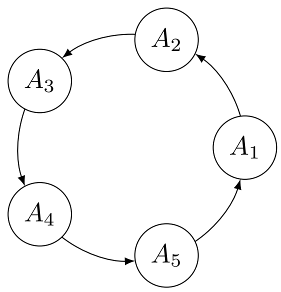
1 Formal Systems

1.1

A formal system has states $A_1, \ldots, A_5$ and rules R. Below is a graph showing all possible transitions between states.



1. Is the derivation $A_1 \Rightarrow_R^* A_5$ possible within this system?
2. If so, provide a derivation.

1.2

A formal system has states $X_1, \ldots, X_7$. The rules R are such that from a state X_i it is possible to derive the state X_{i+1}

1. List the steps in the derivation $X_1 \Rightarrow_R^* X_7$
2. What is the length of this derivation?
3. Is there a derivation $X_4 \Rightarrow_R^* X_3$?

1.3

A string rewriting system has the following rules R.

- $aba \curvearrowright b$
- $aaa \curvearrowright$

Let A be the string $aaaababbbbababababaaaababa$

1. List all the strings B that can be derived from A in one step: $A \Rightarrow_R B$
2. List the strings in one possible derivation of T, $A \Rightarrow_R^* T$, where T is a terminal state (where no further rule applications are possible)

1.4

A string rewriting system is defined with the following rules.

- $(fof) \curvearrowright f$
- $(tof) \curvearrowright t$
- $(fot) \curvearrowright t$
- $(tot) \curvearrowright t$

For each of the following, derive a terminal string (where no further rule applications are possible)

1. $((fot)ot)$
2. $((fof)o(tot))$
3. $(((tof)of)o(fo(fof)))$
4. Give a string from which the string f is derivable by a derivation of length 3

1.5

For the following binary relations, state whether they are: Reflexive, Symmetric, Transitive

1. $=$
2. $>$
3. $\leq$
4. $\neq$
5. older than
6. can be rotated (by some angle) to get

1.6

For the following table of binary operators, state whether they are: Commutative, Associative

1. $+$
2. $\times$
3. $\div$
4. $\times \pmod{n}$
5. $\div \pmod{n}$

2 Modular Arithmetic

2.1

Solve the following modular arithmetic questions

1. $(13 \mod 3) + (23 \mod 4)$
2. $(22 \mod 7) \times (13 \mod 7)$
3. $(14 \mod n + 42 \mod n) \times (13 \mod n)$
4. $11 \div 7 \pmod{13}$
5. $4 \div 11 \pmod{17}$

2.2

For each number below, give its modular inverse (if defined) under a modulus of 8

1. $0^{-1} \pmod 8$
2. $1^{-1} \pmod 8$
3. $2^{-1} \pmod 8$
4. $3^{-1} \pmod 8$
5. $4^{-1} \pmod 8$
6. $5^{-1} \pmod 8$
7. $6^{-1} \pmod 8$
8. $7^{-1} \pmod 8$

2.3

If multiplication distributes over subtraction under modulus then, which of the following statements are true

1. $a(b-c) \equiv a - bc \pmod n$
2. $a(b-c) \equiv ab - ac \pmod n$
3. $a(b-c) \equiv ac - ab \pmod n$
4. $a - bc \equiv ab - ac \pmod n$

2.4

What is the additive inverse of:

1. 24
2. $5 \pmod{17}$
3. $a \mod b$ (give your answer as a formula)

2.5

Write the following as products of their prime factors in the form $f_1 \times f_2 \times f_3 \times \ldots$

1. 68
2. 123
3. 92
4. 44

2.6

Assume a string is encrypted using the encryption function $E(X) = (aX + b) \mod n$, where X is a drawn from the 8-bit ASCII alphabet, and n is the length of that alphabet.

1. Encrypt the string "hello" using the keys a=7, b=4
2. Decrypt the string "panda" using the same keys

2.7

Assume $p = 23$, $g = 7$. Alice generates a public key $A = 11$ using the following formula: $A = g^a \mod p$, where A is the public key and a is a private key.

1. Give a value for Alice's private key
2. You pick a private key $b = 11$. Calculate your public key
3. A shared secret s can be calculated with the formula $s = A^b \mod p$, where A is Alice's public key, and b is your private key. Find s.

