

Mathematics and Problem Solving

Lecture 2

Modular Arithmetic

“Mathematics is the queen of sciences and number theory is the queen of mathematics. She often condescends to render service to astronomy and other natural sciences, but in all relations she is entitled to the first rank.”

Carl Friedrich Gauss

Overview

- Classes of Numbers
- Prime Factorisation
- Modular Arithmetic
 - Properties of Binary Relations
 - Properties of Binary Operators



Classes of Numbers

Numbers

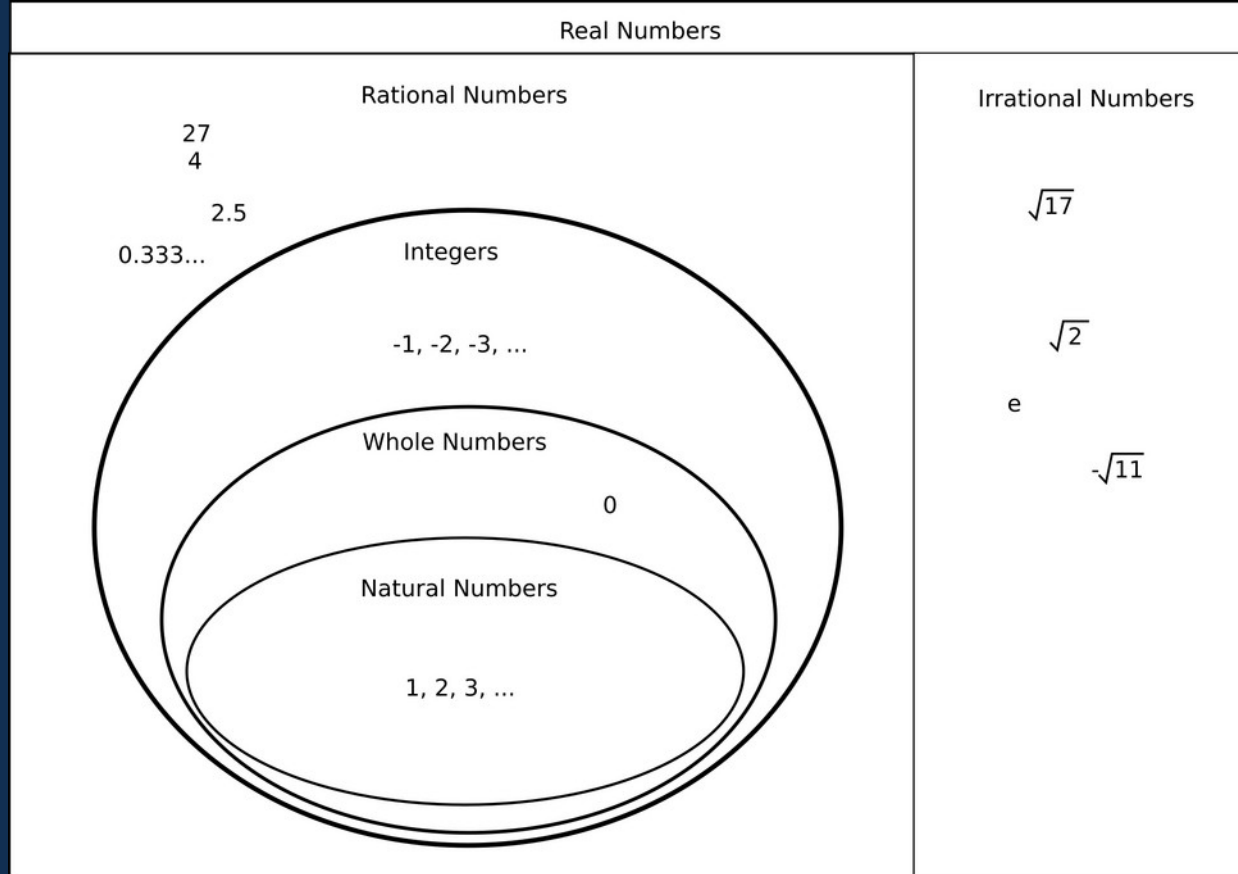
Exercise 1:

What is a number?

