Mathematical Foundations of Elliptic PDEs

Weak Formulation, Functional Analysis, and Variational Principles

Chapter 02 Supplement: Theoretical Foundations

Computational Physics: Numerical Methods

A Rigorous Treatment of Existence, Uniqueness, and Energy Principles

December 3, 2025

Abstract

This document provides the mathematical foundations for the numerical methods developed in Chapter 02. We present the weak (variational) formulation of elliptic boundary value problems, introduce the essential functional analysis framework (Sobolev spaces, trace theorems), and prove the fundamental Lax-Milgram theorem guaranteeing existence and uniqueness of solutions. We then develop the energy minimization interpretation, showing that solving an elliptic PDE is equivalent to minimizing a quadratic energy functional. Finally, we connect these theoretical concepts to the numerical methods implemented in our code, explaining why Galerkin methods (including finite elements and spectral methods) provide natural discretizations of the weak formulation.

Contents

1 Introduction: From Classical to Weak Solutions

1.1 The Model Problem

Throughout this document, we consider the Poisson equation with homogeneous Dirichlet boundary conditions as our model problem:

$$\boxed{\begin{aligned} -\Delta u = f \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, open domain with sufficiently smooth boundary $\partial\Omega$, $f : \Omega \to \mathbb{R}$ is a given source term, and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian operator.

1.2 Limitations of Classical Solutions

A **classical solution** requires $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying (1) pointwise. However:

(i) Classical solutions may not exist for rough data f or irregular domains

(ii) The pointwise interpretation breaks down at corners, interfaces, or singularities

(iii) Numerical methods (finite differences, finite elements) naturally produce approximate solutions that are not C^2

Physical Insight

Consider a membrane deflection problem where a concentrated load (Dirac delta) is applied at a point. The classical formulation $-\Delta u = \delta_{x_0}$ makes no sense pointwise, yet physically the problem has a well-defined solution (a logarithmic singularity in 2D). The weak formulation handles such cases naturally.

1.3 The Key Idea: Integration by Parts

The transition to weak formulations begins with a simple observation. Multiply (1) by a smooth test function v that vanishes on $\partial\Omega$, and integrate over Ω:

$$-\int_\Omega (\Delta u)\, v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x \tag{2}$$

Applying Green's first identity (integration by parts):

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}s}_{=0 \text{ since } v|_{\partial\Omega}=0} = \int_\Omega f\, v \,\mathrm{d}x \tag{3}$$

This leads to the **weak formulation**: Find u such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x \quad \forall v \in V \tag{4}$$

Key Result

The weak formulation (4) requires only **first derivatives** of u (not second), allowing solutions in a larger function space. This is the mathematical basis for finite element methods.

2 Functional Analysis Prerequisites

To make the weak formulation rigorous, we need appropriate function spaces.

2.1 Lebesgue Spaces $L^p(\Omega)$

Definition 2.1 (L^p spaces)**.** For $1 \leq p < \infty$, we define:

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \int_\Omega |u|^p \, \mathrm{d}x < \infty \right\} \tag{5}$$

with norm $\|u\|_{L^p} = \left(\int_\Omega |u|^p \, \mathrm{d}x\right)^{1/p}$.

For $p = 2$, $L^2(\Omega)$ is a Hilbert space with inner product:

$$\langle u, v \rangle_{L^2} = \int_\Omega u \, v \, \mathrm{d}x \tag{6}$$

2.2 Sobolev Spaces

Definition 2.2 (Weak derivative)**.** A function $u \in L^1_{\mathrm{loc}}(\Omega)$ has a **weak derivative** $\partial_i u = w$ if:

$$\int_\Omega u \, \partial_i \phi \, \mathrm{d}x = -\int_\Omega w \, \phi \, \mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega) \tag{7}$$

where $C_0^\infty(\Omega)$ denotes smooth functions with compact support in Ω.

