APUNTS PROPIETAT DE:
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PER QUALSEVOL DUBTE O CONSULTA RESPECTE ELS APUNTS O EL QUE FACI FALTA ESCRIURE A:
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(l'assumpte ha de ser: Apunts ETSETB)

QUALSEVOL ERROR PRESENT S'ATRIBUEIX AL PROFE DE L'ASSIGNATURA QUE ME LA VA IMPARTIR!!

PROBLEMA: Curiositat

$w(x) = x-1 \qquad (0,1)$

$v_{tt} - v_{xx} = 0 \qquad x \in (0,1),\ t>0$

$v(x,0) = w(x) \qquad v(0,t)=0$

$v_t(x,0) = 0 \qquad v(1,t)=0,\ t \geq 0$

$v(x,t) = ?$

$v(x,t) = X(x)\cdot T(t)$

$XT'' - X''T = 0 \Rightarrow \frac{X''}{X} = \frac{T''}{T} = \lambda$

$(D^2-\lambda)X = 0 \qquad (D^2-\lambda)T = 0$

$D^2 = \frac{\partial^2}{\partial \square^2}$

$X(0)\cdot T(t) = 0 \rightarrow X(0)=0$

$X(1)\cdot T(t) = 0 \rightarrow X(1)=0$

$X(x)\cdot T(0) = w(x)$

$X(x)\cdot T'(0) = 0 \rightarrow T'(0)=0$

Considerem $\lambda = 0$

$X'' = 0 \Rightarrow X(x) = ax+b$

$X(0) = b = 0$

$X(1) = a+b = a = 0$

$\rightsquigarrow$ Solució trivial

Considerem $\lambda > 0$

$D = \sqrt{\lambda} \qquad X = A\,\mathrm{ch}\sqrt{\lambda x} + B\,\mathrm{sh}\sqrt{\lambda x}$

$X(0) = A = 0$

$X(1) = B\,\mathrm{sh}\sqrt{\lambda} \Rightarrow B = 0$

Considerem $\lambda < 0$

$D = \sqrt{\lambda} = j\sqrt{-\lambda}$

$X(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)$

$X(0) = A = 0$

$X(1) = B\sin(\sqrt{-\lambda}) = 0$

$\rightarrow B = 0 \rightarrow$ Sol. trivial

$\rightarrow \sin(\sqrt{-\lambda}) = 0 \Rightarrow \sqrt{-\lambda} = k\pi \Rightarrow -\lambda = (k\pi)^2 \Rightarrow \boxed{\lambda = -(k\pi)^2 \quad k \in \mathbb{Z}}$

Funcions pròpies:

$X_k = \sin(k\pi x) \rightarrow v_n = \sin(n\pi x)$

$\Rightarrow (D^2 + (n\pi)^2)T = 0$

$T'(0) = 0$

$T(t) = A\cos(n\pi t) + B\sin(n\pi t)$

$T'(t) = -An\pi\sin(n\pi t) + Bn\pi\cos(n\pi t)$

$T'(0) = Bn\pi \Rightarrow B = 0$

$t_n = \cos(n\pi t) \Rightarrow v_n(x,t) = \sin(n\pi x)\cdot\cos(n\pi t)$

$$v(x,t) = \sum_{n\geq 1} A_n \cdot \sin(n\pi x)\cdot \cos(n\pi t)$$

$$v(x,0) = 1-x = \sum_{n\geq 1} A_n \sin(n\pi x)\cdot 1$$ $\rightsquigarrow$ Per trobar A_n busquem els coef. de Fourier a $L^2(0,1)$

$$A_n = \frac{2}{1}\int_0^1 (x-1)\sin(n\pi x)\,dx = \frac{-2}{n\pi} \Rightarrow \boxed{v(x,t) = \sum_{n\geq 1} \frac{-2}{n\pi}\sin(n\pi x)\cdot\cos(n\pi x)}$$

TEOREMA

Sea $f(x): [-\pi,\pi] \to \mathbb{R}$ continua con $f(-\pi)=f(\pi)$ tq $f' \in L^2(-\pi,\pi)$

$\Downarrow$

La serie de F. converge uniformemente a f en $[-\pi,\pi]$

TEOREMA

En el suposit de les condicions anteriors...

$$f'(x) = \sum_{k\geq 1} k(b_k\cos(kx) - a_k\sin(kx))$$ $\rightsquigarrow$ Serie de Fourier de $f(x)$ derivada dóna $f'(x)$.

TEOREMA.

$f' \in L^2(-\pi,\pi) \Rightarrow f_D \cdot f_I$ EXISTEIXEN

f_D, f_E EXISTEIXEN $\Rightarrow f' \in L^2(-\pi,\pi)$

De la sèrie de Fourier a la Transformada de Fourier

4-4-06

Sigui $x(t) \in L^2(-\frac{T}{2},\frac{T}{2})$, la seva SFC $x(t) = \sum_{-\infty}^{+\infty} c_k e^{jk\frac{2\pi}{T}t}$

a on $c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} x(t)\cdot e^{-jk\frac{2\pi}{T}t}\,dt$

Considerem ara $x(t) \in L^2(-\infty,+\infty)$, és a dir, tq $\int_{-\infty}^{+\infty}|x(t)|^2\,dt < +\infty$

faig que $T\to\infty$ en l'expressió anterior.

Per això defineixo:

$$X_T(f) = \int_{-\frac{T}{2}}^{T/2} x(t)\cdot e^{-j2\pi ft}\,dt$$ Aleshores: $c_k = X_T(f_k)\cdot\Delta f$ $\quad \Delta f = f_{k+1}-f_k = \frac{1}{T}$, $f_k = \frac{k}{T}$

Per tant:

$$x(t) = \sum_{-\infty}^{+\infty} X_T(f_k) \cdot e^{j2\pi f_k t} \cdot \Delta f$$

Així quan $T \to \infty$ i $\Delta f \to 0$

$$\boxed{x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df}$$

TRANSFORMADA INVERSA DE FOURIER

$$\boxed{X(f) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi f t}\, dt}$$

TRANSFORMADA DE FOURIER

PROPIETATS DE LA T.F.

① Simetria:

$$X(-f) = \overline{X(f)}$$

② Linealitat:

$$\mathcal{F}(\alpha x + \beta y) = \alpha \mathcal{F}(x) + \beta \mathcal{F}(y)$$

③ Dualitat:

$$x(t) \longleftrightarrow X(f) \Rightarrow X(t) \longleftrightarrow x(-f)$$

④ Canvi d'escala:

$a > 0$

$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{f}{a}\right)$$

⑤ Desplaçament freqüencial:

$$X(f - f_0) \longleftrightarrow e^{j2\pi f_0 t} \cdot x(t)$$

⑥ Desplaçament temporal:

$$x(t - t_0) \longleftrightarrow e^{-j2\pi f t_0} \cdot X(f)$$

⑦ Modulació:

$$x(t)\cdot\cos(2\pi f_0 t) = \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right]$$

⑧ Derivació:

$$\frac{dx(t)}{dt} \longleftrightarrow j2\pi f\, X(f)$$

⑨ Derivació 1:

$$\frac{dX(f)}{df} \longleftrightarrow -j2\pi t\cdot x(t)$$

Nou espai:

$L(-\infty,+\infty)$ ó $L(\mathbb{R})$

$\hookrightarrow \int_{-\infty}^{+\infty} |x(t)|\,dt < \infty$

CONVOLUCIÓ:

$x, y \in L(-\infty,+\infty)$

$$\boxed{(x * y)(t) = \int_{-\infty}^{+\infty} x(u)\cdot y(t-u)\,du}$$ ⇝ Producte de Convolució

Props

Conmutativa, associativa, distributiva respecte la suma, element neutre.

$$\boxed{(x * y) = \mathcal{F}(x)\cdot\mathcal{F}(y)}$$

Dem: $\mathcal{F}[x*y] = \int_{-\infty}^{+\infty} (x*y)(t)\, e^{-j2\pi f t}\,dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)\cdot y(t-u)\,du\cdot e^{-j2\pi f t}\,dt =$

$= \int_{-\infty}^{+\infty} x(u)\int_{-\infty}^{+\infty} y(t-u)\cdot e^{-j2\pi f t}\,dt\cdot du = \int_{-\infty}^{+\infty} x(u)\cdot e^{-j2\pi f u}\int_{-\infty}^{+\infty} y(t-u)\cdot e^{-j2\pi f(t-u)}\,dt\cdot du =$

$= Y(f)\int_{-\infty}^{+\infty} x(u)\cdot e^{-j2\pi f u}\,du = Y(f)\cdot X(f)$

FORMULA DE PARSEVAL

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df}$$

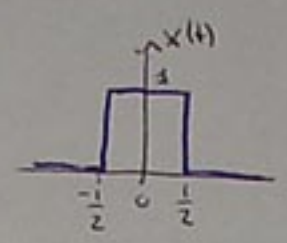
$$\boxed{\int_{-\infty}^{+\infty} x(t)\cdot\overline{y(t)}\, dt = \int_{-\infty}^{+\infty} X(f)\cdot\overline{Y(f)}\, df}$$

DELTA DE DIRAC

$$\boxed{\delta(t) \xleftrightarrow{T.F.} 1}$$

POLS TRIANGULAR

$$x(t) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$



$$X(f) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j2\pi f t}\, dt$$

$$X(f) = \int_{-1/2}^{1/2} e^{-j2\pi f t}\, dt = 2\int_{0}^{1/2} \cos(2\pi f t)\, dt = 2\left[\frac{\sin(2\pi f t)}{2\pi f}\right]_0^{1/2} = \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(f)$$

$$\boxed{\Pi(t) \longleftrightarrow \frac{\sin(\pi f)}{\pi f}}$$

POLS TRIANGULAR 1:

$$x_a(t) = \frac{1}{a}\, x\left(\frac{t}{a}\right) \Rightarrow \quad x_a(t) \longleftrightarrow X(af) = \frac{\sin(\pi a f)}{\pi a f}$$

Ara notem que: $\lim_{a\to 0} x_a(t) = \delta(t)$

Així: $\mathcal{F}[\delta(t)] = \mathcal{F}\left[\lim_{a\to 0} x_a(t)\right] = \lim_{a\to 0} \mathcal{F}[x_a(t)] = \lim_{a\to 0} \frac{\sin(\pi a f)}{\pi a f} = 1$

$\Downarrow$

Tornem a veure que: $\boxed{\delta(t) \longleftrightarrow 1}$

$$x(t) = \int_{-\infty}^{+\infty} X(f)\cdot e^{j2\pi f t}\,df = \int_{-\infty}^{+\infty} \frac{\sin(\pi f)}{\pi f}\cdot e^{j2\pi f t}\,df = 2\int_{0}^{+\infty} \frac{\sin(\pi f)}{\pi f}\, e^{j2\pi f t}\,df =$$

$$= 2\int_{0}^{+\infty} \frac{\sin(\pi f)}{\pi f}\cos(2\pi f t)\,df$$

$$\Downarrow$$

$$x(0) = 2\int_{-\infty}^{+\infty} \frac{\sin(\pi f)}{\pi f}\,df \quad \Rightarrow \quad \int_{-\infty}^{+\infty} \frac{\sin(\pi f)}{\pi f}\,df = \frac{1}{2}$$