Classes of Numbers

- Numbers are grouped into sets called **classes**

Classes of numbers



Integers \mathbb{Z}

- **Integers** \mathbb{Z} – Positive or negative natural numbers or zero
 - $\{-2, -1, 0, 1, 2, \dots\}$
- **Naturals** $\mathbb{N} / \mathbb{N}_1 / \mathbb{N}_+ / \mathbb{N}^*$ – natural counting numbers
 - $\{1, 2, 3, \dots\}$
- **Wholes** \mathbb{N}_0 – natural counting numbers and zero
 - $\{0, 1, 2, 3, \dots\}$

Real Numbers \mathbb{R}

- Any number on a continuous 'number line'
- Two subsets
 - **Rational** \mathbb{Q} – any number which can be written as a fraction
 - **Irrational** – Any decimal number which can't be written as a fraction.
 - A non-terminating and non-repeating decimal.
 - e.g. $\pi = 3.1415926\dots$

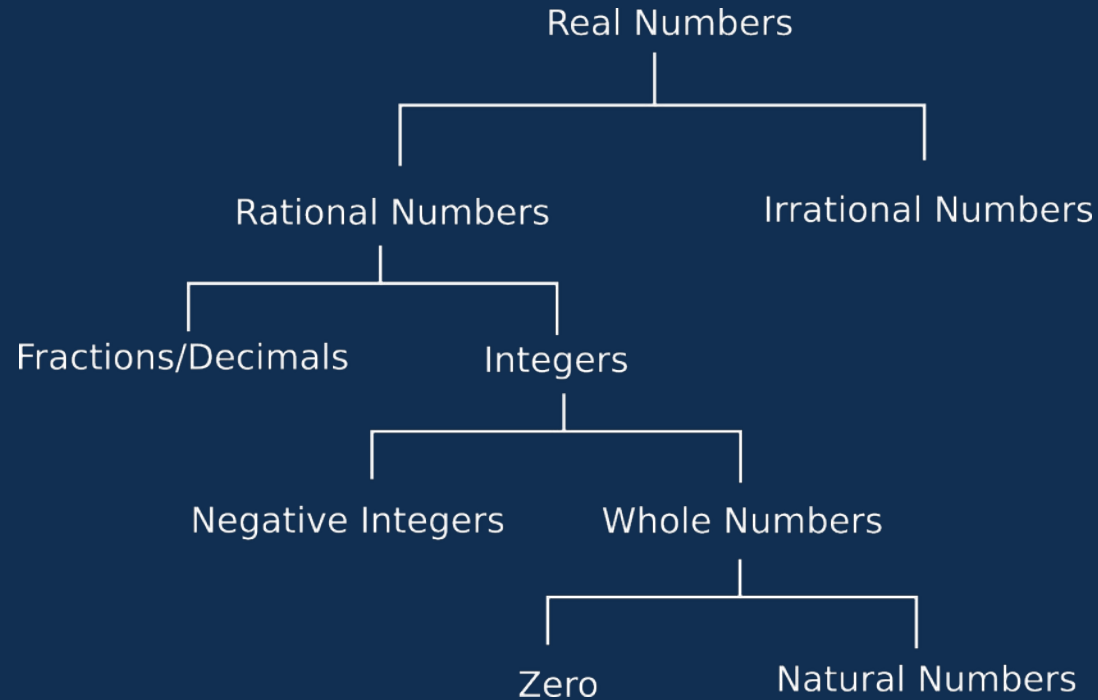
Rational Numbers \mathbb{Q}

- Can be expressed as a fraction (a/b) .
- This set includes the integers, terminating decimals, and repeating decimals.
- Some examples:
 - $2 = 2/1$
 - $3 \frac{1}{4} = 13/4$
 - $-0.25 = -25/100$
 - $1/3 = 0.3333333333333333333333333333$

Irrational Numbers

- Cannot be expressed as a fraction of integers.
- In decimal form, they are the numbers that go on forever without a repeating pattern.
- Some examples:
 - $\sqrt{2} = 1.4142\dots$
 - $\pi = 3.1415\dots$
 - $45.9492\dots$

Classes of numbers – (using a tree map)



Classes of numbers

Exercise 2:

Identify each number below as natural, whole, integer, rational, irrational, or real. More than one answer can apply.

1. $\frac{7}{8}$

2. 0

3. -9

4. $-\frac{4}{5}$

5. π

The background of the slide features a series of white, curved, rib-like structures that sweep across the frame from the bottom left towards the top right. These lines are set against a light gray background that has a subtle gradient, becoming slightly darker towards the right edge. The overall effect is one of dynamic, flowing geometry.

