Mathematics and Problem Solving

Lecture 4

Sequences and Series

> “Mathematics is the language of nature.”
>
> Fibonacci

Overview

- Sigma Notation
- Summation
- Product Notation

Exercise 1:

What is:

1. 1 + 2 + 3 + 4
2. 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + 31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 + 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70 + 71 + 72 + 73 + 74 + 75 + 76 + 77 + 78 + 79 + 80 + 81 + 82 + 83 + 84 + 85 + 86 + 87 + 88 + 89 + 90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 + 100

Sigma Notation

Summation

- Adding up lots of things
- It has a special notation written with the Greek capital letter sigma
  - This allows us to express summations compactly
- We also have tools to solve some such equations easily

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

Solving Exercise 1.2

- Exercise 1.2 would have taken a long time to do by hand
  - (It also took a long time to write out)
- We can express the same thing as a summation
- We can solve some such summations easily

$$\sum_{i=1}^{100} i = \frac{100(1+100)}{2} = 5050$$

Parts of a Summation

$$\sum_{i=1}^{n} a$$

i	-	Variable
=1	-	Lower limit (inclusive)
n	-	Upper Limit (inclusive)
a	-	Formula to sum

Interpreting Summation, Example

$$\sum_{i=1}^{4} i$$

- Let there be a variable *i*
  - At first, *i* = 1
- For each value of *i*
  1) Substitute current value of *i* in the formula
  2) Add formula to running total
  3) Check if *i* is less than 4
     - If so, increment *i* and repeat
     - Else, stop

Interpreting Summation, Example

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  - At first, *i* =1
- For each value of *i*
  1) Substitute current value of *i* in the formula
  2) Add formula to running total
  3) Check if *i* is less than 4
     - If so, increment *i* and repeat
     - Else, stop

Exercise 2:

Solve the following summations

1. $\sum_{i=1}^{10} 1$
2. $\sum_{i=1}^{10} i$
3. $\sum_{i=1}^{5} a$
4. $\sum_{i=1}^{10} i^2$

- Hint:
  - Let there be a variable *i*
    - Let *i* be the lower limit shown (i= …)
  - For each value of *i*
    1) Substitute current value of *i* in the formula
    2) Add formula to running total
    3) Check if *i* is less than upper limit
       - If so, increment *i* and repeat
       - Else, stop

Important to Note

- Solving a summation manually is a bit like a for loop. However,
  - Lower limit (starting value) need not be 0 (or 1, or anything else)
  - Summation is always inclusive of upper limit
- Variable isn't always called *i*
- Formula can be arbitratily complex

Iterating Over Collections

- Can iterate over a collection using indexes
  - If no variable / lower limit / upper limit is given, iterates over entire collection

$$A = \{10, 20, 30, 40\}$$

$$\sum_{i=1}^{4} A_i = 100$$

$$\sum_{i} A_i = 100$$

$$\sum A = 100$$

More Examples

$$\sum_{i=0}^{4} \frac{i}{i+1} = \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} = \frac{163}{60} = 2.7166\bar{6}$$

$$\sum_{i=4}^{6} 2^i x^{2i+1} = 2^4 x^9 + 2^5 x^{11} + 2^6 x^{13} = 16x^9 + 32x^{11} + 64x^{13}$$

$$\sum_{i=1}^{4} f\left(x_i^*\right) = f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + f\left(x_4^*\right)$$

Summing a Constant Sequence

- Lets say we have a constant sequence of numbers, such as:
  - 2, 2, 2, 2, 2, …
- What is the sum of the first n values of this sequence
  - If we have n values, then $n \times 2 = 2n$
- We can express this as a summation
  - We cannot solve this to a number, but we can simplify it to a form that is easier to work with

Exercise 3:

Solve the following summations

1. $\sum_{i=1}^{n} 2$
2. $\sum_{i=1}^{n} a$
3. $\sum_{i=4}^{n} 10$
4. $\sum_{i=0}^{n-1} (a+4)$
5. $\sum_{i=-5}^{n+3} 13k^n$

- Hint:
  - Summations are inclusive of both start and end values
  - Adding up *n* copies of a number *k* is *nk*
  - If upper limit isn't *n*, or lower limit isn't 1, you'll need to work out how many different values *i* can take

Sequences

Sequences and Series

- A sequence is a progression of numbers, e.g.
  - { 1, 10, 4, 2, 4, 77, 6, 4, … }
- If we add them all together, we get a **series**
  - 1 + 10 + 4 + 2 + 4 + 77 + 6 + 4 + …
- Because a series can be arbitrarily long or infinite, we often use sigma notation to express it
- There are special types of sequences that we are going to look at
  - Arithmetic Sequences
  - Geometric Sequences

Notating Sequences

- A sequence can be notated in a number of ways
  - All of the sequences on the right are the same
  - Pay attention to subscripts!
    - $a_{n+1} \neq a_n + 1$
  - Note the similarities to sigma notation in the last one
    - Lower limit (n=1)
    - Upper limit (infinity)

$$\{a_1, a_2, \cdots, a_n, a_{n+1}, \cdots\}$$

$$\{a_n\}$$

$$\{a_n\}_{n=1}^{\infty}$$

Exercise 4:

Give the first 4 values of the following sequences. What is the length of each sequence?

