# **Mathematics and Problem Solving**

Lecture 4

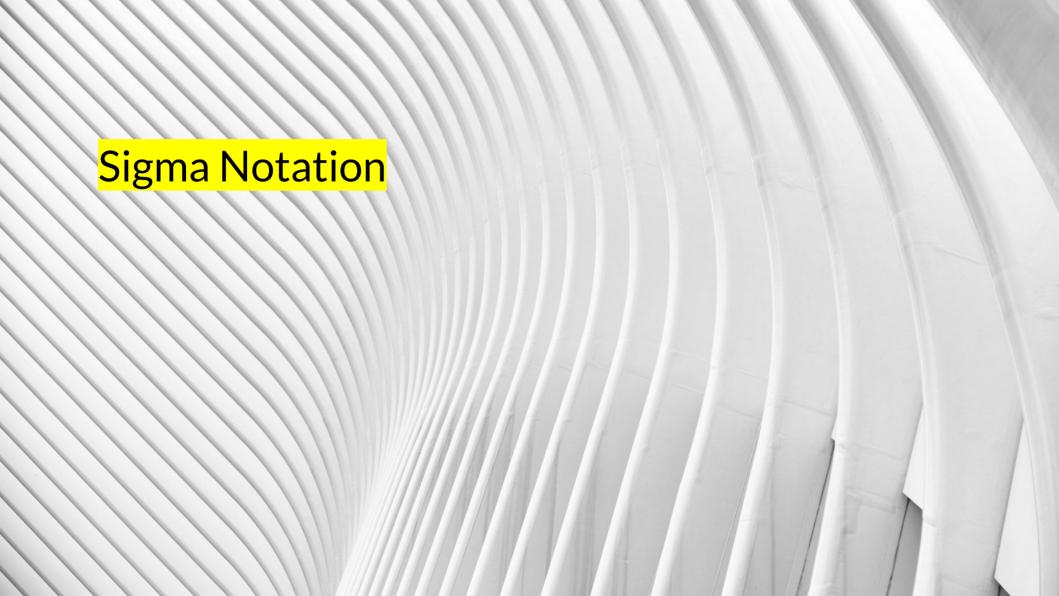
Sequences and Series

"Mathematics is the language of nature."

Fibonacci

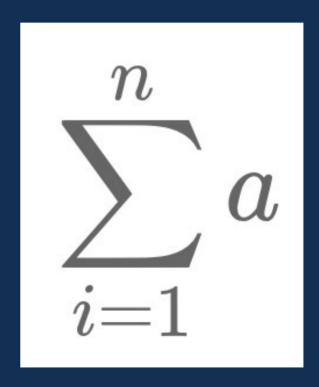
### Overview

- Sigma Notation
- Summation
- Product Notation



### Summation

- Summation is adding up lots of things
- It has a special notation written with the Greek capital letter sigma
  - Sigma Notation
- This allows us to express summations compactly



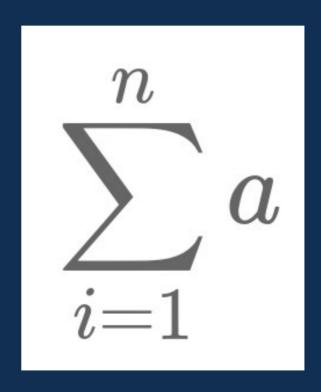
# Sigma and Polynomial form

 We are familiar with seeing addition written in a polynomial

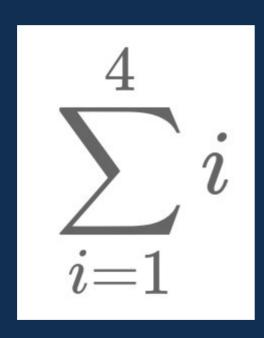
$$\sum_{i=1}^4 i = 1+2+3+4=10$$

### Parts of a Summation

- a
  - formula to sum
- •
- variable
- · =1
  - Lower limit
  - starting point (inclusive)
- n
  - Upper Limit
  - ending point (inclusive)



# Determine the Sum - Example



- Start at i = 1
- For each value of i
  - Substitue every occourance of i for 1 in the formula
  - Formula becomes
    - 1
  - Is i less than 4?
  - Increment 1

## Determine the Sum - Example

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

# Determine the Sum - More Examples

$$egin{aligned} \sum_{i=0}^4 rac{i}{i+1} &= rac{0}{0+1} + rac{1}{1+1} + rac{2}{2+1} + rac{3}{3+1} + rac{4}{4+1} = rac{163}{60} = 2.7166ar{6} \ \sum_{i=4}^6 2^i x^{2i+1} &= 2^4 x^9 + 2^5 x^{11} + 2^6 x^{13} = 16 x^9 + 32 x^{11} + 64 x^{13} \ \sum_{i=1}^4 f\left(x_i^*
ight) &= f\left(x_1^*
ight) + f\left(x_2^*
ight) + f\left(x_3^*
ight) + f\left(x_4^*
ight) \end{aligned}$$

