Mathematics and Problem Solving

Lecture 10

Probability

Overview

- Probability Basics
  - Events
  - Addition Rule
  - Multiplication Rule for Independant Events
  - General Multiplication Rule
- Combinatorics
- Probability Trees
- Law of Total Probability
- Bayes Rule

Events

Experiment

- E.g. pulling a ball out of a bag, rolling a dice
- Outcomes
  - Drawing a particular ball
  - Rolling a particular number
- Events are sets of outcomes
  - Draw a Red ball.
  - Draw a Black ball
  - Roll an even number
  - Roll a 4

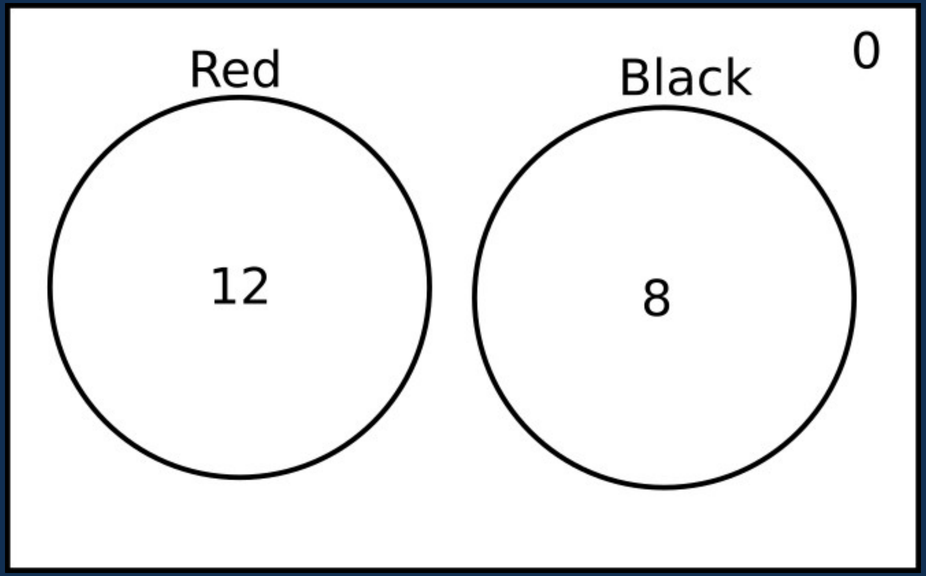
Classical Probability

P(event) = favourable outcomes / total possible outcomes

- Assume a bag of balls
  - 12 Red balls (12 outcomes)
  - 8 Black balls (8 outcomes)
- P(red) = 12 / 20
- P(black) = 8/20

Venn Diagram

- We can represent this experiment with a Venn Diagram
  - Each circle represents an event (set)
  - We give the number of outcomes in each event



Exercise 1:

Represent the information opposite as a Venn diagram.

What is the probability of:

1. A randomly selected employee works remotely
2. A randomly selected company laptop being used by someone on-site

- A company has 100 employees
  - 60 employees work on site
    - 50 of whom use company-provided laptops
  - 40 employees work remotely
    - 20 of whom use company-provided laptops
- Hint:

P(event) = favourable outcomes / total possible outcomes

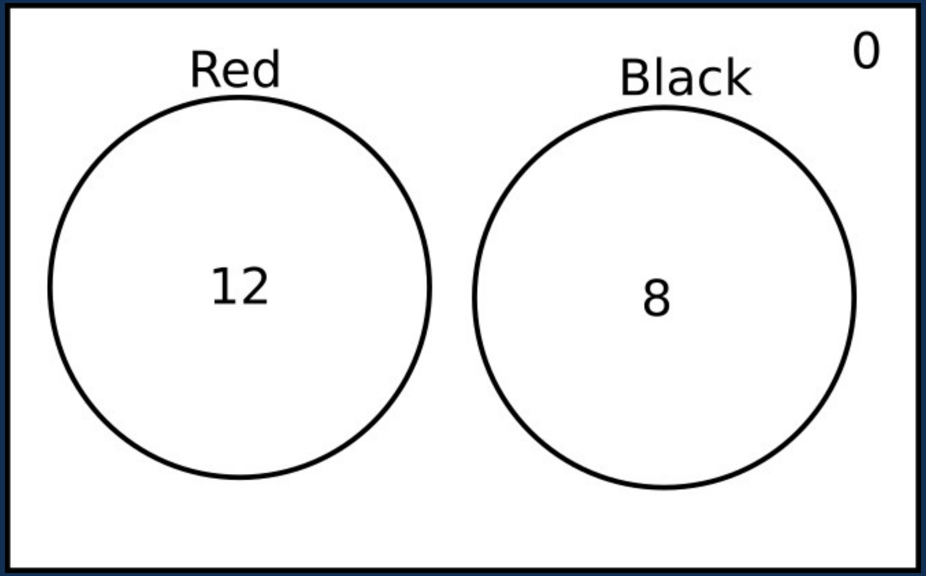
Complement

- Complement of a probaility P(A) is

  $P(A') = 1 - P(A)$

- Chance I don't draw a red ball

  $P(red') = 1 – (12/20) = 8/20$



Exercise 2:

Using the information opposite.

