Mathematics and Problem Solving

Lecture 5

Propositional Logic

“The aim of scientific work is truth. While we internally recognise something as true, we judge, and while we utter judgements, we assert.”

Gottlob Frege

Overview

- Propositions
- Truth Values
- Operators of propositional logic
- Proof in propositional logic

Propositions

Propositions

- We can make statements in language, such as
  - “I am wearing socks”
  - “I like those shoes you are wearing”
  - “If John is taller than Mary then John must also be taller than Bob”
- By abstracting these into propositional logic, we can reason about them using mathematical laws

Atomic Propositions

- Some propositions do not depend on any other proposition
  - They are the “atoms” out of which more complex propositions can be constructed
- Examples of atomic propositions:
  - “The sun rise in the east”
  - “The Earth is flat”
  - “It is raining outside”
- Not atomic:
  - “It’s not raining outside”
  - “The Earth is flat or the sun rises in the east”

Atomic Propositions

Exercise 1:

Which of the following are atomic propositions?

1. “Paul likes pudding”
2. “Justin does not sleep”
3. “The sky is purple”
4. “John is tall and Mary is tall”

Truth Values

- There are two truth values that may be associated with an atomic proposition
  - TRUE
  - FALSE

Truth Values

Exercise 2:

Which propositions are equivalent to TRUE and which are equivalent to FALSE?

1. $11 + 12 = 22$
2. The sky is blue
3. $1 < 0$
4. $12 \times 3 = 36$
5. No dogs live on the moon

Operators

- Atomic propositions are the building blocks of propositional logic.
  - We often represent them using letters, e.g. *p*, *q*, *r*
- You can combine atomic propositions with operators to get compound propositions
  - Arithmetic operators (add, subtract, multiply, ...) apply to numbers
  - Propositional logic operators (and, or, not, …) apply to truth values

Operators

Exercise 3:

Give an example of a proposition that could be represented symbolically as the following

For example,

- p and q = “The sky is blue and ice is hot”

1. p or q
2. (p or q) and r
3. (Not p) and (not q)

Operators

Proposition logic operators apply to truth values

- Resulting in another truth value

The operators are:

1. $\neg$ Negation (not)
2. $\wedge$ Conjunction (and)
3. $\vee$ Disjunction (or)
4. $\Rightarrow$ Implication (implies)
5. $\Longleftrightarrow$ Equivalence (if and only if)

Negation

Negation

- For any proposition p its negation is denoted $\neg p$
  - For example, if p is “the month is October”
  - $\neg p$ might be read “the month is not October”
- If proposition p is TRUE then $\neg p$ must be FALSE.
- The inverse is also true: If proposition p is FALSE then $\neg p$ must be TRUE

Negation

Exercise 4:

Give the negated form of each of the following:

1. FT212 is on the 2nd floor
2. Justin’s office is warm
3. $n < y$
4. My shoes fit my feet

Complement Laws

Law 1.1:

$$(\neg \text{true}) \iff \text{false}$$

$$(\neg \text{false}) \iff \text{true}$$

- Here $\iff$ means that the propositions are logically equivalent (read "if and only if")

Double Negation Law

Law 1.2:

$$(\neg\neg p) \Longleftrightarrow p$$

- The double negation of *p* (the negation of the negation of *p*) is the same as *p*, *i*.e.
  - Not not TRUE is TRUE
  - Not not FALSE is FALSE

Double negation in English

- Let p stand for the proposition “water is wet”
  - $\neg\neg p$ would stand for “water is not not wet”, or “it is not the case that water is not wet”
- Given $(\neg\neg p) \iff p$
  - “Water is not not wet” is equivalent to “water is wet”

Negation Truth Table

- Truth tables represent possible truth values of propositions.
  - All possible combinations of truth values for atomic proposition(s) are listed in the left column(s)
  - The truth values of compound propositions are listed on the right
- A truth table for negation is given to the right

p	$\neg p$
true	false
false	true

Negation

Exercise 5:

Using the laws covered so far, Calculate the truth values for the following:

1. $\neg(3 + 7 = 10)$
2. ¬("*The earth is flat*")
3. $\neg\neg(3 + 1 = 10)$
4. $\neg\,(14 < 15)$
5. $\neg\neg(23 = 8)$

- Hint:
  - Law 1.1: $(\neg\text{true}) \Longleftrightarrow \text{false}$
  - Law 1.2: $(\neg\neg p) \Longleftrightarrow p$

Conjunction

Conjunction

- Equivalent to the English word “and”.
- “Dave likes football and going to the pub”, might be represented as
  - $p \wedge q$

  where
  - p = ‘Dave likes football’
  - q = ‘Dave likes going to the pub’,

Conjunction

- Returns TRUE if and only if both operands are TRUE, otherwise it returns FALSE.
- A truth table can show this clearly
  - Atomic propositions on left
    - All possible combinations of True and False for *p* and *q*
  - Result of conjunction operation on right
  - (I've added highlighting for clarity)

p	q	$\boldsymbol{p \wedge q}$
True	True	**True**
True	False	**False**
False	True	**False**
False	False	**False**

Conjunction

Exercise 6:

Work out the truth value for the following conjunctions:

1. $(1 + 1 = 2) \wedge (1 > 0)$
2. ("All men are mortal") ∧ ("Socrates is a man")
3. $(\neg (1 + 1 = 3)) \wedge (1 > 0)$
4. $(0 \leq 0) \wedge (3 + 2 + 1 = 6)$
5. $(\neg\neg (1 + 1 = 2)) \wedge (1 > 0)$
6. ("All triangles have three sides") ∧ $(\neg 1 > 0)$

p	q	$\boldsymbol{p \wedge q}$
True	True	**True**
True	False	**False**
False	True	**False**
False	False	**False**

Truth Tables

Exercise 7:

The truth table for $(\neg p) \wedge q$ is expressed as shown here

On the basis of this example create truth tables for the following:

1. $p \wedge (\neg q)$
2. $(\neg p) \wedge (\neg q)$

p	q	$\neg p$	$(\neg \boldsymbol{p}) \wedge \boldsymbol{q}$
true	True	false	**false**
true	false	false	**false**
false	true	true	**true**
false	False	True	**false**

Truth Tables

Exercise 8:

1. How many rows would a truth table with 3 atomic propositions have?
2. What about a truth table with n atomic propositions?

Hint:

p	$\neg p$
true	false
false	true

p	q	$\boldsymbol{p \wedge q}$
True	True	**True**
True	False	**False**
False	True	**False**
False	False	**False**

Conjunction

Laws for Conjunction

Conjunction Laws

Exercise 9:

Using the symbols we have seen so far, list statements of propositional logic that are always True

e.g.

- True
- True $\iff$ True
- $p \iff p$

- Hint:
  - Use the symbols
    - True
    - False
    - p
    - q
    - $\iff$ (is equivalent to)
    - $\wedge$ (conjunction)
    - $\neg$ (negation)

Idempotence of Conjunction

- Idempotence means you can repeat an operation multiple times without changing the result

  Law 2.1:

  Conjunction is idempotent. Thus:

$$p \land p \iff p$$

- i.e. any atomic proposition conjoined with itself is logically equivalent to itself

Conjunction Identity

Law 2.2:

The value TRUE conjoined with proposition p is logically equivalent to the truth value of p. Thus:

$$p \land \text{true} \Longleftrightarrow p$$

- In English,
  - "It's raining AND TRUE" is equivalent to "It's raining"

Domination Law

Law 2.3:

The value FALSE conjoined with proposition p is logically equivalent to FALSE. Thus:

$$p \land \text{false} \Longleftrightarrow \text{false}$$

- A conjunction is TRUE if and only if both operands are TRUE
  - If one is FALSE, the result will always be FALSE

Complement Law

Law 2.4:

Any proposition *p* conjoined with its own negation results in FALSE.

$$p \wedge (\neg p) \iff \text{false}$$

- "It's raining AND it's not raining" is a contradiction
  - p cannot be both TRUE and FALSE at the same time