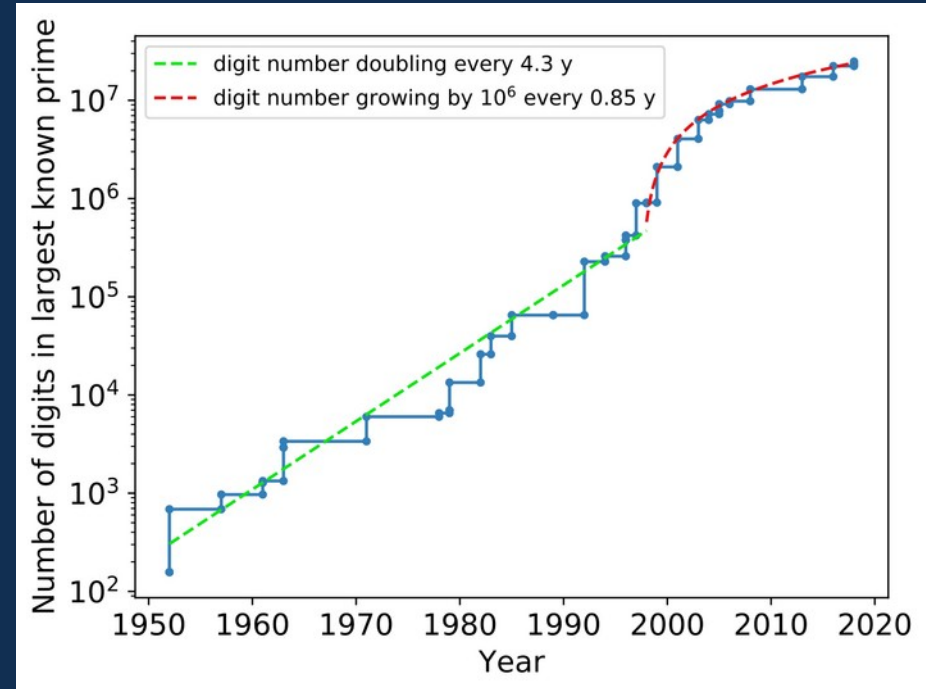
Prime Numbers

Prime Numbers

- A prime number is a whole number greater than 1 whose only factors are 1 and itself.
 - A factor is a whole numbers that can be divided evenly into another number.

Prime Numbers

- Prime numbers are of importance in mathematics
 - They are indivisible units
 - All other numbers can be composed out of prime numbers
- Large prime numbers are essential to many important cryptographic algorithms
 - Current largest is $2^{82,589,933}-1$, which is 24,862,048 digits long



Finding Primes

- The **Sieve of Erathosthenes** is a method for finding primes that dates from the 3rd Century BC
- Write down some integers
 - 2 3 4 5 6 7 8 9 10 11 12 13 14 15
- Start from the left: Remove all divisible by 2 (other than 2)
 - 2 3 4 5 6 7 8 9 10 11 12 13 14 15.
- Go to the next number: Remove all divisible by 3 (other than 3)
 - 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Exercise 3:

- Use the Sieve of Erathosthenes to find all the prime numbers between 2 and 15

Hint:

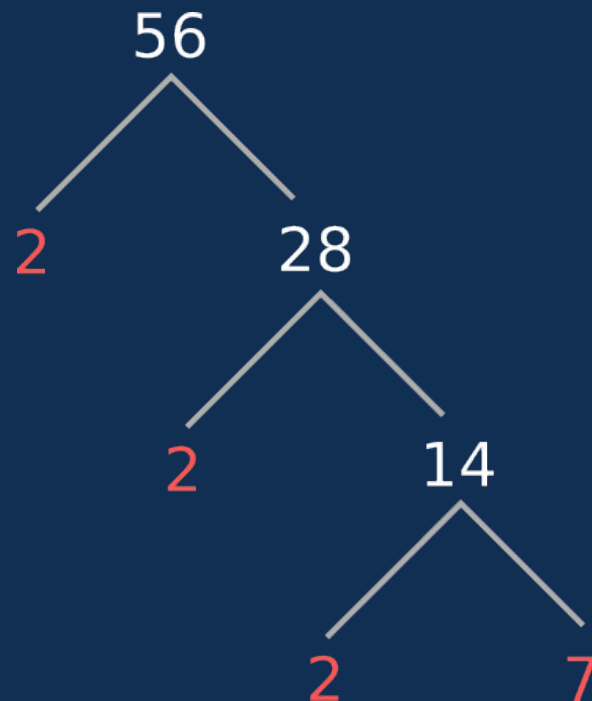
- Write down some integers
- Remove all divisible by the smallest number
2 3 4 5 6 7 8 9 10 11 12 13 14 15.
- Continue with each next non-removed number
2 3 4 5 6 7 8 9 10 11 12 13 14 15
- The numbers that survive to the end are primes

Composite Numbers

- Numbers that have more than two factors are called **composite numbers**, e.g.
 - 12 is composite as it can be factored into 2 and 6
 - 6 is composite as it can be factored into 2 and 3
- Any composite number can be expressed as the product of its prime factors
 - Every number has a unique set of prime factors
- One technique of finding the prime factors of a number is with a Factor Tree