What is the probability of:

1. A randomly selected on-site employee not using a company-provided laptop

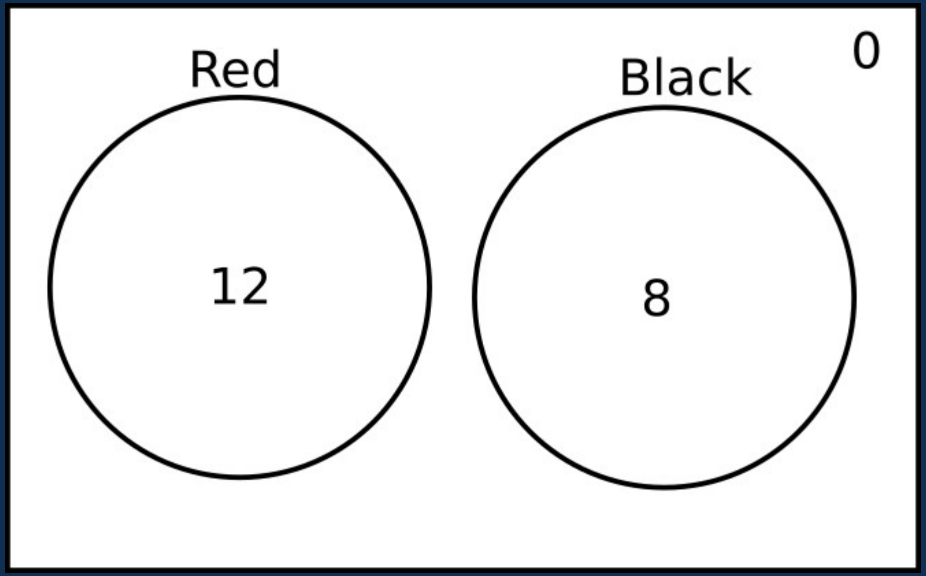
- A company has 100 employees
  - 60 employees work on site
    - 50 of whom use company-provided laptops
  - 40 employees work remotely
    - 20 of whom use company-provided laptops
- Hint:

P(event) = favourable outcomes / total possible outcomes

P(A’) = 1 - P(A)

Mutually Exclusive Events

- Mutually exclusive events cannot both occour at once
  - On a Venn diagram, they do not overlap
  - Their intersection is the empty set
- In the example, Red and Black are mutually exclusive events



Independant Events

- Two events are independent if their probabilities do not change depending on the outcome of the other
- For example
  - Rolling two dice
  - Fliping a coin 10 times
- Events that are not independent are called **dependant events**

Exercise 3:

Which of the following involve dependant or independent events?

1. Generating two random numbers in Python
2. Drawing two cards from a deck and looking at their suit
3. Two people each choosing a random chair in this room
4. The probability of two successive requests to a webserver failing

Summary

- Experiment
  - Outcomes
  - Events
- Venn Diagrams
- Probabilities of Events
  - Favourable outcomes / Total possible outcomes
- Complement
  - $P(A') = 1 - P(A)$

Exercise 4:

You have a deck of 52 cards (4 suits, no jokers). What is the probability of drawing the following.

1. The ace of hearts (🂱)
2. A heart (🂱 🂲 🂳 🂴 🂵 🂶 🂷 🂸 🂹 🂺 🂻 🂽 🂾)
3. A face card (🂻 🂽 🂾 🂫 🂭 🂮 🃋 🃍 🃎 🃛 🃝 🃞)
4. A joker (🃟)

Probability of Union

Adding Probabilities

Exercise 5:

For the deck of cards described opposite, what is the:

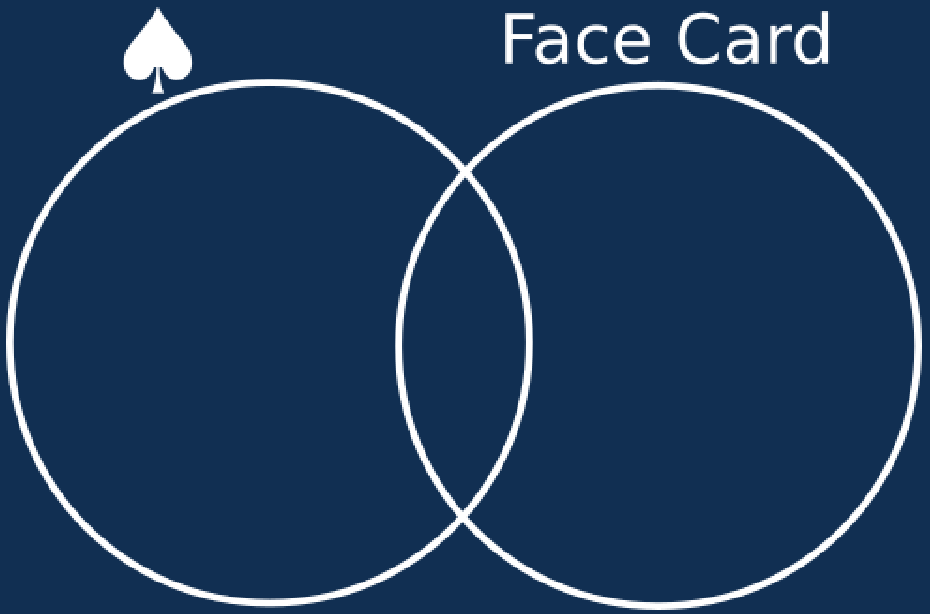
1. Probability of drawing a spade?
2. Probability of drawing a face card?