FUNCIÓ ESGRAÓ

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases} \qquad \mathrm{sign}(t) = \begin{cases} -1 & t<0 \\ 1 & t>0 \end{cases} \qquad e(t) = \begin{cases} e^{-at} & t>0,\ a>0 \\ 0 & t<0 \end{cases}$$

$$\mathrm{sing}(t) = \lim_{a\to 0}\left[e(t) - e(-t)\right]$$

$$\mathcal{F}\left[e(t)\right] = \int_{-\infty}^{+\infty} e^{-at}\cdot u(t)\cdot e^{-j2\pi f t}\,dt = \int_{0}^{+\infty} e^{-at}\cdot e^{-j2\pi f t}\,dt$$

$$\mathcal{F}\left[e(-t)\right] = \int_{-\infty}^{0} e^{at}\cdot e^{-j2\pi f t}\,dt$$

$$\mathcal{F}\left[\mathrm{sing}(t)\right] = \mathcal{F}\left[\lim_{a\to 0} e(t) - e(-t)\right] = \lim_{a\to 0}\left(\mathcal{F}\left[e(t)\right] - \mathcal{F}\left[e(-t)\right]\right) = \lim_{a\to 0}\int_{0}^{+\infty} e^{-at}\cdot e^{-j2\pi f t}\,dt -$$

$$- \int_{-\infty}^{0} e^{at}\cdot e^{-j2\pi f t}\,dt = \lim_{a\to 0}\left[\left.\frac{e^{-(a+j2\pi f)t}}{-a-j2\pi f}\right|_{0}^{+\infty} - \left.\frac{e^{(a-j2\pi f)t}}{a-j2\pi f}\right|_{-\infty}^{0}\right] =$$

$$= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right) = \frac{1}{j\pi f}$$

$$\boxed{\mathrm{sing}(t) \longleftrightarrow \frac{1}{j\pi f}}$$

Però: $u(t) = \frac{1}{2}\left(1 + \mathrm{sing}(t)\right) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

$$\boxed{u(t) \longleftrightarrow \frac{\delta(f)}{2} + \frac{1}{j2\pi f}}$$

Notes:

$$\boxed{f\cdot\delta(f)=0}$$

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \mathcal{F}\left[(x * u)(t)\right] = X(f)\cdot\left[\frac{\delta(f)}{2} + \frac{1}{j2\pi f}\right] =$$

$$= \frac{1}{2}X(0)\cdot\delta(f) + \frac{X(f)}{j2\pi f}$$

Tmc que: $h(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \Rightarrow$ tmc que $h'(t) = x(t)$

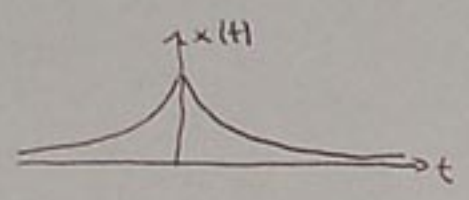
Aleshores:

$$X(f) = j2\pi f\cdot\mathcal{F}[h(t)] = j2\pi f\left[\frac{X(0)\cdot\delta(f)}{2} + \frac{X(f)}{j2\pi f}\right] = X(f) + j\,X(0)\cdot\delta(f)\cdot\pi f\Big|_{f=0} =$$

$$= X(f) \rightsquigarrow \text{Correcte??}$$

PROBLEMA

$x(t) = e^{-3|t|}$



$$X(f) = \int_{-\infty}^{+\infty} e^{-3|t|}\cdot e^{-j2\pi f t}\,dt = \int_{-\infty}^{0} e^{3t}\cdot e^{-j2\pi f t}\,dt + \int_{0}^{+\infty} e^{-3t}\cdot e^{-j2\pi f t}\,dt =$$

$$= \left[\frac{e^{(3-j2\pi f)t}}{3-j2\pi f}\right]_{-\infty}^{0} - \left[\frac{e^{-(3+j2\pi f)t}}{3+j2\pi f}\right]_{0}^{+\infty} = \frac{1}{3-j2\pi f} + \frac{1}{3+j2\pi f} =$$

$$= \boxed{\frac{6}{9+(2\pi f)^2}}$$

$$Y(f) = \frac{3}{5+2\pi f+2\pi^2 f^2} = \frac{6}{10+4\pi f+4\pi^2 f^2} = \frac{6}{9+1+4\pi f+4\pi^2 f^2} = \frac{6}{9+(1+2\pi f)^2} =$$

↳ No ens porta enlloc

$$= \frac{6}{9+4\pi^2\left(f^2+\frac{f}{\pi}+\frac{1}{4\pi^2}\right)} = \frac{6}{9+4\pi^2\left(f+\frac{1}{2\pi}\right)^2} = X\left(f+\frac{1}{2\pi}\right)$$

↳ Es la transformada anterior desplaçada temporalment

$$y(t) = e^{+j2\pi\left(\frac{-1}{2\pi}\right)t} \cdot x(t) = e^{-jt} \cdot x(t) = \boxed{e^{-jt} \cdot e^{-3|t|}}$$

PROBLEMA

$$x''(t) - x(t) = -2e^{-t} \cdot u(t) \longrightarrow \text{Solució de } x(t) = ?$$

$$x(t) = \int_0^{-\infty} e^{-|t-\tau|} \cdot e^{-\tau} \, d\tau \quad ??$$

$$(j2\pi f)^2 \cdot X(f) - X(f) = -2 \frac{1}{1+j2\pi f} \longrightarrow \text{Hem fet la T.F. de l'E.D.O.}$$

$$X(f)\left[-4\pi^2 f^2 - 1\right] = -2 \frac{1}{1+j2\pi f} \Rightarrow X(f) = \frac{2}{(1+j2\pi f)(1+4\pi^2 f^2)}$$

$$X(f) = \mathcal{F}^{-1}\left[\frac{1}{1+j2\pi f}\right] * \mathcal{F}^{-1}\left[\frac{2}{1+4\pi^2 f^2}\right] = \left[e^{-t} \cdot u(t)\right] * \left[e^{-|t|}\right] =$$

$$= \int_{-\infty}^{+\infty} e^{-\tau} \cdot u(\tau) \cdot e^{-|t-\tau|} \, d\tau = \int_0^{+\infty} e^{-\tau} \cdot e^{-|t-\tau|} \, d\tau$$

$\longrightarrow$ Queda demostrat !!

PROBLEMA

$$x(t) = \begin{cases} 0 & t<0 \\ \frac{1}{2} & t=0 \\ e^{-t} & t>0 \end{cases} \qquad ¿\, x(t) = 2\int_0^{+\infty} \frac{\cos(2\pi f t) + 2\pi f \sin(2\pi f t)}{1+(2\pi f)^2} \, df \; ?$$

$$X(f) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi f t} \, dt = \int_0^{+\infty} e^{-t} \cdot e^{-j2\pi f t} \, dt = \int_0^{+\infty} e^{-(1+j2\pi f)t} \, dt = \frac{1}{1+j2\pi f}$$

$$x(t) = \int_{-\infty}^{+\infty} \frac{1}{1+j2\pi f} e^{j2\pi f t} \, df = \int_{-\infty}^{+\infty} \frac{\cos(2\pi f t) + j\sin(2\pi f t)}{1+j2\pi f} \, df = \int_{-\infty}^{+\infty} \frac{(1-j2\pi f)\, e^{j2\pi f t}}{1+(2\pi f)^2} \, df =$$

$\hookrightarrow$ No serveix de res

Eliminarem els senars

$$= \int_{-\infty}^{+\infty} \frac{\cos(2\pi f t) + \cancel{j\sin(2\pi f t)} - \cancel{j2\pi f \cos(2\pi f t)} + 2\pi f \cdot \sin(2\pi f t)}{1+(2\pi f)^2} = \boxed{2\int_0^{+\infty} \frac{\cos(2\pi f t) + 2\pi f \cdot \sin(2\pi f t)}{1+(2\pi f)^2} \, df}$$

$$x(0) = 2\int_0^{+\infty} \frac{1}{1+4\pi^2 f^2}\,df = 2\int_0^{+\infty} \frac{1}{1+(2\pi f)^2}\,df = \frac{1}{\pi}\,\mathrm{arctg}\,(2\pi f)\Big|_0^{+\infty} = \frac{1}{\pi}\cdot\frac{\pi}{2} = \boxed{\frac{1}{2}}$$

PROBLEMA

a) $x(t) = u(t) - u(t-1) = \begin{cases} 1 & 0<t<1 \\ 0 & \text{Altres} \end{cases}$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j2\pi f t}\,dt = \int_0^1 e^{-j2\pi f t}\,dt = \frac{e^{-j2\pi f t}}{-j2\pi f}\Big|_0^1 = \frac{1-e^{-j2\pi f}}{j2\pi f}$$

b) $y(t) = e^{-at}\left[u(t) - u(t-1)\right]$

$$Y(f) = \mathcal{F}\left[e^{-at}\right] * \frac{1-e^{-j2\pi f}}{j2\pi f} \rightarrow \text{Massa complicat}$$

$$Y(f) = \int_0^1 e^{-at}\cdot e^{-j2\pi f t}\,dt = \int_0^1 e^{-(a+j2\pi f)t}\,dt = \frac{e^{-(a+j2\pi f)t}}{a+j2\pi f}\Big|_1^0 = \frac{1-e^{-a-j2\pi f}}{a+j2\pi f}$$

c)

$x(t) = \cos^2(t)$

$$x(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t) = \frac{1}{2}\left[1 + \frac{e^{j2t}+e^{-j2t}}{2}\right] = \frac{1}{4}\left[e^{j2t} + 2 + e^{-j2t}\right]$$

$$X(f) = \frac{1}{4}\left[2\delta(f) + \delta\left(f-\frac{1}{\pi}\right) + \delta\left(f+\frac{1}{\pi}\right)\right]$$

Altre forma...

$$x(t) = \cos(t)\cdot\cos(t) \xrightarrow{T.F.} X(f) = \mathcal{F}\left[\cos(t)\right] * \mathcal{F}\left[\cos(t)\right] = \frac{1}{2}\left[\delta\left(f-\frac{1}{2\pi}\right) + \delta\left(f+\frac{1}{2\pi}\right)\right] * \frac{1}{2}\left[\delta\left(f-\frac{1}{2\pi}\right) + \delta\left(f+\frac{1}{2\pi}\right)\right]$$

$$\mathcal{F}\left[\cos(t)\right] = \frac{1}{2}\left[\delta\left(f-\frac{1}{2\pi}\right) + \delta\left(f+\frac{1}{2\pi}\right)\right]$$

$$= \frac{1}{4}\left[\delta\left(f-\frac{1}{2\pi}\right) * \delta\left(f-\frac{1}{2\pi}\right) + \delta\left(f-\frac{1}{2\pi}\right) * \delta\left(f+\frac{1}{2\pi}\right) + \right.$$

$$\left. + \delta\left(f+\frac{1}{2\pi}\right) * \delta\left(f-\frac{1}{2\pi}\right) + \delta\left(f+\frac{1}{2\pi}\right) * \delta\left(f+\frac{1}{2\pi}\right)\right] =$$

$$= \frac{1}{4}\left[\delta\left(f-\frac{1}{\pi}\right) + \delta(f) + \delta(f) + \delta\left(f+\frac{1}{\pi}\right)\right] =$$

$$= \boxed{\frac{1}{4}\left[2\delta(f) + \delta\left(f-\frac{1}{\pi}\right) + \delta\left(f+\frac{1}{\pi}\right)\right]}$$

d) Amb el $x(t)$ anterior... expressa $X(f) * Y(f)$ en funció de $Y(f)$

$$X(f) * Y(f) = \frac{1}{4}\left[2\delta(f) + \delta(f-\tfrac{1}{\pi}) + \delta(f+\tfrac{1}{\pi})\right] * Y(f) = \boxed{\frac{1}{4}\left[2Y(f) + Y(f-\tfrac{1}{\pi}) + Y(f+\tfrac{1}{\pi})\right]}$$

e) $z(t) = \mathcal{F}^{-1}(X(f) * Y(f))$

$$z(t) = \frac{1}{4}\left[2 + e^{j2t} + e^{-j2t}\right]\cdot y(t) = \left[\frac{1}{2} + \frac{e^{j2t} + e^{-j2t}}{2\cdot 2}\right]\cdot y(t) = \left[\frac{1}{2} + \frac{1}{2}\cos(2t)\right]\cdot y(t) =$$

$$= \cos^2(t)\cdot y(t) = x(t)\cdot y(t)$$

18-04-06

Sèrie discreta de Fourier:

$L^2(0,T)$
$\Downarrow$
Senyals d'energia finita: $\int_0^T |f(x)|^2 dx < +\infty$

Base ortogonal: $\{\phi_n(t)\}_{n\in\mathbb{Z}}$ on $\phi_n = e^{jn\frac{2\pi}{T}t}$

Trio en cada una d'aquestes funcions una mostra de N punts. Així ens mourem a $\mathbb{C}^n$.

