



Mathematics and Problem Solving

Lecture 10

Probability

Overview

- Probability Basics
 - Events
 - Addition Rule
 - Multiplication Rule for Independent Events
 - General Multiplication Rule
- Combinatorics
- Probability Trees
- Law of Total Probability
- Bayes Rule

Disease Screening

- There is a national screening programme for a particular disease
 - 1 in 10,000 people have this disease
 - The test is 99% sensitive (true positive rate)
 - If it is caught early, treatment is effective, however the treatment is invasive and you will need to take a year out of your degree
 - You get a positive test result.

Question:

Your doctor asks whether you want to begin treatment. What do you say?

Disease Screening

- Asking for more information, you are given a leaflet that says:
 - The test correctly diagnoses 99 out of 100 people who have the disease
 - If you don't have the disease there is a 1 in 100 chance of a positive result
 - 1 in 10,000 people have this disease

Question:

Do you begin treatment?

Disease Screening

- We want to know:
 - What is the likelihood that I have the disease?
- By the end of this lecture, you will be able to make an informed answer to the sort of question presented here, and many others

Applied Probability

- Probability is often counter-intuitive
 - This can lead to problems in how people understand
 - Medicine (e.g. screening programmes, vaccines)
 - Criminal evidence (e.g. DNA evidence)
 - Often games (e.g. Civilisation) lie about probabilities to better fit player's expectations
 - How numbers are presented affects perception e.g. "1 in 200" > 0.5%
- Because of this, it's also very easy to make mistakes



Events

Experiment

- An experiment is a procedure for discovering something unknown
 - pulling a ball out of a bag
 - rolling a dice
 - flipping a coin
 - measuring a piece of string
- Experiments have one or more **outcomes**
 - Drawing a particular ball (b_1, b_2, \dots)
 - Rolling a particular number ($1, 2, \dots$)
 - Getting a particular measurement ($l = 12\text{cm}$)

Sample Space

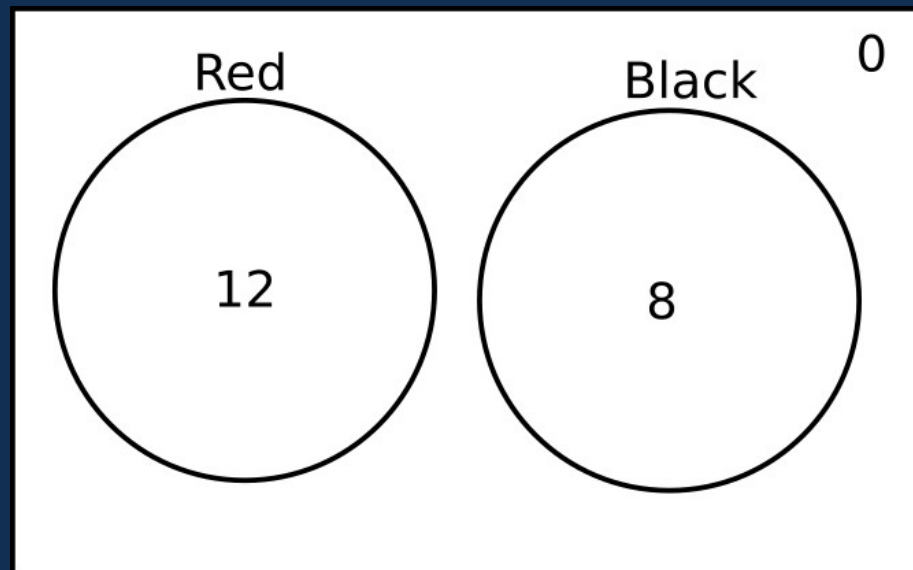
- All the possible outcomes are called the **sample space**
 - The sample space is a set
- E.g. for rolling a 6-sided dice
 - $S = \{1, 2, 3, 4, 5, 6\}$
- The probability that we will get an outcome from the sample space is 1 (certain)
 - $P(S) = 1$

Events

- An event E is a non-empty subset of the sample space
 - It is a set of outcomes
 - $0 < P(E) < 1$
- For example
 - Draw a Red ball ($R = \{ b_1, b_3, b_5 \}$)
 - Draw a Black ball ($B = \{ b_2, b_4 \}$)
 - Roll an even number ($E = \{ 2, 4, 6 \}$)
 - Roll a 4 ($F = \{ 4 \}$)

Venn Diagram

- We can represent an experiment with a Venn Diagram
 - Each circle represents an event (set)
 - We give the number of outcomes in each event



Classical Probability

- Assuming each outcome is equally likely:

$$P(\text{event}) = \text{favourable outcomes} / \text{total possible outcomes}$$

- Assume a bag of balls
 - 12 Red balls (12 outcomes)
 - 8 Black balls (8 outcomes)
- $P(\text{red}) = 12 / 20$
- $P(\text{black}) = 8/20$

Exercise 1:

Represent the information opposite as a Venn diagram.

