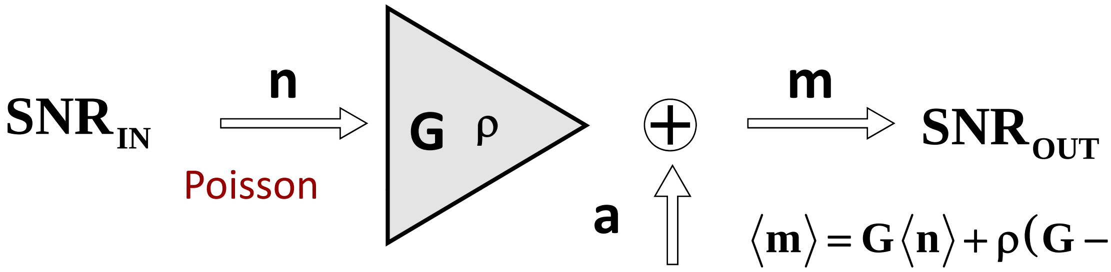
$$\langle \mathbf{a}\rangle=\rho(G-1) \qquad \sigma_a^2=\langle \mathbf{a}\rangle+\langle \mathbf{a}\rangle^2$$

$$G>>1 \qquad \langle \mathbf{n}\rangle>>\rho$$

$$\langle \mathbf{m}\rangle=G\langle \mathbf{n}\rangle+\rho(G-1)$$

$$\sigma_m^2=\underbrace{\cancel{G^2\left(\sigma_n^2-\langle \mathbf{n}\rangle\right)}}_{\text{excess}}+\underbrace{G\langle \mathbf{n}\rangle\cancel{+\rho(G-1)}}_{\text{shot}}+\underbrace{2\rho G(G-1)\langle \mathbf{n}\rangle}_{\text{S-ASE}}+\underbrace{\cancel{\rho^2(G-1)^2}}_{\text{ASE-ASE}}$$

Signal-to-Noise Ratio (SNR)



$$\langle m\rangle = G\langle n\rangle + \rho(G-1)$$

$$SNR_{IN} = \frac{\langle n\rangle^2}{\sigma_n^2} = \langle n\rangle$$

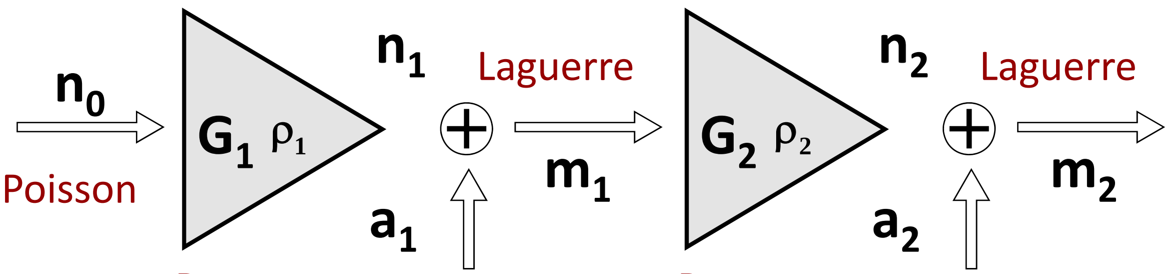
$$SNR_{OUT} = \frac{\langle m\rangle^2}{\sigma_m^2} = \frac{G^2\langle n\rangle^2 + \rho^2(G-1)^2 + 2\langle n\rangle\rho G(G-1)}{G\langle n\rangle + \rho(G-1) + 2\langle n\rangle\rho G(G-1) + \rho^2(G-1)^2}$$

$$\approx \frac{G^2\langle n\rangle^2}{2\rho G(G-1)\langle n\rangle} \approx \frac{\langle n\rangle}{2\rho} = \frac{SNR_{IN}}{2\rho}$$

Noise Figure $\boxed{NF = 2\rho}$ $\rho \geq 1$

Generalized Signal-ASE statistics

Poisson $\langle n_1\rangle = \langle n_0\rangle G_1$ Poisson $\langle n_2\rangle = \langle n_0\rangle G_1 G_2$



