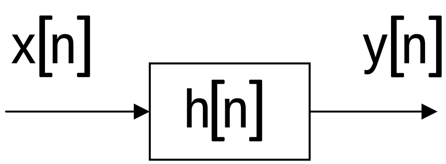
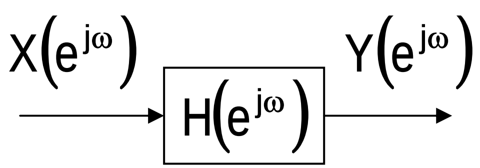
# 4.3: Respuesta frecuencial

- Expresión módulo-argumental
- Interpretación
- Cálculo analítico
- Contribución de ceros y polos



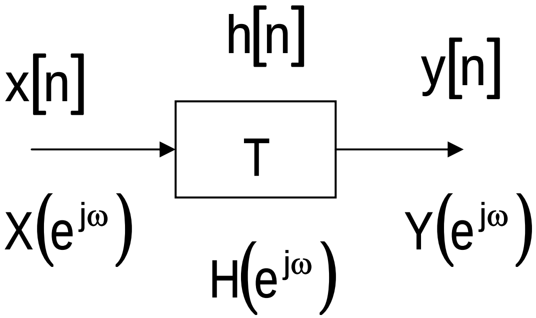

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

# Interpretaciones

- Para sistemas lineales e invariantes, tres interpretaciones:
  - Transformada de Fourier de la respuesta impulsional

$$H\left(e^{j\omega}\right) = TF\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

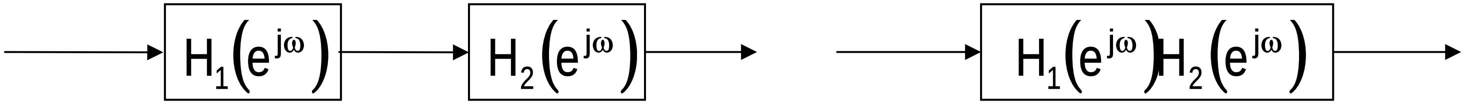



  - Autovalor de la autofunción componente frecuencial

$$x[n] = Ae^{j\omega n} \qquad y[n] = H\left(e^{j\omega}\right)Ae^{j\omega n}$$

  - Cociente de las transformadas de Fourier de la salida y la entrada

$$y[n] = x[n] * h[n] \qquad Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) \cdot H\left(e^{j\omega}\right) \qquad H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$



# Propiedades

- Periodicidad $H(e^{j\omega}) = H(e^{j(\omega + k2\pi)})$
  - Margen frecuencial

$$0 \leq \omega < 2\pi$$

- Sistemas reales $H(e^{j\omega}) = H^*(e^{-j\omega})$
  - Margen frecuencial

$$0 \leq \omega < \pi$$

- En sistemas estables

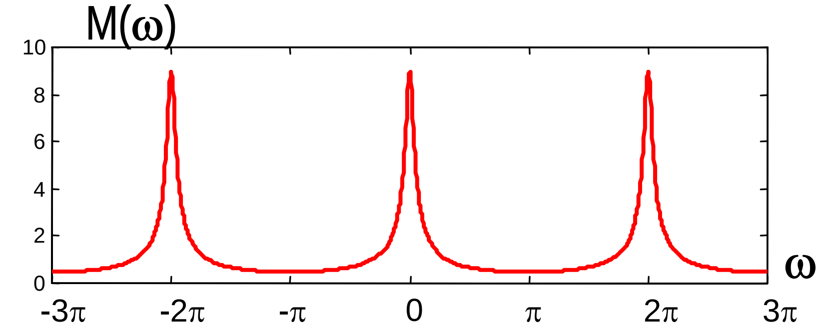
$$\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow \{z / |z| = 1\} \in ROC \qquad H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

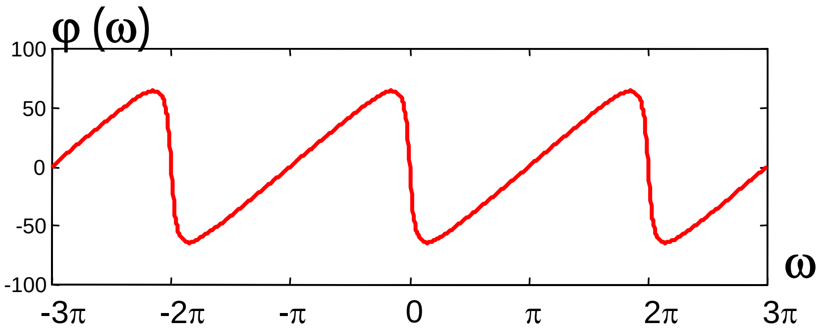
  - respuesta frecuencial acotada continua y con todas las derivadas continuas