What is the probability of:

1. A randomly selected employee works remotely
2. A randomly selected company laptop being used by someone on-site

- A company has 100 employees
 - 60 employees work on site
 - 50 of whom use company-provided laptops
 - 40 employees work remotely
 - 20 of whom use company-provided laptops
- Hint:
$$P(\text{event}) = \text{favourable outcomes} / \text{total possible outcomes}$$

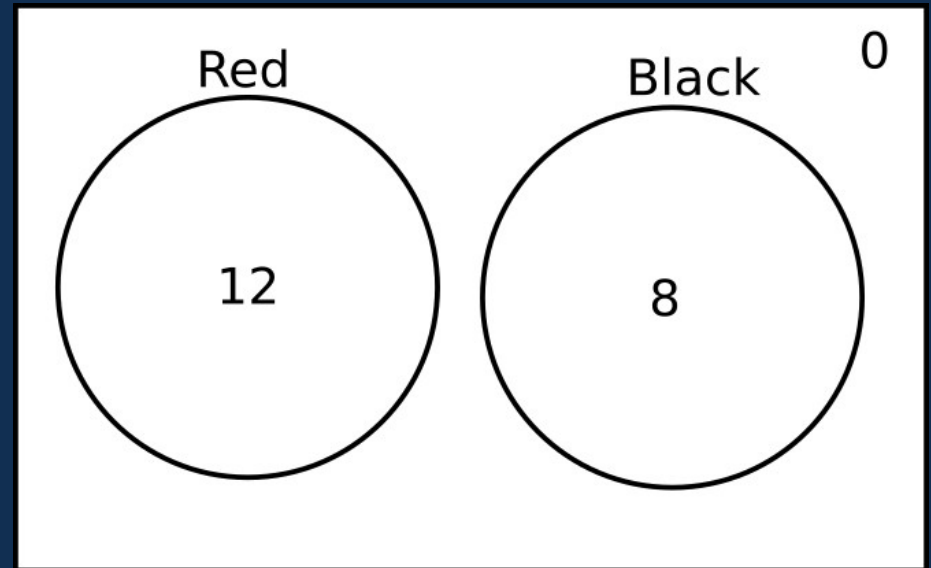
Complement

- Complement of a probability $P(A)$ is

$$P(A') = 1 - P(A)$$

- Chance I don't draw a red ball

$$P(\text{red}') = 1 - (12/20) = 8/20$$



Exercise 2:

Using the information opposite.

What is the probability of:

1. A randomly selected on-site employee not using a company-provided laptop

- A company has 100 employees
 - 60 employees work on site
 - 50 of whom use company-provided laptops
 - 40 employees work remotely
 - 20 of whom use company-provided laptops

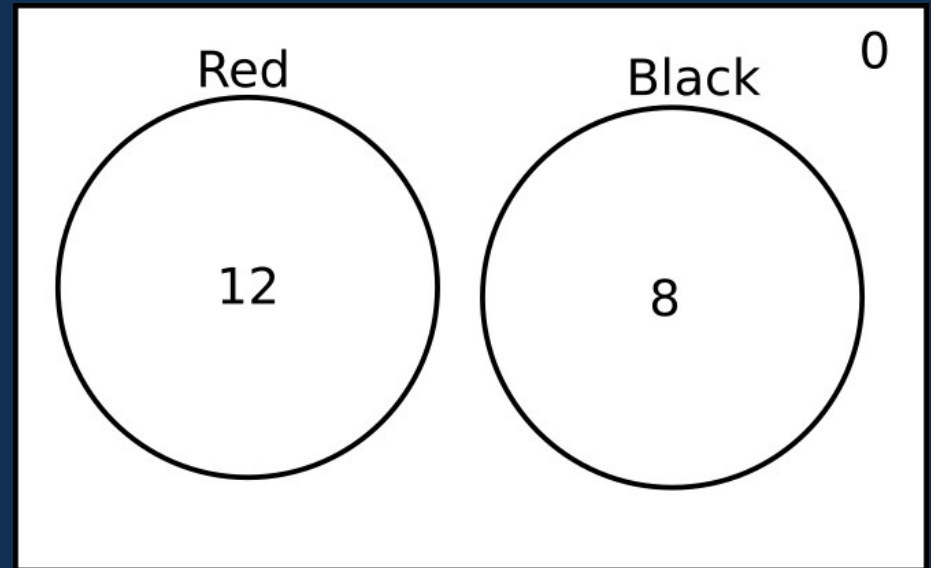
- Hint:

$P(\text{event}) = \text{favourable outcomes} / \text{total possible outcomes}$

$$P(A') = 1 - P(A)$$

Mutually Exclusive Events

- Mutually exclusive events cannot both occur at once
 - On a Venn diagram, they do not overlap
 - Their intersection is the empty set
- In the example, Red and Black are mutually exclusive events



Independant Events

- Two events are independent if their probabilities do not change depending on the outcome of the other
- For example
 - Rolling two dice
 - Flipping a coin 10 times
- Events that are not independent are called **dependant events**

Exercise 3:

Which of the following involve dependant or independent events?


1. Generating two random numbers in Python
2. Drawing two cards from a deck and looking at their suit
3. Two people each choosing a random chair in this room
4. The probability of two successive requests to a webserver failing

Summary

- Experiment
 - Outcomes
 - Events
- Venn Diagrams
- Probabilities of Events
 - Favourable outcomes / Total possible outcomes
- Complement
 - $P(A') = 1 - P(A)$

Exercise 4:

You have a deck of 52 cards (4 suits, no jokers). What is the probability of drawing the following.

1. The ace of hearts ()
2. A heart ()
3. A face card ()
4. A joker ()



Probability of Union

Adding Probabilities

Exercise 5:

For the deck of cards described opposite, what is the:

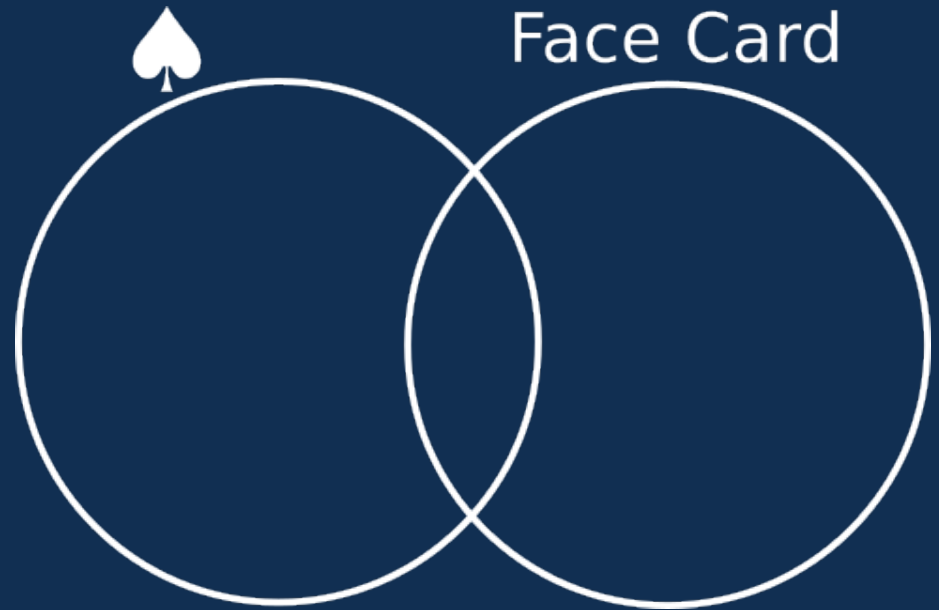
1. Probability of drawing a spade?
2. Probability of drawing a face card?

- I have a deck of cards and I'm making a game
 - 4 suits
 - 13 cards in each suit
 - 3 face cards in each suit
 - 52 cards overall

Adding Probabilities

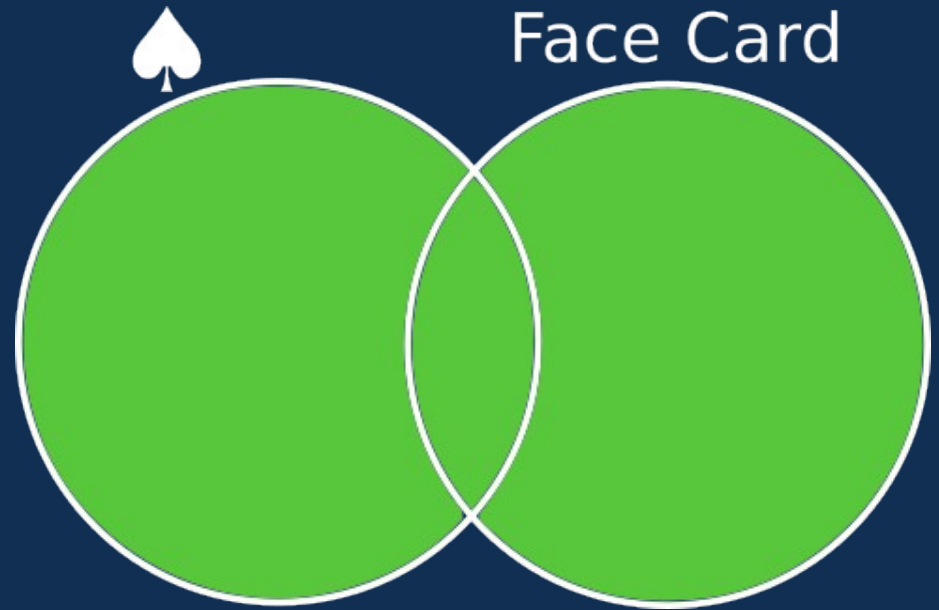
- Whats the probability of drawing either a spade or a face card?
- Assuming
 - $P(\spadesuit) = 1/4$
 - $P(\text{face}) = 3/13$
- The probability of A or B is the probability of the union of A and B

$$P(A \cup B)$$



Union (\cup)

- If events are sets of outcomes, the set-theoretic union of two events is the event that either event occurs
- To find this, we
 - 1) Add up the probabilities of each event
 - 2) Take away the probability that the events co-occur



Addition Rule

Addition Rule:

The combined probability of two events A and B is the sum of the probabilities of those events, less the probability that those events co-occur:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Addition Rule (Example)

- We know
 - $P(\spadesuit) = 1/4 = 13/52$
 - $P(\text{face}) = 3/13 = 12/52$
 - $P(\spadesuit \cap \text{face}) = 3/52$ (because there are 3 spade face cards in the deck)
- $P(\spadesuit \cup \text{face}) = P(\spadesuit) + P(\text{face}) - P(\spadesuit \cap \text{face})$
- $P(\spadesuit \cup \text{face}) = 13/52 + 12/52 - 3/52$
- $P(\spadesuit \cup \text{face}) = 22/52$

Exercise 6:

Two mutually exclusive events have probabilities $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{4}$

What is the probability that A or B occurs?

- Hint:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Probability of Intersection

Intersection

- What is the likelihood that A and B occur?
 - The probability of the intersection of the events A and B

