APUNTS PROPIETAT DE:
MÀRIUS SERRA LÓPEZ

PER QUALSEVOL DUBTE O CONSULTA RESPECTE ELS APUNTS O EL QUE FACI FALTA ESCRIURE A:
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(l'assumpte ha de ser: Apunts ETSETB)

QUALSEVOL ERROR PRESENT S'ATRIBUEIX AL PROFE DE L'ASSIGNATURA QUE ME LA VA IMPARTIR!!

IMPORTANT!!

1) EL DOCUMENT TÉ LA RESOLUCIÓ SUFICIENT COM PER FER ZOOM I VEURE LA LLETRA PETITA MILLOR!

2) L'APARTAT DE COMPLEXES SI ES FAN FORÇA EXERCICIS NO ÉS MOLT COMPLICAT

3) FALTA ALGUN TROCET DE TEORIA!!

4) ACONSELLO AL PROFE: MIQUEL ESCUDERO. SOLEN DIR QUE EL AGUILÓ TAMB ÉS BO, PERÒ JO AMB L'AGUILÓ EM SOBAVA Xdd

5) RECOMANO DONAR UN COP D'ULL A UN LLIBRE QUE ES DIU: MATEMATIQUES DE LA TELECOMUNICACIÓ (EDICIONS UPC)

MARCOS SERRA LÓPEZ

MATEMÀTIQUES DE LA TELECOMUNICACIÓ
EXAMEN FINAL
15 de juny de 2006
Notes: 19/06; **Alegacions:** 19-21/06; **Notes Definitives:** 26/06
20/06

Problema 1
Considereu l'espai $\mathbb{X} = (\mathbb{R}[t], \langle\cdot,\cdot\rangle)$, on el producte escalar $\langle\cdot,\cdot\rangle$ ve definit per

$$\langle p, q\rangle = \int_0^1 t^2 p(t)q(t)dt.$$

Noteu que $\mathbb{X}$ és l'espai de polinomis a coeficients reals (que té dimensió infinita). Es demana:

1. Determineu la millor aproximació de primer grau del polinomi $p(t) = t^2$.
2. Demostreu que el límit, a l'espai $\mathbb{X}$, de la successió de polinomis $q_n(t) = \frac{2n+1}{n}t^3 + 1$ és el polinomi $q(t) = 2t^3 + 1$.
3. Calculeu el polinomi de primer grau que passa pel punt $(1,1)$ i que sigui ortogonal al polinomi t.

Problema 2
Considereu la funció

$$\Lambda(t) = \begin{cases} 1 - |t|, & |t| \le 1, \\ 0, & \text{altrament.} \end{cases}$$

1. Feu una gràfica de $\Lambda(t)$ i dels dos primers termes no nuls de la seva sèrie de Fourier trigonomètrica a l'interval $[-1, 1]$.
2. Fent servir que $\Lambda(t) = (\Pi * \Pi)(t)$, on $\Pi(t) = 1$ si $|t| \le 1/2$ i $\Pi(t) = 0$ altrament, calculeu la transformada de Fourier $Y(f)$ de
$$y(t) = \begin{cases} \Lambda(t-1), & t \ge 0, \\ -\Lambda(t+1), & t \le 0. \end{cases}$$
3. Representeu la funció $z(t) = y(t) * (\delta(t) + \delta(t-4) + \delta(t+4))$ i calculeu la seva transformada de Fourier $Z(f)$. Determineu els zeros de $Z(f)$.

Problema 3

1. Calculeu el residu en $z_0 = 0$ de la funció
$$f(z) = \frac{z-2}{z - \sin z},$$
assenyalant quin tipus de punt és. Indicació: Es pot procedir dividint $z-2$ per la sèrie de $z - \sin z$ fins arribar al coeficient adequat.
2. En el desenvolupament en sèrie de Laurent de la funció
$$g(z) = \frac{1}{z^2+1}$$
en potències de $z-1$, vàlid per a $|z-1| > \sqrt{2}$, calculeu el coeficient de $(z-1)^{-2}$.
3. Calculeu, tot considerant una integració de variable complexa sobre el cercle unitari ($z = e^{i\theta}$), la integral
$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}.$$

1. Calculem auxiliarment $\langle t^n, t^m\rangle = \int_0^1 t^2 t^n t^m dt = \int_0^1 t^{2+n+m} dt = \frac{1}{3+n+m}$. Com que $\langle 1, t\rangle = \frac{1}{4}$, trobem una base ortogonal del subespai dels polinomis de primer grau (o bé plantegem les equacions normals):

   $p_0(t) = 1$

   $p_1(t) = t - \frac{\langle t,1\rangle}{\|1\|^2} 1 = t - \frac{1/4}{1/3} = t - \frac{3}{4}$

   Llavors la millor aproximació és $p^*(t) = c_0 p_0(t) + c_1 p_1(t)$ amb:

   $c_0 = \frac{\langle t^2,1\rangle}{\|1\|^2} = \frac{1/5}{1/3} = \frac{3}{5}$

   $c_1 = \frac{\langle t^2, t-\frac{3}{4}\rangle}{\|t-\frac{3}{4}\|^2} = \frac{1/6-(3/4)(1/5)}{1/5-2\cdot(3/4)(1/4)+(3/4)^2 1/3} = \frac{4}{3}$

   o sigui $p^*(t) = \frac{3}{5}1 + \frac{4}{3}\left(t - \frac{3}{4}\right) = -\frac{2}{5} + \frac{4}{3}t$

2. Per calcular el límit amb la mètrica d'aquest espai només cal calcular:

   $\|q_n(t) - q(t)\|^2 = \left\|\frac{2n+1}{n}t^3 + 1 - (2t^3+1)\right\|^2 = \left\|\frac{1}{n}t^3\right\|^2 = \int_0^1 t^2 \left(\frac{1}{n}t^3\right)^2 dt = \frac{1}{n^2}\int_0^1 t^8 dt = \frac{1}{9n^2} \longrightarrow_{n\longrightarrow\infty} 0$

3. Si el polinomi és $p(t) = at + b$ llavors cal que satisfaci:

   $$\left.\begin{array}{l} \langle at+b, t\rangle = 0 \\ a+b=1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} a\frac{1}{5} + b\frac{1}{4} = 0 \\ a+b=1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} a = 1-b \\ (1-b)\frac{1}{5} + b\frac{1}{4} = 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} a = 1-b \\ b\frac{1}{20} = -\frac{1}{5} \end{array}\right\} \Leftrightarrow$$

   $$\Leftrightarrow \left.\begin{array}{l} a = 5 \\ b = -4 \end{array}\right\}$$

   o sigui el polinomi $p(t) = 5t - 4$.

1. (a) Com que $\Lambda(t)$ és una funció parell, la seva sèrie de Fourier té la forma

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(\pi k t),$$

on $A_0 = \int_0^1 \Lambda(t)dt = 1/2$ i

$$\begin{aligned} A_1 &= 2\int_0^1 \Lambda(t)\cos(\pi t)dt = 2\left(\int_0^1 \cos(\pi t) - \int_0^1 t\cos(\pi t)dt\right) \\ &= 2\left(\frac{\sin(\pi t)}{\pi} - \frac{t\sin(\pi t)}{\pi} - \frac{\cos(\pi t)}{\pi^2}\bigg|_0^1\right) \\ &= \frac{4}{\pi^2}. \end{aligned}$$

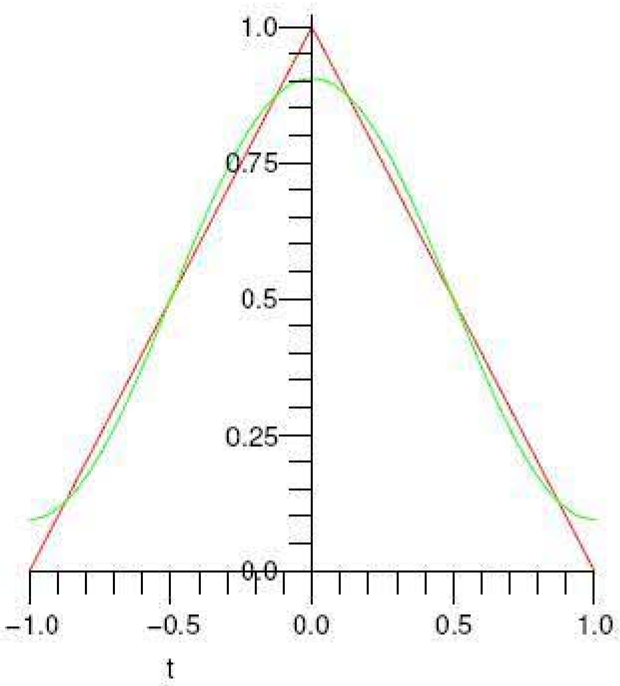


Figura 1: La gràfica de $\Lambda(t)$ i de $f_2(t)$.

Els dos primers termes de la sèrie de Fourier són $f_2(t) = \frac{1}{2} + \frac{4}{\pi^2}\cos(\pi t)$. A la Figura 1 hi ha representades les dues funcions.

(b) Tenim $\Lambda(t) = (\Pi * \Pi)(t)$ que té per transformada de Fourier

$$\hat{\Lambda}(f) = \mathcal{F}(\Lambda)(f) = (\mathcal{F}(\Pi))^2(f) = \left(\frac{\sin(\pi f)}{\pi f}\right)^2.$$

La funció $y(t)$ es pot escriure com $y(t) = \Lambda(t-1) - \Lambda(t+1)$. Fent servir les propietats de la transformada de Fourier,

$$Y(f) = e^{-j2\pi f}\mathcal{F}(\Lambda)(f) - e^{j2\pi f}\mathcal{F}(\Lambda)(f) = -\left(\frac{\sin(\pi f)}{\pi f}\right)^2 2j\sin(2\pi f).$$

(c) La funció $z(t)$ es pot escriure com $z(t) = y(t) * \delta(t) + y(t) * \delta(t-4) + y(t) * \delta(t+4) = y(t) + y(t+4) + y(t-4)$, que té per gràfica la de la figura 1.

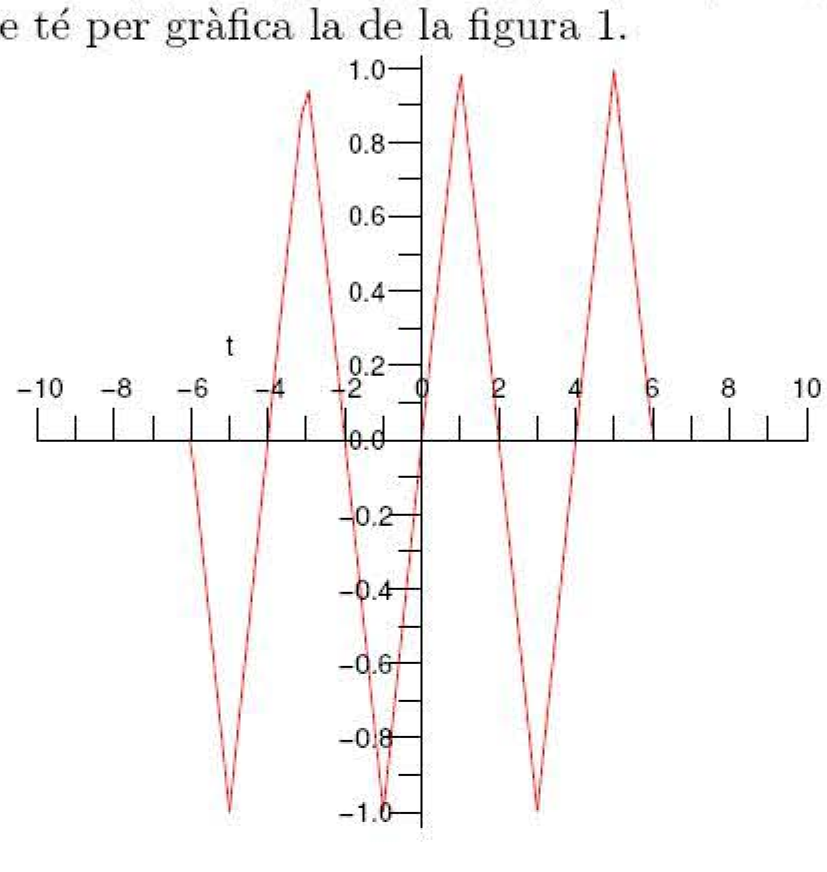


Figura 2: La gràfica de $z(t)$.