Bose-Einstein $\langle a_1\rangle = \rho_1(G_1-1)$ Bose-Einstein $\langle a_2\rangle = \rho_1(G_1-1)G_2 + \rho_2(G_2-1)$

$$\langle m\rangle = \langle n\rangle + \langle a\rangle$$

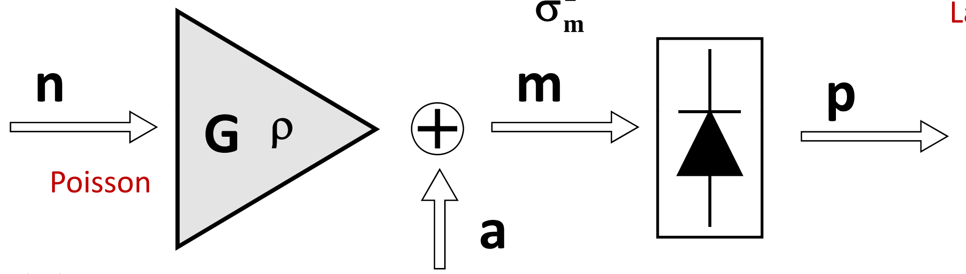
$$\sigma_m^2 = \underbrace{\langle n\rangle}_{\text{shot}} + \langle a\rangle + \underbrace{2\langle n\rangle\langle a\rangle}_{\text{S-ASE}} + \underbrace{\langle a\rangle^2}_{\text{ASE-ASE}}$$

demo
APPENDIX 2

Photodetection Statistics – signal + ASE

Laguerre $\langle m \rangle$ σ_m^2

Laguerre $\langle p \rangle$ σ_p^2



$\langle n \rangle$

$\sigma_n^2 = \langle n \rangle$

$\langle a \rangle = \rho(G-1)$

$\sigma_a^2 = \langle a \rangle + \langle a \rangle^2$

Bose-Einstein

$\langle m \rangle = \langle n \rangle + \langle a \rangle$

$\sigma_m^2 = \underbrace{\langle n \rangle + \langle a \rangle}_{\text{shot}} + 2\langle n \rangle\langle a \rangle + \langle a \rangle^2$

shot S-ASE ASE-ASE

mean $\langle p(t) \rangle = \eta G \langle n(t) \rangle + \eta\rho(G-1)$

variance $\sigma_p^2(t) = \eta G \langle n(t) \rangle + \eta\rho(G-1) + \eta^2 2\rho G(G-1)\langle n(t) \rangle + \eta^2\rho^2(G-1)^2$

$$\sigma_p^2(t) = \eta^2 2\rho G(G-1)\langle n(t) \rangle + \eta^2\rho^2(G-1)^2 + \eta G\langle n(t) \rangle + \eta\rho(G-1)$$

signal shot noise

$$\sigma_{shot-S}^2(t) = \eta G \langle n(t) \rangle$$

$G >> 1$

$\langle n \rangle >> \rho$

ASE shot noise

$$\sigma_{shot-ASE}^2(t) = \eta\rho(G-1)$$

signal-ASE beat noise

$$\sigma_{S-ASE}^2(t) = \eta^2 2\rho G(G-1)\langle n(t) \rangle$$

dominant noise

ASE-ASE beat noise

$$\sigma_{ASE-ASE}^2(t) = \eta^2\rho^2(G-1)^2$$

Photodetection Currents – signal + ASE

$$\boxed{I_{ph}=\left(\frac{q}{T_b}\right)\langle p\rangle}=\left(\frac{q}{T_b}\right)\eta\left[G\langle n\rangle+\rho(G-1)\right]=\left(\frac{q}{T_b}\right)\Re\frac{hf}{q}\left[G\langle n\rangle+\rho(G-1)\right]=$$

$$=\Re\left[G\underbrace{\frac{hf\langle n\rangle}{T_b}}_{P_S}+\underbrace{(G-1)\frac{hf\cdot\rho}{T_b}}_{P_{ASE}=S_{ASE}B_O}\right]=\Re\left[GP_S+S_{ASE}B_O\right]$$

$$\boxed{\sigma_{ph}^2=\left(\frac{q}{T_b}\right)^2\sigma_p^2}=\left(\frac{q}{T_b}\right)^2\left[\eta^2 2\rho G(G-1)\langle n\rangle+\eta^2\rho^2(G-1)^2+\eta G\langle n\rangle+\eta\rho(G-1)\right]$$

$$\sigma_{shot}^2=\sigma_{shot-s}^2+\sigma_{shot-ASE}^2=\left(\frac{q}{T_b}\right)^2\left[\eta G\langle n\rangle+\eta\rho(G-1)\right]=$$