# Expresión módulo-argumental

$$H\left(e^{j\omega}\right) = M(\omega)e^{j\varphi(\omega)}$$

- Módulo $\quad M(\omega) = \left|H\left(e^{j\omega}\right)\right|$
- Fase $\quad \varphi(\omega) = \arg\left[H\left(e^{j\omega}\right)\right]$




- Periodicidad

$$H\left(e^{j\omega}\right) = H\left(e^{j(\omega+k2\pi)}\right) \Rightarrow \begin{cases} M(\omega) = M(\omega + k2\pi) \\ \varphi(\omega) = \varphi(\omega + k2\pi) \end{cases}$$

- Sistemas reales

$$H\left(e^{j\omega}\right) = H^*\left(e^{-j\omega}\right) \Rightarrow \begin{cases} M(\omega) = M(-\omega) \\ \varphi(\omega) = -\varphi(-\omega) \end{cases}$$

Par

Impar

Si prescindimos de su carácter multiforme

# Interpretación de módulo y fase

$$x[n] = A\cos[\omega n + \theta] = \frac{A}{2}e^{j\theta}e^{j\omega n} + \frac{A}{2}e^{-j\theta}e^{-j\omega n}$$

$$x[n] = Ae^{j\omega n} \longrightarrow \boxed{H\left(e^{j\omega}\right)} \longrightarrow y[n] = H\left(e^{j\omega}\right)Ae^{j\omega n}$$

$$y[n] = \frac{A}{2}e^{j\theta}H\left(e^{j\omega}\right)e^{j\omega n} + \frac{A}{2}e^{-j\theta}H\left(e^{-j\omega}\right)e^{-j\omega n}$$

$$= \frac{A}{2}e^{j\theta}M(\omega)e^{j\varphi(\omega)}e^{j\omega n} + \frac{A}{2}e^{-j\theta}M(-\omega)e^{j\varphi(-\omega)}e^{-j\omega n}$$

$$= \frac{A}{2}e^{j\theta}M(\omega)e^{j\varphi(\omega)}e^{j\omega n} + \frac{A}{2}e^{-j\theta}M(\omega)e^{-j\varphi(\omega)}e^{-j\omega n}$$

Sistema real

$$\begin{cases} M(\omega) = M(-\omega) \\ \varphi(\omega) = -\varphi(-\omega) \end{cases}$$

$$y[n] = M(\omega)A\cos[\omega n + \theta + \varphi(\omega)]$$

$$\begin{cases} M(\omega) & \text{escalado} \\ \varphi(\omega) & \text{desfase} \end{cases}$$

# Cálculo analítico

$$M(\omega)=\sqrt{\mathrm{Re}^2\{H(e^{j\omega})\}+\mathrm{Im}^2\{H(e^{j\omega})\}} \qquad \varphi(\omega)=\tan^{-1}\left[\frac{\mathrm{Im}\{H(e^{j\omega})\}}{\mathrm{Re}\{H(e^{j\omega})\}}\right]$$

- Módulo

$$M^2(\omega)=\left|H(e^{j\omega})\right|^2=H(e^{j\omega})H^*(e^{j\omega})=TF\{r_h[n]\}$$

  - Sistema real

$$M^2(\omega)=H(e^{j\omega})H(e^{-j\omega})=H(z)H(1/z)\big|_{z=e^{j\omega}}$$

- Fase

$$\frac{H(e^{j\omega})}{H^*(e^{j\omega})}=\frac{M(\omega)e^{j\varphi(\omega)}}{M(\omega)e^{-j\varphi(\omega)}}=e^{j2\varphi(\omega)} \qquad \varphi(\omega)=\frac{1}{2j}\ln\left[\frac{H(e^{j\omega})}{H^*(e^{j\omega})}\right]$$

  - Sistema real

$$\varphi(\omega)=\frac{1}{2j}\ln\left[\frac{H(e^{j\omega})}{H(e^{-j\omega})}\right]=\frac{1}{2j}\ln\left[\frac{H(z)}{H(1/z)}\right]\Bigg|_{z=e^{j\omega}}$$

