1 Propositional Logic

The following laws of propositional logic are compiled and numbered for the benefit of this course. This is not an exhaustive list.

1.1 Laws of Negation

Law 1.1 not false is true and not true is false

$$(\neg \text{true}) \iff \text{false}$$

$$(\neg \text{false}) \iff \text{true}$$

Law 1.2 two negatives make a positive

$$(\neg\neg p) \iff p$$

1.2 Laws of Conjunction

Law 2.1 a proposition conjoined with itself is equivalent to itself

$$(p \wedge p) \iff p$$

Law 2.2 a proposition conjoined with true is equivalent to itself

$$(p \wedge \text{true}) \iff p$$

Law 2.3 a proposition conjoined with false is equivalent to false

$$(p \wedge \text{false}) \iff \text{false}$$

Law 2.4 a proposition conjoined with its own negation is equivalent to false

$$(p \wedge (\neg p)) \iff \text{false}$$

Law 2.5 conjunction is commutative

$$(p \wedge q) \iff (q \wedge p)$$

Law 2.4 conjunction is associative

$$(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$$

1.3 Laws of Disjunction

Law 3.1 de Morgan's Laws

$$\neg(p \wedge q) \iff ((\neg p) \vee (\neg q))$$

$$\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$$

Law 3.2 disjunction is idempotent

$$(p \vee p) \iff p$$

Law 3.3 a proposition disjoined with false is equivalent to itself

$$(p \vee \text{false}) \iff p$$

Law 3.4 a proposition combined via disjunction with true is equivalent to true

$$(p \vee \text{true}) \iff \text{true}$$

Law 3.5 disjunction is associative

$$p \vee (q \vee r) \iff (p \vee q) \vee r$$

Law 3.6 conjunction is commutative

$$p \vee q \iff q \vee p$$

Law 3.7 a proposition combined via disjunction with its own negation is equivalent to true

$$((\neg p) \vee p) \iff \text{true}$$

Law 3.8 disjunction distributes through conjunction

$$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$$

Law 3.9 conjunction distributes through disjunction

$$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$$

1.4 Laws of Implication

Law 4.1 p implies q is the same as 'not p, or q'

$$(p \implies q) \iff ((\neg p) \vee q)$$

1.5 Laws of Equivalence

Law 5.1 equivalence is associative

$$((p \iff q) \iff r) \iff (p \iff (q \iff r))$$

Law 5.2 equivalence is commutative

$$(p \iff q) \iff (q \iff p)$$

Law 5.3 every proposition is equivalent to itself

$$(p \iff p) \iff \text{true}$$

Law 5.4 no proposition is equivalent to its negation

$$(p \iff (\neg p)) \iff \text{false}$$

Law 5.5 claiming 'p is equivalent to q' is the same as claiming that p implies q and q implies p

$$(p \iff q) \iff ((p \implies q) \wedge (q \implies p))$$

2 Set Theory

The following laws of set theory are compiled and numbered for the benefit of this course. This is not an exhaustive list.

2.1 Laws of Set membership

Law 1.1 for any set S and any element s

$$\neg(s \in S) \iff s \notin S$$

Law 1.2 for any element x

$$x \in \emptyset \iff \text{false}$$

2.2 Laws of Subsets

Law 2.1 for any sets S and T

$$(S \subseteq T \wedge T \subseteq S) \iff S = T$$

Law 2.2 for any sets S

$$(\emptyset \subseteq S)$$

Law 2.3 all sets are a subset of themselves

$$(S \subseteq S)$$

Law 2.4 for any sets S and T

$$\neg(S \subseteq T) \iff S \nsubseteq T$$

Law 2.5 for any sets S and T

$$S \subseteq T \iff (S \subset T \vee S = T)$$

Law 2.6 for any sets S and T

$$S \not\subset T \iff \neg(S \subset T)$$

Law 2.7 for any set S

$$S \not\subset S$$

Law 2.8 for any sets S and T

$$S \subset T \implies T \not\subset S$$

2.3 Laws of Supersets

Law 3.1 for any sets S and T. Stating S is a superset of T is logically equivalent to stating that T is a subset of S

$$S \supseteq T \iff T \subseteq S$$

2.4 Laws of Set Union

Law 4.1 for any element a, and any sets S and T

$$a \in S \cup T \iff (a \in S \vee a \in T)$$

Law 4.2 combining Set S with the empty set ∅, is equivalent to Set S:

$$S \cup \emptyset = S$$

Law 4.3 The set union of any set S combined with itself is equivalent to itself

$$S \cup S = S$$

Law 4.4 Union is commutative

$$S \cup T = T \cup S$$

Law 4.5 Union is associative

$$R \cup (S \cup T) = (R \cup T) \cup S$$

Law 4.6 The union of two sets is always at least as big as each set considered individually