$$P(A \cap B)$$



Multiplication Rule for Independent Events

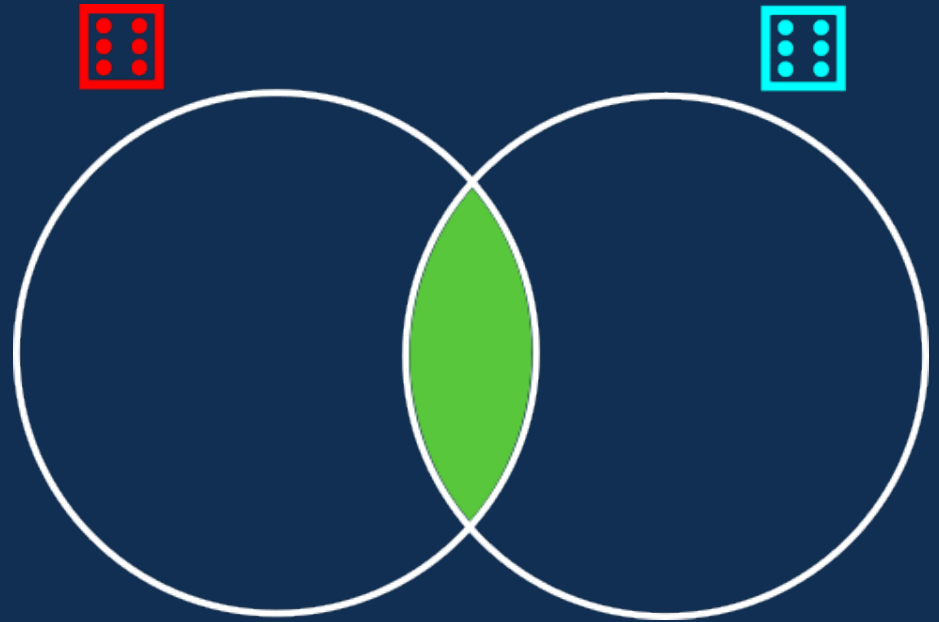
Multiplication Rule for Independent Events:

To find the probability of two independent events co-occurring (their intersection) we multiply their probabilities together

$$P(A \cap B) = P(A) \times P(B)$$

Intersection

- What is the probability of rolling  and  on two dice?
 - $P(\text{1} \cap \text{2}) = P(\text{1}) \times P(\text{2})$
 $= 1/6 \times 1/6$
 $= 1/36$
 - (Note these are independent events)



Exercise 7:

Work out the probabilities for the following **independent events**:

1. Flip a coin twice and get two heads
2. $P(A) = 0.2$; $P(B) = 0.5$; $P(A \cap B) = ?$
3. Draw a red card that is an even number from a 52 card deck



General Multiplication Rule

Conditional Probability

- The chance of event A assuming, or conditional, on event B also happening
 - Written $P(A | B)$
- To find $P(A | B)$ you:
 - 1) find the chance that A and B co-occur
 - 2) divide by the probability of B.

$$P(A|B) = \frac{P(B \cap A)}{P(B)}$$

General Multiplication Rule

General Multiplication Rule:

To find the chance that two events co-occur you multiply their the probability of event A occurring given B by the probability of B.

$$P(A \cap B) = P(A|B) \times P(B)$$

Combinatorics

The background of the slide features a series of white, curved, parallel lines that sweep from the bottom left towards the top right. These lines are set against a light gray background that has a subtle gradient, becoming slightly darker towards the right edge. The overall effect is one of dynamic movement and geometric precision.

Combinatorics

- A fancy name for three things
 - Enumeration
 - Combination
 - Permutation

Factorial

- Formulas for combinatorics make use of factorial
 - Written $n!$
 - $n!$ is n multiplied by every natural number $< n$
- $1! = 1$
- $2! = 2 \times 1$
- $3! = 3 \times 2 \times 1$
- $4! = 4 \times 3 \times 2 \times 1$
- Note that
 - $0! = 1$

Enumeration

- An enumeration is just a listing of a set
- If I have a set $A = \{a, b, c, d\}$
- Its enumeration is
 - $\{a, b, c, d\}$

Combination

- Say we have a set $A = \{a, b, c, d\}$ and we want combinations of 3 elements
- Every subset of A with 3 elements is a combination
 - $\{a, b, c\}$
 - $\{b, c, d\}$
 - $\{a, c, d\}$
 - $\{a, b, d\}$
- Combinations don't have order

Permutation

- Permutations have order
 - (a, b, c)
 - (a, c, b)
- Permutations of 3 elements of the set $A = \{a, b, c, d\}$
 - (b, a, c)
 - (b, c, a)
 - (c, a, b)
 - (c, b, a)
 - (a, b, d)
 - (a, d, b)
 - ...

Exercise 8:

Which of the following are enumerations, combinations, and permutations?

1. I pick some books to take on holiday
2. I select three films to watch
3. I compile a playlist of my favourite music
4. I list all the numbers from 1-100

Calculating Number of Permutations

Exercise 9:

How can we write a formula for permutations?

- Hint:
 - Permutations of 1 out of 4
 - 4
 - Permutations of 2 out of 7
 - 7×6
 - Permutations of 3 out of 5
 - $5 \times 4 \times 3$
 - Permutations of 5 out of 20
 - $20 \times 19 \times 18 \times 17 \times 16$

Calculating Number of Permutations

$${}_n P_k = \frac{n!}{(n - k)!}$$

- Where n is the size of the set and k is the size of the subset we are ordering

Exercise 10:

Work out the answers to the following permutations

1. 1P_1

2. 2P_1

3. 4P_2

4. 5P_3

$${}_nP_k = \frac{n!}{(n - k)!}$$

Calculating Number of Combinations

- Start by treating it as a permutation
 - Gives us (number of combinations) x (number of possible orders)
- Then divide by number of possible orders there could be
- How many orders of k things?
 - ${}^k p_k = k!$

Calculating Number of Combinations

$${}^nC_k = \frac{n!}{k!(n-k)!}$$

- Where n is the size of the set and k is the size of the subset we are picking
- This is also called the Binomial Coefficient and can be written:

$$\binom{n}{k}$$

Comparing Permutations and Combinations

$${}_n P_k = \frac{n!}{(n - k)!}$$

$${}_n C_k = \frac{n!}{k!(n - k)!}$$

- Where n is the size of the set and k is the size of the subset we are picking/ordering

Exercise 11:

Work out the answers to the following combinations

1. 1C_1

2. 2C_1

3. 4C_2

4. 5C_3

Summary

- Factorial
- Permutations and Combinations
 - Two formulas to remember (or look up)

$${}_n P_k = \frac{n!}{(n - k)!}$$

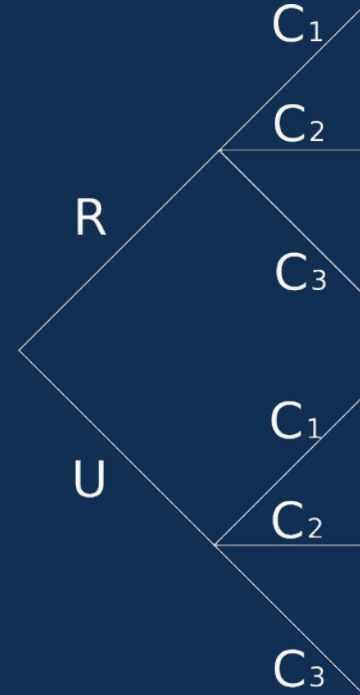
$${}_n C_k = \frac{n!}{k!(n - k)!}$$

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Probability Trees

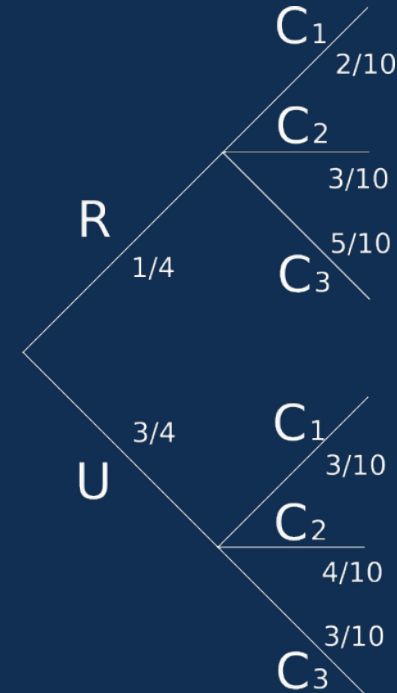
Decision Trees

- A way of representing a sequence of Decisions
- For example, in a MOBA
 - Choose to play Ranked/Unranked
 - Choose character



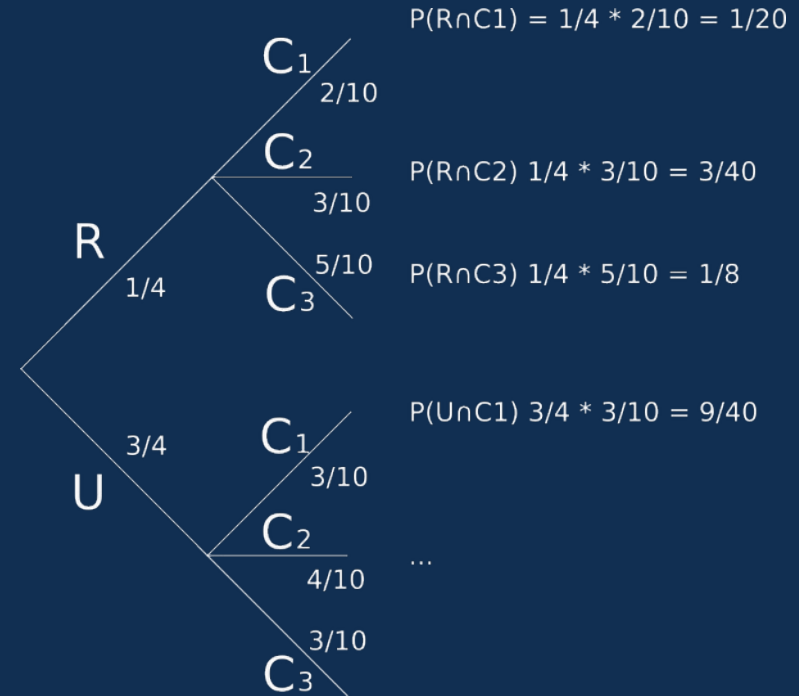
Probability Trees

- Lets say we assign probabilities to each decision
 - Each probability is conditional upon its position in the tree
 - Then we have a probability tree
- We can also write conditional probabilities like this
 - $P(C_1 | R) = 2/10$
 - $P(C_1 | U) = 3/10$



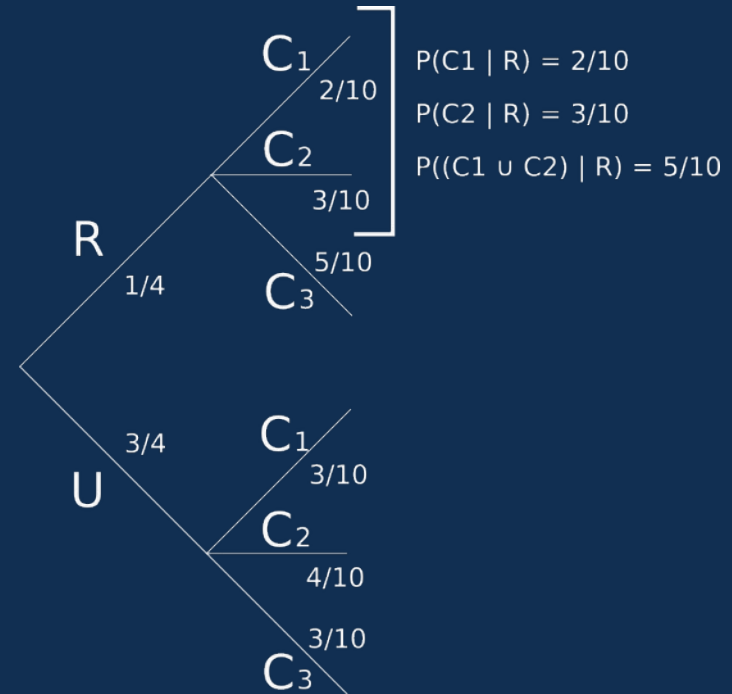
Multiplying Down Branches

- To work out the absolute probability of a set of decisions, we can multiply down the tree
- Remember the Multiplication Rule
 - $P(A \cap B) = P(A|B) \times P(B)$