$$=\left(\frac{q}{T_b}\right)^2\left[\Re\frac{hf}{q}G\langle n\rangle+\Re\frac{hf}{q}\rho(G-1)\right]=2qB_E\Re\left[G\underbrace{\frac{hf\langle n\rangle}{T_b}}_{P_S}+\underbrace{(G-1)\frac{hf\cdot\rho}{T_b}}_{P_{ASE}=S_{ASE}\cdot B_O}\right]$$

$$\frac{1}{T_b}=2B_E=B_O$$

$$\boxed{\sigma_{shot}^2=2qB_E\Re\left(GP_S+S_{ASE}B_O\right)}$$

$$\sigma_{s-ASE}^2=\left(\frac{q}{T_b}\right)^2\eta^2 2\rho G(G-1)\langle n\rangle=\left(\frac{q}{T_b}\right)^2\left(\Re\frac{hf}{q}\right)^2 2\rho G(G-1)\langle n\rangle=$$

$$=2\Re^2 G\underbrace{\frac{hf\langle n\rangle}{T_b}}_{P_S}\underbrace{(G-1)\frac{hf\cdot\rho}{T_b}}_{P_{ASE}=S_{ASE}B_O}=\boxed{2\Re^2 GP_S S_{ASE}B_O}$$

dominant noise

$$\sigma_{ASE-ASE}^2=\left(\frac{q}{T_b}\right)^2\eta^2\rho^2(G-1)^2=\left(\frac{q}{T_b}\right)^2\left(\Re\frac{hf}{q}\right)^2\rho^2(G-1)^2=$$

$$=\Re^2\left(\underbrace{(G-1)\frac{hf\cdot\rho}{T_b}}_{P_{ASE}=S_{ASE}B_O}\right)^2=\boxed{\Re^2 S_{ASE}^2 B_O^2}$$

$$\boxed{\sigma_T^2=4\left(k_B T/R_L\right)F_A B_E}$$

Thermal Noise

Noise Factor

$$SNR \equiv \frac{\langle i_s \rangle^2}{\sigma^2_{tot}} \approx \frac{(\Re G P_S)^2}{4\Re^2 G P_S hf (G-1)\rho B_E} = \frac{P_S}{2hf\,B_E}\frac{G}{2\eta\,\rho(G-1)}$$

Ideal photodetector (η=1, no thermal noise)

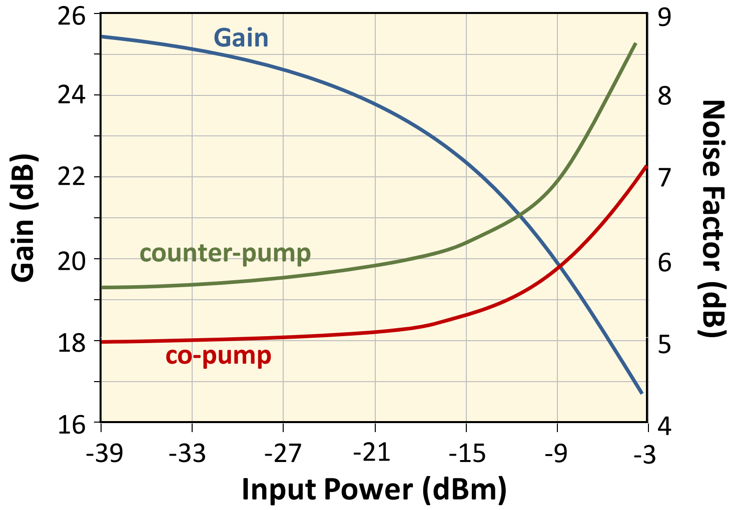
$$SNR_{LQ} \equiv \frac{P_S}{2\,hf\,B_E}$$