Definition 2.3 (Sobolev space $H^1(\Omega)$)**.**

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; \partial_i u \in L^2(\Omega), \; i = 1, \ldots, n \right\} \tag{8}$$

equipped with the norm:

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int_\Omega \left(u^2 + |\nabla u|^2 \right) \mathrm{d}x \tag{9}$$

and inner product:

$$\langle u, v \rangle_{H^1} = \int_\Omega \left(u \, v + \nabla u \cdot \nabla v \right) \mathrm{d}x \tag{10}$$

Theorem 2.4 (H^1 is a Hilbert space)**.** *The Sobolev space $H^1(\Omega)$ is complete with respect to the H^1 norm, hence it is a Hilbert space.*

Definition 2.5 ($H_0^1(\Omega)$: Functions vanishing on the boundary)**.**

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1}} \tag{11}$$

This is the closure of smooth, compactly supported functions in the H^1 norm. Intuitively, these are H^1 functions that "vanish on $\partial\Omega$" in a generalized sense.

2.3 The Poincaré Inequality

Theorem 2.6 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exists a constant $C_P > 0$ depending only on Ω such that:*

$$\|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2} \quad \forall u \in H_0^1(\Omega) \tag{12}$$

Sketch of proof for $\Omega = (0, L)$ in 1D. For $u \in H_0^1(0, L)$, we have $u(0) = 0$, so:

$$u(x) = \int_0^x u'(t)\,\mathrm{d}t \tag{13}$$

By Cauchy-Schwarz:

$$|u(x)|^2 = \left|\int_0^x u'(t)\,\mathrm{d}t\right|^2 \leq x \int_0^x |u'(t)|^2\,\mathrm{d}t \leq L \int_0^L |u'|^2\,\mathrm{d}t \tag{14}$$

Integrating in x:

$$\int_0^L |u|^2\,\mathrm{d}x \leq L^2 \int_0^L |u'|^2\,\mathrm{d}x \tag{15}$$

Hence $C_P = L$ works. The general proof uses similar ideas with the geometry of Ω. □

Corollary 2.7 (Equivalent norm on H_0^1)**.** *On $H_0^1(\Omega)$, the seminorm $|u|_{H^1} = \|\nabla u\|_{L^2}$ is equivalent to the full H^1 norm:*

$$\|\nabla u\|_{L^2} \leq \|u\|_{H^1} \leq (1 + C_P^2)^{1/2} \|\nabla u\|_{L^2} \tag{16}$$

Connection to Numerics

The Poincaré constant C_P depends on the domain size. For $\Omega = (0, L)^n$, we have $C_P \sim L/\pi$. This explains why:

- The condition number of the discrete Laplacian grows as $\mathcal{O}(h^{-2})$ when the mesh is refined
- Larger domains require more iterations for iterative solvers
- Multigrid methods, which coarsen the grid, effectively exploit different Poincaré constants at each level

3 The Weak Formulation

3.1 Abstract Setting

Let V be a Hilbert space. A **weak formulation** of a PDE has the abstract form:

$$\boxed{\text{Find } u \in V \text{ such that } a(u, v) = \ell(v) \quad \forall v \in V} \tag{17}$$

where:

- $a : V \times V \to \mathbb{R}$ is a **bilinear form**
- $\ell : V \to \mathbb{R}$ is a **linear functional**

3.2 Weak Formulation of the Poisson Equation

For problem (1), we set:

- $V = H^1_0(\Omega)$
- $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$ (bilinear form)
- $\ell(v) = \int_\Omega f \, v \, \mathrm{d}x$ (linear functional, requires $f \in L^2$)

Definition 3.1 (Weak solution)**.** A function $u \in H^1_0(\Omega)$ is a **weak solution** of (1) if:

$$a(u, v) = \ell(v) \quad \forall v \in H^1_0(\Omega) \tag{18}$$

Theorem 3.2 (Classical implies weak)**.** *If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution of* (1)*, then u is also a weak solution.*

Proof. For a classical solution, multiply by any $v \in C^\infty_0(\Omega) \subset H^1_0(\Omega)$ and integrate by parts:

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = -\int_\Omega (\Delta u) v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \tag{19}$$

By density of C^∞_0 in $H^1_0(\Omega)$, the equality extends to all $v \in H^1_0(\Omega)$. □