Direm: $t_k = k\frac{T}{N}$, $k=0,\dots,N-1$

Obtindrem: $\phi_n(t_k) = \phi_n(k\frac{T}{N}) = e^{jn\frac{2\pi}{T}\cdot k\frac{T}{N}} = e^{j\frac{2\pi}{N}nk} = \omega^{nk}$, $\omega = e^{j\frac{2\pi}{N}}$

Així doncs el vector: $\phi_n \in \mathbb{C}^N$ vindrà donat per:

$$\phi_n(t) = (1, \omega^n, \omega^{2n}, \dots, \omega^{(N-1)n})$$

En aquest espai $\mathbb{C}^n$, és base ortogonal $\{\phi_0, \phi_1, \dots, \phi_{N-1}\}$. Es tracta de N vectors L.I. d'un espai de N dimensions. A més, $(\phi_n, \phi_m) = 0 \ \forall n \neq m$

En efecte:

$$(\phi_n, \phi_m) = \sum_{k=0}^{N-1} \omega^{nk}\,\overline{\omega^{mk}} = \sum_{k=0}^{N-1} \omega^{(n-m)k} = \frac{1-\omega^{(n-m)}\cdot\omega^{(n-m)(N-1)}}{1-\omega^{(n-m)}} = \frac{1-\omega^{(n-m)N}}{1-\omega^{(n-m)}} =$$

$$= \frac{1-e^{j2\pi(n-m)}}{1-e^{j\frac{2\pi(n-m)}{N}}} = \frac{1-1}{1-\omega^{(n-m)}} = 0$$

$$(\phi_n, \phi_n) = \sum_{k=0}^{N-1} \omega^{nk} \cdot \omega^{-nk} = \sum_{k=0}^{N-1} 1 = N \quad \rightsquigarrow \quad \|\phi_n\|^2 = N$$

Conseqüentment, per a cada $x \in \mathbb{C}^n$...

$$\boxed{x = \sum_{k=0}^{N-1} \underbrace{X(n)}_{\text{coef.}} \cdot \underbrace{\phi_n}_{\text{base}}}$$

Aquests $X(n)$ són els de Fourier:

$$\boxed{X(n) = \frac{(x, \phi_n)}{\|\phi_n\|^2} = \frac{1}{N} \sum_{k=0}^{N-1} x(k) \cdot e^{-j \frac{2\pi}{N} nk}}$$

$\rightsquigarrow$ La component k-èssima és: $\quad x(k) = \sum_{k=0}^{N-1} X(n) \cdot e^{j \frac{2\pi}{N} nk}$

Exemple

$$x(k) = \begin{cases} 1 & k = 0, 1, 2, 3 \\ -1 & k = 4, 5, 6, 7 \end{cases}$$

Sèrie discreta de Fourier?

$N = 8$

$$X(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) \cdot e^{-j \frac{2\pi}{N} nk} = \frac{1}{8} \sum_{k=0}^{7} x(k) e^{-j \frac{2\pi}{8} nk} = \frac{1}{8} \sum_{k=0}^{7} x(k) \cdot e^{-j \frac{\pi}{4} nk} =$$

$$= \frac{1}{8} \sum_{k=0}^{7} x(k) \omega^{-nk} = \frac{1}{8} \left[(1 + \omega^{-n} + \omega^{-2n} + \omega^{-3n}) - (\omega^{-4n} + \omega^{-5n} + \omega^{-6n} + \omega^{-7n}) \right] =$$

$$= \frac{1}{8} \left[(1 + \omega^{-n} + \omega^{-2n} + \omega^{-3n}) - (-1)^n (1 + \omega^{-n} + \omega^{-2n} + \omega^{-3n}) \right] =$$

$\omega^{-4} = -1$, $\omega^{-4n} = (-1)^n$ | $\omega^{-5n} = \omega^{-4n} \cdot \omega^{-n} = (-1)^n \cdot \omega^{-n}$ | $\omega^{-6n} = \omega^{-4n} \cdot \omega^{-2n} = (-1)^n \cdot \omega^{-2n}$ | $\omega^{-7n} = \omega^{-4n} \cdot \omega^{-3n} = (-1)^n \cdot \omega^{-3n}$

$$= \frac{1}{8} \left[[1 - (-1)^n] \underbrace{(1 + \omega^{-n} + \omega^{-2n} + \omega^{-3n})}_{\text{Suma geomètrica}} \right] = \begin{cases} 0 & n = 2k \\ \frac{1}{4} \frac{1 - \omega^{-4n}}{1 - \omega^{-n}} & n = 2k+1 \end{cases} \quad \forall k \in \mathbb{N}$$

$$= \begin{cases} 0 & n = 2k \quad \forall k \in \mathbb{N} \\ \frac{1}{4} \frac{1 - \omega^{-4(2k+1)}}{1 - \omega^{-(2k+1)}} = \frac{1}{2} \frac{1}{1 - \omega^{-(2k+1)}} & \forall k \in \mathbb{N} \end{cases}$$

PROBLEMA $\mathcal{F}\left(e^{j\pi t}\cdot x(t)\cdot x'(t)\right) = \mathcal{F}\left(e^{j\pi t}\right) * \mathcal{F}\left(x(t)\cdot x'(t)\right) =$

$$= \mathcal{F}\left(e^{j\pi t}\right) * \mathcal{F}\left(\frac{1}{2}\frac{d}{dt}x^2(t)\right) = \delta\left(f-\frac{1}{2}\right) * \frac{1}{2}\left[j2\pi f\,\mathcal{F}\left(\underset{x(t)\cdot x(t)}{x^2(t)}\right)\right] =$$

$$= j\pi\,\delta\left(f-\frac{1}{2}\right) * f\left[X(f) * X(f)\right] = j\pi\cdot\left(f-\frac{1}{2}\right)\left[X*X\right]\left(f-\frac{1}{2}\right)$$

Seguint l'altre camí: $\rightarrow \mathcal{F}(x\cdot x') = \mathcal{F}(x) * j2\pi f\mathcal{F}(x)$

$$= j2\pi\,\delta\left(f-\frac{1}{2}\right) * \left[X(f) * f X(f)\right] = j2\pi\left[X(s) * sX(s)\right]\left(f-\frac{1}{2}\right) =$$

$$= 2\pi j\int_{-\infty}^{+\infty}\left(f-\frac{1}{2}-s\right)X\left(f-\frac{1}{2}-s\right)X(s)\,ds = j2\pi\left[\int_{-\infty}^{+\infty}\left(f-\frac{1}{2}\right)X\left(f-\frac{1}{2}-s\right)X(s)\,ds - \int_{-\infty}^{+\infty} s\,X\left(f-\frac{1}{2}-s\right)X(s)\,ds\right] =$$

$$= 2\pi j\left[\left(f-\frac{1}{2}\right)\left[X*X\right]\left(f-\frac{1}{2}\right) - \left[X*sX\right]\left(f-\frac{1}{2}\right)\right] = \left[X*sX\right]\left(f-\frac{1}{2}\right) = \frac{1}{2}\left(f-\frac{1}{2}\right)\left[X*X\right]\left(f-\frac{1}{2}\right)$$

Transformada discreta de Fourier:

Sigui una funció $x(t)=0$ si $t\notin[0,T]$. Aproximem la seva T.F. dividint $[0,T]$ en N parts i $t_k = k\frac{T}{N}$.

$$\boxed{X(n) = \sum_{k=0}^{N-1} x(k)\cdot\omega^{-nk} \qquad \text{on } \omega = e^{j\frac{2\pi}{N}}}$$

$$\boxed{x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X(n)\,\omega^{nk}}$$

→ Alerta que aquí el subindex és "n" i no "k"

PROBLEMA T.D.F $X(n) = \sin\frac{2\pi}{N}n + \cos\frac{2\pi}{N}n$, $N \geq 2001$

$$x(k) = \frac{1}{N}\sum_{k=0}^{N-1} X(n)\,\omega^{nk}$$

$$X(n) = \frac{e^{j\frac{2\pi}{N}n} - e^{-j\frac{2\pi}{N}n}}{2j} + \frac{e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}}{2} = \frac{\omega^n - \omega^{-n}}{2j} + \frac{\omega^n + \omega^{-n}}{2}$$

$$X(n) = \frac{1}{2}\left[\omega^n + \omega^{-n} + j\left(\omega^{-n} - \omega^n\right)\right] = \sum_{k=0}^{N-1} x(k)\cdot\omega^{-nk}$$

$$x(k) = \frac{1}{2N}\sum_{n=0}^{N-1}\left[\omega^{n} + \omega^{-n} + j\left(\omega^{-n} - \omega^{n}\right)\right]\cdot\omega^{nk}$$

$$x(k) = \frac{1}{2N}\sum_{n=0}^{N-1}\left[\omega^{n(1+k)} + \omega^{n(k-1)} + j\left(\omega^{n(k-1)} - \omega^{n(k+1)}\right)\right] =$$

$$x(k) = \frac{1}{2N}\sum_{n=0}^{N-1}\left[\omega^{n(1+k)}\left[1-j\right] + \omega^{n(k-1)}\left[1+j\right]\right]$$

$$x(k) = \frac{1-j}{2N}\sum_{n=0}^{N-1}\omega^{n(1+k)} + \frac{1+j}{2N}\sum_{k=0}^{N-1}\omega^{n(k-1)}$$

$$\cancel{x(k) = \frac{(1-j)\,\omega^{n}}{2N}\sum_{k=0}^{N-1}(\omega^{n})^{k} + \frac{(1+j)\,\omega^{-n}}{2N}\sum_{k=0}^{N-1}(\omega^{n})^{k}}$$

$$x(k) = \frac{(1-j)}{2N}\,\frac{1-\omega^{N(1+k)}}{1-\omega^{(1+k)}} + \frac{1+j}{2N}\,\frac{1-\omega^{N(k-1)}}{1-\omega^{(k-1)}} =$$

$$\begin{cases} 0 & \text{si } \omega^{k-1}\neq 1,\ \omega^{k+1}\neq 1 \\ \text{Quan } \omega^{k-1}=1? \ \text{Quan } \omega^{k+1}=1? & \\ k-1=0 \Rightarrow \underline{k=1} \qquad k+1=0 \Rightarrow \cancel{k=-1} & \\ k-1=N \Rightarrow \cancel{k=N+1} \qquad k+1=N \Rightarrow \underline{k=N-1} & \\ 0 \quad \text{Altrament} & \\ \frac{1+j}{2N}\sum_{k=0}^{N-1}1 = \frac{1}{2}+\frac{j}{2} & k=1 \\ \frac{1-j}{2N}\sum_{k=0}^{N-1}1 = \frac{1}{2}-\frac{j}{2} & k=N-1 \end{cases}$$

19-9-06

PROBLEMA

$(1,1,1,1)$, $(1,j,-1,-j)$, $(1,-1,1,-1)$, $(1,-j,-1,j)$

$$X(n) = \sum_{k=0}^{N-1} x(k)\,\omega^{-nk} \qquad x(k) = \frac{1}{N}\sum_{n=0}^{n=N-1} X(n)\,\omega^{nk}$$

$\phi_n = (1, \omega^{n}, \omega^{2n}, \omega^{3n})$, $n=0,1,2,3$ $\qquad \omega = e^{j\frac{2\pi}{4}} = e^{j\frac{\pi}{2}} = j$

$t_k = k\frac{T}{N}$

$$\underset{\substack{\| \\ 0}}{(\phi_n, \phi_m)} \overset{?}{} = \sum_{k=0}^{N-1}\omega^{nk}\cdot\overline{\omega^{mk}} = \sum_{k=0}^{N-1}\omega^{(n-m)k} = \sum_{k=0}^{N-1} j^{(n-m)k} = \begin{cases} 4 & n=m \\ \dfrac{j^{(n-m)4}-1}{j^{(n-m)}-1} = 0 & n\neq m \end{cases}$$

$\forall m\neq n$

Determineu les coordenades d'aquesta base, del vector: $(1,3,3,1)$.