  - Se ha de considerar el carácter multiforme de la fase
  - Nota: ha de incluirse en la fase el signo de la constante multiplicativa

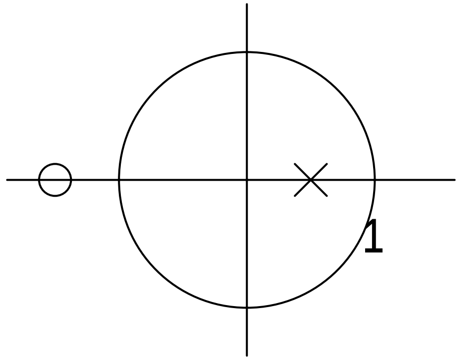
# Ejemplo módulo

$$H(z) = K\frac{1+bz^{-1}}{1-az^{-1}}$$

$$K = 1$$

$$a = 0.5$$

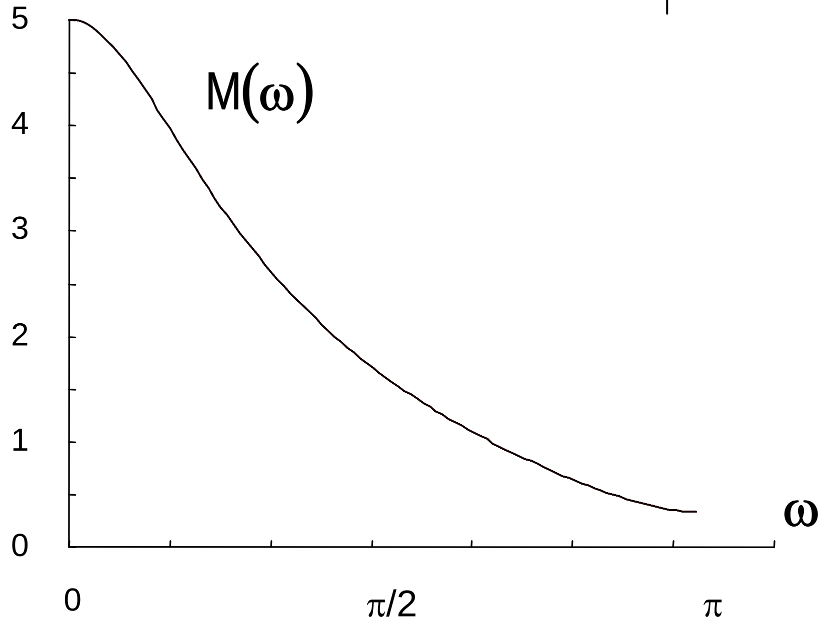
$$b = 1.5$$




$$H(z)H(z^{-1}) = K^2\frac{1+bz^{-1}}{1-az^{-1}}\frac{1+bz}{1-az} =$$

$$= K^2\frac{1+b(z+z^{-1})+b^2}{1-a(z+z^{-1})+a^2}$$

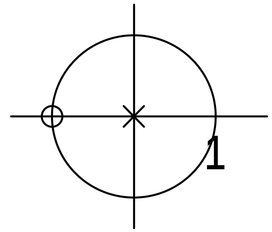
$$M^2(\omega) = K^2\frac{1+2b\cos\omega+b^2}{1-2a\cos\omega+a^2}$$