3.3 Properties of the Bilinear Form

Definition 3.3 (Continuity/Boundedness)**.** A bilinear form $a : V \times V \to \mathbb{R}$ is **continuous** (or bounded) if there exists $M > 0$ such that:

$$|a(u, v)| \leq M \, \|u\|_V \, \|v\|_V \quad \forall u, v \in V \tag{20}$$

Definition 3.4 (Coercivity/Ellipticity)**.** A bilinear form $a : V \times V \to \mathbb{R}$ is **coercive** (or V-elliptic) if there exists $\alpha > 0$ such that:

$$a(v, v) \geq \alpha \, \|v\|^2_V \quad \forall v \in V \tag{21}$$

Proposition 3.5. *For the Poisson problem with $V = H^1_0(\Omega)$ and $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$:*

1. *a is* ***symmetric****: $a(u, v) = a(v, u)$*
2. *a is* ***continuous*** *with $M = 1$ (using the H^1 seminorm)*
3. *a is* ***coercive*** *with $\alpha = (1 + C_P^2)^{-1}$ (using the Poincaré inequality)*

Proof. **Symmetry**: Obvious from the definition.

Continuity: By Cauchy-Schwarz,

$$|a(u, v)| = \left| \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \right| \leq \|\nabla u\|_{L^2} \, \|\nabla v\|_{L^2} \leq \|u\|_{H^1} \, \|v\|_{H^1} \tag{22}$$

Coercivity: Using Poincaré's inequality (12):

$$a(v, v) = \|\nabla v\|^2_{L^2} \geq \frac{1}{1 + C_P^2} \left(\|v\|^2_{L^2} + \|\nabla v\|^2_{L^2} \right) = \frac{1}{1 + C_P^2} \, \|v\|^2_{H^1} \tag{23}$$

□

4 The Lax-Milgram Theorem

The Lax-Milgram theorem is the fundamental result guaranteeing existence and uniqueness of solutions to the weak formulation.

Theorem 4.1 (Lax-Milgram). *Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ a continuous, coercive bilinear form, and $\ell : V \to \mathbb{R}$ a continuous linear functional. Then the problem*

$$\text{Find } u \in V \text{ such that } a(u,v) = \ell(v) \quad \forall v \in V \tag{24}$$

has a ***unique solution*** $u \in V$*. Moreover:*

$$\|u\|_V \leq \frac{1}{\alpha} \|\ell\|_{V'} \tag{25}$$

where α is the coercivity constant and $\|\ell\|_{V'} = \sup_{v \neq 0} \frac{|\ell(v)|}{\|v\|_V}$.

Proof. The proof uses the Riesz representation theorem and the Banach fixed-point theorem.

Step 1: Riesz representation. For fixed $u \in V$, the map $v \mapsto a(u,v)$ is a continuous linear functional on V. By the Riesz representation theorem, there exists a unique $Au \in V$ such that:

$$a(u,v) = \langle Au, v \rangle_V \quad \forall v \in V \tag{26}$$

Similarly, there exists $F \in V$ such that $\ell(v) = \langle F, v \rangle_V$ for all v.

The problem becomes: Find u such that $Au = F$.

Step 2: Properties of A. The operator $A : V \to V$ satisfies:

- *Boundedness*: $\|Au\|_V \leq M \|u\|_V$ (from continuity of a)
- *Coercivity*: $\langle Au, u \rangle_V = a(u,u) \geq \alpha \|u\|_V^2$

Step 3: Fixed-point formulation. For any $\rho > 0$, finding u with $Au = F$ is equivalent to finding a fixed point of:

$$T_\rho u = u - \rho(Au - F) \tag{27}$$

We show T_ρ is a contraction for appropriate ρ:

$$\begin{aligned}
\|T_\rho u - T_\rho w\|_V^2 &= \|(u-w) - \rho A(u-w)\|_V^2 && (28)\\
&= \|u-w\|_V^2 - 2\rho \langle A(u-w), u-w \rangle_V + \rho^2 \|A(u-w)\|_V^2 && (29)\\
&\leq \|u-w\|_V^2 - 2\rho\alpha \|u-w\|_V^2 + \rho^2 M^2 \|u-w\|_V^2 && (30)\\
&= (1 - 2\rho\alpha + \rho^2 M^2) \|u-w\|_V^2 && (31)
\end{aligned}$$

The coefficient $1 - 2\rho\alpha + \rho^2 M^2 < 1$ when $0 < \rho < 2\alpha/M^2$. Taking $\rho = \alpha/M^2$, we get:

$$\|T_\rho u - T_\rho w\|_V \leq \sqrt{1 - \alpha^2/M^2} \|u-w\|_V \tag{32}$$

By the Banach fixed-point theorem, T_ρ has a unique fixed point, which is the unique solution u of $Au = F$.

