Homework 2

1) Multivariate Normal

Given: N_1 samples from $N(\mu_1, \Sigma)$

N_2 samples from $N(\mu_2, \Sigma)$

$\Sigma_1 = \Sigma_2 = \Sigma$

Show: $P(G=2|x), P(G=1|x)$

if sample more likely to come from class $G=2$, then...

$$x^T \Sigma^{-1}(\mu_2 - \mu_1) > \frac{1}{2}(\mu_2 + \mu_1)^T \Sigma^{-1}(\mu_2 - \mu_1) - \log(N_2/N_1)$$

Given Conditional Probability... =>

$$P(G=2|x) > P(G=1|x)$$

Bayes rule $\Rightarrow \dfrac{P(G=2|x)P(x)}{P(G=2)} > \dfrac{P(G=1|x)P(x)}{P(G=1)}$

$f_2(x)$ and $f_1(x)$ are likelihood, N_2 and N_1 are prior probabilities $\Rightarrow \dfrac{f_2(x)N_2}{\sum P(G=2|\cancel{x})N_i} > \dfrac{f_1(x)N_1}{\sum P(G=1|\cancel{x})N_i}$

$$\Rightarrow f_2(x)N_2 > f_1(x)N_1$$

Multivariate normal PDF $N(\mu_1, \Sigma)$ & $N(\mu_2, \Sigma)$ $\Rightarrow \dfrac{1}{\sqrt{(2\pi)^d|\Sigma_2|}} \exp\left(-\dfrac{(x-\mu_2)^T\Sigma^{-1}(x-\mu_2)}{2}\right) \times N_2 > \dfrac{1}{\sqrt{(2\pi)^d|\Sigma_1|}} \exp\left(-\dfrac{(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)}{2}\right) \times N_1$

Assume $\Sigma_1 = \Sigma_2 = \Sigma$ in LDA $\Rightarrow \dfrac{1}{\sqrt{(2\pi)^d|\Sigma|}} \exp\left(-\dfrac{(x-\mu_2)^T\Sigma^{-1}(x-\mu_2)}{2}\right) \times N_2 > \dfrac{1}{\sqrt{(2\pi)^d|\Sigma|}} \exp\left(-\dfrac{(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)}{2}\right) \times N_1$

Take out $\dfrac{1}{\sqrt{2\pi^d \Sigma}}$ now that $\Sigma_1 = \Sigma_2 = \Sigma$ $\Rightarrow \exp\left(-\dfrac{(x-\mu_2)^T\Sigma^{-1}(x-\mu_2)}{2}\right)N_2 > \exp\left(-\dfrac{(x-\mu_1)^T\Sigma^{-1}(x-\mu_2)}{2}\right)N_1$

Take log $\Rightarrow$ $-\frac{1}{2}(x-N_2)^T\Sigma^{-1}(x-N_2)+\log(N_2) > -\frac{1}{2}(x-N_1)^T\Sigma^{-1}(x-N_1)+\log(N_1)$

Distribute transpose

$$\underbrace{-\frac{1}{2}x^T\Sigma^{-1}x}\ -\frac{1}{2}x^T\Sigma^{-1}N_2\ \underbrace{-\frac{1}{2}N_2^T\cdot\Sigma^{-1}x}+\frac{1}{2}N_2^T\Sigma^{-1}N_2+\log(N_2) >$$

$$-\frac{1}{2}x^T\Sigma^{-1}x-\frac{1}{2}x^T\Sigma^{-1}N_1-\frac{1}{2}N_1^T\cdot\Sigma^{-1}x+\frac{1}{2}N_1^T\Sigma^{-1}N_1+\log(N_1)$$

$x^T\Sigma^{-1}N = N^T\Sigma^{-1}x$

as it is scaler

&

Σ^{-1} is symmetric so

$\Sigma^{-1}=\Sigma$ ...

$$\Rightarrow -\frac{1}{2}x^T\Sigma^{-1}x - x^T\Sigma^{-1}N_2+\frac{1}{2}N_2^T\Sigma^{-1}N_2+\log(N_2) >$$

$$\underbrace{-\frac{1}{2}x^T\Sigma^{-1}x}_{\text{similar term cancels out}} - \underbrace{x^T\Sigma^{-1}N_1}+\underbrace{\frac{1}{2}N_1^T\Sigma^{-1}N_1}+\log(N_1)$$

hence we can subtract

$-\frac{1}{2}x^T\Sigma^{-1}N-\frac{1}{2}N^T\Sigma^{-1}x = -x^T\Sigma^{-1}N$ $\quad(-\frac{1}{2}x^T\Sigma^{-1}x+\frac{1}{2}x^T\Sigma^{-1}x)$

can be either way

Bring equation to one side & simplify

$$\Rightarrow \overbrace{x^T\Sigma^{-1}(N_2-N_1)}+\overbrace{\frac{1}{2}(N_2+N_1)^T\Sigma^{-1}(N_2-N_1)}+\log(N_2)-\log(N_1) > 0$$

$\log(N_2)-\log(N_1) = \log\left(\frac{N_2}{N_1}\right)$

and moving equation to look like result

$$\boxed{\Rightarrow x^T\Sigma^{-1}(N_2-N_1) > \frac{1}{2}(N_2+N_1)^T\Sigma^{-1}(N_2-N_1)-\log\left(\frac{N_2}{N_1}\right)}$$

$\blacksquare$