$x = \sum_{0=n}^{3} c_n\cdot\phi_n$ $\quad$ ϕ_n és base ortogonal, per tant els c_n són els coeficients de Fourier del vector en aquesta base.

$$c_n = \frac{(x, \phi_n)}{\|\phi_n\|^2} = \frac{1}{4}\sum_{k=0}^{3} x(k)\cdot\overline{\phi_n} = \frac{1}{4}\sum_{k=0}^{3} x(k)\cdot\omega^{-nk} =$$

"ene"

$$\omega^{-1} = \frac{1}{j} = -j$$

$$\omega^{-nk} = \left((-j)^n\right)^k$$

$$= \frac{1}{4}\sum_{k=0}^{3} x(k)\left((-j)^n\right)^k$$

$$c_0 = \frac{1+3+3+1}{4} = 2$$

$$c_1 = \frac{1+3\cdot(-j)-3+1(j)}{4} = \frac{-1}{2} - \frac{1}{2}j$$

$$c_2 = \frac{1+3(-1)+3-1}{4} = 0$$

$$c_3 = \frac{1+3(j)+3(-1)+1(-j)}{4}$$

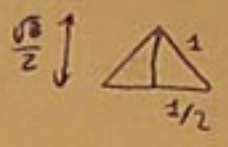
$$c_3 = \frac{-1}{2} + \frac{1}{2}j$$

$$X(n) = 4c_n = (8, -2-2j, 0, -2+2j)$$

PROBLEMA

$$x(k) = -\sin\left(\frac{\pi k}{3}\right), \quad k = 0, \dots, 5$$

$$\omega = e^{j\frac{2\pi}{6}} = e^{j\frac{\pi}{3}} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$$



$$\omega^{6k} = 1, \quad \omega^{3k} = \begin{cases} -1 & k = 2n-1 \\ 1 & k = 2n \end{cases} \quad n \in \mathbb{N}$$

$$\sin\left(\frac{\pi k}{3}\right) = \frac{-e^{j\frac{\pi k}{3}} + e^{-j\frac{\pi k}{3}}}{2j} = \frac{-\omega^k + \omega^{-k}}{2j} = j\,\frac{\omega^k - \omega^{-k}}{2}$$

$$X(n) = \frac{j}{2}\left[\sum_{k=0}^{5}\omega^{(1-n)k} - \sum_{k=0}^{5}\omega^{-(n+1)k}\right] = \frac{j}{2}\left[\frac{\overset{=1}{\omega^{(1-n)6}} - 1}{\omega^{1-n} - 1} - \frac{\overset{=1}{\omega^{-(n+1)6}} - 1}{\omega^{-(n+1)} - 1}\right] =$$

$$\begin{cases} 0 & \text{sempre que } \omega^{-(1+n)} \neq 1, \ \omega^{1-n} \neq 1 \\ & \Downarrow \\ & 1-n = 0 \Rightarrow n = 1 \\ & 1+n = 0 \Rightarrow \cancel{n = -1} \\ & 1-n = N = 6 \Rightarrow \cancel{n = -5} \\ & 1+n = N = 6 \Rightarrow n = 5 \end{cases}$$

$\omega^{1-n} = 1$, $\omega^{-1-n} = 1$

$$X(n) = \begin{cases} 0 & n = 0, 2, 3, 4 \\ \frac{j}{2}\left[\sum_{k=0}^{5} 1 - \cancel{\sum_{k=0}^{5}\omega^{-2k}}^{\,0}\right] = j3 & n = 1 \\ \frac{j}{2}\left[\cancel{\sum_{k=0}^{5}\omega^{-4k}}^{\,0} - \sum_{k=0}^{5}\omega^{-6k}\right] = \frac{j}{2}\left[-\sum_{k=0}^{5} 1\right] = -j3 & n = 5 \end{cases}$$

PROBLEMA

$$x(t) = e^{jt/2} \qquad -\pi < t < \pi \qquad L^2(-\pi, \pi)$$

$$x(t) = \sum_{-\infty}^{+\infty} c_k e^{jkt}$$

$$c_k = \frac{(x(t), e^{jkt})}{\|e^{jkt}\|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jt/2}\cdot e^{-jkt}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jt(-k+\frac{1}{2})}\,dt = \frac{1}{\pi}\int_{0}^{\pi}\cos\left(\left(\tfrac{1}{2}-k\right)t\right)dt =$$

$$= \frac{1}{\pi}\,\frac{\sin\left(\left(\frac{1}{2}-k\right)t\right)}{\frac{1}{2}-k}\Bigg|_0^{\pi} = \frac{1}{\pi}\,\frac{\sin\left(\frac{\pi}{2}-k\pi\right)}{\frac{1}{2}-k} = \frac{1}{\pi\left(\frac{1}{2}-k\right)}(-1)^k$$

$$\sin\left(\frac{\pi}{2}-k\pi\right) = \sin\frac{\pi}{2}\cdot\cos(k\pi) - \overset{0}{\sin(k\pi)}\cdot\cos\frac{\pi}{2} = \cos(k\pi) = (-1)^k$$

$$\boxed{e^{jt/2} = \frac{2}{\pi}\sum_{-\infty}^{+\infty} \frac{(-1)^k}{1-2k} e^{jkt}}$$

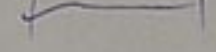
$$\boxed{\begin{aligned} c_k &= \tfrac{1}{2}(a_k - j b_k) \\ \overline{c_k} &= c_{-k} = \tfrac{1}{2}(a_k + j b_k) \end{aligned}}$$

$$c_k = \frac{2(-1)^k}{1-2k} \qquad \overline{c_k} = \frac{2(-1)^k}{1+2k} \rightsquigarrow \frac{1}{(-1)^k} = (-1)^k$$

$$a_k = c_k + \overline{c_k} = 2(-1)^k\left[\frac{1}{1-2k} + \frac{1}{1+2k}\right] = 2(-1)^k\left[\frac{2}{1-4k^2}\right] = \frac{4(-1)^k}{1-4k^2}$$

$$b_k = \frac{\overline{c_k} - c_k}{j} = \frac{2(-1)^k}{j}\left[\frac{1}{1+2k} - \frac{1}{1-2k}\right] = -\frac{2(-1)^k}{j}\left[\frac{4k}{1-k^2 4}\right] = \frac{2(-1)^k}{j(k^2 4 - 1)}$$

$$\frac{a_0}{2} = \frac{2}{\pi}$$



$$y(t) = \begin{cases} e^{jt/2} & |t| < \pi \\ 0 & |t| > \pi \end{cases} \qquad \frac{4\cos(2\pi^2 f)}{1-4\pi f}$$

$$Y(f) = \int_{-\infty}^{+\infty} y(t)\cdot e^{-j2\pi f t}\,dt = \int_{-\pi}^{\pi} e^{jt(-2\pi f + \frac{1}{2})}\,dt = 2\int_0^{\pi} \cos\left(t\left(\tfrac{1}{2} - 2\pi f\right)\right)dt =$$

$$= 2\left[\frac{\sin\left(\frac{t}{2} - 2\pi f t\right)}{1/2 - 2\pi f}\right]_0^{\pi} = \frac{1}{\frac{1}{4} - \pi f}\sin\left(\frac{\pi}{2} - 2\pi^2 f\right) = \boxed{\frac{4\cos(2\pi^2 f)}{1-4\pi f}}$$

$$\Rightarrow \sin\left(\frac{\pi}{2} - 2\pi^2 f\right) = \sin\left(\frac{\pi}{2}\right)\cdot\cos(2\pi^2 f) - \sin(2\pi^2 f)\cdot\cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0} = \cos(2\pi^2 f)$$

$$\int_{-\infty}^{+\infty}\left|\frac{\cos 2\pi^2 f}{1-4\pi f}\right|^2 df = \frac{1}{16}\int_{-\infty}^{+\infty}|y(t)|^2\,dt = \frac{1}{16}\int_{-\pi}^{\pi} e^{jt/2}\cdot e^{-jt/2}\,dt = \boxed{\frac{\pi}{8}}$$

VARIABLE COMPLEXE

9-5-06

(26)

Funció harmònica conjugada:

$g(x,y) = y^3 - 3x^2y$

$h(x,y) = x^3 - 3xy^2$ $\Rightarrow$ Busquem el harmònic conjugat de $g(x,y)$

Per. Cauchy-Rieman:

$\boxed{u_x = -6xy} = v_y \Rightarrow v = \int -6xy\, dy = -3xy^2 + p(x)$

$\boxed{u_y = 3y^2 - 3x^2} = -v_x$

$v = -\int (3y^2 - 3x^2)\, dx = -3y^2x + x^3$

$v_x = -3y^2 + 3x^2$ $\qquad$ $v_x = -3y^2 + p'(x)$

$p'(x) = 3x^2 \Rightarrow p(x) = x^3 + K$

$$\boxed{f(z) = y^3 - 3x^2y + j\underbrace{(x^3 - 3xy^2)}_{v}}$$

Fem el mateix amb l'altra...

> * Dues funcions senceres coincideixen sobre un obert qualsevol, llavors ho fan sobre el pla complex.

$f(z) = jz^3 = j(x+jy)^3 = j\left[x^3 + 3jx^2y - 3xy^2 - jy^3\right] = y^3 - 3x^2y + j(x^3 - 3xy^2)$

PROBLEMA $f(z) = u + jv$ analítica, quina es pot afirmar que tb ho és:

a) $u^2 + v^2$ b) $u \cdot v$ c) u/v d) u^2

$\nabla^2 u = u_{xx} + u_{yy} = 0$

$\nabla^2 v = v_{xx} + v_{yy} = 0$

$0 \overset{?}{=} \nabla u \cdot v = \dfrac{\partial^2(uv)}{\partial x \partial x} + \dfrac{\partial^2(uv)}{\partial y \partial y}$

Eduard Vinazquez

6'5

$\dfrac{\partial(uv)}{\partial x} = u \cdot v_x + u_x \cdot v$

$\dfrac{\partial(uv)}{\partial y} = u \cdot v_y + u_y \cdot v$

$\dfrac{\partial^2(uv)}{\partial^2 x} = u_x \cdot v_x + u \cdot v_{xx} + u_{xx} \cdot v + v_x \cdot u_x =$

$= u \cdot v_{xx} + v \cdot u_{xx} + 2u_x v_x$

$\dfrac{\partial^2(uv)}{\partial^2 y} = u \cdot v_{yy} + v \cdot u_{yy} + 2u_y v_y$

Condició inicial de $\nabla^2 u = \nabla^2 v = 0$

$$\nabla^2(uv) = 2(u_x v_x + u_y v_y) + u(\cancel{v_{xx}} + v_{yy}) + v(\cancel{u_{xx}} + u_{yy})$$

A més a més per Cauchy-Reaman: $u_x = v_y$, $u_y = -v_x$

$$\nabla^2(uv) = 2(u_x v_x + u_y v_y) = 2(v_y \cdot v_x + (-v_x)\cdot v_y) = 0 \rightarrow \text{És analítica}$$

$\boxed{\text{PROBLEMA}}$ $\quad tg\, z = j$

$$j = \frac{\sin z}{\cos z} = \frac{\dfrac{e^{jz} - e^{-jz}}{2j}}{\dfrac{e^{jz} + e^{-jz}}{2}} = \frac{(e^{+jz} - e^{-jz})}{j(e^{jz} + e^{-jz})} = \frac{1}{j}\,\frac{e^{j2z} - 1}{e^{j2z} + 1}$$

$$\Downarrow$$

$$e^{j2z} + 1 = 1 - e^{j2z} \Rightarrow 2e^{j2z} = 0 \rightarrow \text{No pot ser}$$

La funció logarítmica:

Donat z, diem que w és logaritme de z sii $\underline{\underline{e^w = z}}$

$w = u + jv$ $\qquad z = e^{u+jv}$

$$z = r(\cos\theta + j\sin\theta) = r\cdot e^{j\theta} \quad , \arg z = \theta \text{ on } |\theta| < \pi$$

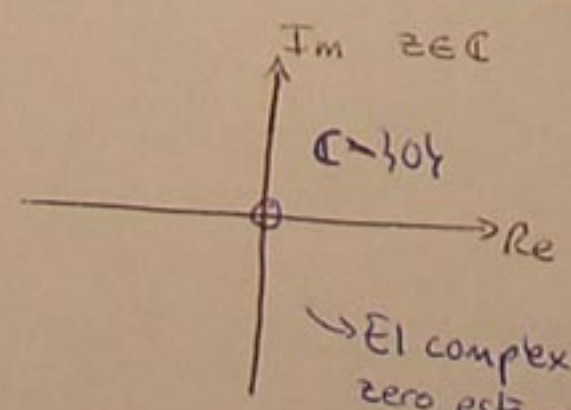
$$\Downarrow$$

$$e^u = r \quad , \quad e^{jv} = e^{j\theta} \Rightarrow u = \ln r \quad , \quad v = \theta + 2k\pi$$

→ Compte amb el detall!!

