Proofs

# Theorem. 20 + 21 + … + 2h = 2h+1 – 1.

# The proof is by induction on h.

1. For h = 0, the formula reduces to 20 = 21 – 1, which is true since 20 = 1 and 21 = 2.
2. For the inductive case, we assume that the formula holds for h in the range 0..k and prove it for h = k+1. We have

20 + 21 + … + 2k+1

= <arithmetic, since 0 ≤ k>

20 + 21 + … + 2k + 2k+1

= <inductive assumption>

2k+1 – 1 + 2k+1

= <arithmetic>

2k+2 – 1

Q.E.D. (meaning Quit.End.Done)

# Theorem. The number of leaves in a binary tree of height h is at most 2h.

# The proof is by induction on h.

# Theorem. A binary tree with n leaves has height at least log2 n.

# The proof is by induction on h.

Theorem. A height-balanced binary tree with height h has at least 2^((h/2)-1) internal nodes.

The proof is by induction on h, h >= 1.

Let n(h) be the minimum number of nodes in a height-balanced binary tree of height h.

By investigating all trees of height 1, and 2, we see that n(1) = 1, and n(2) = 2.

Hence, for h in 1..2, the theorem holds.

Consider a height-balanced binary tree t with height h ≥ 3 and a minimum number of nodes.

T has a root and two subtrees. Since t has a minimum number of nodes, so does its subtrees.

Since t has height h, at least one subtree has height h-1. Since t is height-balanced, the other subtree has height h-1 or h-2. Since t has a minimum number of nodes, the other tree will have height h-2 (not h-1). Thus, we have:

n(h)

= < the previous paragraph>

1 + n(h+1) + n(h-2)

> <arith>

n(h+1) + n(h-2)

> <arith>

2 n(h-2)

>= 2 2^(h-2

Theorem: n(h) >= 2(h/2) – 1

Proof by induction on h.

For h = 0, true.

For h = 1, true

Induction: Depending on whether h+1 is even or odd

even

n(h+1) need to get to >= 2(h+1)/2 – 1.

> by lemma 2

2n(h+1 - 2)

>= by induction >= 2 2(h-2)/2 – 1 =

2(h+1-2)/2 – 1

Theorem. For any i > 0, n(h) > 2^i n(h – 2i).

Choose an i so that h-2i is 1 or 2. In which case n(h) = 2^i or n(h) = 2^(i+1)

Choose i = ⌈h/2⌉-1.

If i is even, we have

n(h) > 2^i n(h – 2i) =

SINCE n(h) >= 2(h/2) – 1

log(n(h)) >