La seva transformada de Fourier és

$$\begin{aligned} Z(f) &= Y(f) + e^{j8\pi f}Y(f) + e^{-j8\pi f}Y(f) = Y(f)(1 + 2\cos(8\pi f)) \\ &= -\left(\frac{sin(\pi f)}{\pi f}\right)^2 2j\sin(2\pi f)(1 + 2\cos(8\pi f)). \end{aligned}$$

La transformada $Z(f)$ s'anul·la als zeros de $\sin(2\pi f)$ ($f = k/2,\ k \in \mathbb{Z}$) i quan $\cos(8\pi f) = -1/2$, és a dir, per a $f = (1/12) + k/4,\ k \in \mathbb{Z}$ i $f = (1/6) + k/4,\ k \in \mathbb{Z}$.

$$-z+z \quad \Big| \ \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots$$

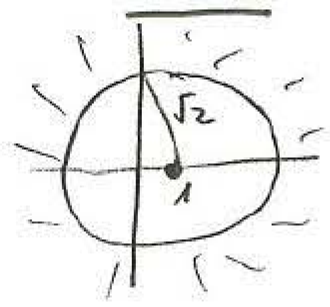
$$-\frac{z\cdot 3!}{z^3} + \frac{3!}{z^2} - \frac{3!}{10}\,\frac{1}{z} + \cdots$$

$$\frac{z - \frac{z^2}{10}\cdots}{z - \frac{z^2}{10}\cdots}$$

$$\frac{-z + \frac{z^3}{20} + \cdots}{-\frac{z^2}{10} + \frac{z^3}{20}\cdots}$$

Per tant: $z=0$ és un pol d'ordre 3 i el residu en ell és $\boxed{-\frac{3}{5}}$

P3.2



$$\frac{1}{z^2+1} = \frac{j}{2}\left(\frac{1}{z+j} - \frac{1}{z-j}\right)$$

$$|z-1| > \sqrt{2}$$

$$\left|\frac{\sqrt{2}}{z-1}\right| < 1$$

(i) $$\frac{1}{z+j} = \frac{1}{j+1+z-1} = \frac{1}{z-1}\,\frac{1}{1+\frac{j+1}{z-1}}$$

$$= \frac{1}{z-1}\left(1 - \frac{j+1}{z-1} + \left(\frac{j+1}{z-1}\right)^2 - \cdots\right)$$

(ii) $$\frac{1}{z-j} = \frac{1}{-j+1+z-1} = \frac{1}{z-1}\,\frac{1}{1+\frac{1-j}{z-1}}$$

$$= \frac{1}{z-1}\left(1 - \frac{1-j}{z-1} + \left(\frac{1-j}{z-1}\right)^2 - \cdots\right)$$

$$\boxed{C_{-2} = \frac{j}{2}\left[-j-1+1-j\right] = 1}$$

3.3

$$I = \int_{|z|=1} \frac{1}{jz}\,\frac{dz}{2+\frac{z+1/z}{2}} = \frac{2}{j}\oint_C \frac{dz}{z^2+4z+1} = -2j\oint_C \frac{\frac{1}{(z+2+\sqrt{3})}}{z+2-\sqrt{3}}\,dz$$

$$= -2j\; 2\pi j\; g(-2+\sqrt{3}) = 4\pi\,\frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \boxed{\frac{2\pi}{\sqrt{3}}}$$

TIC.

MATEL | Despatx: C3-205C

Horaris: Martes: 10-13h ; Miercoles: 10-13h

Nota: 60 EF + 40 A

Profe: Miquel Escudero

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① Propiedades del Producto Escalar

21-02-06

1) $(f, g+h) = (f, g) + (f, h)$

2) $(f, \lambda g) = \bar{\lambda}(f, g)$

3) $(f, 0) = (0, g) = 0$

4) $(f, h) = (g, h) \Rightarrow f \subset g$

⇒ Todo espacio Euclideo es un espacio normado.

⇒ Todo espacio normado es métrico

② Espais mètrics

$X \times X \longrightarrow \mathbb{R}$

$$\boxed{d(x, y) = \sqrt{(x-y, x-y)}}$$

Props

a) $d(x,y) \leq d(x,z) + d(z,y)$

b) $d(x,y) = d(y,x)$

c) $d(x,y) \geq 0$; $d(x,y) = 0 \Leftrightarrow x = y$

Desigualdad Cauchy-Schwartz

$$\boxed{|(x,y)| \leq \|x\| \cdot \|y\|}$$

Dem

$z = r \cdot e^{j\theta} = r(\cos\theta + j\sin\theta) = x + jy$

$(x,y) = |(x,y)| \cdot e^{j\theta}$

$0 \leq (x + te^{j\theta}y, x + te^{j\theta}y) = (x, x + te^{j\theta}y) + (te^{j\theta}y, x + te^{j\theta}y) =$

$= (x,x) + (x, te^{j\theta}y) + (te^{j\theta}y, x) + (te^{j\theta}y, te^{j\theta}y) = (x,x) + te^{-j\theta}(x,y) + t^2(y,y) +$

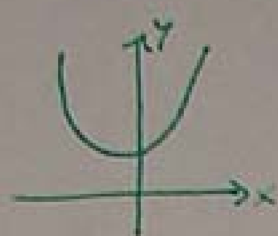
$+ te^{j\theta}(y,x) = \|x\|^2 + t\left[e^{j\theta}(y,x) + e^{-j\theta}(x,y)\right] + t^2\|y\|^2 \geq 0$

$$e^{j\theta}(y,x) + e^{-j\theta}(x,y) = e^{j\theta}\cdot|(y,x)|e^{-j\theta} + e^{-j\theta}|(x,y)|\cdot e^{j\theta} = 2|(x,y)|$$

No hauria de ser $|(x,y)|e^{-j\theta}$ ja que $(x,y) = \overline{(y,x)}$

;

$$\|x\|^2 + 2|(x,y)|t + t^2\|y\|^2 \geq 0$$



$$y = ax^2 + bx + c$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x = \frac{-2|(x,y)| \pm \sqrt{4|(x,y)|^2 - 4\|x\|^2\cdot\|y\|^2}}{2\|y\|^2}$$

$$4|(x,y)|^2 - 4\|x\|^2\cdot\|y\|^2 \leq 0$$

$$|(x,y)|^2 \leq \|x\|^2\|y\|^2 \implies \boxed{|(x,y)| \leq \|x\|\cdot\|y\|}$$

Demostració de:

$$\|f+g\| \leq \|f\| + \|g\|$$

$$\|f+g\|^2 = (f+g, f+g) = (f,f) + (g,g) + (f,g) + (g,f) = \|f\|^2 + \|g\|^2 +$$

$$+ (f,g) + \overline{(f,g)} = \|f\|^2 + \|g\|^2 + 2Re[(f,g)] \leq \|f\|^2 + \|g\|^2 + 2|(f,g)| \leq$$

$$\leq \|f\|^2 + \|g\|^2 + 2\|f\|\cdot\|g\| = (\|f\| + \|g\|)^2$$

PROBLEMA

- $\|x+y\|^2 + \|x-y\|^2 \overset{?}{=} 2\|x\|^2 + 2\|y\|^2$

$$\left.\begin{array}{l} \|x+y\|^2 = (x+y, x+y) = (x,x) + (y,y) + (x,y) + (y,x) \\ \|x-y\|^2 = (x-y, x-y) = (x,x) + (y,y) - (x,y) - (y,x) \end{array}\right\} \text{Sumem} \Rightarrow 2\|x\|^2 + 2\|y\|^2$$

INEQ

1) $|(x,y)| \leq \|x\| \cdot \|y\|$ $\to$ Dem: $0 \leq (x+te^{j\theta}y, x+te^{j\theta}y)$

2) $\|x+y\| \leq \|x\| + \|y\|$

3) $\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$

4) $|\|u\| - \|v\|| \leq \|u-v\|$

5) NO CAUCHY $\Rightarrow$ NO CONV

$\|x_n - x_m\| \xrightarrow[n,m\to\infty]{} 0$

6) $\frac{1}{K} \cdot \frac{1}{K+1} = \frac{1}{K} - \frac{1}{K+1}$

7) $e^x = \sum_{K\geq 0} \frac{x^K}{K!}$

8) L.I. $\Rightarrow$ Gram Schmit $\Rightarrow$ ORTOGONAL

$$\phi_n = f_n - \sum_{K=1}^{n-1} \frac{(f_n, \phi_K)}{\|\phi_K\|^2}\phi_K \qquad \phi_1 = f_1$$

9) COEF. FOURIER

$$c_i = \frac{(f, \phi_i)}{\|\phi_i\|^2} \to \text{Dem: } 0 = (f - \sum_1^n c_K\phi_K, \phi_i)$$

10) DESIG. BESSEL

$$\|f\|^2 \geq \sum_{K\geq 1} |c_K|^2 \cdot \|\phi_K\|^2$$ $\to$ Si c_K COEF. FOURIER $\to$ Dem: $\|f - \sum_{K=1}^n c_K\phi_K\|^2$

$c_K \cdot \|\phi_K\|^2 = (f, \phi_K)$

11) TEO. RIESZ-FISCHER

$$\sum_{K\geq 1} d_K \cdot \phi_K \text{ CONV} \Leftrightarrow \sum_{K\geq 1} |d_K|^2 \|\phi_K\|^2 \text{ CONV}$$

12) K convex + tancat + Hilbert $\Rightarrow$ té un únic vector de norma mínima.

BASES ORTO:

$L^2(0,\pi) \Rightarrow \{1, \cos(Kx)\}$ i $\{\sin(Kx)\}$

$L^2(-\ell,\ell) \Rightarrow \{1, \cos(K\frac{\pi}{\ell}x), \sin(K\frac{\pi}{\ell}x)\} \to a_K = \frac{1}{\ell}, b_K = \frac{1}{\ell}, a_0 = \frac{1}{2\ell}$

$L^2(0,\ell) \Rightarrow \{1, \cos(K\frac{\pi}{\ell}x)\} \to a_K = \frac{2}{\ell}$ $a_0 = \frac{2}{\ell}$

$\{\sin(K\frac{\pi}{\ell}x)\} \to b_K = \frac{2}{\ell}$

$L^2(0,T) \Rightarrow \{1, \cos(K\frac{2\pi}{T}x), \sin(K\frac{2\pi}{T}x)\} \to a_K = \frac{2}{T} = b_K$ $a_0 = \frac{2}{T}$

$\{1, \cos\}$

$\{\sin\}$

GAMA!!!

E.v. dim finita amb p.e

$\Downarrow$

Euclidi

$\Downarrow$

Hilbert

$L^2(a,b)$ Hilbert

Continues $\Rightarrow$ NO Hil

Totes suc. cauchy és conv $\Rightarrow$ complert

HILBERT $\Leftrightarrow$ Complert + p.e.

Euclidi $\Rightarrow$ normat $\Rightarrow$ mètric

Conv $\underset{\not\Leftarrow}{\Rightarrow}$ Cauchy

$(x,y) = 0 \Rightarrow$ ortogonals

Suma d'una serie $\to$ $\|f\|^2 = \sum |c_K|^2 \|\phi_K\|^2$

$\to$ Directament: $f(x) = \sum c_K \cdot \phi_K$

SUCC. TOTAL $\Rightarrow$ únic vect. ortog. és zero

$\hookrightarrow$ Suc total + ortogonal $\Rightarrow$ base ortogonal

Pels complexes:

$L^2(-\pi,\pi)$ $c_K = \frac{1}{2\pi}\int$

$L^2(-\ell,\ell)$ $c_K = \frac{1}{2\ell}\int$

$L^2(0,T)$ $c_K = \frac{1}{T}\int$

Pàg 18

Conv. Uniforme

$$\big| \|u\| - \|v\| \big| \leq \|u-v\|$$

$$\|u\| = \|u+v-v\| \leq \|u-v\| + \|v\| \Rightarrow \|u\| - \|v\| \leq \|u-v\|$$

$$\|v\| = \|v-u+u\| \leq \|v-u\| + \|u\| \Rightarrow \|v\| - \|u\| \leq \|v-u\| \Rightarrow -(\|u\| - \|v\|) \leq \underset{\|u-v\|}{\underset{\shortparallel}{\|v-u\|}}$$

H. Debesgue

22-02-06

$L^2(a,b) = f : (a,b) \to \mathbb{C}$ tq $\displaystyle\int_a^b |f(x)|^2 dx < +\infty \Rightarrow$ ÉS DE HILBERT

* Un altre producte escalar:

$$(f, g) = \int_a^b f(x) \cdot \overline{g(x)}\, dx$$

* Convergència en mitjana quadratica:

$$\{f_n\} \to f \Rightarrow \int_a^b |f_n(x) - f(x)|^2 dx \to 0$$

Convergència de successions:

Si X és esp. vect. i $\{f_n\}$ una successió de vectors de X

$$\{f_n\} \to f \Leftrightarrow d(f_n, f) = \|f_n - f\| \xrightarrow[n\to\infty]{} 0$$

$\forall \varepsilon > 0 \quad \exists N$ tq $\quad n \geq N \quad \|f_n - f\| < \varepsilon$

* Sabem que tota successió convergent és de Cauchy i compleixen:

$$\|f_n(x) - f_m(x)\| \xrightarrow[n,m\to\infty]{} 0$$

⇝ Si no és de Cauchy (no compleix el criteri anterior) ja sabrem que no és convergent.

