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FIBER-OPTIC COMMUNICATIONS

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OPTICAL COMMUNICATIONS GROUP

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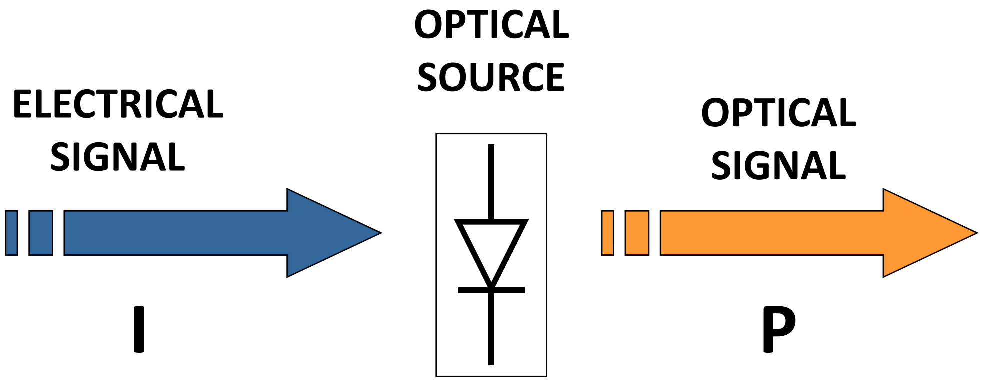
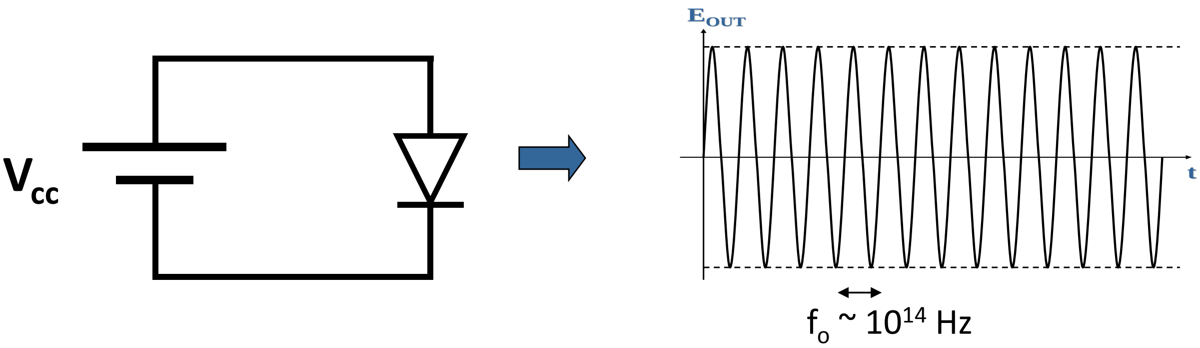
INTRODUCCTION TO OPTICAL SOURCES



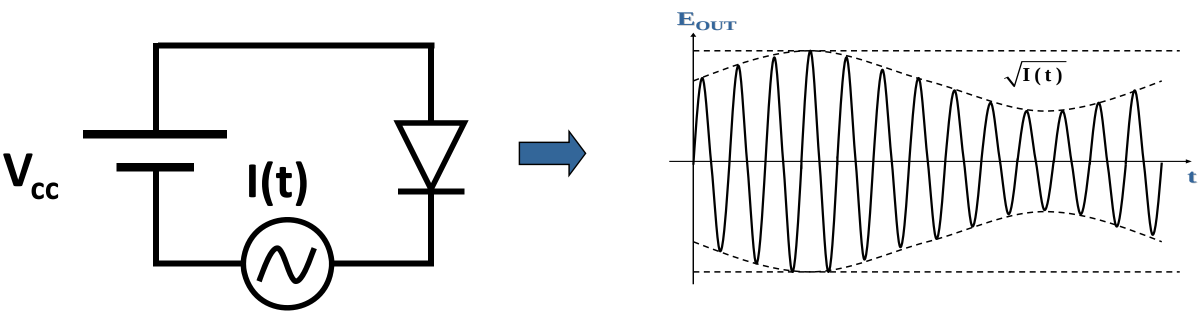
QUADRATIC CONVERSION

$$I \propto P$$

Continuous Wave (CW)



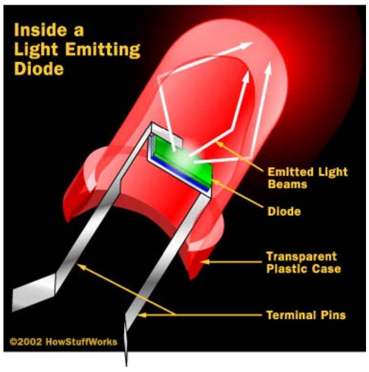
Intensity Modulation (IM)



DESIRABLE CHARACTERISTICS

- ✓ HIGH E/O CONVERSION EFFICIENCY
- ✓ WORKING TEMPERATURE AND STABILITY
- ✓ EMISSION FREQUENCY
- ✓ HIGH MODULATION SPEED
- ✓ LINEAR LIGHT-CURRENT RESPONSE
- ✓ HIGH SPECTRAL PURITY (LASER)
- ✓ FIBER COMPATIBILITY (COUPLING)
- ✓ SMALL SIZE AND CONSUMPTION (INTEGRATION)
- ✓ REDUCED COST

TYPES AND APPLICATIONS

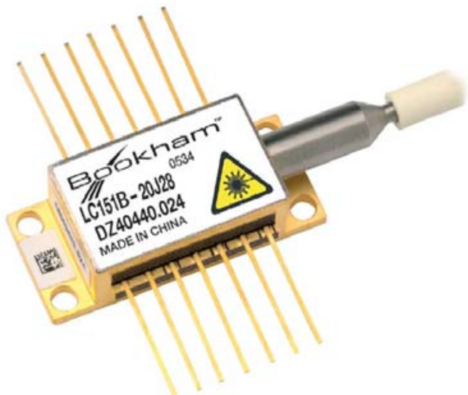


LED DIODE

- ❑ Visible → visualization
- ❑ near IR → telecom

LASER DIODE

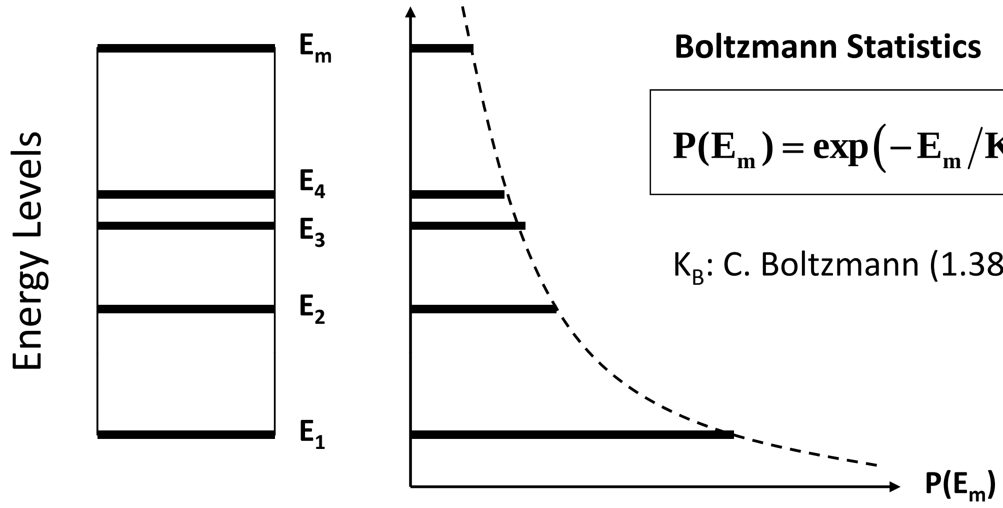
- ❑ Visible
  - → industry
  - → medicine
  - → space telecom
- ❑ near IR → telecom



LIGHT-MATTER INTERACTION

Atomic / Molecular Energy Level

"The energy level of an isolated atom / molecule is discrete due to Pauli Exclusion Principle.



Boltzmann Statistics

$$P(E_m) = \exp\left(-E_m / K_B T\right)$$

K_B: C. Boltzmann ($1.38 \cdot 10^{-23}$ J/K)

Light Absorption / Emission Processes

"Any given material shows a particular light absorption characteristic. Some of them, under specific conditions, have the capacity of light emission".

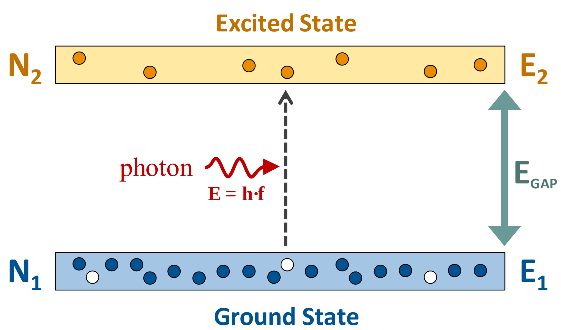
hf → photon energy

hf ≈ E_2 - E_1 = E_g (direct GAP)

h: C. Planck ($6{,}63 \cdot 10^{-34}$ J·s)

f: light frequency

STIMULATED ABSORPTION

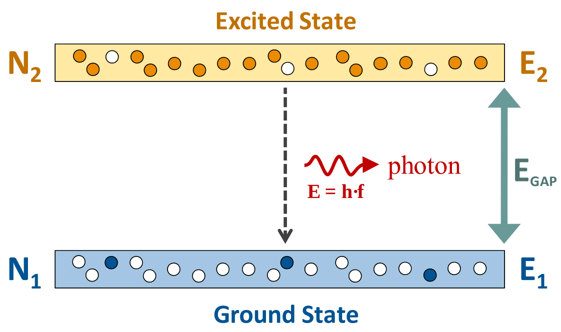


"The incident photon is absorbed by an electron which increments its energy level"

⇩

Photodetectors

SPONTANEOUS EMISSION



"An excited electron releases energy in the form of a photon with random frequency, phase, and direction"

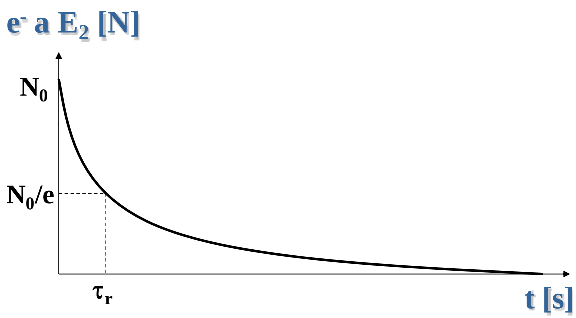
⇩

Incoherent light (LED)

Bose-Einstein statistics

$$\sigma_m^2 = \langle m \rangle \left(\langle m \rangle + 1 \right)$$

Recombination Lifetime

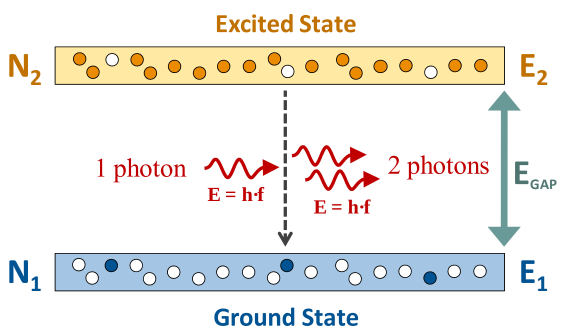


"Average time to return to ground state"

$$\boxed{N = N_0 e^{-\frac{t}{\tau_r}}} \longleftrightarrow \frac{\partial N}{\partial t} = -\frac{N}{\tau_r}$$

τ_r: Carrier Lifetime

STIMULATED EMISSION



"An incident photon forces an excited electron to release its energy in the form of a new photon with exactly the same frequency, phase, and direction"

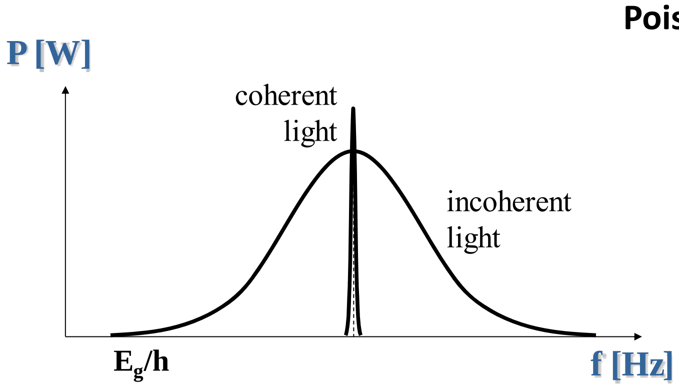
⇩

Coherent Light (LÀSER)

Poisson Statistics

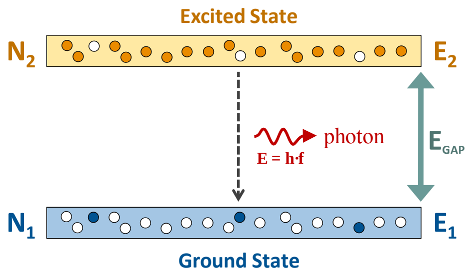
$$\sigma_m^2 = \langle m \rangle$$

spectra

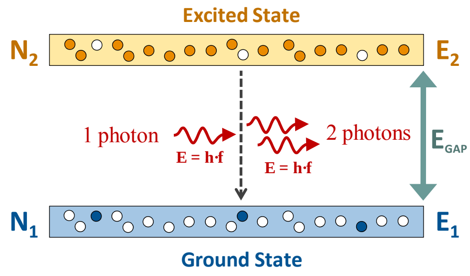


Thermal Equilibrium Condition

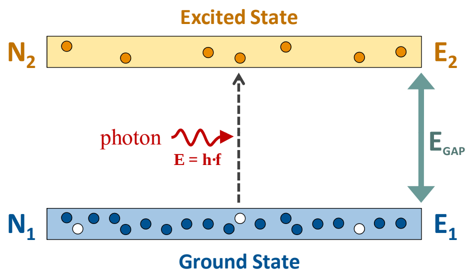
SPONTANEOUS EMISSION



STIMULATED EMISSION



STIMULATES ABSORPTION



$$hf >> K_B T$$

"No external energy interchange"

$$\boxed{\text{Absorptions Rate} = \text{Emissions Rate}}$$

E_i: Energy N_i: Carriers Density

Einstein Relations

$$\tau_r \equiv \frac{1}{A} \quad [s] \qquad \rho \equiv \frac{P}{v \cdot Wd} = S \cdot hf \quad \left[\frac{J}{m^3}\right]$$

Spontaneous E. Rate $\quad r_{sp} \equiv AN_2 = N_2/\tau_r$

Stimulated A. Rate $\quad r_a \equiv B_{12}\rho N_1 = avS \cdot N_1$

Stimulated E. Rate $\quad r_{st} \equiv B_{21}\rho N_2 = avS \cdot N_2$

Thermal Equilibrium

$$r_a = r_{sp} + r_{st} \quad [m^{-3}s^{-1}]$$

$$B_{12} = B_{21} = \frac{av}{hf} \quad \left[\frac{m^3}{J \cdot s}\right]$$

$$\frac{N_2}{N_1} = \exp\left(-E_g/K_B T\right) = \exp\left(-hf/K_B T\right)$$

Boltzmann Statistics

$$B_{12}\rho N_1 = AN_2 + B_{21}\rho N_2$$

$$\rho = \frac{AN_2}{B_{12}N_1 - B_{21}N_2} = \frac{A/B_{21}}{B_{12}N_1/B_{21}N_2 - 1} = \frac{A/B_{21}}{(B_{12}/B_{21})\exp(hf/K_B T) - 1}$$

A: Sp. Em. Coef. B_{12}: St. Ab. Coef. B_{21}: St. Em. Coef. ρ: E.S.D. radiation

Blackbody Radiation (Planck's Formula)

$$\rho \equiv \frac{8\pi h (f/v)^3}{\exp(hf/K_B T) - 1}$$ [J·m^-3]

$$\rho = \frac{A/B_{21}}{(B_{12}/B_{21})\exp(hf/K_B T) - 1} = \frac{8\pi h (f/v)^3}{\exp(hf/K_B T) - 1}$$

$$\left.\begin{array}{l} A = B_{21} 8\pi h (f/v)^3 \\ B_{12} = B_{21} = B \end{array}\right\} \rightarrow \boxed{\frac{r_{st}}{r_{sp}} = \frac{B_{21}\rho N_2}{AN_2} = \left[\exp(hf/K_B T) - 1\right]^{-1} << 1}$$

→ P(Esp) >> P(Est)

Not interesting

Net Stimulated Emission Rate

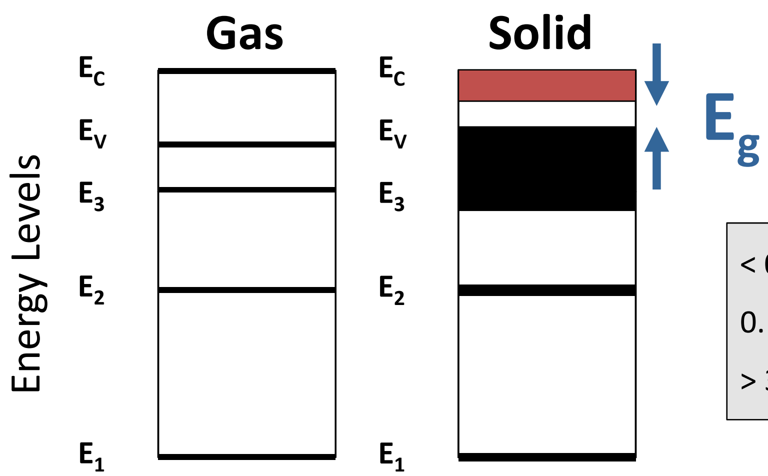
$$\boxed{r_e \equiv r_{st} - r_a = (N_2 - N_1)\rho \cdot B}$$ [m^-3 s^-1]

→ $N_2 > N_1$

Population Inversion

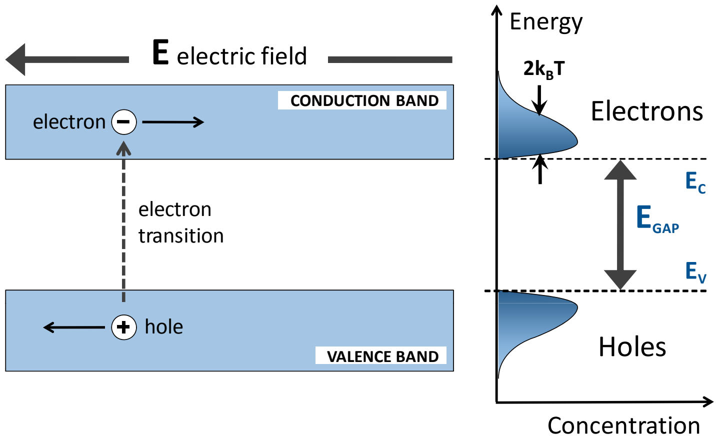
SEMICONDUCTORS PRINCIPLES

1. The electrons are located in discrete energy states being the last two the Valence / Conduction Bands separated by an energy GAP



< 0.1 eV → Conductors
0.1-3 eV → Semiconductors
> 3 eV → Isolators

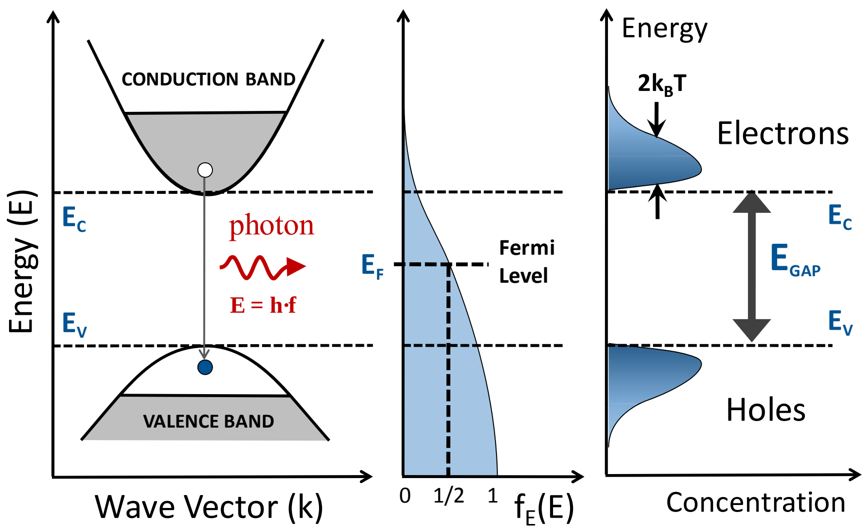
Depending on the energy GAP, the materials are divided among: isolators, conductors and <u>semiconductors</u>.



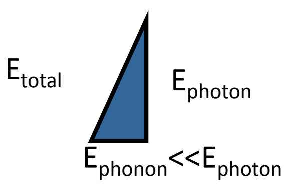
2. Electrons in the CB are not tied to any particular atom so they are free to move along the semiconductor.

3. When an electron liberates from its atom and moves to CB leaves a hole in the VB which is called to have positive charge.

4. An electron placed in CB may return to VB occupying a hole an releasing its energy that can be in the form of a **photon**. This process is known as **electron-hole recombination**.



Direct GAP



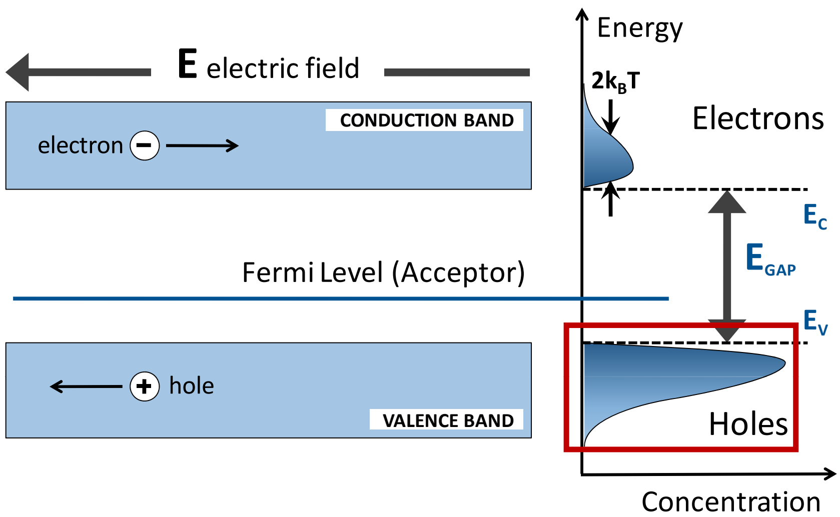
Fermi-Dirac Distribution

$$f(E) = \left\{1 + \exp\left[\left(E - E_f\right)/K_B T\right]\right\}^{-1}$$

E_f : Fermi Level

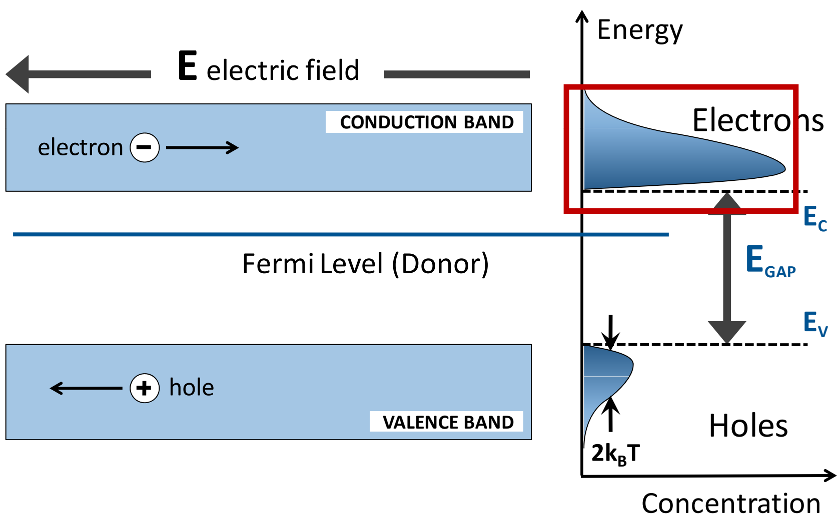
P-type Semiconductor

Some “acceptor” doping atoms are added which take electrons from the Conduction Band. A positive carrier flux is produced.