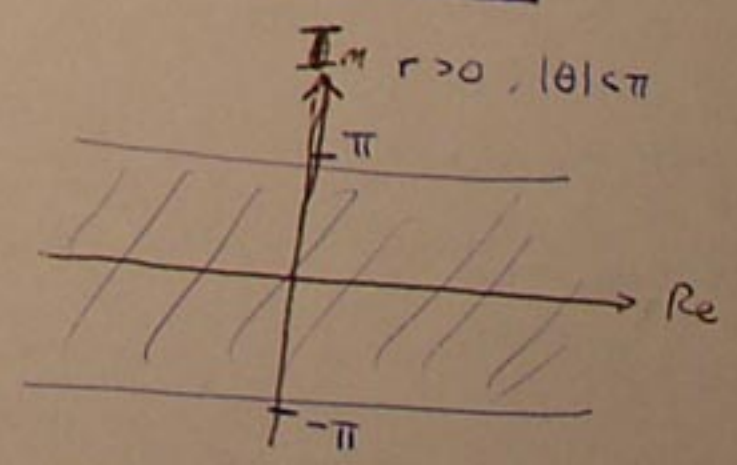
$$\boxed{\ln z = \ln r + j(\theta + 2k\pi)} \rightarrow \boxed{k \in \mathbb{Z}}$$

$\rightarrow$ $\boxed{\text{El valor principal } k=0}$



→ El complexe zero està malèit, no es pot aplicar el $\ln z$

$\xrightarrow{f}$



Grafica de $f(z)$

Cauchy-Rieman en forma polar:

$$\boxed{\begin{aligned} u_r &= \frac{1}{r}\cdot v_\theta \\ u_\theta &= -r\cdot v_r \end{aligned}}$$

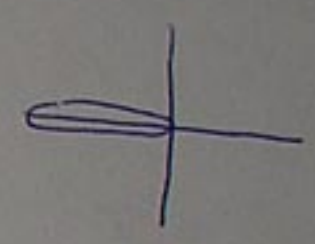
Notes a la llibreteta !!

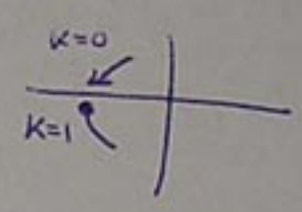
És analítica el $\ln z$?

$$\ln z = \ln r + j(\theta + 2k\pi) \; ; \; k \in \mathbb{Z}$$

$$\left.\begin{aligned} u(r,\theta) &= \ln r \\ v(r,\theta) &= \theta + 2k\pi \end{aligned}\right\} \quad \begin{aligned} u_r = \frac{1}{r} v_\theta &\Rightarrow \frac{1}{r} = \frac{1}{r} + \emptyset \; \checkmark \\ u_\theta = -r\cdot v_r &\Rightarrow \emptyset = -r\cdot\emptyset = \emptyset \; \checkmark \end{aligned}$$

No és analítica per a $\theta = -\pi$, ja que hi ha un petit salt en el lloc marcat. Comencem a $\theta = -\pi$ fins a $\theta = 0'9999\,(+\pi)$ i quan fem $k=1$, sortim a $\theta = \pi$.





PROBLEMA $\sin z = 3$

10-5-06

$$\frac{e^{jz} - e^{-jz}}{2j} = 3 \Rightarrow e^{jz} - e^{-jz} - 6j = 0 \Rightarrow e^{2jz} - 1 - 6j e^{jz} = 0$$

$$(e^{jz})^2 - 6j(e^{jz}) - 1 = 0 \rightarrow \text{eq. segon grau}$$

$$e^{jz} = \frac{6j \pm \sqrt{-36 - 4(-1)}}{2} \quad \begin{cases} e^{jz} = \dfrac{6j + j4\sqrt{2}}{2} = 3j + j2\sqrt{2} \\ e^{jz} = 3j - j2\sqrt{2} \end{cases}$$

$$e^{jz} = j(3 \pm 2\sqrt{2}) \Rightarrow jz = \ln j(3 \pm 2\sqrt{2}) = \ln(3 \pm 2\sqrt{2}) + j\left(\frac{\pi}{2} + 2k\pi\right)$$

$$\boxed{z = \left(\frac{\pi}{2} + 2k\pi\right) - j\ln(3 \pm 2\sqrt{2})}$$

* Les eq. de Cauchy-Rieman són condicions necessaries però no suficients per la derivabilitat. Si a més, les derivades parcials: v_x, u_x, v_y, u_y són continues en un punt, asegurem que existeix la derivada.

FUNCIÓ DERIVADA:

$$f'(x,y) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}$$

Diferencial en polar:

$$f(z) = u(x,y) + j\,v(x,y)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$z = r \cdot e^{j\theta} \rightsquigarrow$ en polars

$$f'(z) = \lim_{\Delta r \to 0} \frac{\big(u(r+\Delta r,\theta) + j v(r+\Delta r,\theta)\big) - \big(u(r,\theta) + j v(r,\theta)\big)}{e^{j\theta}\,\Delta r} =$$

$$= \frac{1}{e^{j\theta}}\left[\lim_{\Delta r \to 0} \frac{u(r+\Delta r,\theta) - u(r,\theta)}{\Delta r} + j \lim_{\Delta r \to 0} \frac{v(r+\Delta r,\theta) - v(r,\theta)}{\Delta r}\right] =$$

$$= \frac{1}{e^{j\theta}}\left[\frac{\partial u}{\partial r} + j\frac{\partial v}{\partial r}\right]$$

$$\boxed{f'(z) = \frac{1}{e^{j\theta}}\frac{\partial u}{\partial r} + j\frac{1}{e^{j\theta}}\frac{\partial v}{\partial r}}$$

PROBLEMA $\cos x = \sqrt{26}$

$$\frac{e^{jx} + e^{-jx}}{2} = \sqrt{26} \Rightarrow e^{jx} + \frac{1}{e^{jx}} - 2\sqrt{26} = 0 \Rightarrow e^{j2x} + 1 - 2\sqrt{26}\,e^{jx} = 0$$

$$e^{jx} = \frac{2\sqrt{26} \pm \sqrt{4 \cdot 26 - 4}}{2}$$

$$\nearrow e^{jx} = \frac{2\sqrt{26} + 2 \cdot 5}{2} = \sqrt{26} + 5$$

$$\searrow e^{jx} = \sqrt{26} - 5$$

$$jx = \ln(\sqrt{26} \pm 5) \Rightarrow x = -j\ln(\sqrt{26} \pm 5) = -j\left[\ln(\sqrt{26} \pm 5) + j(2k\pi)\right] =$$

$$\boxed{x = 2k\pi - j\ln(\sqrt{26} \pm 5), \quad k \in \mathbb{Z}}$$

FUNCIÓ POTENCIAL:

$$f(z) = z^c \quad \text{on } c \in \mathbb{C} \quad \Rightarrow \quad \boxed{z^c = e^{c \cdot \ln z}}$$

$|e^{jz}|$ no té perquè ser 1
↗ modul

PROBLEMA Expressa en forma potencial 1^{-j}

$$1^{-j} = e^{-j\ln 1}$$

→ però $\ln 1$ es considera un complex → $1^{-j} = e^{-j[\ln 1 + j2k\pi]}$

$$1^{-j} = e^{-j\overset{0}{\ln 1} + 2k\pi} = e^{2k\pi} \Rightarrow \boxed{1^{-j} = e^{2k\pi}}$$

$$(-2)^{\sqrt{2}} = e^{\sqrt{2}\ln(-2)} = e^{\sqrt{2}[\ln(2) + j(\pi + 2k\pi)]} = e^{\sqrt{2}\ln(2)} \cdot e^{\sqrt{2}j\pi(2k+1)} =$$

$$= 2^{\sqrt{2}} \cdot e^{j\sqrt{2}\pi j(2k+1)}$$

$$(j)^{2/3} = e^{\frac{2}{3}\ln j} = e^{\frac{2}{3}[\ln 1 + j(\frac{\pi}{2} + 2k\pi)]} = e^{j\frac{2}{3}(\frac{\pi}{2} + 2k\pi)} = e^{j\frac{2\pi}{3}(\frac{1}{2} + 2k)}$$

Fem l'arrel cúbica i dona:

$k=0 \rightarrow e^{j\pi/3}$

$k=1 \rightarrow e^{j5\pi/3} = e^{-j\pi/3}$

$k=2 \rightarrow e^{j3\pi} = e^{j\pi}$

Fixem-nos que si fessim $k=3$ tornaria a repetir-se valors...

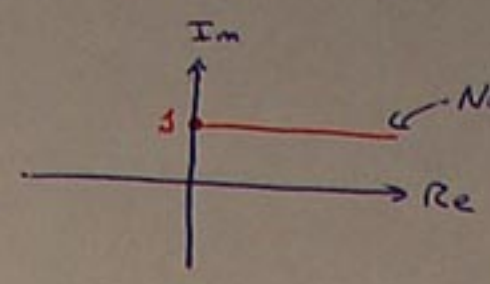
$k=3 \rightarrow e^{j\frac{3\pi}{3}} = e^{j\pi/3}$

PROBLEMA On és analítica: $f(z) = \ln(z-j)$; $0 \leq \theta < 2\pi$

$z = x + jy$

$\Downarrow$

$z - j = x + j(y-1)$



No analítica $\begin{cases} x \geq 0 \\ y = 1 \end{cases}$

En aquest cas és en el semipla dret ($x \geq 0$) perquè θ varia entre 0 i 2π i no entre $-\pi$ i π.

