# Mathematics and Problem Solving

## Lecture 4

### Sequences and Series

> "Mathematics is the language of nature."
>
> Fibonacci

# Overview

- Sigma Notation
- Summation
- Product Notation

# Sigma Notation

# Summation

- Summation is adding up lots of things
- It has a special notation written with the Greek capital letter sigma
  - Sigma Notation
- This allows us to express summations compactly

$$\sum_{i=1}^{n} a$$

# Sigma and Polynomial form

- We are familiar with seeing addition written in a polynomial

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

# Parts of a Summation

- *a*
  - *formula to sum*
- *i*
  - variable
- *=1*
  - Lower limit
  - starting point (inclusive)
- *n*
  - Upper Limit
  - ending point (inclusive)

$$\sum_{i=1}^{n} a$$

# Determine the Sum - Example

$$\sum_{i=1}^{4} i$$

- Start at i =1
- For each value of i
  - Substitue every occourance of i for 1 in the formula
  - Formula becomes
    - 1
  - Is i less than 4?
  - Increment 1

# Determine the Sum - Example

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

# Determine the Sum – More Examples

$$\sum_{i=0}^{4} \frac{i}{i+1} = \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} = \frac{163}{60} = 2.7166\overline{6}$$

$$\sum_{i=4}^{6} 2^i x^{2i+1} = 2^4 x^9 + 2^5 x^{11} + 2^6 x^{13} = 16x^9 + 32x^{11} + 64x^{13}$$

$$\sum_{i=1}^{4} f\left(x_i^*\right) = f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + f\left(x_4^*\right)$$

# Exercise

**Exercise 1:**

Solve the following summations

1. $\sum_{i=1}^{10} 1$
2. $\sum_{i=1}^{10} i$
3. $\sum_{i=1}^{5} a$
4. $\sum_{i=1}^{10} i^2$

# Summing a Constant Sequence

- Lets say we have a constant sequence of numbers, such as:
  - 2, 2, 2, 2, 2, …
- What is the sum of the first n values of this sequence
  - If we have n values, then n x 2 = 2n
- We can express this as a summation
  - We cannot solve this to get a number, but we can simplify it to a polynomial form that is easier to work with

**Exercise 2:**

Solve the following summations

1. $\sum_{i=1}^{n} 2$
2. $\sum_{i=1}^{i} a$
3. $\sum_{i=4}^{n} 10$

- Hint:
  - Summations are inclusive of both start and end values
  - Adding up *n* copies of a number *k* is *nk*

# Sequences

# Sequences and Series

- A sequence is a progression of numbers, e.g.
  - { 1, 10, 4, 2, 4, 77, 6, 4, … }
- If we add them all together, we get a **series**
  - { 1 + 10 + 4 + 2 + 4 + 77 + 6 + 4 + … }
- Because a series can be arbitrarily long or infinite, we often use sigma notation to express it
- There are special types of sequences that we are going to look at
  - Arithmetic Sequences
  - Geometric Sequences

# Notating Sequences

- A sequence can be notated in a number of ways
- Pay attention to subscripts!
  - $a_{n+1} \neq a_n + 1$

$$\{a_1, a_2, \cdots, a_n, a_{n+1}, \cdots\}$$

$$\{a_n\}$$

$$\{a_n\}_{n=1}^{\infty}$$

**Exercise 3:**

Give the first 4 values of the following sequences

1. $\{n^2 + 3\}_{n=1}^{\infty}$
2. $\{\frac{n}{n+1}\}_{n=1}^{\infty}$

# Terminology about Sequences

- A sequence $\{a_n\}$ is **increasing** if, *for every n*
  - $a_n < a_{n+1}$
  - { 1, 23, 112, 158 } is an increasing sequence
- A sequence $\{a_n\}$ is **decreasing** if, *for every n*
  - $a_n > a_{n+1}$
  - { 202, 150, 3, 0.1} is a decreasing sequence
- Both increasing and decreasing sequences are **monotonic**

# Bounded Sequences

- If there is a number $m \leq a_n$ for every n (less or equal to any number in the sequence)
  - m is a **lower bound** of the sequence
  - The sequence is **bounded below**
- If there is a number $m \geq a_n$ for every n
  - m is an **upper bound** of the sequence
  - The sequence is **bounded above**
- A sequence bounded both below and above is called **bounded**

**Exercise 4:**

For the finite sequence { 4, 2, 3, 2 }

1. Is 2 a lower bound?
2. Is 1 a lower bound?
3. Is 3 an upper bound?
4. Is the sequence bounded?
5. Is the sequence monotonic?