- I have a deck of cards and I'm making a game
  - 4 suits
  - 13 cards in each suit
  - 3 face cards in each suit
  - 52 cards overall

Adding Probabilities

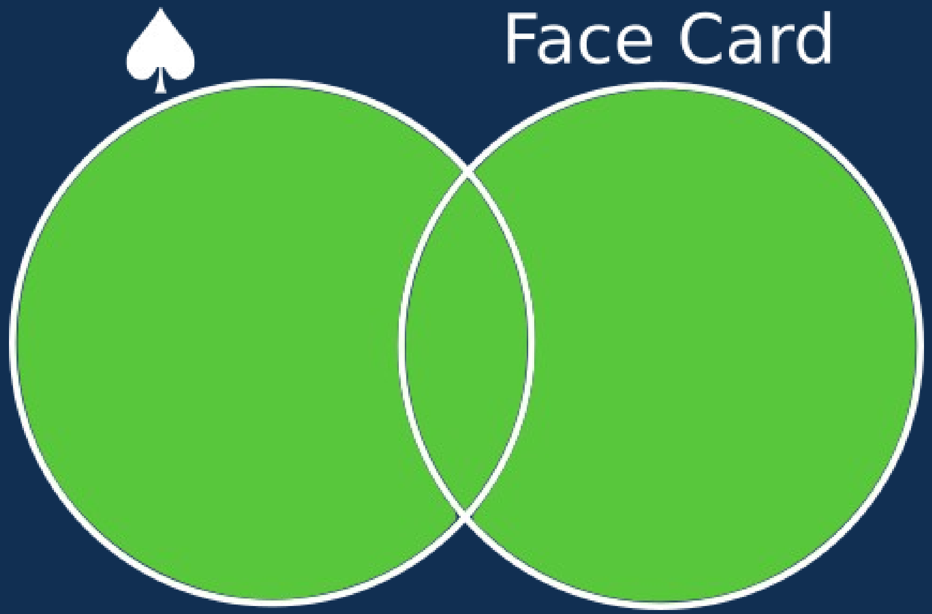
- Whats the probability of drawing either a spade or a face card?
- Assuming
  - P(♠) = 1/4
  - P(face) = 3/13
- The probability of A *or* B is the probability of the union of A and B

$$P(A \cup B)$$



Union (∪)

- If events are sets of outcomes, the set-theoretic union of two events is the event that either event occours
- To find this, we
  1) Add up the probabilities of each event
  2) Take away the probability that the events co-occour



Addition Rule

Addition Rule:

The combined probability of two events A and B is the sum of the probabilities of those events, less the probability that those events co-occour:

$$P(A \cup B) = P(A) + P(B) + P(A \cap B)$$

Addition Rule (Example)

- We know
  - P(♠) = 1/4 = 13/52
  - P(face) = 3/13 = 12/52
  - P(♠ ∩ face) = 3/52 (because there are 3 spade face cards in the deck)
- P(♠ ∪ face) = P(♠) + P(face) - P(♠ ∩ face)
- P(♠ ∪ face) = 13/52 + 12/52 – 3/52
- P(♠ ∪ face) = 22/52

Exercise 6:

Two mutually exclusive events have probabilities P(A) = ½ and P(B) = ¼

What is the probability that A or B occours?

- Hint:

  $P(A \cup B) = P(A) + P(B) + P(A \cap B)$

Probability of Intersection

Intersection

- What is the liklihood that A and B occour?
  - The probability of the intersection of the events A and B

$$P(A \cap B)$$

Multiplication Rule for Independant Events

Multiplication Rule for Independent Events:

To find the probability of two independent events co-occuring (their intersection) we multiply their probabilities together

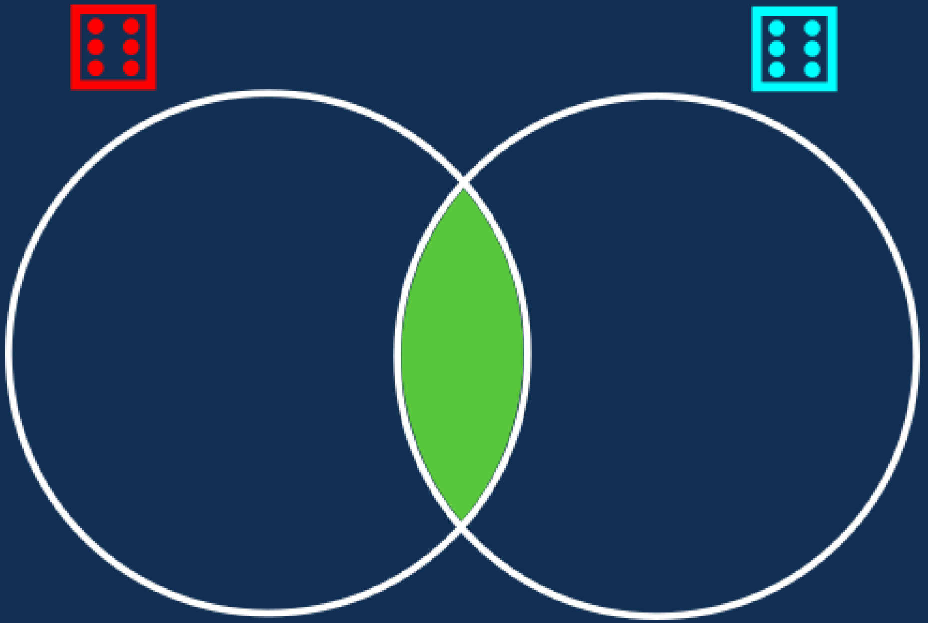
$$P(A \cap B) = P(A) \times P(B)$$

Intersection

- What is the probability of rolling ⚅ and ⚅ on two dice?
  - P(⚅∩⚅) = P(⚅) x P(⚅)