2.8

The string "example" has been encrypted using the following algorithm: $E(X) = (X + k) \mod n$, where k is the key and n is the length of the alphabet. You know the original (plaintext) string begins in "a".

1. Give a value for k

2.9

In what circumstances is the following true. Express as an inequality involving a, b and n

1. $[(a+b) \mod n] < [a \mod n]$

3 Number Systems

3.1

Convert all the following values into each of: Binary, Octal, Decimal, and Hexadecimal

1. $(01100011)_2$
2. $(0257)_8$
3. $(611)_{10}$
4. $(19F)_{16}$

3.2

Answer the following binary arithmetic questions. All numbers given are in binary. Representations use the number of bits shown. Give your answers in binary.

1. $0101011 + 1101011$
2. $1101 - 1011$
3. $00010100 - 10101000$
4. 1101×1010
5. 11101001×00101001
6. $01001011 \div 00010111$ (report both dividend and quotient)
7. $1011 \div 0101$ (report both dividend and quotient)

3.3

Give the Two's Complement for each of the following 8-bit binary numbers

1. 10000000
2. 10100111
3. 01101011

3.4

Calculate the answers to the following, working with 8-bit signed binary numbers **stored as sign and magnitude**

1. What is the additive inverse of 01101111?
2. $00110001 + 11001101$
3. $10010101 - 00101010$

3.5

Determine whether the following pairs of numbers are co-prime

1. $(3, 7)$
2. $(14, 39)$
3. $(11, 23)$

3.6

Assume numbers are represented in a way that can store 511 unique numbers. What is the One's Complement of 432?

3.7

What is the radix complement of the 5 digit decimal number 52342?

3.8

Assume base 32 is written with the alphabet [0-9A-V] Convert the base 4 number 321223123212 into base 32

3.9

A 2341 digit number in base 536 can store how many different values?

3.10

You have the number 2348923947234. How many digits are required to store this number in base 132?

4 Sequences and Summation

4.1

For each of the following, state whether the sequence is arithmetic or geometric, and give the next 3 values in the sequence

1. 1, 7, 13
2. 0.9, 0.6, 0.4
3. 2k, 6k, 18k

4.2

Give the first 5 elements of the following sequences:

1. $\{\frac{n^2}{n-1}\}_{n=1}^{\infty}$
2. $\{i^3\}_{i=3}^{\infty}$
3. $a_n = a_{n-1} + 3$ where $a_1 = 22$
4. $a_n = 14 \times 3^{n-1}$

4.3

A geometric sequence has 10 terms and a common ratio of $\frac{1}{10}$ and it's final term is 10^{-8}.

1. Give an explicit formula to define this sequence
2. Is this sequence increasing, decreasing, monotonic, and/or bounded? (List all that apply)
3. Is -10 a lower bound for this sequence?
4. Is 1 an upper bound for this sequence?

4.4

Solve the following summations

1. $\sum_{i=1}^{5} 2$
2. $\sum_{i=4}^{7} i$
3. $\sum_{i=1}^{3} 2^i$
4. $\sum A$ where $A = \{n^2\}_{n=1}^{7}$

4.5

Simplify the following summations

1. $\sum_{i=1}^{n} c$
2. $\sum_{i=1}^{n}(i+2)$
3. $\sum_{i=1}^{n}(i^2+3i)$
4. $\sum_{i=1}^{n}\sum_{j=1}^{i} j$
5. $\sum_{i=1}^{n} A_i$ where $A_n = A_{n-1} + \frac{1}{2}$ and $A_1 = \frac{1}{2}$

4.6

Solve the following products

1. $\prod_{k=1}^{3}(2k+1)$
2. $\prod_{k=7}^{17} 2$
3. $\prod A$ where $A = \{\frac{1}{k}\}_{k=1}^{4}$

4.7

Simplify the following products

1. $\prod_{k=1}^{n} c$
2. $\prod_{k=1}^{n}(3c+5)$

4.8

Consider the sequence $S = \{\frac{n}{n+1}\}_{n=1}^{\infty}$

1. Give a lower bound m that also in this sequence.
2. What is a number that this sequence will approach, but never quite reach?
3. Is this sequence increasing, decreasing, monotonic, and/or bounded? (List all that apply)

5 Propositional Logic

5.1

When p is true and q is false, state whether the following statements are true or false

1. $\neg\neg\neg q$
2. $(p \wedge q) \vee p$
3. $(p \vee \neg q) \implies p$
4. $q \implies (p \Leftrightarrow q)$

5.2

Construct a truth table for each of the following. For each, state whether the statement is always true, or if not give a case where it is false.