⇝ Ser de Cauchy no suposa ser convergent.

Completesa:

- Quan en un espai tota successió de Cauchy és convergent, llavors l'espai és complet.

- $\mathcal{X}$ és un espai de Hilbert si és un espai dotat de producte escalar i és complet amb la mètrica corresponent.
- Un espai Euclidi és de Hilbert

ANGLES:

$$\boxed{\cos\theta = \frac{|(x,y)|}{\|x\|\cdot\|y\|}} \leq 1$$

ORTOGONALITAT:

$$\theta = 90 \Rightarrow \cos\theta = 0$$
$$\Downarrow$$
$$\boxed{(x,y) = 0}$$

- Una successió $\{\phi_n\}$ és ortogonal sii $\{\phi_n, \phi_m\} = 0 \quad \forall n \neq m$
- Una successió és ortonormal sii $\forall n \quad \|\phi_n\| = 1$

PROBLEMES

$L^2(-\pi, \pi) \Rightarrow$ Espai de Hilbert

$\Downarrow$

$\{e^{jnx}\}_{n\in\mathbb{N}}$ és successió ortogonal

$\Rightarrow$ Hem de demostrar que $\forall n,m \quad (e^{jnx}, e^{jmx}) = 0$

DEM

$$(e^{jnx}, e^{jmx}) = \int_{-\pi}^{\pi} e^{jnx}\cdot e^{-jmx}\,dx = \int_{-\pi}^{\pi} e^{j(n-m)x}\,dx = \left.\frac{e^{j(n-m)x}}{j(n-m)}\right|_{-\pi}^{\pi} =$$

$$= \frac{e^{j(n-m)\pi} - e^{-j(n-m)\pi}}{j(n-m)} = \frac{2\sin((n-m)\pi)}{(n-m)} = 0 \quad \forall n \neq m$$

És tb ortonormal?

$\rightsquigarrow \quad \|e^{jnx}\|^2 = (e^{jnx}, e^{jnx}) = \int_{-\pi}^{\pi} e^{j\,nx} \cdot e^{-jnx} \cdot dx = 2\pi$ No ho és!!

Per fer-la ortonormal $\Rightarrow \left\{ \frac{1}{\sqrt{2\pi}} e^{jnx} \right\}_{n \in \mathbb{N}}$

PROBLEMES

$$(x,y) = \int_a^b \left(x'(t) \cdot \overline{y'(t)} + \alpha \cdot x(t) \cdot \overline{y(t)} \right) dt$$

És un producte escalar? Que passa si $\alpha = 0$?

1) $(x,y) = \overline{(y,x)}$

Sabent que producte de conjugats és conjugat del producte.
I que conjugat de una suma és suma de conjgats.
S'obté fàcilment:

$$\overline{(y,x)} = \int_a^b \left(\overline{x'(t)} \cdot y'(t) + \alpha \cdot \overline{x(t)} \cdot y(t) \right) dt$$

2) $(x+y, z) = (x,z) + (y,z)$

3) $(\alpha x, y) = \alpha (x,y)$

Per la linealitat de la integral es compleix.

4) $(x,x) \geq 0$; $(x,x) = 0 \Rightarrow \vec{x} = 0$

$$(x,x) = \int_a^b \left(x'(t) \cdot \overline{x'(t)} + \alpha\, x(t) \cdot \overline{x(t)} \right) dt = \int_a^b \left(|x'(t)|^2 + \alpha\, |x(t)|^2 \right) dt \geq 0$$

$(x,x) = 0 \Rightarrow |x'(t)|^2 + \alpha |x(t)|^2 = 0 \Rightarrow \boxed{|x(t)| = |x'(t)| = 0}$

* Però si $\alpha = 0$

$|x'(t)| = 0 \Rightarrow x(t) = k \rightsquigarrow k$ <u>no té perquè ser zero</u>

PROBLEMES

Tenim les successions: $x_K = \frac{1}{K}$, $y_K = \frac{1}{K+1}$

El seu producte escalar val?

$$(x_K, y_K) = \sum_{K\geq 1} x_K \cdot \overline{y_K} = \sum_{K\geq 1} \frac{1}{K}\cdot\frac{1}{K+1} = \sum_{K\geq 1}\left(\frac{1}{K} - \frac{1}{K+1}\right) =$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)\cdots = 1 - \lim_{K\to\infty} \frac{1}{K+1} = \boxed{1}$$

PROBLEMES

$x_K = \frac{1}{K!}$ $\quad y_K = \frac{1}{2^K}$

$$e^x = \sum_{K\geq 0} \frac{x^K}{K!} = 1 + \sum_{K\geq 1}\frac{x^K}{K!}$$

$$(x_K, y_K) = \sum_{K\geq 1} \frac{1}{K!}\cdot\frac{1}{2^K} = \sum_{K\geq 1} \frac{\left(\frac{1}{2}\right)^K}{K!} = \boxed{e^{1/2} - 1}$$

PROBLEMES

$\mathbb{X} = \{ax^2 + bx + c \,;\ a,b,c \in \mathbb{R}\}$ $\rightsquigarrow$ espai dels polinomis $\mathbb{R}$ de grau 2 o inferior.

$$(p,q) = \int_0^\infty e^{-t}\cdot p(t)\cdot q(t)\,dt$$

És un espai de Hilbert amb aquest producte?

1) $(p,q) = \overline{(q,p)}$ $\rightsquigarrow$ Estem treballant amb $\mathbb{R}$, llavors $\overline{(q,p)} = (q,p)$
Per la propietat commutativa es veu que ho compleix.

2) $(p+q, r) = (p,r) + (q,r)$

$$(p+q, r) = \int_0^\infty e^{-t}\big((p+q)(t)\cdot r(t)\big)\,dt = \int_0^\infty \left(e^{-t} p(t)\cdot r(t) + e^{-t}\cdot q(t)\cdot r(t)\right)dt \quad \checkmark$$

3) $(\alpha p, q) = \alpha(p,q)$ $\rightsquigarrow$ linealitat de la integral

$\frac{1}{1}\,\frac{1}{2}\cdot\frac{1}{2!}\cdot\frac{1}{2^2}$

4) $(p \cdot p) \geq 0$; $(p \cdot p) = 0 \Rightarrow p = 0$

$$(p \cdot p) = \int_0^\infty e^{-t} \cdot p(t) \cdot p(t)\, dt \geq 0 \rightsquigarrow p(t)^2 \geq 0 \text{ i } e^{-t} > 0$$

$$(p \cdot p) = 0 \quad e^{-t} \neq 0 \ \forall t \Rightarrow p(t)^2 = 0 \Rightarrow \boxed{p(t) = 0}$$

PROBLEMA

$$p_k(t) = t^k$$

$$(p_n \cdot p_m) = \int_0^\infty e^{-t} \cdot t^n \cdot t^m\, dt = \int_0^\infty e^{-t} \cdot t^{n+m}\, dt = \Gamma(n+m+1) = (n+m)!$$

Funció Gamma

7-03-06

SUCCESSIÓ TRIGONOMÈTRICA:

$\{1, \cos(nx), \sin(nx)\} \Rightarrow$ És ortogonal a $L^2(-\pi, \pi)$

S'ha de complir: $(\phi_n, \phi_m) = 0 \ \forall n \neq m$

* $$(1, \cos(nx)) = \int_{-\pi}^{\pi} \cos(nx)\, dx = 2\int_0^{\pi} \cos(nx)\, dx = 2\left[\frac{\sin(nx)}{n}\right]_0^{\pi} = 0$$

* $$(1, \sin(nx)) = \int_{-\pi}^{\pi} \sin(nx)\, dx = 0$$

funció imparell

* $$(\cos(nx), \cos(mx)) = \int_{-\pi}^{\pi} \cos(nx) \cdot \cos(mx)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos(n+m)x + \cos(n-m)x\right) dx =$$

$$\begin{aligned} & \cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b \\ + \ & \cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b \\ \hline & \cos(a+b) + \cos(a-b) = 2 \cos a \cdot \cos b \end{aligned}$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0$$

$$* \ (\cos(nx), \sin(mx)) = \int_{-\pi}^{\pi} \cos(nx)\cdot\sin(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin(m+n) + \sin(m-n)\right]dx = 0$$

$$\begin{aligned} &\sin(a+b) = \sin a\cdot\cos b + \sin b\cdot\cos a \\ + \ &\sin(a-b) = \sin a\cdot\cos b - \sin b\cdot\cos a \\ \hline &\sin(a+b) + \sin(a-b) = 2\sin a\cdot\cos b \end{aligned}$$

$$* \ (\cos(nx), \sin(nx)) = \int_{-\pi}^{\pi} \cos(nx)\cdot\sin(nx)\,dx = 0$$

↳ parell × senar = senar

$$* \ (\sin(nx), \sin(mx)) = \int_{-\pi}^{\pi} \sin(nx)\cdot\sin(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(n+m)x - \cos(n-m)x\right] =$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} - \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0$$

$\Rightarrow$ Efectivament és ortogonal!!

a $L^2(-\pi, \pi)$

GRAM-SCHMIDT

Donada una successió de vectors linealment independents ($\{f_n\}$ L.I.) existeix una altra successió ortogonal $\{\phi_n\}$ t.q. genera el mateix s.e.v.

$$\boxed{\phi_n = f_n - \sum_{k=1}^{n-1} \frac{(f_n, \phi_k)}{\|\phi_k\|^2}\phi_k} \quad \text{on } \phi_1 = f_1$$

$\boxed{\text{PROBLEMA}}$ És ortogonal la base $\{1, t, t^2\}$. Si no ho és troba'n una.

$$(1,t) = \int_0^\infty e^{-t} \cdot t \, dt = \Gamma(2) = 1! \neq 0 \rightsquigarrow \text{No és ortogonal}$$

$\Rightarrow$ ORTOGONALITZEM

* $\phi_0 = f_0 = \underline{\underline{1}}$

* $\phi_1 = f_1 - \dfrac{(f_1, \phi_0)}{\|\phi_0\|^2} \phi_0 = t - \dfrac{(t, 1)}{\|1\|^2} \cdot 1 = \underline{\underline{t - 1}}$

$$\|1\|^2 = (1,1) = \int_0^\infty e^{-t} dt = \Gamma(1) = 0! = 1$$

* $\phi_2 = f_2 - \dfrac{(f_2, \phi_1)}{\|\phi_1\|^2} \phi_1 - \dfrac{(f_2, \phi_0)}{\|\phi_0\|^2} \phi_0 = t^2 - \dfrac{(t^2, t-1)}{\|t-1\|^2}(t-1) - \dfrac{(t^2, 1)}{\|1\|^2} \cdot 1 \Rightarrow$

$$\begin{cases} (t^2, t-1) = \displaystyle\int_0^\infty e^{-t}(t^3 - t^2)\, dt = \int_0^\infty e^{-t} \cdot t^3 dt - \int_0^\infty e^{-t} \cdot t^2 dt = \Gamma(4) - \Gamma(3) = 3! - 2! = 6 - 2 = 4 \\ (t^2, 1) = 2! = 2 \\ (t-1, t-1) = \displaystyle\int_0^\infty e^{-t}(t^2 + 1 - 2t)\, dt = \int_0^\infty e^{-t} t^2 dt + \int_0^\infty e^{-t} \cdot dt - 2\int_0^\infty e^{-t} \cdot t \, dt = \Gamma(3) + \Gamma(1) - 2\Gamma(2) = \\ \qquad = 2! + 0! - 2 \cdot 1! = 2 + 1 - 2 = 1 \end{cases}$$

$\Rightarrow$

$$\phi_2 = t^2 - \frac{4t - 4}{1} - \frac{2}{1} = t^2 - 4t + 2$$

$\underline{\{1, t-1, t^2 - 4t + 2\} \Rightarrow \text{base ortogonal}}$

TEORIA

X euclidi $\Rightarrow$ Hilbert

$\dim X = n$

$\{\phi_1, \phi_2, \phi_3, \dots, \phi_n\} \Rightarrow$ base ortogonal

Sigui $f, g \in X$ qualssevols...

$$\left.\begin{array}{l} f = \sum_{k=1}^{n} c_k \cdot \phi_k \\ g = \sum_{k=1}^{n} d_k \cdot \phi_k \end{array}\right\}$$

$$\boxed{(f, g) = \left(\sum_{k=1}^{n} c_k \cdot \phi_k , \sum_{k=1}^{n} d_k \cdot \phi_k\right) = \sum_{k=1}^{n} c_k \cdot \overline{d_k} \, \|\phi_k\|^2}$$

$$\boxed{(f, f) = \sum_{1}^{n} c_k \cdot \overline{c_k} \cdot \phi_k \cdot \overline{\phi_k} = \sum_{1}^{n} \|c_k\|^2 \|\phi_k\|^2 = \|f\|^2}$$

$$0 = \left(f - \sum_{1}^{n} c_k \cdot \phi_k , \phi_i\right) = (f, \phi_i) - \left(\sum_{1}^{n} c_k \cdot \phi_k , \phi_i\right) = (f, \phi_i) - c_i \|\phi_i\|^2$$

COEFICIENTS DE FOURIER

$$\Rightarrow \boxed{c_i = \frac{(f, \phi_i)}{\|\phi_i\|^2}}$$

Per tant: 8-03-06

$$\boxed{f = \sum_{k=1}^{n} \frac{(f, \phi_k)}{\|\phi_k\|^2} \phi_k}$$

▷ Sigui $f \in X$ un espai de Hilbert i que té $\{\phi_1, \phi_2, \dots, \phi_n, \dots\}$ com a successió ortogonal (és el mateix que la base en dimensió finita, però per a dimensió infinita).