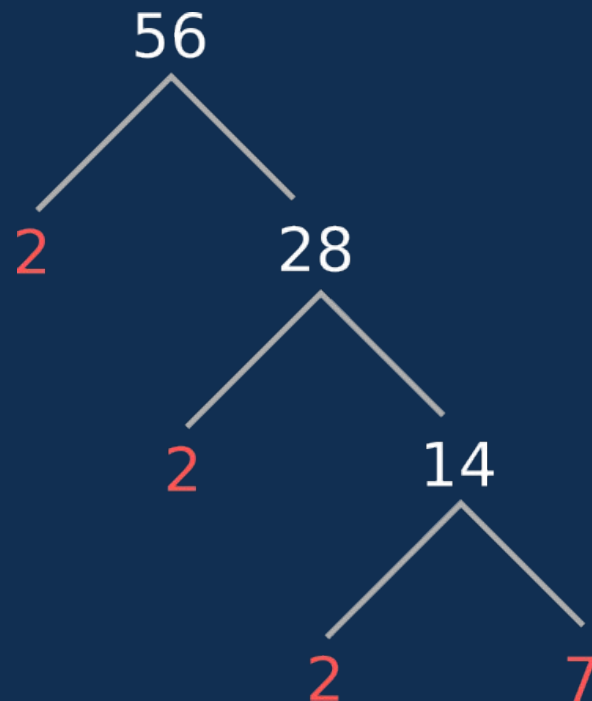
Factor Trees

- This is a the factor tree of 56
- Red digits are prime factors of 56
- $56 = 2 \times 2 \times 2 \times 7$



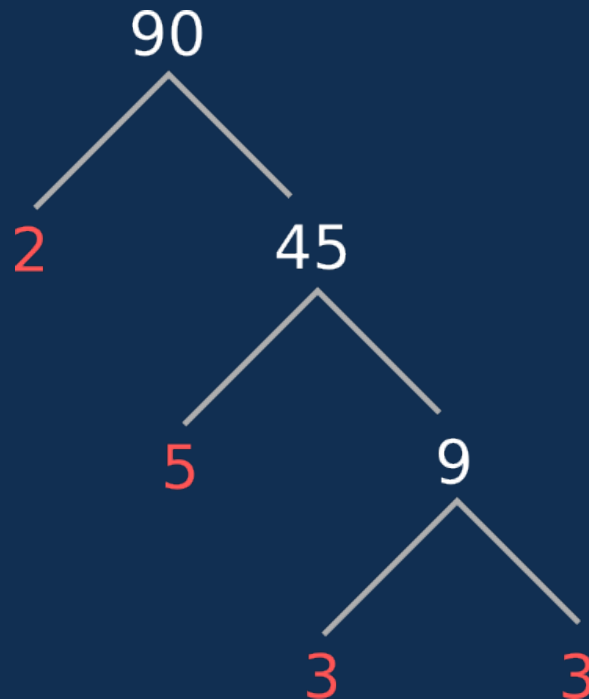
Factor Trees

- Write 56 as a product of its prime factors
 - List the first few primes
2, 3, 5, 7, 11, 13, 17, 19, 23, 29
 - Find the prime that factors 56 $\rightarrow 2$
 - Is 28 prime? No, continue.
 - Find the prime that factors 28 $\rightarrow 2$
 - Is 14 prime? No, continue.
 - Find the prime that factors 14 $\rightarrow 2$
 - Is 7 prime? Yes, stop



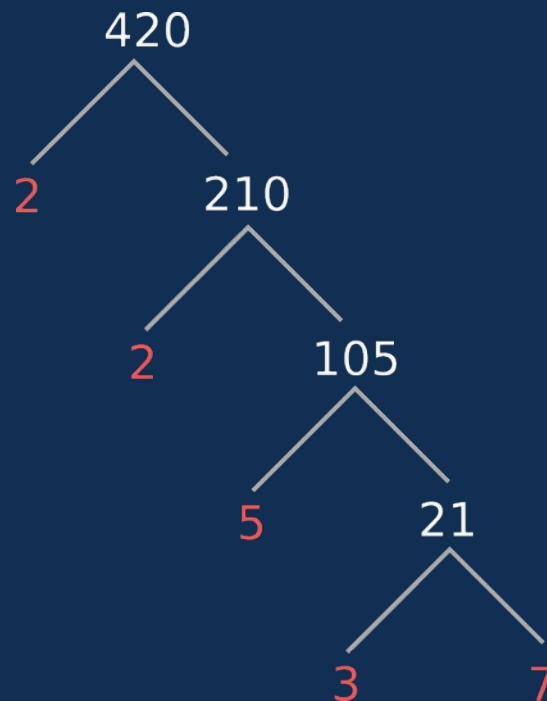
Factor Trees

- Write 90 as a product of its prime factors
- $90 = 2 \times 5 \times 3 \times 3$



Factor Trees

- Write 420 as a product of its prime factors
- $420 = 2 \times 2 \times 5 \times 3 \times 7$



Factor Trees

Exercise 4:

Draw a factor tree and use it to write each of the following as the product of its prime factors.