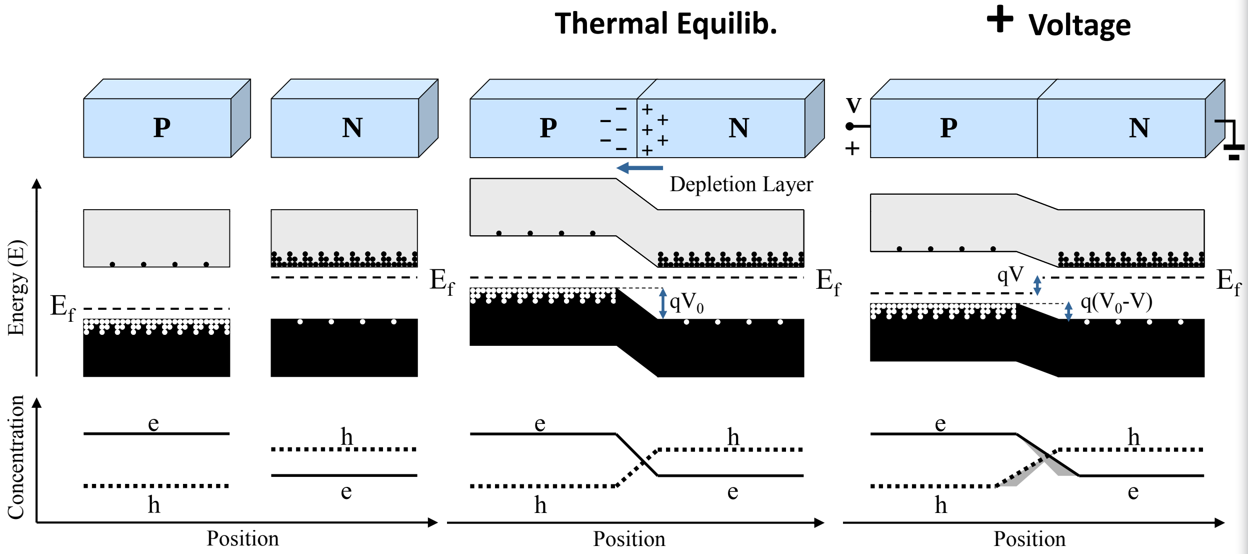


N-type Semiconductor

Some “donor” doping atoms are added which give electrons to the Conduction Band. A negative carrier flux is produced.



P-N Union Homojunction



In Thermal Equilibrium, the Fermi Level has to be continuous along the PN junction

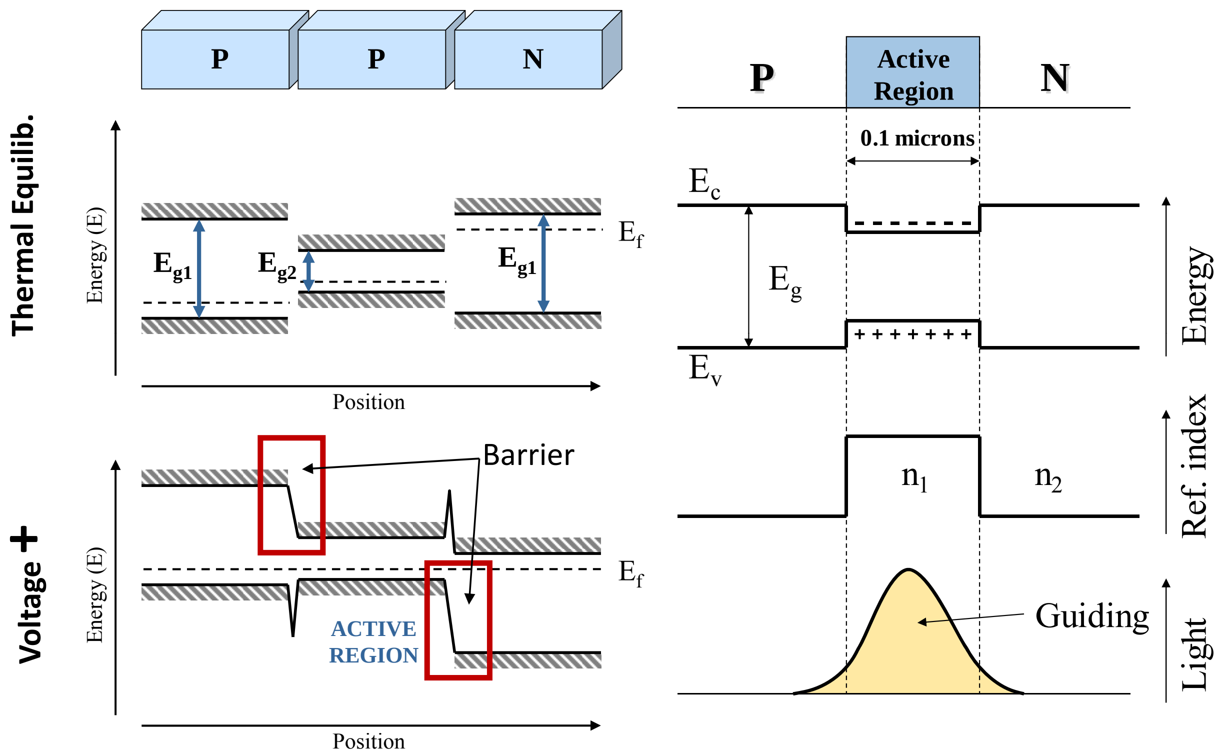
Active Region 1-10 microns

$$I = I_S\left[\exp\left(\frac{qV}{K_B T}\right) - 1\right]$$

Diode

I_S : Saturation Current

P-N Union Heterojunction



Materials for Optical Sources Fabrication

Material	E_g (eV)	λ_g (μm)	GAP
Ge	0.66	1.88	I
Si	1.11	1.15	I
AlP	2.45	0.52	I
AlAs	2.16	0.57	I
AlSb	1.58	0.75	I
GaP	2.26	0.55	I
GaAs	1.42	0.87	D
GaSb	0.73	1.70	D
InP	1.35	0.92	D
InAs	0.36	3.5	D
InSb	0.17	7.3	D

NO METALS

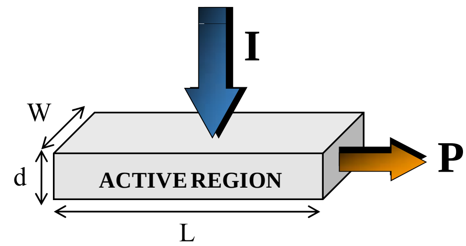
III	IV	V	VI	VII
B	C	N	O	F
Al	Si	P	S	Cl
Ga	Ge	As	Se	Br
In	Sn	Sb	Te	I
Tl	Pb	Bi	Po	At

$$E_g = h \cdot f_g = h\frac{c}{\lambda_g}$$

- Binaries → GaAs (1st window)
- Ternaries → $Al_xGa_{1-x}As$ (1st window)
  → $In_xGa_{1-x}As$ (2nd & 3rd window)
- Quaternaries → $In_xGa_{1-x}As_yP_{1-y}$ (1st ,2nd & 3rd window)

QUANTUM EFFICIENCY

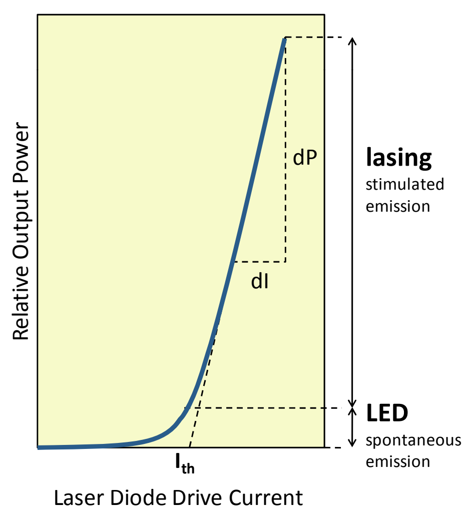
$$\eta \equiv \frac{\langle N^o fot/seg \rangle}{\langle N^o e-h/seg \rangle} = \frac{P_{OUT}/hf}{I/q}$$



$$\boxed{P_{OUT} = \eta \frac{hf}{q} I}$$ [W]

0.8

light-current characteristic



LED → η ~ 6%

LASER → η ~ 70%

q: electron charge ($1{,}6 \cdot 10^{-19}$ C)

Internal / External Quantum Efficiency

$$\eta \equiv \frac{\langle N^o fot/seg \rangle_{out}}{\langle N^o e-h/seg \rangle_{total}} = \overbrace{\frac{\langle N^o fot/seg \rangle_{out}}{\langle N^o fot/seg \rangle_{generated}}}^{\eta_e} \overbrace{\frac{\langle N^o fot/seg \rangle_{generated}}{\langle N^o e-h/seg \rangle_{total}}}^{\eta_i}$$

$$\boxed{\eta \equiv \eta_i \cdot \eta_e}$$

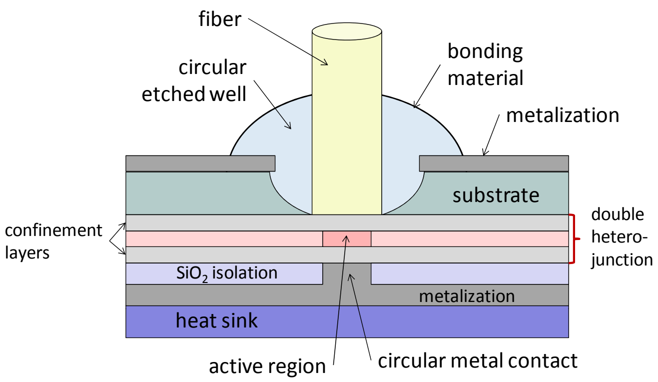
Si → $\eta_i \sim 10^{-5}$

AsGa → $\eta_i \sim 0.7$

Inefficiency Causes

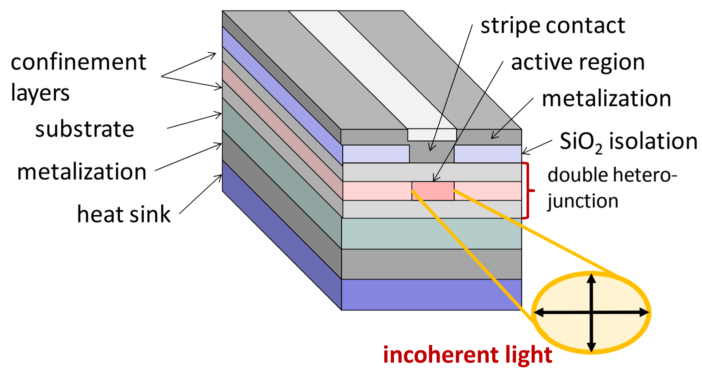
- Emitted light omnidirectionality
- Non-radiative recombinations → thermal energy
- Stimulated absorption in the active region
- Reflection in the source-air transition
- Phonon

LED (Light Emiting Diode)

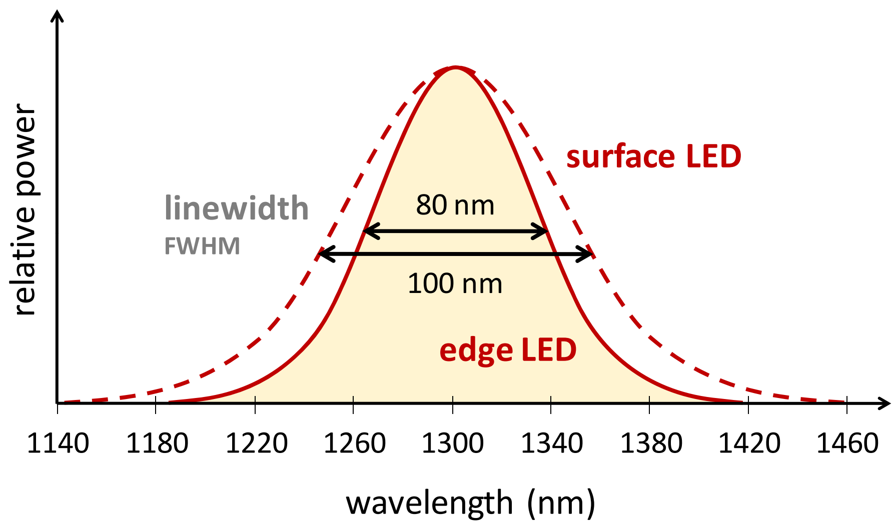


Surface-Emitting LED

Edge-Emitting LED



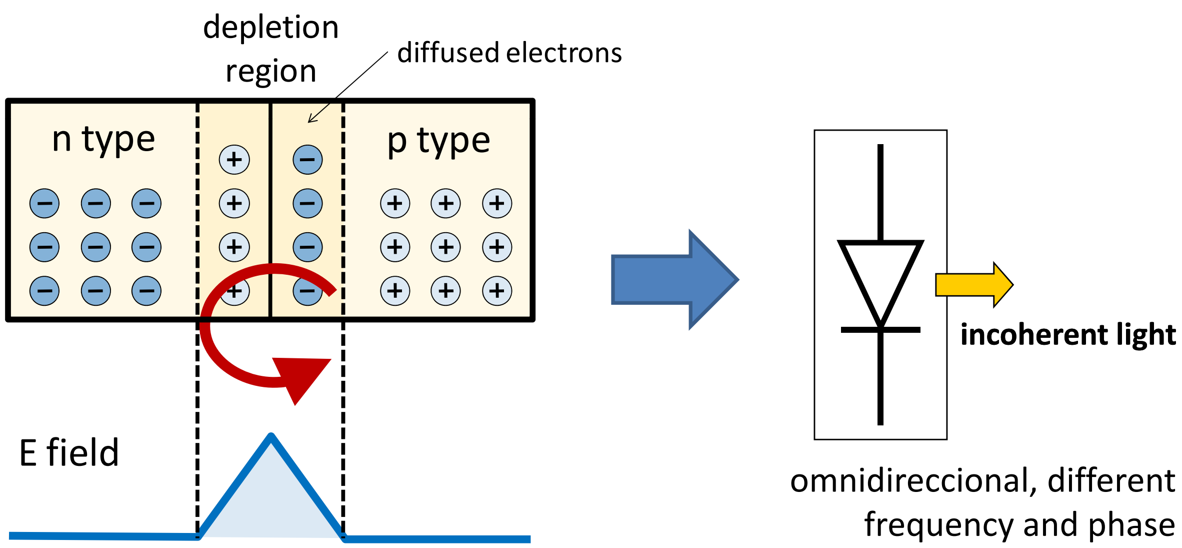
LED characteristic figures



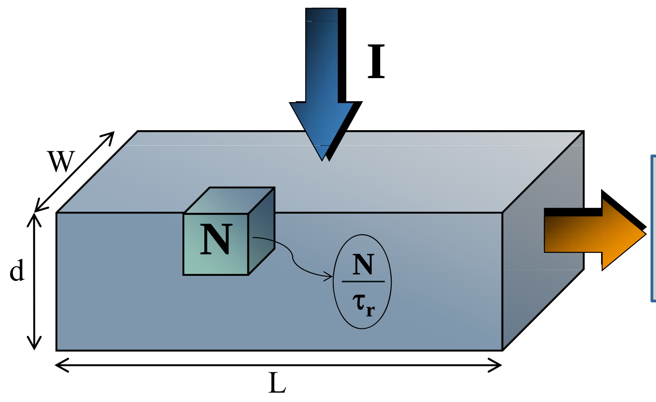
- BW up to 100 MHz → R_B up to 100 Mb/s
- $\Delta\lambda$ huge → 100 nm
- P_{OUT} very small → -20 dBm

WORKING PRINCIPLE

"LED source is a diode (PN junction) directly polarized which emits light by **spontaneous emission** (incoherent light) thanks to an electron-hole recombination process"



Carrier Injection – Optical Power



quantum efficiency

$$\eta \equiv \frac{\langle N^o fot/seg \rangle}{\langle N^o e-h/seg \rangle} = \frac{P_{OUT}/hf}{I/q}$$

$$\boxed{P = \eta \frac{N}{\tau_r} V \cdot hf \equiv \frac{Joules}{s}}$$

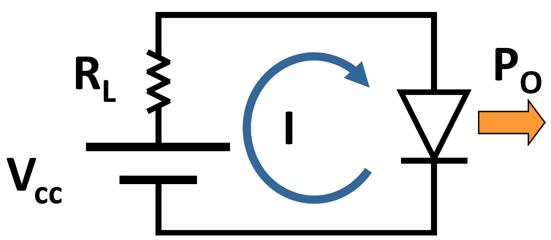
$$\frac{N}{\tau_r} \equiv \frac{recomb/m^3}{s} \Longrightarrow \frac{N}{\tau_r} V \equiv \frac{recomb}{s} \Longrightarrow \eta \frac{N}{\tau_r} V \equiv \frac{fotons}{s}$$

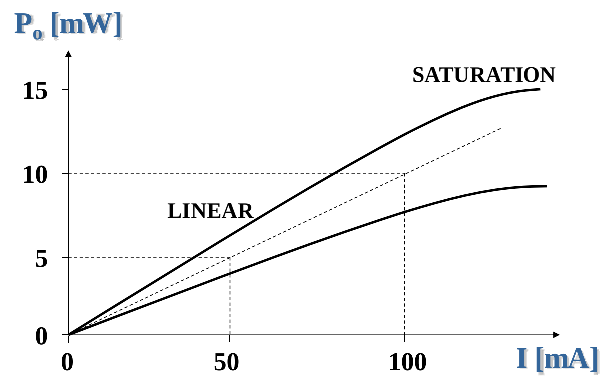
$$P_{OUT} = \eta \frac{hf}{q} I = \eta \frac{N}{\tau_r} V \cdot hf \Longrightarrow \boxed{\frac{N}{\tau_r} = \frac{I}{qV}}$$

equilibrium state

Light – Current characteristic

"Representation of the optical power emitted by the source as a function of the polarization electrical current intensity"





$$\boxed{P_{OUT} = \eta \frac{hf}{q} I} \quad [W]$$

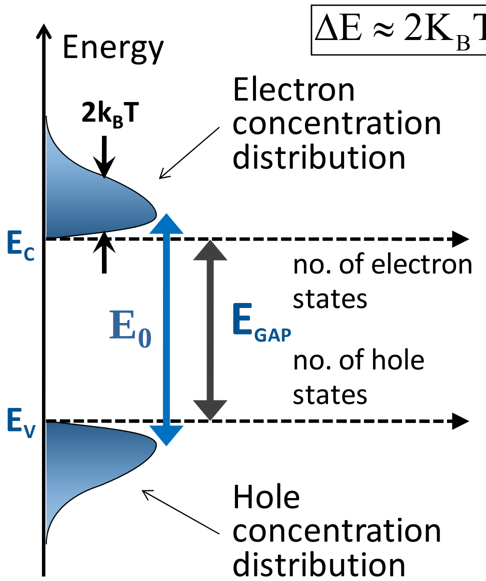
typical efficiency : 0.05 mW/mA

AsGa → $\eta_i \sim 0.7$ $\qquad \frac{hf}{q} \approx 0.8$

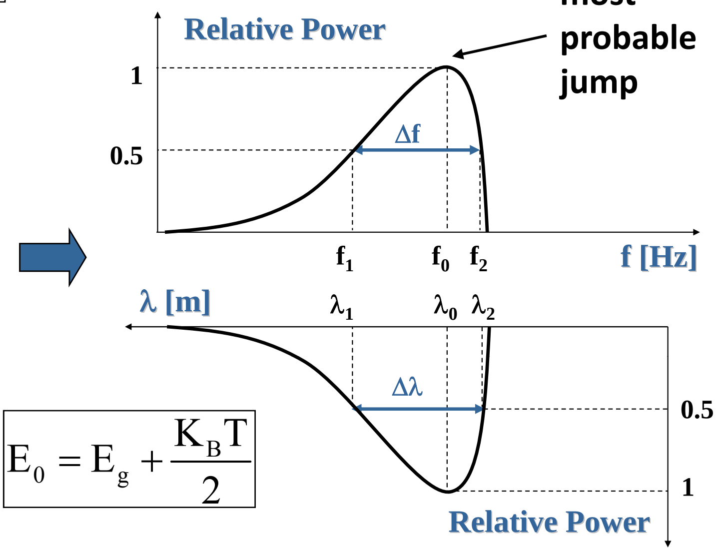
Power Spectral Density

"One of the main characteristics of LED diodes is its spectral width due to the fact that the light is incoherent (spontaneous emission)"

$\boxed{\Delta E \approx 2K_BT}$ Direct band-to-band



$$\boxed{E_0 = E_g + \frac{K_BT}{2}}$$



Peak wavelength - λ_0 (most probable jump):

$$E_0 = E_g + \underbrace{\frac{K_BT}{2}}_{\text{thermal}} = hf_0 = h\frac{c}{\lambda_0} \quad \Rightarrow \quad \boxed{\lambda_0 = \frac{hc}{E_g + K_BT/2}} \approx \frac{hc}{E_g}$$

[J]

Spectral width - $\Delta\lambda$:

$$\boxed{\Delta\lambda \equiv \lambda_2 - \lambda_1} = \frac{c}{f_2} - \frac{c}{f_1} = \frac{hc}{E_2} - \frac{hc}{E_1} = hc\left[\frac{E_1 - E_2}{E_1E_2}\right]$$

$\Delta E << E_0$

$E_c \approx E_0$

$$= hc\frac{\Delta E}{\left(E_c - \frac{\Delta E}{2}\right)\left(E_c + \frac{\Delta E}{2}\right)} = hc\frac{\Delta E}{E_c^2 - \left(\frac{\Delta E}{2}\right)^2} \approx hc\frac{\Delta E}{E_0^2}$$

$$\approx hc\frac{2K_BT}{E_g^2} \approx \boxed{\frac{2K_BT}{hc}\lambda_0^2}$$

$\Delta E \approx 2K_BT \quad E_0 = E_g + \frac{K_BT}{2} \approx E_g$

$$\Delta\lambda \approx \frac{2K_BT}{hc}\lambda_0^2 = \underbrace{\frac{2K_BT}{hc}\lambda_0}_{0.03-0.06}\lambda_0 \Rightarrow \Delta\lambda \rightarrow$$ 30 – 100 nm

ΔE_{LED}~ 3-4 K_BT/q

Temperature Effect :

$$E_0 = E_g(T) + \frac{K_BT}{2} \rightarrow \lambda_0(T)$$

$$\Delta\lambda(T) \approx \frac{2K_BT}{hc}\lambda_0^2(T)$$

$\Delta\lambda_{LED}$~ 0.3-0.4 nm/ºC

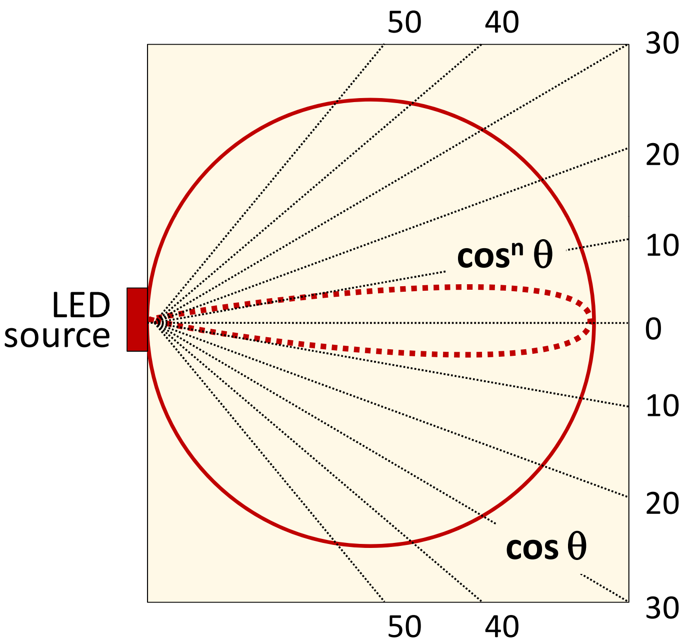
Incoherent Light :

spontaneous emission → photons with random frequency, phase, and direction (incoherent light)

Bose-Einstein statistics $$\sigma_m^2 = \langle m\rangle(\langle m\rangle + 1)$$

LED LOSSES

Radiation Diagram



Lambert's Law

$$I(\theta) = I_0\cos\theta$$
$$P(\theta) = P_0\cos\theta$$

$$P_T = 2\pi\int_0^{\pi/2} P(\theta)\sin\theta\partial\theta$$

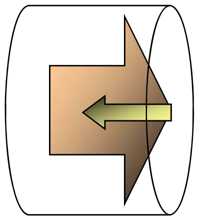
$$P_i = 2\pi\int_0^{\theta_a} P(\theta)\sin\theta\partial\theta$$

$$\boxed{\eta_c \equiv \frac{P_i}{P_T} = \sin^2\theta_a = \left[\frac{NA}{n_0}\right]^2}$$

