Mathematics and Problem Solving

Lecture 9

Graph Theory

Overview

- What is a Graph
- Terminology
- Representing Graphs

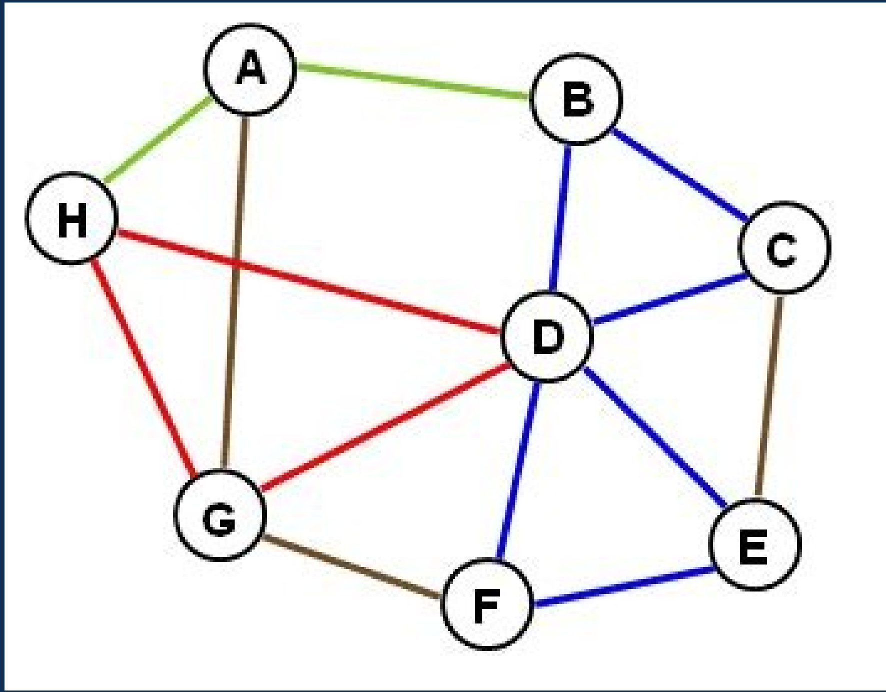
Why do we study graph theory ?

- Graph gives us the tool to formally study structures in graphical representations
  - Databases
  - Task planning
  - Pattern recognition
  - Tasks optimization
  - Scheduling
  - Data mining – Clustering
  - Solve shortest path problems

What is a Graph?

What is a graph?

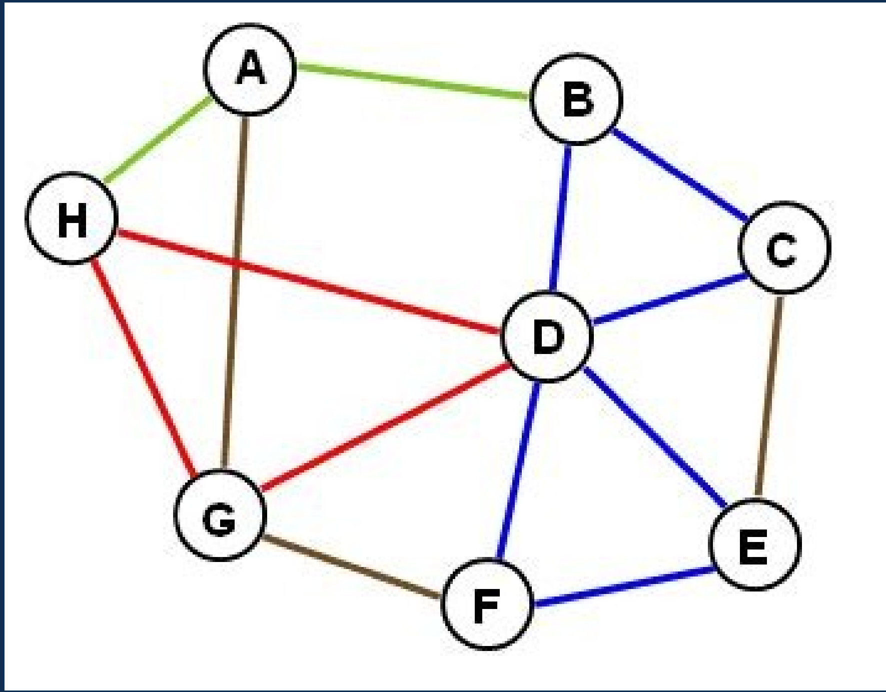
- A **graph** is a set of **vertices** connected by **edges**
- In other words, it's
  - Circles and lines
  - Dots and arcs
  - Boxes and arrows
  - Junctions and pipes
  - Cities and roads
  - Rooms and doors
  - ...etc.



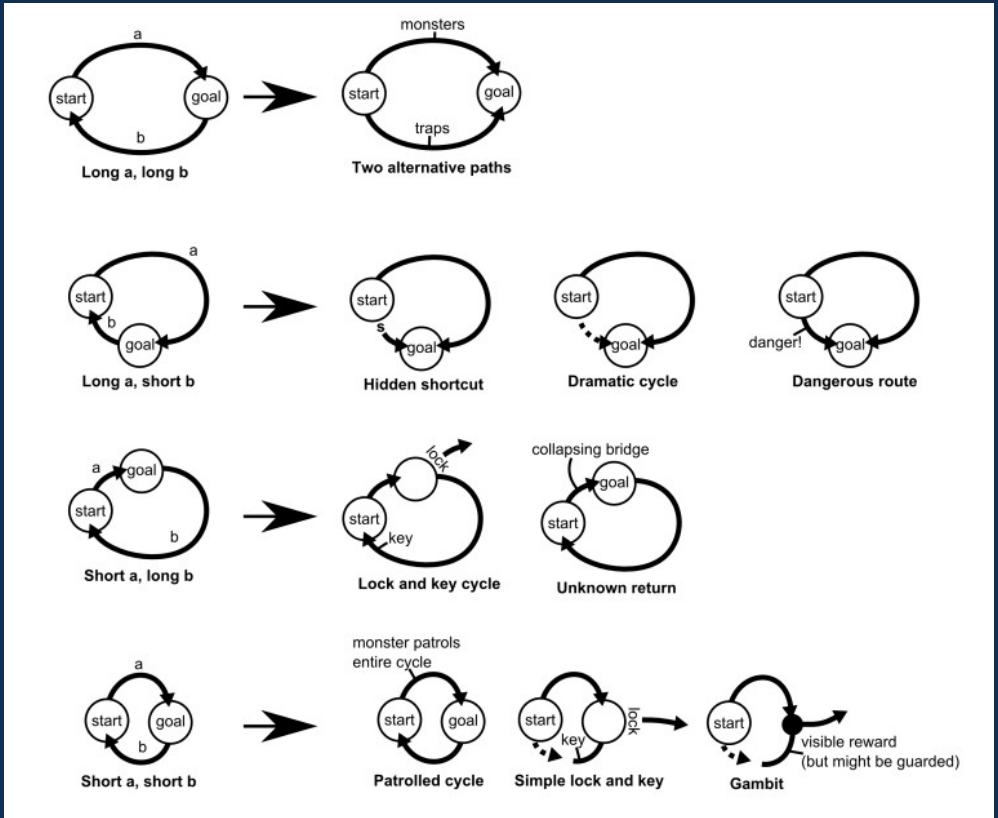
Exercise 1:

How might you use a graph to solve the following problems? What are the nodes? What are the edges?