1. $p \implies (q \implies (r \implies p))$
2. $p \vee \neg r \implies (\neg(r \wedge p)$

5.3

Identify the atomic propositions in the following sentences and assign them each a letter (e.g. b = "the bus is late") Then express this as a statement of propositional logic using the notation taught in class

1. If my bike is not working or the bus is late, then I am late for class
2. I am happy if and only if I am riding my bike

5.4

For the following, state whether they are tautologies, contraditions, or contingiencies

1. $p \implies (\neg p \vee p)$
2. $p \vee q \implies p \wedge q$
3. $\neg p \vee \neg\neg p$

5.5

Prove the following by Equational Reasoning. Format your proof as in the lecture slides

1. $p \implies (p \vee \neg p)$
2. $\neg p \wedge \text{true} \iff \neg p$

For this, use only the laws given in the lecture slides reproduced below:

- Law 1.1: $(\neg\text{true}) \iff \text{false}$
- Law 1.2: $(\neg\neg p) \iff p$
- Law 2.1: $(p \wedge p) \iff p$
- Law 2.2: $(p \wedge \text{true}) \iff p$
- Law 2.3: $(p \wedge \text{false}) \iff \text{false}$
- Law 2.4: $(p \wedge (\neg p)) \iff \text{false}$
- Law 2.5: $(p \wedge q) \iff (p \wedge q)$
- Law 2.6: $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$
- Law 3.1: $\neg(p \wedge q) \iff ((\neg p) \vee (\neg q))$
  $\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$
- Law 3.2: $(p \vee p) \iff p$
- Law 3.3: $(p \vee \text{false}) \iff p$
- Law 3.4: $(p \vee \text{true}) \iff \text{true}$
- Law 3.5: $p \vee (q \vee r) \iff (p \vee q) \vee r$
- Law 3.6: $p \vee q \iff q \vee p$
- Law 3.7: $((\neg p) \vee p) \iff \text{true}$
- Law 3.8: $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
- Law 3.9: $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
- Law 4.1: $(p \implies q) \iff (\neg p \vee q)$
- Law 5.1: $((p \iff q) \iff r) \iff (p \iff (q \iff r))$
- Law 5.2: $(p \iff q) \iff (q \iff p)$
- Law 5.3: $(p \iff p) \iff \text{true}$
- Law 5.4: $(p \iff (\neg p)) \iff \text{false}$

6 Set Theory

6.1

Define the following sets by extension

1. The set of natural numbers between 6 and 11 (not inclusive)
2. The set of letters in the phrase "formal systems, logic, and semantics"
3. The set of sets with exactly one subset

6.2

State whether each of the following is a singleton, the empty set, or neither of these

1. The set of Real numbers, less the Natural numbers
2. The set of even primes
3. The set of digits used in binary
4. The set red clubs in a standard deck of cards

6.3

Give the extension of the following sets

1. $\{a, b, c\} \cap \{c, d\}$
2. $\{a, c\} \cup \{b, c, d\}$
3. $\{a, b, c\} \setminus \{b, c, d\}$
4. $\{a\} \cap (\{b\} \cup \{a, b, c\})$
5. $(\{c\} \cup \{b\} \cup \{a\}) \cap (\{b, c, a\} \cup \emptyset)$

6.4

For the given sets, state whether the following propositions are true or false
$A = \{a, b, c\}$
$B = \{a, d\}$
$C = \{c\}$

1. $A \supset C$
2. $(C \cup A) \subseteq B$
3. $c \in (A \cap B)$
4. $a \notin (A \setminus B)$
5. $(B \cap \{d, a\}) \supseteq B$

6.5

For the given sets, give the extension of the following sets
$A = \{a, b, c\}$
$B = \{a, d\}$
$C = \{c\}$

1. $(A \setminus B) \cap C$
2. $(B \cap C) \cup A$

6.6

Let 0 (zero) be represented as $\emptyset$. Assume the common arithmetic operations ($+$, $\times$, etc.) work as normal. For any $a, b \in \mathbb{N}$ such that $a + 1 = b$, then $a \in b$. In this way we can define the numbers:

- $0 = \emptyset$
- $1 = \{\emptyset\}$
- $2 = \{\{\emptyset\}\}$
- etc.