1. $\{n^2 + 3\}_{n=1}^{\infty}$
2. $\{\frac{n}{n+1}\}_{n=1}^{\infty}$
3. $\{n + 3\}_{n=4}^{17}$
4. $\{k + \frac{n}{2}\}_{n=k}^{k+4}$

Terminology about Sequences

- A sequence $\{a_n\}$ is **increasing** if, *for every n*
  - $a_n < a_{n+1}$
  - { 1, 23, 112, 158 } is an increasing sequence
- A sequence $\{a_n\}$ is **decreasing** if, *for every n*
  - $a_n > a_{n+1}$
  - { 202, 150, 3, 0.1} is a decreasing sequence

Exercise 5:

Are the following sequences increasing or decreasing?

1. $\{ 1, 2, 5, 7, 8, 9 \}$
2. $\{ 10, 7, 4, 4, 2 \}$
3. $\{ n, 2n, 3n \}$

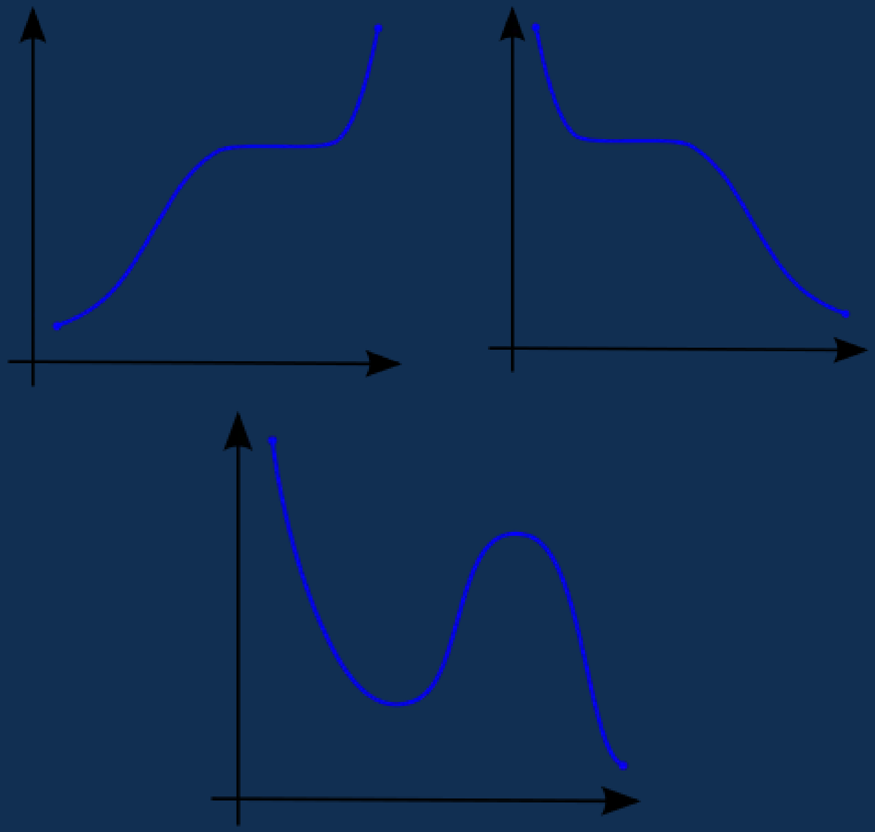
Monotonicity

- Both increasing and decreasing sequences are **monotonic**
  - Monotonically increasing
    - e.g. {1, 2, 3}
  - Monotonically decreasing
    - e.g. {3, 2, 1}
- A sequence is monotonic if doesn't change direction, so we have two more types:
  - Monotonically non-decreasing
    - e.g. { 1, 2, 2, 3}
  - Monotoically non-increasing
    - e.g. {3, 2, 2, 1}

Exercise 6:

Are the functions shown

- Monotonically increasing
- Monotonically decreasing
- Monotonically non-increasing
- Monotonically non-decreasing
- Not monotonic



Exercise 7:

Are the following sequences monotonic, if so, what kind?

1. { 3, 2 , 4, 5 }
2. { 3, 4, 4, 4, 8 }
3. { 43, 22, 22, 1 }
4. { 4, 2, 3, 1 }

- Hint:
  - Monotonically increasing
  - Monotonically decreasing
  - Monotonically non-increasing
  - Monotonically non-decreasing

Bounded Sequences

- If there is a number $m \leq a_n$ for every n (less or equal to any number in the sequence)
  - m is a **lower bound** of the sequence
  - The sequence is **bounded below**
- If there is a number $m \geq a_n$ for every n
  - m is an **upper bound** of the sequence
  - The sequence is **bounded above**
- A sequence bounded both below and above is called **bounded**

Exercise 8:

For the finite sequence { 4, 2, 3, 2 }

1. Is 2 a lower bound?
2. Is 1 a lower bound?
3. Is 3 an upper bound?
4. Is the sequence bounded?
5. Is the sequence monotonic?