1. Creating levels for a video game
2. Planning a delivery route
3. Playing Tic-Tac-Toe
4. Machine translation

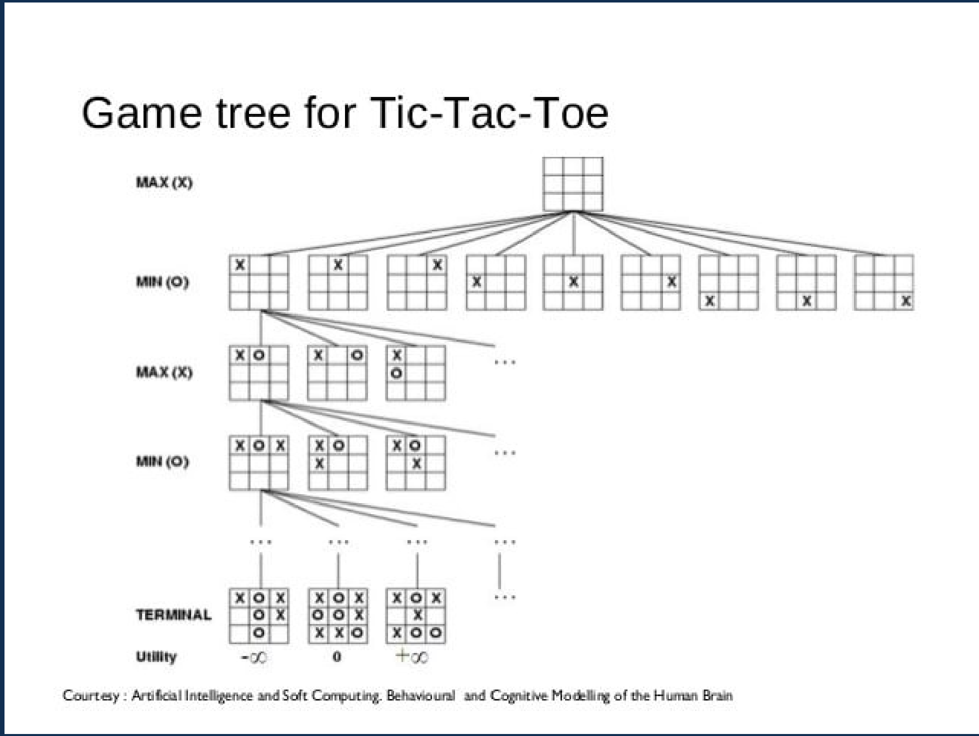


- Game level design patterns using cyclic graphs
  - Here each edge is marked either “short” or “long”
- Used in *Unexplored 2* by Ludomotion



https://ctrl500.com/tech/handcrafted-feel-dungeon-generation-unexplored-explores-cyclic-dungeon-generation/

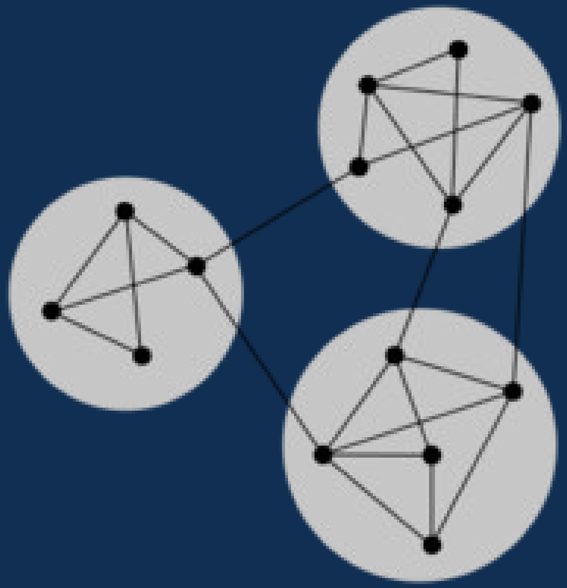
- Game AI with MINIMAX search
- Construct a game state tree
  - Terminal states have +/- utility
  - You take moves to maximise utility
  - Assume oponent takes moves to minimise utility



Terminology

Connected

- If a graph is connected there is a path from every point to every other point
- Nodes are immediately connected if they share an edge



Connected

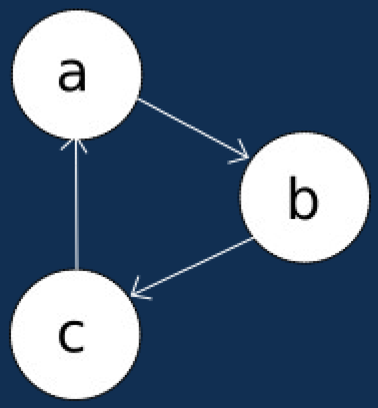
Exercise 2:

Which of the following are connected graphs?

1. A graph of possible derivations in a formal system
2. The road network in the UK
3. A social network
4. The Internet

Directed Graph (Digraph)

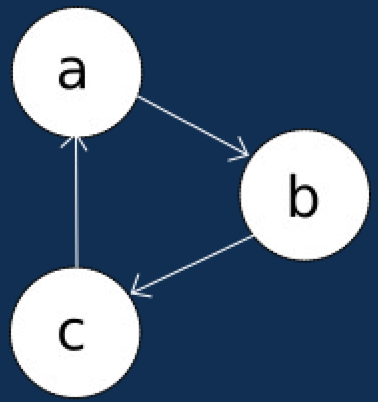
- A graph is **directed** if the edges have a **direction**
  - Represented with arrow heads
- Otherwise it is **undirected**
- Directed graphs might represent
  - The flow of water through pipes
  - A one-way relationship such as derivation: $A \Rightarrow_R B$



Exercise 3:

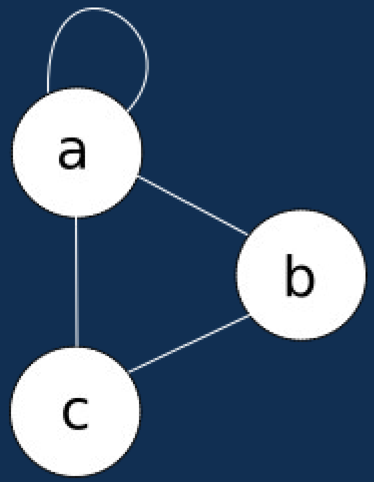
Which of the following are digraphs?