### Exercise

### Exercise 1:

Solve the following summations

1. 
$$\sum_{i=1}^{10} 1$$
  
2.  $\sum_{i=1}^{10} i$ 

2. 
$$\sum_{i=1}^{10} i$$

3. 
$$\sum_{i=1}^{5} a$$

4. 
$$\sum_{i=1}^{10} i^2$$

# Summing a Constant Sequence

- Lets say we have a constant sequence of numbers, such as:
  - 2, 2, 2, 2, 2, ...
- What is the sum of the first n values of this sequence
  - If we have n values, then  $n \times 2 = 2n$
- We can express this as a summation
  - We cannot solve this to get a number, but we can simplify it to a polynomial form that is easier to work with

### Exercise 2:

Solve the following summations

- 1.  $\sum_{i=1}^{n} 2$
- 2.  $\sum_{i=1}^{i} a$
- 3.  $\sum_{i=4}^{n} 10$

### Hint:

- Summations are inclusive of both start and end values
- Adding up n copies of a number k
   is nk



## Sequences and Series

- A sequence is a progression of numbers, e.g.
  - { 1, 10, 4, 2, 4, 77, 6, 4, ... }
- If we add them all together, we get a **series** 
  - <u>- {1+10+4+2+4+77+6+4+...}</u>
- Because a series can be arbitrarily long or infinite, we often use sigma notation to express it
- There are special types of sequences that we are going to look at
  - Arithmetic Sequences
  - Geometric Sequences

# **Notating Sequences**

- A sequence can be notated in a number of ways
- Pay attention to subscripts!
  - $a_{n+1} \neq a_n + 1$

$$\{a_1, a_2, \cdots, a_n, a_{n+1}, \cdots\}$$

$$\{a_n\}$$

$$\{a_n\}_{n=1}^{\infty}$$

### Exercise 3:

Give the first 4 values of the following sequences

- 1.  $\{n^2 + 3\}_{n=1}^{\infty}$
- 2.  $\{\frac{n}{n+1}\}_{n=1}^{\infty}$

# Terminology about Sequences

- A sequence {a<sub>n</sub>} is increasing if, for every n
  - $a_n < a_{n+1}$
  - { 1, 23, 112, 158 } is an increasing sequence
- A sequence {a<sub>n</sub>} is decreasing if, for every n
  - $a_n > a_{n+1}$
  - { 202, 150, 3, 0.1} is a decreasing sequence
- Both increasing and decreasing sequences are monotonic

## Bounded Sequences

- If there is a number  $m \le a_n$  for every n (less or equal to any number in the sequence)
  - m is a lower bound of the sequence
  - The sequence is bounded below
- If there is a number  $m \ge a_n$  for every n
  - m is an upper bound of the sequence
  - The sequence is **bounded above**
- A sequence bounded both below and above is called bounded

### Exercise 4:

For the finite sequence { 4, 2, 3, 2 }

- 1. Is 2 a lower bound?
- 2. Is 1 a lower bound?
- 3. Is 3 an upper bound?
- 4. Is the sequence bounded?
- 5. Is the sequence monotonic?

#### • Hint:

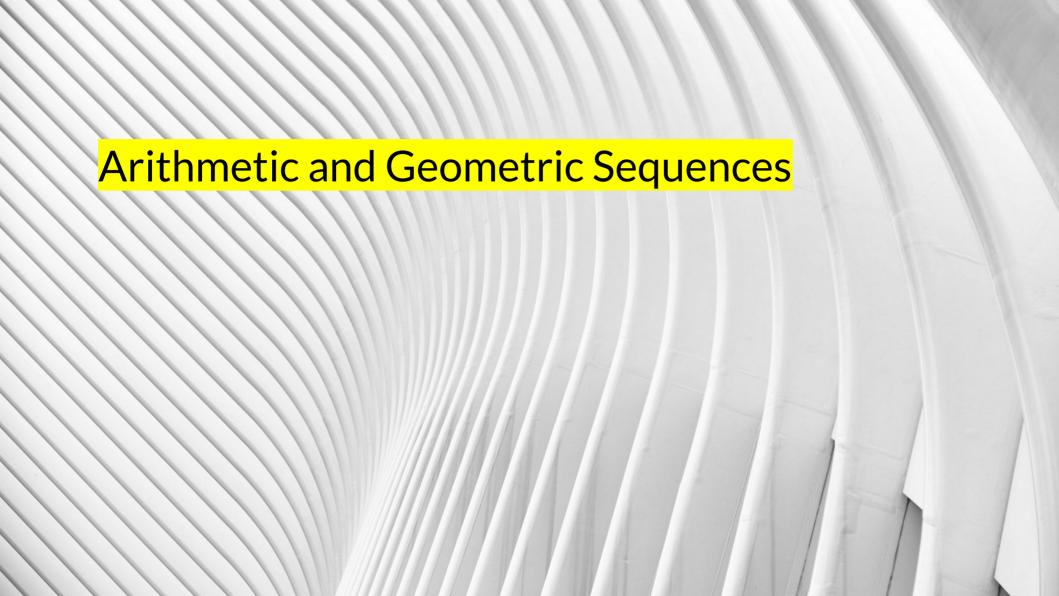
- m is a lower bound if  $m \le a_n$  for every n
- m is a upper bound if  $m \ge a_n$  for every n