Step 4: Stability estimate. From $a(u,u) = \ell(u)$ and coercivity:

$$\alpha \|u\|_V^2 \leq a(u,u) = \ell(u) \leq \|\ell\|_{V'} \|u\|_V \tag{33}$$

Hence $\|u\|_V \leq \|\ell\|_{V'} / \alpha$. □

Key Result

For the Poisson problem (1) with $f \in L^2(\Omega)$:

1. There exists a **unique weak solution** $u \in H_0^1(\Omega)$
2. The solution depends **continuously** on the data: $\|u\|_{H^1} \leq C\,\|f\|_{L^2}$
3. The constant C depends on the domain through the Poincaré constant

Connection to Numerics

The Lax-Milgram proof via fixed-point iteration is exactly the **Richardson iteration**:

$$u^{(k+1)} = u^{(k)} + \rho(F - Au^{(k)}) \tag{34}$$

with optimal parameter $\rho = \alpha/M^2$. This connects to:

- **Jacobi/SOR methods**: Variants with different splitting of A
- **Conjugate Gradient**: Optimal Krylov method for symmetric positive-definite A
- **Condition number**: $\kappa(A) = M/\alpha$ determines convergence rate

5 Energy Minimization Principle

When the bilinear form is symmetric, the weak formulation has a beautiful interpretation as an energy minimization problem.

5.1 The Energy Functional

Definition 5.1 (Energy functional)**.** For a symmetric, continuous, coercive bilinear form a and linear functional ℓ, define the **energy functional** $J : V \to \mathbb{R}$:

$$J(v) = \frac{1}{2}a(v,v) - \ell(v) \tag{35}$$

For the Poisson problem:

$$\boxed{J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x} \tag{36}$$

Theorem 5.2 (Equivalence of weak and minimization problems)**.** *Let a be symmetric, continuous, and coercive. Then $u \in V$ solves the weak problem* (17) *if and only if u minimizes J over V:*

$$a(u,v) = \ell(v)\ \forall v \in V \quad \Longleftrightarrow \quad u = \arg\min_{v\in V} J(v) \tag{37}$$

Proof. ($\Rightarrow$) **Weak solution minimizes energy:**

Let u solve the weak problem. For any $v \in V$ and $t \in \mathbb{R}$:

$$J(u+tv) = \frac{1}{2}a(u+tv, u+tv) - \ell(u+tv) \tag{38}$$

$$= \frac{1}{2}a(u,u) + t\,a(u,v) + \frac{t^2}{2}a(v,v) - \ell(u) - t\,\ell(v) \tag{39}$$

$$= J(u) + t\underbrace{[a(u,v) - \ell(v)]}_{=0} + \frac{t^2}{2}a(v,v) \tag{40}$$

$$= J(u) + \frac{t^2}{2}a(v,v) \geq J(u) \tag{41}$$

by coercivity $(a(v,v) \geq 0)$. Hence u is a minimizer.

($\Leftarrow$) Minimizer solves weak problem:

Let u minimize J. For any $v \in V$, the function $g(t) = J(u+tv)$ has a minimum at $t = 0$, so $g'(0) = 0$:

$$g'(0) = a(u,v) - \ell(v) = 0 \quad \forall v \in V \tag{42}$$

This is exactly the weak formulation. □

5.2 Physical Interpretation

Physical Insight

The energy functional (36) has a direct physical meaning:

$$J(u) = \underbrace{\frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x}_{\text{Stored elastic energy}} - \underbrace{\int_\Omega f\,u\,\mathrm{d}x}_{\text{Work done by load } f} \tag{43}$$