Adding Across Branches

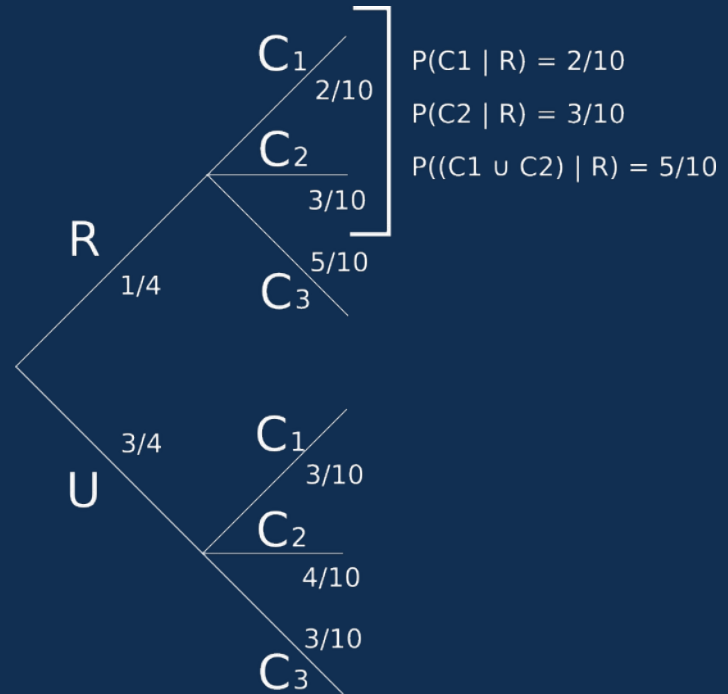
- Branches are mutually exclusive, so we can add them together to get their combined probability
- Remember the Addition Rule
 - $P(A \cup B) = P(A) + P(B) + P(A \cap B)$



Exercise 12:

What are the following probabilities:

1. $P(C_1 | U)$
2. $P(U \cap C_3)$
3. $P(R \cap U)$
4. $P(U \cap (C_1 \cup C_2))$



Disease Screening

- We can use Probability trees to visualise testing for disease, for e.g. screening programmes.

- Prevalence

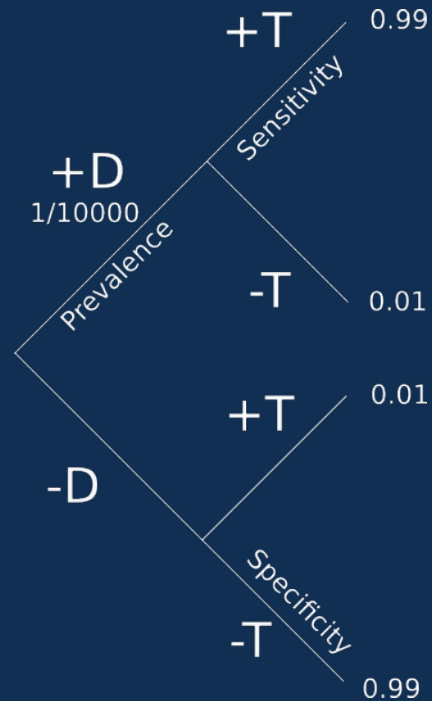
$$P(+D)$$

- Sensitivity

$$P(+T \mid +D)$$

- Specificity

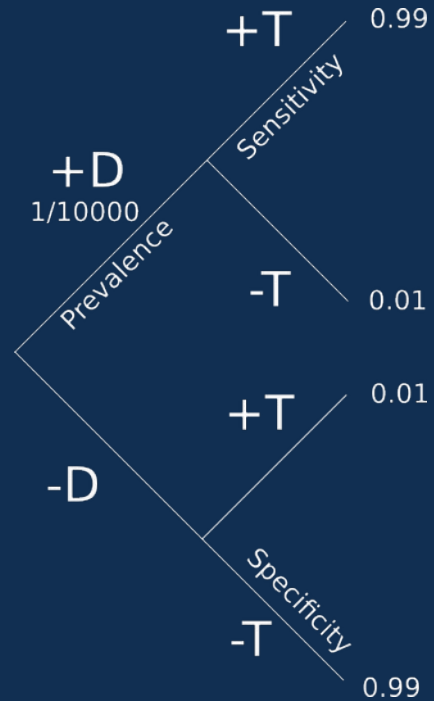
$$P(-T \mid -D)$$



Exercise 13:

Assume a random individual is tested. What is the likelihood of a:

1. True positive
2. False positive
3. True negative
4. False negative

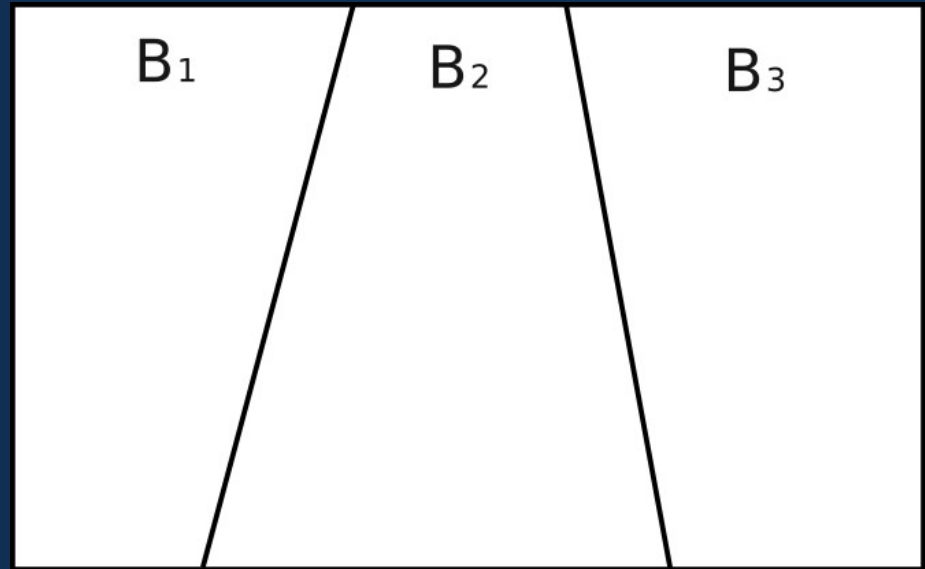




Law of Total Probability

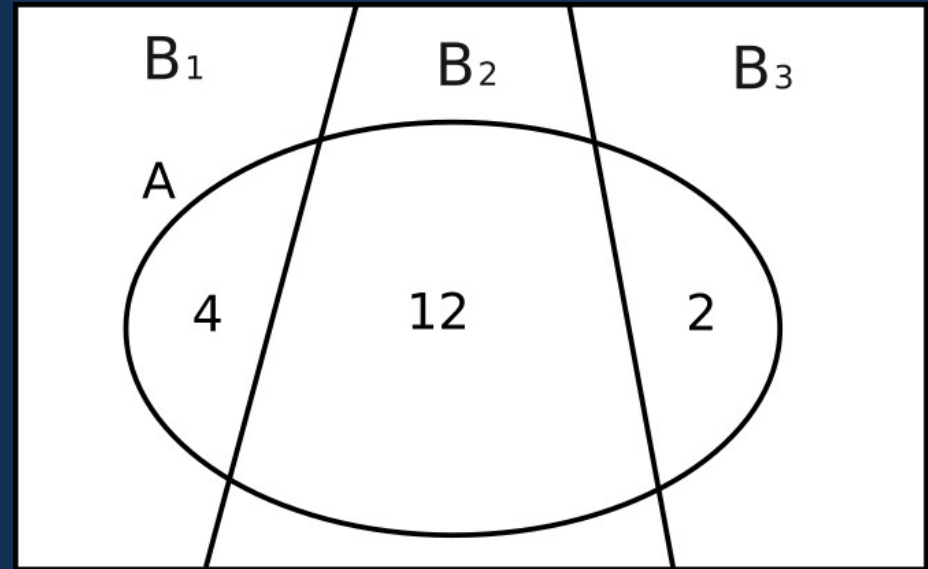
Partition

- A partition is a way of subdividing a set into partitions B_1, B_2, \dots, B_n that are:
 - **disjoint** (i.e. there is no overlap between B_i and B_j , or in other words $B_i \cap B_j = \emptyset$)
 - **collectively exhaustive** (i.e. together they contain everything in the set)



Summing Conditional Probabilities

- Events may be divided across these partitions
- We can reason about the event A as the sum of conditional probabilities
 - $P(A | B_1) +$
 - $P(A | B_2) +$
 - $P(A | B_3)$



Law of Total Probability

Law of Total Probability:

If B_1, B_2, B_3, \dots is a partition of the sample space S , then for any event A :

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i) \times P(B_i)$$

Exercise 14:

You have 2 bags of balls that each contain 70 balls.

- Bag 1 has 50 red balls and 20 black balls
- Bag 2 has 30 red balls and 40 black balls

You pick a bag at random and pick a random ball from that bag. What is the probability that you pick a red ball?

- Hint:

$$P(A \cap B) = P(A|B) \times P(B)$$

$$P(A) = \sum_i P(A \cap B_i)$$



Bayes Rule

Bayesian Reasoning

- Say we had a scientific hypothesis
 - We gather evidence
 - We want to know how certain we should be of the hypothesis
- We want to give a value to the formula:

$$P(H | E)$$

- Probability of hypothesis H given evidence E
- What do we need to know to calculate this?

What we (might) know

- Lets say we know
 - $P(H)$, the prior probability of hypothesis H
 - $P(E)$, the likelihood that the evidence is true
 - $P(E | H)$, the likelihood of the evidence given the hypothesis
- This is enough to work out $P(H | E)$
 - Called the posterior probability of hypothesis H

Revision Example

- For example, assuming:
 - Half of students revise
 - Half of students pass their exam
 - Of those who revise, 80% passed their exam
- What is the likelihood that a student who passed their exam had revised?

Revision Example

- What is the likelihood that a student who passed their exam had revised?
 - Prior probability of hypothesis
 - $P(H) = 50\%$ (Half of students revise)
 - Probability of evidence
 - $P(E) = 50\%$ (Half of students pass their exam)
 - Probability of evidence given hypothesis
 - $P(E | H) = 80\%$ (Of those who revise, 80% passed their exam)
- We need to rearrange this for $P(H | E)$

Exercise 15:

Using the formulas opposite,
rewrite:

$$P(H | E)$$

in terms of

- $P(H)$
- $P(E)$
- $P(E | H)$

- Hint:

$$P(A \cap B) = P(A|B) \times P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(B \cap A)}{P(B)}$$

$$P(A) = \sum_i P(A \cap B_i)$$

Bayes Rule

The diagram illustrates the components of Bayes' Rule. The formula is centered, with four arrows pointing to its parts from descriptive text labels. The label 'Prior Probability' points to $P(H)$. The label 'Likelihood of the evidence 'E' if the Hypothesis 'H' is true' points to $P(E|H)$. The label 'Posterior Probability of 'H' given the evidence' points to $P(H|E)$. The label 'Prior probability that the evidence itself is true' points to $P(E)$.