PROBLEMA $z^{j\pi}$ complex:

a) $e^{j\pi} \cdot e^{-2k\pi^2}$

b) un dels valors és $e^{-4\pi^2}(\cos(\pi \ln 2) + j\sin(\pi \cdot \ln 2))$

c) " " " " " " $-$ "

$\hookrightarrow$ signe negatiu

d) $e^{-2k\pi^2} \cdot e^{jk\pi}$; $k \in \mathbb{Z}$

$$z^{j\pi} = e^{j\pi \ln z} = e^{j\pi[\ln z + j2k\pi]} = e^{-2k\pi^2} \cdot e^{j\pi \ln z}$$

$\rightsquigarrow$ Podem dir que no és ni la primera ni l'última

$$z^{j\pi} = e^{-2\pi^2 k}(\cos(\pi \ln z) + j\sin(\pi \ln z)) \rightsquigarrow \boxed{\text{La correcta és la b}}$$

PROBLEMA $\ln(je)$

$$\ln(je) = \ln e + j\left(\frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$$

$$\boxed{\ln(je) = 1 + j\left(\frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z}}$$

PROBLEMA $w = \operatorname{arctg}(z)$

$$z = \operatorname{tg} w = \frac{\sin w}{\cos w} = \frac{e^{jw} - e^{-jw}}{j(e^{jw} + e^{-jw})} = \frac{e^{j2w} - 1}{j(e^{j2w} + 1)}$$

$$jz(e^{j2w} + 1) = e^{j2w} - 1 \Rightarrow 1 + jz = e^{j2w}(1 - jz) \Rightarrow e^{j2w} = \frac{1 + jz}{1 - zj}$$

$$j2w \cdot \ln e = \ln(1 + jz) - \ln(1 - jz) \Rightarrow j2w[\ln e + j2k\pi] = j2w - 4k\pi w = w(j2 - k4\pi)$$

$$w = \frac{1}{2j} \ln\left(\frac{1+jz}{1-jz}\right) \cdot \frac{1}{1 + j2k\pi}$$ ó com que $e^{j2w} = e^{j2w + 2k\pi j} \Rightarrow w = \frac{1}{2j} \ln\left(\frac{1+jz}{1-jz}\right) - k\pi$

16-05-06

PROBLEMA Si $f(z)=\sqrt{z}$ és analítica en $\mathbb{C}\setminus\{x+jy : x\leq 0, y=0\}$

aleshores $g(z)=\sqrt{e^z+1}$ on és analítica?

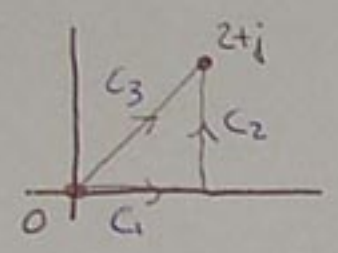
INTEGRACIÓ

$$\left[\int_C f(z)\,dz = \int_C (u+jv)(dx+j\,dy) = \int_C u\,dx - v\,dy + j\int_C v\,dx + u\,dy\right]$$

$\vec{F_R} = (u,-v) \rightarrow$ part real

$\vec{F_I} = (v,u) \rightarrow$ part imaginaria $\Rightarrow \int_C \vec{F_R}\,d\vec{r} + \int_C \vec{F_I}\,d\vec{r}$

Exemple



$$\int_C z^2\,dz$$

$z = x+jy$

$z^2 = (x+jy)^2 = x^2-y^2+j2xy$

$\triangleright\ u = x^2-y^2$

$v = 2xy$

$\longrightarrow$ Una forma és fer l'integral pent servir aquesta descomposició.

$$\int_C z^2\,dz = \int_C u\,dx - v\,dy + j\int_C v\,dx + u\,dy = \int_C (x^2-y^2)\,dx - (2xy)\,dy + j\int_C 2xy\,dx + (x^2-y^2)\,dy$$

$\rightarrow$ Mirem les corbes:

$C_1: y=0,\ x\in[0,2]$ $\qquad C_2: x=2,\ y\in[0,j]$ $\qquad C_3: (2+j)t,\ t\in[0,1]$

$$\int_0^2 x^2\,dx + \int_0^1 -4y\,dy + j\int_0^1 (4-y^2)\,dx = \frac{8}{3} + (-2) + j\left(4-\frac{1}{3}\right) = \frac{2}{3} + j\frac{11}{3}$$

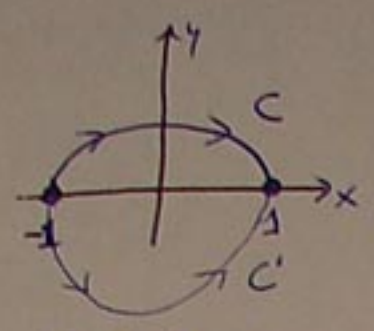
Altre forma:

$$\int_{C_3} z^2\,dz = \int_{C_3} (2+j)^2 t^2 (2+j)\,dt = (2+j)^3 \left.\frac{t^3}{3}\right]_0^1 = \frac{8}{3} + 4j + 2(-1) + \frac{(-j)}{3} = \frac{2}{3} + j\frac{11}{3}$$

Veiem que la integral no depén del camí seguit!!

Exemple: Parametritzem la corba: $x^2+y^2=1$ $\quad x(t)=\cos(t) \quad y(t)=\sin(t)$

$t\in[0,\pi] \quad t: \pi \xrightarrow[C]{} 0$



$z=x+jy$

$\bar{z}=x-jy$

$u=x$

$v=-y$

$$\int_C \bar{z}\,dz = \int_C (u,-v)(dx,dy) + j\int_C (v,u)(dx,dy) =$$

$$= \int_C (x,+y)(dx,dy) + j\int (-y,x)(dx,dy) =$$

$$= \int_0^{\pi} -\sin t\cdot\cos t\,dt + \sin t\cdot\cos t\,dt + j\int_0^{\pi}(\sin^2 t+\cos^2 t)\,dt = -j\pi$$

Si en comptes de haver escollit C, haguessim triat C', el resultat hauria sigut: $j\pi$ (per tant la integral sí que depén del camí!!).

$$\boxed{\oint_C \bar{z}\,dz = j2\pi}$$

Altres formes:

$z=e^{j\theta}$

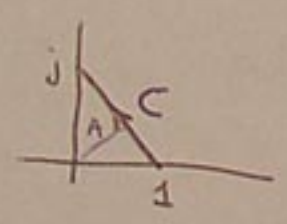
$\bar{z}=e^{-j\theta}$

$dz=j\,e^{j\theta}$

$$\int_C \bar{z}\,dz = \int_{\pi}^{0} e^{-j\theta}\cdot j\cdot e^{j\theta}\,d\theta = \int_{\pi}^{0} j\,d\theta = \pi j$$

PROBLEMA

$$\left|\int_C \frac{dz}{z^4}\right| \leq \int_C \left|\frac{dz}{z^4}\right| \leq \int_C \frac{|dz|}{|z^4|} \leq \int 4\,dz = 4\sqrt{2}$$



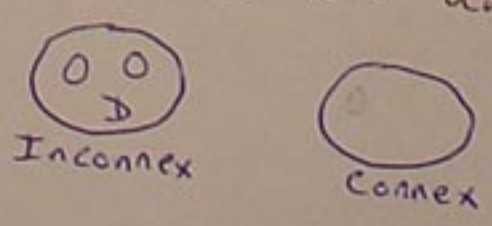
longitud(C) $=\sqrt{2}$

$\frac{1}{\sqrt{2}} = \frac{1}{|z|} \Leftarrow |z| \geq A = \sqrt{2}$

Perquè una integral no depengui del camí ha de ser irrotacional!!

Teorema de Cauchy - Goursat

Si $f(z)$ és analítica en un domini simplement connex D (sense forats).



Aleshores per a tota corba de Jordan C contingudа a D

$$\left[\oint_C f(z)\,dz = 0\right]$$

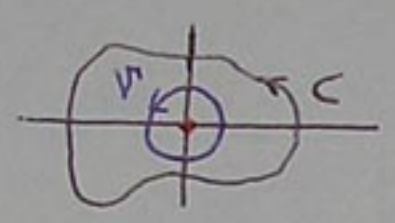
Dem:

$$\oint_C f(z)dz = \oint_C u\,dx - v\,dy + j\oint v\,dx + u\,dy \underset{\text{TEO. GREEN}}{=} \iint_D (-v_x - u_y)\,dx\,dy + j\iint_D (u_x - v_y)\,dx\,dy \underset{\text{Cauchy Rieman}}{=} 0$$

Principi de deformació dels camins:

$$\boxed{\oint_{C_1} f = \oint_{C_2} f}$$

PROBLEMA



$$\oint_C \frac{1}{z^n}dz = \oint_r \frac{1}{z^n}dz \underset{\substack{z=e^{j\theta} \\ dz = je^{j\theta}d\theta}}{=} \int_0^{2\pi} \frac{1}{e^{jn\theta}}\, j e^{j\theta} d\theta = \int_0^{2\pi} j e^{j\theta(1-n)} d\theta =$$

$$= j\left.\frac{e^{j\theta(1-n)}}{j(1-n)}\right]_0^{2\pi} = 0 \quad \rightsquigarrow \text{ Si } \underline{\underline{n=1}} \Rightarrow \oint_r \frac{1}{z^n}dz = j2\pi$$

Fórmula integral de Cauchy:

Si $f(z)$ és analítica en un domini simplement connex D i C una corba de Jordan contingudа en D.

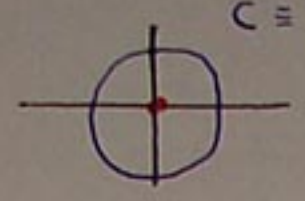
↳ tancada i simple

Si z_0 és un punt interior a C, es verifica:

$$\boxed{f(z_0) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z - z_0}dz}$$

PROBLEMA

a)



$C \equiv |z| = 1$

$$\int_C \bar{z}\,dz = \int_C \frac{1}{z}\,dz = 2\pi j \cdot f(0) = 2\pi j$$

$z_0 = 0$ ← Mirar fórmula de Cauchy

$f(z) = 1$

$\sqrt{z\bar{z}} = 1$

$z \cdot \bar{z} = 1$

$\bar{z} = \frac{1}{z}$

b)

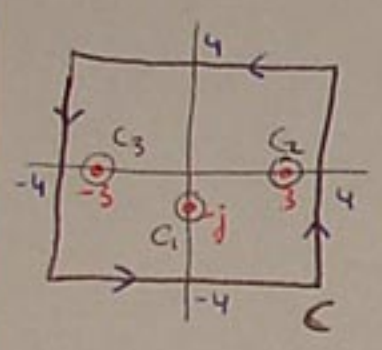
$C \equiv |z| = 2$

$$\int_C \frac{z}{(9-z^2)(z+j)}\,dz = \int_C \frac{z/(9-z^2)}{z+j}\,dz = j2\pi f(-j) = \frac{\pi}{5}$$

$f(z) = \frac{z}{9-z^2}$ → analítica → $f(-j) = \frac{-j}{9-(-1)} = \frac{-j}{10}$

$z_0 = -j$

c)



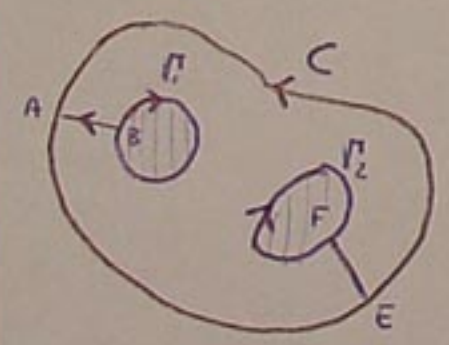
$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \int_{C_1} \frac{\frac{z}{9-z^2}}{z+j}\,dz + \int_{C_2} \frac{\frac{-z}{(z+j)(z+3)}}{z-3}\,dz + \int_{C_3} \frac{\frac{-z}{(z+j)(z-3)}}{z+3}\,dz =$$

$$= 2\pi j \left[\frac{-j}{9-(-j)^2} + \frac{-3}{(3+j)(6)} + \frac{-3}{(-3+j)6} \right] =$$

$$= 2\pi j \left[\frac{-j}{10} - \frac{1}{2(3+j)} + \frac{-1}{2(-3+j)} \right] = 0$$

TEORIA:

17-05-06



$C_1 = C \cup \Gamma_1 \cup \Gamma_2 \cup \overline{AB} \cup \overline{BA} \cup \overline{EF} \cup \overline{FE}$

$\oint_{C_1} f = 0$ per Cauchy-Groussat

Aleshores

$$\oint_C f - \oint_{\Gamma_1} f - \oint_{\Gamma_2} f = 0$$

Per tant

$$\oint_C f = \sum_{i=1}^{2} \oint_{\Gamma_i} f$$

F.I. de Cauchy

$$\boxed{f(z_0) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_0}\,dz}$$

Extensió a derivades

$$\boxed{\underbrace{f^{(n)}}_{\text{derivada n-èssima}}(z_0) = \frac{n!}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

Definició de punt singular (o singularitat) de f: z_0 és punt singular de f si f no és analítica en z_0, però qualsevol entorn seu conté algun punt en el qual f és analítica. En cas contrari, diem que z_0 és un punt singular aïllat.