Refractive Indices Mismatch (reflection)

$$\mathbf{P_{IN}} = 2\pi\int_0^{\theta_a}(1-\mathbf{R})\mathbf{P}(\theta)\sin\theta\cdot\partial\theta$$

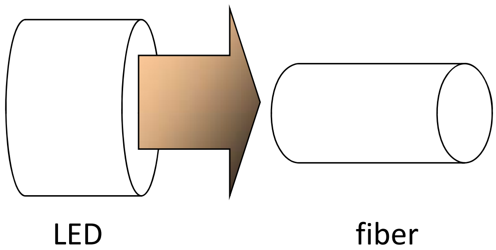
$$\mathbf{R} = \left(\frac{\mathbf{n_{ZA}} - \mathbf{n_0}}{\mathbf{n_{ZA}} + \mathbf{n_0}}\right)^2 \longleftarrow \text{Fresnel's Law}$$



LED/fiber Effective Area Mismatch

$$\mathbf{P_{IN}} = 2\pi\int_0^{\theta_a}\mathbf{L}\mathbf{P}(\theta)\sin\theta\cdot\partial\theta$$

$$\mathbf{L} = \left(\frac{\phi_{\mathbf{fibra}}}{\phi_{\mathbf{LED}}}\right)^2 \longleftarrow \varphi_{\mathbf{LED}} > \varphi_{\mathbf{fiber}}$$



LED DYNAMICS

"The way the carrier equilibrium is restored after a current fluctuation can be modeled by what is known as LED's **rate equation**"

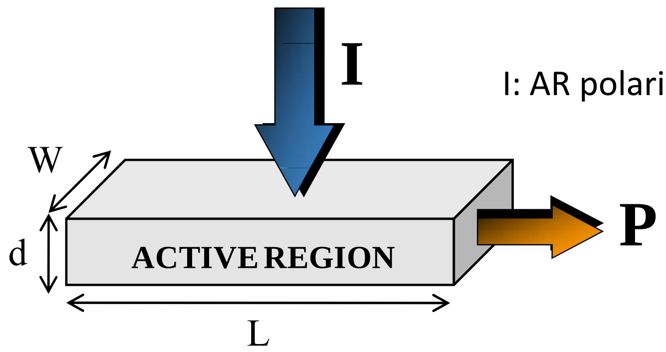
$$\left\{\begin{matrix}\text{carrier}\\ \text{density}\\ \text{variation}\end{matrix}\right\} = \left\{\begin{matrix}\text{carrier}\\ \text{density}\\ \text{generation}\\ \text{rate}\end{matrix}\right\} - \left\{\begin{matrix}\text{carrier density}\\ \text{recombination}\\ \text{rate}\end{matrix}\right\}$$

$$\frac{\partial \mathbf{N}}{\partial \mathbf{t}} \qquad \mathbf{I} \qquad \mathbf{N},\tau_{\mathbf{r}}$$

I: electrical current

τ_r: carrier lifetime

N: AR carrier density



I: AR polarization current

LED's rate equation

current density $J \equiv \frac{I}{\text{àrea}} = \frac{I}{WL}$

carrier generation $\equiv \frac{I}{q \cdot V} = \frac{J}{q \cdot d}$

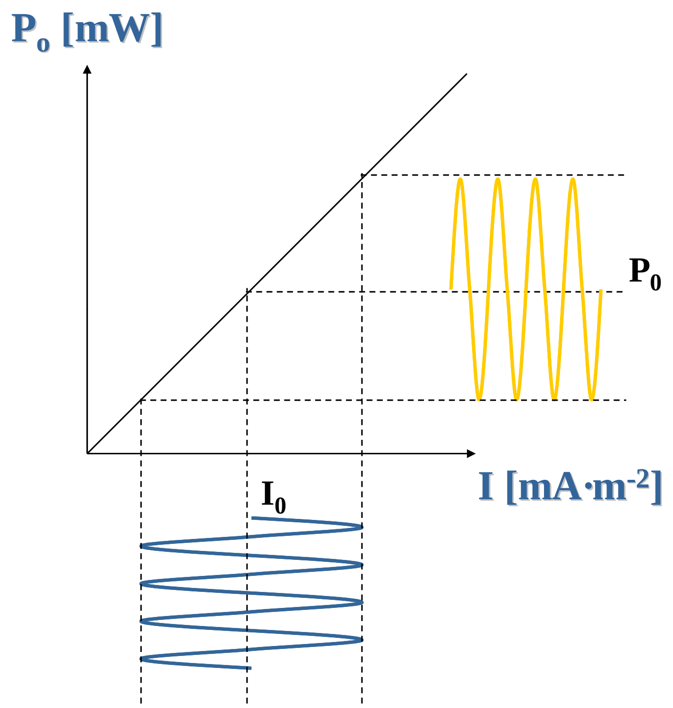
carrier recombination $\equiv \frac{N}{\tau_r}$

$$\boxed{\frac{\partial N}{\partial t} = \frac{I}{qV} - \frac{N}{\tau_r}}$$ [$m^{-3}s^{-1}$]

Unstimulated state $t_0 \begin{cases} N = N_0 \\ I = 0 \end{cases}$

$$\frac{\partial N}{\partial t} = -\frac{N}{\tau_r} \rightarrow N = N_0 e^{-t/\tau_r} \xrightarrow{t \to \infty} 0$$

LED's modulation - sinusoidal modulation



sinusoidal stimulus

$$I(t) \equiv I_0 \left[1 + m_I e^{j(\omega_0 t + \phi)}\right]$$

$$N(t) \equiv N_0 \left[1 + m_N e^{j(\omega_0 t + \phi - \theta_N)}\right]$$

$$P(t) \equiv P_0 \left[1 + m_N e^{j(\omega_0 t + \phi - \theta_N)}\right]$$

I_0: DC electrical component

Optical Power

$$\boxed{P(t) = \eta \frac{N(t)}{\tau_r} V \cdot hf}$$

Modulation Signal

$$\boxed{I(t) = I_0\left[1 + m_I e^{(j\omega_0 t+\phi)} u(t)\right]}$$

$$N(t) = \underbrace{\frac{I_0 \tau_r}{qV}}_{N_0}\left\{\left[1-e^{-t/\tau_r}\right] + \underbrace{\frac{m_I}{1+j\omega_0\tau_r}}_{m_N} e^{j\phi}\left[e^{j\omega_0 t} - e^{-t/\tau_r}\right]u(t)\right\}$$

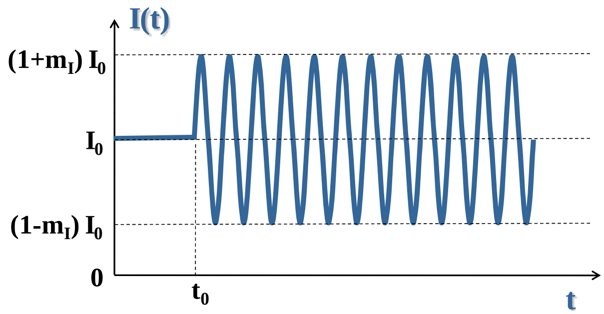
modulation index

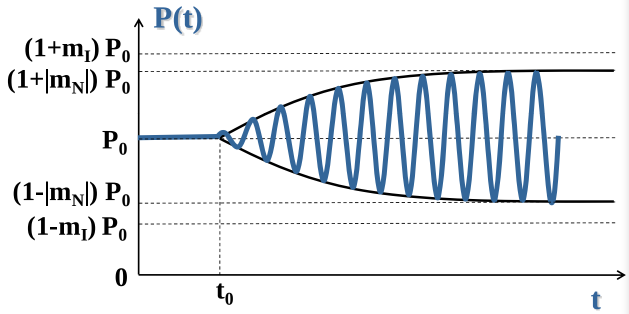
$$\left|m_N\right| = \frac{m_I}{\sqrt{1+(\tau_r\omega_0)^2}}$$

$$\theta_N = tg^{-1}\{-\tau_r\omega_0\}$$

$$P(t) = \eta\frac{N(t)}{\tau_r} V \cdot hf$$

$$\boxed{P(t) = \underbrace{\eta\frac{hf}{q}I_0}_{P_0}\left\{\left[1-e^{-t/\tau_r}\right] + \underbrace{\frac{m_I}{1+j\omega_0\tau_r}}_{m_N} e^{j\phi}\left[e^{j\omega_0 t} - e^{-t/\tau_r}\right]u(t)\right\}} \xrightarrow{t\to\infty} P_0\left\{1 + m_N e^{(j\omega_0 t+\phi)}\right\}$$





LED's Transfer Function

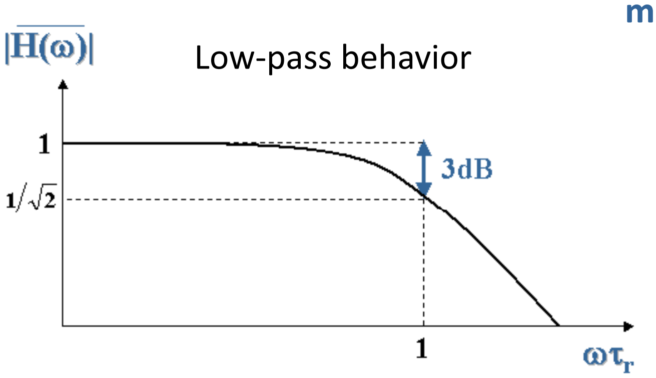
$$H(\omega_0) \equiv \frac{\Delta P}{\Delta I}$$

$$P(t) = P_0\left[1 + \frac{m_I}{1+j\omega_0\tau_r}e^{j\omega_0 t}\right] \rightarrow \Delta P = P_0\frac{m_I}{1+j\omega_0\tau_r}e^{j\omega_0 t}$$

$$I(t) = I_0\left[1 + m_I e^{j\omega_0 t}\right] \rightarrow \Delta I = I_0 m_I e^{j\omega_0 t}$$

$$P_0 = \eta\frac{I_0}{q}hf$$

$$\boxed{H(\omega_0) = \eta\frac{hf}{q}\frac{1}{1+j\omega_0\tau_r}}$$



modulation cutoff frequency

$$\left|H(\omega_0)\right| = \eta\frac{hf}{q}\frac{1}{\sqrt{1+(\omega_0\tau_r)^2}}$$

$$\omega_0 = \frac{1}{\tau_r} \rightarrow \boxed{f_{3dB} = \frac{1}{2\pi\tau_r}}$$

typically: 10–100 MHz

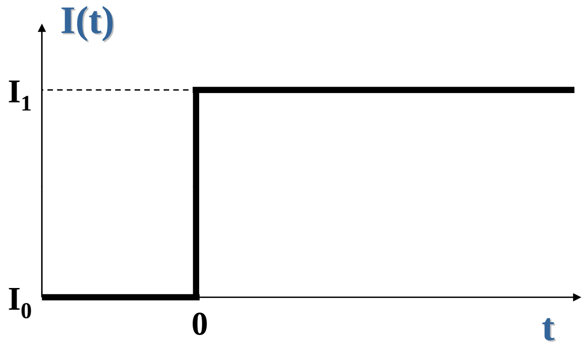
Digital Modulation

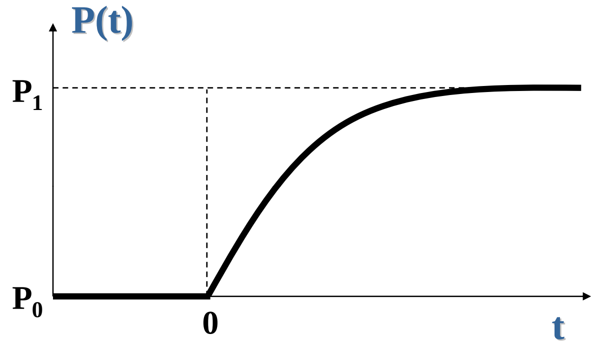
graó de corrent

$$I(t) \equiv I_0 + \left[I_1 - I_0\right] \cdot u(t)$$

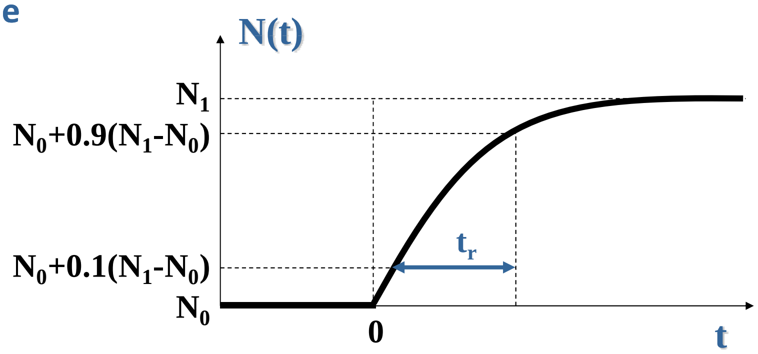
$$N(t) = \underbrace{\frac{I_0}{qV}\tau_r}_{N_0} + \underbrace{\frac{I_1 - I_0}{qV}\tau_r}_{N_1 - N_0}\left(1 - e^{-t/\tau_r}\right) \cdot u(t)$$

$$P(t) = \underbrace{\eta\frac{I_0}{q}hf}_{P_0} + \underbrace{\eta\frac{I_1 - I_0}{q}hf}_{P_1 - P_0}\left(1 - e^{-t/\tau_r}\right) \cdot u(t)$$





Response Time



$$N(t) = N_0 + \left(N_1 - N_0\right)\left(1 - e^{-t/\tau_r}\right)$$

$$N_f - N_0 = \left(N_1 - N_0\right)\left(1 - e^{-t_f/\tau_r}\right)$$

$$e^{-t_f/\tau_r} = 1 - \frac{N_f - N_0}{N_1 - N_0} = \frac{N_1 - N_f}{N_1 - N_0}$$

$$t_f = \tau_r \ln\left(\frac{N_1 - N_0}{N_1 - N_f}\right) = \tau_r \ln\left(\frac{P_1 - P_0}{P_1 - P_f}\right)$$

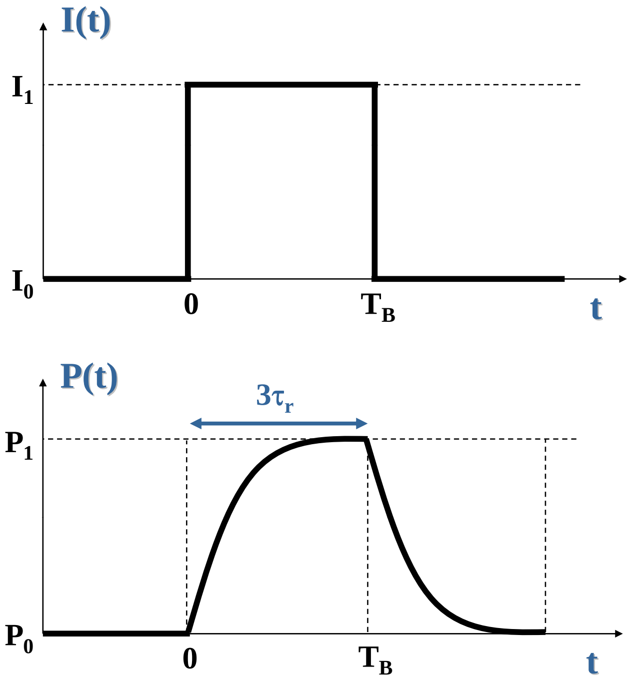
$$t_{0.1} = \tau_r \ln\left(\frac{N_1 - N_0}{N_1 - \left(0.1\left(N_1 - N_0\right) + N_0\right)}\right)$$

$$t_{0.9} = \tau_r \ln\left(\frac{N_1 - N_0}{N_1 - \left(0.9\left(N_1 - N_0\right) + N_0\right)}\right)$$

$$t_r = t_{0.9} - t_{0.1} = \tau_r \ln\left(0.9/0.1\right)$$

$$f_{3dB} = \frac{1}{2\pi\tau_r} = \frac{1}{2\pi t_r}\ln\left(0.9/0.1\right)$$

Maximum Modulation Speed



LED's response time limits the modulation speed

$$f_{3dB,LED} = \frac{1}{2\pi\tau_r} \qquad f_{3dB,NRZ} = \frac{R_B}{2}$$

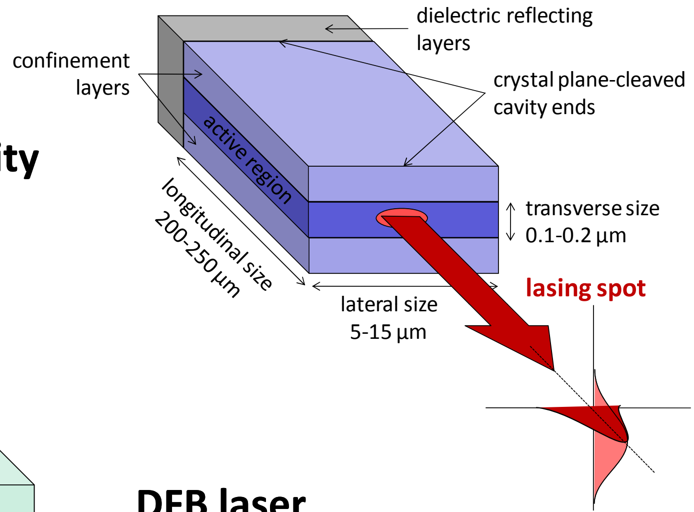
$$f_{3dB,LED} \geq f_{3dB,NRZ}$$

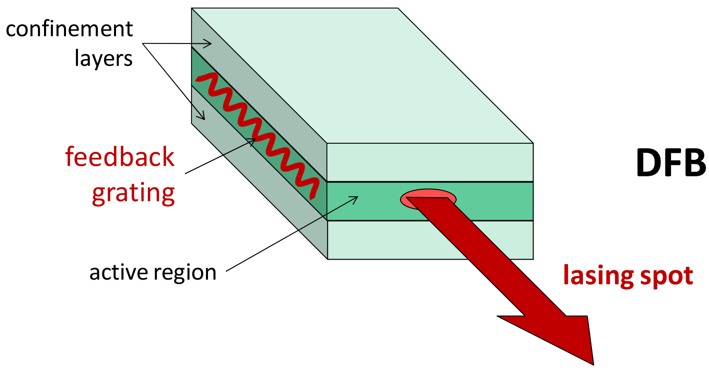
$$\frac{1}{2\pi\tau_r} \geq \frac{R_B}{2} \rightarrow \boxed{R_B \leq \frac{1}{\pi\tau_r}}$$

typically: $\tau_r \sim 10\text{ns}$

LASER (Light Amplification by Estimulated Emission of Radiation)

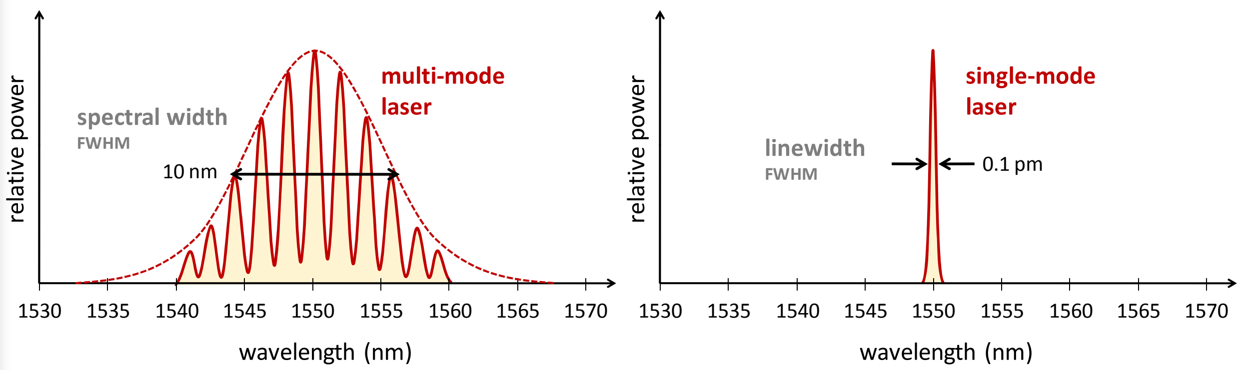
Fabry-Perot resonant cavity





DFB laser

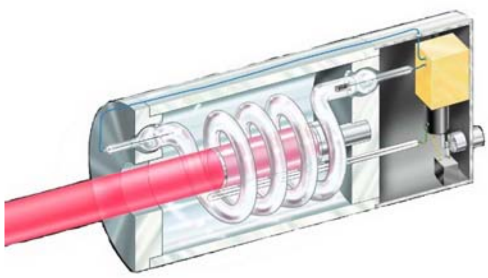
LASER Main Figures

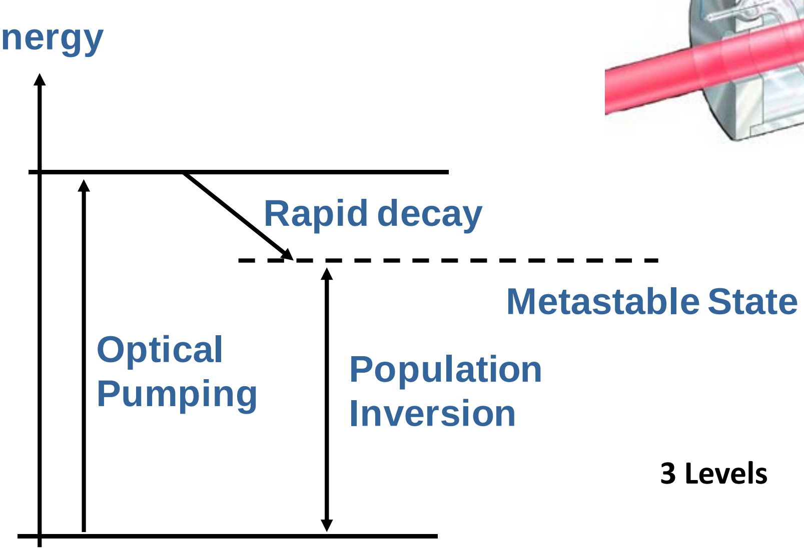


- BW up to 10 GHz → R_B up to 10 Gb/s
- $\Delta\lambda$ very narrow → 10 MHz (0.08 pm)
- P_{OUT} high → 3 dBm