Prenem:

$$\left\| f - \sum_{1}^{n} c_k \cdot \phi_k \right\|^2 = \left(f - \sum_{1}^{n} c_k \cdot \phi_k , f - \sum_{1}^{n} c_k \cdot \phi_k\right) = (f, f) +$$

$$- \left(f, \sum_{1}^{n} c_k \cdot \phi_k\right) - \left(\sum_{1}^{n} c_k \cdot \phi_k , f\right) + \left(\sum_{1}^{n} c_k \cdot \phi_k , \sum_{1}^{n} c_k \cdot \phi_k\right) = \|f\|^2 - \sum_{1}^{n} \overline{c_k} (f, \phi_k) - \sum_{1}^{n} c_k (\phi_k, f) + \sum_{1}^{n} c_k \overline{c_k} \|\phi_k\|^2$$

→ Tenint en compta:

$$c_k = \frac{(f, \phi_k)}{\|\phi_k\|^2} \Rightarrow c_k \cdot \|\phi_k\|^2 = (f, \phi_k)$$

$$\ldots = \|f\|^2 - \sum_1^n c_k (\phi_k, f) - \sum_1^n \overline{c_k} (f, \phi_k) + \sum_1^n c_k \overline{c_k} \|\phi_k\|^2$$

$$\Downarrow$$

$$\|f - \sum_1^n c_k \phi_k\|^2 = \|f\|^2 - \sum_1^n c_k (\phi_k, f)$$

Observem:

$$(\phi_k, f) = \overline{(f, \phi_k)} = \overline{c_k \cdot \|\phi_k\|^2} \underset{\overline{a \cdot b} = \overline{a} \cdot \overline{b}}{=} \overline{c_k} \cdot \underbrace{\|\phi_k\|^2}_{\text{És real}}$$

$$\|f - \sum_1^n c_k \phi_k\|^2 = \|f\|^2 - \sum_1^n c_k \cdot \overline{c_k} \|\phi_k\|^2 = \|f\|^2 - \sum_1^n |c_k|^2 \cdot \|\phi_k\|^2 \geq 0$$

$$\Downarrow$$

$$\boxed{\|f\|^2 \geq \sum_1^n |c_k|^2 \|\phi_k\|^2}$$

Això només passa si c_k són els coef. de Fourier.

↳ successió monotona no decreixent i acotada superiorment $\Bigg\} \Rightarrow$ És convergent: per tant té límit.

Passant al límit $(n \to \infty)$

DESIGUALTAT DE BESSEL

$$\boxed{\sum_{k \geq 1} |c_k|^2 \cdot \|\phi_k\|^2 \leq \|f\|^2}$$

$$\lim_{n \to \infty} \sum_{k=1}^n \sim = \sum_{k=1}^\infty \sim$$

TEOREMA DE RIESZ - FISCHER

Sigui $\{\phi_n\}_{n\in\mathbb{N}}$ ortogonal de Hilbert a l'espai X
↳ successió

i $\{d_n\}_{n\in\mathbb{N}}$ una successió d'escalars.

$$\boxed{\sum_{k\geq 1} d_k \cdot \phi_k \ \text{CONV} \iff \sum_{k\geq 1} |d_k|^2 \cdot \|\phi_k\|^2 \ \text{CONV}}$$

↳ Per tant com que ja sabiem que $\sum_{k\geq 1} |d_k|^2 \|\phi_k\|^2$ és CONV, llavors sabem que $\sum_{k\geq 1} d_k \cdot \phi_k$ CONV.

DEM

Considerem:

$$s_n = \sum_{k=1}^{n} d_k \cdot \phi_k$$

$$t_n = \sum_{k=1}^{n} |d_k|^2 \|\phi_k\|^2$$

Agafem: $\|s_n - s_m\|^2 = \|\sum_{m+1}^{n} d_k \phi_k\|^2 = \left(\sum_{m+1}^{n} d_k \phi_k , \sum_{m+1}^{n} d_k \cdot \phi_k\right) = \sum_{m+1}^{n} |d_k|^2 \|\phi_k\|^2 =$

$= t_n - t_m$

↳ Podem dir que $\{s_n\}$ és de Cauchy $\iff$ $\{t_n\}$ ho és

$\Rightarrow$ $\{t_n\}$ és una succ. d'escalars, per tant pertany a $\mathbb{R}$ que és complet.

$\Rightarrow$ $\{s_n\}$ és una succe. d'elements de X, que és de Hilbert, i per tant complet.

$\Rightarrow$ Tota succ. de Cauchy en un espai complert és CONV.

CONTRADICCIONS:

Estem a $L^2(-\pi,\pi)$

Considero la successió ortogonal $\{\sin(nt)\}$

Per $x(t)=\cos(t) \in L^2(-\pi,\pi)$ prenem:

$$\sum_{n\geq 1} c_k \cdot \phi_k = \sum_{n\geq 1} \frac{(\cos(t), \sin(kt))}{\|\sin(kt)\|^2} \sin(kt) = 0 \rightsquigarrow \text{Però el } \cos t \neq 0$$

en la majoria de punts de $t \in [-\pi,\pi]$

$\Rightarrow$ Sembla que la serie no sempre aproxima la funció!!

SOLUCIÓ:

▷ Haurem d'imposar que $\{\phi_n\}$ sigui no només ORTOGONAL sinó que també allò que es diu TOTAL. D'aquesta forma ortogonal i total tindrem bases ortogonals.

▷ Els elements d'un subespai euclidi pertanyen a un e.v. (conjunt) tancat.

SUCCESSIÓ TOTAL:

▷ L'únic vector ortogonal a tots els seus elements és el zero.

$$\boxed{M \oplus M^{\perp} = \mathcal{X}}$$

14-3-06

ALGEBRA

$$M^{\perp} = \{y \in \mathcal{X} \,;\, (y,x)=0 \,;\, x \in M\}$$

IDEMPOTÈNCIA

$$(M^{\perp})^{\perp} = M$$

▷ $\mathcal{X}$ de Hilbert $\rightsquigarrow$ una successió ortogonal i total s'anomena: base ortogonal

TEOREMA DE PARSEVAL

Sigui $\{\phi_k\}_{k\in\mathbb{N}}$ una base ortogonal de $\mathcal{X}$ (Hilbert)

Per tot $x, y \in \mathcal{X}$ es verifica:

$$x = \sum_{k\geq 1} c_k \phi_k \qquad (x,y) = \sum_{k\geq 1} c_k \bar{d}_k \|\phi_k\|^2$$

$$y = \sum_{k\geq 1} d_k \cdot \phi_k \qquad \|x\|^2 = \sum_{k\geq 1} |c_k|^2 \cdot \|\phi_k\|^2$$

TEO. D'EQUIVALÈNCIES

Sigui $\{\phi_k\}_{k\in\mathbb{N}}$ una successió ortogonal de Hilbert (X)

Són equivalents les següents afirmacions:

① $\{\phi_k\}$ és una successió total

$\Downarrow$ implica

② $\{\phi_k\}$ és una base ortogonal

$\Downarrow$

③ $\forall f \in X: \quad f = \sum_{k\geq 1} c_k \cdot \phi_k \Rightarrow f = \sum_{k\geq 1} \frac{(f, \phi_k)}{\|\phi_k\|^2} \phi_k$

$\Downarrow$

④ $\|f\|^2 = \sum_{k\geq 1} |c_k|^2 \|\phi_k\|^2$

DEM ④ $\Rightarrow$ ①

$(z, \phi_k) = 0, \ \forall k \overset{?}{\Rightarrow} z = 0$

Si $(z, \phi_k) = 0$ a través de ④ tenim que $\|z\| = 0$

Així doncs $z = 0$ ⇝ keda demostrat

▷ Ja sabem que la successió trigonomètrica és ortogonal en aquest espai

$\{1, \cos(kx), \sin(kx)\}_{k\in\mathbb{N}} \quad (L^2(-\pi, \pi))$

- Es pot demostrar que tb és total
- És una base ortogonal a $L^2(-\pi, \pi)$

Total suposa que verifica:

$$\left.\begin{array}{l} (f, 1) = 0 \\ (f, \cos(kx)) = 0 \\ (f, \sin(kx)) = 0 \quad \forall k \in \mathbb{N} \end{array}\right\} \Rightarrow f = 0$$

SÈRIE DE FOURIER TRIGONOMÈTRICA

$$\boxed{f \sim \frac{a_0}{2} + \sum_{k \geq 1} a_k \cdot \cos(kx) + b_k \cdot \sin(kx)}$$

$$\frac{a_0}{2} = \frac{(f, 1)}{\|1\|^2}, \quad a_k = \frac{(f, \cos(kx))}{\|\cos(kx)\|^2}, \quad b_k = \frac{(f, \sin(kx))}{\|\sin(kx)\|^2}$$

Calculem les normes:

$$\|1\|^2 = \int_{-\pi}^{\pi} dx = x\Big|_{-\pi}^{\pi} = 2\pi$$

$$\|\cos(kx)\|^2 = \int_{-\pi}^{\pi} \cos^2(kx)\, dx = \int_{-\pi}^{\pi} \left[\frac{1}{2} + \frac{1}{2}\cos(2kx)\right] dx = \frac{1}{2}\left[x + \frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi} = \pi$$

$$\|\sin(kx)\|^2 = \int_{-\pi}^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos(2kx)\right] dx = \frac{1}{2}\left[x - \frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi} = \pi$$

COEFICIENTS DE LA SÈRIE TRIGONOMÈTRICA DE FOURIER

$$\boxed{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(kx)\, dx, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(kx)\, dx}$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2(f, 1)}{\|1\|^2}}$$

PROBLEMA

a)

$L^2(-\pi, \pi)$ $\quad f(x) = x, \quad x \in [-\pi, \pi]$ $\Rightarrow$ Trobar els coef. de Fourier de la sèrie trigon.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot dx = 0$$

funció impar

Com que $x \cdot \sin(kx)$ és parell $\Rightarrow \frac{2}{\pi}\int_0^{\pi} x \sin(kx)\, dx$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin(kx)\, dx = \left\{\begin{array}{ll} u = x & du = dx \\ dv = \sin(kx) & v = \frac{-\cos(kx)}{k} \end{array}\right\} = \frac{-x \cdot \cos(kx)}{\pi k}\Bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(kx)}{k}\, dx =$$

$$= \frac{(-1)^{k+1}\cdot\pi}{\pi k} + \frac{(-1)^{k+1}\cdot\pi}{\pi k} + \cancelto{0}{\left[\frac{\sin(kx)}{k^2}\right]_{-\pi}^{\pi}} = \frac{2(-1)^{k+1}}{k}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\cos(kx)\,dx = \left\{\begin{matrix} u=\frac{x}{\pi} & du=\frac{dx}{\pi} \\ dv=\cos(kx)dx & v=\frac{\sin(kx)}{k}\end{matrix}\right\} = \left.\frac{\cancel{\sin(kx)}^{0}\cdot x}{\pi k}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(kx)}{k\pi}dx =$$

Imparell segur que dóna zero

$$= \frac{+1}{k^2\pi}\Big[\cos(kx)\Big]_{-\pi}^{\pi} = \frac{1}{k^2\pi}\Big[\cos(k\pi)-\cos(-k\pi)\Big] =$$

$$= \frac{1}{k^2\pi}\Big[\cos(k\pi)-\cos(k\pi)\Big] = 0$$

funció par

b) $f(x) = |x|$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cdot\cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi}x\cdot\cos(kx)\,dx = \left\{\begin{matrix} u=\frac{2x}{\pi} & du=\frac{2}{\pi}dx \\ dv=\cos(kx)dx & v=\frac{\sin(kx)}{k}\end{matrix}\right\} =$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cdot\sin(kx)\,dx = 0$$

funció imparell

$$= \left.\frac{2}{k\pi}\cdot x\cdot\cancel{\sin(kx)}^{0}\right]_0^{\pi} - \frac{2}{\pi k}\int_0^{\pi}\sin(k\pi)\,dx =$$

$$= \frac{+2}{\pi k}\left[\frac{\cos(k\pi)}{k}\right]_0^{\pi} \quad \frac{2}{k^2\pi}\Big[(-1)^k-1\Big]$$

$$\Rightarrow = \begin{cases} 0 & k=2m \\ \frac{-4}{(2m+1)^2\pi} & k=2m+1 \end{cases}$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{2}{2\pi}\int_0^{\pi}x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{2\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}$$

$$\boxed{|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m\geq 1}\frac{1}{(2m+1)^2}\cos\big((2m+1)x\big)}$$

c)