    = 1/6 x 1/6

    = 1/36
  - (Note these are independent events)



Exercise 7:

Work out the probabilities for the following **independent events**:

1. Flip a coin twice and get two heads
2. $P(A) = 0.2$; $P(B) = 0.5$; $P(A \cup B) = ?$
3. Draw a red card or a even number from a 52 card deck

General Multiplication Rule

Conditional Probability

- The chance of event A assuming, or conditional, on event B also happening
  - Written P(A | B)
- To find P(A | B) you:
  1) find the chance that A and B co-occour
  2) divide by the probability of B.

$$P(A|B) = \frac{P(B \cap A)}{P(B)}$$

General Multiplication Rule

General Multiplication Rule:

To find the chance that two events co-occour you multiply their the probability of event A occouring given B by the probability of B.

$$P(A \cap B) = P(A|B) \times P(B)$$

Combinatorics

Combinatorics

- A fancy name for three things
  - Enumeration
  - Combination
  - Permuation

Factorial

- Formulas for combinatorics make use of factorial
  - Written n!
  - n! is n multiplied by every natural number <n

- 1! = 1
- 2! = 2 x 1
- 3! = 3 x 2 x 1
- 4! = 4 x 3 x 2 x 1
- Note that
  - 0! = 1

Enumeration

- An enumeration is just a listing of a set
- If I have a set A = {a, b, c, d}
- Its enumeration is
  - {a, b, c, d}

Combination

- Say we have a set A = {a, b, c, d} and we want combinations of 3 elements
- Every subset of A with 3 elements is a combination
  - {a, b, c}
  - {b, c d}
  - {a, c, d}
  - {a, b, d}
- Combinations don’t have order

Permutation

- Permutations have order
- Permutations of 3 elements of the set A = {a, b, c, d}

  - (a, b, c)
  - (a, c, b)
  - (b, a, c)
  - (b, c, a)
  - (c, a, b)
  - (c, b, a)
  - (a, b, d)
  - (a, d, b)
  - ...

Exercise 8:

Which of the following are enumerations, combinations, and permutations?

1. I pick some books to take on holiday
2. I select three films to watch
3. I compile a playlist of my favourite music
4. I list all the numbers from 1-100

Calculating Number of Permutations

Exercise 9:

How can we write a formula for permutations?

- Hint:
  - Permutations of 1 out of 4
    - 4
  - Permutations of 2 out of 7
    - 7 x 6
  - Permutations of 3 out of 5
    - 5 x 4 x 3
  - Permutations of 5 out of 20
    - 20 x 19 x 18 x 17 x 16

Calculating Number of Permutations

$$^{n}P_{k} = \frac{n!}{(n-k)!}$$

- Where *n* is the size of the set and *k* is the size of the subset we are ordering

Exercise 10:

Work out the answers to the following permutations

1. ${}^{1}P_{1}$
2. ${}^{2}P_{1}$
3. ${}^{4}P_{2}$
4. ${}^{5}P_{3}$

$$ {}^{n}P_{k} = \frac{n!}{(n-k)!} $$

Calculating Number of Combinations

- Start by treating it as a permutation
  - Gives us (number of combinations) x (number of possible orders)
- Then divide by number of possible orders there could be
- How many orders of k things?
  - ${}^{k}p_{k} = k!$

Calculating Number of Combinations

$$^{n}C_k = \frac{n!}{k!(n-k)!}$$

- Where *n* is the size of the set and *k* is the size of the subset we are picking
- This is also called the Binomial Coefficient and can be written: $\binom{n}{k}$

Comparing Permutations and Combinations

$$^{n}P_{k} = \frac{n!}{(n-k)!}$$

$$^{n}C_{k} = \frac{n!}{k!(n-k)!}$$

- Where *n* is the size of the set and *k* is the size of the subset we are picking/ordering

Exercise 11:

Work out the answers to the following combinations

1. ${}^{1}C_{1}$
2. ${}^{2}C_{1}$
3. ${}^{4}C_{2}$
4. ${}^{5}C_{3}$

Summary

- Factorial
- Permutations and Combinations
  - Two formulas to remember (or look up)

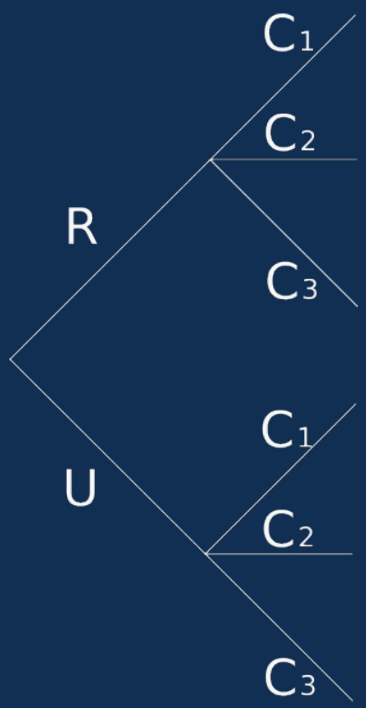
$$^{n}P_k = \frac{n!}{(n-k)!}$$

$$^{n}C_k = \frac{n!}{k!(n-k)!}$$

Probability Trees

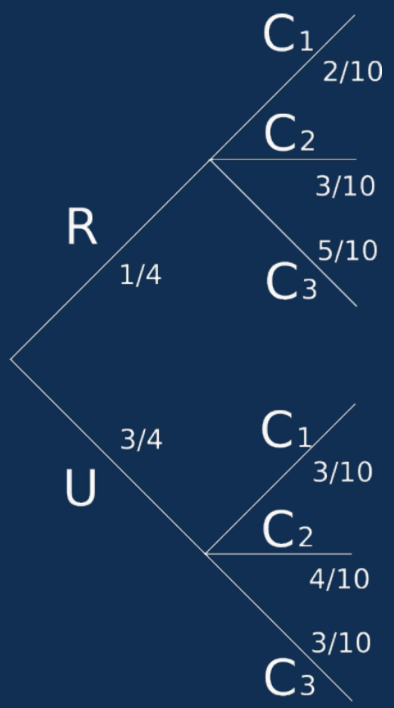
Decision Trees

- A way of representing a sequence of Decisions
- For example, in a MOBA
  - Choose to play Ranked/Unranked
  - Choose character