Exercise 10:

Applying the laws covered so far, calculate the truth values for the following logical propositions:

1. $(\neg \text{ true}) \land \text{false}$
2. $(\neg \text{ true}) \land \text{false} \land (\neg \text{ false})$
3. $(\neg \text{ false}) \land p \land \text{true}$
4. $(\neg \text{ true}) \land \text{true}$
5. $(\neg \text{ true}) \land p \land (\neg \text{ false})$

- Hint:
  - Law 1.1: $(\neg\text{true}) \Longleftrightarrow \text{false}$
  - Law 1.2: $(\neg\neg p) \Longleftrightarrow p$
  - Law 2.1: $(p \land p) \Longleftrightarrow p$
  - Law 2.2: $(p \land \text{true}) \Longleftrightarrow p$
  - Law 2.3: $(\text{p} \land \text{false}) \Longleftrightarrow \text{false}$
  - Law 2.4: $(\text{p} \land (\neg\text{p})) \Longleftrightarrow \text{false}$

Commutativity of Conjunction

Law 2.5:

Conjunction is commutative

$$p \wedge q \Longleftrightarrow (q \wedge p)$$

- Commutativity means order of the operands (p and q) does not make a difference

Associativity of Conjunction

Law 2.6:

Conjunction is associative. Thus for three atomic propositions p, q, r:

$$p \wedge (q \wedge r) \Longleftrightarrow (p \wedge q) \wedge r$$

- Associativity means the order of performing the operations (the order of the brackets) does not make a differererence.
  - Compare to addition: a + (b + c) = (a + b) + c

Exercise 11:

Applying the Laws covered so far, calculate the truth values for the following logical propositions:

1. $(\neg (true) \wedge p) \wedge q$
2. $p \wedge (false \wedge q)$

- Hint:
  - Law 1.1: $(\neg \text{true}) \iff \text{false}$
  - Law 1.2: $(\neg\neg p) \iff p$
  - Law 2.1: $(p \wedge p) \iff p$
  - Law 2.2: $(p \wedge \text{true}) \iff p$
  - Law 2.3: $(p \wedge \text{false}) \iff \text{false}$
  - Law 2.4: $(p \wedge (\neg p)) \iff \text{false}$
  - Law 2.5: $(p \wedge q) \iff (p \wedge q)$
  - Law 2.6: $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$

Disjunction

Disjunction

- Disjunction is equivalent to the English word “or” (though not exclusive or)
- A disjunction is TRUE if and only if at least one operand is TRUE.
- For example, the conditions for wearing a coat might be represented:
  - $(p \vee q)$

  where

  - p stands for ‘it is raining’
  - q stands for ‘its cold outside’,

p	q	$\boldsymbol{p \vee q}$
true	true	**true**
true	false	**true**
false	true	**true**
false	false	**false**

Exercise 11:

Work out the truth value for the following disjunctions:

1. (1 + 1 = 2) ∨ (1 > 0)
2. (1 < 0) ∨ (3 > 0)
3. (¬ (1 + 1)) = (3 ∨ 1 > 0)
4. (0 ≤ 0) ∨ (3 + 2 + 1 = 6)
5. (¬¬(1 + 1 = 2)) ∨ (1 > 0)
6. (3 + 4 ≠ 7) ∨ (¬ (1 > 0))

p	q	$\boldsymbol{p} \vee \boldsymbol{q}$
true	true	**true**
true	false	**true**
false	true	**true**
false	false	**false**

Truth Tables

Exercise 12:

Create truth tables for the following:

1. $\neg(p \vee q)$
2. $(\neg p) \vee (\neg q)$

- Hint:
  - Write columns for each atomic proposition
  - Every combination of truth value assignments should have a row
    - There will be 2^n rows, where *n* is number of propositions
  - Give a column for each step in the calculation

Disjunction

Laws for Disjunction

Disjunction Laws

Exercise 13:

List statements of propositional logic – involving disjunction – that are always true

e.g. $p \vee p$

- Hint:
  - Use laws of conjunction to inspire you
    - Law 2.1: $(p \wedge p) \Longleftrightarrow p$
    - Law 2.2: $(p \wedge \text{true}) \Longleftrightarrow p$
    - Law 2.3: $(p \wedge \text{false}) \Longleftrightarrow \text{false}$
    - Law 2.4: $(p \wedge (\neg p)) \Longleftrightarrow \text{false}$
    - Law 2.5: $(p \wedge q) \Longleftrightarrow (p \wedge q)$
    - Law 2.6: $(p \wedge q) \wedge r \Longleftrightarrow p \wedge (q \wedge r)$

De Morgan’s Laws

Law 3.1:

Conjunction and Disjunction are related by De Morgan’s Laws:

$$\neg(p \wedge q) \iff ((\neg p) \vee (\neg q))$$

$$\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$$

- In English,
  - “Not raining and not snowing” is logically equivalent to ‘not raining or snowing’
  - “Not raining or snowing” is logically equivlant to “not raining and not snowing”

Idempotence of Disjunction

Law 3.2:

Disjunction is idempotent.

$$p \vee p \iff p$$

- Any proposition disjoined with itself is logically equivalent to itself.
  - ‘It’s raining or it’s raining’ is logically equivalent to ‘It’s raining’
- Remember, idempotence means an operation can repeated multiple times without changing the result

Disjunction Identity

Law 3.3:

The value FALSE disjoined with proposition p is logically equivalent to p. Thus:

$$p \lor \text{false} \iff p$$

- In English,
  - "It's raining or FALSE" is equivalent to "It's raining"

Domination Law

Law 3.4:

The value TRUE combined via disjunction with proposition p is logically equivalent to *true*. Thus:

$$p \vee true \Longleftrightarrow true$$

- "It's raining or TRUE" is logically equivalent to TRUE
  - only one operand to disjunction needs to be TRUE for a disjunction to be TRUE

Associativity of Disjunction

Law 3.5:

Disjunction is associative. Thus for three atomic propositions p, q, r:

$$p \lor (q \lor r) \Longleftrightarrow (p \lor q) \lor r$$

- Associativity means that the order in which the operations are performed (determined by the brackets) does not change the final result

Commutativity of Disjunction

Law 3.6:

Disjunction is commutative. Thus:

$$p \vee q \Longleftrightarrow q \vee p$$

- Commutativity means relative positions of operands (p and q) doesn't make a difference
- "It's raining or It's snowing" is equivalent to "It's snowing or it's raining"

Complement Law

Law 3.7:

The disjunction of any proposition p with its own negation results in TRUE

$$(\neg p) \vee p \Longleftrightarrow \text{true}$$

- “Today is Monday or Today is not Monday” is always true.
  - One or other must be true, and if at least one is true, the disjunction is true

Absorption Law

Law 3.8:

Disjunction distributes through conjunction. For propositions p, q and r:

$$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$$

- "I'm want salad, or fish and Chips" is logically equivalent to

  "I want salad or fish, and salad or chips"
- Compare with distributivity of multiplication over addition:
  - k (a + b) = ka + kb

Absorption Law

Law 3.9:

Conjunction distributes through disjunction. For propositions p, q and r:

$$p \wedge (q \vee r) \Longleftrightarrow (p \wedge \ q) \vee (p \wedge r)$$

- “I want salad, and fish or chips” is logically equivalent to

  “I want salad and fish, or salad and chips”
- Compare with distributivity of multiplication over addition:
  - k (a + b) = ka + kb

Exercise 14:

Using the laws covered so far, calculate the truth values for the following logical propositions:

1. $(p \vee \text{false}) \vee \text{true}$
2. $(\neg p) \vee (\text{false} \vee \text{true})$