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases} \rightsquigarrow \text{funció contínua}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x^2\,dx\right] = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(kx)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\cdot\cos(kx)\,dx = \left\{\begin{matrix} u = x^2 & du = 2x\,dx \\ dv = \cos(kx)dx & v = \dfrac{\sin(kx)}{k} \end{matrix}\right\} =$$

$$= \frac{1}{k\pi}\left.\cancelto{0}{x^2\cdot\sin(kx)}\right]_0^{\pi} - \frac{2}{k\pi}\int_0^{\pi} x\cdot\sin(kx)\,dx = \left\{\begin{matrix} u = x & du = dx \\ dv = -\sin(kx)dx & v = \dfrac{\cos(kx)}{k} \end{matrix}\right\} =$$

$$= \frac{2}{k\pi}\left[\frac{x\cdot\cos(kx)}{k}\right]_0^{\pi} - \frac{2}{k^2\pi}\int_0^{\pi}\cos(kx)\,dx = \frac{2}{k^2\pi}\left[\pi(-1)^k - 0\right] - \frac{2}{k^2\pi}\left[\cancelto{0}{\frac{\sin(kx)}{k}}\right]_0^{\pi} =$$

$$= \frac{2(-1)^k}{k^2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(kx)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\sin(kx) = \left\{\begin{matrix} u = x^2 & du = dx\cdot 2x \\ dv = \sin(kx)dx & v = \dfrac{-\cos(kx)}{k} \end{matrix}\right\} =$$

$$= \frac{-1}{k\pi}\left. x^2\cdot\cos(kx)\right]_0^{\pi} + \frac{1}{k\pi}\int_0^{\pi} 2x\cos(kx)\,dx = \frac{-1}{k\pi}\left[\pi^2(-1)^k\right] + \frac{1}{k\pi}\left[\cancelto{0}{\frac{2x}{k}\sin(kx)}\right]_0^{\pi} - \frac{2}{k\pi}\int_0^{\pi}\frac{\sin(kx)}{k}\,dx =$$

$$\left\{\begin{matrix} u = 2x & du = 2dx \\ dv = \cos(kx)dx & v = \dfrac{\sin(kx)}{k} \end{matrix}\right.$$

$$= \frac{\pi(-1)^{k+1}}{k} + \left.\frac{2\cos(kx)}{k^3\pi}\right]_0^{\pi} = \frac{\pi(-1)^{k+1}}{k} + \frac{2\left[(-1)^k - 1\right]}{k^3\pi}$$

$$f(x) = \frac{\pi^2}{6} + 2\sum_{k\geq 1}\frac{(-1)^k}{k^2}\cos(kx) + \sum_{k\geq 1}\left[\frac{\pi}{k}(-1)^{k+1} + \frac{2}{k^3\pi}\left((-1)^k - 1\right)\right]\sin(kx) \rightsquigarrow \text{CONTINUARÀ}$$

15-03-06

CONJUNTS:

$A \subset X$ és CONVEX $\Leftrightarrow$ $\forall x, y \in A$ tq $\{\lambda x + (1-\lambda)y \,;\, \lambda \in [0,1]\} \subset A$ dos punts es poden unir per una recta que es troba dins a A completament.

A és CONNEX si dos punts d'ell es poden unir per un camí, rectilini o no.

- Tot subespai vectorial és CONVEX.
- Els conjunts tancats tenen tots els seus punts d'acumulació.

TEOREMA:

Un conjunt convex i tancat de X de Hilbert té un únic vector de norma mínima.

Colorari:

$\forall x \in X \quad \exists_1 \, y \in A$ tq $\|x-y\| \leq \|x-z\| \quad \forall z \in A \qquad A \subset X$

Existeix un únic

Descomposició ortogonal:

Sigui M un subespaivectorial de X (Hilbert) aleshores: $X = M \oplus M^{\perp}$

PROBLEMA

$X = \mathbb{R}^m$

$C = \{ x = (x^1, x^2, \dots, x^m)), \sum_{i=1}^{m} x^i = 1 \}$

a) C és convex i tancat?

b) Demostren que el vector de noma mínima ($\vec{y}$) de C ve donat per $y^k = \frac{1}{m} \; \forall k$.

C tancat

$\{x_n\} \in C \quad x_n \longrightarrow x \overset{?}{\in} C$

$x_n = (x_n^1, x_n^2, \dots, x_n^m) \longrightarrow x = (x^1, x^2, \dots, x^m)$

En $\mathbb{R} \rightarrow$

$$\begin{aligned} x_n^1 &\rightarrow x^1 \\ x_n^2 &\rightarrow x^2 \\ &\vdots \\ x_n^m &\rightarrow x^m \end{aligned}$$

Sumat tb convergeix

$$\underbrace{\sum_{i=1}^{m} x_n^i}_{\parallel} \rightarrow \sum_{i=1}^{m} x^i$$

$1 \longrightarrow 1$ → Per tant es compleix $x \in C$ i que C és tancat.

Sempre és cert

b)

$y=(y^1, y^2, \ldots, y^m)$; $\sum_{i=1}^{m} y^2 = 1$

$\|y\|$ mínima $\Leftrightarrow$ $\|y\|^2$ mínima

$\|y\|^2 = \langle y, y\rangle = \langle (y^1, y^2, \ldots, y^m), (y^1, y^2, \ldots, y^m)\rangle =$

$= (y^1)^2 + (y^2)^2 + \cdots + (y^m)^2 - \lambda\left[y^1 + y^2 + \cdots + y^m - 1\right]$ $\rightsquigarrow$ multiplicadors de Lagrange

$\frac{\partial f}{\partial y^1} = 0$, $\frac{\partial f}{\partial y^2} = 0$, $\ldots$, $\frac{\partial f}{\partial y^m} = 0$, $\frac{\partial f}{\partial \lambda} = 0$

$\frac{\partial f}{\partial y^1} = 2y^1 - \lambda$ $\quad$ $\frac{\partial f}{\partial y^2} = 2y^2 - \lambda$ $\quad$ $\frac{\partial f}{\partial y^m} = 2y^m - \lambda$ $\quad$ $\frac{\partial f}{\partial \lambda} = y^1 + y^2 + \cdots + y^m - 1$

$y^1 = y^2 = \cdots = y^m = \frac{1}{m}$

PROBLEMA $\rightsquigarrow$ continuació

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases} \rightsquigarrow f(x) = \frac{\pi^2}{6} + 2\sum_{k\geq 1} \frac{(-1)^k}{k^2} \cdot \cos(kx) + \sum_{k\geq 1}\left[\frac{\pi}{k}(-1)^{k+1} + \frac{2}{k^3\pi}\left((-1)^k - 1\right)\right]\sin(kx)$$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \cdots = \frac{\pi^2}{6}$

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$ $\rightsquigarrow$ fixem-nos que la tercera successió sorgeix de la suma de les dues anteriors.

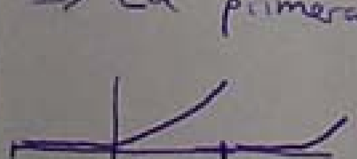
$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

$\Rightarrow$ La segona:

$f(x=0) = 0 = \frac{\pi^2}{6} + 2\sum_{k\geq 1}\frac{(-1)^k}{k^2}$ $\rightsquigarrow$ $\frac{\pi^2}{6} = -2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$

$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$\Rightarrow$ La primera:



En els punts de discontinuitat la serie de fourier suma: $\frac{f(x^+)+f(x^-)}{2}$

$\Rightarrow \overset{x=\pi}{\frac{f(\pi^+)+f(\pi^-)}{2}} = \frac{0+\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{k\geq 1}\frac{\overbrace{(-1)^{2k}}^{=1}}{k^2} \Rightarrow \frac{\pi^2}{6} = \sum_{k\geq 1}\frac{1}{k^2}$

$\boxed{\text{PROBLEMA}}$

$$I = \int_{-\pi}^{\pi} \left(\sin x - (\alpha + \beta x + \gamma x^2) \right)^2 dx \quad \text{mínim}$$

$$I = \| \sin x - (\alpha + \beta x + \gamma x^2) \|^2 \quad \text{en } L^2(-\pi, \pi)$$

Fem que els coef. de la sèrie finita $\{1, x, x^2\}$ aproximi a $\sin x$

GRAM-SCHMIT

$$\psi_0 = 1$$

$$\psi_1 = x - \frac{(x,1)}{\|1\|^2} \cdot 1 = x$$

$$\psi_2 = x^2 - \frac{(x^2,1)}{\|1\|^2} 1 - \frac{(x^2,x)}{\|x\|^2} \cdot x = x^2 - \frac{\frac{2\pi^3}{3}}{2\pi} - x \frac{0}{\frac{2 \cdot \pi^3}{3}} = x^2 - \frac{\pi^2}{3}$$

$$(x,1) = \int_{-\pi}^{\pi} x \, dx = 0 \qquad \|1\|^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$(x^2,1) = \int_{-\pi}^{\pi} x^2 dx = 2\int_0^{\pi} x^2 dx = \frac{2\pi^3}{3} = (x,x) = \|x\|^2$$

$$(x^2,x) = \int_{-\pi}^{\pi} x^3 dx = 0$$

Ara busquem els coef. de Fourier d'aquesta successió ortogonal: $\{1, x, x^2 - \frac{\pi^2}{3}\}$

$$a_0 = \frac{(\sin x, 1)}{\|1\|^2} \qquad a_1 = \frac{(\sin x, x)}{\|x\|^2} \qquad a_2 = \frac{(\sin x, x^2 - \frac{\pi^2}{3})}{\|x^2 - \frac{\pi^2}{3}\|^2}$$

$$\sin x \sim a_0 \cdot 1 + a_1 \cdot x + a_2 \left(x^2 - \frac{\pi^2}{3}\right)$$

$$\Downarrow$$

$$\alpha = a_0 - \frac{a_2 \pi^2}{3} \qquad \beta = a_1 \qquad \gamma = a_2$$

$$(\sin x, 1) = \int_{-\pi}^{\pi} \sin x \, dx = 0 \quad \Rightarrow a_0 = 0$$

$$(\sin x, x^2 - \frac{\pi^2}{3}) = \int_{-\pi}^{\pi} \underbrace{x^2 \cdot \sin x}_{0} \, dx - \frac{\pi^2}{3} \int_{-\pi}^{\pi} \underbrace{\sin x}_{0} \, dx = 0 \quad \Rightarrow a_2 = 0$$

$$\boxed{\alpha = 0} \qquad \boxed{\gamma = 0}$$

$$(\sin x, x) = \int_{-\pi}^{\pi} x\cdot \sin x\,dx = 2\int_{0}^{\pi} x\cdot \sin x\,dx = \left\{ \begin{matrix} u=2x & du=2dx \\ dv=\sin x\,dx & v=-\cos x \end{matrix} \right\} =$$

$$= -2x\cdot\cos x\Big]_{0}^{\pi} + 2\int_{0}^{\pi} \cos x\,dx = 2\pi$$

$$a_1 = \frac{(\sin x, x)}{\|x\|^2} = \frac{2\pi}{\frac{2\pi^3}{3}} = \frac{3}{\pi^2} \Longrightarrow \boxed{\beta = \frac{3}{\pi^2}}$$

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* Una altra forma de resoldre'l:

$$I = \int_{-\pi}^{\pi} \left(\sin x - (\alpha+\beta x+\gamma x^2)\right)^2 dx$$ ⇝ Es pot considerar la norma del espai: $L^2(-\pi,\pi)$