### Exercise 5:

Consider the sequence

$$\{\frac{n}{n+1}\}_{n=1}^{\infty}$$

- 1. Give an upper / lower bound if such exist
- 2. Is it increasing/decreasing?



- An Arithmetic Sequence is a list of numbers with an:
  - Initial Term
  - Common Difference
- For example (initial term in red)
  - { 1, 2, 3, 4, 5, 6, ... } (common difference 1)
  - { 5, 10, 15, 20, 25, ... } (common difference 5)
  - { 3, 6, 9, 12, 15, ... } (common difference 3)

### Exercise 6:

What is the initial term and common difference for these sequences?

- 1. { 1, 2, 3, 4, 5, ... }
- 2. { 99, 104, 109, 114, ... }

• We can define an arithmetic sequence recursively

$$a_{n} = a_{n-1} + d$$

Or give an explicit formula

$$a_n = a_1 + d(n-1)$$

#### Exercise 7:

Find the last term of the sequence that starts with 5, has a common difference of 0.25 and has 10 terms.

#### Hint:

- We can define an arithmetic sequence recursively
  - $a_n = a_{n-1} + d$
- Or give an explicit formula
  - $a_n = a_1 + d(n-1)$

#### Exercise 8:

- 1. What is the implicit formula for the sequences
  - {2,3,4,5,...}
  - {-1,-3,-5,-7,...}
- 2. What is the explicit formula for the sequences
  - {4, 5, 6, 7, ...}
  - { 10, 20, 30, 40, ... }

#### Hint:

- $a_n = a_{n-1} + d$
- $-a_n = a_1 + d(n-1)$

## Geometric Sequence

- An Arithmetic Sequence is a list of numbers with an:
  - Initial Term
  - Common Ratio
- For example
  - { **10**, 40, 160, 640, ... } (common ratio 4)

### Exercse 9:

What is the initial term and common ratio for these sequences?

- 1. { 1,3,9,18, ... }
- 2. { 1, 0.5, 0.25, 0.125, ... }

### Geometric Sequence

We can define a geometric sequence recursively

$$g_n = g_{n-1} \times r$$

Or explicitly

$$g_n = g_1 \times r^{n-1}$$

# Geometric Sequence

#### Exercise 10:

- 1. What is the implicit formula for the sequences
  - {2,3,4,5,...}
  - {-1,-3,-5,-7,...}
- 2. What is the explicit formula for the sequences
  - {4, 5, 6, 7, ...}
  - { 10, 20, 30, 40, ... }

#### Hint:

$$g_n = g_{n-1} \times r$$

$$g_n = g_1 \times r^{n-1}$$

# Fibonacci Sequence

One well known sequence is the Fibonacci sequence, defined as follows:

$$F_n = F_{n-1} + F_{n-2}$$

• Where  $F_0 = 0$  and  $F_1 = 1$ 

# Fibonacci Sequence

- The Fibonacci Sequence can be expressed as an explicit formula
- Where  $\varphi$  is the golden ratio
  - 1.6180339887...
- And  $\psi$  is  $-\phi^{-1}$ , the negative multiplicative inverse of the golden ratio
  - -0.6<del>1</del>80339887...

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

### **Exercise 11:**

I define a sequence using the following explicit definition

$$s_n = (0.5 \times 10^{x-1}) + 2(x-1)$$

- 1. Write out the first 5 elements
- 2. Decompose this into two sequences
- 3. How could you write a recursive definition for this sequence?

#### ' Hint:

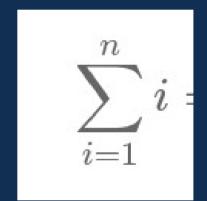
- The number of digits in a base 10 number is given by
  - log<sub>10</sub> n
- Remember mod
- To round down (floor) we can write

$$\mathrm{floor}(x) = \lfloor x \rfloor$$



### Sum of first n numbers

- We often need to sum the first n natural numbers: 1, 2, 3, 4, ..., n
  - -1+2+3+4+...+n
- Problem: arbitrarily long
  - Easy to write with sigma notation!
- But how do we calculate the sum for a given n?