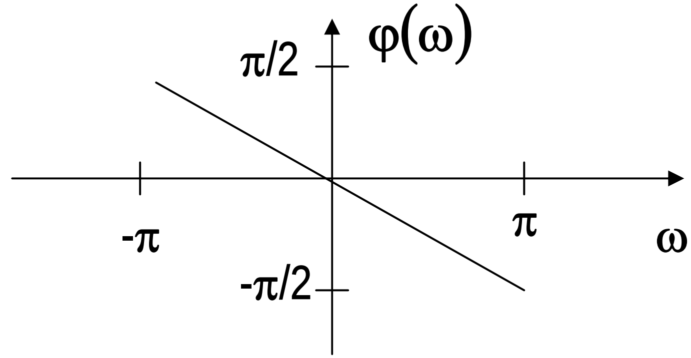

Generalizable de forma inmediata: $M(\omega)$ es función de $\cos\omega$

# Ejemplo fase

$$H(z)=1+z^{-1}$$



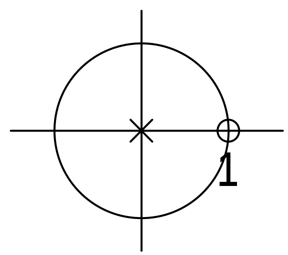

$$\frac{H(z)}{H(z^{-1})}=\frac{1+z^{-1}}{1+z}=\frac{z^{-1}(z+1)}{1+z}=z^{-1}$$



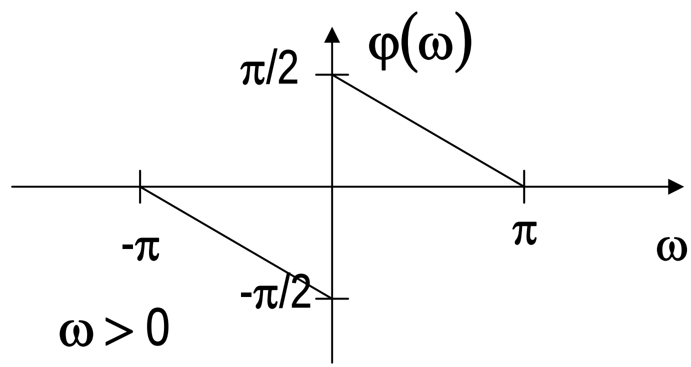

$$\varphi(\omega)=\frac{1}{2j}\ln\left[\frac{H(z)}{H(1/z)}\right]\Bigg|_{z=e^{j\omega}}=\frac{1}{2j}\ln\left[e^{-j\omega}\right]=-\frac{\omega}{2}$$

Saltos de fase de $\pi$ en los ceros

$$H(z)=1-z^{-1}$$




$$\frac{H(z)}{H(z^{-1})}=\frac{1-z^{-1}}{1-z}=\frac{z^{-1}(z-1)}{1-z}=-z^{-1}$$




$$\varphi(\omega)=\frac{1}{2j}\ln\left[e^{j(\pi+k2\pi)}e^{-j\omega}\right]=-\frac{\omega}{2}+\frac{\pi}{2}+k\pi\quad k=\begin{cases}0 & \omega>0\\ -1 & \omega<0\end{cases}$$

# Contribución de ceros y polos

$$H\left(e^{j\omega}\right)=\frac{b_o\prod_{k=1}^{Q}\left(1-c_k e^{-j\omega}\right)}{a_o\prod_{k=1}^{P}\left(1-p_k e^{-j\omega}\right)}=\frac{b_o}{a_o}\frac{\prod_{k=1}^{Q}e^{-j\omega}\left(e^{j\omega}-c_k\right)}{\prod_{k=1}^{P}e^{-j\omega}\left(e^{j\omega}-p_k\right)}=\frac{b_o}{a_o}e^{-j(Q-P)\omega}\frac{\prod_{k=1}^{Q}\left(e^{j\omega}-c_k\right)}{\prod_{k=1}^{P}\left(e^{j\omega}-p_k\right)}$$

Módulo

$$M(\omega)=\left|\frac{b_o}{a_o}\right|\frac{\prod_{k=1}^{Q}\left|e^{j\omega}-c_k\right|}{\prod_{k=1}^{P}\left|e^{j\omega}-p_k\right|}$$

Ganancia

$$G(\omega)=20\log[M(\omega)]=20\log\left|\frac{b_o}{a_o}\right|+\sum_{k=1}^{Q}20\log\left|e^{j\omega}-c_k\right|-\sum_{k=1}^{P}20\log\left|e^{j\omega}-p_k\right|$$

Fase

$$\varphi(\omega)=\arg\left[\frac{b_o}{a_o}\right]+\sum_{k=1}^{Q}\left\{-\omega+\arg\left[e^{j\omega}-c_k\right]\right\}-\sum_{k=1}^{P}\left\{-\omega+\arg\left[e^{j\omega}-p_k\right]\right\}$$

$$\varphi(\omega)=\arg\left[\frac{b_o}{a_o}\right]+(P-Q)\omega+\sum_{k=1}^{Q}\arg\left[e^{j\omega}-c_k\right]-\sum_{k=1}^{P}\arg\left[e^{j\omega}-p_k\right]$$