- Hint:
  - m is a lower bound if $m \leq a_n$ for every n
  - m is a upper bound if $m \geq a_n$ for every n

**Exercise 5:**

Consider the sequence

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

1. Give an upper / lower bound if such exist
2. Is it increasing/decreasing?

# Arithmetic and Geometric Sequences

# Arithmetic Sequences

- An Arithmetic Sequence is a list of numbers with an:
  - Initial Term
  - Common Difference
- For example (initial term in red)
  - { 1, 2, 3, 4, 5, 6, … } (common difference 1)
  - { 5, 10, 15, 20, 25, … } (common difference 5)
  - { 3, 6, 9, 12, 15, … } (common difference 3)

**Exercise 6:**

What is the initial term and common difference for these sequences?

1. { 1, 2, 3, 4, 5, … }
2. { 99, 104, 109, 114, … }

# Arithmetic Sequences

- We can define an arithmetic sequence recursively

$$a_n = a_{n-1} + d$$

- Or give an explicit formula

$$a_n = a_1 + d(n-1)$$

# Arithmetic Sequences

**Exercise 7:**

Find the last term of the sequence that starts with 5, has a common difference of 0.25 and has 10 terms.

- Hint:
  - We can define an arithmetic sequence recursively
    - $a_n = a_{n-1} + d$
  - Or give an explicit formula
    - $a_n = a_1 + d(n-1)$

# Arithmetic Sequences

**Exercise 8:**

1. What is the implicit formula for the sequences
   - { 2, 3, 4, 5, … }
   - { -1, -3, -5, -7, … }
2. What is the explicit formula for the sequences
   - { 4, 5, 6, 7, … }
   - { 10, 20, 30, 40, … }

Hint:

- $a_n = a_{n-1} + d$
- $a_n = a_1 + d(n-1)$

# Geometric Sequence

- An Arithmetic Sequence is a list of numbers with an:
  - Initial Term
  - Common Ratio
- For example
  - { 10, 40, 160, 640, … } (common ratio 4)

**Exercse 9:**

What is the initial term and common ratio for these sequences?

1. { 1,3,9,18, … }
2. { 1, 0.5, 0.25, 0.125, … }

# Geometric Sequence

- We can define a geometric sequence recursively

$$g_n = g_{n-1} \times r$$

- Or explicitly

$$g_n = g_1 \times r^{n-1}$$

# Geometric Sequence

**Exercise 10:**

1. What is the implicit formula for the sequences
   - { 2, 3, 4, 5, … }
   - { -1, -3, -5, -7, … }
2. What is the explicit formula for the sequences
   - { 4, 5, 6, 7, … }
   - { 10, 20, 30, 40, … }

Hint:

$$g_n = g_{n-1} \times r$$

$$g_n = g_1 \times r^{n-1}$$

# Fibonacci Sequence

- One well known sequence is the Fibonacci sequence, defined as follows:

$$F_n = F_{n-1} + F_{n-2}$$

- Where $F_0 = 0$ and $F_1 = 1$

# Fibonacci Sequence

- The Fibonacci Sequence can be expressed as an explicit formula
- Where φ is the golden ratio
  - 1.6180339887…
- And ψ is $-\varphi^{-1}$, the negative multiplicative inverse of the golden ratio
  - -0.6180339887...

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

**Exercise 11:**

I define a sequence using the following explicit definition

$s_n = (0.5 \times 10^{x-1}) + 2(x-1)$

1. Write out the first 5 elements
2. Decompose this into two sequences
3. How could you write a recursive definition for this sequence?