1. 72
2. 80
3. 75

Clock Puzzle



Clock Puzzle

- Imagine we have a clock face
 - Except the number 12 is replaced with 0
 - Moreover, we can only represent whole hours (not fractions of an hour / minutes)
- I want to be able to answer arbitrary arithmetic questions, such as
 - $11 + 9 + 3 + 0 + 2$
 - $10 - 3 - 6 - 11$
 - $10 \times 4 \times 9$
 - $7 \div 5$
- The answer must be a **whole number** on the clock



Clock Puzzle

- To express this as a formula, we have to extend our notation, because

$$11 + 9 + 3 + 0 + 2 = 25 \neq 1$$

- We could use a
 - New operator, i.e. $+$ - \times , ...
 - New function, i.e. $f()$, $\sin()$, $\cos()$, ...
 - New binary relation, i.e. $=$, $<$, $>$, ...
- Whatever we use, we need to define what it does as precisely as possible



Clock Puzzle

Exercise 5:

1. Define (a) new operator(s) / function(s) / relation(s) / something else(?) to let us express these questions in algebra
2. Describe what it does as precisely as possible
 - First write out a procedure / step-by-step instructions
 - Then try giving a definition it in a purely symbolic form using formulas
3. Begin with $+$, $-$ and \times , leave \div until last!



Modular Arithmetic

The background of the slide features a series of white, curved, parallel lines that sweep across the frame from the bottom left towards the top right. These lines are set against a light gray background that has a subtle gradient, becoming slightly darker towards the right edge. The overall effect is a sense of depth and movement, reminiscent of architectural details or a stylized landscape.

Modular Arithmetic

- Modular Arithmetic is a system of arithmetic that makes use of only a **finite set of integers**
- Analogue clocks only have 12 numbers. We are familiar with arithmetic on a clock face
 - 10 o'clock + 3 hours = 1 o'clock
 - Not 13 o'clock, 13 isn't a number in our set
 - The numbers “wrap around”



Why Modular Arithmetic?

- Why would you perform arithmetic on a finite set of integers?
 - Cyclic systems
 - Times, Dates, Numerology
 - Alphabets (finite set of letters)
 - Cryptography
 - Arithmetic algorithms on numbers in bit-representations
 - 8 bits stores only integers 0-255

What is a finite set of integers?

- With a modulus of n we use the numbers 0 - n
 - We would represent a clock face as 0 - 11
- For example, with a modulus of 6 we use the numbers
 - $\{0, 1, 2, 3, 4, 5\}$
- All other numbers are equivalent (or **congruent**) to one of these numbers
 - E.g. if we do $5+3 = 8$, that is congruent to 2

Modulo Operation

- We write the modulo operation “ $a \bmod b$ ”
 - E.g. $(5+3) \bmod 6 = 2$
- Mod finds the remainder of a division
- In many programming languages we use the symbol %
 - Often programming languages will return fractional values for %. We will be sticking to the mathematical version

Modular Arithmetic

- When finding $n \bmod m$, remember your answer must be in the range $0 \rightarrow (m-1)$
- To calculate the value of $n \bmod m$,
 - If n is positive, subtract multiples of m
 - If n is negative, add multiples of m
 - Until you get a number between 0 and $(m-1)$ (inclusive)
- Examples
 - $17 \bmod 5 = 2$
 - $20 \bmod 3 = 2$
 - $-3 \bmod 11 = 8$
 - $25 \bmod 5 = 0$

Modular Arithmetic

Exercise 6:

Solve the following

1. $7 \bmod 11$
2. $11 \bmod 11$
3. $-1 \bmod 11$
4. $-11 \bmod 11$



Congruance

Congruence

- Modular arithmetic introduces a new **binary relation** called **Congruance** (\equiv)
 - The following can be read “a and b are congruent modulo n”:

$$a \equiv b(\text{mod } n)$$

- If, when you “wrap around” a and b , you get the same value, then a and b are congruant, e.g.
 - 12 o'clock and 0 o'clock
 - 1 o'clock and 13 o'clock
 - 15 o'clock and 3 o'clock

Congruences

Congruance

Two numbers are congruant ($a \equiv b \pmod{n}$) if

$$(a \bmod n) = (b \bmod n)$$

If $a \equiv b$, the difference between a and b will be a multiple of n

$$a - b = kn, \text{ for some value of } k$$

Examples of Congruence

- $4 \equiv 9 \equiv 14 \equiv 19 \equiv -1 \equiv -6 \pmod{5}$
- $73 \equiv 4 \pmod{23};$
- $21 \equiv -9 \pmod{10}$

Reflexivity

Congruance is Reflexive:

Every number (that exists under the modulus) is congruant to itself, mod n:

$$a \equiv a \pmod{n}$$

Symmetry

Congruance is Symmetric:

If a is congruant to b , then b must be cogruant to a

$a \equiv b \pmod{n}$ if $b \equiv a \pmod{n}$ for all a , b , and n .