# Contribución de un cero (I)

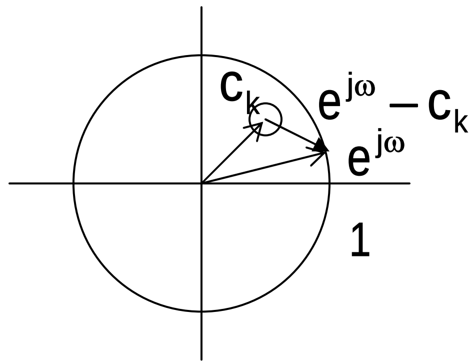
- Cero dentro del círculo de radio unidad

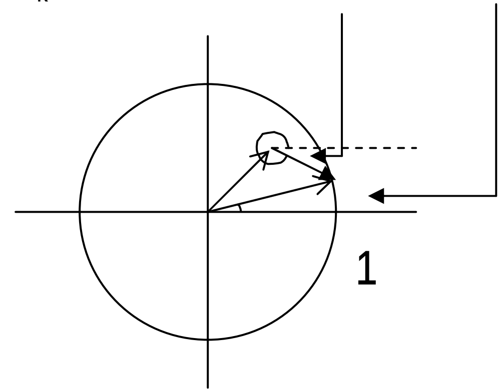
$$M_{c_k}(\omega) = \left|e^{j\omega} - c_k\right| = 1 + r^2 - 2r\cos(\omega - \theta)$$

$$\varphi_{c_k}(\omega) = \arg\left[e^{j\omega} - c_k\right] - \omega$$

$$c_k = re^{j\theta}$$

$$\theta = \pi/4$$



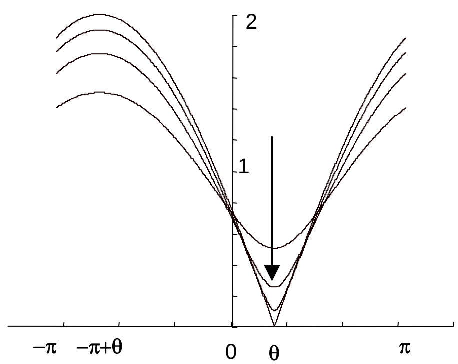



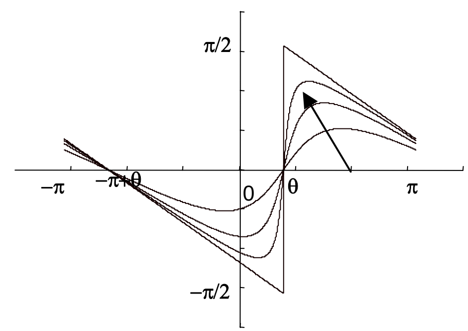

$r = 0.5$

$r = 0.75$

$r = 0.9$

$r = 1$




Excursión de fase:

$$\Delta\varphi(\omega) = \varphi(\pi) - \varphi(-\pi) = 0$$

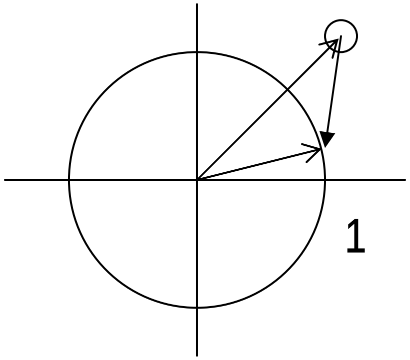
# Contribución de un cero (II)

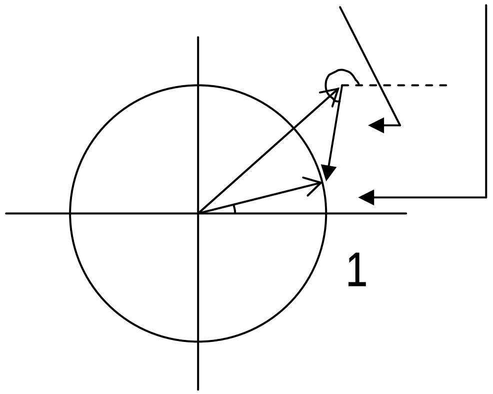
- Cero fuera del círculo de radio unidad

$$M_{c_k}(\omega) = \left| e^{j\omega} - c_k \right| \qquad \varphi_{c_k}(\omega) = \arg\left[ e^{j\omega} - c_k \right] - \omega$$