Per multiplicadors de Lagrange:

$$\frac{\partial I}{\partial \alpha} = 0 \Rightarrow \frac{\partial}{\partial \alpha}\int_{-\pi}^{\pi} \left(\sin x - (\alpha+\beta x+\gamma x^2)\right)^2 dx = \int_{-\pi}^{\pi} \frac{\partial}{\partial \alpha}\left(\sin x - (\alpha+\beta x+\gamma x^2)\right)^2 dx =$$

$$= \int_{-\pi}^{\pi} -2\left(\sin x - (\alpha+\beta x+\gamma x^2)\right) dx = -2\int_{-\pi}^{\pi}(-\alpha - \gamma x^2)\,dx = 2\alpha x + \frac{2}{3}\gamma x^3\Big]_{-\pi}^{\pi} = 0$$

$$\Longrightarrow 2\alpha\pi + 2\alpha\pi + \frac{2}{3}\gamma\pi^3 + \frac{2}{3}\gamma\pi^3 = 4\alpha\pi + \frac{4}{3}\gamma\pi^3 = 0 \Rightarrow \boxed{\alpha = \frac{-1}{3}\gamma\pi^2}$$

$$\frac{\partial I}{\partial \beta} = \int_{-\pi}^{\pi} \frac{\partial}{\partial \beta}\left(\sin x - (\alpha+\beta x+\gamma x^2)\right)^2 dx = 2\int_{-\pi}^{\pi} x\left(\sin x - (\alpha+\beta x+\gamma x^2)\right) dx =$$

$$= 2\int_{-\pi}^{\pi}(x\sin x - \beta x^2)\,dx = 2\int_{-\pi}^{\pi} x\sin x\,dx - 2\int_{-\pi}^{\pi}\beta x^2\,dx = \left\{ \begin{matrix} u=x & du=dx \\ dv=\sin x\,dx & v=-\cos x \end{matrix} \right\} =$$

$$= -2x\cos x\Big]_{-\pi}^{\pi} + 2\int_{-\pi}^{\pi}\cos x\,dx - \frac{2\beta}{3}x^3\Big]_{-\pi}^{\pi} = -2\pi(-1) + 2(-\pi)(-1) + 4\Big[\cancel{\sin x}^{\,0}\Big]_{0}^{\pi} - \frac{2\beta}{3}\pi^3 + \frac{2}{3}\beta(-\pi^3) =$$

$$= 4\pi - \frac{4}{3}\beta\pi^3 = 0 \Rightarrow \boxed{\beta = \frac{3}{\pi^2}}$$

$$\frac{\partial I}{\partial \gamma} = \int_{-\pi}^{\pi} \frac{\partial}{\partial \gamma}(\sin x - (\alpha + \beta x + \gamma x^2))^2 dx = 2\int_{-\pi}^{\pi} x^2(\sin x - (\alpha + \beta x + \gamma x^2))dx =$$

$$= 2\int_{-\pi}^{\pi}(-\alpha x^2 - \gamma x^4)dx = 2\left[\frac{-\alpha x^3}{3} - \gamma\frac{x^5}{5}\right]_{-\pi}^{\pi} = 2\left[\frac{-\alpha\pi^3}{3} - \gamma\frac{\pi^5}{5} - \frac{\alpha\pi^3}{3} - \gamma\frac{\pi^5}{5}\right] =$$

$$= -2\left[\frac{2}{3}\alpha\pi^3 + \frac{2}{5}\gamma\pi^5\right] = 0 \Rightarrow \boxed{\alpha = -\frac{3}{5}\gamma\pi^2} \Rightarrow \boxed{\alpha = \gamma = 0}$$

Si existeix una $f \in L^2(0,\pi)$ tq

$L^2(0,\pi)$

$$\left.\begin{array}{l}(f,1)=0\\ \forall k \quad (f,\cos(kx))=0\end{array}\right\}$$ es tracta de veure que això implica que $f=0$

↳ $\boxed{\text{Això ho fem per veure que } \{1, \cos(kx)\} \text{ és base ortogonal (per tant total)}}$ a $L^2(0,\pi)$

* Funció perllongació parell.

$$\tilde{f}(x) = \begin{cases} f(-x) & -\pi < x < 0 \\ f(x) & 0 < x < \pi \end{cases}$$ $\rightsquigarrow$ és el mateix f d'abans

$\tilde{f}(x) \in L^2(-\pi,\pi)$

Sabem que si:

A $\Rightarrow$

$$\left.\begin{array}{l}(\tilde{f},1)=0\\ (\tilde{f},\cos(kx))=0\\ (\tilde{f},\sin(kx))=0\end{array}\right\}$$

aleshores $\tilde{f}$ val zero ($\tilde{f}=0$) perquè $\{1, \cos(kx), \sin(kx)\}_{k\in\mathbb{N}}$ és una base ortogonal $\Rightarrow$ total de l'espai $L^2(-\pi,\pi)$. A més a més si $\tilde{f}=0 \Rightarrow \boxed{f=0}$

Anem a comprobar A...

$$(\tilde{f},1) = \int_{-\pi}^{\pi}\tilde{f}(x)dx = 2\int_{0}^{\pi} f(x)dx = 0$$

↑ funció parell

↑ Recordem que $f(x)=g(x)$

$$(\tilde{f}, \cos(kx)) = \int_{-\pi}^{\pi} \tilde{f}(x)\cdot\cos(kx)\,dx = 2\int_0^{\pi} f(x)\cdot\cos(kx)\,dx \underset{\substack{f(x)=g(x)\\ \text{i } (g,\cos(kx))=0}}{=} 0$$

$$(\tilde{f}, \sin(kx)) = \int_{-\pi}^{\pi} \tilde{f}(x)\cdot\sin(kx)\,dx \underset{\text{funció imparell}}{=} 0$$

$\Rightarrow$ Per tant: $\boxed{\{1, \cos(kx)\}_{k\in\mathbb{N}} \text{ és base ortogonal de } L^2(0,\pi)}$

Ara anem a veure que $\boxed{\{\sin(kx)\}_{k\in\mathbb{N}} \text{ és base ortogonal de } L^2(0,\pi)}$

Dem:

Hem de veure que:
- és una successió ortogonal ✓ $\Longrightarrow$ Ja s'ha vist!!
- és " " total

TOTAL?

$$\underbrace{\forall k \ (f, \sin(kx)) = 0}_{\text{Suposició}} \overset{?}{\Longrightarrow} f = 0$$

Funció de $L^2(-\pi,\pi)$:
$$\tilde{f}_1(x) = \begin{cases} -f(-x) & -\pi < x < 0 \\ f(x) & 0 < x < \pi \end{cases}$$

Si succeeix que:

$(\tilde{f}_1, 1) = 0$, $(\tilde{f}_1, \cos(kx)) = 0$, $(\tilde{f}_1, \sin(kx)) = 0$

Com que $\{1, \cos(kx), \sin(kx)\}_{k\in\mathbb{N}}$ és base ortogonal (total) $\Rightarrow$ $\tilde{f} = 0 \Rightarrow f = 0$

Suposant que els 3 productes escalars anteriors siguin tots nuls.

$$(\tilde{f}_1, 1) = \int_{-\pi}^{\pi} \tilde{f}_1\,dx \underset{\text{imparell}}{=} 0 \qquad (\tilde{f}_1, \cos(kx)) = \int_{-\pi}^{\pi} \tilde{f}_1\cdot\cos(kx)\,dx \underset{\text{imparell}}{=} 0$$

$$(\tilde{f}_1, \sin(kx)) = \int_{-\pi}^{\pi} \tilde{f}_1(x)\cdot\sin(kx)\,dx = 2\int_0^{\pi} f(x)\cdot\sin(kx) \underset{\text{Suposició inicial}}{=} 0$$

$\boxed{\text{PROBLEMA}}$

$f(x) = x(\pi - x)$ Desenvolupeu-la en serie de sinus a $x \in [0,\pi]$

$f(x) \sim \sum_{k \geq 1} b_k \cdot \sin(kx)$ a l'espai $L^2(0,\pi)$

$$b_k = \frac{2}{\pi}\int_0^\pi f(x)\cdot \sin(kx)\,dx \Leftarrow b_k = \frac{(f, \sin(kx))}{\|\sin(kx)\|^2} \quad \text{on } \|\sin(kx)\|^2 = \frac{\pi}{2}$$

$$b_k = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\sin(kx)\,dx = \underbrace{\frac{2\pi}{\pi}\int_0^\pi x\cdot\sin(kx)\,dx}_{A} - \underbrace{\frac{2}{\pi}\int_0^\pi x^2\cdot\sin(kx)\,dx}_{B}$$

A)
$$\int_0^\pi 2x\sin(kx)\,dx = \left\{\begin{array}{ll} u = 2x & du = 2\,dx \\ dv = \sin(kx)\,dx & v = \frac{-\cos(kx)}{k}\end{array}\right\} = \left.\frac{-2x\cos(kx)}{k}\right|_0^\pi + \frac{2}{k}\int_0^\pi \cos(kx)\,dx =$$

$$= \frac{-2\pi(-1)^k}{k} + \frac{2}{k^2}\Big[\cancel{\sin(kx)}^{\,0}\Big]_0^\pi = \frac{2\pi(-1)^{k+1}}{k}$$

B)
$$\int_0^\pi \frac{-2}{\pi}x^2\cdot\sin(kx)\,dx = \left\{\begin{array}{ll} u = \frac{-2}{\pi}x^2 & du = \frac{-4}{\pi}x \\ dv = \sin(kx)\,dx & v = \frac{-\cos(kx)}{k}\end{array}\right\} =$$

$$= \left.\frac{2}{\pi k}x^2\cos(kx)\right|_0^\pi - \frac{4}{k\pi}\int_0^\pi x\cdot\cos(kx)\,dx = \left\{\begin{array}{ll} u = x & du = dx \\ dv = \cos(kx)\,dx & v = \frac{\sin(kx)}{k}\end{array}\right\} =$$

$$= \frac{2}{\pi k}\left[\pi^2(-1)^k\right] - \frac{4}{k\pi}\left[\left.\frac{x\,\cancel{\sin(kx)}^{\,0}}{k}\right|_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)\,dx\right] = \frac{2\pi}{k}(-1)^k + \frac{4}{k^2\pi}\left[\frac{-\cos(kx)}{k}\right]_0^\pi =$$

$$= \frac{2\pi}{k}(-1)^k - \underbrace{\frac{4}{k^3\pi}\left[(-1)^k - 1\right]}_{0,\ k\in 2m \quad \frac{+8}{k^3\pi},\ k\in 2m+1} = -\frac{4}{k^3\pi}\left[(-1)^k - 1\right] - \frac{2\pi}{k}(-1)^{k+1}$$

Hem multiplicat
$(-1)\cdot(-1)\cdot\frac{2\pi}{k}(-1)^k \Rightarrow$ Ara desapareix amb el terme de A)

$$b_k = \frac{2\pi(-1)^{k+1}}{k} + \frac{2\pi(-1)^k}{k} - \underbrace{\frac{4}{k^3\pi}\left[(-1)^k - 1\right]}$$

$0 \Leftarrow k \in 2m$ on $m \in \mathbb{N}$

$\frac{-8}{k^3\pi} \Leftarrow k \in 2m-1$

$$b_k = \frac{2\pi(-1)^{k+1}}{k} - \frac{(-1)\,2\pi(-1)^k}{k} - \frac{4}{k^3\pi}\left[(-1)^k - 1\right]$$

$$b_k = \cancel{\frac{2\pi(-1)^{k+1}}{k}} - \cancel{\frac{2\pi(-1)^{k+1}}{k}} - \frac{4}{k^3\pi}\left[(-1)^k - 1\right]$$

$$b_k = \frac{8}{(2m-1)^3\pi} \quad \text{on } m \in \mathbb{N} \quad \Rightarrow \quad \boxed{f(x) \sim \frac{8}{\pi}\sum_{m\geq 1}\frac{1}{(2m-1)^3}\sin\big((2m-1)x\big)}$$

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Mínim de: $\displaystyle\int_0^\pi \left(f(x) - c_1\cdot\sin x\right)^2 dx$?

$$\|f(x) - c_1\cdot\sin x\|^2 = \|x(\pi - x) - \tfrac{8}{\pi}\sin x\| =: (\Delta)$$

$$(f(x) - c_1\cdot\sin x,\, f(x) - c_1\cdot\sin x) = \|f\|^2 + |c_1|^2 \overset{\|\sin x\|^2}{} - c_1(\sin x, f) - \bar{c}_1(f, \sin x) =$$

$$\underset{(*)}{=} \|f\|^2 - \bar{c}_1(f, \sin x) = \|f\|^2 - |c_1|^2\cdot\|\sin x\|^2$$

$(*)$ $\displaystyle c_1 = \frac{(f, \sin x)}{\|\sin x\|^2} \Rightarrow c_1\cdot\|\sin x\|^2 = (f, \sin x) \Rightarrow \bar{c}_1(f, \sin x) = \bar{c}_1\left[c_1\|\sin x\|^2\right] =$

$$= |c_1|^2\,\|\sin x\|^2$$

$$c_1(\sin x, f) = c_1\,\overline{(f, \sin x)} = c_1\left[\bar{c}_1\cdot\overline{\|\sin x\|^2}\right] = \underset{\bar{c}_1(f,\sin x)}{|c_1|^2\|\sin x\|^2}$$

$$(\Delta) = \|x(\pi - x)\|^2 - \frac{8}{\pi}\cdot\frac{8}{\pi}\cdot\frac{\pi}{2} = \boxed{\frac{\pi^5}{30} - \frac{32}{\pi}}$$

$$\|\sin x\|^2 = \int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}$$

$$\|x(\pi - x)\|^2 = \int_0^\pi x^2(\pi - x)^2\,dx = \underbrace{\int_0^\pi \pi^2 x^2\,dx}_{\frac{\pi^5}{3}} + \underbrace{\int_0^\pi x^4\,dx}_{\frac{\pi^5}{5}} - \underbrace{\int_0^\pi 2\pi x^3\,dx}_{-\frac{2\pi^5}{4} = \frac{\pi^5}{2}} = \frac{\pi^5}{30}$$