1. Sat-Nav directions
2. A graph of game states in a game of chess
3. A computer network
4. A snakes and ladders game board



Degree of a Node

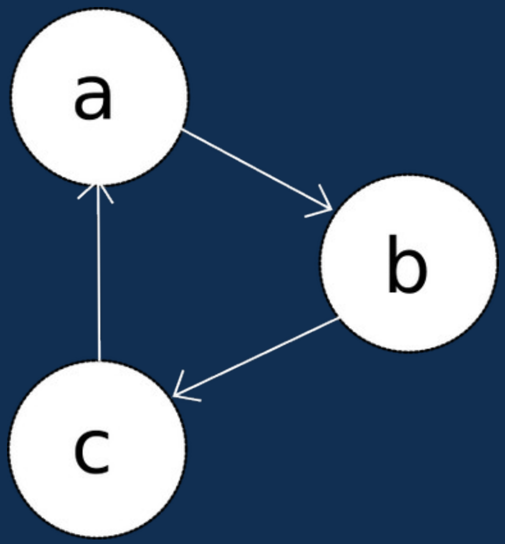
- The **degree** of a node is the number of edge connections it has
  - The numbers of edges incident to it
  - Loops count for 2
- For example
  - deg(a) = 4
  - deg(b) = 2
  - deg(c) = 2



Indegree and Outdegree

- For a node in a directed graph
  - **indegree** is the number of edges pointing to a node
  - **outdegree** is the number of edges pointing away from it

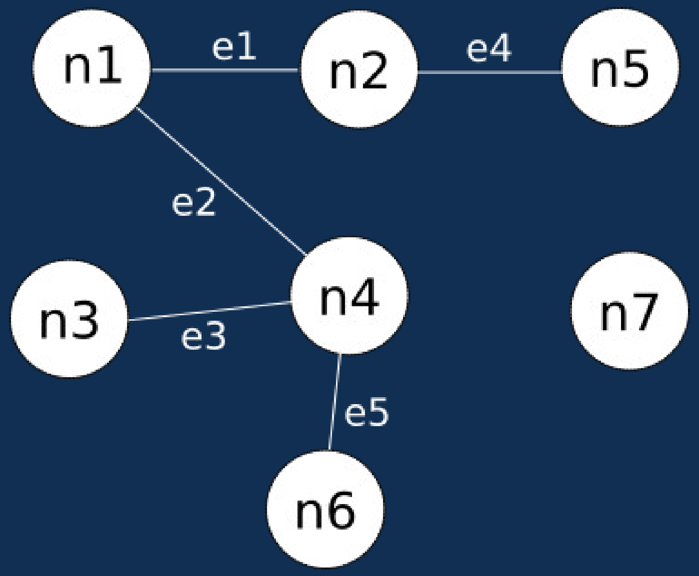
degree = indegree + outdegree



Exercise 4:

What is the degree of the following nodes:

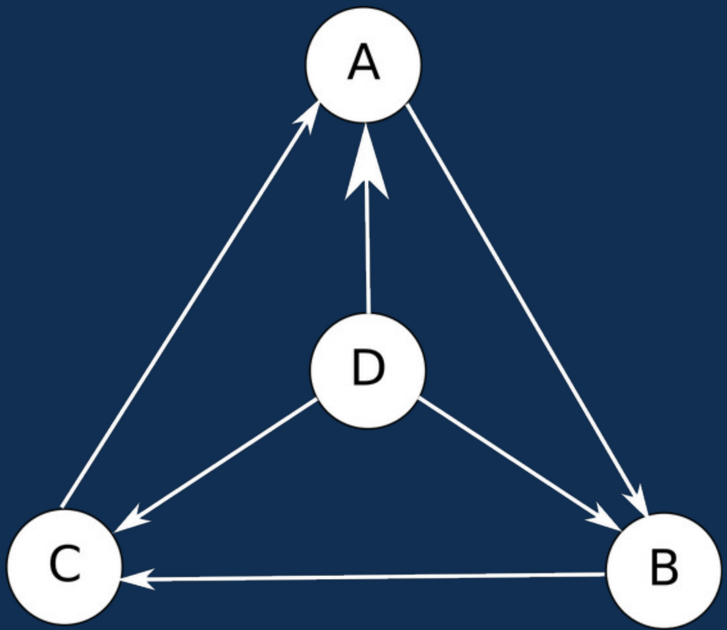
1. n1
2. n3
3. n4
4. n7



Exercise 5:

What is the in- and out-degree of the following nodes?

1. A
2. B
3. C
4. D



Some “types” of node for directed graph

- Source node is a node with

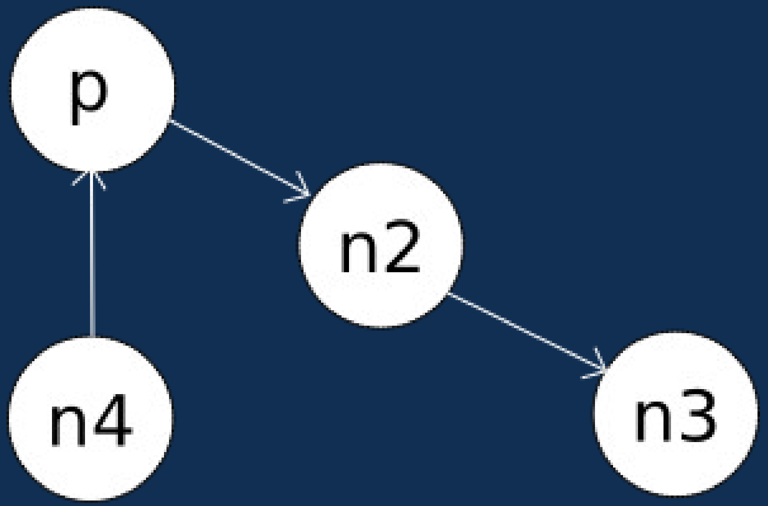
  indegree = 0.

- Sink node is a node with

  outdegree = 0.

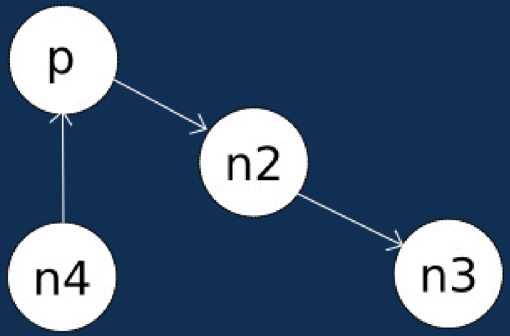
- Transfer node is node with

  indegree ≠ 0 and outdegree ≠ 0.



Exercise 6:

Which of the nodes for the graph below are sources, sinks, and transfer nodes?