- Hint:
  - The number of digits in a base 10 number is given by
    - $\log_{10} n$
  - Remember mod
  - To round down (floor) we can write

$$\text{floor}(x) = \lfloor x \rfloor$$

# Summation of Series

# Sum of first n numbers

- We often need to sum the first n natural numbers: 1, 2, 3, 4, …, n
  - 1 + 2 + 3 + 4 + … + n
- Problem: arbitrarily long
  - Easy to write with sigma notation!
- But how do we calculate the sum for a given n?

$$\sum_{i=1}^{n} i =$$

# Sum of first n numbers

- We can write this in a more familiar form
- Where does this come from
  - Find the average of first value 1 and n, the last value, which is (n+1)/2
  - Multiply this by the length of the series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

# Sum of an Arithmetic Series

- We can generalise this for a formula for the sum of the first *n* values of a sequence *a*
  - Find average value:
    - $(a_1 + a_n)/2$
  - Multiply by length of sequence *n*
- We can always use substitute the formula for the *n*th term of the sequence in place of $a_n$

$$\sum_{i=1}^{n} a = n\left(\frac{a_1 + a_n}{2}\right)$$

# Sum of a Geometric Series

- We can also define the sum of the first *n* values of a Geometric Series
  - *a* is the initial value
  - *r* is the common ratio

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

# Sum of first n squares

- It is also common to want to sum the first n square numbers
  - You can derive this using simultaneous equations
  - Or just look up the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

# Arithmetic Properties of Summation

# Arithmetic Properties of Summation

- When you cannot resolve a summation to a number, we often need to transform it
  - Factor constants out of a summation
  - Break up summation across a sum or difference
- By transforming it, we can often get it to a state where it is easier to work with
  - e.g. so we can substitute one of the formulas we've seen

# Factor Constants out of Summation

- Due to the distributivity of multiplication over addition
  - n(a+b) = na + nb
- We can factor constants out of a series
  - 1c + 2c + 3c = c ( 1 + 2 + 3)
- This lets us transform summations into more manageable forms

$$\sum_{i=1}^{n} ci = c \sum_{i=1}^{n} i$$

# Break up Summation across a Sum

- Due to the associativity of addition
  - (a + b) + c = a + (b + c)
- We can split summations across sums
  - (1 + a) + (2 + b) + (3 + c)

    = (1 + 2 + 3) + (a + b + c)
- This lets us transform summations into more manageable forms

$$\sum_{i=1}^{n}(a+b) = \sum_{i=1}^{n} a + \sum_{i=1}^{n} b$$

**Exercise 12:**

Simplify the following summations

1. $\sum_{i=1}^{n} 3ci$
2. $\sum_{i=1}^{n} 3i + 4$
3. $\sum_{i=1}^{n} 2i^2 + 3i$

Hint:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

# Product Notation

# Product Notation

- To add a sequence we can use sigma notation and use summation
- What if we want to multiply a sequence? e.g.
  - $1 \times 2 \times 3 \times \ldots \times n-1 \times n$
- Product Notation
  - Uses the Greek capital letter pi, $\Pi$

# Parts of a Product

- *a*
  - *formula to multiply*
- *i*
  - variable
- *=1*
  - Lower limit
  - starting point (inclusive)
- *n*
  - Upper Limit
  - ending point (inclusive)

$$\prod_{i=1}^{n} a$$

# Example Product

- Multiply the first 5 elements of the sequence a:
  - $a_1 \times \ldots \times a_5$
- Lower limit = 1
- Upper Limit = 5
- Variable = *k*
- Formula to multiply = $a_k$

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5 .$$

# Product of a constant

- The product of *n* values of a constant *c* is given by the $n^{th}$ power of *c*
- The product of, e.g.
  - $10 \times 10 \times 10 \times 10 = 10^4 = 10000$
  - $c \times c \times c = c^3$

$$\prod_{k=1}^{n} c = c^n$$

# Product Notation

- We can also define a product recursively
  - The left part gives a base case - the product of one element is that element
  - The product of multiple elements (m → n) is the product of the first (m → n-1) elements multiplied by the *n*th element

$$\prod_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

**Exercise 13:**

Compute the following products:

1. $\prod_{k=1}^{4} 2$
2. $\prod_{k=1}^{4} k$
3. $\prod_{k=1}^{3} \frac{k}{k+1}$
4. $\prod_{k=0}^{7} 2^k$

# Summary

# Summary

- Sigma Notation – Summation
- Product Notation
- Sequences
- Summation of a Series