Solve the following and express the answer in set notation

1. $\emptyset + \emptyset$
2. $\{\{\emptyset\}\} \times \{\{\emptyset\}\}$
3. $\{\{\{\{\emptyset\}\}\}\} + \{\{\{\emptyset\}\}\} - \{\emptyset\}$

State whether the following are true

1. $9 \in 10$
2. $(\{\{\emptyset\}\} + \{\emptyset\}) \in (\{\{\{\emptyset\}\}\})$

7 Reading Week

8 Set Theory 2

8.1

Give the cardinality of the following sets

1. $\{7, 8, 8\}$
2. $\bigcap\{\{a, b\}, \{b, a, d\}, \{d, f, a, b\}\ \}$
3. $\mathbb{P}(\{a, b, c\})$
4. $\mathbb{N}^0 \setminus \mathbb{N}^+$
5. $\mathbb{Q} \setminus \mathbb{R}$

8.2

Give the cardinality of the following sets, given:

- $\#A = 4$
- $\#B = 6$
- $\mathbb{P}(C) = 8$
- $\#(A \cap B) = 1$
- $C \subseteq A$
- $\#(C \cap B) = \emptyset$

1. $A \cup B$
2. $\mathbb{P}(A \setminus B)$
3. $C \cap A$
4. $B \cup C$
5. $A \times B$

8.3

Give the extension of the following sets.

1. $\mathbb{P}(\{1, 2, 3, 4\})$
2. $\bigcup \mathbb{P}(\{a, b, c\})$
3. $\{a, b\} \times \{a, b, c, d\}$
4. $\mathbb{P}(\emptyset)$
5. $(\{1\} \times \{a, b\}) \cap (\{1, 2\} \times \{b, a\})$

8.4

Give the extension of the following sets.

1. $\{n : \mathbb{N} \mid n \mod 3 = 1 \land n < 13\}$
2. $\{n : \mathbb{N} \mid 4 < n < 7\}$
3. $\{n : \mathbb{N} \bullet n \mod 7\}$
4. $\{a : \mathbb{R}; b : \mathbb{R} \mid a^2 = b \land b^2 = a \bullet a\}$
5. $\{a : \mathbb{N}^+; b : \mathbb{N}^+ \mid a + b < 3 \bullet (a, b)\}$

8.5

Define by extension the set containing the smallest four elements of the following sets:

1. $\{n : \mathbb{N}^+ \bullet n^n\}$
2. $\{n : \mathbb{N}^+ \bullet \frac{n}{n+1}\}$

8.6

Given the table below, give the extensions of the following sets

Name	Age	Group
Alice	18	A
Bob	17	B
Eve	19	A
Mary	22	B

1. $\{x : Group \times Name\}$
2. $\{x : Name \times Age \mid x.2 \geq 18 \bullet x.1\}$
3. $\{a : Name \times Group; b : Name \times Group \mid a.2 = b.2 \bullet \{a.1, b.1\}\}$

8.7

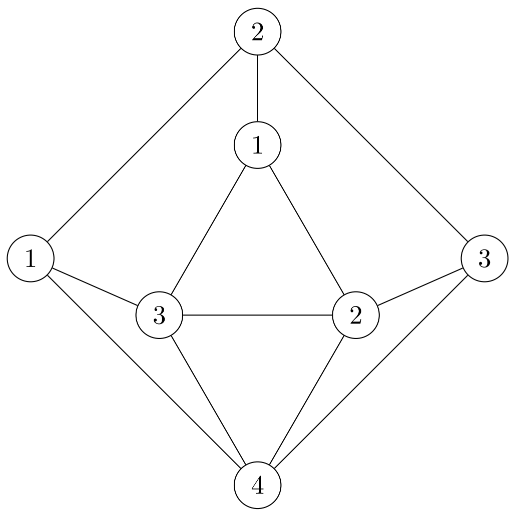
For the table given above, define the following sets by set comprehension:

1. The set of all names
2. The set of groups containing someone under the age of 18

9 Graph Theory

9.1

Consider the following graph.