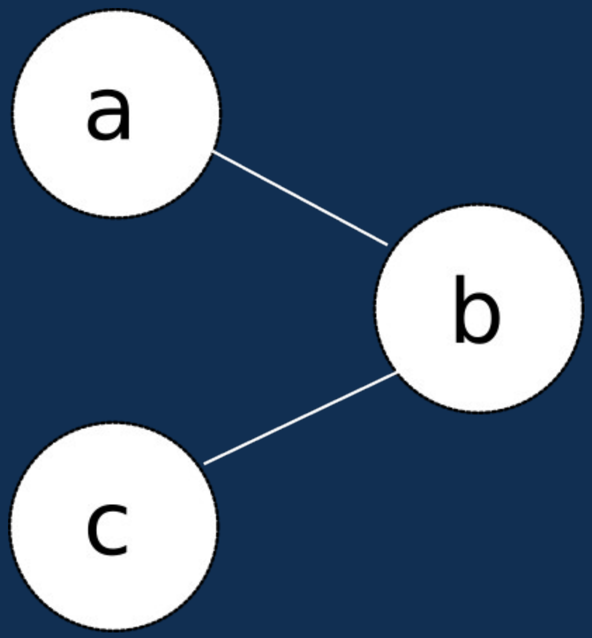
- Hint:
  - Source node is a node with indegree = 0.
  - Sink node is a node with outdegree = 0.
  - Transfer node is node with indegree ≠ 0 and outdegree ≠ 0.

Terminology

Paths

Immediately Connected

- Two nodes are **immediately connected** if there is an edge between the two nodes.
- In the example:
  - *a* and *b*
  - *b* and *c*



Path

- A **path** is a sequence of immediately connected vertices:

  $N_i, \ldots N_j$

  N_i is immediately connected to N_{i+1}

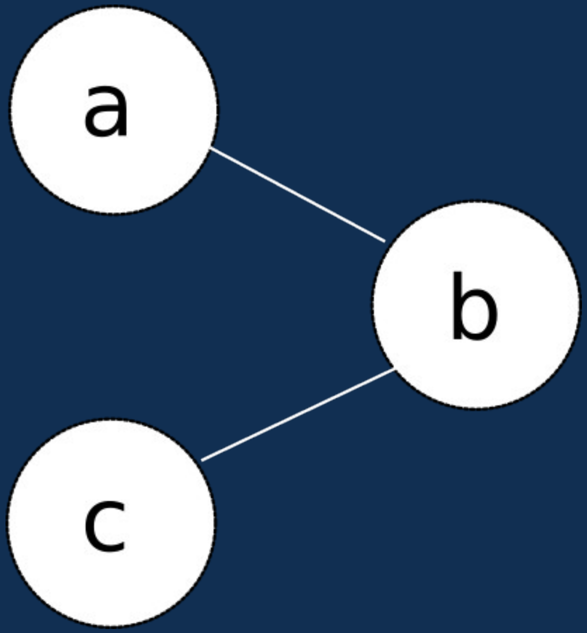
  N_{i+1} is immediately connected to N_{i+2}

  ...

  N_{j-1} is immediately connected to N_j

- In the example there is a path

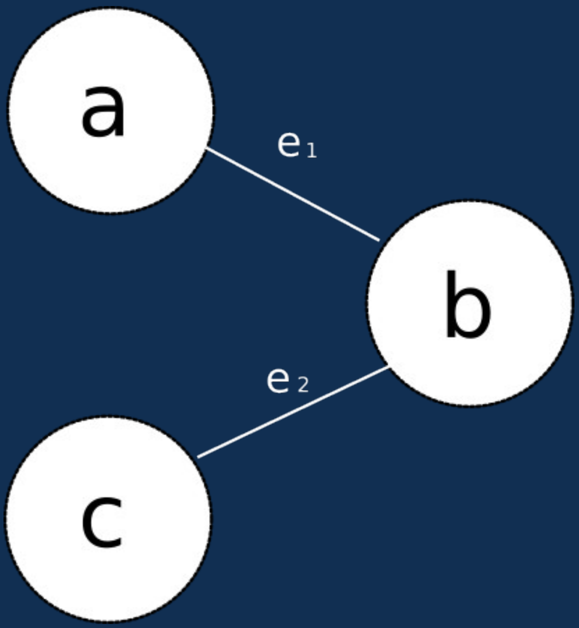
  a, b, c



Representing Paths with Edges

- A path may also be defined as a sequence of edges, such that
  - for any pair of edges, e_i and e_{i+1}
  - e_i and e_{i+1} share a common node.
- In the example, there is a path (represented by edges):

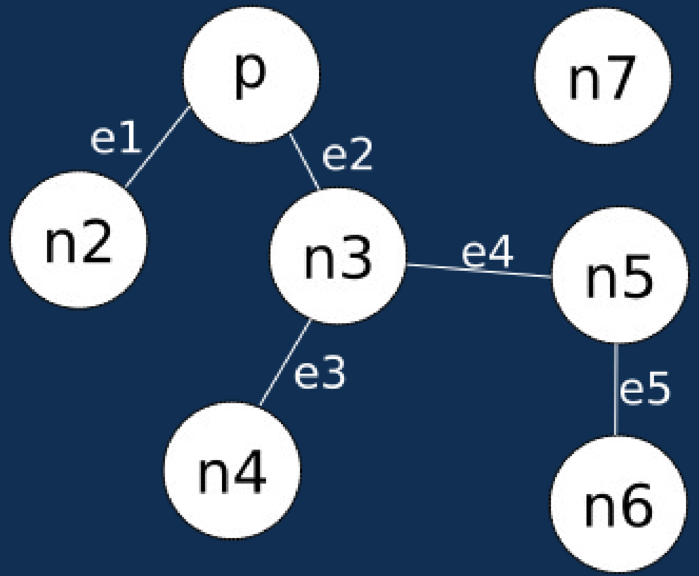
e_1, e_2



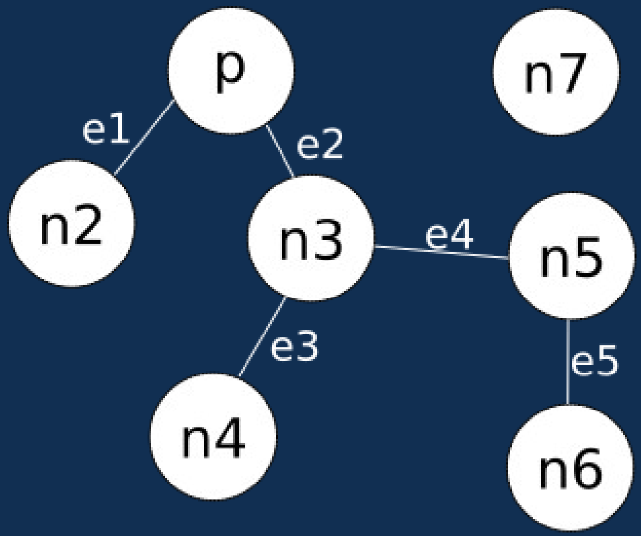
Exercise 6:

Identify a path in the graph shown. Express it as both a

1. sequence of vertices
2. sequence of edges



Example Paths



There are several paths in the example graph.