TYPES OF LASERS

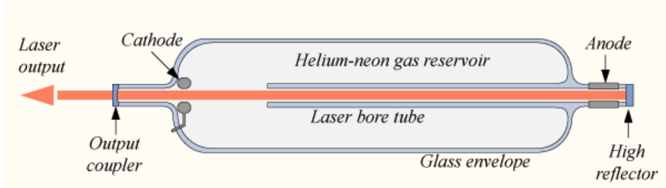
Solid-State Lasers: Ruby

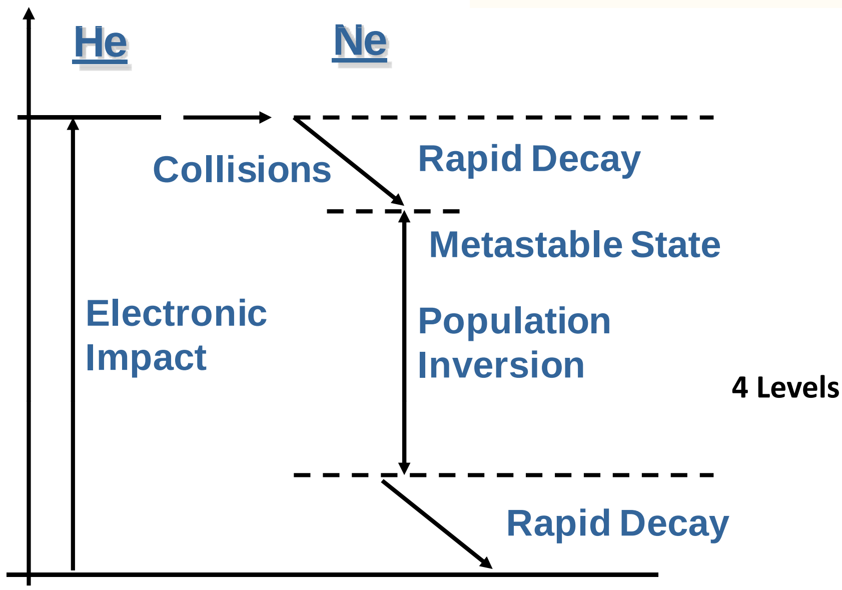




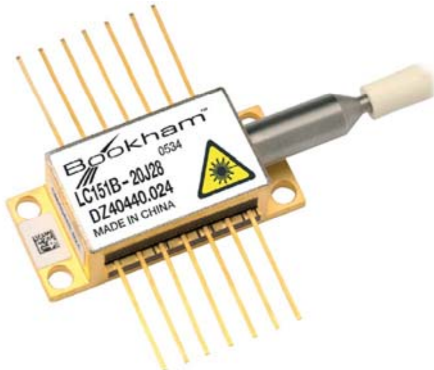
3 Levels

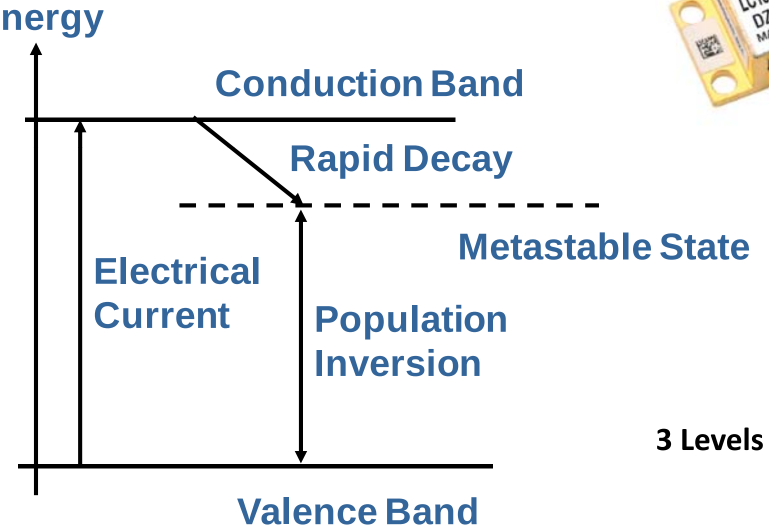
Gas Lasers: He-Ne





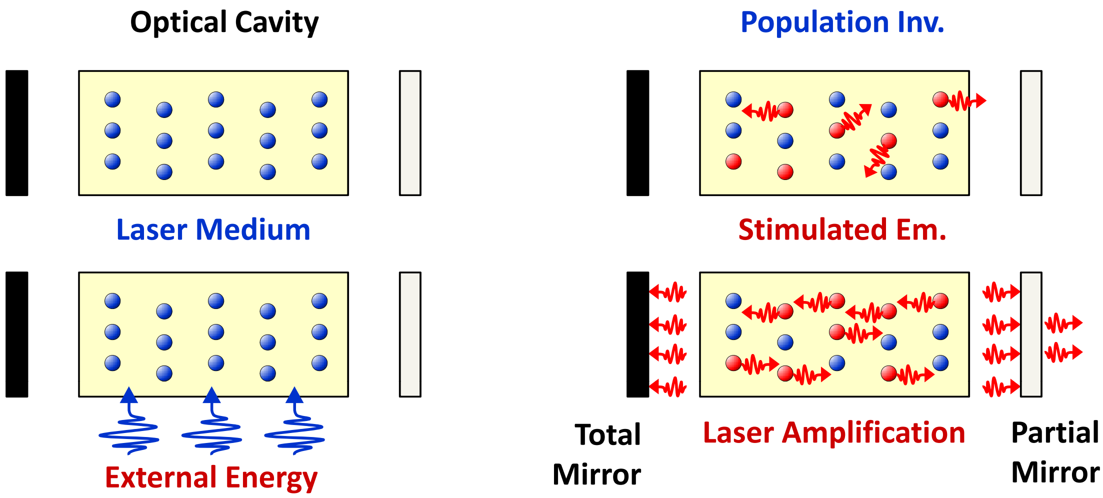
Semiconductor Lasers: GaAs





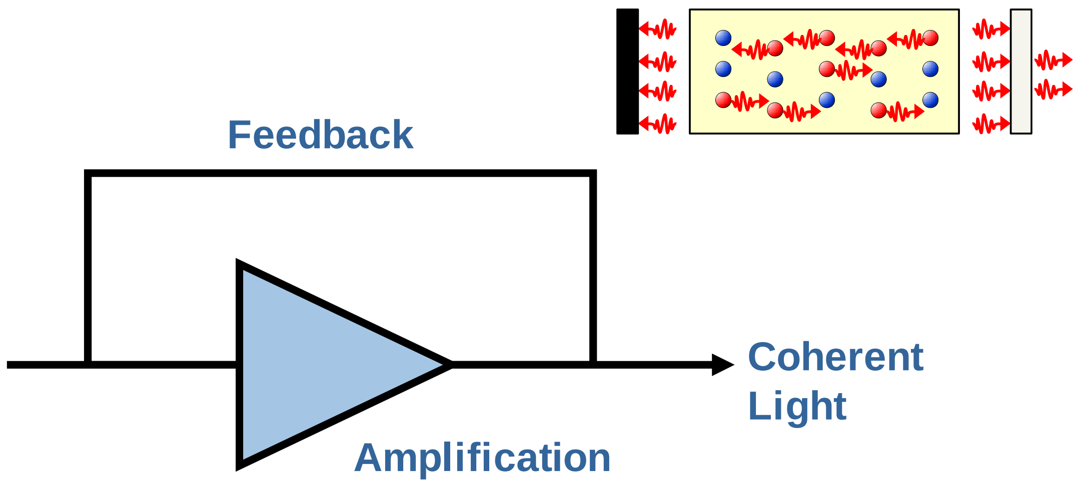
WORKING PRINCIPLE

"The LASER source consists of an optical resonant cavity based on the **stimulated emission** process and provides coherent light"



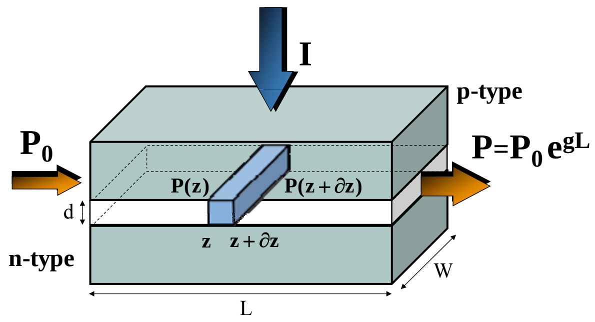
Equivalent Model

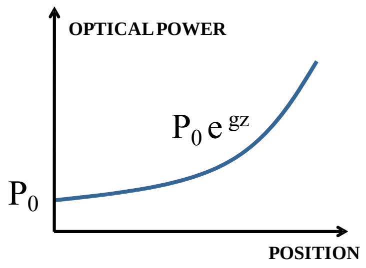
"The LASER can be modeled as an amplification system with feed-back"



Condition → Gain > Losses

MATERIAL GAIN



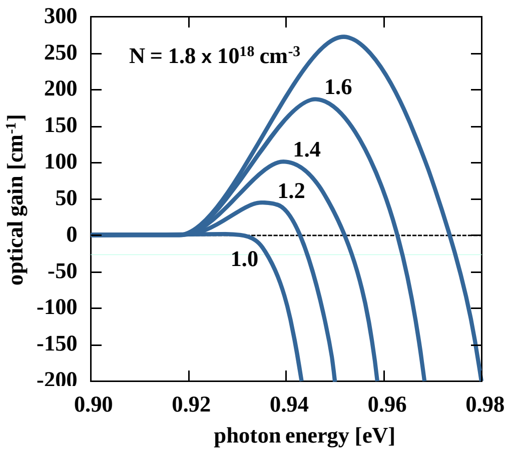


$$\frac{\partial P}{\partial z} = gP$$

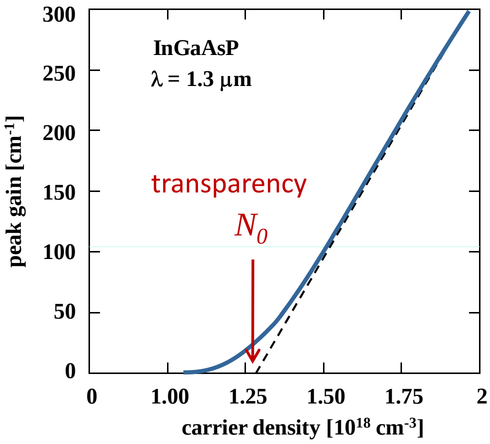
$$\Updownarrow$$

$$P(z) = P_0 e^{gz}$$

$g \equiv$ material unity gain [m^{-1}]



$$\lambda = \frac{hc}{q} E[eV]$$

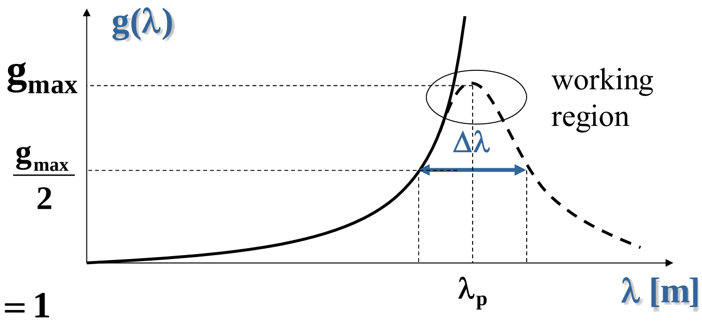


material unity gain [m^{-1}]

$$\boxed{g = (N_2 - N_1)\frac{\lambda^2}{\tau_r 8\pi n^2}\upsilon(f)}$$

$$g \propto \lambda^2$$

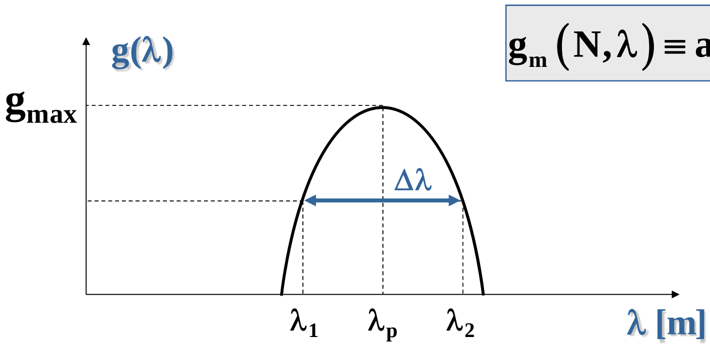
lineshape function $\upsilon(f) \longleftarrow \int_0^\infty \upsilon(f)\,\partial f = 1$



Material Gain per unit length

mathematical model

$$g_m(N,\lambda) \equiv a(N - N_0) - \gamma(\lambda - \lambda_p)^2 \quad [m^{-1}]$$

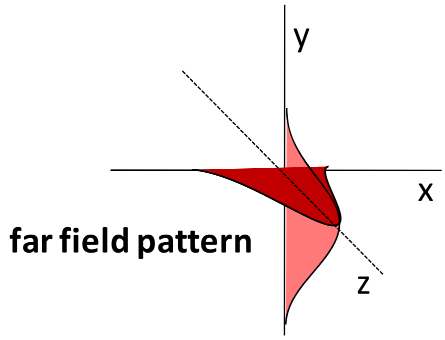


a: gain coefficient
γ: curvature factor
λ_p: peak wavelength
N_0: transparence level
g_p: peak gain

Confinement Factor

"Energy fraction inside the active region "

$$\Gamma \equiv \frac{E|_{AR}}{E|_{Total}} \leq 1$$



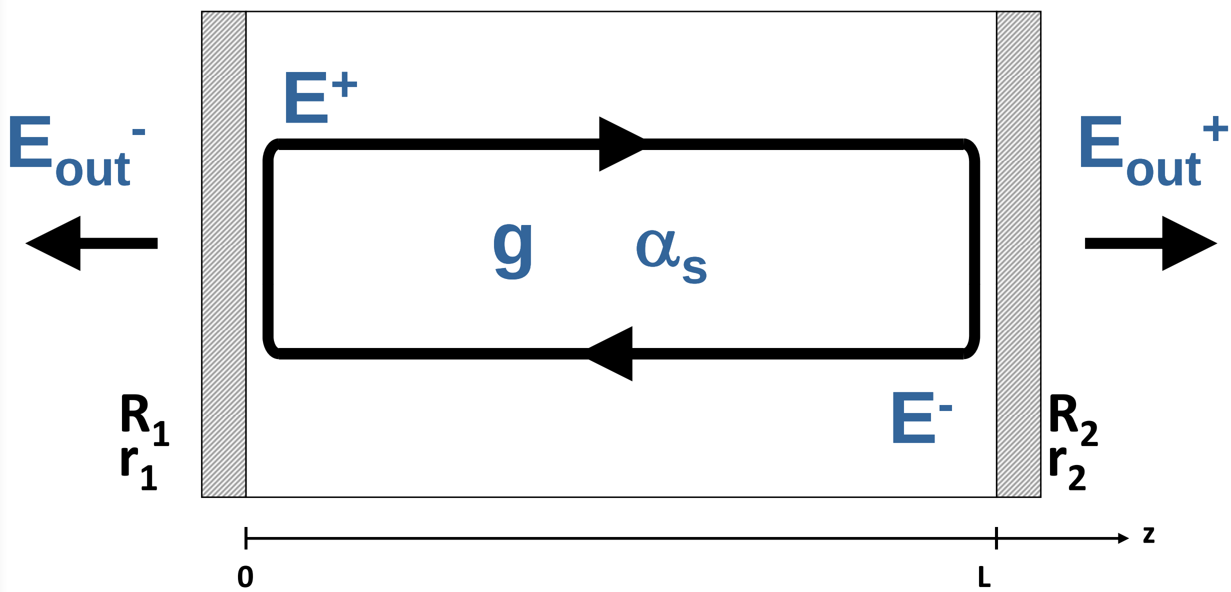
$$g(\lambda) \equiv \Gamma g_m(\lambda) = \Gamma a(N - N_0) - \Gamma\gamma(\lambda - \lambda_p)^2$$

Net Material Gain per unit length

$$\boxed{g_n(\lambda) \equiv g(\lambda) - \alpha_s = \Gamma a(N - N_0) - \Gamma\gamma(\lambda - \lambda_p)^2 - \alpha_s}$$

OSCILLATION CONDITION

FABRY-PEROT CAVITY



R_i: Reflectivity
r_i: Reflectance

g: Material Gain Coef. [m^{-1}]
α_s: Scattering Loss Coef. [m^{-1}]

Propagation Equations (plane wave)

$$E^+(z) = E_0^+ e^{\frac{1}{2}(g-\alpha_s)z} e^{-j\beta z} e^{-j\omega t}$$

$$E^-(z) = E_L^- e^{\frac{1}{2}(g-\alpha_s)(L-z)} e^{-j\beta(L-z)} e^{-j\omega t}$$

Boundary Conditions

$$E^+(0) = r_1 E^-(0)$$

$$E^-(L) = r_2 E^+(L)$$

$$E^+(0) = E_0^+ \cancel{e^{-j\omega t}} = r_1 E^-(0) = r_1\left(E_L^- e^{\frac{1}{2}(g-\alpha_s)L} e^{-j\beta L} \cancel{e^{-j\omega t}}\right) \rightarrow E_0^+ = r_1 E_L^- e^{\frac{1}{2}(g-\alpha_s)L} e^{-j\beta L}$$

$$E^-(0) = E_L^- e^{\frac{1}{2}(g-\alpha_s)L} e^{-j\beta L} e^{-j\omega t}$$

$$E^-(L) = E_L^- \cancel{e^{-j\omega t}} = r_2 E^+(L) = r_2\left(E_0^+ e^{\frac{1}{2}(g-\alpha_s)L} e^{-j\beta L} \cancel{e^{-j\omega t}}\right) \rightarrow E_L^- = r_2 E_0^+ e^{\frac{1}{2}(g-\alpha_s)L} e^{-j\beta L}$$

$$E^+(L) = E_0^+ e^{\frac{1}{2}(g-\alpha_s)L} e^{-j\beta L} e^{-j\omega t}$$

$$\cancel{E_0^+} = r_1 r_2 \cancel{E_0^+} e^{(g-\alpha_s)L} e^{-j2\beta L} \rightarrow \boxed{r_1 r_2 e^{(g-\alpha_s)L} e^{-j2\beta L} = 1}$$

Module Oscillation Condition

$$1 = r_1 r_2 e^{(g_{th}-\alpha_s)L}$$

g_{th}: threshold gain

α_c: cavity losses

α_t: total losses

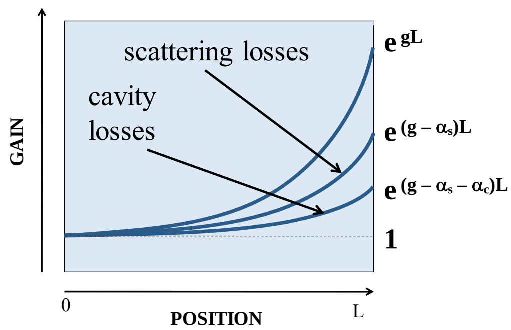
$$\frac{1}{r_1 r_2} = e^{(g_{th}-\alpha_s)L} \rightarrow \ln\left(\frac{1}{r_1 r_2}\right) = (g_{th}-\alpha_s)L$$

$$g_{th} = \alpha_s + \frac{1}{L}\ln\left(\frac{1}{r_1 r_2}\right) = \alpha_s + \underbrace{\frac{1}{2L}\ln\left(\frac{1}{R_1 R_2}\right)}_{\alpha_c} \equiv \alpha_t$$

$$R_i = |r_i|^2$$

Threshold Gain [m^{-1}]

$$g \geq g_{th} = \alpha_s + \frac{1}{2L}\ln\left(\frac{1}{R_1 R_2}\right)$$



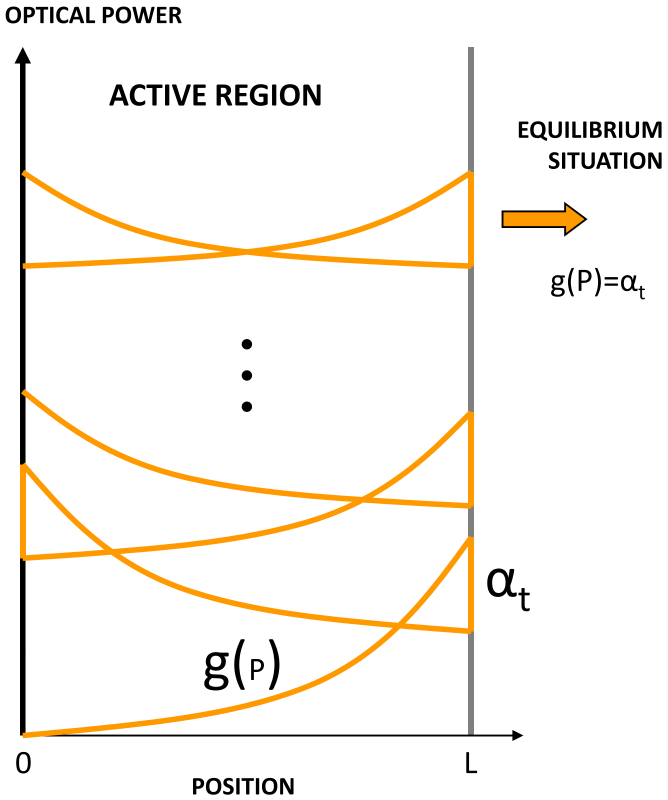
Gain Saturation

$$\boxed{g(P)=\frac{g_0}{1+P/P_{sat}}}$$

$g<\alpha_t \rightarrow$ Unstable Situation (No Oscillation)

$g=\alpha_t \rightarrow$ Stable Situation (Oscillation)

$g>\alpha_t \rightarrow$ Unstable Situation (Saturation)



Phase Oscillation Condition

$$1=e^{-j2\beta L}$$

$$2\beta L=m\cdot 2\pi$$

$$2\frac{2\pi}{\lambda}nL=m\cdot 2\pi \rightarrow L=m\frac{\lambda_m}{2n}=m\frac{c}{2nf_m}$$

$$\boxed{f_m=m\frac{c}{2nL}}$$

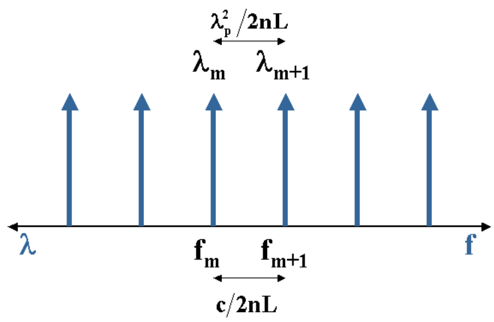
Cavity Resonance Frequencies

Oscillation Modes (longitudinal)

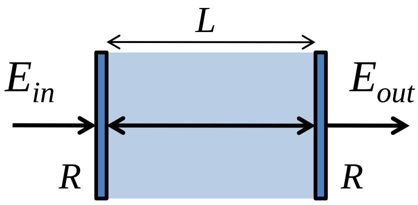
$$\Delta f=\frac{c}{2nL}$$

$$\frac{\Delta\lambda}{\lambda_p}\approx\frac{\Delta f}{f_p}$$

$$\Delta\lambda\approx\Delta f\cdot\frac{\lambda_p^2}{c}=\frac{\lambda_p^2}{2nL}$$



Fabry-Perot Interferometer (Etalon)



$$E_{out} = (1-R^2)e^{-j\beta L}E_{in} + R^2 e^{-j2\beta L}(1-R^2)e^{-j\beta L}E_{in} + R^4 e^{-j4\beta L}(1-R^2)e^{-j\beta L}E_{in} + \vdots$$

$$E \equiv \frac{4R^2}{(1-R^2)^2}$$

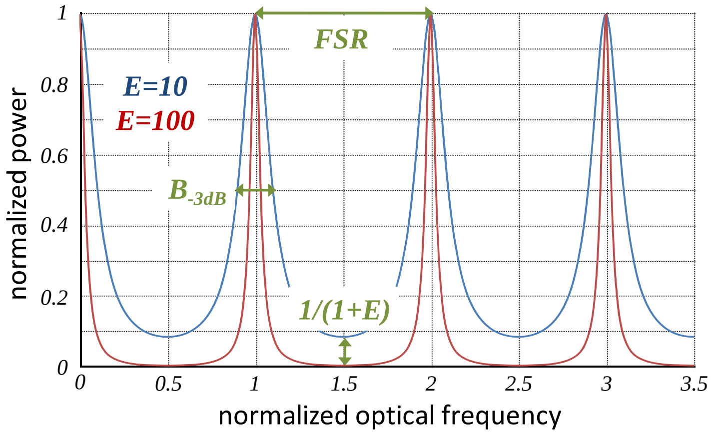
$$E_{out} = (1-R^2)e^{-j\beta L}E_{in}\sum_{k=0}^{\infty} R^{2k}e^{-j2k\beta L} = \overbrace{\frac{(1-R^2)e^{-j\beta L}}{(1-R^2 e^{-j2\beta L})}}^{H(f)} E_{in} \rightarrow \boxed{|H(f)|^2 = \frac{1}{1+E\sin^2(\beta L)}}$$

Contrast $C \equiv |H|^2_{\max} / |H|^2_{\min} = 1+E$

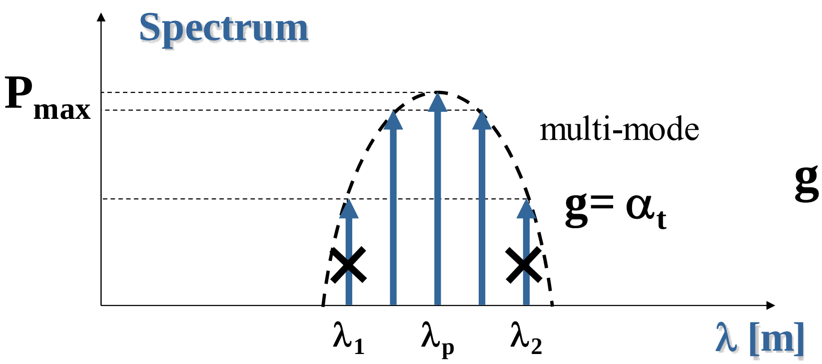
Free-Spectral Range $FSR = \dfrac{c}{2nL}$

3dB-Bandwidth $B_{-3dB} = \dfrac{c}{\pi n L\sqrt{E}}$

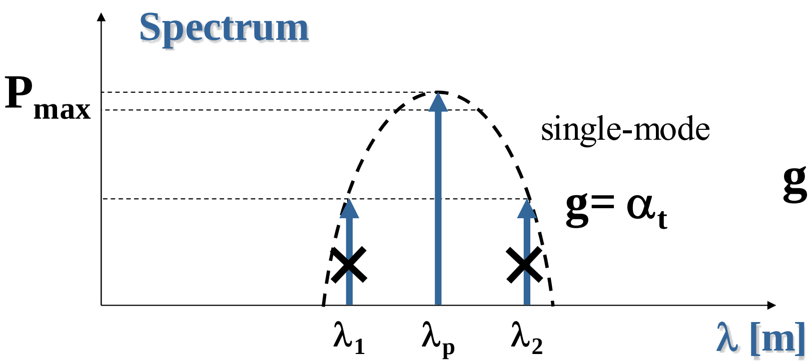
Finesse $F = \dfrac{FSR}{B_{-3dB}} = \dfrac{\pi}{2}\sqrt{E}$