$$\boxed{F \equiv \frac{2\rho(G-1)}{G}}$$

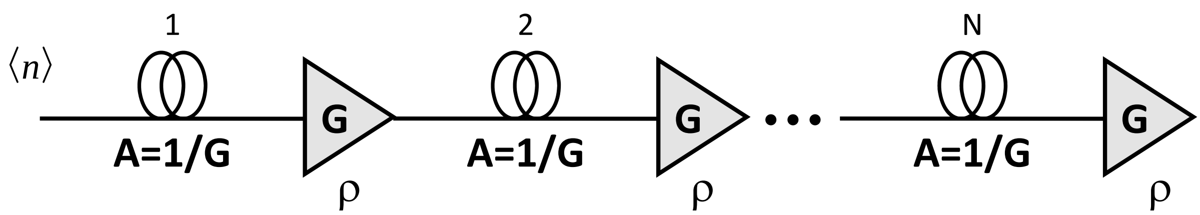
$$F \xrightarrow{G>>} 2\rho$$

$$F \xrightarrow{\rho=1} 2\ (3dB)$$

Quantum Limit



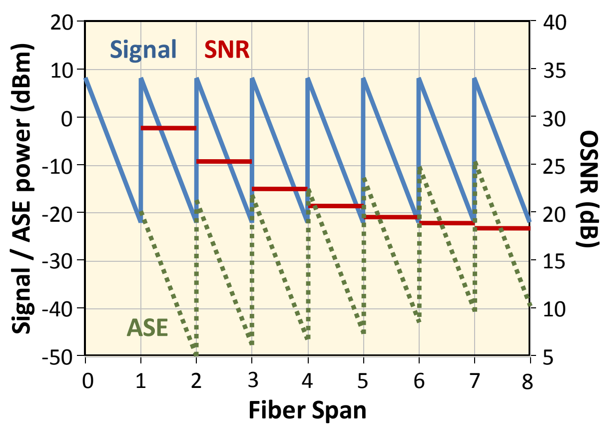
SNR degradation through transmission



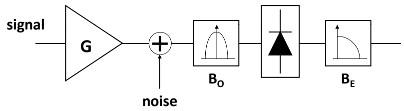
$$\mu = \langle n \rangle + N\rho G$$

$$\sigma^2 = \langle n \rangle + N\rho G + 2\langle n \rangle N\rho G + N^2\rho^2 G^2$$

$$\boxed{SNR = \frac{\mu^2}{\sigma^2} \approx \frac{\langle n \rangle^2}{2\langle n \rangle N\rho G} = \frac{\langle n \rangle}{2N\rho G}}$$



Pre-Amplified Receiver Sensitivity



$$\mu_1 \approx \eta G \langle n \rangle + \eta\rho(G-1) \qquad \mu_0 = \eta\rho(G-1)$$

$$\sigma_1 \approx \sqrt{\eta^2 2\rho G^2 \langle n \rangle + \eta^2\rho^2 G^2} \qquad \sigma_0 \approx \eta\rho G$$

$$Q_{OA} = \frac{\mu_1 - \mu_0}{\sigma_1 + \sigma_0} \approx \frac{\eta G \langle n \rangle}{\sqrt{\eta^2 2\rho G^2 \langle n \rangle + \eta^2\rho^2 G^2} + \eta\rho G} > Q$$

Gaussian Statistics

$$\langle n \rangle > 2\rho\left(Q^2 + Q\right) \quad \leftarrow \quad \text{OSNR} \equiv \frac{\langle n \rangle}{\rho}$$

$$\boxed{\langle n_a \rangle > \rho\left(Q^2 + Q\right)} \xrightarrow{Q = 6,\ \rho = 1} \langle n_a \rangle = 42 \left[\frac{\text{fot}}{\text{bit}}\right]$$

$$\langle n \rangle = \frac{P}{hf} T_b \longrightarrow P_a > hf\,\rho B_O\left[Q^2 + Q\right] \equiv S_{AO}$$

sensitivity AO + PIN $\quad \rho\left(Q^2 + Q\right) \equiv S_{OA}$

sensitivity PIN / APD $\quad \frac{1}{2\eta}\left(Q^2 F(M) + 2Q\frac{\sigma_t}{M}\right) \equiv S_{APD}$

Pre-Amplified Receivers Sensitivity Improvement

$$S_{OA} < S_{APD}$$

$$\rho\left(Q^2 + Q\right) < \frac{1}{2\eta}\left(Q^2 F(M) + 2Q\frac{\sigma_t}{M}\right)$$

$$\rho(Q+1) < \frac{1}{2\eta}\left(QF(M) + 2\frac{\sigma_t}{M}\right) \longrightarrow \boxed{\rho < \frac{1}{2\eta}\frac{1}{Q+1}\left(QF(M) + 2\frac{\sigma_t}{M}\right)}$$

$$Q_{PIN}=\frac{\mu_1-\mu_0}{\sigma_1+\sigma_0}\approx\frac{\mu_1}{2\sigma_0}=\frac{\eta\langle n\rangle}{2\sigma_t}$$

$$Q_{AO}=\frac{\mu_1-\mu_0}{\sigma_1+\sigma_0}\approx\frac{\eta G\langle n\rangle}{\sqrt{\eta^2 2\rho G^2\langle n\rangle+\eta^2\rho^2G^2}+\eta\rho G}=\frac{\langle n\rangle}{\sqrt{2\rho\langle n\rangle+\rho^2}+\rho}$$