- Path 1: n2, p, n3, n5
- Path 2: p, n3, n4

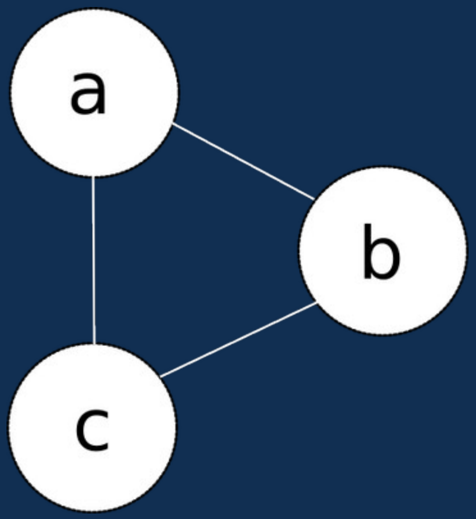
- Or represented with edges:
  - Path 1: e1, e2, e3
  - Path 2: e2, e3

Cycle

- A cycle is a path whose first and last vertices are the same
- The path

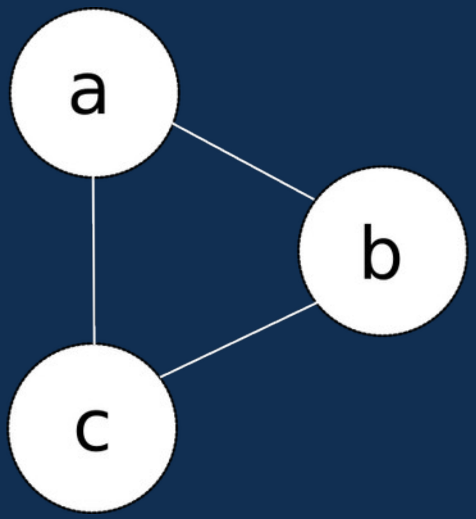
  a, b, c, a

  is a cycle
- A graph that contains one or more cycles is called **cyclic**
  - Otherwise it's called **acyclic**



Simple Paths and Cycles

- A path (or cycle) is **simple** if it has no repeated vertices
  - Except for first and last in the case of a cycle
- A simple cycle:
  - a, b, c, a
- A cycle that is not simple:
  - a, b, c, a



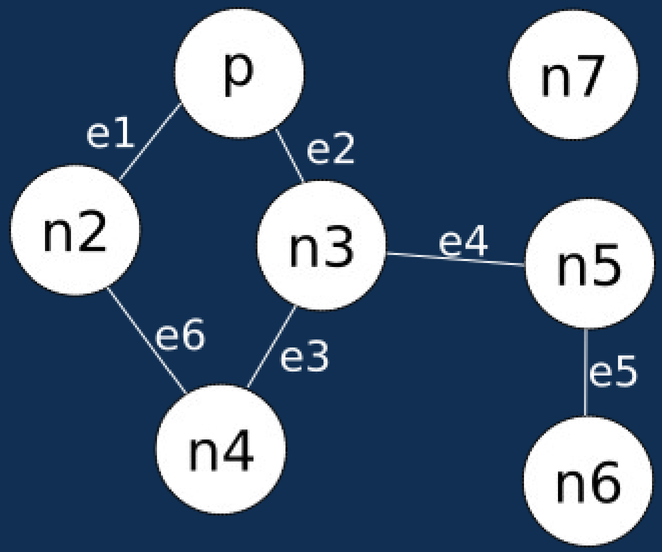
Path Length

- The **length** of a path (or cycle) is the number of edges

Path Length

Exercise 7:

1. Identify a simple path of length 6
2. Identify a simple cycle
3. Is it possible to connect every node with one path?

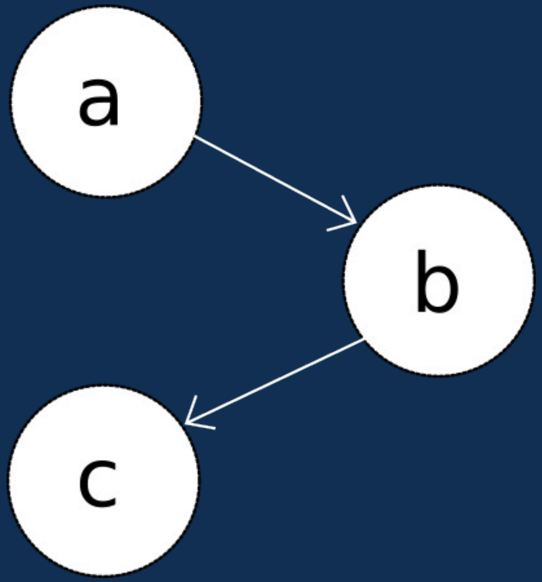


Directed Paths

- A sequence of vertices connected by directed edges is a **directed path**
  - A directed path:

    a b c
  - A path that is **not** directed

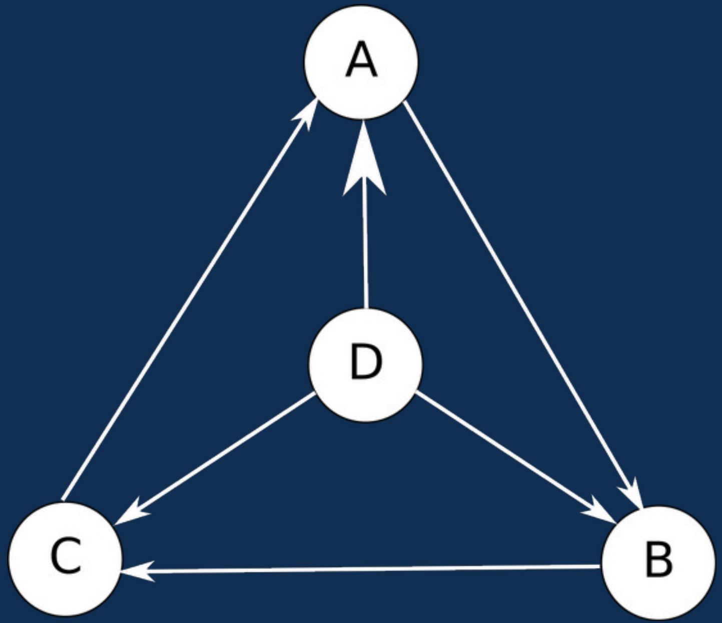
    c b a



Exercise 8:

List all the directed simple paths in this graph

Identify which ones are cycles.