Combined Effect

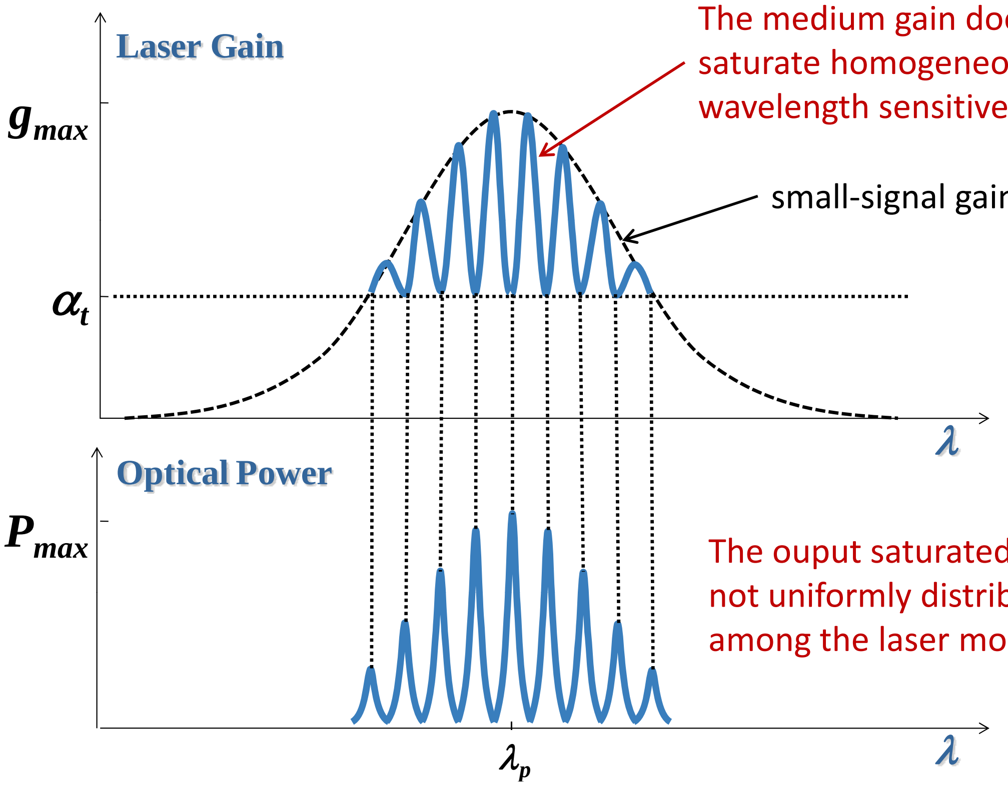


$$g(\lambda) \geq g_{th} \rightarrow \Delta\lambda$$



$$g'(\lambda) \geq g_{th} \rightarrow \Delta\lambda'$$

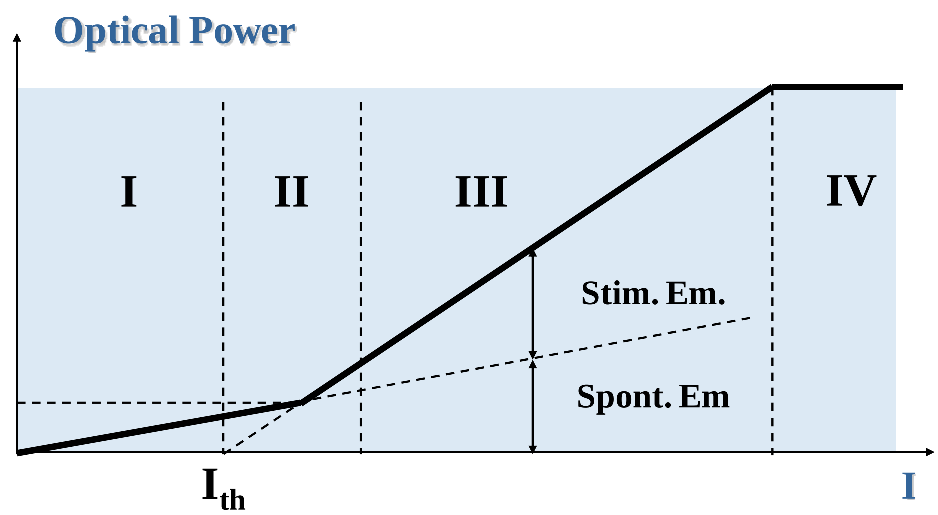
Inhomogeneouly Broadened Medium (Spectral Hole Burning)



The medium gain does not saturate homogeneously, it is wavelength sensitive.

The ouput saturated power is not uniformly distributed among the laser modes.

LIGHT-CURRENT CHARACTERISTIC



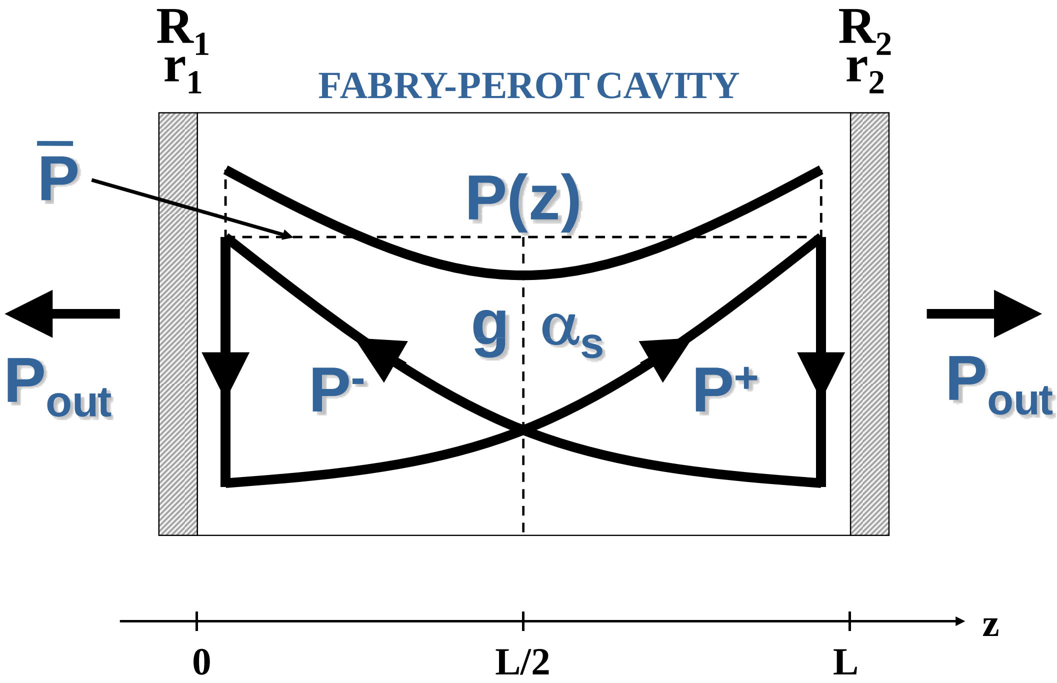
I → LED-like light, Spontaneous Emission

II → Amplified LED-like light, Amplified Spont. Em.

III → Laser Effect , Coherent Light, Spontaneous Em.

IV → Saturation

Optical Power in the Laser Cavity



$$g_n = g - \alpha_s \qquad R_1 = R_2 = R \qquad 1 = r_1 r_2 e^{g_n L} = R e^{g_n L}$$

$$P(z) = P^+(z) + P^-(z) = P_0\left[e^{g_n z} + e^{g_n[L-z]}\right] = P_0 e^{g_n L/2}\left[e^{g_n[z-L/2]} + e^{g_n[L/2-z]}\right]$$

$$P^+(z) = P_0 e^{g_n z} \qquad e^{g_n[z-L/2]} \approx 1 + g_n[z-L/2]$$

$$P^-(z) = P_0 e^{g_n[L-z]} \qquad e^{g_n[L/2-z]} \approx 1 + g_n[L/2-z] = 1 - g_n[z-L/2]$$

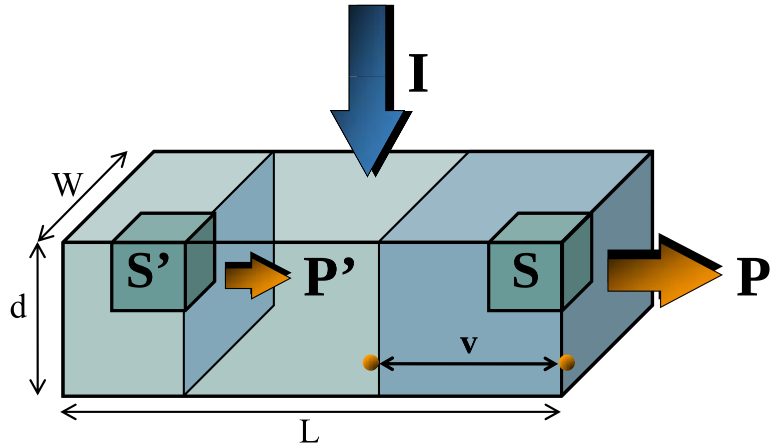
$$P(z) \approx P_0 e^{g_n L/2}\left[1 + \cancel{g_n[z-L/2]} + 1 - \cancel{g_n[z-L/2]}\right] = 2P_0 e^{g_n L/2} \equiv \overline{P}$$

$$P_{out} = P_0 e^{g_n L}(1-R) \approx \frac{\overline{P}}{2e^{g_n L/2}} e^{g_n L}(1-R) = \frac{\overline{P}}{2} e^{g_n L/2}(1-R) = \frac{\overline{P}}{2}\frac{1-R}{\sqrt{R}}$$

$$R e^{g_n L} = 1 \rightarrow e^{g_n L} = \frac{1}{R} \rightarrow e^{g_n L/2} = \frac{1}{\sqrt{R}}$$

$$\overline{P} \approx \frac{2\sqrt{R}}{1-R} P_{out} \xrightarrow{\text{m modes}} \boxed{P_{out} \approx \sum_m \frac{1-R}{2\sqrt{R}} \overline{P}_m}$$

Photon Density – Optical Power Relationship



$$\frac{\textbf{Fotons}}{\textbf{sec.}} = \frac{P\,[J/s]}{hf\,[J]}$$

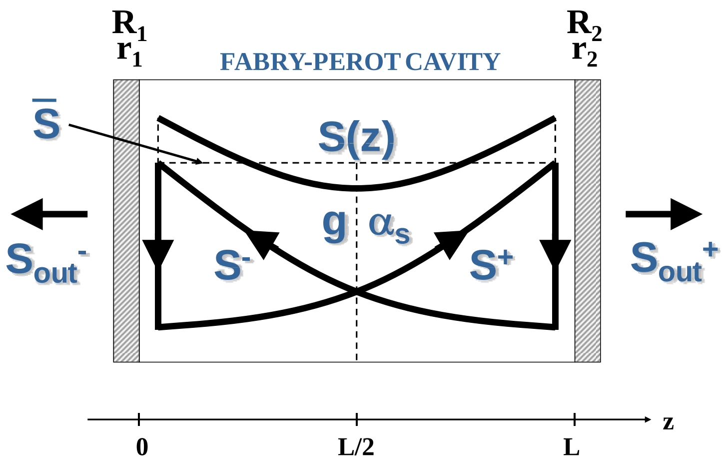
Photon Density in the Active Region

$$S\,[m^{-3}] = \frac{P\,[J/s]}{hf\,[J] \times \frac{c}{n}\,[m/s] \times Wd\,[m^2]}$$

$$\Rightarrow \quad P(z) = P_0\, e^{g_n z} \quad \downarrow \quad S(z) = S_0\, e^{g_n z}$$

Photon Density in the Laser Cavity

$S(z) \approx ct$



Optical Power

$$\bar{P} = \bar{S} \cdot v \cdot Wd \cdot hf$$

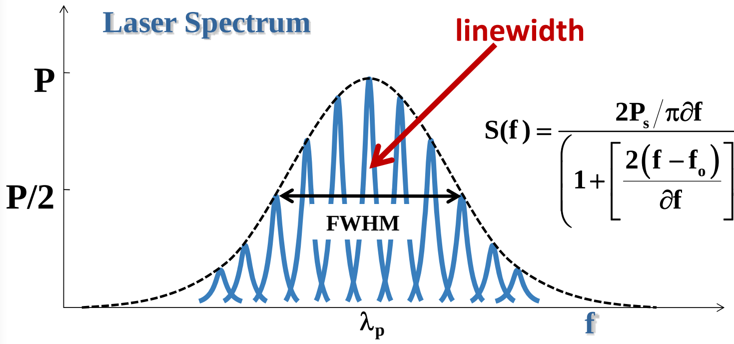
$$P_{out} \approx \left[(1-R)/2\sqrt{R}\right] \bar{P} = \left[(1-R)/2\sqrt{R}\right] \bar{S} \cdot v \cdot Wd \cdot hf$$

$$P_m \approx \left[(1-R)/2\sqrt{R}\right] \bar{S}_m \cdot v \cdot Wd \cdot hf_m$$

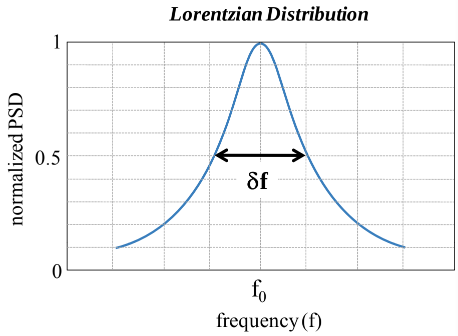
Total Optical Power

$$P_{out} = \sum_m P_m \quad \xrightarrow{f_m \approx f_p} \quad \boxed{P_{out} \approx \sum_m \frac{1-R}{2\sqrt{R}} \cdot \bar{S}_m \cdot v \cdot Wd \cdot hf_p}$$

OPTICAL POWER SPECTRUM



$$S(f) = \frac{2P_s/\pi\partial f}{1 + \left[\frac{2(f - f_o)}{\partial f}\right]^2}$$



Finesse $F \equiv \dfrac{\Delta f}{\partial f}$

1. $\Delta f = \dfrac{c}{2nL}$ $\quad \Delta\lambda \approx \dfrac{\lambda_p^2}{2nL}$ resonance frequencies

2. $g \geq g_{th} = \alpha_s + \dfrac{1}{2L} \ln\left(\dfrac{1}{R_1 R_2}\right)$ gain condition

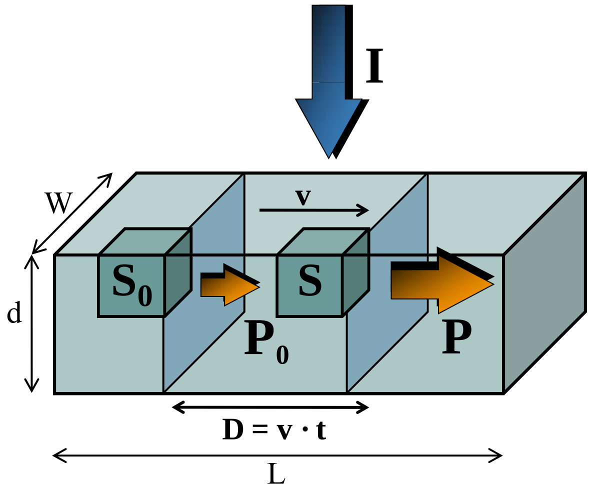
LASER DYNAMICS

"Carrier and Photons concentration can be modeled using two coupled rate equations"

{carrier variation} = {carrier generation rate} - {spont. emission rate} - {stimul. emission rate}

{photon variation} = {stimul. emission rate} - {photon absorp. rate} + {spont. emission fraction}

Temporal evolution of Photon Density



$$\frac{\partial P}{\partial z} = gP \rightarrow \frac{\partial S}{\partial z} = gS$$

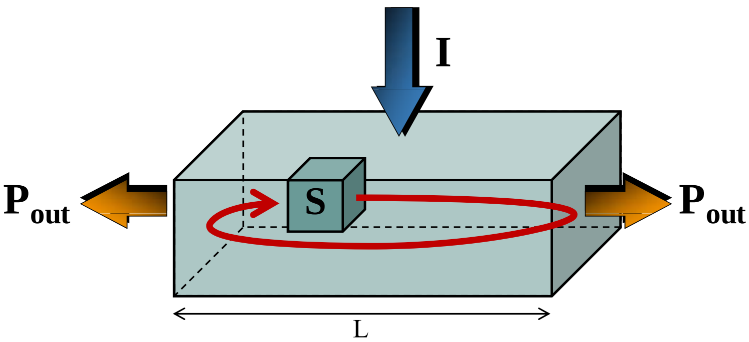
$$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial z}\frac{\partial z}{\partial t} = gS \cdot v$$

$$\boxed{\begin{aligned}\frac{\partial P}{\partial t} &= vg\,P \\ \frac{\partial S}{\partial t} &= vg\,S\end{aligned}}$$

$$P = P_0\,e^{gD} = P_0\,e^{g\cdot v\cdot t} \longrightarrow \frac{\partial P}{\partial t} = g\,v\,P_0\,e^{g\cdot v\cdot t} = g\,v\,P$$

$$S = \frac{P}{hf\cdot v\cdot Wd} \longrightarrow \frac{\partial S}{\partial t} = \frac{1}{hf\cdot v\cdot Wd}\frac{\partial P}{\partial t} = \frac{1}{hf\cdot v\cdot Wd}vgP = vgS$$

Photon Variation in the Cavity



$$S = S_0\,e^{(g-\alpha_s)2L}R^2 = S_0\,e^{(g-\alpha_s)2L}e^{-2\ln\frac{1}{R}} = S_0\,e^{\left(g-\alpha_s-\frac{1}{L}\ln\frac{1}{R}\right)2L} = S_0\,e^{(g-\alpha_t)2L}$$

$$R^2 = e^{2\ln R} = e^{-2\ln\frac{1}{R}} \qquad \alpha_t \equiv \alpha_s - \frac{1}{L}\ln\frac{1}{R}$$

$$S(z) = S_0\,e^{(g-\alpha_t)d} \xrightarrow{d=v\cdot t} S(t) = S_0\,e^{(g-\alpha_t)v\cdot t}$$

$$\Rightarrow \boxed{\frac{\partial S(t)}{\partial t} = v(g-\alpha_t)\cdot S_0\,e^{(g-\alpha_t)v\cdot t} = v(g-\alpha_t)\cdot S(t)}$$

LASER'S RATE EQUATIONS

Carriers $\Rightarrow$ $$\frac{\partial N}{\partial t} = \frac{I}{qV} - \frac{N}{\tau_r} - v\sum_i g_i S_i \quad [m^{-3}s^{-1}]$$

Photons $\Rightarrow$ $$\frac{\partial S_i}{\partial t} = v \cdot g_i S_i - v \cdot \alpha_t S_i + \beta \frac{N}{\tau_r} \quad [m^{-3}s^{-1}]$$

photon lifetime $$\tau_p \equiv \frac{1}{v \cdot \alpha_t} \quad [s]$$

N: carrier density in the AR
S: photon density in the AR
g_i: net espont. emission gain
sub-index i: mode # i

I: electrical current intensity
τ_r: carrier lifetime
β: spontaneous emission coeff.
α_t: cavity total losses

Static Behavior

$$\frac{\partial N}{\partial t} = \frac{I}{qV} - \frac{N}{\tau_r} - v \cdot gS = 0$$

$$\frac{\partial S}{\partial t} = v \cdot gS - v \cdot \alpha_t S + \cancel{\beta \frac{N}{\tau_r}} = 0$$

$\Leftarrow$ single-mode cavity

stationary regime $\frac{\partial}{\partial t} = 0$

$\beta <<$

$$v \cdot gS - v \cdot \alpha_t S = 0 \rightarrow g = \alpha_t$$

$$\lambda_{emission} = \lambda_p$$

$$\Gamma a (N - N_0) = \alpha_t \rightarrow N = N_0 + \frac{\alpha_t}{\Gamma a}$$

$$g = \Gamma a (N - N_0) - \cancel{\Gamma\gamma(\lambda - \lambda_p)^2}$$

$$\tau_p \equiv \frac{1}{v \cdot \alpha_t}$$

$+$ $$\frac{I}{qV} - \frac{N}{\tau_r} - v \cdot \alpha_t S = 0$$

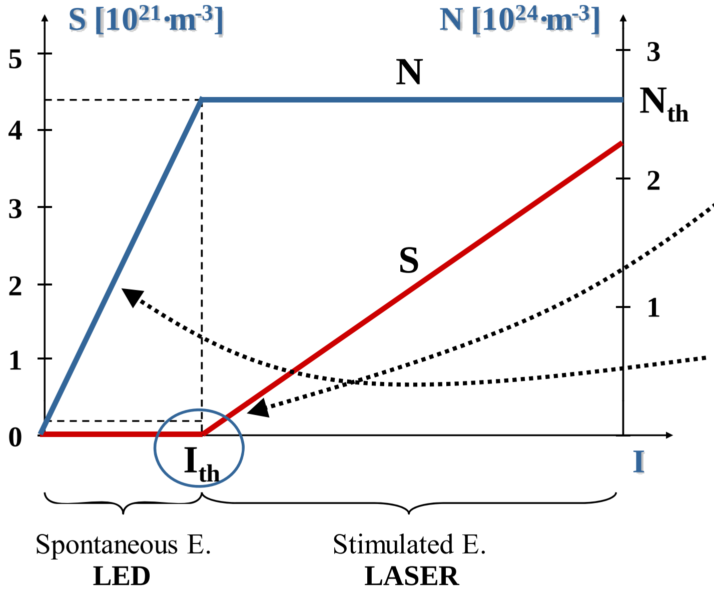
$$S = \frac{I}{v \cdot \alpha_t qV} - \frac{N}{v \cdot \alpha_t \tau_r} = \frac{I}{qV}\tau_p - \frac{\tau_p}{\tau_r} N = \frac{I}{qV}\tau_p - \frac{\tau_p}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right]$$

Carriers Constant with J

$$N = N_0 + \frac{\alpha_t}{\Gamma a} \equiv N_{th} \quad [m^{-3}]$$

Photons Lineal with I

$$S = \frac{I}{qV}\tau_p - \frac{\tau_p}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right] \quad [m^{-3}]$$



inflexion point

$S = 0$

$N = N_{th}$

LED behavior

$$\frac{I}{qV} = \frac{N}{\tau_r} \rightarrow \frac{I_{th}}{qV} = \frac{N_{th}}{\tau_r}$$

I_{th}: threshold current

N_{th}: threshold carrier density

Laser Activation Condition

$$\frac{I_{th}}{qV} = \frac{N_{th}}{\tau_r} \rightarrow \boxed{I_{th} = \frac{qV}{\tau_r}N_{th} = \frac{qV}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right]}$$

Threshold Current

$$I \geq \frac{qV}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right] = \underbrace{\frac{qV}{\tau_r}N_0}_{\text{T.Medi}} + \underbrace{\frac{qV}{\tau_r}\frac{\alpha_t}{\Gamma a}}_{\text{P. Totals}}$$

The minimum current has to compensate for the Medium Transparency and Total Losses

Photon Density

$$S = \frac{I}{qV}\tau_p - \frac{\tau_p}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right] = \frac{I}{qV}\tau_p - \frac{I_{th}}{qV}\tau_p = \frac{\tau_p}{qV}\left[I - I_{th}\right]$$

$$I_{th} = \frac{qV}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right] \rightarrow \frac{1}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right] = \frac{I_{th}}{qV}$$

Output Optical Power

$$P_{out} = \left(\left(1-R\right)/2\sqrt{R}\right)\cdot S\cdot v\cdot W\cdot d\cdot hf$$

$$\boxed{P_{out} = \left(\left(1-R\right)/2\sqrt{R}\right)\frac{hf}{q\alpha_t L}\left(I-I_{th}\right)}$$

$$I \geq I_{th} \qquad \tau_p \equiv \frac{1}{v\cdot\alpha_t}$$

$$S \approx \frac{\tau_p}{qV}\left(I-I_{th}\right)$$

$$N = N_{th}$$

active region length influence

$$J = \frac{I}{WL}$$

$$P_{out} = \left(\left(1-R\right)/2\sqrt{R}\right)\frac{hf}{q\alpha_t \cancel{L}}W\cancel{L}\left(J-J_{th}\right) \xrightarrow{L\to 0} 0$$