$$S \subseteq S \cup T$$

2.5 Laws of Set Intersection

Law 5.1 where a given element a is in the intersection of sets S and T is must be an element of both sets

$$a \in S \cap T \iff (a \in S \wedge a \in T)$$

Law 5.2 the intersection of a given set S with the empty set Ø is always the empty set

$$S \cap \emptyset = \emptyset$$

Law 5.3 the intersection of set S with itself is always S

$$S \cap S = S$$

Law 5.4 Intersection is commutative

$$S \cap T = T \cap S$$

Law 5.5 Intersection is associative

$$R \cap (S \cap T) = (R \cap S) \cap T$$

Law 5.6 The intersection of any given sets is always at least as small as one of the given sets

$$S \cap T \subseteq S$$

Law 5.7 union distributes through Intersection and Intersection distributes through distribution

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

2.6 Laws of Set Difference

Law 6.1 if a is an element of the Set difference of Sets $S \setminus T$ then S is a member of the former and not the latter

$$a \in S \setminus T \iff (a \in S \wedge a \notin T)$$

Law 6.2 Set S intersected with the empty set is equivocal to set S

$$S \setminus \emptyset = S$$

Law 6.3 The empty set intersected with set S is equivocal to the empty set

$$\emptyset \cap S = \emptyset$$

Law 6.4 The intersection of any set with itself is equal to the empty set

$$S \setminus S = \emptyset$$

Law 6.5 The difference in Set R and the union or sets S and T is equivocal to the union of the difference in set R

and S and R and T. A similar propery holds for Intersection.

$$R \setminus (S \cup T) = (R \setminus S) \cap (R \setminus T)$$

$$R \setminus (S \cap T) = (R \setminus S) \cup (R \setminus T)$$

2.7 Laws of equality

Law 7.1 When two different sets have exactly the same elements, they are equal

$$x \in S \iff x \in T$$

2.8 Laws of cardinality

Law 8.1 the cardinality of the empty set is 0

$$\#\emptyset = 0$$

Law 8.2 the cardinality of the union of S and T is equal to the cardinality of S minus the cardinality of the intersection of S and T

$$\#(S \cap T) = \#S - \#(S \setminus T)$$

Law 8.3 the cardinality of the union of S and T is equal to the cardinality of S plus the cardinality of T minus the cardinality of the intersection of S and T

$$\#(S \cup T) = \#S + \#T - \#(S \cap T)$$

Law 8.4 the Cardinality of the intersection of S and T is equal to the cardinality of S minus the cardinality of the intersection of S and T

$$\#(S \setminus T) = \#S - \#(S \cap T)$$

Law 8.5 the Cardinality of the cartesian product of S and R is the product of the cardinalities of S and R

$$\#(S \times T) = \#S \times \#T$$

2.9 Laws of Power Sets

Law 9.1 Set S is an element of the power set of T if and only if S is a subset of T

$$S \in \mathbb{P}(T) \iff S \subseteq T$$

Law 9.2 the empty set is an element of the power set of a any given set S

$$\emptyset \in \mathbb{P}(S)$$

Law 9.3 for any given set S, S is an element of the power set of itself

$$S \in \mathbb{P}(S)$$

Law 9.4 the cardinality of the power set of set S is equal to two to the power of the cardinality of S

$$\#(\mathbb{P}(S)) = 2^{\#(S)}$$

Law 9.5 for a given set R which is an element of the power set of S, the intersection of R and S is equal to R.

$$R \cap S = R, \text{ where } R \in \mathbb{P}(S)$$

2.10 Laws of Complements

Law 10.1 For a given set T ⊆ S, the compliment of T is equal to the set difference in S and T

$$T^- = S \setminus T, \text{ where } T \in \mathbb{P}(S)$$

Law 10.2 For a given set T ⊆ S, The union of T and its compliment is equal to the S

$$T \cup T^- = S, \text{ where } T \in \mathbb{P}(S)$$

Law 10.3 For a given set T ⊆ S, The intersection of T and its compliment is equal to the empty set.

$$T \cap T^- = \emptyset, \text{ where } T \in \mathbb{P}(S)$$

2.11 Laws of Generalised Operations

Law 11.1 for any set of sets A and any element a, $a \in \bigcup A$ if, and only if, there is some set $S \in A$ such that $a \in S$

$$a \in \bigcup A \iff \exists S \in A . a \in S$$

Law 11.2 for any set of sets A and any element a, $a \in \bigcap A$ if, and only if, for every set $S \in A$ it is the case that $a \in S$

$$a \in \bigcap A \iff \forall S \in A . a \in S$$