PROBLEMA

$$\sum_{k\geq 1} \frac{1}{(2k-1)^6} \quad ?$$

Parseval:

$$\|f\|^2 = \sum_{k\geq 1} |c_k|^2 \cdot \|\varphi_k\|^2$$

$$(f,g) = \sum_{k\geq 1} c_k \cdot \overline{d_k} \cdot \|\varphi_k\|^2$$

$$\|x(\pi-x)\|^2 = \sum_{k\geq 1} \frac{64}{\pi^2} \frac{1}{(2k-1)^6} \|\sin(2k-1)x\|^2$$

$$\Downarrow$$

$$\|x(\pi-x)\|^2 = \frac{\pi^5}{30}, \qquad \|\sin(2k-1)x\|^2 = \frac{\pi}{2}$$

$$\boxed{\sum_{k\geq 1} \frac{1}{(2k-1)^6} = \frac{\pi^5}{30}\cdot\frac{\pi^2}{64}\cdot\frac{2}{\pi}}$$

Nou espai vectorial de Hilbert:

$$\boxed{L^2(-\ell,\ell) \rightsquigarrow \text{base ortogonal: } \left\{1, \cos\left(k\frac{\pi}{\ell}x\right), \sin\left(k\frac{\pi}{\ell}x\right)\right\}}$$

Dem:

$$L^2(-\ell,\ell) \qquad t = \frac{\pi x}{\ell} \qquad L^2(-\pi,\pi)$$

$$\left\{1, \cos\frac{k\pi x}{\ell}, \sin\frac{k\pi x}{\ell}\right\} \rightsquigarrow \{1, \cos kt, \sin kt\}$$

Els seus coef. de Fourier són:

$$\boxed{a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cdot\cos\left(k\frac{\pi}{\ell}x\right)dx} \qquad \boxed{b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cdot\sin\left(k\frac{\pi}{\ell}x\right)dx}$$

$$\boxed{a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx}$$ $\rightsquigarrow$ Correcte?

Altres espais:

$L^2(0,\ell)$ $\quad \left\{1, \cos\left(k\frac{\pi}{\ell}x\right)\right\}_{k\in\mathbb{N}} \to$ base ortogonal

$$f(x) \approx \frac{a_0}{2} + \sum_{k\geq 1} a_k\cdot\cos\left(k\frac{\pi}{\ell}x\right) \rightsquigarrow a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)dx$$

$$a_k = \frac{2}{\ell}\int_0^{\ell} f(x)\cdot\cos\left(k\frac{\pi}{\ell}x\right)dx$$

$L^2(0,\ell)$ $\quad \left\{ \sin\left(k\frac{\pi}{\ell}x\right) \right\}_{k\in\mathbb{N}} \to$ base ortogonal

$$f(x) \sim \sum_{k\geq 1} b_k \cdot \sin\left(k\frac{\pi}{\ell}x\right) \quad \to \quad b_k = \frac{2}{\ell}\int_0^{\ell} f(x)\cdot \sin\left(k\frac{\pi}{\ell}x\right)dx$$

$L^2(0,T)$

$$\left\{ 1,\ \cos\left(k\frac{2\pi}{T}x\right),\ \sin\left(k\frac{2\pi}{T}x\right)\right\}_{k\in\mathbb{N}}$$

$$\left\{ 1,\ \cos\left(k\frac{2\pi}{T}x\right)\right\}_{k\in\mathbb{N}}$$

$$\left\{ \sin\left(k\frac{2\pi}{T}x\right)\right\}_{k\in\mathbb{N}}$$

Dem:

$[0, T] \xrightarrow{\ t = \frac{2\pi}{T}x - \pi\ } [-\pi, \pi]$

$$\left\{ 1,\ \cos(kt - k\pi),\ \sin(kt-k\pi)\right\}_{k\in\mathbb{N}}$$

$$\shortparallel$$

$$\left\{ 1,\ \cos(kt)\cdot\underbrace{\cos(+k\pi)}_{(-1)^k} + \sin(kt)\cdot \overset{0}{\sin(k\pi)},\ \sin(kt)\cdot\underbrace{\cos(k\pi)}_{(-1)^k} - \overset{0}{\sin(k\pi)}\cos(kt)\right\}_{k\in\mathbb{N}}$$

$$\shortparallel$$

$$\left\{ 1,\ (-1)^k\cos(kt),\ (-1)^k\sin(kt)\right\}_{k\in\mathbb{N}}$$ $\rightsquigarrow$ Les constants $(-1)^k$ no importen, segueix sent una base ortogonal.

$L^2(0,T)$

$$f(x) \sim \frac{a_0}{2} + \sum_{k\geq 1} a_k\cdot\cos\left(k\frac{2\pi}{T}x\right) + b_k \sin\left(k\frac{2\pi}{T}x\right)$$

$$a_k = \frac{2}{T}\int_0^T f(x)\cos\left(k\frac{2\pi}{T}x\right)dx \qquad b_k = \frac{2}{T}\int_0^T f(x)\cdot\sin\left(k\frac{2\pi}{T}x\right)dx$$

$$\left\|\cos\left(k\frac{2\pi}{T}x\right)\right\|^2 = \int_0^T \cos^2\left(k\frac{2\pi}{T}x\right)dx = \int_0^T \frac{1+\cos\left(k\frac{4\pi}{T}x\right)}{2}dx = \frac{T}{2}$$

$$\frac{a_0}{2} = \frac{1}{T}\int_0^T f(x)\,dx$$ $\rightsquigarrow$ Correcte ?

SERIES COMPLEXES:

$L^2(-\pi,\pi)$ $\quad \{e^{jkx}\}_{k\in\mathbb{Z}} \longrightarrow$

$$\boxed{f(x) \sim \sum_{-\infty}^{+\infty} c_k \cdot e^{jkx} \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-jkx}\,dx}$$

$$\|e^{jkx}\|^2 = \int_{-\pi}^{\pi} e^{jkx}\cdot e^{-jkx}\,dx = 2\pi$$

$L^2(-\ell,\ell)$ $\quad \{e^{jk\frac{\pi}{\ell}x}\}_{k\in\mathbb{Z}} \longrightarrow$

$$\boxed{f(x) \sim \sum_{-\infty}^{+\infty} c_k \cdot e^{jk\frac{\pi}{\ell}x} \qquad c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\cdot e^{-jk\frac{\pi}{\ell}x}\,dx}$$

$L^2(0,T)$ $\quad \{e^{jk\frac{2\pi}{T}x}\}_{k\in\mathbb{Z}} \longrightarrow$

$$\boxed{f(x) \sim \sum_{-\infty}^{+\infty} c_k \cdot e^{jk\frac{2\pi}{T}x} \qquad c_k = \frac{1}{T}\int_{0}^{T} f(x)\cdot e^{-jk\frac{2\pi}{T}x}}$$

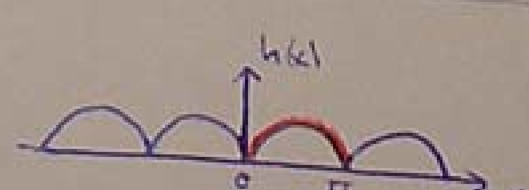
PROBLEMA

$f(x) = \sin^2(x) \qquad -\infty < x < +\infty \quad \rightsquigarrow L^2(-\infty,\infty)$ ó $L^2(\mathbb{R})$

$\boxed{f(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)}$ → Desenvolupament en sèries de Fourier

$g(x) = \cos^2(x) \quad \rightarrow \boxed{f(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)}$

$h(x) = |\sin x|$



$$|\sin(x)| = \frac{a_0}{2} + \sum_{k\geq 1} a_k\cdot\cos(kx) \qquad L^2(0,\pi)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{-2}{\pi}\Big[\cos x\Big]_0^{\pi} = \frac{1+1}{\pi/2} = \frac{4}{\pi} \rightsquigarrow \frac{a_0}{2} = \frac{2}{\pi}$$

$k=1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cdot\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(2x)\,dx = \frac{1}{\pi}\left[\frac{-\cos(2x)}{2}\right]_0^{\pi} = \frac{1}{2\pi}\Big[1-1\Big] = 0$$

$\underline{k \neq 1}$

$$a_k = \frac{1}{\pi}\int_0^{\pi}\left[\sin(1+k)x + \sin(1-k)x\right]dx = \frac{-1}{\pi}\left[\frac{\cos(1+k)x}{1+k} + \frac{\cos(1-k)x}{1-k}\right]_0^{\pi} =$$

$$= \frac{1}{\pi}\left[\frac{\cos(k-1)x}{k-1} - \frac{\cos(k+1)x}{k+1}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{(-1)^{k-1}-1}{k-1} - \frac{(1)^{k+1}-1}{k+1}\right]$$

$\underline{\text{Si } k = 2n \text{ on } n \in \mathbb{N}}$

$$a_k = \frac{1}{\pi}\left[\frac{-2}{2n-1} - \frac{-2}{2n+1}\right] = \frac{-2}{\pi}\left[\frac{2n+1-(2n-1)}{4n^2-1}\right] = \frac{-4}{\pi(4n^2-1)}$$

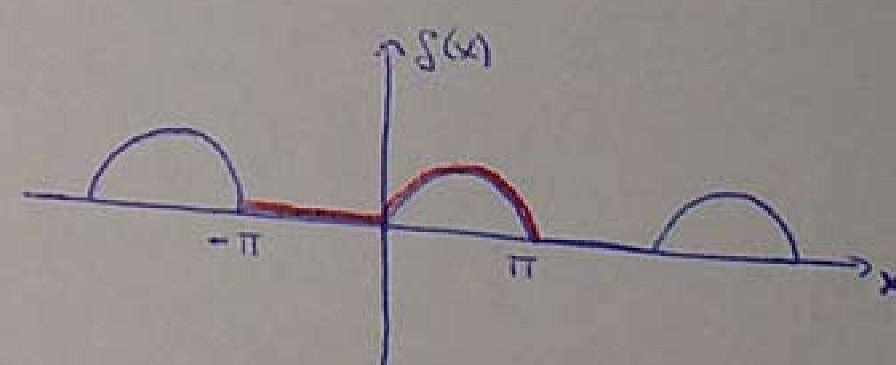
$\underline{\text{Si } k = 2n-1 \text{ on } n \in \mathbb{N}}$

$$a_k = 0$$

$$\boxed{|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1}\frac{1}{4n^2-1}\cos(2nx)}$$

PROBLEMA

$$g(x) = \begin{cases} \sin x & \sin x > 0 \\ 0 & \sin x < 0 \end{cases}$$



$$g(x) = \frac{a_0}{2} + \sum_{k\geq 1} a_k\cos(kx) + b_k\sin(kx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi} \rightsquigarrow \frac{a_0}{2} = \frac{1}{\pi}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cdot\cos kx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\cdot\cos kx = \begin{cases}\dfrac{2}{\pi(1-4n^2)} & k = 2n \\ 0 & k = 2n-1\end{cases} \quad \text{on } n \in \mathbb{N}$$

$$b_k = \frac{1}{\pi}\int_0^{\pi}\sin x\cdot\sin(kx)\,dx = \begin{cases}\frac{1}{2} & k = 1 \\ 0 & k \neq 1\end{cases}$$

$$g(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{2}{\pi}\sum_{k\geq 1} \frac{1}{1-4n^2}\cos(2nx)$$

28-03-06

PROBLEMA

$f(x) \leftarrow 0<x<\pi$

Bases de $L^2(0,\pi)$: a) $\{1, \cos(kx)\}_{k\in\mathbb{N}}$

b) $\{\sin(kx)\}_{k\in\mathbb{N}}$

c) $\{e^{j2kx}\}_{k\in\mathbb{Z}}$

a) $$f(x) = \frac{a_0}{2} + \sum_{k\geq 1} a_k \cdot \cos(kx)$$

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$a_k = \frac{2}{\pi}\int_0^\pi x\cdot\cos(kx)dx = \left\{\begin{array}{ll} u=x & du=dx \\ dv=\cos(kx)dx & v=\frac{\sin(2kx)}{k}\end{array}\right\} =$$

$$= \frac{2}{\pi}\left[\cancel{\frac{x\sin(kx)}{k}}^{\,0}\Big|_0^\pi - \int_0^\pi \frac{\sin(kx)}{k}dx\right] =$$

$$= \frac{2}{\pi}\left[\frac{-\cos(kx)}{k^2}\right]_0^\pi = 2\frac{(-1)^k+1}{k^2\pi} = \begin{cases} 0 & k=2n \\ \frac{4}{k^2\pi} & k=2n-1 \end{cases}$$

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n\geq 1}\frac{4}{(2n-1)^2\pi}\cos((2n-1)x)}$$

b) $$\boxed{f(x) = 2\sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\cdot\sin(kx)}$$

PROBLEMA

$$S_1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Agafem l'apartat b) d'abans i evaluem per $x=\frac{\pi}{2}$ ...