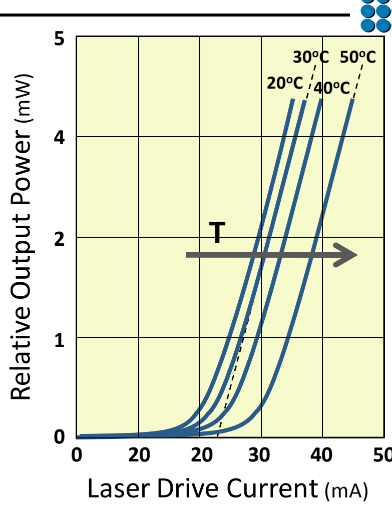
$$J_{th} = \frac{qd}{\tau_r}\left[N_0 + \frac{\alpha_t}{\Gamma a}\right] \xrightarrow{L\to 0} \infty$$

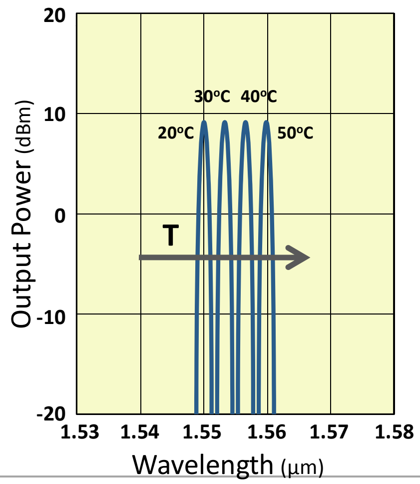
$$\alpha_t \equiv \alpha_s + \frac{1}{L}\ln\frac{1}{R}$$

Temperature Effect

Threshold Current & Optical Power

$$T\uparrow \longrightarrow I_{th}\uparrow\;,\;\; P_{out}\downarrow$$





Wavelenth / frequency

$$T\uparrow \longrightarrow \lambda_c\uparrow\;,\;\; f_c\downarrow$$

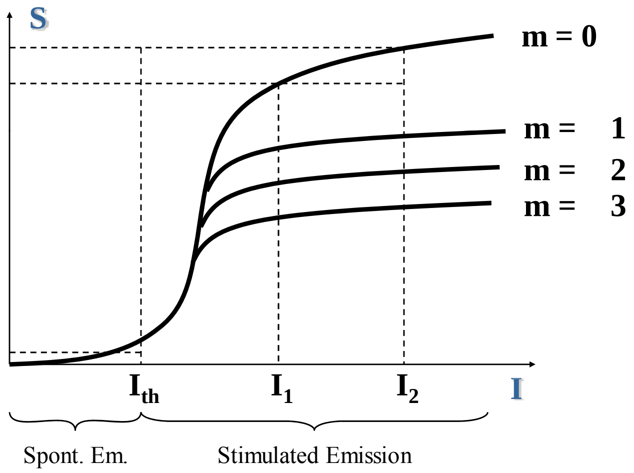
Modal Condition of a Laser Diode

1. **gain profile**

$$g_m(\lambda) \equiv a(N - N_0) - \gamma(\lambda - \lambda_p)^2$$

$$|\lambda - \lambda_p| \uparrow \Rightarrow g_m(\lambda) \downarrow \Rightarrow S_m \downarrow$$

"The sharper the gain profile the more single-mode"

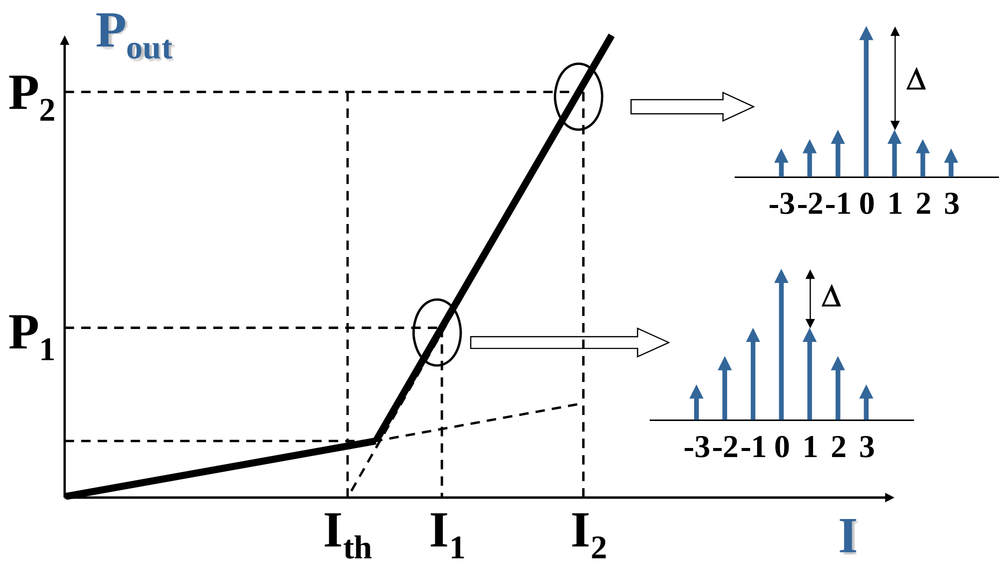


2. **bias current**

$I < I_{th}$ ⇨ Spontaneous Em. (LED effect)

$I > I_{th}$ ⇨ The efficiency depends on the mode (LASER)

"The Side-Mode Suppression Ration increases with the bias current. The higher the current the more single-mode"



PRACTICAL SINGLE MODE CRITERION ⇨ $SMSR \equiv 10\log\frac{P_o}{P_1}\begin{cases}\geq 20dB \leftarrow 3a \\ \geq 13dB \leftarrow 2a\end{cases}$

3. β parameter

"Determines the spontaneous emission fraction in the LD"

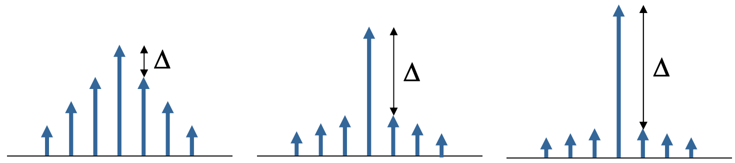
$$\frac{\partial S_i}{\partial t} = v \cdot g_i S_i - v \cdot \alpha_t S_i + \beta \frac{N}{\tau_r}$$

S_{SAT} , P_{SAT} $\propto \beta$

$\beta \downarrow \rightarrow$ single-modality $\uparrow$

P_{out} , I = ct

$\beta = 10^{-3}$ $\beta = 10^{-4}$ $\beta = 10^{-5}$



4. cavity length

$$\Delta\lambda \approx \frac{\lambda_p^2}{2nL} \Rightarrow$$

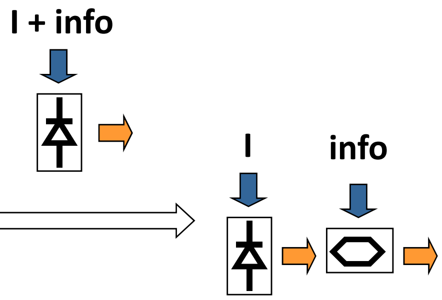
L $\downarrow \rightarrow$ single-modality $\uparrow$

trade-off with optical power

LASER'S DIRECT MODULATION

Modulation Techniques

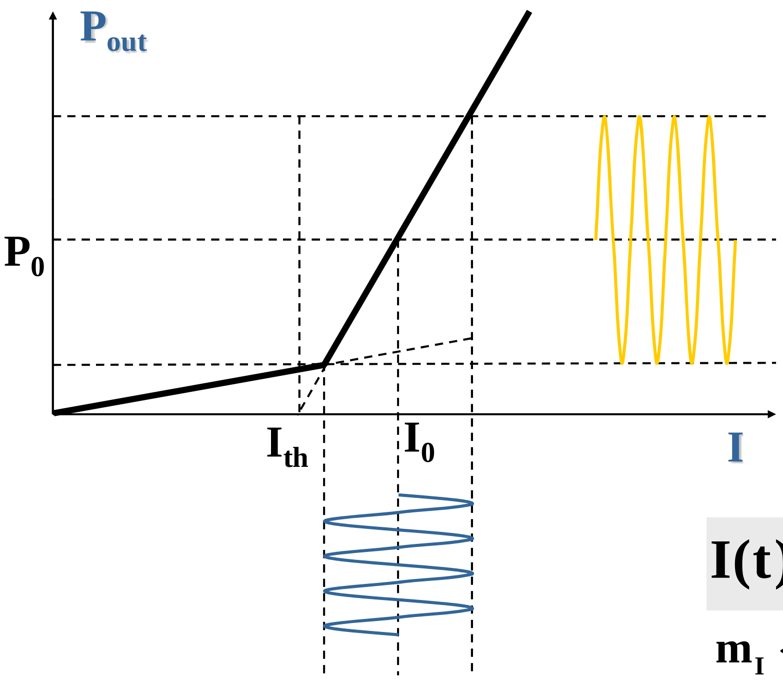
- **Intensity Modulation** (direct detection) ⇒ optical power (IM)
- **Field Modulation** (coherent detection) ⇒ amplitude (ASK), frequency (FSK), phase (PSK)
- **Direct Modulation** ⇒
- **External Modulation** ⇒



Sinusoidal Modulation

$$\frac{\partial N}{\partial t} = \frac{I}{qV} - \frac{N}{\tau_r} - v \cdot g \cdot S$$

$$\frac{\partial S}{\partial t} = v \cdot g \cdot S - v \cdot \alpha_t S$$



$$I(t) \equiv I_0 \left[1 + m_I e^{j\omega_0 t} \right]$$

$$m_I << 1$$

Small Signal !!

Response to a Sinusoidal Signal in Permanent Regime

$$I(t) \equiv I_0 \left[1 + m_I e^{j\omega_0 t} u(t) \right]$$

Static Relations

$$S = \frac{\tau_p}{qV}(I - I_{th})$$

$$N = N_{th}$$

Carriers

$$N(t) = N_{th} \left[1 + \underbrace{\frac{I_0}{I_{th}} \frac{m_I}{\tau_r} \frac{j\omega_0}{\omega_c^2 - \omega_0^2 + j2\alpha\omega_0}}_{m_N} e^{j\omega_0 t} \right]$$

$$N_{th} = I_{th} \frac{\tau_r}{qV}$$

α: laser damping factor → ~10^9

ω_c: laser resonance frequency → ~ 10^{12}

$$2\alpha \equiv \frac{1}{\tau_r} + v\Gamma a \cdot S_0$$

$$\omega_c^2 \equiv \frac{v\Gamma a \cdot S_0}{\tau_p}$$

Photons

$$S(t) = S_0 \left[1 + \underbrace{\frac{I_0}{I_0 - I_{th}} m_I \frac{\omega_c^2}{\omega_c^2 - \omega_0^2 + j2\alpha\omega_0}}_{m_S} e^{j\omega_0 t} \right]$$

$$S_0 = \frac{\tau_p}{qV}(I_0 - I_{th})$$

Laser Output Power in Permanent Regime

$$P(t) = \frac{1-R}{2\sqrt{R}} \cdot v \cdot Wd \cdot hf \cdot S(t)$$

$$I(t) = I_0\left[1 + m_I e^{(j\omega_0 t)} u(t)\right]$$

$$\boxed{P(t) = P_0\left[1 + \underbrace{\frac{I_0}{I_0 - I_{th}} m_I}_{m_S} \frac{\omega_c^2}{\omega_c^2 - \omega_0^2 + j2\alpha\omega_0} e^{j\omega_0 t}\right]}$$

$$P_0 = \frac{1-R}{2\sqrt{R}} \cdot v \cdot Wd \cdot hf \cdot S_0$$

$$S_0 = \frac{\tau_p}{qV}\left(I_0 - I_{th}\right)$$

$$\tau_p \equiv \frac{1}{v \cdot \alpha_t}$$

$$\boxed{P(t) = \frac{1-R}{2\sqrt{R}} \frac{1}{\alpha_t L} \frac{hf}{q}\left[\left(I_0 - I_{th}\right) + I_0 m_I \underbrace{\frac{\omega_c^2}{\omega_c^2 - \omega_0^2 + j2\alpha\omega_0}}_{M(\omega_0)} e^{j\omega_0 t}\right] = P_{DC} + \Delta P(t)}$$

$$P_{DC} = \frac{1-R}{2\sqrt{R}} \frac{1}{\alpha_t L} \frac{hf}{q}\left(I_0 - I_{th}\right)$$

$$\Delta P(t) = \frac{1-R}{2\sqrt{R}} \frac{1}{\alpha_t L} \frac{hf}{q} I_0 m_I M(\omega_0) e^{j\omega_0 t}$$

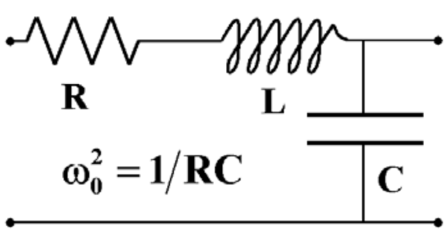
Laser Transfer Function (Small Signal)

$$\frac{\Delta P}{\Delta I} = \frac{\frac{1-R}{2\sqrt{R}} \frac{1}{\alpha_t L} \frac{hf}{q} I_0 m_I M(\omega_0) e^{j\omega_0 t}}{I_0 m_I e^{j\omega_0 t}} = \frac{1-R}{2\sqrt{R}} \frac{1}{\alpha_t L} \frac{hf}{q} M(\omega_0)$$

$$\Delta P(t) = \frac{1-R}{2\sqrt{R}} \frac{1}{\alpha_t L} \frac{hf}{q} I_0 m_I M(\omega_0) e^{j\omega_0 t}$$

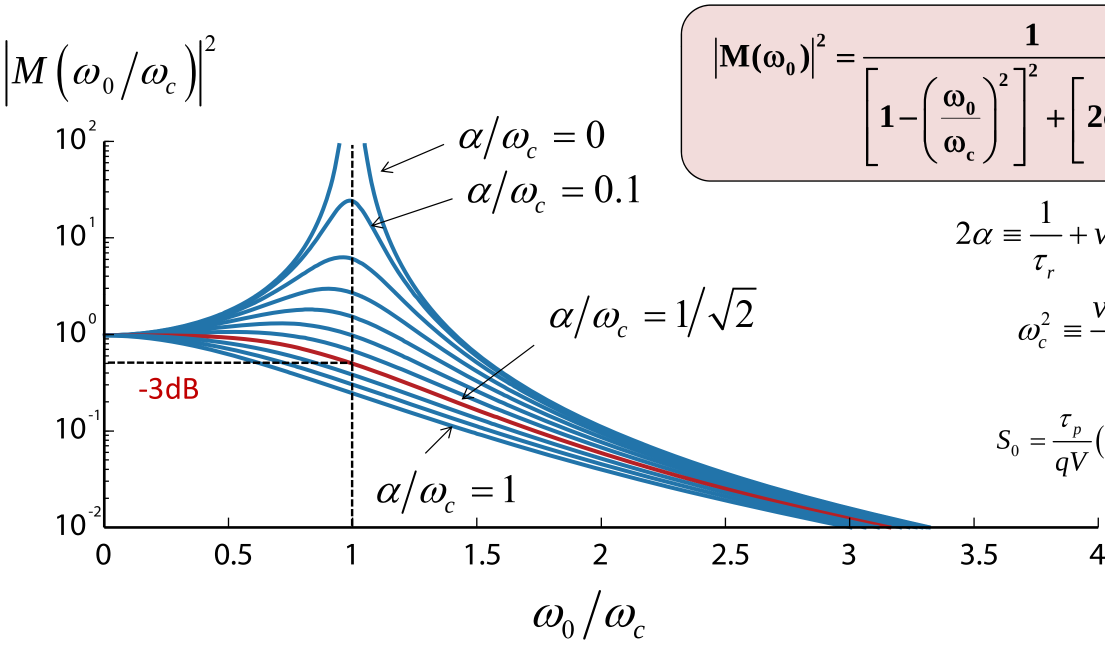
$$\Delta I(t) = m_I I_0 e^{j\omega_0 t}$$

$$H(\omega_0) = M(\omega_0) = \frac{\omega_c^2}{\omega_c^2 - \omega_0^2 + j2\alpha\omega_0}$$



$$\boxed{\left|M(\omega_0)\right|^2 = \frac{1}{\left[1 - \left(\frac{\omega_0}{\omega_c}\right)^2\right]^2 + \left[2\alpha \frac{\omega_0}{\omega_c^2}\right]^2}}$$

2nd order low-pass filter



$$|\mathbf{M}(\omega_0)|^2 = \frac{1}{\left[1-\left(\frac{\omega_0}{\omega_c}\right)^2\right]^2 + \left[2\alpha\frac{\omega_0}{\omega_c^2}\right]^2}$$

$$2\alpha \equiv \frac{1}{\tau_r} + v\Gamma a \cdot S_0$$

$$\omega_c^2 \equiv \frac{v\Gamma a \cdot S_0}{\tau_p}$$

$$S_0 = \frac{\tau_p}{qV}(I_0 - I_{th})$$

Trade-off ⇒

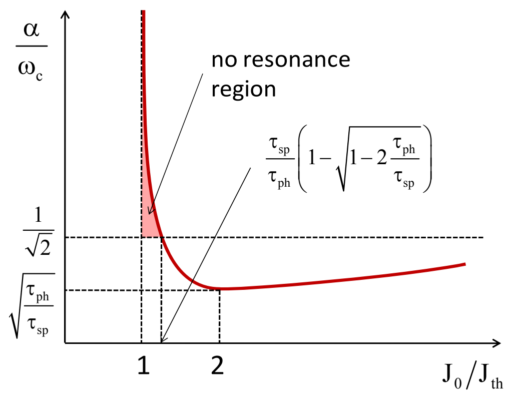
$\alpha/\omega_c < 1/\sqrt{2} \rightarrow$ resonance

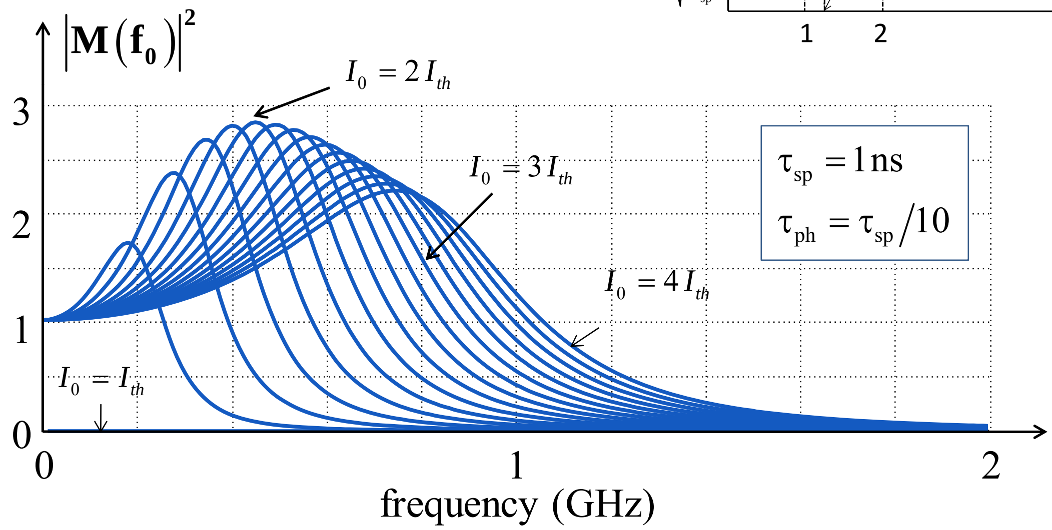
$\alpha/\omega_c > 1/\sqrt{2} \rightarrow$ $\mathbf{BW < \omega_c}$

$$\mathbf{BW \equiv \omega_c}$$

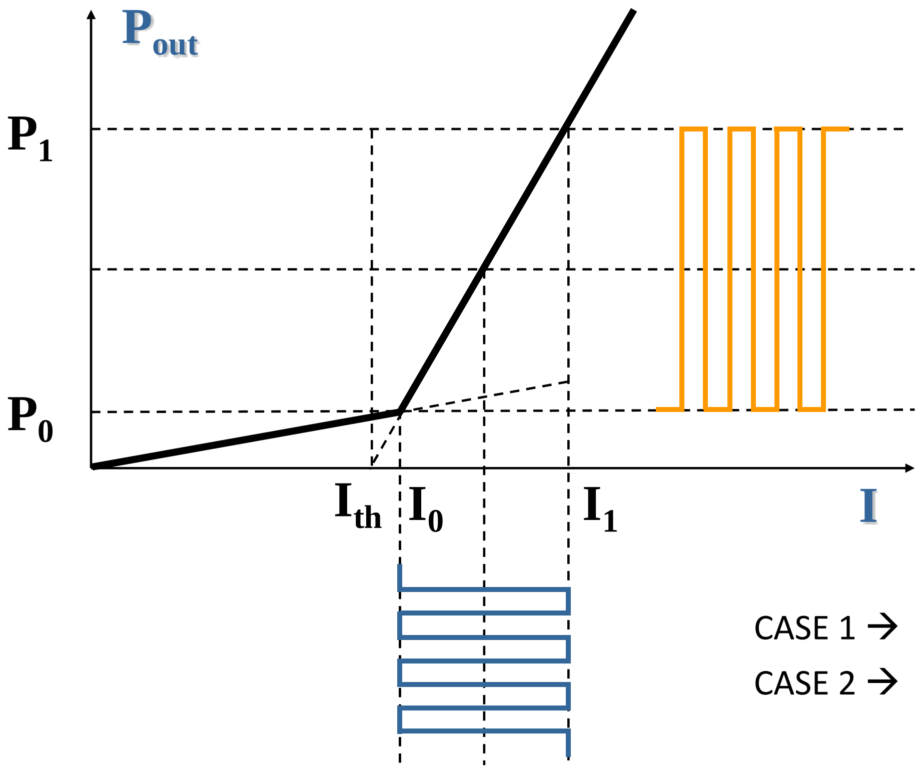
$$N_0 = 0 \Rightarrow \quad \alpha = \frac{1}{2\tau_{sp}}\left(\frac{J_0}{J_{th}}\right) \qquad \omega_c^2 = \frac{1}{\tau_{sp}\tau_{ph}}\left(\frac{J_0}{J_{th}} - 1\right)$$

⇒





Digital Modulation



CASE 1 → $I_0 < I_{th} < I_1$

CASE 2 → $I_{th} < I_0 < I_1$

Digital Modulation – Current Step

$$I(t) = I_0 + \underbrace{(I_1 - I_0)u(t)}_{\Delta I(t)}$$

inside the laser working regime (particular case of Appendix 2 when $\omega_0=0$, & $m_I=(I_1 - I_0)/I_0$)