Pre-Amplified Receivers BER Improvement

$$Q_{AO}>Q_{PIN}$$

$$\frac{\langle n\rangle}{\sqrt{2\rho\langle n\rangle+\rho^2}+\rho}>\frac{\eta\langle n\rangle}{2\sigma_t}\longrightarrow\langle n\rangle<2\frac{\sigma_t}{\eta}\left(\frac{\sigma_t}{\eta\rho}-1\right)$$

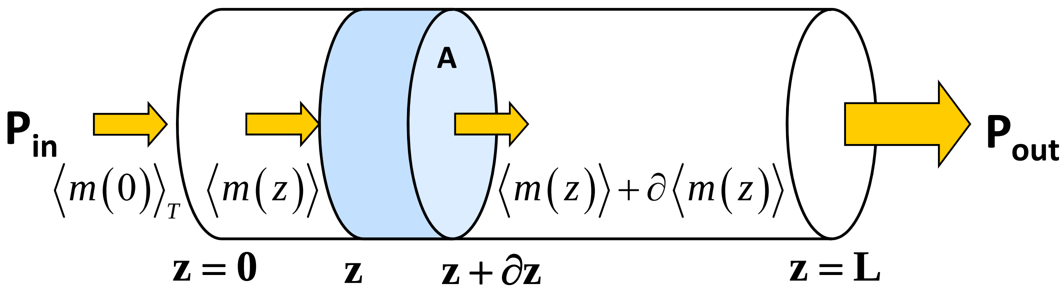
$$\boxed{\langle n_a\rangle<\frac{\sigma_t}{\eta}\left(\frac{\sigma_t}{\eta\rho}-1\right)}$$

$$\langle n\rangle=\frac{P}{hf}T_b\longrightarrow P_a<\frac{hf}{T_b}\frac{\sigma_t}{\eta}\left(\frac{\sigma_t}{\eta\rho}-1\right)=\frac{q}{T_b}\frac{\sigma_t}{\Re}\left(\frac{\sigma_t}{\eta\rho}-1\right)=\frac{\sigma_T}{\Re}\left(\frac{\sigma_T}{\eta\rho}\frac{T_b}{q}-1\right)$$

APPENDIX 1

OA statistical model

Amplifier Statistical Model



$A \equiv$ Probability per photon, per second of generating a new photon by stimulated emission

$B \equiv$ Probability per photon, per second of absorbing /scatter a photon by stimulated absorption/scatt.

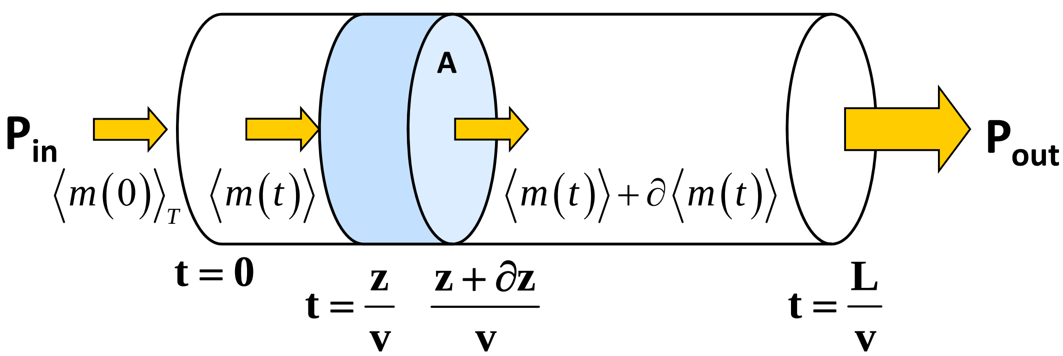
$C \equiv$ Probability per second of generating a new photon by spontaneous emission

laser theory parameter identification

$$A = v \cdot \Gamma a N$$

$$B = \underbrace{v \cdot \Gamma a N_0}_{B_R} + \underbrace{v \cdot \alpha_s}_{B_{NR}}$$