$$\frac{\pi}{2} = 2\sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\sin(kx) \rightsquigarrow k=2n-1$$

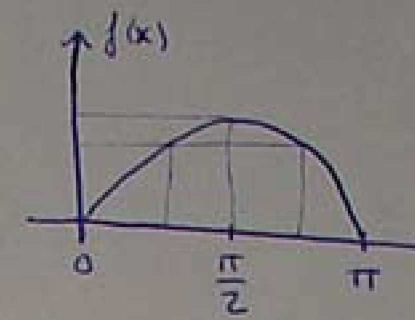
$$\frac{\pi}{2} = 2\sum_{n\geq 1}\frac{(-1)^{2n}}{2n-1}(-1)^{n+1}$$

$$S_1 = \sum_{n\geq 1}\frac{\overbrace{(-1)^{2n}}^{1}}{2n-1}(-1)^{n+1} = \boxed{\frac{\pi}{4}}$$

PROBLEMA

Demostreu que a $L^2(0,\pi)$ amb $f(x)=f(\pi-x)$ ...

a) $f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1} a_{2n}\cos(2nx)$



$f(0)=f(\pi)$
$f(\pi)=f(0)$
$f(\frac{\pi}{2})=f(\frac{\pi}{2})$

b) $f(x) \sim \sum_{n\geq 1} b_{2n-1}\cdot \sin((2n-1)x)$

a) Agafem la base $\{1, \cos(kx)\}$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cdot\cos(kx)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cdot\cos(kx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\cdot\cos(kx)dx \underset{x=\pi-t}{=}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cdot\cos(kx)dx + \frac{-2}{\pi}\int_{\frac{\pi}{2}}^{0} f(\pi-t)\cdot\cos(k\pi-kt)\,dt =$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos(kx)dx + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(\pi-t)\cdot\cos(k(\pi-t))dt \xrightarrow[\text{si } k=2n-1]{} \cos((2n-1)(\pi-t)) = (-1)\cos((2n-1)t)$$

$$\underset{f(x)=f(\pi-t)}{=} \frac{2}{\pi}\int_0^{\pi/2} f(x)\cdot\cos((2k-1)x)\,dx - \frac{2}{\pi}\int_0^{\pi/2} f(x)\cdot\cos((2k-1)x)\,dx = 0 \rightsquigarrow \text{S'anul·la si } k=2n-1,\ n\in\mathbb{N}$$

b)
$$b_k = \frac{2}{\pi}\int_0^\pi f(x)\cdot\sin(kx)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cdot\sin(kx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\cdot\sin(kx)\,dx \underset{\pi-t=x}{=}$$

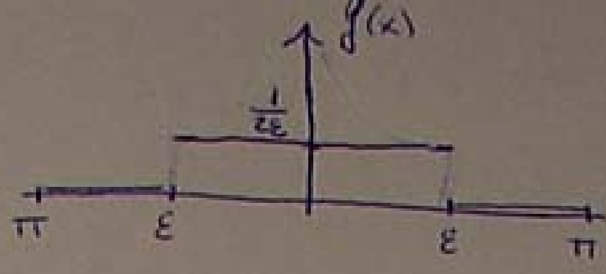
$$= \frac{2}{\pi}\int_0^{\pi/2} f(x)\cdot\sin(kx)dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} f(\pi-t)\cdot\sin(k\pi-kt)\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin(kx)dx + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(\pi-t)\cdot\sin(k\pi-kt)dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin(2nx)dx - \frac{2}{\pi}\int_0^{\pi/2} f(x)\sin(2nx)\,dx = 0 \rightsquigarrow \text{S'anul·la si } k=2n,\ n\in\mathbb{N}$$

$\sin(2\pi-2t)=\sin(-2t)=-\sin(2t)$ i $f(x)=f(\pi-t)$
$\Downarrow$
$\sin(2n\pi-2nt)=\sin(-2nt)=-\sin(2nt)$

PROBLEMA

$$f(x) = \begin{cases} \dfrac{1}{2\varepsilon} & |x| < \varepsilon \\ 0 & \varepsilon < |x| < \pi \end{cases}$$



$\rightsquigarrow L^2(-\pi, \pi)$

Veiem que: $f(x) = \dfrac{a}{2} + \sum\limits_{k \geq 1} a_k \cdot \cos(kx) + b_k \cdot \sin(kx)$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(kx)\, dx \underset{\substack{\uparrow \\ \text{imparell}}}{=} 0$$

$\Downarrow$

Desenvolupem en $L^2(0,\pi)$ amb la base: $\{1, \cos(kx)\}_{k \in \mathbb{N}}$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\varepsilon} \frac{1}{2\varepsilon}\, dx = \frac{1}{\pi}$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos(kx)\, dx = \frac{2}{\pi} \int_0^{\varepsilon} f(x) \cdot \cos(kx) = \frac{2}{\pi} \int_0^{\varepsilon} \frac{1}{2\varepsilon} \cos(kx)\, dx = \frac{1}{\pi\varepsilon} \left[\frac{\sin(kx)}{k} \right]_0^{\varepsilon} =$$

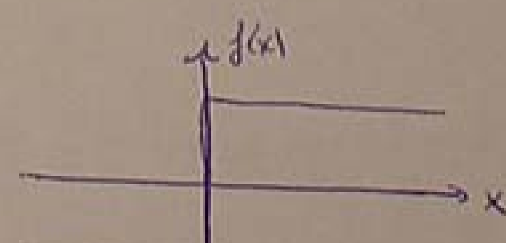
$$= \frac{\sin(k\varepsilon)}{\pi k \varepsilon}$$

$$\boxed{f(x) \sim \frac{1}{2\pi} + \sum_{k \geq 1} \frac{\sin(k\varepsilon)}{\pi k \varepsilon} \cos(kx)} \rightsquigarrow \boxed{\lim_{\varepsilon \to 0} f(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k \geq 1} \cos(kx)}$$

↳ Funció delta de Dirac desenvolupada en sèries de Fourier

PROBLEMA

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

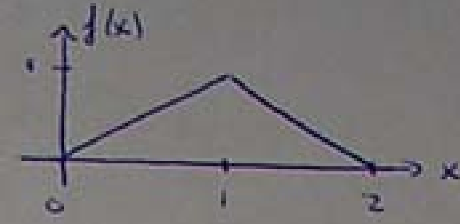


$f(x) = \operatorname{sing}(x) \rightsquigarrow L^2(-1, 1)$

$\rightsquigarrow$ Veiem que els $a_k = 0 \Rightarrow L^2(0,1)$ $\{\sin(k\pi x)\}_{k \in \mathbb{N}}$

$$b_k = \frac{2}{1} \int_0^1 \sin(k\pi x)\, dx = -2 \left[\frac{\cos(k\pi x)}{k\pi} \right]_0^1 = -2\, \frac{(-1)^k - 1}{k\pi} = \begin{cases} 0 & k = 2n \\ \dfrac{4}{k\pi} & k = 2n-1 \end{cases} \quad \text{on } n \in \mathbb{N}$$

$$f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases}$$



$\rightsquigarrow L^2(0,2)$

$$f(x) = \sum_{k\geq 1} b_k \cdot \sin\left(k\frac{\pi}{2}x\right)$$

$$b_k = \frac{2}{2}\int_0^2 f(x)\cdot \sin\left(\frac{k\pi x}{2}\right)dx = \underbrace{\int_0^1 x\cdot\sin\left(\frac{k\pi x}{2}\right)dx}_{A} + \underbrace{\int_1^2 (2-x)\sin\left(\frac{k\pi x}{2}\right)dx}_{B}$$

A) $$= \left\{ \begin{array}{ll} u=x & du=dx \\ dv=\sin\left(\frac{k\pi x}{2}\right)dx & v=\dfrac{-\cos\left(\frac{k\pi x}{2}\right)}{\frac{k\pi}{2}} \end{array} \right\} = \frac{-\cos\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}} + \int_0^1 \frac{\cos\left(\frac{k\pi x}{2}\right)}{\frac{k\pi}{2}}dx =$$

$$= \frac{-\cos\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}} + \frac{1}{\frac{k\pi}{2}}\left[\frac{\sin\left(\frac{k\pi x}{2}\right)}{\frac{k\pi}{2}}\right]_0^1 = \frac{2}{k\pi}\left[\frac{\sin\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}} - \cos\left(\frac{k\pi}{2}\right)\right]$$

B) $$2\int_1^2 \sin\left(\frac{k\pi x}{2}\right)dx = 2\left[\frac{-\cos\left(\frac{k\pi x}{2}\right)}{\frac{k\pi}{2}}\right]_1^2 = 2\left[\frac{-\cos(k\pi)+\cos\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}}\right]$$

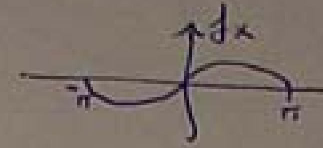
$$-\int_1^2 x\sin\left(\frac{k\pi x}{2}\right)dx = \frac{2\cos(k\pi)-\cos\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}} - \int_1^2 \frac{\cos\left(\frac{k\pi x}{2}\right)dx}{\frac{k\pi}{2}} =$$

$$= \frac{2\cos(k\pi)-\cos\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}} - \left[\frac{\sin\left(\frac{k\pi x}{2}\right)}{\left(\frac{k\pi}{2}\right)^2}\right]_1^2 = \frac{2}{k\pi}\left[2\cos(k\pi)-\cos\left(\frac{k\pi}{2}\right) - \frac{\sin(k\pi)-\sin\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}}\right]$$

$$b_k = \frac{2}{k\pi}\left[-2\cos\left(\frac{k\pi}{2}\right) + 2\cos(k\pi) - \cancelto{0}{\frac{2\sin(k\pi)}{k\pi}} + \frac{2\sin\left(\frac{k\pi}{2}\right)}{k\pi}\right] = \begin{cases} \dfrac{2}{n\pi}\left[(-1)^{n+1}+(-1)^{2n}\right] & k=2n \\ \underbrace{\dfrac{4}{(2n-1)\pi}\left[(-1)^{2n-1}+\dfrac{(-1)^{n+1}}{(2n-1)\pi}\right]}_{k=2n-1} \end{cases}$$

PROBLEMA

$f(x) = \sin x \qquad |x| < \pi$



$$f(x) = \sum_{-\infty}^{+\infty} c_k e^{jkx} \quad \Rightarrow \quad f(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$c_k = \frac{(\sin x, e^{jkx})}{\|e^{jkx}\|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x \cdot e^{-jkx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x\left(\cos(kx) - j\sin(kx)\right)dx =$$

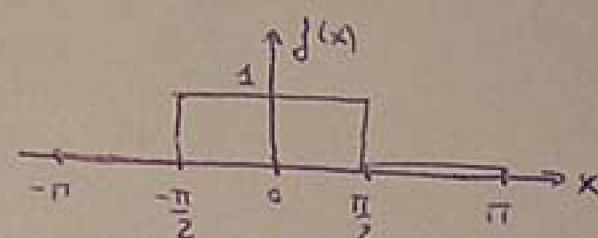
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(x)\cdot\cos(kx) - \frac{1}{2\pi}j\int_{-\pi}^{\pi} \sin(x)\cdot\sin(kx)\,dx = \boxed{\frac{1}{2}(a_k - jb_k) = c_k}$$

$$\boxed{c_{-k} = \overline{c_k} = \frac{1}{2}(a_k + jb_k)}$$

PROBLEMA

29-3-06

$$f(x) = \begin{cases} 1 & |x| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| < \pi \end{cases}$$



$\rightsquigarrow L^2(-\pi, \pi)$

$$f(x) = \sum_{-\infty}^{+\infty} c_k \cdot e^{jkx}$$

Que passa si $\rightsquigarrow e^{-jk\frac{\pi}{2}} = ¿\,??$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-jkx}\,dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-jkx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-jkx}}{-jk}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{-1}{jk2\pi}\left[e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right] =$$

$$= \frac{\sin(k\frac{\pi}{2})}{k\pi} = \begin{cases} \frac{(-1)^{n+1}}{k\pi} & k = 2n-1 \\ 0 & k = 2n \end{cases}$$

$$f(x) = \frac{1}{k\pi}\sum_{-\infty}^{+\infty} (-1)^{n+1}\cdot e^{jkx} \quad , \; k = 2n-1 \;\; \forall n \in \mathbb{N}$$

$$\downarrow$$

$$\boxed{f(x) = \frac{1}{\pi}\sum_{-\infty}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)} e^{j(2n-1)x}}$$

↳ No del tot correcte

"c_0" No sabem quant val, perquè per a $k=0$ $\frac{(-1)^{n+1}}{k\pi} \to \infty$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\pi/2} dx = \frac{1}{2} \quad \Rightarrow \quad \boxed{f(x) = \frac{1}{2} + \frac{1}{\pi}\sum \frac{(-1)^{n+1}}{(2n-1)} e^{j(2n-1)x}}$$