### Sum of first n numbers

- We can write this in a more familiar form
- Where does this come from
  - Find the average of first value 1 and n, the last value, which is (n+1)/2
  - Multiply this by the length of the series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

## Sum of an Arithmetic Series

- We can generalise this for a formula for the sum of the first n values of a sequence a
  - Find average value:
    - $(a_1 + a_n)/2$
  - Multiply by length of sequence *n*
- We can always use substitute the formula for the nth term of the sequence in place of  $a_n$

$$\sum_{i=1}^n a = n(\frac{a_1 + a_n}{2})$$

## Sum of a Geometric Series

- We can also define the sum of the first n values of a Geometric Series
  - a is the initial value
  - r is the common ratio

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

# Sum of first n squares

- It is also common to want to sum the first n square numbers
  - You can derive this using simultaneous equations
  - Or just look up the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$



# **Arithmetic Properties of Summation**

- When you cannot resolve a summation to a number, we often need to transform it
  - Factor constants out of a summation
  - Break up summation across a sum or difference
- By transforming it, we can often get it to a state where it is easier to work with
  - e.g. so we can substitute one of the formulas we've seen

## **Factor Constants out of Summation**

- Due to the distributivity of multiplication over addition
  - n(a+b) = na + nb
- We can factor constants out of a series
  - 1c + 2c + 3c = c (1 + 2 + 3)
- This lets us transform summations into more manageable forms

$$\sum_{i=1}^{n} ci = c \sum_{i=1}^{n} i$$

# Break up Summation across a Sum

Due to the associativity of addition

$$-$$
 (a + b) + c = a + (b + c)

 We can split summations across sums

$$- (1+a) + (2+b) + (3+c)$$
$$= (1+2+3) + (a+b+c)$$

This lets us transform summations into more manageable forms

$$\sum_{i=1}^{n} (a+b) = \sum_{i=1}^{n} a + \sum_{i=1}^{n} b$$

#### Exercise 12:

Simplify the following summations

1. 
$$\sum_{i=1}^{n} 3ci$$

2. 
$$\sum_{i=1}^{n} 3i + 4$$

3. 
$$\sum_{i=1}^{n} 2i^2 + 3i$$

Hint:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = rac{n(n+1)(2n+1)}{6}$$

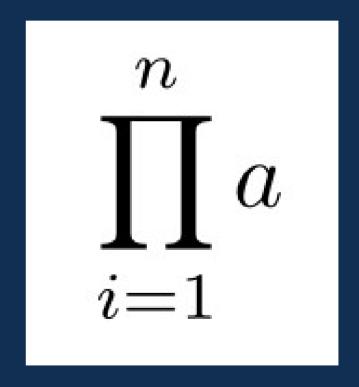


### **Product Notation**

- To add a sequence we can use sigma notation and use summation
- What if we want to multiply a sequence? e.g.
  - -1x2x3x...xn-1xn
- Product Notation
  - Uses the Greek capital letter pi,  $\Pi$

## Parts of a Product

- a
  - formula to multiply
- •
- variable
- · =1
  - Lower limit
  - starting point (inclusive)
- · n
  - Upper Limit
  - ending point (inclusive)



# Example Product

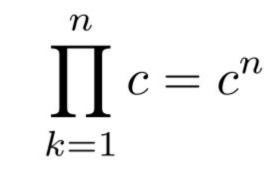
Multiply the first 5 elements of the sequence a:

- Lower limit = 1
- Upper Limit = 5
- Variable = k
- Formula to multiply =  $a_k$

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5.$$

### Product of a constant

- The product of n values of a constant c is given by the n<sup>th</sup> power of c
- The product of, e.g.
  - $10 \times 10 \times 10 \times 10 = 10^4 = 10000$
  - $C \times C \times C = C^3$



### **Product Notation**

- We can also define a product recursively
  - The left part gives a base case the product of one element is that element
  - The product of multiple elements (m → n) is the product of the first (m → n-1) elements multiplied by the  $n^{th}$  element

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

#### Exercise 13:

Compute the following products:

1. 
$$\prod_{k=1}^{4} 2$$

2. 
$$\prod_{k=1}^{4} k$$

3. 
$$\prod_{k=1}^{3} \frac{k}{k+1}$$

4. 
$$\prod_{k=0}^{r} 2^{k}$$



# Summary

- Sigma Notation Summation
- Product Notation
- Sequences
- Summation of a Series