Amplifier Statistical Model



first order moment

$$\frac{\partial\langle m(t)\rangle}{\partial t} = (A-B)\langle m(t)\rangle + C$$

$$\langle m(t)\rangle = \langle m(0)\rangle e^{(A-B)t} + \frac{C}{A-B}\left\{e^{(A-B)t} - 1\right\}$$

$$\langle m(z)\rangle = \langle m(0)\rangle e^{\frac{A-B}{v}z} + \frac{C}{A-B}\left\{e^{\frac{A-B}{v}z} - 1\right\}$$

$$\langle m(L)\rangle = \langle m(0)\rangle e^{\frac{A-B}{v}L} + \frac{C}{A-B}\left\{e^{\frac{A-B}{v}L} - 1\right\} = \langle m(0)\rangle G + \frac{C}{A-B}(G-1)$$

$$G \equiv e^{\frac{A-B}{v}L}$$ single-passing gain

$$\langle m(L)\rangle = \langle m(0)\rangle G + \frac{C}{A-B}(G-1) = \langle m(0)\rangle G + \frac{C}{A}\rho(G-1)$$

$$\rho \equiv \frac{A}{A-B}$$ spontaneous emission parameter

$0 \le B \le A \rightarrow \rho \ge 1$

single-mode cavity → C=A

$$\boxed{\langle m(L)\rangle = \langle m(0)\rangle G + \rho(G-1)}$$

second order moment (variance)

$$v\frac{\partial\langle m^2\rangle}{\partial z} = 2(A-B)\langle m^2\rangle + (A+B+2C)\langle m\rangle + C$$

$$G \equiv e^{\frac{A-B}{v}L}$$

demo APPENDIX 1

$$\sigma_m^2 = \langle m^2\rangle - \langle m\rangle^2$$

$$\rho \equiv \frac{A}{A-B}$$

$$\sigma_m^2(L) = G^2\left\{\sigma_{m(0)}^2 - \langle m(0)\rangle\right\} + G\langle m(0)\rangle + \frac{C}{A-B}(G-1) +$$

$$+2\frac{A}{A-B}G(G-1)\langle m(0)\rangle + \frac{C}{A}\left(\frac{A}{A-B}\right)^2(G-1)^2$$

single-mode cavity → C=A

$$\boxed{\sigma_m^2(L) = G^2\left\{\sigma_{m(0)}^2 - \langle m(0)\rangle\right\} + G\langle m(0)\rangle + \rho(G-1) + 2\rho G(G-1)\langle m(0)\rangle + \rho^2(G-1)^2}$$

demo

$$\langle m^2\rangle = \sigma_m^2 + \langle m\rangle^2$$

$$\langle m\rangle^2 = \langle m(0)\rangle^2 G^2 + \left(\frac{C}{A-B}\right)^2 (G-1)^2 + 2\langle m(0)\rangle \frac{C}{A-B} G(G-1)$$

$$\langle m^2\rangle = G^2\left\{\sigma_{m(0)}^2 - \langle m(0)\rangle\right\} + G\langle m(0)\rangle + \frac{C}{A-B}(G-1) + 2\langle m(0)\rangle \frac{A}{A-B}G(G-1) + \frac{C}{A}\left(\frac{A}{A-B}\right)^2 (G-1)^2 + $$
$$+\langle m(0)\rangle^2 G^2 + \left(\frac{C}{A-B}\right)^2 (G-1)^2 + 2\langle m(0)\rangle \frac{C}{A-B} G(G-1)$$

$$\frac{\partial\langle m^2\rangle}{\partial z} = \left\{\sigma_{m(0)}^2 - \langle m(0)\rangle\right\}\frac{\partial G^2}{\partial z} + \langle m(0)\rangle \frac{\partial G}{\partial z} + \frac{C}{A-B}\frac{\partial G}{\partial z} + 2\langle m(0)\rangle \frac{A}{A-B}\left(\frac{\partial G^2}{\partial z} - \frac{\partial G}{\partial z}\right) +$$
$$+\frac{C}{A}\left(\frac{A}{A-B}\right)^2\left(\frac{\partial G^2}{\partial z} - 2\frac{\partial G}{\partial z}\right) + \langle m(0)\rangle^2 \frac{\partial G^2}{\partial z} + \left(\frac{C}{A-B}\right)^2\left(\frac{\partial G^2}{\partial z} - 2\frac{\partial G}{\partial z}\right) +$$
$$+2\langle m(0)\rangle \frac{C}{A-B}\left(\frac{\partial G^2}{\partial z} - \frac{\partial G}{\partial z}\right)$$

