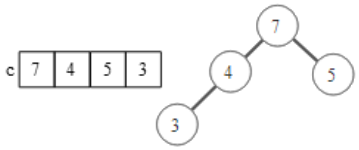
Heap sort

We assume you understand heaps, both min-heaps and max-heaps. We show you how a max-heap can be used to sort an array of size `n` in time O(`n` log `n`) and space O(1).

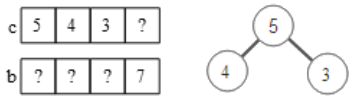
The basic idea

We illustrate using an array `b`[0..3] that contains (3, 5, 7, 4). Sorting the array has two steps.

1. **Heapify**: Build a max-heap `c`[0..3] of the array elements. As you know, this can be done with a loop that inserts `b`[0], `b`[1], `b`[2], and `b`[3] into the max-heap, bubbling up each value as it is inserted. To the right, we show both the final array `c` and the tree that it contains.



2. **Poll the heap**: Poll the values from the heap, one by one, and store them in array `b` starting at the end and working forward. Each poll removes and returns the largest value, which is in `c`[0], so the values are polled in the order 7, 5, 4, 3. To the right, we show the state after the first value is polled and stored in `b`[3]. The 7 is in `b`[3]. The ? in `b`[0..2] indicates that we don't care what those values are. The heap is now in `c`[0..2]. You also see the heap in tree form.



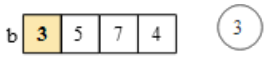
Analysis of space and time

Because of the extra array `c`, this algorithm take space O(`n`). Below, we show how to eliminate `c` so that the space is O(1). Step 1 takes time O(`n` log `n`). Step 2 does also. So the total time is O(`n` log `n`).

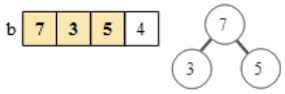
Eliminating array c

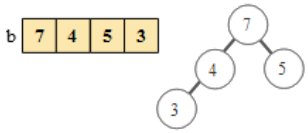
There is no need for array `c`! Instead, perform all the operations in array `b`, making this sort an in-place or *insitu* sort. We consider first making the max-heap.

1. **Heapify**: `b`[0] can already be considered a max-heap of one value. We show this to the right, with heap value 3 boldfaced. Now consider `b`[1] = 5 to be in the heap and bubble it up. The second diagram to the right shows the state after bubbling `b`[1] up.



Next, consider `b`[2] = 7 to be in the heap and bubble it up, as shown in the first diagram to the right. Finally, do the same for `b`[3] = 4, the result being the second diagram to the right.



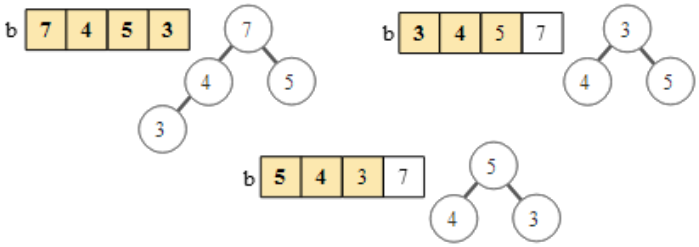


You see that it is easy to heapify array `b`: Just bubble up each successive element. The algorithm appears below. It uses method `bubbleUp`, which appears to the right..

```
// Heapify b: Make b into a max-heap
// invariant: b[0..k-1] is a max-heap AND
//            b is a permutation of its initial value
for (int k= 1; k < b.length; k= k + 1) {
   // Make b[0..k] into a max-heap by
   // bubbling b[k] up
   bubbleUp(b, k);
}
```

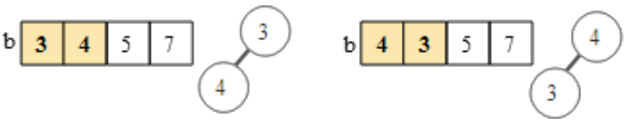
```
/** Bubble b[k] up so that b[0..k] is a max-heap. <br>
  * Precondition: b[0..k] is a max-heap except that
  *               b[k] might be > its parent. */
public static void bubbleUp(int[] b, int k) {
   int p= (k - 1) / 2;   // p is the parent of k
   // inv: p is the parent of p AND original b[0..k-1]
   //.      is a max-heap except: b[k] might be > b[p].
   while (k > 0 && b[k] > b[p]) {
      Swap b[k] and b[p]
      k= p;
      p= (k - 1) / 2;
   }
}
```

2. **Polling the heap**: Suppose `b` is the max-heap shown in the first diagram to the right, showing both `b` and its tree form. The first step in polling is to swap `b[0]` with the last element of the heap, in this case `b[3]`, and to consider that last value to be no longer in the heap. The second diagram to the right shows the state after this step. *Note that the value 7 is now in its final position*.



The next step in polling is to bubble `b[0]` down. The result is shown in the third diagram (the lower one). You can see the largest value in the heap, 5, is in the root.

If the heap is polled again, the 5 will be placed in its final position (see first diagram to the right) and `b[0]` will be bubbled down (second diagram to the right).



Polling one more time complete the sorting of `b`.

You could probably write the invariant of the polling loop yourself at this point. Here it is.

> **Invariant of polling loop**:
> `b` is a permutation of its initial value AND
> `b[0..k]` is a max-heap AND
> `b[k+1..]` is sorted AND
> `b[0..k]` ≤ `b[k+1..]`

With this invariant, we write the second step of heapsort. It uses a method `bubbleDown`, which appears to the right.

```
// Sort max-heap b[0..]
// invariant: See above
for (int k= b.length - 1; k > 0; k= k - 1) {
   Swap b[0] and b[k];
   bubbleDown(b, k);
}
```

```
/** Bubble b[0] down to its heap position in b[0..k-1].
  * Precondition: b[0..k-1] is a max-heap except
  * that b[0] may have a larger child */
public static void bubbleDown(int[] b, int k) {
   int h= 0;
   // in: b[0..k-1] is a max-heap except that
   // b[h] may be larger than one of its children.
   while (h < k) {
      // c= larger child of h (return if h has no children)
      int c= 2 * h + 1;                  // h's smallest child
      if (c > k - 1) return;
      if (c < k - 1 && b[c] < b[c + 1]) c= c + 1;

      if (b[h] >= b[c]) return;  // b[h] is in the right place

      Swap b[n] and b[c];
      h= c;
   }
}
```