1. Is it directed?
2. Is it connected?
3. Is it cyclic?
4. What is the maximum degree of the graph?
5. What is the degree of node 3?
6. Create an adjacency matrix for the given graph
7. List the loops in this graph as a set

9.2

A graph G is defined as $G = \{V, E\}$, where $V = \{a, b, c\}$, and $E = V \times V$

1. Is it directed?
2. Is it connected?
3. Is it cyclic?
4. What is the maximum degree of the graph?
5. What is the degree of node b?

9.3

Below is a database for a social network

Name	Age	Group
Alice	18	A
Bob	17	B
Eve	19	A
Mary	22	B

A graph of connections is given by $G = (V, E)$ where $V = \{x : Name\}$,

The graph has directed edges between those pairs of users who share a group

1. Give an extensional definition of E
2. Give an intensional definition of E using set comprehension

9.4

A formal string-rewriting system is defined with the the rule $a \rightsquigarrow abb$, and the starting string a.

1. Is $aaabbbbbb$ a string in this system? If so, draw a graph of the derivation $a \Rightarrow^* aaabbbbbb$
2. Is there any derivation possible in this system that would result in a cyclic graph?

9.5

A formal string-rewriting system is defined with the rules below. The starting string is a

- $a \rightsquigarrow aba$
- $bab \rightsquigarrow b$

1. Draw a graph, of the first 5 unique strings that can be constructed in this system, where edges represent applications of one of the grammatical rules
2. Is this graph directed?
3. Is this graph cyclic?

10 Descriptive Statistics

10.1

For the following types of data, say whether they are nominal, ordinal, or numeric, and give an appropriate measure of central tendancy to use to describe such data

1. Duration of daily commute
2. Frequency in Hz
3. DEFCON level
4. Brands of espresso maker

10.2

For the following data, calculate the mean, median, and mode

1. 4, 2, 5, 2, 34, 2, 4
2. 1, 4, 6, 2, 4, 3, 1

10.3

The following data represents a sample of scores collected from two levels of a mobile game. Calculate the range, inter-quartile range, and an appropriate form of standard deviation

1. 1, 52, 3, 53, 23, 5, 3, 6
2. 5, 2, 14, 25, 14, 11, 5

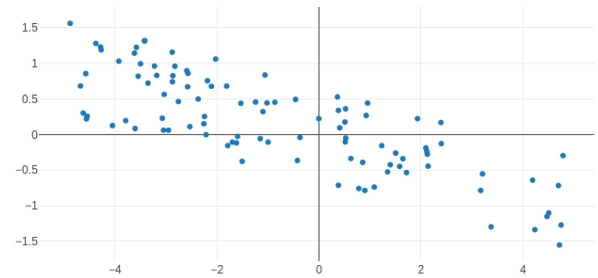
10.4

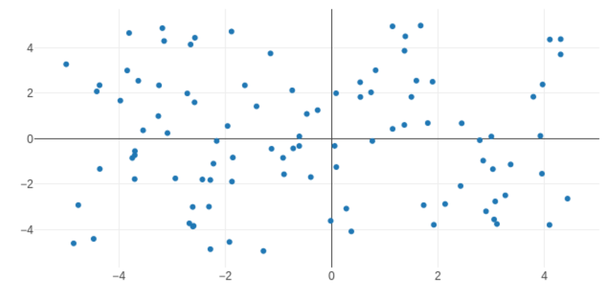
I recruit undergraduate students for a study on video game immersion, which is measured with a questionnaire. Half of the participants are asked to drink 3 cups of coffee before the experiment. Half are asked to not drink any coffee before the experiment.

1. What is my dependant variable?
2. What is my independant variable?
3. What is my sample?

10.5

What is the approximate correlation of the data shown? Pick the closest out of $\{-1, -0.5, 0, 0.5, 1\}$

1. 

2. 