$$G = e^{\frac{A-B}{v}z} \qquad \rightarrow \frac{\partial G}{\partial z} = \frac{A-B}{v}G \qquad \rightarrow \frac{\partial G^2}{\partial z} = 2\frac{A-B}{v}G^2$$

demo

$$\langle m^2\rangle = G^2\left\{\sigma_{m(0)}^2 - \langle m(0)\rangle\right\} + G\langle m(0)\rangle + \frac{C}{A-B}(G-1) + 2\langle m(0)\rangle \frac{A}{A-B}G(G-1) +$$
$$+\frac{C}{A}\left(\frac{A}{A-B}\right)^2 (G-1)^2 + \langle m(0)\rangle^2 G^2 + \left(\frac{C}{A-B}\right)^2 (G-1)^2 + 2\langle m(0)\rangle \frac{C}{A-B} G(G-1)$$

$$v\frac{\partial\langle m^2\rangle}{\partial z} = 2(A-B)\left\{\begin{array}{l}
\left\{\sigma_{m(0)}^2 - \langle m(0)\rangle\right\}G^2 + \langle m(0)\rangle\frac{G}{2} + \frac{C}{A-B}\frac{G}{2} + 2\langle m(0)\rangle\frac{A}{A-B}\left(G^2 - \frac{G}{2}\right) + \\
+\frac{C}{A}\left(\frac{A}{A-B}\right)^2\left(G^2 - G\right) + \langle m(0)\rangle^2 G^2 + \left(\frac{C}{A-B}\right)^2\left(G^2 - G\right) + \\
+2\langle m(0)\rangle\frac{C}{A-B}\left(G^2 - \frac{G}{2}\right) \\
\pm\langle m(0)\rangle\frac{G}{2} \pm \frac{C}{A-B}\frac{G}{2} \mp \frac{C}{A-B} \mp \langle m(0)\rangle\frac{A}{A-B}G \\
\pm\frac{C}{A}\left(\frac{A}{A-B}\right)^2(1-G) \pm \left(\frac{C}{A-B}\right)^2(1-G) \mp \langle m(0)\rangle\frac{C}{A-B}G
\end{array}\right\}$$

demo

$$v\frac{\partial\langle m^2\rangle}{\partial z}=2(A-B)\langle m^2\rangle-$$

$$-2(A-B)\left\{\begin{array}{c}\langle m(0)\rangle\frac{G}{2}+\frac{C}{A-B}\frac{G}{2}-\frac{C}{A-B}-\langle m(0)\rangle\frac{A}{A-B}G\\ -\frac{C}{A}\left(\frac{A}{A-B}\right)^2(G-1)-\left(\frac{C}{A-B}\right)^2(G-1)-\langle m(0)\rangle\frac{C}{A-B}G\end{array}\right\}=$$