Carriers

$$\boxed{\Delta N(t) = \frac{I_1 - I_0}{qV}\frac{e^{-\alpha t}}{\Omega}\sin(\Omega t)u(t)}$$

$$\Omega^2 \equiv \omega_c^2 - \alpha^2$$

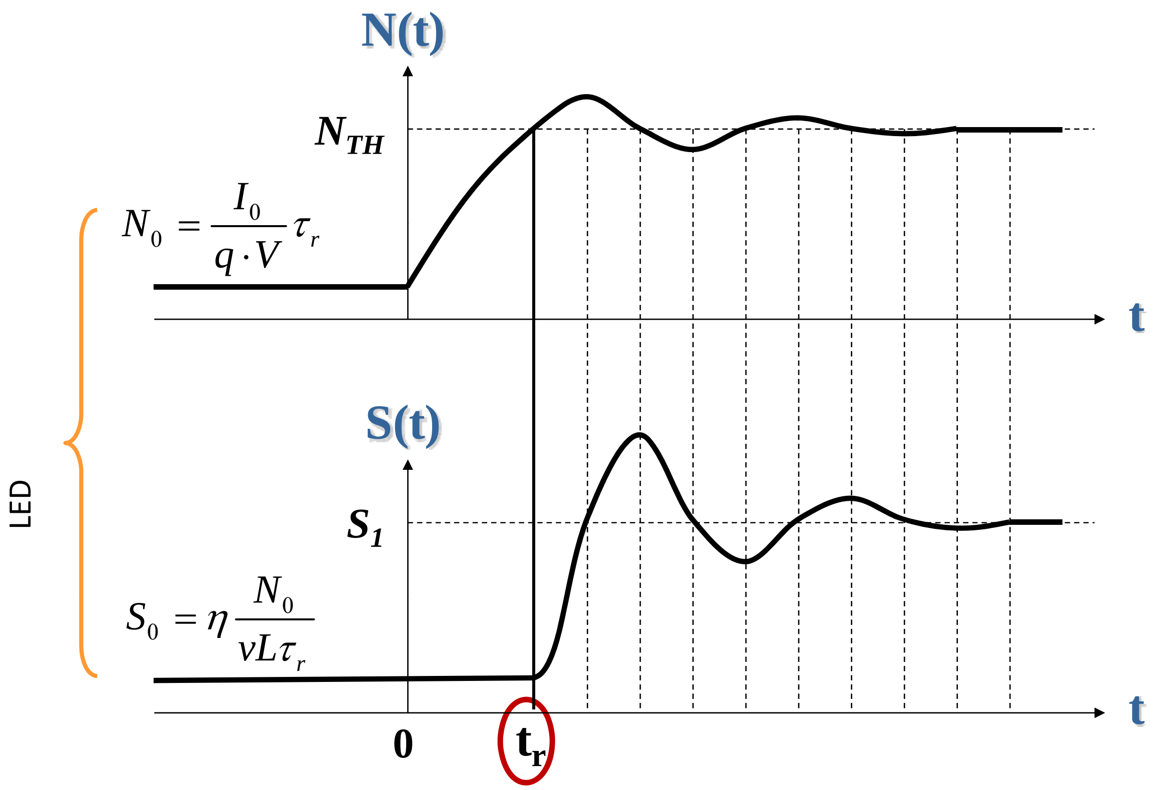
Photons

$$\Delta S(t) = v\Gamma a \cdot S_0 \frac{I_1 - I_0}{qV}\frac{1}{\omega_c^2}\left[1 - \frac{\alpha}{\sqrt{\omega_c^2 - \alpha^2}}e^{-\alpha t}\sin(\Omega t) - e^{-\alpha t}\cos(\Omega t)\right]u(t) \approx$$

$$\boxed{\Delta S(t) \approx v\Gamma a \cdot S_0 \frac{I_1 - I_0}{qV}\frac{1}{\omega_c^2}\left[1 - e^{-\alpha t}\cos(\Omega t)\right]u(t)}$$

damped sinusoidal oscillations shifted 90º

$$I_0 < I_{th} < I_1$$



Only Spontaneous Emission (LED)

$$I(t) = I_0 + (I_1 - I_0)u(t)$$

$$N(t) = \frac{\tau_r}{qV}I_0 + \frac{\tau_r}{qV}[I_1 - I_0](1 - e^{-t/\tau_r})u(t) \qquad 0 \le t \le t_r$$

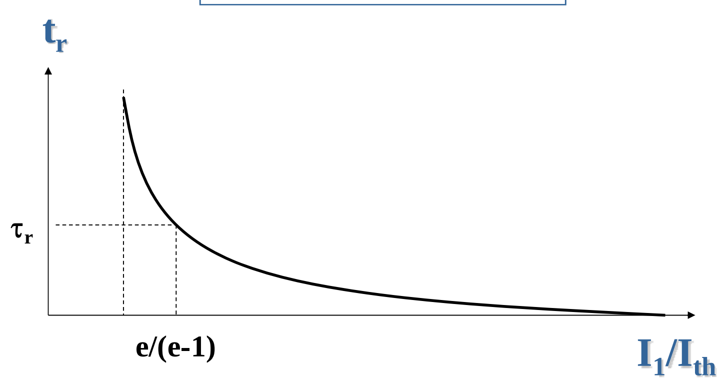
Response Time

$$t_r = \tau_r \ln \frac{I_1 - I_0}{I_1 - I_{th}}$$

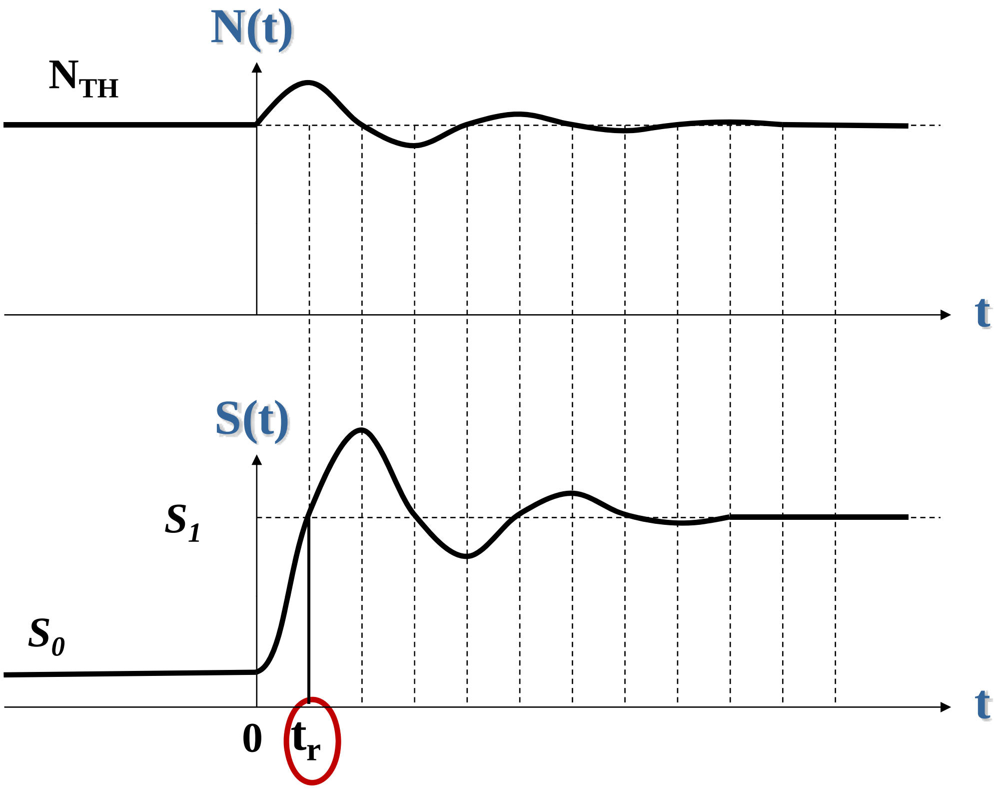
$$I_0 = 0 \rightarrow \boxed{t_r = \tau_r \ln \frac{I_1/I_{th}}{I_1/I_{th} - 1}}$$

$$t_r\big|_{LED} = 2.19\tau_r$$

In this case we have not a fixed t_r as we had with a LED



$$I_{th} < I_0 < I_1$$



Always Stimulated Emission (LASER)

First order approximations of carriers and photons functions in the rise time

$$N(t) \approx N_{TH} + \frac{I_1 - I_0}{q \cdot V} t \qquad 0 \le t \le t_r$$

$$S(t) \approx S_0 \exp\left[\frac{v\Gamma a}{2qV}(I_1 - I_0)t^2\right] \qquad 0 \le t \le t_r$$

$$\downarrow S(t) = S_1$$

Response Time

$$\boxed{t_r \approx \left[\frac{2qV}{v\Gamma a}\frac{\ln\left((I_1 - I_{th})/(I_0 - I_{th})\right)}{I_1 - I_0}\right]^{1/2}}$$

$\Leftarrow$ No τ_r dependence

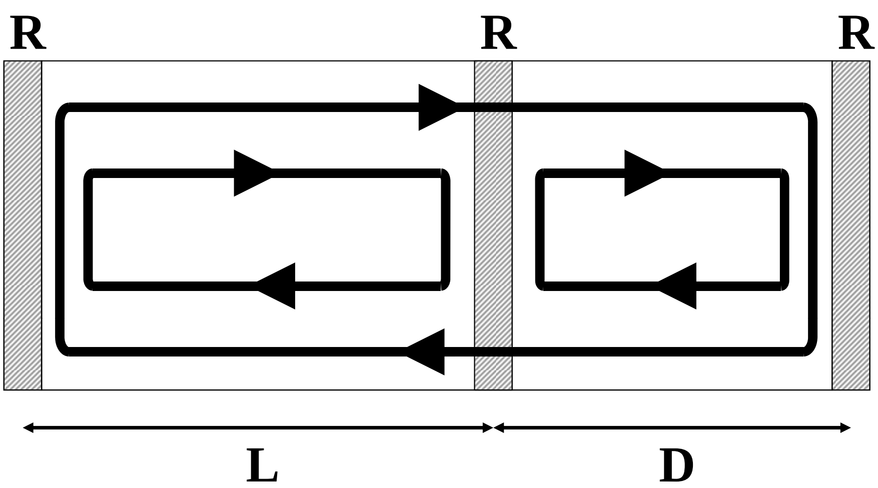
The response time in this case is much faster.

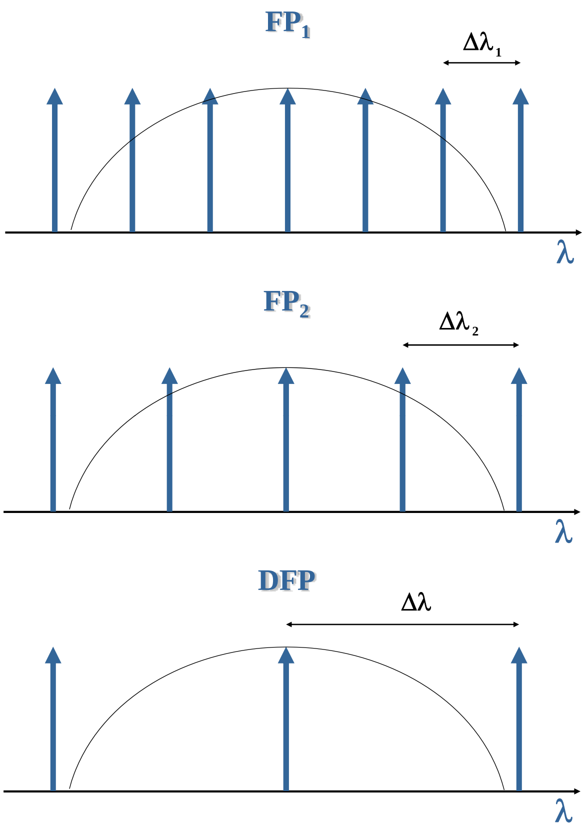
$I_1 - I_0 = \alpha \cdot I_{th} \rightarrow I_1 \uparrow \rightarrow t_r \downarrow$

MODERN LASER STRUCTURES

Coupled Cavities

COUPLEDFABRY-PEROT CAVITIES





$$\Delta\lambda_1 = \frac{\lambda^2}{2Ln_1}$$

$$\Delta\lambda_2 = \frac{\lambda^2}{2Dn_2}$$

$$\Delta\lambda = \frac{\lambda^2}{2\left|Dn_2 - Ln_1\right|} = \frac{\lambda^2}{2n\left|D - L\right|}$$

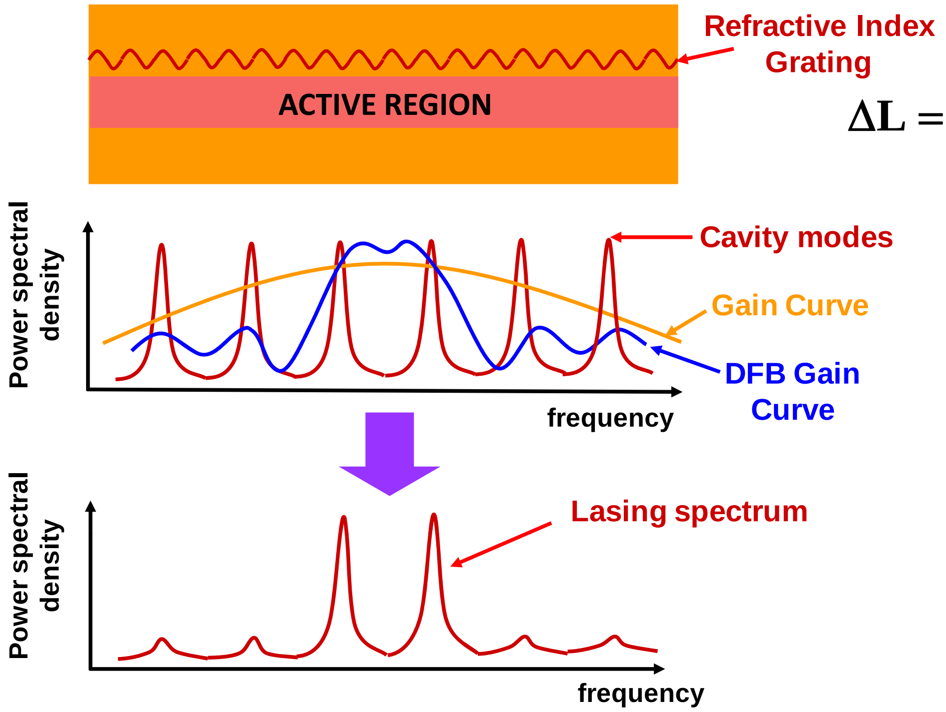
$$n_2 = n_1 = n$$

$\partial\lambda \approx 30\text{MHz}$

tunning ~ 5 nm

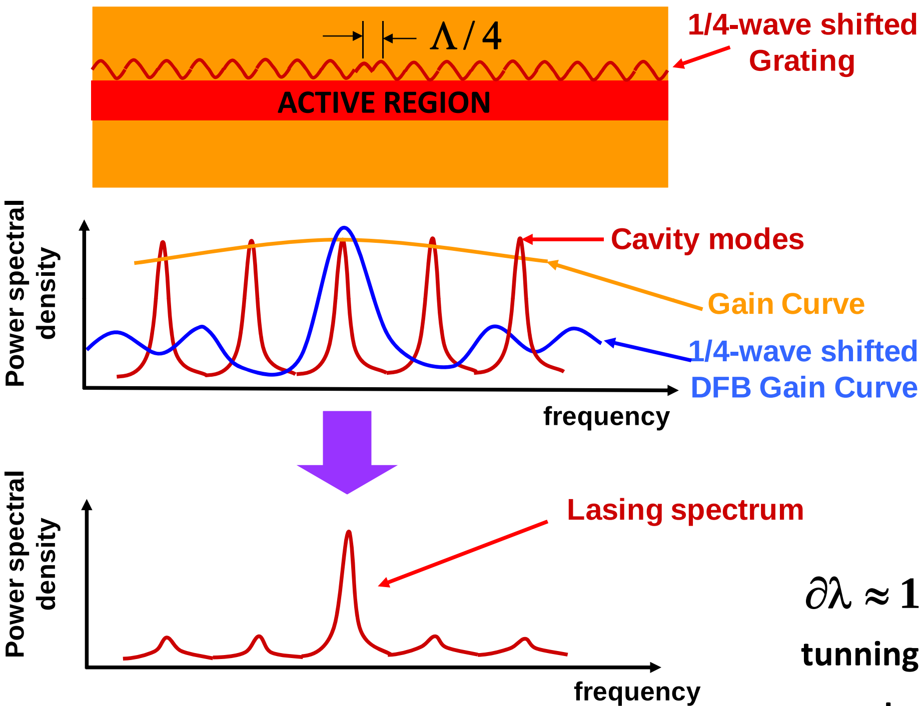
speed ~ 1 μs

Distributed Feed-Back (DFB) Lasers



$$\Delta L = m \frac{\lambda}{2n}$$

Distributed Feed-Back (DFB) Lasers

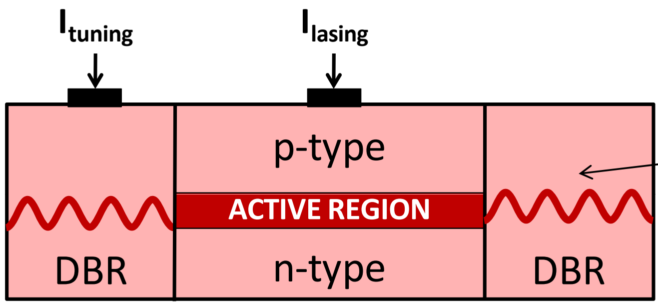


$$\partial\lambda \approx 10\text{MHz}$$

tunning ~ 5 nm

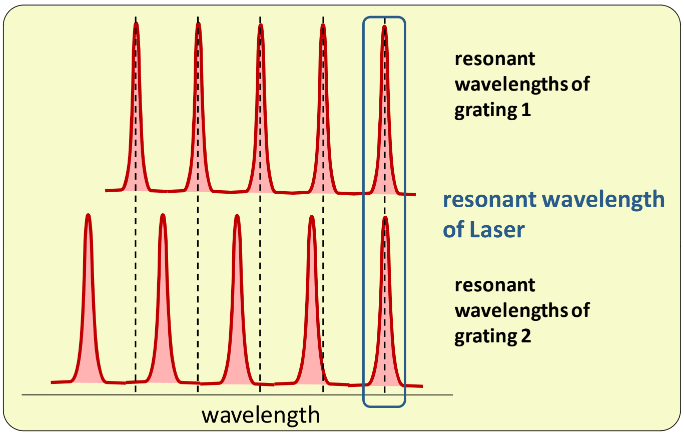
speed ~ 1 µs

Distributed Bragg Reflector (DBR) Lasers



wavelength dependent mirrors

$$\Delta L = m\frac{\lambda}{2n}$$

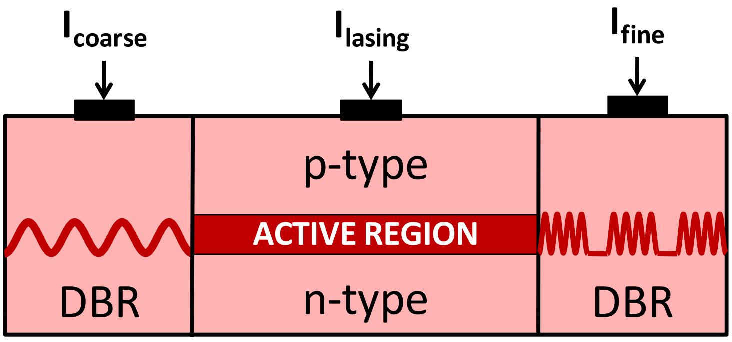


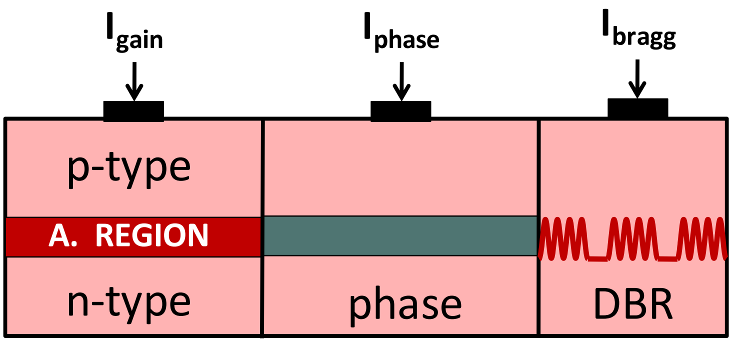
$$\partial\lambda \approx 30\text{MHz}$$

tunning ~ 15 nm

speed ~ 10 ns

Sampled Grating DBR (SG-DBR) Lasers – 3 sections



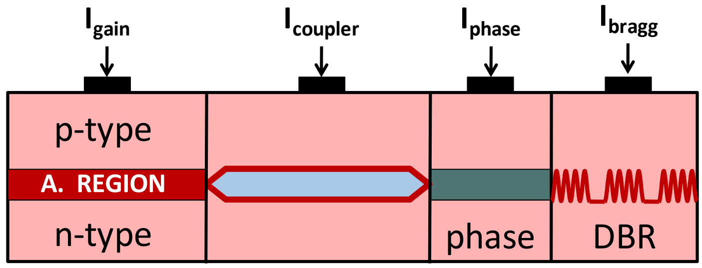


$$\partial\lambda \approx 10\text{MHz}$$

tunning ~ 30 nm

speed ~ 0.1 ns

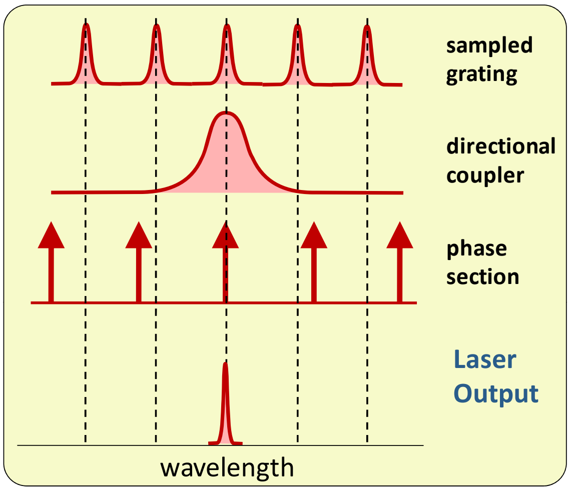
Grating Coupler Sampled Reflector (GCSR) Lasers – 4 sections

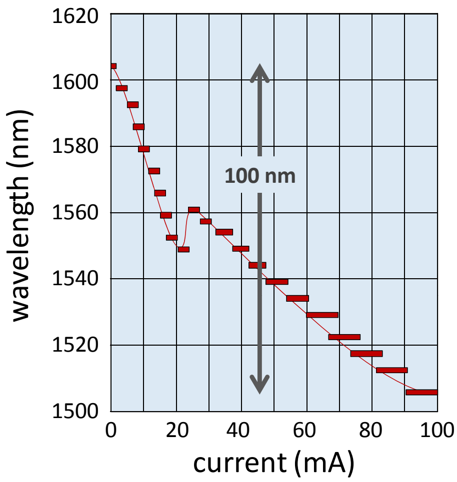


$\partial\lambda \approx 1$ MHz

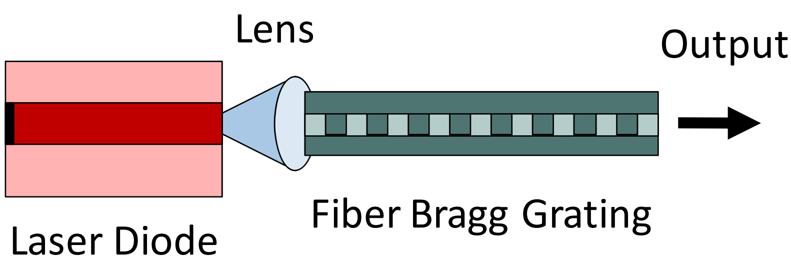
tunning ~ 100 nm

speed ~ 0.1 ns



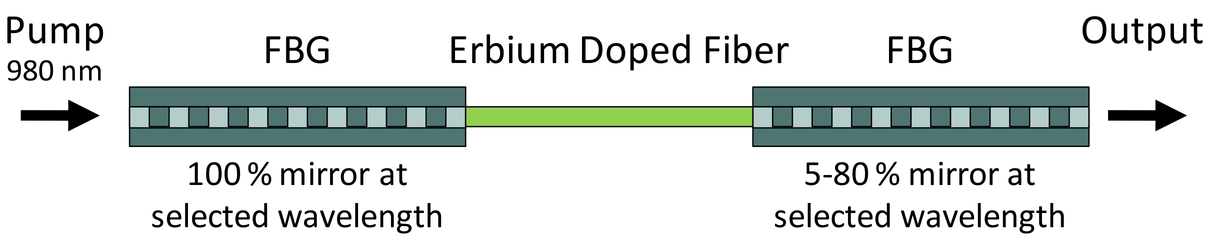


External Cavity Lasers



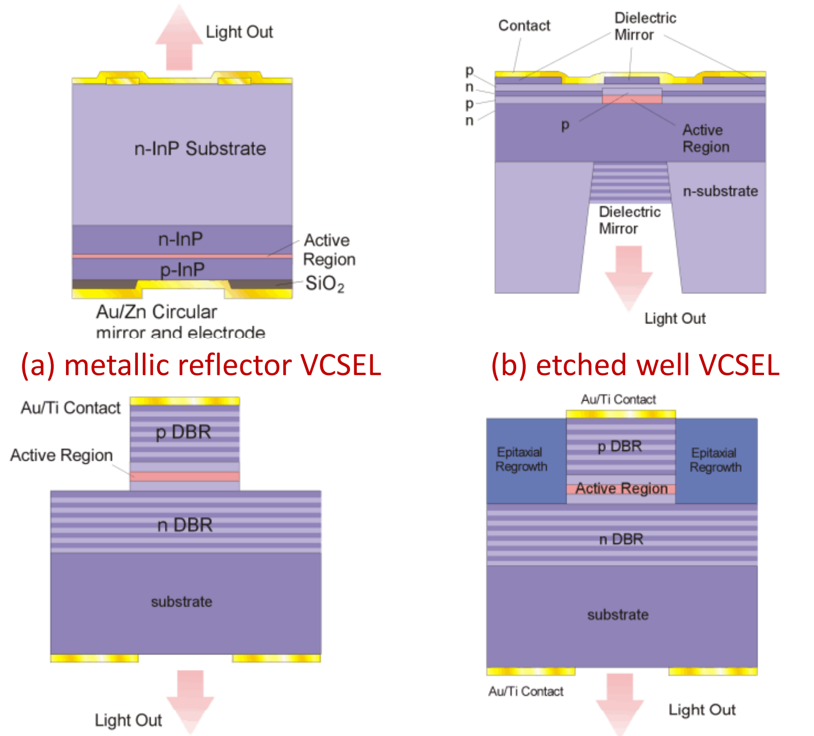
$\partial\lambda \approx 50$KHz

Fiber Lasers (non semi-conductor)