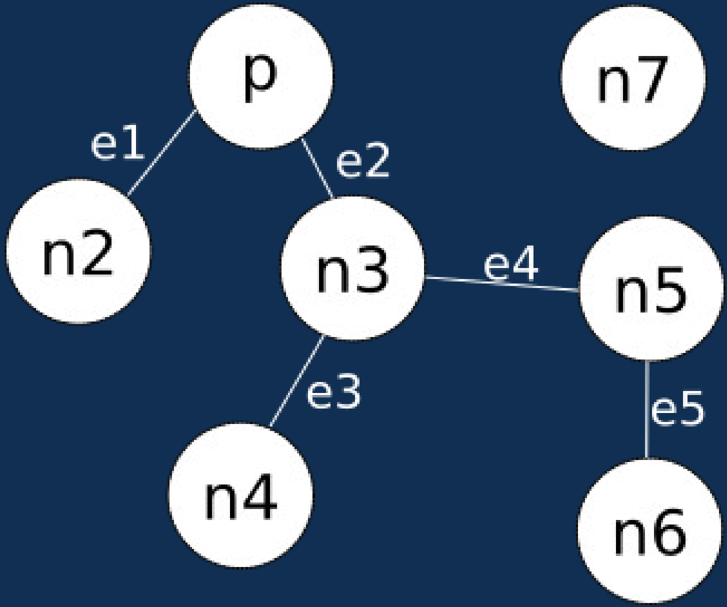


Connectedness

- Node n_i is **connected** to node n_j if there is a path from n_i to n_j
  - An undirected graph is connected if there is a path from every node to every other node
- Node n_i is **strongly connected** to node n_j if there is a directed path from n_i to n_j
  - A digraph is strongly connected if there is a directed path from every node to every other node

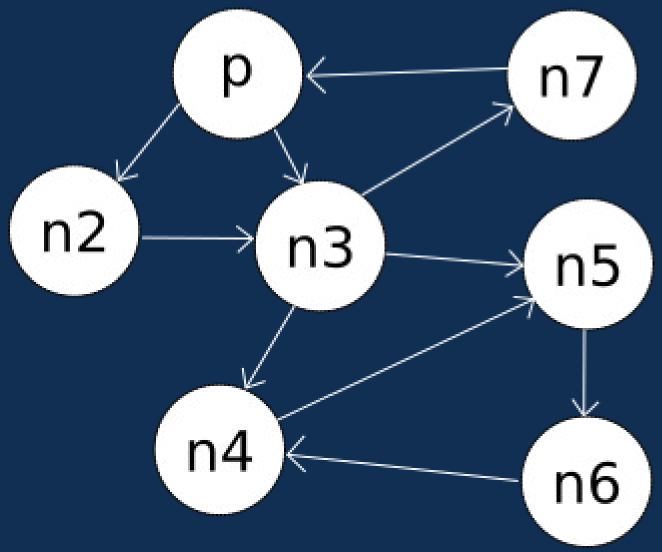
Component

- A component of a graph is defined as the maximal set of connected nodes.
  - In the example graph:
    - {p, n2, n3, n4, n5, n6} form a component
    - {n7} also forms a component



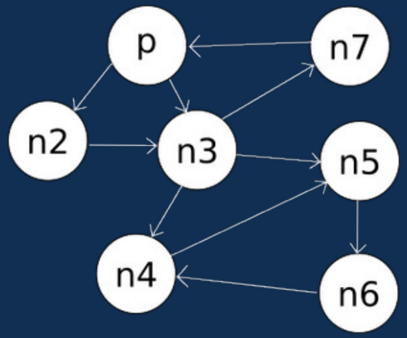
Strongly Connected Component

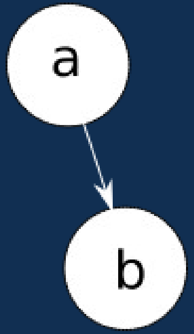
- In a digraph, a strongly connected component is maximal set of strongly connected nodes.
  - In the example graph:
    - {p, n2, n3, n7} form a strongly connected component
    - {n4, n5, n6} forms a strongly connectedcomponent



Condensation graph

- A condensation graph is formed by replacing each component in the original graph by a “condensed” single node. There are no edges.
  - Condensation graph of a graph gives us a view of the “complexity” through the number of components.
- We can consense a digraph as well, to see the relationships between its strongly connected components

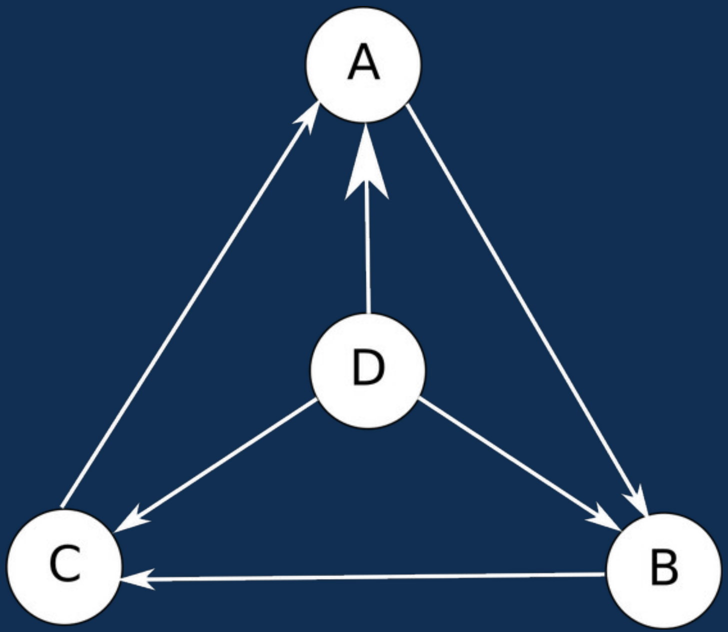




Exercise 9:

Look at the graph shown

1. Identify the strongly connected components
2. Draw its condensation graph



Terminology

- There is a lots more terminology about graphs
  - And some different ways these terms can be defined
- For this course, use the terminology and definitions given
- For example
  - Walks
  - Trails
  - Paths
  - Circuits
  - Multigraph
  - Pseudograph
  - Bouquet
  - …

Definitions – Graph Type

Type	Edges	Multiple Edges Allowed ?	Loops Allowed ?
Simple Graph	undirected	No	No
Multigraph	undirected	Yes	No
Pseudograph	undirected	Yes	Yes
Directed Graph	directed	No	Yes
Directed Multigraph	directed	Yes	Yes

