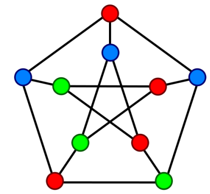
Graph coloring

A coloring of an undirected graph is an assignment of a color to each node so that adjacent nodes have different colors. The graph to the right, taken from Wikipedia, is known as the *Petersen* graph, after Julius Petersen, who discussed some of its properties in 1898. It has been colored with 3 colors. It can't be colored with one or two.

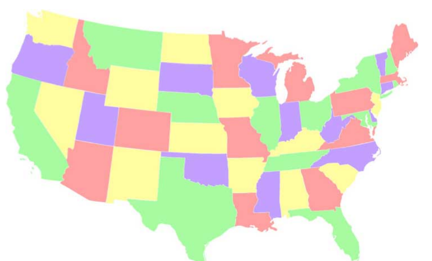


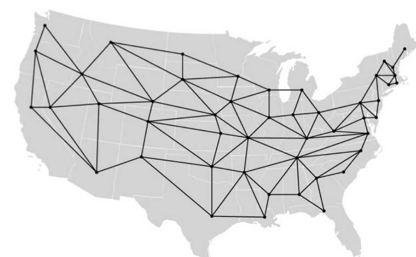
The Petersen graph has both K_5 and bipartite graph $K_{3,3}$, so it is not planar. That's all you have to know about the Petersen graph. But if you are at all interested in what mathematicians and computer scientists do, visit the Wikipedia page for *Petersen graph*.

This discussion on graph coloring is important not so much for what it says about the four-color theorem but what it says about proofs by computers, for the proof of the four-color theorem was just about the first one to use a computer and sparked a lot of controversy.

Kempe's flawed proof that four colors suffice to color a planar graph

Thoughts about graph coloring appear to have sprung up in England around 1850 when people attempted to color maps, which can be represented by planar graphs in which the nodes are countries and adjacent countries have a directed edge between them. Francis Guthrie conjectured that four colors would suffice.

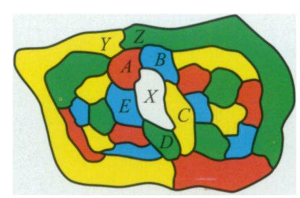




In 1879, Alfred Kemp, a barrister in London, published a proof in the *American Journal of Mathematics* that only four colors were needed to color a planar graph. Eleven years later, P.J. Heawood found a flaw in the proof. He and Kempe and others tried to fix the flaw but were unable to do so, and the theorem remained only a conjecture until almost one hundred years later. Appel and Haken, who proved the theorem in 1976, said that ``Kempe's argument was extremely clever, and although his "proof" turned out not to be complete, it contained most of the basic ideas that eventually led to the correct proof one century later.''

Of course, five colors are needed to color a map, because the water has to be blue (joke).

Alan Sipka of Alma College gave a presentation titled "Kempe's flawed proof that four colors suffice," at the MAA MathFest 2013 in Hartford, CT on 1 August 2013. It's eminently readable —if you have the time, read it! It can be found in JavaHyperText entry "four-color theorem". In it, Alan gives the map, shown to the right, that Heawood used as a counterexample to Kempe's proof. Of course, this map can be colored with four colors, but not using the process given by Kempe's flawed proof.



Appel and Haken's proof

In 1976, Kenneth Appel and Wolfgang Haken proved that only four colors were needed to color a planar graph. Parts of the proof appeared later in 1977 in the *Illinois Journal of Mathematics*. Their proof was the first proof that required use of the computer, and it provoked a lot of controversy about whether it was a real proof.

The proof rested on checking that 1,936 special graphs had a certain property. How could anyone know that these 1,936 graphs, and not perhaps 1 or 2 others, were needed for the proof? That's an amazing number of items for anyone to wrap their head around. How many people could actually study them and verify that just these 1,936 graphs were needed?

Further, Appel and Haken used a computer to check that those 1, 936 graphs had that property! How could that be a proof if people themselves couldn't understand it but had to rely on a computer?

David Gries actually found a flaw in the computer program, which was written in the assembly language of the IBM 7090 computer. But it could easily be corrected, and it was the safe kind —it might say that a graph didn't have the special property even though it did, not the other way around. Others also found this error.

Since then, the Appel-Haken proof has been simplified by reducing the number of special cases, and people have accepted proofs by computer. In fact, a complete formal proof of the four-color theorem has been done in the Coq theorem prover. A paper about it by Georges Gonthier appeared in 2008 in *Notices of the AMS*.

Actually, people have been working on computer systems for theorem provers since the 190's. One of the largest and most influential computer proof system is NuPrl, developed at Cornell by Professor Bob Constable and his coworkers since the 1970s. Here is its web page: http://www.nuprl.org. Hundreds of theorems in mathematics and computer science have been proven, and the proofs are available, in NuPRL. Much of NuPrl uses "constructive proofs", which means that from the proof an algorithm can be extracted.