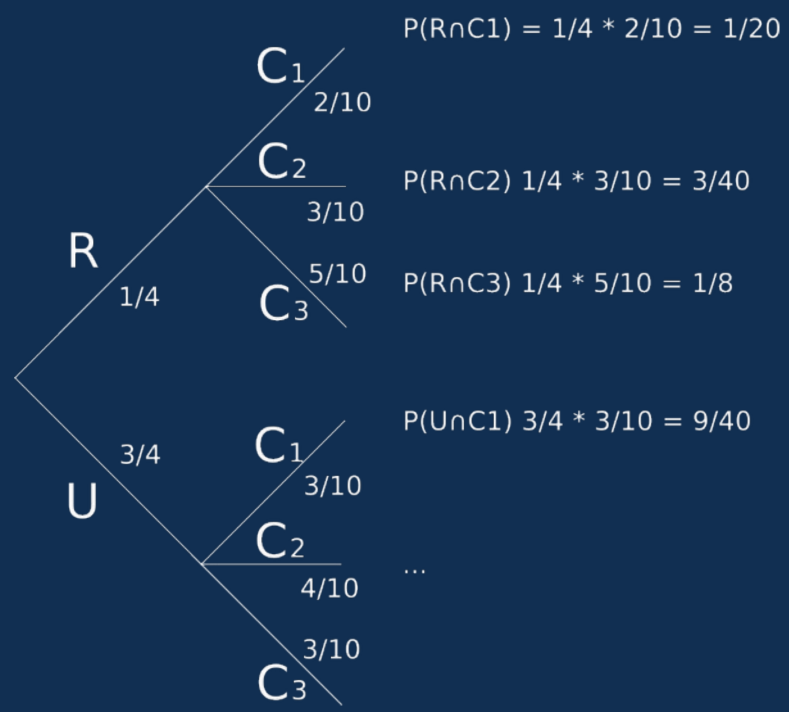
Probability Trees

- Lets say we assign probabilities to each decision
  - Each probability is conditional upon its position in the tree
  - Then we have a probability tree
- We can also write conditional probabilities like this
  - $P(C_1 \mid R) = 2/10$
  - $P(C_1 \mid U) = 3/10$



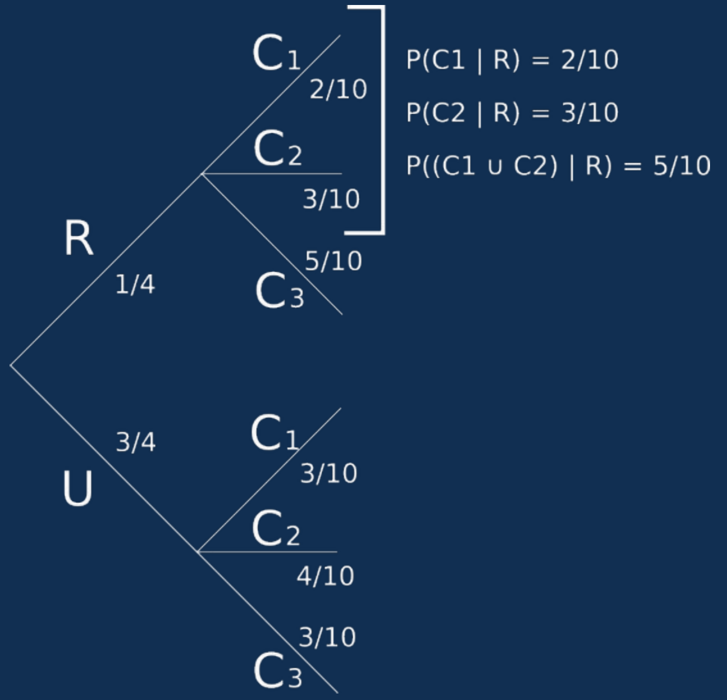
Multiplying Down Branches

- To work out the absolute probability of a set of decisions, we can multiply down the tree
- Rememeber the Multiplication Rule
  - $P(A \cap B) = P(A|B) \times P(B)$