$$c_k = re^{j\theta}$$

$$\theta = \pi/4$$



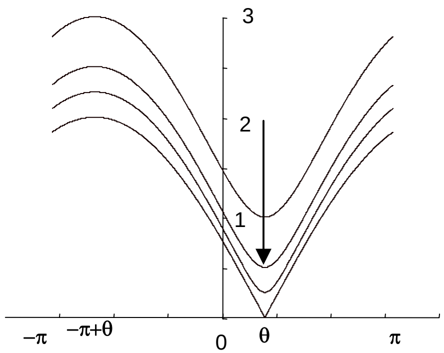



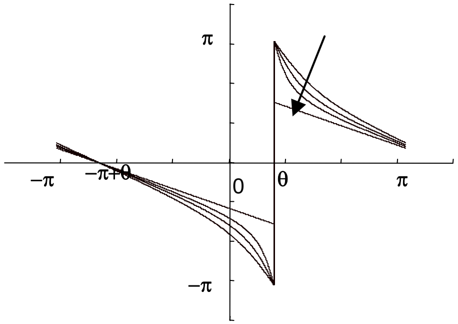

$r = 2$

$r = 1.5$

$r = 1.25$

$r = 1$




$r > 1$

No hay salto de fase

Excursión de fase:

$$\Delta\varphi(\omega) = \varphi(\pi) - \varphi(-\pi) = -2\pi$$

# Contribución de un polo

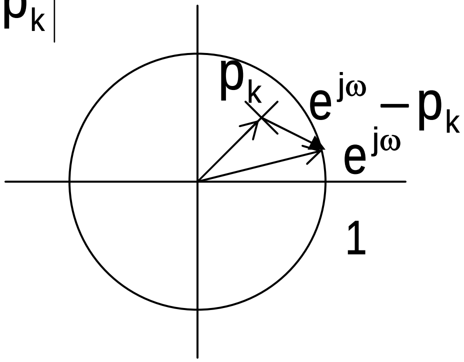
- Polo dentro del círculo de radio unidad

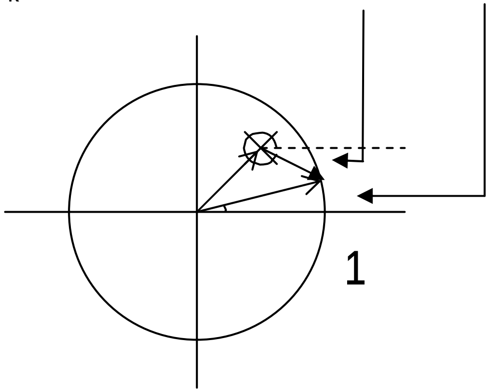
$$M_{p_k}(\omega) = \frac{1}{\left| e^{j\omega} - p_k \right|}$$

$$\varphi_{p_k}(\omega) = -\arg\left[ e^{j\omega} - p_k \right] + \omega$$

$$p_k = re^{j\theta}$$

$$\theta = \pi / 4$$



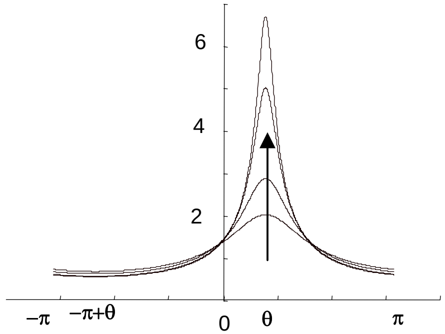



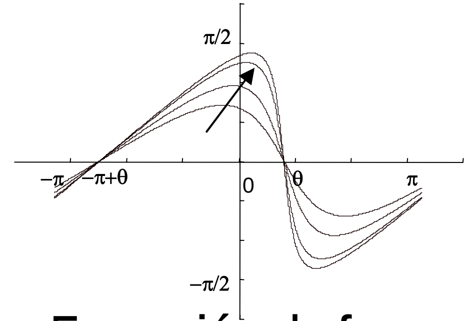

$$r = 0.5$$

$$r = 0.65$$

$$r = 0.8$$

$$r = 0.85$$




Excursión de fase:

$$\Delta\varphi(\omega) = \varphi(\pi) - \varphi(-\pi) = 0$$

# Sistemas reales

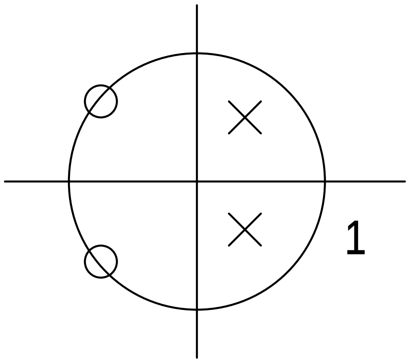
- En sistemas reales

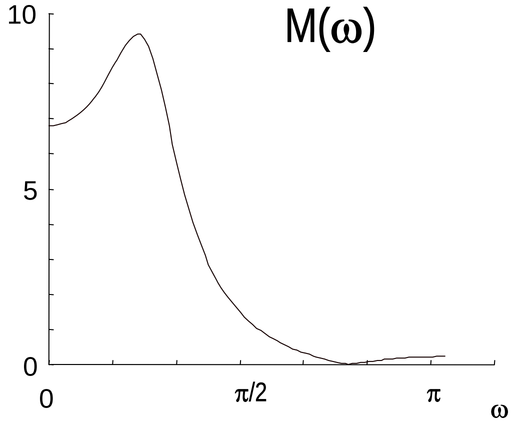
$$M_{c_k}(\omega) = \left|e^{j\omega} - c_k\right| = M_{c_k{}^*}(-\omega) \qquad \varphi_{c_k}(\omega) = -\omega + \arg\left[e^{j\omega} - c_k\right] = 2\pi - \varphi_{c_k{}^*}(-\omega)$$

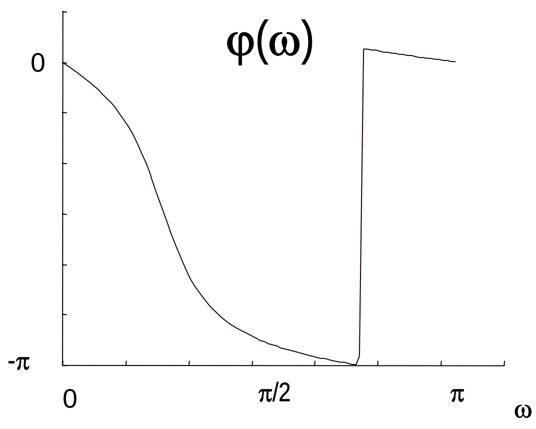
$$M_{p_k}(\omega) = \frac{1}{\left|e^{j\omega} - p_k\right|} = M_{p_k{}^*}(-\omega) \qquad \varphi_{p_k}(\omega) = \omega - \arg\left[e^{j\omega} - p_k\right] = 2\pi - \varphi_{p_k{}^*}(-\omega)$$

- Ejemplo

$$H(z) = \frac{1 + \sqrt{2}z^{-1} + z^{-2}}{1 - 0.75\sqrt{2}z^{-1} + 0.5625z^{-2}} = \frac{\left(1 - e^{j3\pi/4}z^{-1}\right)\left(1 - e^{-j3\pi/4}z^{-1}\right)}{\left(1 - 0.75e^{j\pi/4}z^{-1}\right)\left(1 - 0.75e^{-j\pi/4}z^{-1}\right)}$$



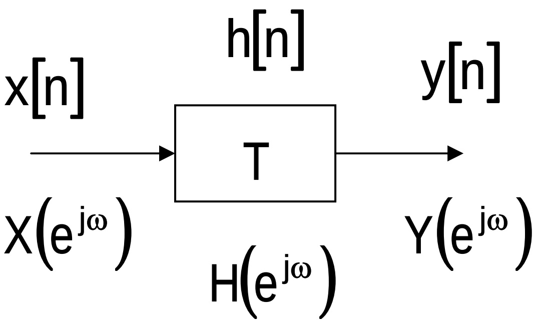



# Resumen

- Respuesta frecuencial

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$




- Expresión módulo argumental $\quad H(e^{j\omega}) = M(\omega)e^{j\varphi(\omega)}$
- Sistemas reales

$$H(e^{j\omega}) = H^*(e^{-j\omega}) \Rightarrow \begin{cases} M(\omega) = M(-\omega) & \text{Par} \\ \varphi(\omega) = -\varphi(-\omega) & \text{Impar} \end{cases}$$

$$x[n] = A\cos[\omega n + \theta] \qquad y[n] = M(\omega)A\cos[\omega n + \theta + \varphi(\omega)]$$

$$M^2(\omega) = H(z)H(1/z)\Big|_{z=e^{j\omega}} \qquad \varphi(\omega) = \frac{1}{2j}\ln\left[\frac{H(z)}{H(1/z)}\right]\Bigg|_{z=e^{j\omega}}$$