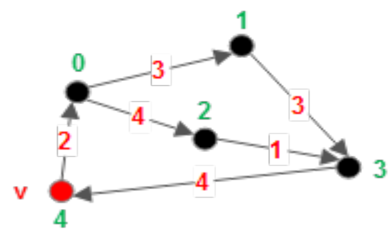
Extending the shortest-path algorithm to calculate shortest paths
David Gries

The shortest-path algorithm calculates the distance of the shortest path from start node `v` to every node in a graph. We now extend the algorithm to calculate the shortest paths themselves.

This practice often works well: Start with a fairly simple basic algorithm and then extend it to calculate more information.

Consider the graph to the right. It has `n` = 5 nodes, with node numbers in 0..4, given in green. Red node `v` = 4 is the start node. The edge weights are given in red. Here are the shortest paths from v to all nodes:



(`v`, 0)	distance 2
(`v`, 0, 1)	distance 5
(`v`, 0, 2)	distance 6
(`v`, 0, 2, 3)	distance 7 —path (`v`, 0, 1, 3) has distance 8
(`v`)	distance 0

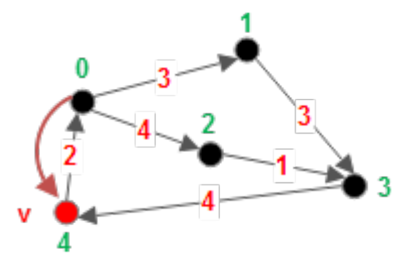
The first problem is to decide how to save the shortest paths. What data structure should be used? The obvious approach is to store information in start node `v`. For example, use an array `c` such that for each node `w`, `c`[`w`] contains the neighbor of `v` on the shortest path to `w`. Array `c` is shown to the right. It doesn't matter what is in `c`[4] since that is the start node and the shortest path contains exactly 1 node, `v`. All the other elements of `c` are 0 because all shortest paths from `v` to other nodes go from `v` to 0.

`c`[0] = 0
`c`[1] = 0
`c`[2] = 0
`c`[3] = 0
`c`[4] = ?

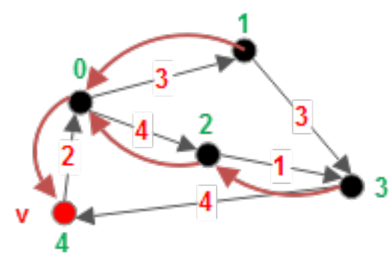
If we continue with this approach, we probably need an array for each node. This approach requires a *lot* of space, and it will probably require a lot of code to create these arrays. There *must* be a better approach. One that requires a lot less space, hopefully one value per node.

Use backpointers!

Instead, we do the following. Consider the shortest path from `v` to 0: (`v`, 0). The node *preceding* 0 on this path is node `v`. We therefore draw a *backpointer* from 0 to `v`, shown in the diagram to the right as a curved arrow.



We do this for all nodes except the start node. For each node `w` except `v`, draw a curved arrow from `w` to the previous node on the shortest path from `v` to `w`. This is shown in the second diagram to the right.



Thus, for each node `w`*, the backpointers give the reverse of the shortest path from* `v` *to* `w`. That's neat!

We can maintain these backpointers in an array `bp`. Thus, we have two arrays:

- `d`[`w`] contains the distance of the shortest path from `v` to `w`.
- `bp`[`w`] contains the previous node on the shortest path from `v` to `w`.

For this graph, here are arrays `d` and `bp`:

`d`[0]: 2	`bp`[0]: 4
`d`[1]: 5	`bp`[1]: 0
`d`[2]: 6	`bp`[2]: 0
`d`[3]: 7	`bp`[3]: 2
`d`[4]: 0	`bp`[4]: ?

We can construct the shortest path from `v` to any node `w` using the backpointers in time proportional to the distance of the shortest path. That's pretty good. And, for a graph with `n` nodes, only `O`(`n`) space is needed for the backpointers.

Our next task is to modify the shortest path algorithm to create backpointer array `bp`.

To the right, we give the invariant and theorem of the shortest-path algorithm. Below is the algorithm. It sets d[w] for each node w reachable from start node v to the distance of the shortest path from v to w.

```
F= {v}; d[v]= 0; S= {};
// invariant: P1, P2, and P3
while (F != {}) {
   f= a node in F with minimum d value;
   Remove f from F and add it to S;
   for each w with (f, w) an edge {
      if (w not in S or F) {
          d[w]= d[f] + wgt(f, w);
          add w to F;
      }
     else if d[f] + wgt(f, w) < d[w] {
           d[w]= d[f] + wgt(f, w);
         }
   }
}
```

Invariant
P1. For node s in settled set S, at least one shortest v-s path contains only settled nodes and d[s] is its distance.

P2. For node f in *frontier set* F, at least one v-f path contains only *settled* nodes, except for f, and d[f] is the minimum distance from v to f over all paths from v to f that contain only settled nodes except for the last one, f.

P3. Edges leaving S end in the frontier.

Theorem. For f in *frontier* set F with minimum d value (over nodes in F), d[f] is the shortest-path distance from v to f.

We modify the algorithm to fill in elements of array bp. There are two situations in which a new shortest-path is formed. We investigate them.

1. Case w is not in S or F. Here, w is in the far-out set and is placed in the frontier set. The new shortest path (so far) from v to w is (v, ..., f, w). Therefore, f is the backpointer for w, and the statement bp[w]= f; is needed.
2. Case w is in S or F and d[f] + wgt(f, w) < d[w]. Here, the new shortest path (so far) from v to w is (v, ..., f, w). Therefore, f is the backpointer for w, and the statement bp[w]= f; is needed.

The modified algorithm is given below, with the additional assignments shown in red. Wow! Isn't that simple? Isn't that neat?

```
F= {v}; d[v]= 0; S= {};
// invariant: P1, P2, and P3
while (F != {}) {
   f= a node in F with minimum d value;
   Remove f from F and add it to S;
   for each w with (f, w) an edge {
      if (w not in S or F) {
          d[w]= d[f] + wgt(f, w); bp[w]= f;
          add w to F;
      }
     else if (d[f] + wgt(f, w) < d[w]) {
           d[w]= d[f] + wgt(f, w); bp[w]= f;
         }
   }
}
```

Invariant
P1. For node s in settled set S, at least one shortest v-s path contains only settled nodes and d[s] is its distance.

P2. For f in *frontier set* F, at least one v-f path contains only *settled* nodes, except for f, and d[f] is the minimum distance from v to f over all such paths from v to f.

P3. Edges leaving S end in the frontier.

P4. For w in S or F, bp[w] is the backpointer on the shortest known path from v to w.

Theorem. For f in *frontier* set F with minimum d value (over nodes in F), d[f] is the shortest-path distance from v to f.

This version of the shortest path algorithm uses three sets: settled set S, frontier set F, and the far-off set, which contains all nodes that are not in S or F. It uses two arrays: d and b. Implementing this algorithm in Java would require finding a good implementation of F (use a heap) and a new data structure to efficiently maintain the information d[w] and bp[w] for each node in S or F. Set S is not be needed.