PROBLEMA $\quad C \equiv 4e^{j\theta},\ 0<\theta<2\pi$

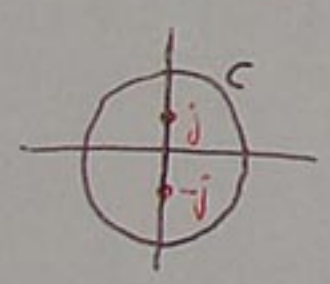
$$\int_C \frac{1}{z^2-4z+3}\,dz = \int_C \frac{1}{(z-1)(z-3)}\,dz = \int_C \left(\frac{1/2}{z-3} - \frac{1/2}{z-1}\right)dz =$$

$$z = \frac{4\pm\sqrt{16-12}}{2} = 2\pm 1 = \begin{cases}3\\1\end{cases}$$

$$= \frac{1}{2}\left[\int_C \frac{1}{z-3}\,dz - \int_C \frac{1}{z-1}\,dz\right] = \frac{1}{2}\,2\pi j\,\big(f(3)-g(1)\big) = 0$$

$f(z)=1$
$g(z)=1$

PROBLEMA



$$\oint_C \frac{e^z}{z^2+1}\,dz = \oint_C e^z\left(\frac{1/2j}{z+j} - \frac{1/2j}{z-j}\right)dz$$

$$\frac{A}{z+j} + \frac{B}{z-j} = \frac{1}{z^2+1} \Rightarrow A(z-j) + B(z+j) = 1$$

$$z=j \Rightarrow B = \frac{1}{2j} = \boxed{\frac{-j}{2} = B}$$

$$z=-j \Rightarrow A = \frac{1}{-2j} = \boxed{\frac{j}{2} = A}$$

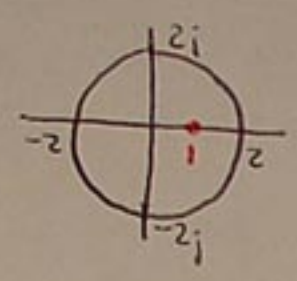
$$= \frac{1}{2} j \left[\oint \frac{e^z}{z+j} dz - \oint \frac{e^z}{z-j} dz \right] = \frac{j}{2} \cdot 2\pi j \left(f(-j) - f(j) \right) =$$

$$= -\pi \left(f(-j) - f(j) \right) = -\pi \left(e^{-j} - e^{j} \right) = \pi \left(e^{j} - e^{-j} \right) = 2\pi j \, \frac{e^{j} - e^{-j}}{2j} = 2\pi j \sin(1)$$

$f(z) = e^z$

PROBLEMA

$C \equiv |z| = 2$



$$\oint_C \frac{z^3 + 2z}{(z-1)^3} dz$$

Utilitzem la formula integral de Cauchy en extensió a derivades !!

$f(z) = z^3 + 2z$

$z_0 = 1$

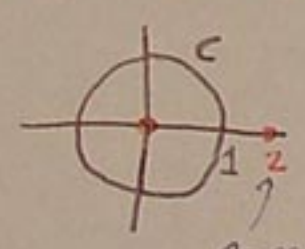
$n = 2$

$f'(z) = 3z^2 + 2$

$f''(z) = 6z$

$$= \frac{f^{(2)}(z_0) \cdot 2\pi j}{n!} = \frac{6 \cdot 2\pi j}{2} = 6\pi j$$

PROBLEMA



no es troba dins C

$z^2 - 2z = z(z-2)$

$(z^2 - 2z)^3 = z^3 (z-2)^3$

$$\oint_C \frac{1}{(z^2 - 2z)^3} dz = \oint_C \frac{1/(z-2)^3}{z^3} dz = \frac{f^{(2)}(0) \cdot 2\pi j}{n!} = \frac{-6\pi j}{2 \cdot 2} = \frac{-3\pi j}{8}$$

$z_0 = 0$

$n = 2$

$f(z) = (z-2)^{-3}$

$f'(z) = -3(z-2)^{-4}$

$f''(z) = 12(z-2)^{-5} = -\frac{12}{32} = \frac{-3}{8}$

PROBLEMA $C \equiv |z| = 1$ $\quad z = e^{j\theta}, \; 0 \leq \theta < 2\pi$

$$\int_C \frac{\cos z}{z^n} dz = \int_C \frac{z \cdot \cos z}{z^{n+1}} dz = \frac{2\pi j}{n!} \cdot f^{(n)}(0) = (*)$$

$f'(z) = \cos z + z(-\sin z) = \cos z - z \sin z$

$f''(z) = -\sin z - (\sin z + z \cos z) = -2\sin z - z\cos z$

$f'''(z) = -2\cos z - (\cos z + z(-\sin z)) = -3\cos z + z \sin z$

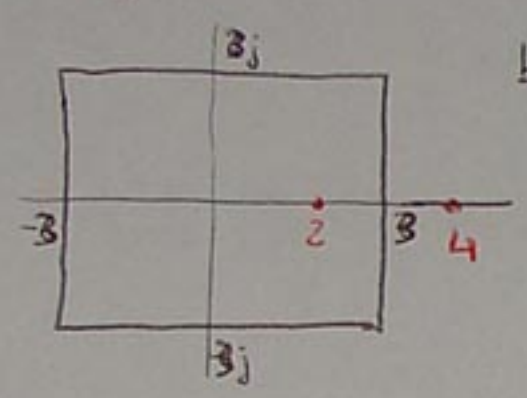
$$f^{(n)}(z) = n \cdot \overset{\text{derivada}}{\cos^{(n-1)} z} + z \cdot \overset{\text{derivada}}{(\cos z)^{(n)}}$$

$$(*) = \begin{cases} 0 & n = 2k \\ \dfrac{\pm n\, 2\pi j}{n!} & n = 2k-1 \end{cases} \qquad k \in \mathbb{Z}$$

PROBLEMA

a)

$$\int_C \frac{2z^2 - z - 2}{z-2}\, dz = 2\pi j \cdot f(2) = 2\pi j\,(8-2-2) = 8\pi j$$



b)

$$\int_C \frac{2z^2 - z - 2}{z-4}\, dz = 0$$

la singularitat es troba fora C

PROBLEMA Continuació...

$$\int_0^{2+j} z^2\, dz = \left[\frac{z^3}{3}\right]_0^{2+j} = \frac{(2+j)^3}{3} = \frac{2}{3} + j\frac{11}{3}$$

$$\int_{|z|=1} \frac{1}{z}\, dz = \int_{z_1}^{z_2} \frac{1}{z}\, dz = \ln z \Big|_{z_1}^{z_2} = -\ln z_1 + \ln z_2 = \ln \cancel{|z_2|}^{\,0} + j\pi - \left(\ln \cancel{|z_1|}^{\,0} + j(-\pi)\right) =$$

$z = e^{j\theta}$ $\quad |\theta| < \pi$

$z_1 = -\pi$, $z_2 = \pi$

$$= j2\pi$$

PROBLEMA

$0 \le \arg(z) < 2\pi$ $\qquad \ln z = \ln|z| + j\arg(z)$ $\qquad \ln \frac{z_1}{z_2} = \ln z_1 - \ln z_2$

a) Sempre

b) Mai

c) Només si $\arg(z_1) \ge \arg(z_2)$

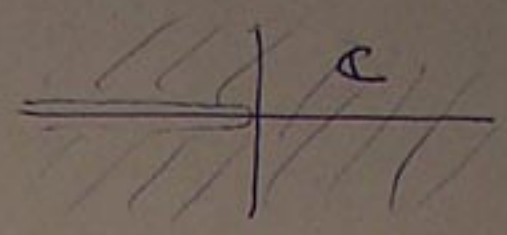
d) " " $\arg(z_1) + \arg(z_2) < 2\pi$

$$\ln \frac{z_1}{z_2} = \ln\left|\frac{z_1}{z_2}\right| + j\left(\arg z_1 - \arg z_2\right) = \ln|z_1| - \ln|z_2| + j\left(\arg z_1 - \arg z_2\right)$$

PROBLEMA

$f(z)$ és analítica

$f(z) = \sqrt{z}$ $\quad \mathbb{C} \setminus \{x+jy : x \leq 0, y=0\}$



On és analítica $f(z) = \sqrt{e^z+1}$?

$$e^z+1 = e^{x+jy}+1 = e^x \cdot e^{jy}+1 = e^x(\cos y + j\sin y)+1 = (1+e^x\cos y) + j\sin y \cdot e^x$$

$$\sqrt{z} = \sqrt{x+jy}$$

Per analogia...

$$\left.\begin{array}{l} 1+e^x\cos y \geq 0 \\ e^x \sin y = 0 \end{array}\right\} \rightarrow \underline{\underline{y = k\pi}}, \ k \in \mathbb{Z}$$

$$\Downarrow$$

$$1+e^x \cdot \cos(k\pi) \geq 0 \Rightarrow 1+e^x(-1)^k \geq 0 \Rightarrow 1-e^x(-1)^{k+1} \geq 0$$

$$\Rightarrow 1 \geq e^x \cdot (-1)^{k+1} \Rightarrow x \geq 0$$

$$-1 \leq e^x(-1)^k \Rightarrow \underline{\underline{x \geq 0}}$$

PROBLEMA

$f(z) = z^c, \quad z \neq 0$

$|f(z)| = |z|^c$

a) Només si $c \in \mathbb{R}$

b) " " c imaginari pur

c) " " $c \in \mathbb{Z}$

d) Sempre

$$|z^c| = |z^{a+jb}| = |z^a \cdot z^{jb}| = |z^a| \cdot |z^{jb}| =$$

$$\overset{(\Delta)}{=} |z|^a \cdot |z^{jb}| = e^{a\ln|z|} \cdot |z^{jb}| = (*)$$

al final

$$z^{jb} = e^{jb\ln z} = e^{jb[\ln|z| + j\arg z + 2k\pi]} = \underbrace{e^{jb\ln|z|}}_{=1} \cdot e^{-b(\arg z + 2k\pi)}$$

$\rightarrow$ perquè $b \in \mathbb{R}$

$$(*) = e^{a\ln|z|} \cdot |e^{-b(\arg z + 2k\pi)}| = e^{a\ln|z|} \cdot e^{-b(\arg z + 2k\pi)}$$

$$|z|^c = |z|^{a+bj} = |z|^a \cdot |z|^{jb} = e^{a\ln|z|} \cdot e^{jb\ln|z|}$$

$b=0$

$$\overset{(\Delta)}{|z^a|} = |e^{a\ln z}| = |e^{a(\ln|z| + j(\arg z + 2k\pi))}| = (e^{a\ln|z|}) = |z|^a \Leftrightarrow a \in \mathbb{R}$$

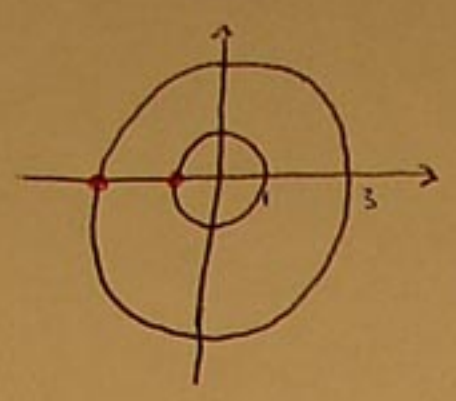
PROBLEMA

24-5-06

3B

$1<|z|<3$

$$f(z)=\frac{1}{(z+1)(z+3)}$$



$$f(z)=\frac{1/2}{z+1}-\frac{1/2}{z+3}$$

$$f(z)=\frac{1}{2}\frac{1}{z}\frac{1}{1+\frac{1}{z}}-\frac{1}{2}\frac{1}{3}\frac{1}{1+\frac{z}{3}}$$

$$\frac{1}{1+z}=1-z+z^2-z^3\dots \quad , \quad \frac{1}{1-z}=1+z+z^2+z^3\dots$$

$$f(z)=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\dots\right]=$$

$$=\underbrace{\dots-\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}}_{\text{Part principal}}\underbrace{-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\dots}_{\text{part analítica}}$$

$$\Rightarrow f(z)=\frac{1}{2}\sum_{n\geq 1}(-1)^{n+1}\frac{1}{z^n}-\frac{1}{6}\sum_{n\geq 0}(-1)^n\frac{z^n}{3^n}$$