Transitivity

Congruance is Transitive:

If a is congruant to b , and b is congruant to c , then a must be congruant to c

If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$

Binary Relations

Exercise 7:

For the following relations, are they reflexive, symmetric, and/or transitive?

1. within 10 metres of
2. slower than
3. as tall or taller than
4. is preferred (by me) to
5. trusts



Properties of Modular Arithmetic

Properties of Modular Arithmetic

- Addition, subtraction, and multiplication work as you expect from regular arithmetic

$$[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$$

$$[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$$

$$[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$$

Identities

Modular Identities:

$$(0 + w) \bmod n = w \bmod n$$

$$(1 \times w) \bmod n = w \bmod n$$

- An identity is an equation which is true no matter what values are chosen

Commutativity

Commutative Laws:

Addition and multiplication are commutative under modulus

$$(w + x) \bmod n = (x + w) \bmod n$$

$$(w \times x) \bmod n = (x \times w) \bmod n$$

Associativity

Associative Laws:

Addition and multiplication are associative under modulus

$$[(w + x) + y] \bmod n = [w + (x + y)] \bmod n$$

$$[(w \times x) \times y] \bmod n = [w \times (x \times y)] \bmod n$$

Distributivity

Distributive Law:

Multiplication distributes over addition under modulus

$$[w \times (x + y)] \bmod n = [(w \times x) + (w \times y)] \bmod n$$

Properties of Arithmetic

Exercise 8:

I invent a mathematical operator \mathbb{A} , the following are valid for all a, b, c :

- $a \mathbb{A} b = b \mathbb{A} a$
- $a \mathbb{A} (b + c) = a \mathbb{A} b + a \mathbb{A} c$
- $a \mathbb{A} (b \mathbb{A} c) \neq (a \mathbb{A} b) \mathbb{A} c$

Is \mathbb{A} :

1. Commutative?
2. Associative?
3. Distributive over addition?

The background of the slide features a series of white, curved, parallel lines that sweep across the frame from the bottom left towards the top right. These lines are set against a light gray background that has a subtle gradient, becoming slightly darker towards the right edge. The overall effect is a sense of depth and movement.

Additive Inverse

Additive Inverse

- The **additive inverse** of a number is whatever you need to add to get 0
- In regular arithmetic, the additive inverse of a number w is $-w$, because

$$w + (-w) = 0$$

- However, we do not have negative numbers in modular arithmetic, the additive inverse will still exist
 - It will be positive
 - Relys on the “wrapping” effect to get to 0
 - Whatever you need to add to get to the modulus

Additive Inverse

Additive Inverse:

For each $w \in \mathbb{Z}_n$, there exists a z such that

$$w + z \equiv 0 \pmod{n}$$

Additive Inverse, Examples

Exercise 9:

The additive inverse is what you need to add to get 0

Find the additive inverse of the following

1. $1 \pmod{2}$
2. $4 \pmod{10}$
3. $7 \pmod{11}$



Multiplicative Inverse

Multiplicative Inverse

- The multiplicative inverse of a (written a^{-1}) is whatever you need to multiply by a to get 1
 - In regular arithmetic this is $1/a$, e.g.
 - $4^{-1} = \frac{1}{4}$
- The multiplicative inverse works differently under modulus. Why?
 - Because we don't have fractions!

Multiplicative Inverse

Multiplicative Inverse:

For each $w \in \mathbb{Z}_n$, the multiplicative inverse a is defined where w and n are co-prime such that:

$$aw \equiv 1 \pmod{n}$$

- The multiplicative inverse is defined only if the greatest common divisor of w and n is 1 ($\text{GCD}(w, n) = 1$)

Multiplicative Inverse

Exercise 10:

The multiplicative inverse is the number you need to multiply by to get 1

Find the multiplicative inverse of the following:

1. 6
2. $7 \pmod{13}$
3. $5 \pmod{7}$
4. 2
5. $3 \pmod{13}$

Division

- You can perform division by multiplying by the multiplicative inverse, e.g.
 - $4 \div 2 = 4 \times 2^{-1}$
 - $10 \div 3 = 10 \times 3^{-1}$
- We can apply this same principle to perform division under modulus

Modular Division

- What is $5 \div 3 \bmod 11$?
 - We need to multiply 5 by the inverse of 3 mod 11
 - When you multiply a number by its inverse, the answer is 1.
- The inverse of 3 mod 11 is 4 since $3 \cdot 4 = 1 \bmod 11$
- Thus $5 \div 3 \bmod 11 = 5 \times 4 \bmod 11 = 9 \bmod 11$

Modular Division

Exercise 11:

To divide under modulus, find the modular inverse of the divisor with respect to the modulus and multiply by the dividend.