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$

Prior Probability

Likelihood of the evidence 'E' if the Hypothesis 'H' is true

Posterior Probability of 'H' given the evidence

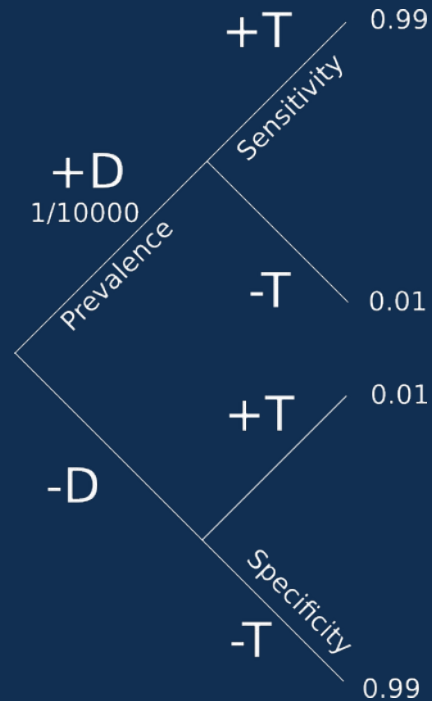
Prior probability that the evidence itself is true

Disease Screening

- Probability is often counter intuitive.
- Imagine you have been screened for a disease (e.g. cancer)
 - You get a positive result
 - What is the probability that you have the disease?

Question:

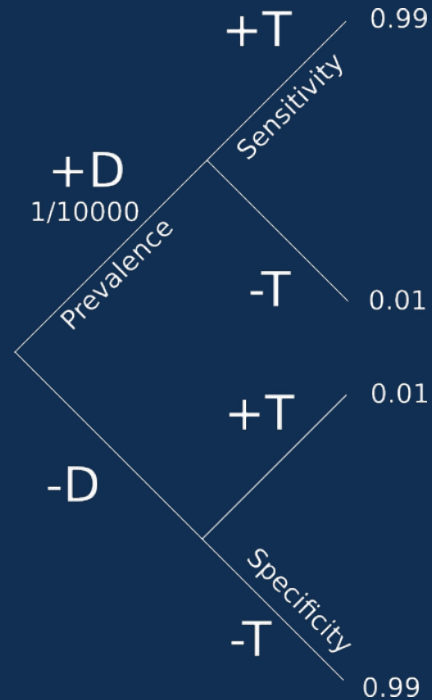
Given the statistics opposite, would you be willing to undergo potentially harmful treatment after a positive test?



Disease Screening

- We want to work out $P(H | E)$, we know:
 - $P(H) = 1/10000$ (prior probability = prevalence)
 - $P(E | H) = 0.99$ (sensitivity)
 - $P(E) = ?$

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$



- Our hypotheses (Has disease, doesn't have disease) is a partition
 - Disjoint
 - Collectively exhaustive
- We can rewrite $P(E)$ using the Law of Total Probability

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$

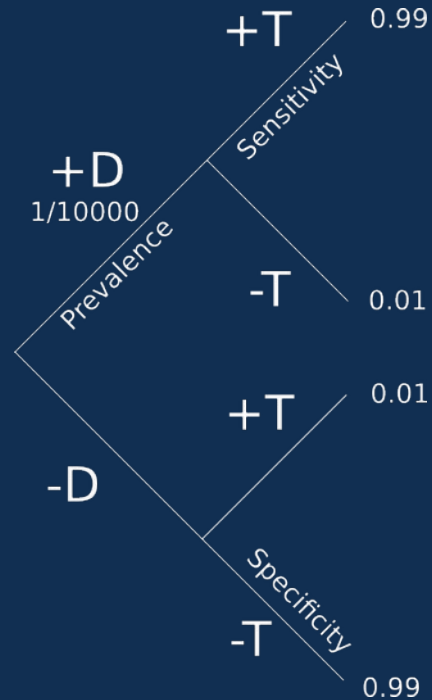
$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i) \times P(B_i)$$

$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$

Disease Screening

- We want to work out $P(H_1 | E)$, we know:
 - $P(H_1) = 1/10000$
 - $P(H_2) = 9999/10000$
 - $P(E | H_1) = 0.99$
 - $P(E | H_2) = 0.01$

$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$



Exercise 16:

Work out $P(H_1 | E)$,:

- $P(H_1) = 1/10000$
- $P(H_2) = 9999/10000$
- $P(E | H_1) = 0.99$
- $P(E | H_2) = 0.01$

Would you undergo potentially harmful treatment?

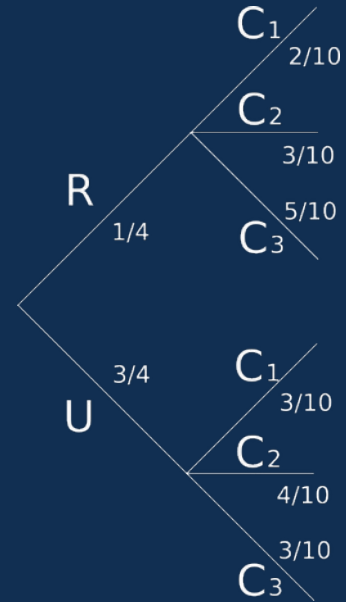
- Hint

$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$

Exercise 17:

A player of a MOBA chooses to play Ranked/Unranked and then picks one of three characters to play. The probabilities are shown in the probability tree opposite

You see someone playing character C_3 . What is the likelihood they are playing ranked?



$$P(H_1|E) = \frac{P(E|H_1) \times P(H_1)}{\sum_i P(E|H_i) \times P(H_i)}$$



Summary

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- Probability Basics
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