Adding Across Branches

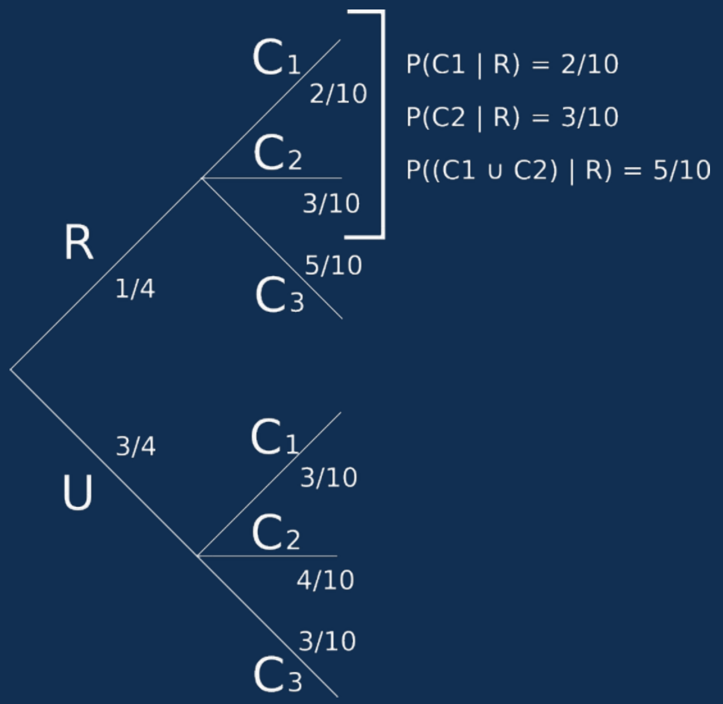
- Branches are mutually exclusive, so we can add them together to get their combined probability
- Remember the Addition Rule
  - $P(A \cup B) = P(A) + P(B) + P(A \cap B)$



Exercise 12:

What are the following probabilities:

1. $P(C_1 \mid U)$
2. $P(U \cap C_3)$
3. $P(R \cap U)$
4. $P(U \cap (C_1 \cup C_2))$



Applied Probability

- Probability is often counter-intuitive
  - This can lead to problems in how people understand
    - Medicine (e.g. screening programes, vaccines)
    - Criminal evidence (e.g. DNA evidence)
  - Often games (e.g. Civilisation) lie about probailities to better fit player's expectations
  - How numbers are presented affects perception e.g. "1 in 200" > 0.5%
- Because of this, it's also very easy to make mistakes without realising it

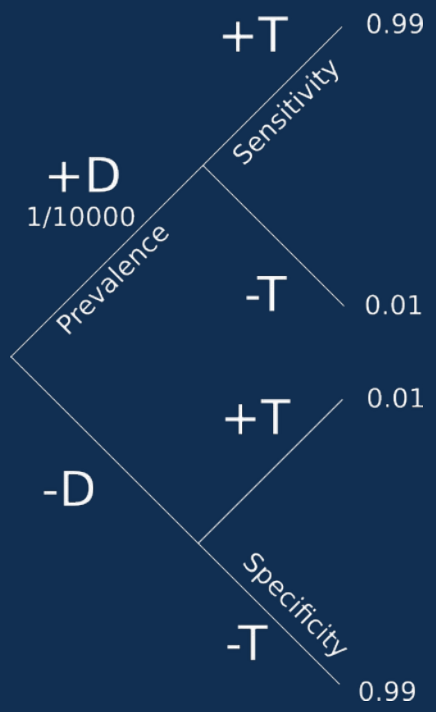
Disease Screening

- We can use Probaility trees to visualise testing for disease, for e.g. screening programmes.
  - Prevalence

    P(+D)
  - Sensitivity

    P(+T | +D)
  - Specificity

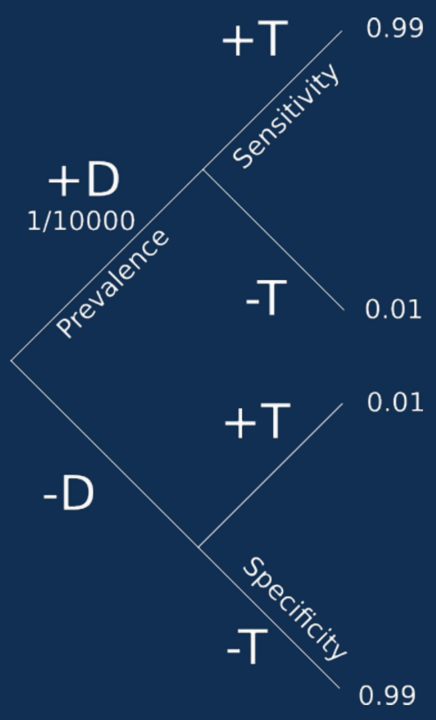
    P(-T | -D)



Exercise 13:

Assume a random individual is tested. What is the liklihood of a:

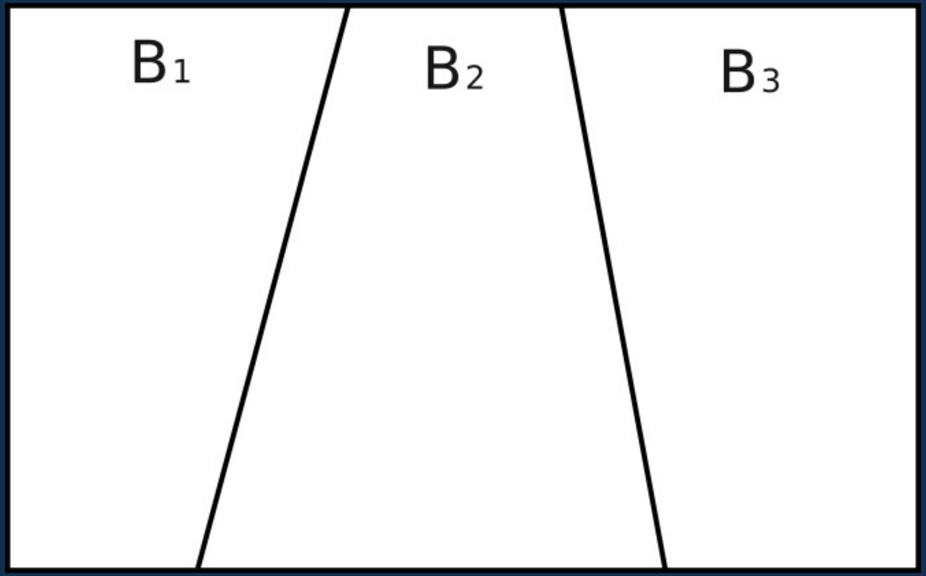
1. True positive
2. False positive
3. True negative
4. False negative