$$v\frac{\partial\langle m^2\rangle}{\partial z}=2(A-B)\langle m^2\rangle-$$

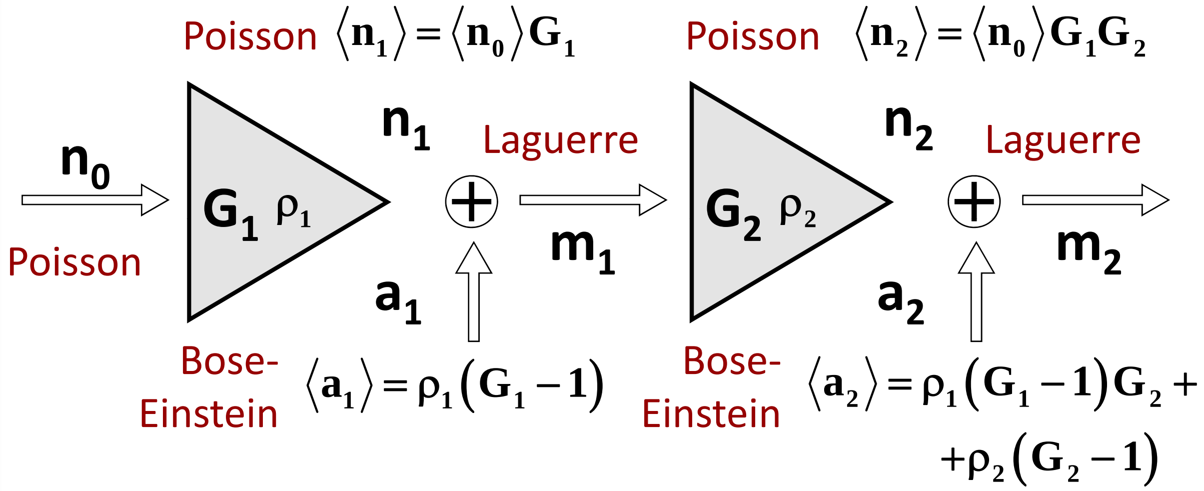
$$-2(A-B)\left\{\langle m(0)\rangle G\left\{-\frac{A}{A-B}-\frac{C}{A-B}+\frac{1}{2}\right\}+\frac{C}{A-B}(G-1)\left\{-\frac{A}{A-B}-\frac{C}{A-B}+\frac{1}{2}\right\}-\frac{1}{2}\frac{C}{A-B}\right\}=$$

$$v\frac{\partial\langle m^2\rangle}{\partial z}=2(A-B)\langle m^2\rangle+(A+B+2C)\underbrace{\left\{\langle m(0)\rangle G+\frac{C}{A-B}(G-1)\right\}}_{\langle m\rangle}+C$$

APPENDIX 2

Generalized Signal-ASE statistics

Generalized Signal-ASE statistics



$$\langle \mathbf{m}\rangle = \langle \mathbf{n}\rangle + \langle \mathbf{a}\rangle$$

$$\sigma_{\mathrm{m}}^2 = \underbrace{\langle \mathbf{n}\rangle + \langle \mathbf{a}\rangle}_{\text{shot}} + \underbrace{2\langle \mathbf{n}\rangle\langle \mathbf{a}\rangle}_{\text{S-ASE}} + \underbrace{\langle \mathbf{a}\rangle^2}_{\text{ASE-ASE}}$$

demo

$$\begin{aligned}
\sigma_{m,2}^2 &= G_2^2\left\{\sigma_{m,1}^2 - \langle m_1\rangle\right\} + G_2\langle m_1\rangle + \rho_2(G_2-1) + 2\langle m_1\rangle\rho_2 G_2(G_2-1) + \rho_2^2(G_2-1)^2 = \\
&= G_2^2\left\{2\langle n_0\rangle\rho_1 G_1(G_1-1) + \rho_1^2(G_1-1)^2\right\} + \\
&+ \underbrace{G_2\left\{\langle n_0\rangle G_1 + \rho_1(G_1-1)\right\} + \rho_2(G_2-1)}_{\langle n_0\rangle G_1G_2 + \rho_1(G_1-1)G_2 + \rho_2(G_2-1)} + \\
&+ 2\left\{\langle n_0\rangle G_1 + \rho_1(G_1-1)\right\}\rho_2 G_2(G_2-1) + \\
&+ \rho_2^2(G_2-1)^2 \\
&= \langle n_0\rangle G_1G_2 + \rho_1(G_1-1)G_2 + \rho_2(G_2-1) + \\
&+ 2\langle n_0\rangle G_1G_2\left\{\rho_1(G_1-1)G_2 + \rho_2(G_2-1)\right\} + \\
&+ \underbrace{\rho_1^2(G_1-1)^2 G_2^2 + 2\rho_1(G_1-1)\rho_2 G_2(G_2-1) + \rho_2^2(G_2-1)^2}_{\left\{\rho_1(G_1-1)G_2 + \rho_2(G_2-1)\right\}^2}
\end{aligned}$$

$$\sigma_{m,2}^2 = \overbrace{\langle n_0\rangle G_1G_2}^{senyal} + \overbrace{\rho_1(G_1-1)G_2 + \rho_2(G_2-1)}^{ASE} + \overbrace{2\langle n_0\rangle G_1G_2\left\{\rho_1(G_1-1)G_2 + \rho_2(G_2-1)\right\}}^{senyal-ASE} + \overbrace{\left\{\rho_1(G_1-1)G_2 + \rho_2(G_2-1)\right\}^2}^{ASE-ASE}$$