Membrane interpretation: For a stretched membrane under load f:

- $u(x,y)$ = vertical displacement
- $|\nabla u|^2$ = local stretching strain energy
- The equilibrium configuration minimizes total potential energy

Electrostatics interpretation: For electrostatic potential ϕ:

- $|\nabla \phi|^2 = |\mathbf{E}|^2$ = electric field energy density
- $-\rho\phi$ = interaction energy with charges
- Nature "chooses" the configuration minimizing total energy

5.3 Uniqueness via Strict Convexity

Proposition 5.3 (Strict convexity of J). *When a is coercive, J is **strictly convex**:*

$$J(\theta u + (1-\theta)v) < \theta J(u) + (1-\theta)J(v) \tag{44}$$

for all $u \neq v$ and $0 < \theta < 1$.

Proof. Direct calculation using $a(u - v, u - v) > 0$ when $u \neq v$. □

Corollary 5.4. *A strictly convex function on a convex set has at most one minimizer. Combined with existence (from Lax-Milgram), this gives another proof of uniqueness.*

6 Variational Principles and Galerkin Methods

6.1 The Rayleigh-Ritz Method

The minimization principle suggests a powerful approximation strategy:

1. Choose a finite-dimensional subspace $V_h \subset V$
2. Minimize J over V_h instead of V:

$$u_h = \arg\min_{v_h \in V_h} J(v_h) \tag{45}$$

Theorem 6.1 (Galerkin orthogonality)**.** *If $u \in V$ solves the continuous problem and $u_h \in V_h$ solves the discrete problem, then:*

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h \tag{46}$$

The error $u - u_h$ is a-orthogonal to the approximation space.

Proof. From the weak formulations:

$$a(u, v_h) = \ell(v_h) \tag{47}$$

$$a(u_h, v_h) = \ell(v_h) \tag{48}$$

Subtracting gives $a(u - u_h, v_h) = 0$. □

6.2 Céa's Lemma: Best Approximation

Theorem 6.2 (Céa's lemma)**.** *The Galerkin approximation u_h is quasi-optimal:*

$$\boxed{\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V} \tag{49}$$

where M and α are the continuity and coercivity constants.

Proof. For any $v_h \in V_h$, using coercivity and Galerkin orthogonality:

$$\alpha \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) \tag{50}$$

$$= a(u - u_h, u - v_h) + \underbrace{a(u - u_h, v_h - u_h)}_{=0 \text{ by } (46)} \tag{51}$$

$$\leq M \|u - u_h\|_V \|u - v_h\|_V \tag{52}$$

Dividing by $\|u - u_h\|_V$ gives the result. □

Key Result

Céa's lemma states that the Galerkin solution is within a factor $M/\alpha = \kappa(A)$ of the **best possible approximation** in V_h. The problem of solving the PDE is reduced to approximation theory!

6.3 Finite Element Connection

Connection to Numerics

The finite element method is a Galerkin method with piecewise polynomial basis functions:

1. **Mesh**: Divide Ω into elements (triangles, quadrilaterals)
2. **Basis**: Choose piecewise polynomials (linear, quadratic, etc.)
3. **Assembly**: Compute the stiffness matrix $K_{ij} = a(\phi_j, \phi_i)$
4. **Solve**: The linear system $Ku = f$ is the discrete weak formulation

The finite difference discretization of Chapter 02 can be viewed as a finite element method with:

- Rectangular elements
- Bilinear basis functions
- Lumped mass matrix approximation

7 Extensions and Generalizations

7.1 Non-Homogeneous Boundary Conditions

For $u = g$ on $\partial\Omega$ (with $g \neq 0$):

1. Find a **lifting** $u_g \in H^1(\Omega)$ with $u_g|_{\partial\Omega} = g$ (trace theorem guarantees existence)
2. Write $u = u_0 + u_g$ where $u_0 \in H_0^1(\Omega)$
3. Solve for u_0: $a(u_0, v) = \ell(v) - a(u_g, v)$ for all $v \in H_0^1(\Omega)$

7.2 Neumann and Robin Boundary Conditions

For Neumann conditions $\frac{\partial u}{\partial n} = g_N$ on $\Gamma_N \subset \partial\Omega$:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \ell(v) = \int_\Omega f v \, \mathrm{d}x + \int_{\Gamma_N} g_N v \, \mathrm{d}s \tag{53}$$

The boundary term appears naturally in the weak formulation!