- Law 3.1: $\neg(p \wedge q) \Longleftrightarrow ((\neg p) \vee (\neg q))$
  $\neg(p \vee q) \Longleftrightarrow ((\neg p) \wedge (\neg q))$
- Law 3.2: $(p \vee p) \Longleftrightarrow p$
- Law 3.3: $(p \vee \text{false}) \Longleftrightarrow p$
- Law 3.4: $(p \vee \text{true}) \Longleftrightarrow \text{true}$
- Law 3.5: $p \vee (q \vee r) \Longleftrightarrow (p \vee q) \vee r$
- Law 3.6: $p \vee q \Longleftrightarrow q \vee p$
- Law 3.7: $((\neg p) \vee p) \Longleftrightarrow true$
- Law 3.8: $p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$
- Law 3.9: $p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$

Implication

Implication (⇒)

- Implication is a conditional
  - p ⇒ q
  - Read "If *p* then *q*", or "*p* implies *q*"
- e.g. "If I am drinking tea then I am happy"
- Importantly when the antecedent (here *p*) is false, the statement becomes **vacuously true**

p	q	$p \Rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	true

Vacuous Truth

- Vacuous truth is a weird idea
  - The statement “all dogs on the moon are blue” is (vacuously) true
- In an implication, there is an antecedent and a consequent
  - If **antecedent**, then **consequent**
- If the antecedent is false, the statement is always (vacuously) true

p	q	$p \Rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

Implication

Exercise 15:

Complete the provided truth table for

1. $(p \wedge q) \Rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T			
T	F			
F	T			
F	F			

Implication

Laws for Implication

Exercise 16:

Do either of the following apply to implication?

1. Commutativity
2. Associativity

- Hint:

p	q	$p \Rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	true

Conditional Identity

Law 4.1:

$$(p \Rightarrow q) \Longleftrightarrow (\neg p \vee q)$$

- In English
  - "If it the canteen serves fish today, I will have the vegan option" is logically equivalent to

    "either the canteen will not serve fish today OR I will have the vegan option"

Implication

Exercise 17:

Calculate the truth value of the following propositions using the laws explored so far:

1. $(p \wedge q) \Rightarrow q$
2. $(p \wedge \text{false}) \Rightarrow \text{true}$

Equivlance

Equivalence ($\iff$)

- Equivalence is the claim that two propositions are logically equivalent, written $p \iff q$
  - Often read "p if and only if q"
  - Sometimes written "p iff q"
- An equivalence is true when **both** truth values are **the same**
  - e.g.this is true:
    - "I am wearing socks" $\iff$ "The sun rises in the east"
  - This is false:
    - "I am wearing socks" $\iff$ "The UK is in the southern hemisphere"

Equivalence

- An equivalence is TRUE when both p and q match:

p	q	$p \iff q$
True	True	True
True	False	False
False	True	False
False	False	True

Equivalence

Laws for Equivalence

Associativity of Equivalence

Law 5.1:

Equivalence is associative:

$$((p \iff q) \iff r) \iff (p \iff (q \iff r))$$

Commutativity of Equivalence

Law 5.2:

Equivalence is commutative:

$$(p \iff q) \iff (q \iff p)$$

Equivalence Laws

Law 5.3:

Every proposition is equivalent to itself:

$$(p \iff p) \iff \text{true}$$

Equivalence Laws

Law 5.4:

No proposition is equivalent to its negation:

$$(p \iff (\neg p)) \iff \mathit{false}$$

Conditional Identity

Law 5.5:

Claiming propositions p and q are equivalent is the same as claiming that p implies q and q implies p

$$(p \iff q) \iff ((p \Rightarrow q) \wedge (q \Rightarrow p))$$

Precidence of Logical Operators

Precedence

- Excepting the presence of brackets, the following operators are applied in the following order:
  1. $\neg$ Negation
  2. $\wedge$ Conjunction
  3. $\vee$ Disjunction
  4. $\Rightarrow$ Implication
  5. $\Longleftrightarrow$ Equivalence
- For example $\neg p \wedge q$ means $(\neg p) \wedge q$ and not $\neg(p \wedge q)$.