Solve the following

1. $4 \div 2 \pmod{5}$
2. $7 \div 5 \pmod{9}$

Test for Coprimeness

- Before we can perform modular division (or the modular inverse) we must check that divisor (w) is co-prime with the modulus (n)

$$a \div w \pmod{n}$$

- This is because we need to find the modular inverse of w , which is only defined when

$$\text{GCD}(w, n) = 1$$

The background of the slide features a series of white, curved, rib-like structures that sweep across the frame from the bottom left towards the top right. These lines are set against a light gray background that has a subtle gradient, becoming slightly darker towards the right edge. The overall effect is one of dynamic movement and architectural elegance.

Euclidean Algorithm

Coprime Integers

- It is sometimes interesting to know whether two integers are co-prime, meaning their greatest common divisor is 1
- Greatest Common Divisors (GCD) of two integers is the largest integer that divides both
 - If $\text{GCD}(A, B) = 1$ then A and B are coprime

Euclidean algorithm

- Euclidean Algorithm is a way of finding the GCD of two numbers
 - Take two numbers, a and b
 - Find $b \bmod a$
 - Repeat with
 - $a = b$
 - $b = b \bmod a$
 - Until $b = 0$
 - The value of a is the GCD

```
int Euclid(int a, int b)
{
    if (b == 0)
        return a;
    else
        return Euclid(b, b % a);
}
```


Representing Computation with a Table

- We can draw a table to understand its computation, e.g. for `Euclid(6, 15)`

```
1. int Euclid(int a, int b) {  
2.     if (b == 0)  
3.         return a;  
4.     else  
5.         return Euclid(b, b % a); }
```

line	a	b	b=0	b mod a	return
1	6	15			
2			F		
5				15	
1	15	3			
2			F		
5				0	
1	3	0			
2			T		
3					3

Finding the Greatest Common Divisor

Exercise 12:

Using the Euclidean Algorithm, find

1. GCD(42, 78)

```
1. int Euclid(int a, int b) {  
2.     if (b == 0)  
3.         return a;  
4.     else  
5.         return Euclid(b, b %  
        a); }
```

The background of the slide is an abstract pattern of numerous white, curved lines that sweep across the frame from the bottom left towards the top right. These lines are set against a light gray background that features a subtle gradient, becoming slightly darker towards the right edge. The overall effect is one of dynamic movement and depth.

Creating a Cipher

Ciphers

A cipher is a way of encrypting text to make it harder to read

To define a cipher

- We first associate each letter with a number ($a = 0, b=1, c=2$, etc.)
- Then define a mathematical function of type $\mathbb{N}_n \rightarrow \mathbb{N}_n$ that takes a number and returns a different number
 - The numbers have to remain smaller than the length of our alphabet
 - In English $n = 26$, so our numbers must remain in the range 0-25
- Then we convert that number back into a letter ($0=a, 1=b, 2=c$, etc.)

Ciphers

Exercise 13:

Using modular arithmetic, create a new cipher

Try to define two functions

- one that encrypts a character
- one that decrypts a character

Give formulas for both

Use it to encrypt a short sentence

Hint:

To define a cipher

- We first associate each letter with a number (a = 0, b=1, c=2, etc.)
- Then define a mathematical function of type $\mathbb{N}_n \rightarrow \mathbb{N}_n$ that takes a number and returns a different number
 - The numbers have to remain smaller than the length of our alphabet
 - In English $n = 26$, so our numbers must remain in the range 0-25
- Then we convert that number back into a letter (0=a, 1=b, 2=c, etc.)

Ciphers

The background of the slide features a series of white, curved, rib-like structures that sweep across the frame from the bottom left towards the top right. These lines are set against a light gray background that has a subtle gradient, becoming slightly darker towards the right edge. The overall effect is one of dynamic, flowing motion.

Ciphers

- A cipher is a way to encrypt text to make it hard for other people to read
- The most familiar cipher is the monoalphabetic substitution cypher
 - replaces every occurrence of a letter with the same letter each time
- Ciphers often involve modular arithmetic
 - (Need to transform inputs to outputs within same finite alphabet)
- We'll see two examples
 - Caesar Cipher (shift cypher)
 - Affine Cipher

Caesar Cipher

- Earliest known substitution cipher
 - Invented by Julius Caesar
 - Each letter is replaced by the letter three positions further down the alphabet.