10.6

Use the least squares method to provide a formula for the line of best fit $y = a + bx$ through the data below. The formula for the least squares method is reproduced below to help you.

$$b = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2}$$

$$a = \overline{y} - b\overline{x}$$

X	Y
1	5
2	7
3	10
4	9

11 Probability

11.1

You have 3 fair six-sided dice. What is the probability of

1. Rolling 3 sixes
2. Rolling at least 2 sixes

11.2

Let $P(A) = 0.25$, $P(B) = 0.5$, and $P(C) = 0.75$. You know that events B and C are independant, and that $P(A \mid B) = 0.7$. Calculate the following

1. $P(A \cap B)$
2. $P(B \cup A)$
3. $P(B \cup C)$

11.3

Calculate the following

1. $4!$
2. $0!$
3. ${}^{4}P_{3}$
4. ${}^{34}P_{23}$
5. ${}^{11}C_{11}$
6. ${}^{15}C_{7}$

11.4

1. How many combinations of four 8-bit binary numbers are possible?
2. I define an ordering over the set {a, b, c, d}. How many possible orderings can I define?

11.5

A_1, and A_2 partition a probability space. B_1, B_2, and B_3 partition the event A_1

There is an event C, such that $P(C) = \frac{1}{2}$, and $P(C \mid A_2) = 0$

There is an event D with 4 outcomes in B_1, 3 outcomes in B_2, and 0 outcomes in B_3. You know that $P(D \mid A_1 = \frac{1}{4})$

1. What is $\sum_{i=1}^{3} P(C \mid B_i)$?
2. What is $P(D \mid A_2)$

11.6

You have 3 bags of balls that each contain 80 balls, their colours are described in the table below.

Bag	Red Balls	Black Balls
Bag 1	65	15
Bag 2	23	57
Bag 3	23	57

You pick a bag at random and pick a random ball from that bag. What is the probability that you pick a red ball?

11.7

What is the liklihood that a student who passed their exam attended lectures?

- 80% of students attend lectures
- 80% of students pass their exam
- Of those who attend lectures, 95% pass their exam

11.8

You have 3 hypotheses that are disjoint and collectively exhaustive, H_1, H_2, H_3. The prior probability of each being true, and conditional probabilities that each is true given the event E are given in the table below.

H_n	$P(H_n)$	$P(E \mid H_n)$
H_1	1/4	3/4
H_2	1/3	1/2
H_3	1/8	2/7

Calculate posterior probabilities for all three hypotheses, given event E has been observed.

12 Inferrential Statistics

12.1

The binomial distribution is given by

$$P(X = x) =^n C_x.p^x.(1-p)^{(n-x)}$$

Let X be a binomially distributed variable, with $n = 6$ and $p = 0.2$

1. What is $P(X = 2)$?
2. What is the most likely value of X?
3. What is $\sum_{i=1}^{n} P(X = i)$?

12.2

Let Y be a continuous variable uniformly distributed between 0 and 1.

1. What is $P(Y = 0.5)$?
2. What is $P(Y > 0.5)$?

12.3

Assuming a conventional experiment-wise value for $\alpha = 0.05$, and the Bonferroni adjustment for multiple testing ($\alpha_{adjusted} = \frac{\alpha}{m}$), which of the experiments below report at least 1 statistically significant result.

1. $p = 0.04$
2. $p = 0.12$
3. An experiment with two hypotheses, $p = 0.03$ and $p = 0.04$
4. An experiment with three hypotheses, $p = 0.01$ $p = 0.24$ and $p = 0.0025$

12.4

You run a series of experiments. The test statistics and degrees of freedom (if appropriate) are given below. For each, state whether the result is significant with $\alpha = 0.05$

1. $z = 0.24$
2. $z = 2.1$
3. $t = 0.04$, $df = 33$
4. $t = 4.31$, $df = 8$

12.5

I sample means of groups of size $n = 10$, with a mean of $\mu = 0$ and a standard deviation $\sigma = 2$. I get 0.4.

Calculate the z score

12.6

You want to compare an observed difference between group means against an expected difference of 0. What do you use?

1. Z test
2. One sample t-test
3. Paired t-test
4. Two sample t-test

12.7

You take 5 friends to two ice cream shops. You get them to rate each shop out of 10. The data is given below.

Name	Shop A	Shop B
John	2	8
Mary	3	8
Bill	4	10
Sue	2	10
Lisa	1	7

What is the t statistic for the hypothesis that people prefer Shop B to Shop A?