Tautologies, Contradictions, and Contingiencies

Tautology

- A tautology is a proposition that is always TRUE no matter the values of the atomic propositions, e.g.
  - $p \Longleftrightarrow p$
  - TRUE
  - $p \vee \neg p$

Contradiction

- A contradiction is a proposition that is always FALSE no regardless of the truth values of it's atomic propositions
  - $p \Longleftrightarrow \neg p$
  - FALSE
  - $p \land \neg p$

Contingency

- A proposition whose truth value may be either TRUE or FALSE depending on the truth values associated with it.

Tautologies, contradictions and contingencies

Exercise 18:

Which of the following is a **tautology**, a **contradiction** and a **contingency**?

1. $p \Rightarrow (q \Rightarrow p)$
2. $(p \vee \neg p) \Rightarrow (q \wedge \neg q)$
3. $(p \vee q) \Rightarrow (p \wedge q)$

- Hint:
  - Tautology = always true
  - Contradiction = always false
  - Contingiency = sometimes true and sometimes false

Proof

Proof in Propositional Logic

- A proof in propositional logic is a series of statements of propositional logic
  - Each statement is a theorem (i.e. something that is proved)
- The statements present a step-by-step argument
  - They must be connected by **inference rules** (the laws of our logic)
- Eventually we relate all of our statements to an axiom
  - We either start with an axiom and derive theorems from it; or
  - Start with the thing to prove and work backwards to an axiom

    (^^ what we will use)

Working forwards from Axioms

- Lets say we have two axioms
  - Rain
  - Rain ⇒ Wet Lawn
- We then can make use of some of our laws of propositional logic to derive new true statements
  - Law 4.1: $(p \Rightarrow q) \Longleftrightarrow ((\neg p) \vee q)$
  - Law 1.1: $(\neg \text{true}) \Longleftrightarrow \text{false}$
  - Law 3.3: $(p \vee \text{false}) \Longleftrightarrow p$

Example Proof

1 Rain → Wet Lawn (Axiom)

2 ¬Rain ∨ WetLawn (Law 4.1, 1)

3 Rain (Axiom)

4 false ∨ Wet Lawn (Law 1.1, 2, 3)

5 Wet Lawn (Law 3.3, 4)*

*Technically, we first need to switch the order using the commutativity of ∨ before Law 3.3 can apply

Hilbert-style Proof

- A Hilbert style proof is a sequence of expressions
  - Start from axioms
  - Work forwards
- Each expression is a theorem
  - It's an axiom
  - It's the conclusion of an inference rule, justified by previous theorems
- For each line, we give a reference to either
  - The axiom
  - The name of the inference rule, and reference to the previous theorems we need for the rule

Example Proof

1	$p \rightarrow q$	Axiom
2	$\neg p \vee q$	Law 4.1, 1
3	p	Axiom
4	$false \vee q$	Law 1.1, 2, 3
5	q	Law 3.3, 4

Equational Reasoning

- It's often easier to start with what we want to prove and work backwards to an axiom
  - If starting from axioms it's not clear what direction to go in
    - Why do we apply Law 4.1 at line 2? Where are we going?
- We still need to infer each statement using the laws of propositional logic
  - So long as we eventually get to an axiom

Finding a Proof

- How do we prove the following statement?

  $p \Rightarrow p$

- We need to know the laws of our logic
  - Having a written copy to reference is a good idea
  - Being able to think logically, practice
- We work to transform/simplify the statement to get it closer to something we can prove

Equational Reasoning

Example Proof

1 $p \Rightarrow p$

= (Law 4.1: Conditional Identity)

2 $\neg p \vee p$

= (Law 3.7: Complement Law)

3 *true*

- For Reference
  - Law 4.1: $(p \Rightarrow q) \Longleftrightarrow (\neg p \vee q)$
  - Law 3.7: $((\neg p) \vee p) \Longleftrightarrow$ true
- As the last line is a theorem, if we have followed our inference rules, everything before must be a theorem

Note how we write the inference rule used between deriving each line

Equational reasoning

Exercise 19:

Prove the following using Equational Reasoning:

1. $(p \Longleftrightarrow \text{true}) \Longleftrightarrow p$

- Hint:
  - Law 5.1: $((p \Longleftrightarrow q) \Longleftrightarrow r) \Longleftrightarrow (p \Longleftrightarrow (q \Longleftrightarrow r))$
  - Law 5.2: $(p \Longleftrightarrow q) \Longleftrightarrow (q \Longleftrightarrow p)$
  - Law 5.3: $(p \Longleftrightarrow p) \Longleftrightarrow \text{true}$

Example Proof

Example Proof

1 $(p \iff \text{true}) \iff p$

= (Law 5.2)

2 $(\text{true} \iff p) \iff p$

= (Law 5.1)

3 $\text{true} \iff (p \iff p)$

= (Law 5.2)

4 $(p \iff p) \iff \text{true}$

(Which is Law 5.3)

- For Reference:
  - Law 5.1: $((p \iff q) \iff r) \iff (p \iff (q \iff r))$
  - Law 5.2: $(p \iff q) \iff (q \iff p)$
  - Law 5.3: $(p \iff p) \iff \text{true}$

Another Example Proof

Example Proof

1 $(p \iff \text{true}) \iff p$

= [Law 5.5]

2 $((p \Rightarrow \text{true}) \land (\text{true} \Rightarrow p)) \iff p$

= [Law 4.1]

3 $(((\neg p) \lor \text{true}) \land ((\neg \text{true}) \lor p)) \iff p$

= [Law 3.4]

4 $(\text{true} \land ((\neg \text{true}) \lor p)) \iff p$

= [Law 1.1]

5 $(\text{true} \land (\text{false} \lor p)) \iff p$

= [Law 3.6]

6 $(\text{true} \land (p \lor \text{false})) \iff p$

= [Law 3.3]

7 $(\text{true} \land p) \iff p$

=[Law 2.5]

8 $p \land \text{true} \iff p$

= [Law 2.2]

9 $p \iff p$

= [Law 2.1]

10 true

Get both sides to match

Another approach is to work on the left side side of the statement, using the same Equational Reasoning format until you get it to match the right side

- Because we know $(p \iff p) \iff \text{true}$

Exercise 20:

Try this for the following:

1. $((p \land \neg q) \Rightarrow q) \iff (\neg p \lor q)$

- Law 1.1: $(\neg \text{true}) \iff \text{false}$
- Law 1.2: $(\neg\neg p) \iff p$
- Law 2.1: $(p \wedge p) \iff p$
- Law 2.2: $(p \wedge \text{true}) \iff p$
- Law 2.3: $(p \wedge \text{false}) \iff \text{false}$
- Law 2.4: $(p \wedge (\neg p)) \iff \text{false}$
- Law 2.5: $(p \wedge q) \iff (p \wedge q)$
- Law 2.6: $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$

- Law 3.1: $\neg(p \wedge q) \iff ((\neg p) \vee (\neg q))$

  $\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$
- Law 3.2: $(p \vee p) \iff p$
- Law 3.3: $(p \vee \text{false}) \iff p$
- Law 3.4: $(p \vee \text{true}) \iff \text{true}$
- Law 3.5: $p \vee (q \vee r) \iff (p \vee q) \vee r$
- Law 3.6: $p \vee q \iff q \vee p$
- Law 3.7: $((\neg p) \vee p) \iff \text{true}$
- Law 3.8: $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
- Law 3.9: $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$

- Law 4.1: $(p \Rightarrow q) \iff (\neg p \vee q)$
- Law 5.1:

  $((p \iff q) \iff r) \iff (p \iff (q \iff r))$
- Law 5.2:

  $(p \iff q) \iff (q \iff p)$
- Law 5.3: $(p \iff p) \iff \text{true}$
- Law 5.4: $(p \iff (\neg p)) \iff \text{false}$

Summary

Propositional Logic

- Propositions
  - Atomic Propositions
  - Truth Values
- Truth Tables
- Tautologies
- Contradictions
- Contingiencies
- Operators
  - Negation
  - Conjunction
  - Disjunction
  - Implication
  - Equivalence
- Proof
  - Equational Reasoning