Output Power ~10 W

Vertical Cavity Surface Emitting Lasers (VCSELs)



(a) metallic reflector VCSEL

(b) etched well VCSEL

(c) air post VCSEL

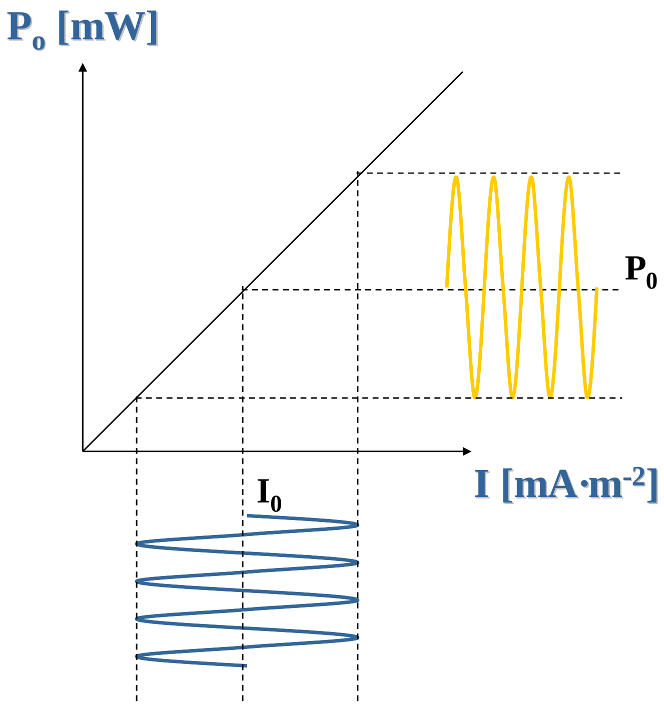
(d) burned regrouwth VCSEL

Cheap singlemode Lasers @ 1st , 2nd , and 3rd window

APPENDIX 1

LED modulation

LED's modulation - sinusoidal modulation



sinusoidal stimulus

$$I(t) \equiv I_0\left[1 + m_I e^{j(\omega_0 t+\phi)}\right]$$

$$N(t) \equiv N_0\left[1 + m_N e^{j(\omega_0 t+\phi-\theta_N)}\right]$$

$$P(t) \equiv P_0\left[1 + m_N e^{j(\omega_0 t+\phi-\theta_N)}\right]$$

I_0: DC electrical component

Optical Power

$$\boxed{P(t) = \eta \frac{N(t)}{\tau_r} V \cdot hf}$$

Pure sinusoidal stimulus response

$$\frac{\partial N(t)}{\partial t} + \frac{N(t)}{\tau_r} = \frac{I(t)}{qV}$$

TF

$$I(t) = m_I I_0 e^{j(\omega_0 t+\phi)} u(t)$$
$$\Updownarrow$$
$$I(\omega) = m_I I_0 e^{j\phi}\left[\pi\partial(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}\right]$$

$$j\omega N(\omega) - \cancel{N(t=0^+)} + \frac{N(\omega)}{\tau_r} = \frac{m_I I_0}{qV} e^{j\phi}\left[\pi\partial(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}\right]$$

$$N(\omega) = \frac{m_I I_0}{qV} e^{j\phi}\left[\pi\partial(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}\right]\frac{1}{j\omega + \frac{1}{\tau_r}}$$

$$N(\omega)=\frac{m_I I_0}{qV}e^{j\phi}\left[\underbrace{\pi\partial\left(\omega-\omega_0\right)\frac{1}{j\omega+\frac{1}{\tau_r}}}_{\partial(\omega-\omega_0)\frac{1}{j\omega_0+\frac{1}{\tau_r}}}+\underbrace{\frac{1}{j\left(\omega-\omega_0\right)}\frac{1}{j\omega+\frac{1}{\tau_r}}}_{\left[\frac{1}{j(\omega-\omega_0)}-\frac{1}{j\omega+\frac{1}{\tau_r}}\right]\frac{1}{j\omega_0+\frac{1}{\tau_r}}}\right]$$

$$=\frac{m_I I_0}{qV}e^{j\phi}\frac{1}{j\omega_0+\frac{1}{\tau_r}}\left[\underbrace{\pi\partial\left(\omega-\omega_0\right)+\frac{1}{j\left(\omega-\omega_0\right)}}_{F^{-1}\left\{e^{j\omega_0 t}u(t)\right\}}-\underbrace{\frac{1}{j\omega+\frac{1}{\tau_r}}}_{F^{-1}\left\{e^{-t/\tau_r}u(t)\right\}}\right]$$

TF^{-1} $\downarrow$

$$\boxed{N(t)=\frac{I_0\tau_r}{qV}\frac{m_I e^{j\phi}}{1+j\omega_0\tau_r}\left[e^{j\omega_0 t}-e^{-t/\tau_r}\right]u(t)\xrightarrow{t\to\infty}\underbrace{\frac{I_0\tau_r}{qV}}_{N_0}\underbrace{\frac{m_I}{1+j\omega_0\tau_r}}_{m_N}e^{j(\omega_0 t+\phi)}}$$

Step stimulus response

$$I(t)=m_I I_0 e^{(j\omega_0 t+\phi)}u(t)\xrightarrow[m_I=1]{\omega_0=\phi=0}I_0 u(t)$$

$$\Updownarrow$$

$$I(\omega)=m_I I_0 e^{j\phi}\left[\pi\partial\left(\omega-\omega_0\right)+\frac{1}{j\left(\omega-\omega_0\right)}\right]\xrightarrow[m_I=1]{\omega_0=\phi=0}I_0\left[\pi\partial\left(\omega\right)+\frac{1}{j\omega}\right]$$

$$N(t)=\frac{I_0\tau_r}{qV}\frac{m_I e^{j\phi}}{1+j\omega_0\tau_r}\left[e^{j\omega_0 t}-e^{-t/\tau_r}\right]u(t)\xrightarrow[m_I=1]{\omega_0=\phi=0}\frac{I_0\tau_r}{qV}\left[1-e^{-t/\tau_r}\right]u(t)$$

$$\xrightarrow{t\to\infty}\frac{I_0\tau_r}{qV}$$

Pure sinusoidal stimulus response

$$\boxed{I(t)=I_0\left\{1+m_I e^{(j\omega_0 t+\phi)}\right\}u(t)}$$

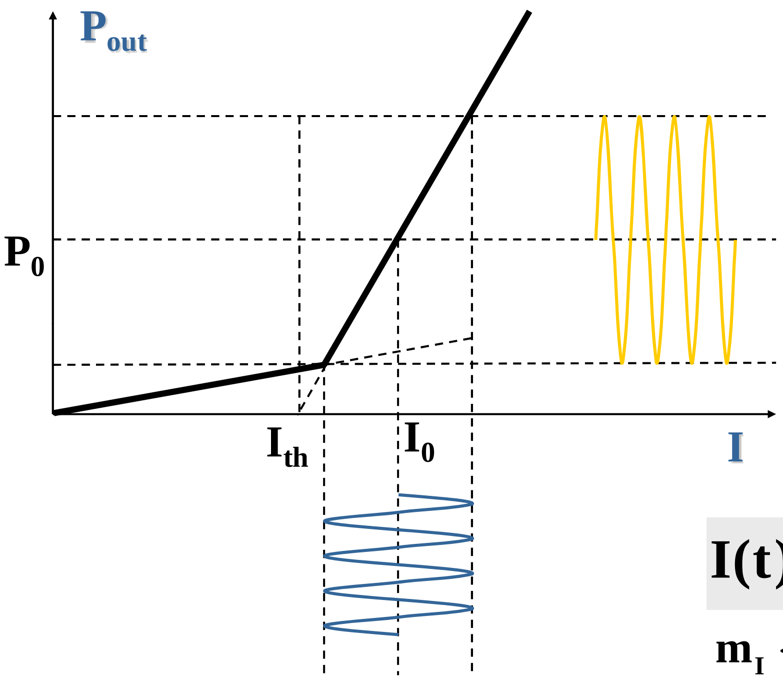
$$N(t)=\frac{I_0\tau_r}{qV}\left\{\left[1-e^{-t/\tau_r}\right]+\frac{m_I e^{j\phi}}{1+j\omega_0\tau_r}\left[e^{j\omega_0 t}-e^{-t/\tau_r}\right]\right\}u(t)$$

$$\xrightarrow{t\to\infty}\boxed{\frac{I_0\tau_r}{qV}\left\{1+\underbrace{\frac{m_I}{1+j\omega_0\tau_r}}_{m_N}e^{j(\omega_0 t+\phi)}\right\}}$$

APPENDIX 2

LASER modulation

Sinusoidal Modulation



$$\frac{\partial N}{\partial t} = \frac{I}{qV} - \frac{N}{\tau_r} - v \cdot gS = 0$$

$$\frac{\partial S}{\partial t} = v \cdot gS - v \cdot \alpha_t S = 0$$

$$I(t) \equiv I_0 \left[1 + m_I e^{j\omega_0 t} \right]$$

$$m_I << 1$$

Small Signal !!

$$I(t) \equiv I_0\left[1 + m_J e^{j\omega_0 t} u(t)\right] \equiv I_0 + \Delta I(t)$$ modulation current

stationary regime

static behavior

$$S = \frac{\tau_p}{qV}\left(I - I_{th}\right)$$
$$N = N_{th}$$

$$\omega_0 \to 0$$
$$I \equiv I_0\left[1 + m_J\right]$$
$$S = \frac{\tau_p}{qV}\left(I - I_{th}\right) = \underbrace{\frac{\tau_p}{qV}\left(I_0 - I_{th}\right)}_{S_0} + \underbrace{\frac{\tau_p}{qV} m_J I_0}_{\Delta S(0)}$$

sinusoidal regime

$$S \equiv S_0 + \Delta S(t)$$
$$N \equiv N_{th} + \Delta N(t)$$

S_0: DC photon density
N_{th}: DC carrier density

sinusoidal oscillation

Carrier Rate Equation

$$S \equiv S_0 + \Delta S(t)$$
$$N \equiv N_{th} + \Delta N(t)$$
$$g|_{\lambda_p} = \Gamma a\left(N - N_0\right)$$

$$\frac{\partial N}{\partial t} = \frac{I}{qV} - \frac{N}{\tau_r} - vgS \qquad \leftarrow g(N) = g\left(N_{th} + \Delta N\right) = \underbrace{\Gamma a\left(N_{th} - N_0\right)}_{\alpha_t} + \Gamma a \cdot \Delta N$$

$$\frac{\partial \Delta N}{\partial t} = \frac{I_0 + \Delta I}{qV} - \frac{N_{th} + \Delta N}{\tau_r} - v\left[\alpha_t + \Gamma a \cdot \Delta N\right] \cdot \left(S_0 + \Delta S\right)$$

$$N_{th} = N_0 + \frac{\alpha_t}{\Gamma a}$$

$$N_{th} = I_{th}\frac{\tau_r}{qV} \qquad S_0 = \frac{\tau_p}{qV}\left(I_0 - I_{th}\right)$$

$$\frac{\partial \Delta N}{\partial t} = \underbrace{\frac{I_0 - I_{th}}{qV}}_{S_0/\tau_p} + \frac{\Delta I}{qV} - \frac{\Delta N}{\tau_r} - v\left[\alpha_t + \Gamma a \cdot \Delta N\right] \cdot \left(S_0 + \Delta S\right)$$

$$\tau_p \equiv \frac{1}{v \cdot \alpha_t}$$

$$\frac{\partial \Delta N}{\partial t} = \cancel{\frac{S_0}{\tau_p}} + \frac{\Delta I}{qV} - \frac{\Delta N}{\tau_r} - \left[\cancel{\frac{S_0}{\tau_p}} + \frac{\Delta S}{\tau_p} + v\Gamma a \cdot \Delta N \cdot S_0 + \underbrace{\cancel{v\Gamma a \cdot \Delta N \cdot \Delta S}}_{\text{neglible}}\right]$$

$$\boxed{\frac{\partial \Delta N}{\partial t} = -\Delta N\left(\frac{1}{\tau_r} + v\cdot\Gamma a\cdot S_0\right) - \frac{\Delta S}{\tau_p} + \frac{\Delta I}{qV}} \quad (1) \quad \xrightarrow{\frac{\partial}{\partial t}}$$

Photon Rate Equation

$$S \equiv S_0 + \Delta S(t) \qquad \tau_p \equiv \frac{1}{v\cdot\alpha_t}$$

$$g = \alpha_t + \Gamma a\cdot\Delta N$$

$$\frac{\partial S}{\partial t} = v\cdot g\cdot S - v\cdot\alpha_t S + \cancel{\beta\frac{N}{\tau_r}} \approx \frac{S}{\tau_p}\left(\frac{g}{\alpha_t} - 1\right)$$

$$\frac{\partial \Delta S}{\partial t} = \frac{(S_0 + \Delta S)}{\tau_p}\left(\frac{\cancel{\alpha_t} + \Gamma a\cdot\Delta N}{\alpha_t} - \cancel{1}\right) = v\Gamma a\cdot S_0\Delta N + \underbrace{v\Gamma a\cdot\cancel{\Delta S}\cdot\Delta N}_{\text{negligible}}$$

$$\boxed{\frac{\partial \Delta S}{\partial t} \approx v\Gamma a\cdot S_0\Delta N} \quad (2)$$

Derivate (1) and substute in (2)

$$\frac{\partial^2 \Delta N}{\partial t^2} + \frac{\partial \Delta N}{\partial t}\left(\frac{1}{\tau_r} + v\Gamma a\cdot S_0\right) + \frac{v\Gamma a\cdot S_0}{\tau_p}\Delta N = \frac{1}{qV}\frac{\partial \Delta I}{\partial t}$$

α: laser damping factor → ~10^9

ω_c: laser resonance frequency → ~ 10^{12}

$$2\alpha \equiv \frac{1}{\tau_r} + v\Gamma a\cdot S_0$$

$$\omega_c^2 \equiv \frac{v\Gamma a\cdot S_0}{\tau_p}$$

Carrier Oscillation Equation

$$\boxed{\frac{\partial^2 \Delta N(t)}{\partial t^2} + 2\alpha\frac{\partial \Delta N(t)}{\partial t} + \omega_c^2\Delta N(t) = \frac{1}{qV}\frac{\partial \Delta I(t)}{\partial t}}$$

$$\alpha << \omega_0$$

Sinusoidal Oscillation

$$\frac{\partial^2 \Delta N(t)}{\partial t^2} + 2\alpha \frac{\partial \Delta N(t)}{\partial t} + \omega_c^2 \Delta N(t) = \frac{1}{qV}\frac{\partial \Delta I(t)}{\partial t}$$

$$\Delta I(t) = m_I I_0 e^{j\omega_0 t} u(t)$$
$$\Updownarrow$$
$$\Delta I(\omega) = m_I I_0 \left[\pi\partial(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)} \right]$$

$$\Big\downarrow \text{TF}$$

$$-\omega^2 \Delta N(\omega) + j2\alpha\omega \Delta N(\omega) + \omega_c^2 \Delta N(\omega) = \frac{1}{qV} j\omega \Delta I(\omega)$$

$$\Delta N(\omega)\left[\underbrace{\omega_c^2 + j2\alpha\omega - \omega^2}_{\omega_c^2 - \alpha^2 + (\alpha+j\omega)^2} \right] = \frac{1}{qV} m_I I_0 j\omega \left[\pi\partial(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)} \right]$$

$$\Omega^2 \equiv \omega_c^2 - \alpha^2$$

$$\Delta N(\omega) = \frac{m_I I_0}{qV} j\omega \left[\pi\partial(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)} \right] \frac{1}{\Omega^2 + (\alpha+j\omega)^2}$$

$$\Delta N(\omega) = \frac{m_I I_0}{qV} j\omega \left[\underbrace{\pi\partial(\omega-\omega_0)\frac{1}{\Omega^2 + (\alpha+j\omega)^2}}_{\partial(\omega-\omega_0)\frac{1}{\Omega^2+(\alpha+j\omega_0)^2}} + \underbrace{\frac{1}{j(\omega-\omega_0)} \frac{1}{\Omega^2 + (\alpha+j\omega)^2}}_{\left[\frac{1}{j(\omega-\omega_0)} - \frac{(\alpha+j\omega_0)+(\alpha+j\omega)}{\Omega^2+(\alpha+j\omega)^2}\right]\frac{1}{\Omega^2+(\alpha+j\omega_0)^2}} \right]$$

$$\Delta N(\omega) = \frac{m_I I_0}{qV} \frac{j\omega}{\Omega^2 + (\alpha+j\omega_0)^2} \left[\underbrace{\pi\partial(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}}_{\frac{1}{2\pi}e^{j\omega_0 t}u(t)} - \underbrace{\frac{(\alpha+j\omega_0)+(\alpha+j\omega)}{\Omega^2 + (\alpha+j\omega)^2}}_{\frac{\alpha+j\omega_0}{\Omega^2+(\alpha+j\omega)^2} + \frac{\alpha+j\omega}{\Omega^2+(\alpha+j\omega)^2}} \right]$$

$$\Big\downarrow \text{TF}^{-1}$$

$$-\frac{\alpha+j\omega_0}{\Omega} \underbrace{\frac{\Omega}{\Omega^2 + (\alpha+j\omega)^2}}_{e^{-\alpha t}\sin(\Omega t)u(t)} - \underbrace{\frac{\alpha+j\omega}{\Omega^2 + (\alpha+j\omega)^2}}_{e^{-\alpha t}\cos(\Omega t)u(t)}$$

$$\Delta N(t)=\frac{m_I I_0}{qV}\frac{1}{\Omega^2+\left(\alpha+j\omega_0\right)^2}\frac{\partial}{\partial t}\left\{e^{j\omega_0 t}u(t)-\frac{\alpha+j\omega_0}{\Omega}e^{-\alpha t}\sin(\Omega t)u(t)-\right.$$
$$\left.-e^{-\alpha t}\cos(\Omega t)u(t)\right\}$$

$$\frac{\partial}{\partial t}\left\{e^{j\omega_0 t}u(t)\right\}=j\omega_0 e^{j\omega_0 t}u(t)+e^{j\omega_0 t}\partial(t)=j\omega_0 e^{j\omega_0 t}u(t)+\partial(t)$$

$$\frac{\partial}{\partial t}\left\{\frac{\alpha+j\omega_0}{\Omega}e^{-\alpha t}\sin(\Omega t)u(t)\right\}=\left[-\alpha\frac{\alpha+j\omega_0}{\Omega}e^{-\alpha t}\sin(\Omega t)+\left(\cancel{\alpha}+j\omega_0\right)e^{-\alpha t}\cos(\Omega t)\right]u(t)$$
$$+\underbrace{\frac{\alpha+j\omega_0}{\Omega}e^{-\alpha t}\sin(\Omega t)\partial(t)}_{0}$$

$$\frac{\partial}{\partial t}\left\{e^{-\alpha t}\cos(\Omega t)u(t)\right\}=\left[\cancel{-\alpha e^{-\alpha t}\cos(\Omega t)}-\Omega e^{-\alpha t}\sin(\Omega t)\right]u(t)+\underbrace{e^{-\alpha t}\cos(\Omega t)\partial(t)}_{\partial(t)}$$

$$\Delta N(t)=\frac{m_I I_0}{qV}\frac{1}{\Omega^2+\left(\alpha+j\omega_0\right)^2}\left\{\begin{array}{l} j\omega_0 e^{j\omega_0 t}u(t)+\cancel{\partial(t)}-\cancel{\partial(t)}+ \\ +\left[\begin{array}{r}\alpha\dfrac{\alpha+j\omega_0}{\Omega}e^{-\alpha t}\sin(\Omega t)-j\omega_0 e^{-\alpha t}\cos(\Omega t) \\ +\Omega e^{-\alpha t}\sin(\Omega t)\end{array}\right]u(t)\end{array}\right\}$$

$$\Delta N(t)=\frac{m_I I_0}{qV}\frac{1}{\Omega^2+\left(\alpha+j\omega_0\right)^2}\left\{j\omega_0 e^{j\omega_0 t}+\cancel{\left[\alpha\frac{\alpha+j\omega_0}{\Omega}+\Omega\right]e^{-\alpha t}}\sin(\Omega t)-\right.$$
$$\left.-\cancel{j\omega_0 e^{-\alpha t}}\cos(\Omega t)\right\}u(t)$$

Permanent Regime

$$\boxed{\left.\Delta N(t)\right|_{t\to\infty}=\frac{m_I I_0}{qV}\frac{j\omega_0}{\Omega^2+\left(\alpha+j\omega_0\right)^2}e^{j\omega_0 t}=\frac{I_0}{qV}m_I\frac{j\omega_0}{\omega_c^2-\omega_0^2+j2\alpha\omega_0}e^{j\omega_0 t}}$$

$$\Omega^2\equiv\omega_c^2-\alpha^2$$

Carriers

Photons

$$\Delta N(t) = \frac{m_I I_0}{qV}\frac{1}{\omega_c^2-\omega_0^2+j2\alpha\omega_0}\frac{\partial}{\partial t}\left\{\left[e^{j\omega_0 t} - \frac{\alpha + j\omega_0}{\Omega}e^{-\alpha t}\sin(\Omega t) - e^{-\alpha t}\cos(\Omega t)\right]u(t)\right\}$$

$$\frac{\partial \Delta S(t)}{\partial t} = v\Gamma a \cdot S_0 \Delta N(t)$$

$$\Delta S(t) = v\Gamma a \cdot S_0 \frac{m_I I_0}{qV}\frac{1}{\omega_c^2-\omega_0^2+j2\alpha\omega_0}\left[e^{j\omega_0 t} - \cancel{\frac{\alpha + j\omega_0}{\Omega}e^{-\alpha t}\sin(\Omega t)} - \cancel{e^{-\alpha t}\cos(\Omega t)}\right]u(t)$$

Permanent Regime

$$\boxed{\Delta S(t)\big|_{t\to\infty} = v\Gamma a \cdot S_0 \frac{I_0}{qV}\frac{m_I}{\omega_c^2-\omega_0^2+j2\alpha\omega_0}e^{j\omega_0 t} = \frac{I_0}{qV}\tau_p m_I \frac{\omega_c^2}{\omega_c^2-\omega_0^2+j2\alpha\omega_0}e^{j\omega_0 t}}$$

$$\omega_c^2 \equiv \frac{v\Gamma a}{\tau_p}S_0$$

Response to a Sinusoidal Signal in Permanent Regime

$$I(t) \equiv I_0\left[1 + m_I e^{j\omega_0 t}u(t)\right]$$

Static Relations

$$S = \frac{\tau_p}{qV}(I - I_{th})$$

$$N = N_{th}$$

Carriers

$$\boxed{N(t) = N_{th}\left[1 + \underbrace{\frac{I_0}{I_{th}}\frac{m_I}{\tau_r}\frac{j\omega_0}{\omega_c^2-\omega_0^2+j2\alpha\omega_0}}_{m_N}e^{j\omega_0 t}\right]}$$

$$N_{th} = I_{th}\frac{\tau_r}{qV}$$

α: laser damping factor → ~10^9

ω_c: laser resonance frequency → ~ 10^{12}

$$2\alpha \equiv \frac{1}{\tau_r} + v\Gamma a \cdot S_0$$

$$\omega_c^2 \equiv \frac{v\Gamma a \cdot S_0}{\tau_p}$$

Photons

$$\boxed{S(t) = S_0\left[1 + \underbrace{\frac{I_0}{I_0 - I_{th}}m_I\frac{\omega_c^2}{\omega_c^2-\omega_0^2+j2\alpha\omega_0}}_{m_S}e^{j\omega_0 t}\right]}$$

$$S_0 = \frac{\tau_p}{qV}(I_0 - I_{th})$$