PROBLEMA

$$f(z)=\frac{1+z}{1-\cos z}$$

Punts singulars:

$\cos z=1$

$$\cos(x+jy)=\cos x\cdot\cos jy-\sin x\cdot\sin jy=\cos x\cdot\cosh y-j\sin x\cdot\sinh y$$

$$\cos x\cdot\cosh y-j\sin x\cdot\sinh y=1$$

$$\Downarrow$$

$\cos x\cdot\cosh y=1$, $j\sin x\cdot\sinh y=0$

$$\Downarrow$$

$\boxed{y=0}$ ó $x=k\pi \longrightarrow (-1)^k\cdot\cosh y=1 \longrightarrow y=0$ (condició anterior)

$\cos x=1$

$$\Downarrow$$

$\boxed{x=2k\pi}$

Desenvolupem al voltant del zero:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \dots$$

$$\begin{array}{l|l}
1 + z & \frac{z^2}{2} - \frac{z^4}{24} \\ \hline
-1 + \frac{z^2}{12} & \boxed{\frac{2}{z^2} + \frac{2}{z} + \frac{1}{6}} \\ \hline
/ \frac{z^2}{12} + z & \\
+\frac{z^3}{12} - z & \\ \hline
+\frac{z^3}{12} + \frac{z^2}{12} / & \\
-\frac{z^2}{12} \dots & \\ \hline
/ \dots &
\end{array}$$

TEORIA

▷ Si f té un pol simple (d'ordre 1) en z_0:

$$f(z) = \frac{c_{-1}}{z - z_0} + c_0 + c_1 (z - z_0) + \dots$$

$$(z - z_0) \cdot f(z) = c_{-1} + c_0 (z - z_0) + c_1 (z - z_0)^2$$

$$\boxed{\lim_{z \to z_0} (z - z_0) \cdot f(z) = c_{-1}}$$

▷ Si f té un pol d'ordre 2 en z_0:

$$f(z) = \frac{c_{-2}}{(z - z_0)^2} + \frac{c_{-1}}{z - z_0} + c_0 + c_1 (z - z_0) + \dots$$

$$(z - z_0)^2 \cdot f(z) = c_{-2} + c_{-1} (z - z_0) + c_0 (z - z_0)^2 + c_1 (z - z_0)^3$$

$$\boxed{\lim_{z \to z_0} \frac{d}{dz} \left[(z - z_0)^2 \cdot f(z) \right] = c_{-1}}$$

D Si f té pol d'ordre n en z_0:

FORMULA GENERAL:

$$\boxed{c_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n \cdot f(z) \right]}$$

PROBLEMA

$\text{Res}(f, 0) = c_{-1}$

$$f(z) = \frac{z}{(e^z-1)^2}$$

Si $z_0 = 0$ fos un pol simple, llavors: $\lim_{z \to 0} z \cdot f(z) \neq 0$

Ho compleix ??

$$\lim_{z \to 0} \frac{z^2}{(e^z-1)^2} \underset{\text{Hôpital}}{=} \lim_{z \to 0} \frac{2z}{2(e^z-1)e^z} \underset{\text{Hôpital}}{=} \lim_{z \to 0} \frac{\cancel{2}}{\cancel{2}(e^z-1)e^z + \cancel{2}e^{2z}} = 1 \neq 0$$

PROBLEMA

$$f(z) = \frac{\text{sh}\, z}{z - j\pi}$$

A $z = j\pi$ té:

a) singularitat essencial
b) singularitat evitable
c) pol d'ordre 1
d) pol d'ordre 2

La singularitat ~~essencial~~ evitable és com tenir un pol d'ordre zero. El ~~evitable~~ essencial és com tenir un pol d'ordre infinit.

$$\text{sh}(j\pi) = \frac{e^{j\pi} - e^{-j\pi}}{2} = j \sin(\pi) \qquad \text{ch}(j\pi) = \frac{e^{j\pi} + e^{-j\pi}}{2} = \cos(\pi)$$

$$\lim_{z \to j\pi} \frac{\text{sh}\, z}{z - j\pi} = \lim_{z \to j\pi} \frac{j \sin(\pi)}{z - j\pi} = \frac{0}{0} \rightsquigarrow \text{Hôpital} \qquad \lim_{z \to j\pi} \frac{\text{sh}\, z}{z - j\pi} = \lim_{z \to j\pi} \frac{\text{ch}\, z}{1} = \cos \pi = -1 \neq 0$$

Sèrie de Laurent:

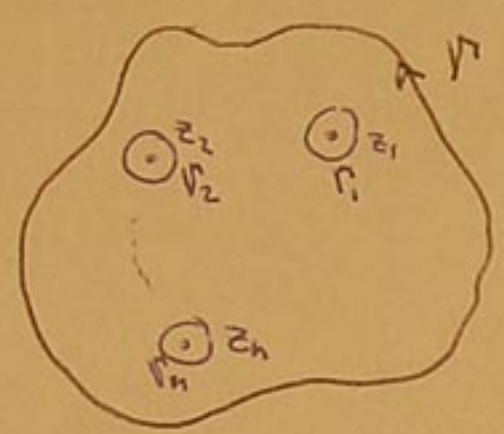
$$f(z) = \sum_{-\infty}^{+\infty} c_n \cdot (z-a)^n$$

$$c_n = \frac{1}{2\pi j} \oint_{\Gamma} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\underset{\text{Residu}}{c_{-1}} = \frac{1}{2\pi j} \oint_{\Gamma} f(z)\,dz \Rightarrow \oint_{\Gamma} f(z)\,dz = 2\pi j \cdot \mathrm{Res}(f, a)$$

Teorema dels residus:

Si f té punts singulars aïllats $z_1, z_2, \dots, z_n$ situats en un domini en el qual f és analítica



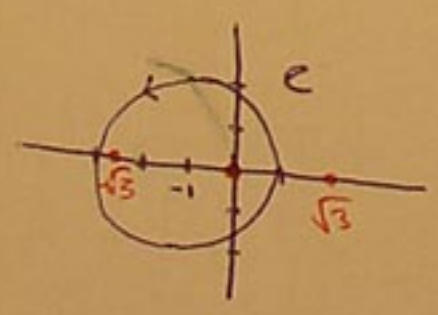
$$\boxed{\oint_{\Gamma} f = \sum_{i=1}^{n} \oint_{\Gamma_i} f = 2\pi j \sum_{i=1}^{n} \mathrm{Res}(f, z_i)}$$

PROBLEMA

$C \equiv z+1 = 2e^{j\theta}$

$0<\theta<2\pi$

$$\oint_C \frac{\sin z}{z^3 - 3z} dz$$



$$z^3 - 3z = z(z^2-3) = z(z-\sqrt{3})(z+\sqrt{3})$$

$\Rightarrow$ f té a l'interior de C dos singularitats aïllades que $z_0 = -\sqrt{3}$ és pols simple $\Rightarrow$

$z_0 = -\sqrt{3}$ pols simple? $\Rightarrow$ $\lim_{z \to -\sqrt{3}} (z+\sqrt{3}) \cdot f(z) = \lim_{z \to -\sqrt{3}} \frac{\sin z}{z(z-\sqrt{3})} = \cancel{\frac{0}{2\sqrt{3}}} = \frac{-\sin(\sqrt{3})}{-\sqrt{3}(-2\sqrt{3})} = \frac{-\sin(\sqrt{3})}{6} \neq 0$

Tot i així $z_0 = 0$ no és pol simple.

Què és?

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z}{z^3-3z} = \lim_{z\to 0} \frac{1}{z^2-3} = \frac{-1}{3} \neq 0 \rightsquigarrow z_0=0 \text{ és singularitat evitable}$$

$$\oint_C \frac{\sin z}{z^3-3z}\,dz = 2\pi j\left[\operatorname{Res}(f,0) + \operatorname{Res}(f,-\sqrt{3})\right] = 2\pi j\,\frac{-\sin\sqrt{3}}{6} = -\frac{\sin\sqrt{3}}{3}\pi j$$

PROBLEMA

Desenvolupar z^{-3} i z^3 en sèries de Laurent a $1<|z|<2$ $\nearrow |\frac{1}{z}|<1$ $\searrow |z|<2$

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2} = \frac{-1}{z}\,\frac{1}{1+\frac{1}{z}} + \frac{1}{1+\frac{z}{2}} =$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right) + \left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\dots\right)$$

$$c_3 = -\frac{1}{8} \qquad c_{-3} = -1 \qquad c_{-1} = -1$$

PROBLEMA

$$\operatorname{Res}\left(e^{zt}\cdot \operatorname{tg} z,\ \frac{3\pi}{2}\right)$$

$z_0 = \frac{3\pi}{2}$ no és singularitat evitable

Mirem si és un pol d'algun ordre:

$$\lim_{z\to\frac{3\pi}{2}} \left(z-\frac{3\pi}{2}\right)\frac{e^{zt}\cdot\sin z}{\cos z} \underset{\text{Hôpital}}{=} \lim_{z\to\frac{3\pi}{2}} \frac{e^{zt}\cdot\sin z}{\cos z} + \left(z-\frac{3\pi}{2}\right)\left[\frac{(te^{zt}\cdot\sin z + \cos z\cdot e^{zt})\cos z}{\cos^2 z}\right.$$

$$\left. + \frac{\sin z\,(e^{zt}\cdot\sin z)}{\cos^2 z}\right]$$

$$-e^{\frac{3\pi t}{2}} \neq 0$$

PROBLEMA

$$\int_0^{2\pi} \frac{dt}{1+a\,\mathrm{sm}\,t} \qquad a\in\mathbb{R} \quad |a|<1$$

$z = e^{jt}$

$|z| = 1$

$dz = j e^{jt} dt \Rightarrow dt = \frac{1}{jz} dz$

$$\mathrm{sm}\,t = \frac{e^{jt}-e^{-jt}}{2j} = \frac{z-\frac{1}{z}}{2j} = \frac{z^2-1}{2jz}$$

$$= \oint_C \frac{\frac{1}{jz}}{1+a\frac{z^2-1}{2jz}}\,dz = 2\oint_C \frac{dz}{2jz+az^2-a} =$$

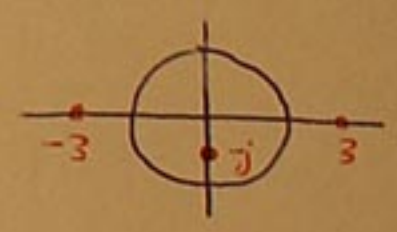
$$= \frac{2}{a}\,2\pi j\,\mathrm{Res}\left(f, \frac{-j}{z}\right) = \frac{2\pi}{\sqrt{1-a^2}}$$

PROBLEMES

30-5-06

$$\oint_C \frac{z}{(9-z^2)(z+j)}\,dz \qquad C \equiv |z| = 2$$

Recorregut en sentit positiu ↺



→ Pol. Teorema del Residu

$$= 2\pi j \cdot \mathrm{Res}(f, -j) = 2\pi j \lim_{z\to -j} (z+j)\cdot f(z) =$$

$$= 2\pi j \lim_{z\to -j} (z+j)\frac{z}{(9-z^2)(z+j)} = 2\pi j\left(\frac{-j}{10}\right) = \frac{\pi}{5}$$

* També es podia utilitzar la Formula Integral de Cauchy