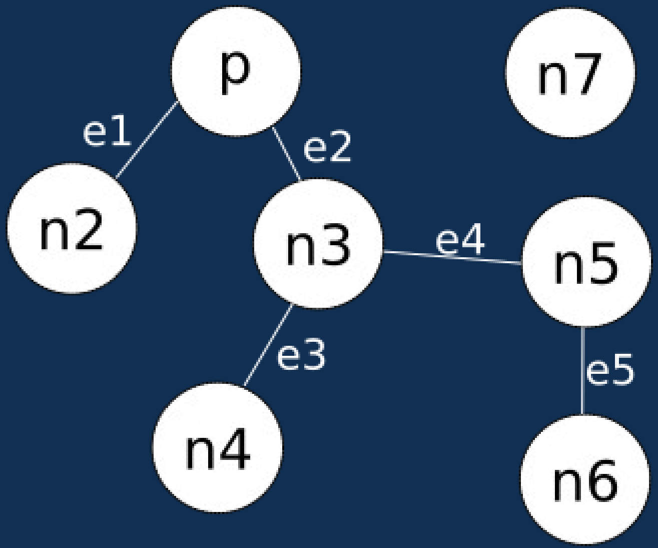
Representation

Representing a Graph

- When working with a graph mathematically, we want a formal representation
  - Allows us to express properties about our graph using the tools of Set Theory, Logic, etc.
  - Allows generalisation over possible graphs
  - Allows us to describe algorithms on graphs
- Different representations can express different sorts of graph

Set-theoretic representation

- A simple graph can be represented as an ordered pair G = (V, E) composed of :
  - a set of vertices, V, and
  - a set of edges, E, which connects the nodes.
- For the example
  - V = { p, n2, n3, n4, n5, n6, n7 }
  - E = {e1, e2, e3, e4, e5 }



Representing Vertices

- Vertices are usually represented as labels if their content doesn't matter
  - V = { a, b, c }
- We could also represent the data of each node, for example with numbers:
  - V = {2, 54, 25}
  - Or define the labels above, e.g. a = 2

Question:

The following set of vertices is not allowed, why? How could we fix it?

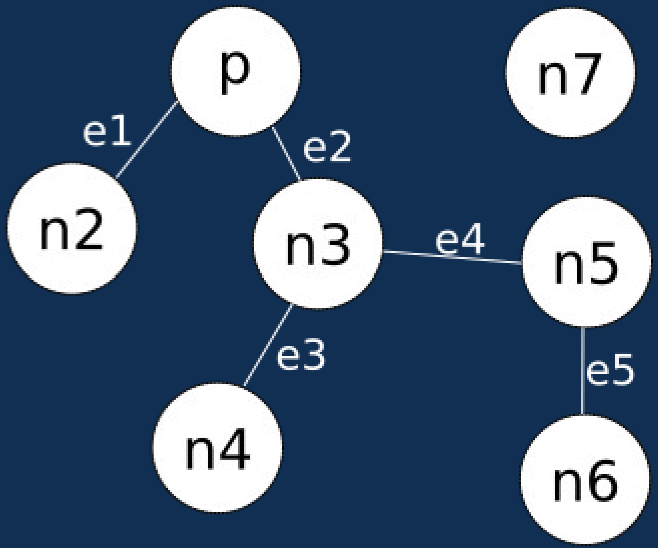
- V = { 12, 42, 42 }

Representing Edges

- There are different ways of representing edges
- Edges are commonly represented as sets or tuples
  - E1 = {n2, p}
  - E1 = (n2, p)

Question:

What difference does this make?



Exercise 10:

1. Draw the **undirected graph** $G_1 = (V_1, E_1)$ where

   $V_1 = \{ a, b, c, d \}$

   $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\} \}$

2. Draw the **directed graph** $G_2 = (V_2, E_2)$ where

   $V_2 = \{ a, b, c, d \}$

   $E_2 = \{(a, b), (a, c), (a, d) \}$

Defining Edges

- So far we've seen an extensional definition of the set of edges
  - We can also describe it as follows

$$E \subseteq \{ \{x, y\} \mid x, y \in V \quad \wedge \quad x \neq y \}$$

- Or if you prefer, we could express the set comprehension with a term

$$E \subseteq \{ x: V; y: V \mid x \neq y \bullet \{x, y\} \}$$

- And if we want directed edges, we just change the term to a tuple

$$E \subseteq \{ x: V; y: V \mid x \neq y \bullet (x, y) \}$$

Representation

Multigraphs

Exercise 11:

Try to give an extensional definition for the graph shown. What problem do you encounter?

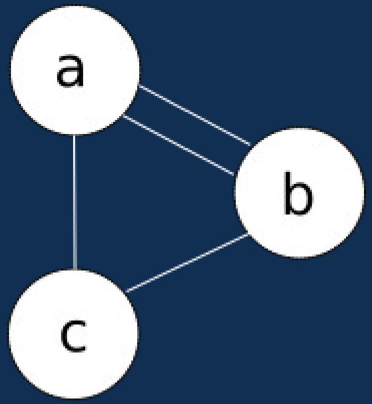
Hint:

- An extensional definition is listing all the elements of a set

$$G = (V, E)$$

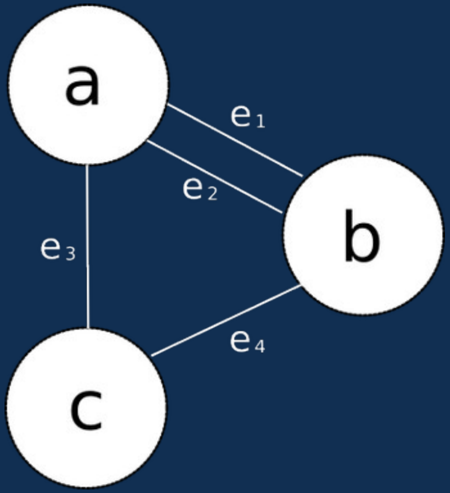
$$V = \{ \dots \}$$

$$E = \{ \dots \}$$



Multiple Edges (Multigraph)

- With a graph (V, E) there is no way to represent multiple edges
  - We need to change our representation
- Let's invent a function, ϕ, that tells us, for each edge, what vertices it joins
  - $G = (V, E, \phi)$
- For example,
  - $\phi(e_1) = \{a, b\}$
  - $\phi(e_2) = \{a, b\}$



Multigraphs

- We can define the *type* of the function ϕ.
  - Functions map inputs in a domain (edges) to a range (2-sets of vertices)

$$\phi : \mathrm{E} \rightarrow \mathrm{P}$$

$$\textit{where } \mathrm{P} = \{ \{ x, y \} \mid x, y \in \mathrm{V} \ \wedge \ x \neq y \}$$

- Here P is the set of sets {x, y}, where x and y are
  - vertices in our graph ($\in$ V)
  - not equal to one another

Functions in Set Theory

- Function, like everything else in Set Theory, are sets
- A function is a set of pairs (A, B), such that
  - A is an element of the domain
  - B is an element of the range
- For example the successor function, $S : \mathbb{N} \rightarrow \mathbb{N}$, is the set of pairs
  - $S = \{(1, 2), (2, 3), (3, 4), \ldots, (n, n+1), \ldots\}$

