Exercises on proofs of $f(x) \in O(g(n))$

> **Definition**. Function $f(n) \in O(g(n))$ iff there exist positive constants c and N such that for all n, $n \geq N$, $f(n) \leq c * g(n)$.

Our proofs of these theorems are given at the end of this document.! There are other ways to prove these theorems; the constants c and N that we come up with are not the only possibilities. Wee prefer out proof format for two reasons: (1) with each step, it explains why the step is logical. (2) Each proof has a natural progression, changing $f(n)$ into $c\ g(n)$.

Theorem 1. $20n \in O(n)$.

Theorem 2. $n \in O(20n)$.

Theorem 3. $n + 50 \in O(n)$.

Theorem 4. $\log n \in O(n)$.

Theorem 5. If $f(n) \in O(n)$ and $h(n) \in O(n)$ then $f(n) + g(n) \in O(n)$.
That is, if two functions are in $O(n)$, so is their sum.

Theorem 6. $n + \log n \in O(n)$.

Theorem 7. If $f(n) \in O(g(n))$ and $h(n) \in O(g(n))$ then $f(n) + g(n) \in O(g(n))$.

Theorem 8. $5n^2 + 50n \in O(n^2)$.

Theorem 9. $n\ (n-1)/2 \in O(n^2)$.

Theorem 10. $80\ 2^n + 60n^3 \in O(2^n)$.

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Theorem 1. $20n \in O(n)$. We change $20n$ into $c\ n$, finding N and c as we go.

$20n$
$=$ <Choose $N = 1$ and $c = 20$>
$c\ n$ for $n \geq N$

Theorem 2. $n \in O(20n)$. We change n into $c(20n)$, finding N and c as we go.

n
$<$ <Choose $N = 1$. For all $n \geq 1$, $n < 2n$>
$20\ n$ for $n \geq N$
$<$ <Choose $c = 1$>
$c\ (20n)$ for $n \geq N$

Theorem 3. $n + 50 \in O(n)$. We change $n + 50$ into $c(20n)$, finding N and c as we go.

$n + 50$
$\leq$ <Choose $N = 50$. For all $n \geq 50$, $50 \leq n$>
$n + n$ for $n \geq N$
$<$ <Arithmetic, and choose $c = 2$>
$c\ n$ for $n \geq N$

Theorem 4. $\log n \in O(n)$. We change $\log n$ into $c\ n$, finding N and c as we go.

$\log n$
$\leq$ <Choose $N = 1$. If $n = 2^y$, $\log n = y$. From this, we know that for $n \geq 1$, $\log n \leq n$>
n for $n \geq N$
$<$ <Choose $c = 1$>
$c\ n$ for $n \geq N$

Theorem 5. If $f(n) \in O(n)$ and $h(n) \in O(n)$ then $f(n) + h(n) \in O(n)$.
Proof. Since $f(n) \in O(n)$, there exist positive $N1$ and $c1$ such that $f(n) \leq c1\ n$ for all $n \geq N1$.
Since $h(n) \in O(n)$, there exist positive $N2$ and $c2$ such that $h(n) \leq c2\ n$ for all $n \geq N2$. We calculate:

$f(n) + h(n)$
$\leq$ <Above-mentioned fact about $f(n) \in O(n)$>
$c1\ n + h(n)$ for $n \geq N1$
$\leq$ <Above-mentioned fact about $h(n) \in O(n)$>
$c1\ n + c2\ n$ for $n \geq N1$ and $n \geq N2$
$\leq$ <Choose $N = \max(N1, N2)$ >
$c1\ n + c2\ n$ for $n \geq N$
$\leq$ <Arithmetic, and choose $c = c1 + c2$ >
$c\ n$ for $n \geq N$

Theorem 6. $n + \log n \in O(n)$.
Proof. You can prove this as we did most of the previous ones. However, you can also apply theorem 5, since by theorems 1 and 2, $n \in O(n)$, and by theorem 4, $\log n \in O(n)$.

Theorem 7. If $f(n) \in O(g(n))$ and $h(n) \in O(g(n))$ then $f(n) + g(n) \in O(g(n))$.
Proof. This proof will be exactly the same as the proof of Theorem 5, with all occurrences of $O(n)$ replaced by $O(g(n))$.

Theorem 8. $5n^2 + 50n \in O(n^2)$. We change $5n^2 + 50n$ into $c\ n^2$, finding N and c as we go.

$5n^2 + 50n$
$\leq$ <Choose $N = 1$, because $50\ n <= 50n^2$ for $n \geq 1$>

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$5n^2 + 50n^2$ for $n \geq N$

$\leq$ <Arithmetic>

$55n^2$ for $n \geq N$

$\leq$ <Choose $c = 55$>

cn^2 for $n \geq N$

Theorem 9. $n(n-1)/2 \in O(n^2)$. We change $5n^2 + 50n$ into $c\ n^2$, finding N and c as we go.

$n(n-1)/2$

$\leq$ <Arithmetic>

$n^2/2 - n/2$

$\leq$ <Arithmetic>

n^2

$\leq$ <Choose $N = 1$ and $N = 1$>

cn^2 for $n \geq N$

Theorem 10. $80\ 2^n + 60n^3 \in O(2^n)$. We change $80\ 2^n + 60n^3$ into $c\ 2^n$, finding N and c as we go. .

$80\ 2^n + 60n^3$

$\leq$ <For $n = 10$, $n^3 = 1000 < 1024 = 2^n$. For $n > 10$, $2^n > n^3$. Choose $N = 10$.>

$80\ 2^n + 60\ 2^n$ for $n \geq N$

$\leq$ <Arithmetic>

$140\ 2^n$ for $n \geq N$

$\leq$ <Choose $c = 140$>

$c\ 2^n$ for $n \geq N$

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