For Robin conditions $\frac{\partial u}{\partial n} + \beta u = g_R$:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\Gamma_R} \beta u v \, \mathrm{d}s \tag{54}$$

7.3 Variable Coefficients

For $-\nabla \cdot (\kappa \nabla u) = f$ with variable coefficient $\kappa(x) > 0$:

$$a(u, v) = \int_\Omega \kappa \nabla u \cdot \nabla v \, \mathrm{d}x \tag{55}$$

Lax-Milgram applies if $0 < \kappa_{\min} \leq \kappa(x) \leq \kappa_{\max}$:

- Coercivity: $\alpha = \kappa_{\min}/(1 + C_P^2)$
- Continuity: $M = \kappa_{\max}$
- Condition number: $\kappa(A) \sim \kappa_{\max}/\kappa_{\min} \cdot h^{-2}$

8 Summary and Connections

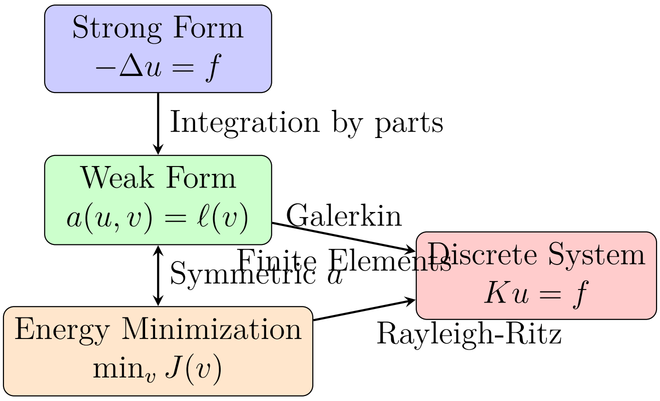

Figure 1: Connections between different formulations of elliptic PDEs

8.1 Key Theorems Summary

Theorem	Statement	Numerical Impact
Lax-Milgram	Existence & uniqueness of weak solution	Well-posedness
Poincaré	$\|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2}$	Condition number
Céa's Lemma	$\|u - u_h\| \leq \frac{M}{\alpha} \inf \|u - v_h\|$	Convergence rate

8.2 Why This Matters for Computation

1. **Well-posedness**: Lax-Milgram guarantees the discrete system has a unique solution
2. **Error estimates**: Céa's lemma + approximation theory gives $\|u - u_h\| = \mathcal{O}(h^k)$ for k-th order elements
3. **Condition number**: $\kappa = M/\alpha \sim h^{-2}$ explains why:
   - Iterative methods need $\mathcal{O}(h^{-2})$ iterations without preconditioning
   - Multigrid achieves $\mathcal{O}(n)$ by using multiple scales
4. **Energy principle**: The solution minimizes a quadratic → the discrete system is SPD → CG is optimal

Appendix: Proof of Key Inequalities

A.1 Cauchy-Schwarz Inequality

For any inner product space:

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\| \tag{56}$$

with equality iff u and v are linearly dependent.

A.2 Young's Inequality

For $a, b \geq 0$ and $\epsilon > 0$:

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2 \tag{57}$$

This is crucial for deriving coercivity estimates when lower-order terms are present.

A.3 Trace Inequality

For $u \in H^1(\Omega)$ with smooth $\partial\Omega$:

$$\|u\|_{L^2(\partial\Omega)} \leq C_T \|u\|_{H^1(\Omega)} \tag{58}$$

This justifies boundary integrals in the weak formulation.

References

[1] L.C. Evans, *Partial Differential Equations*, 2nd ed., AMS, 2010.

[2] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Springer, 2008.

[3] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, SIAM Classics, 2002.

[4] D. Braess, *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, 3rd ed., Cambridge, 2007.

[5] W. Hackbusch, *Elliptic Differential Equations: Theory and Numerical Treatment*, Springer, 2017.