Functions in Set Theory

Exercise 12:

Let *f* be a function of type $\mathbb{N}_4 \to \mathbb{N}_4$

$$f(x) = x^2 \bmod 4$$

1. Give an extensional definition of *f*

Hint:

- $\mathbb{N}_4 = \{0, 1, 2, 3\}$
- Functions in set theory are sets of pairs
  - { ($input_1$, $output_1$), ($input_2$, $output_2$) … }
- An extensional definition lists all the elements

Multigraph

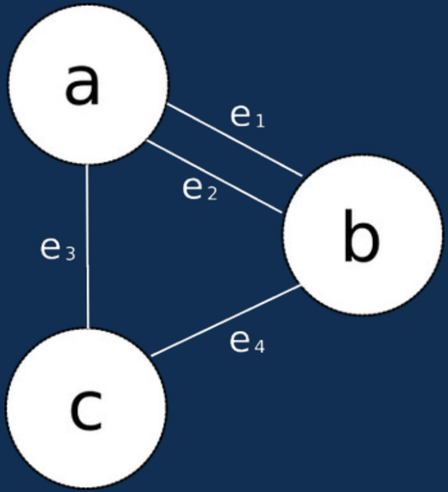
- We now have the tools to give an extensional definition of our multigraph

$$G = (V, E, \phi)$$

$$V = \{a, b, c\}$$

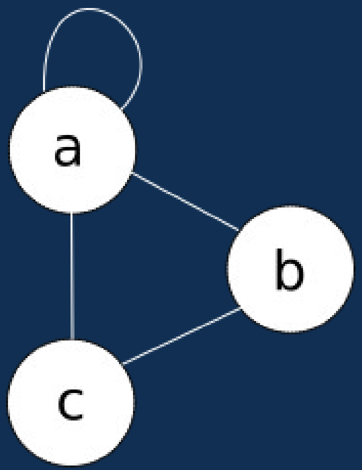
$$E = \{e_1, e_2, e_3, e_4\}$$

$$\phi = \{(e_1, \{a, b\}), (e_2, \{a, b\}), (e_3, \{a, c\}), (e_4, \{b, c\})\}$$



Exercise 13:

Try to give a formal representation for the graph shown. What problem do you encounter?



Hint:

- It’s a simple graph (not a multigraph) so we don’t need ϕ here

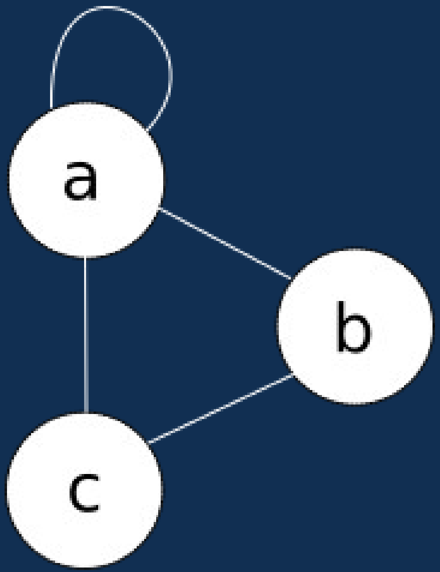
$$G = (V, E)$$

$$V = \{ \dots \}$$

$$E = \{ \dots \}$$

Loops

- Representing looping edges
  - Can’t use a 2-set, as all elements in a set are unique
    - { a, a }
  - Don’t want to use a tuple, as it’s unordered
    - (a, a)
- We have to change our representations to allow edges to be singletons { a } representing a loop



Edges that Permit Loops

- To permit loops we need to drop the requirement that $x \neq y$

$$E \subseteq \{ \{x, y\} \mid x, y \in V \}$$

- Or using a term...

$$E \subseteq \{ x: V; y: V \bullet \{x, y\} \}$$

Exercise 14:

Allowing loops in a *multigraph* means we need to change our definition of ϕ. What does it need to become?

Hint:

- An undirected multigraph has a function ϕ of type

$$\phi : E \rightarrow \{ \{ x, y \} \mid x, y \in V \ \wedge \ x \neq y \}$$

Directed Multigraph Permitting Loops

- If we want a *directed* graph that
  - allow loops
  - allows multiple edges between nodes
- We can adapt our previous definition to that shown on the right

$$G = (V, E, \phi)$$

$$V = \{ \dots \}$$

$$E = \{ \dots \}$$

$$\phi : E \rightarrow \{ (x, y) \mid x, y \in V \}$$

Exercise 15:

Give an extensional definition of the graph opposite

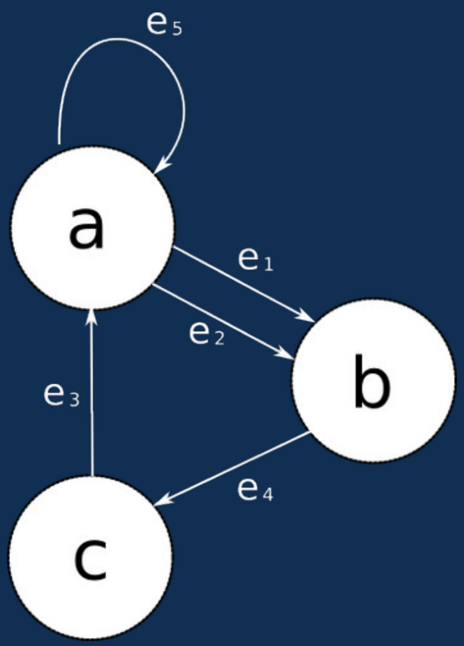
Hint:

$$G = (V, E, \phi)$$

$$V = \{ a, ... \}$$

$$E = \{ e_1, ... \}$$

$$\phi = \{ (e_1, (a, b)), ... \}$$



Graphs and Set Comprehensions

Exercise 16:

1. Draw a graph of the relationships between users, orders, and items
2. Intensionally define the set of edges connecting Items to Items (which have a offer price associated with them)
3. Intensionally define the set of edges connecting Users to Orders

User ID	Name	Address	Orders
5	John	1 A St	{13}
6	Mary	2 B St	{14,15}

Order ID	User	Shipping Cost	Items Cost	Items
13	5	4.00	22.50	{3, 4}
14	6	4.00	22.50	{3, 4}
15	6	2.55	4.50	{4}

Item ID	Name	Price	Offer Items
3	Shoe	20.00	{ (4, 2.50) }
4	Laces	4.50	{ }

Summary

Summary

- What is a Graph
- Terminology
  - Node/vertex, edge
  - (Strongly) Connected, Directed (digraph), (In/Out)degree,
  - (Simple) Path, Cycle,
  - Multigraph, Loops
- Representing Graphs
  - Set theoretic definitions