Law of Total Probability

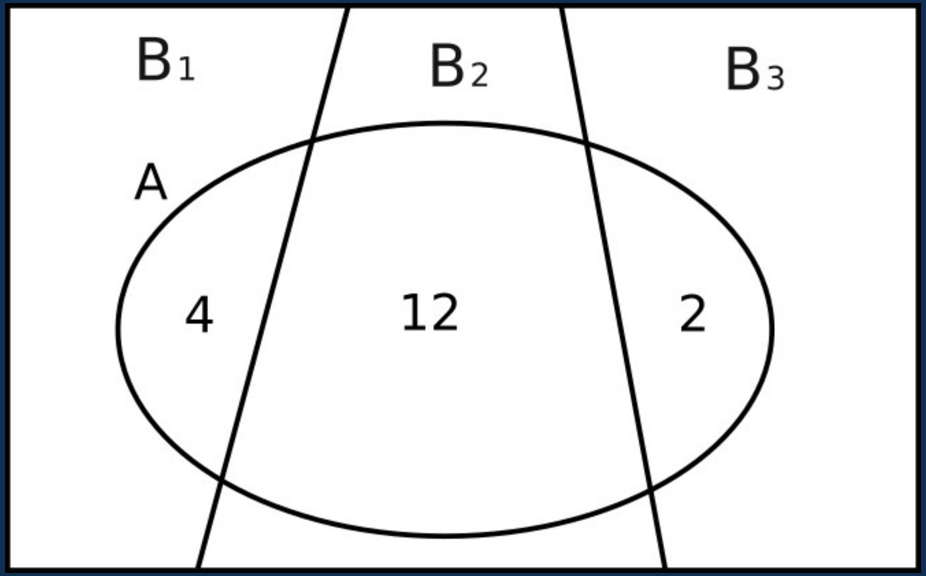
Partition

- A partition is a way of subdividing a set into partitions $B_1, B_2, \ldots, B_n$ that are:
  - **disjoint** (i.e. there is no overal between B_i and B_j, or in other words $B_i \cap B_j = \emptyset$)
  - **collectively exhaustive** (i.e. together they contain everything in the set)



Summing Conditional Probabilities

- Events may be divided across these partitions
- We can reason about the event A as the sum of conditional probabilities
  - $P(A \mid B_1)$ +
  - $P(A \mid B_2)$ +
  - $P(A \mid B_3)$



Law of Total Probability

Law of Total Probability:

If B_1, B_2, B_3, … is a partition of the sample space S, then for any event A:

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i) \times P(B_i)$$

Exercise 14:

You have 2 bags of balls that each contain 70 balls.

- Bag 1 has 50 red balls and 20 black balls
- Bag 2 has 30 red balls and 40 black balls

You pick a bag at random and pick a random ball from that bag. What is the probability that you pick a red ball?

- Hint:

$$P(A \cap B) = P(A|B) \times P(B)$$

$$P(A) = \sum_i P(A \cap B_i)$$

Bayes Rule

Bayesian Reasoning

- Say we had a scientific hypothesis
  - We gather evidence
  - We want to know how certain we should be of the hypothesis
- We want to give a value to the formula:

$$P\,(H \mid E)$$

  - Probability of hypothesis *H* given evidence *E*
- What do we need to know to calculate this?

What we (might) know

- Lets say we know
  - P(H), the prior probability of hypothesis H
  - P(E), the liklihood that the evidence is true
  - P(E | H), the liklihood of the evidence given the hypothesis
- This is enough to work out P(H | E)
  - Called the posterior probability of hypothesis H

Revision Example

- For example, assuming:
  - Half of students revise
  - Half of students pass their exam
  - Of those who revise, 80% passed their exam
- What is the liklihood that a student who passed their exam had revised?

Revision Example

- What is the liklihood that a student who passed their exam had revised?
  - Prior probability of hypothesis
    - P(H) = 50% (Half of students revise)
  - Probability of evidence
    - P(E) = 50% (Half of students pass their exam)
  - Probability of evidence given hypothesis
    - P(E | H) = 80% (Of those who revise, 80% passed their exam)
- We need to rearrange this for P(H | E)

Exercise 15:

Using the formulas opposite, rewrite:

$$P(H \mid E)$$

in terms of

- $P(H)$
- $P(E)$
- $P(E \mid H)$

- Hint:

$$P(A \cap B) = P(A|B) \times P(B)$$

$$P(A \cup B) = P(A) + P(B) + P(A \cap B)$$

$$P(A|B) = \frac{P(B \cap A)}{P(B)}$$

$$P(A) = \sum_i P(A \cap B_i)$$

Bayes Rule

Prior Probability

Likelihood of the evidence 'E' if the Hypothesis 'H' is true

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$

Posterior Probability of 'H' given the evidence

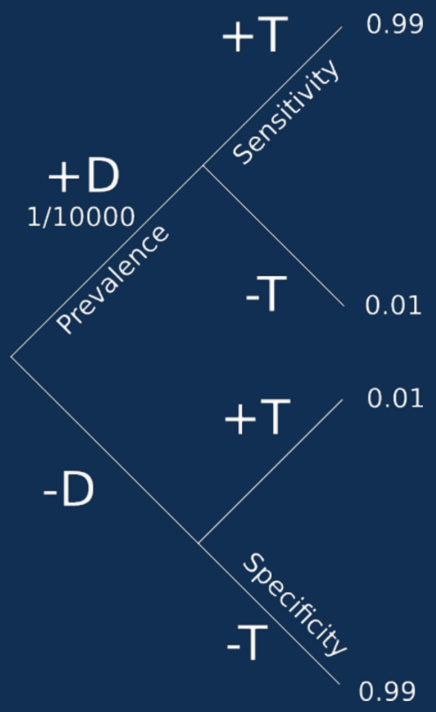
Priori probability that the evidence itself is true

Disease Screening

- Probability is often counter intuitive.
- Imagine you have been screened for a disease (e.g. cancer)
  - You get a positive result
  - What is the liklihood that you have the disease?

Question:

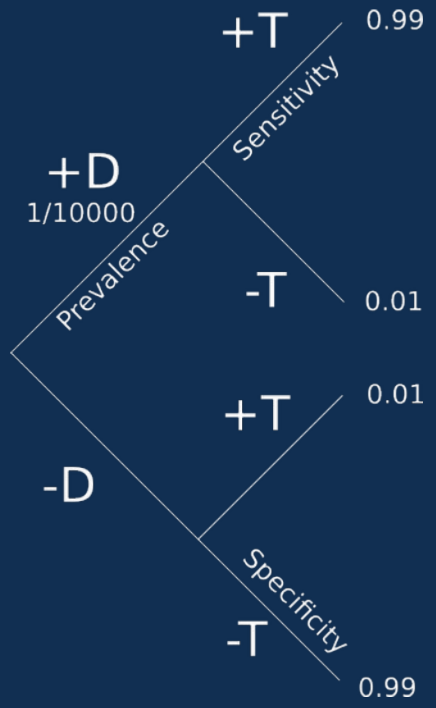
Given the statistics opposite, wold you be willing to undergo potentially harmful treatment after a positive test?



Disease Screening

- We want to work out P(H | E), we know:
  - P(H) = 1/10000 (prior probabiltiy = prevalence)
  - P(E | H) = 0.99 (sensitivity)
  - P(E) = ?

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$



- Our hypotheses (Has disease, doesnt have disease) is a partition
  - Disjoint
  - Collectively exhaustive
- We can rewrite P(E) using the Law of Total Probability

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$

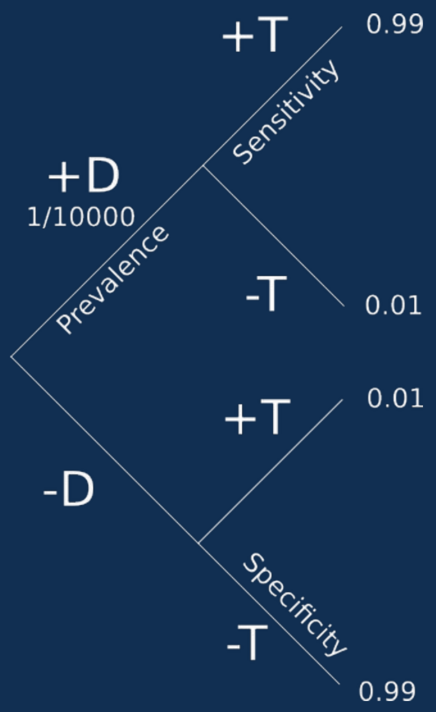
$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i) \times P(B_i)$$

$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$

Disease Screening

- We want to work out $P(H_1 \mid E)$, we know:
  - $P(H_1) = 1/10000$
  - $P(H_2) = 9999/10000$
  - $P(E \mid H_1) = 0.99$
  - $P(E \mid H_2) = 0.01$

$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$



Exercise 16:

Work out $P(H_1 \mid E)$,:

- $P(H_1) = 1/10000$
- $P(H_2) = 9999/10000$
- $P(E \mid H_1) = 0.99$
- $P(E \mid H_2) = 0.01$

Would you undergo potentially harmful treatment?

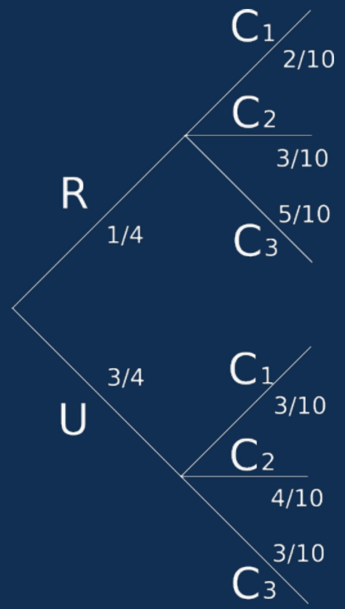
- Hint

$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$

Exercise 17:

A player of a MOBA chooses to play Ranked/Unranked and then picks one of three characters to play. The probabilities are shown in the probability tree opposite

You see someone playing character C_3. What is the liklihood they are playing ranked?



$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$

Summary

Summary

- Probability Basics
  - Events
  - Addition Rule
  - Multiplication Rule for Independant Events
  - General Multiplication Rule
- Combinatorics
- Probability Trees
- Law of Total Probability
- Bayes Rule