Exercise 14:

Decode the phrase

1. KHOOR

a	b	c	d	e	f	g	h	i	j	k	l	m
D	E	F	G	H	I	J	K	L	M	N	O	P
n	o	p	q	r	s	t	u	v	w	x	y	z
Q	R	S	T	U	V	W	X	Y	Z	A	B	C

Caesar Cipher

- Defined mathematically as a pair of functions, for encryption and decryption

$$E(X) = (X + k) \bmod 26$$

$$D(X) = (X - k) \bmod 26$$

- It has a single key, k , which historically was $k=3$

Affine Cipher

- The Affine Cipher is defined by the pair of functions

$$E(X) = (aX + b) \bmod 26$$

$$D(X) = a^{-1}(X - b) \bmod 26$$

- It has two keys, a and b
- Note that to decrypt we need to calculate the modular inverse (a^{-1})
 - Because of this the value of a must be co-prime with the modulus 26

Affine Cipher

Exercise 15:

Decrypt the following string, using
keys $a=3$, $b=5$

YDSRLVSR

- The Affine Cipher is defined by the pair of functions
 - $E(X) = (aX + b) \bmod 26$
 - $D(X) = a^{-1}(X - b) \bmod 26$
- Note that to decrypt we need to calculate the modular inverse (a^{-1})
 - Because of this the value of a must be co-prime with the modulus 26



Public Key Cryptography

Conundrum

- If you meet a person in advance of encoding messages you can
 - Agree on a secure key
 - Or establish a One Time Pad, etc.
- How do you communicate over an insecure channel with someone with whom you've never met, when you haven't already exchanged keys?

Diffie Hellman Key Exchange

- Method of exchanging keys over an insecure channel
- Commonly used
 - TLS
 - IPsec
 - SSH
 - PHP
- Works using modular arithmetic

Diffie Hellman Key Exchange

- Alice and Bob agree upon
 - modulus (p) (prime usually at least 2048 bits)
 - base (g)
- Alice and Bob choose private keys
 - Alice (a)
 - Bob (b)

4153687576287365984259382475698437658276348
7912837582736592873684273684728938572983759
2834759348759384759284759287395872495872987
3958729835792875982795837529876348273685729
8435793487958279385792873954877239759283759
2478593867045986792384737826735267354762356
8734869386945673456827659498063849024875809
6039479027945982730187439759284620950293759
2870495029380589209839458720948602984912837
5029480193710924801935810379958109375019385
0791395710937597019385089103951073058710393
7019347019380918039840918049810938019850139
8401983509183501983091079180395810395190395
1809358109385019840193580193840198340918093
851098309180019

▲ what 2048 bits looks like as a decimal!

Diffie Hellman Key Exchange

- Alice generates a public key and sends to Bob

$$A = g^a \bmod p$$

- Bob generates a public key and sends to Alice

$$B = g^b \bmod p$$

- Public
 - modulus (p) (large prime)
 - base (g)
- Private
 - Alice's private key (a)
 - Bob's private key (b)

Diffie Hellman Key Exchange

- Alice calculates the shared secret

$$s = B^a \bmod p$$

- Bob calculates the shared secret

$$s = A^b \bmod p$$

- Public

- modulus (p) (large prime)
- base (g)
- Alice's public key (A)
- Bob's public key (B)

- Private

- Alice's private key (a)
- Bob's private key (b)

Diffie Hellman Key Exchange

- Alice and Bob now both know s without it ever being transmitted
- It is not practical for an attacker to calculate s , a , or b from the information they can intercept
 - So long as large numbers are used
- s can now be used as the key for a symmetrical encryption algorithm
- Public
 - modulus (p) (large prime)
 - base (g)
 - Alice's public key (A)
 - Bob's public key (B)
- Private
 - Alice's private key (a)
 - Bob's private key (b)
 - Shared secret (s)

Exercise 16:

Assume $p = 17$, $g = 5$. Alice generates a public key $A = 6$ using the following formula: $A = g^a \bmod p$, where A is the public key and a is a private key.

1. What is Alice's private key?
2. Devise a simple algorithm to crack a private key that would work for small values of p and g



Summary

Summary

- Numbers
 - Classes of Numbers
 - Prime Numbers
- Modular Arithmetic
 - Properties of Binary Relations
 - Properties of Binary Operators
 - Ciphers

Further Reading

- Read about Modular Arithmetic and Number Theory
 - <http://www.math.umbc.edu/~campbell/NumbThy/Class/BasicNumbThy.html>