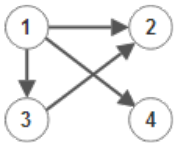
Graph representation

There are essentially two ways to represent a graph: as an *adjacency matrix* and as an *adjacency list*.



We describe the two representations using the 4-node directed graph shown to the right; you can figure out the same thing for an undirected graph yourself.

Adjacency matrix representation

To represent an `n`-node graph with node numbers in 1..`n`, use an `n` x `n` matrix `m`, where `m[u, v]` = 1 if the graph has an edge (`u`, `v`) and `m[u, v]` = 0 if there is no such edge.

	1	2	3	4
1	0	1	1	1
2	0	0	0	0
3	0	1	0	0
4	0	0	0	0

To the right is adjacency matrix `m` for the above 4-node graph. Since the graph has an edge from node 3 to node 2, `m[3, 2]` = 1. In the same way, three elements of row 1 of `m` are 1 and all the other matrix elements are 0.

When implementing an adjacency matrix in Java, one could use a 2-dimensional boolean array `m`, in which `m[u][v]` is true if there is a directed edge from node `u` to node `v` and false otherwise.

One has to be careful because Java arrays start with 0. Here are two possibilities to deal with the difference between a matrix starting with row 1 and a Java array starting with index 0: (1) Declare the array of size [0..`n`][0..`n`] and don't use row 0 and column 0. (2) Give the nodes numbers in 0..`n`-1 instead of in 1..`n`.

Adjacency list representation

The adjacency list representation of a graph maintains a list or set of nodes. The list element for a node `u` contains a list of nodes that are adjacent to it, i.e. a list of nodes `v` such that (`u`, `v`) is an edge.

To the right is the adjacency list representation of the 4-node graph given above. Three directed edges leave node 1; therefore, the list for node 1 contains the sinks of these three edges: 4, 2, and 3, in no particular order. One directed edge leaves node 3, so the list for node 3 contains its sink, 2. The lists for nodes 2 and 4 are empty because no edges leave these nodes.

1 (4, 2, 3)
2 ()
3 (2)
4 ()

What data structures might we use in Java for the adjacency list representation? First, the list of nodes could be given in an array, an `ArrayList`, a `LinkedList`, a `Set` —in many different ways. Second, the element for each node can be a `List` of nodes or node numbers —e.g. an `ArrayList` or a `LinkedList`. The important point is that processing the edges leaving a node —or the sinks of those edges— should take time proportional to the length of the list. Later, will use this property in discussing the time to perform various operations on a graph.

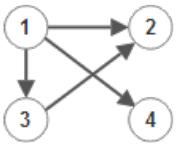
In an OO implementation of a graph. There would typically be classes `Graph`, `Node`, and `Edge`. Class `Graph` would have methods to retrieve (1) the `Nodes` as a `Set` or `List` and (2) the `Edges` as a `Set` or `List`. Class `Node` would have methods to retrieve (1) the `Edges` leaving the `Node` and (2) the `Nodes` at the other end of `Edges` leaving the `Node`. You can figure out what methods class `Edge` could have.

When deciding on an OO implementation of a graph, one must take into consideration the space requirements as well as the time requirements.

Space and time considerations

We are about to discuss the space and time requirements for the two graph representations. Some students are inclined to try to memorize the space and time requirements, and they will ask on the Piazza for the course (or whatever the discussion board is) if they forget. But this is plain rote memorization and not at all useful. Don't do it! Instead, understand the two representations of graphs and figure out the space and time requirements whenever required. It's not that difficult.

Consider a directed graph with n nodes and e edges. To the right, we again give the 4-node graph from above and its two representations, to make it easier for you to understand what is said below.



	1	2	3	4
1	0	1	1	1
2	0	0	0	0
3	0	1	0	0
4	0	0	0	0

1 (4, 2, 3)
2 ()
3 (2)
4 ()

Just which representation one uses depends on what kind of graph it is and how it will one used.

The table below gives the space requirements of each graph representation and gives the time complexities for two operations on the graph. Analysis to help you understand the entries in the table are given below he table.

	Adjacency list	Adjacency list for sparse graphs	Adjacency list for dense graphs	Adjacency matrix
Space	1: O(n+e)	2: O(n)	2: $O(n^2)$	3: $O(n^2)$
Enumerate all edges	4: O(n+e)	4: O(n)	4: $O(n^2)$	5: $O(n^2)$
Is there an edge (u, v)?	6: O(outdegree of node u)	6: O(outdegree of node u)	6: O(outdegree of node u)	7: O(1)

1: The adjacency list representation requires space O(n) for the list of nodes and, for each node, space proportional to the outdegree of the node for its adjacency list. Since there are *e edges*, the adjacency list representation requires O(n+e) space.

The n in O(n+e) is important. Suppose e is 0 —there are no edges. Then there are n empty lists of edges, each of which takes O(1) space, so the total space is O(n).

2: A sparse graph has O(n) edges, so the space requirement O(n+e) reduces to O(n). A dense graph has $O(n^2)$ edges, so the space requirement is $O(n^2)$.

3: The adjacency matrix has n^2 entries, so it takes space $O(n^2)$. But it requires only n^2 bits, while an adjacency list for a dense graph requires many more because each entry in a list is an integer and the list could be implemented as a linked list.

4: To enumerate all edges when an adjacency list is used requires looking at each of the n nodes and for each looking through the list of its edges. Since there are a total of e edges, this takes time O(n+e). For a sparse graph, with O(n) edges, this can be simplified to O(n). For a dense graph, with $O(n^2)$ edges, this can be simplified to $O(n^2)$.

5: Enumerating the edges when using an adjacency matrix requires looking at each entry of the n x n matrix.

6: When using an adjacency list, determining whether (u, v) is an edge may require looking at all entries in u's adjacency list.

7: When using an adjacency matrix, determining whether (u, v) is an edge requires looking only at entry m[u, v].