- Hint:
  - m is a lower bound if $m \leq a_n$ for every n
  - m is a upper bound if $m \geq a_n$ for every n

Exercise 9:

Consider the sequence

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

1. Give an upper / lower bound if such exist
2. Is it monotonic? If so, what type?

Sequences

Arithmetic Sequences

Arithmetic Sequences

- An Arithmetic Sequence is a list of numbers with an:
  - Initial Term
  - Common Difference
- For example (initial term in red)
  - { 1, 2, 3, 4, 5, 6, … } (common difference 1)
  - { 5, 10, 15, 20, 25, … } (common difference 5)
  - { 3, 6, 9, 12, 15, … } (common difference 3)

Exercise 10:

What is the initial term and common difference for these sequences?

1. { 1, 2, 3, 4, 5, … }
2. { 99, 104, 109, 114, … }
3. { 0.25n, 0.5n, 0.75n, n }

Arithmetic Sequences

- We can define an arithmetic sequence recursively

$$a_n = a_{n-1} + d$$

- Or give an explicit formula

$$a_n = a_1 + d(n-1)$$

Arithmetic Sequences

Exercise 11:

Find the last term of the sequence that starts with 5, has a common difference of 0.25 and has 10 terms.

Give both recursive and explicit formulas to define the sequence

- Hint:
  - We can define an arithmetic sequence recursively
    - $a_n = a_{n-1} + d$
  - Or give an explicit formula
    - $a_n = a_1 + d(n-1)$

Arithmetic Sequences

Exercise 12:

1. What is the implicit formula for the sequences
   - { 2, 3, 4, 5, … }
   - { -1, -3, -5, -7, … }
2. What is the explicit formula for the sequences
   - { 4, 5, 6, 7, … }
   - { 10, 20, 30, 40, … }

Hint:

- $a_n = a_{n-1} + d$
- $a_n = a_1 + d(n-1)$

Sequences

Geometric Sequences

Geometric Sequence

- A Geometric Sequence is a list of numbers with an:
  - Initial Term
  - Common Ratio
- For example
  - { 10, 40, 160, 640, … } (common ratio 4)

Exercse 13:

What is the initial term and common ratio for these sequences?

1. { 1,3,9,27, … }
2. { 1, 0.5, 0.25, 0.125, … }

Geometric Sequence

- We can define a geometric sequence recursively

$$g_n = g_{n-1} \times r$$

- Or explicitly

$$g_n = g_1 \times r^{n-1}$$

Geometric Sequence

Exercise 14:

1. What is the implicit formula for the sequences
   - { 2, 3, 4, 5, … }
   - { -1, -3, -5, -7, … }
2. What is the explicit formula for the sequences
   - { 4, 5, 6, 7, … }
   - { 10, 20, 30, 40, … }

Hint:

$g_n = g_{n-1} \times r$

$g_n = g_1 \times r^{n-1}$

Exercise 15:

Write a formula to sum of the first *n* elements of the following infinite sequences

1. $\{ 5, 7, 9, 11, \ldots \}$
2. $\{4, 8, 16, 32, \ldots \}$

- Hint:

$a_n = a_{n-1} + d$

$a_n = a_1 + d(n-1)$

$g_n = g_{n-1} \times r$

$g_n = g_1 \times r^{n-1}$

$$\sum_{i=1}^{n} a$$

Summation of Series

Common Formulas

- There are formulas for finding the sums of various common series
- You don’t need to remember them, but
  - Remember they exist
  - Look them up if necessary (you might need to use them)

Sum of first n numbers

- We often need to sum the first n natural numbers: 1, 2, 3, 4, …, n
  - 1 + 2 + 3 + 4 + … + n
- Problem: arbitrarily long
  - Easy to write with sigma notation!
- But how do we calculate the sum for a given n?

$$\sum_{i=1}^{n} i$$

Gauss' Method

- Named after Karl Freidrick Gauss
- Sum the numbers 1-100
- Imagine they were paired
  - 1 + 100 = 101
  - 2 + 99 = 101
  - 3 + 98 = 101
  - 4 + 97 = 101
- What does each pair sum to?
  - 101
- How many pairs?
  - 50
- 101 x 50 = 5050

Another Method

- Sum the numbers 1-100
- Imagine they were paired
  - 1 + 100
  - 2 + 99
  - … etc.
- What is the average of a pair?
  - (1 + 100)/2 = 50.5
- How many numbers?
  - 100
- 50.5 x 100 = 5050

Sum of first n numbers

- We can give a formula for this sum
- Where does this come from?
  - Find the average of first and last values,
    - (n+1)/2
  - Multiply this by the length of the series
    - n

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Exercise 16:

What is the sum of the first 20 natural numbers?

- Hint:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Sum of an Arithmetic Series

- We can generalise this for a formula for the sum of the first *n* values of an arithmetic sequence
  - Find average value:
    - $(A_1 + A_n)/2$
  - Multiply by length of sequence *n*
- In place of A_n we can substitute a formula for the *n*th term of the sequence

$$\sum_{i=1}^{n} A_i = \frac{n(A_1 + A_n)}{2}$$

Sum of a Geometric Series

- We can also define the sum of the first *n* values of a Geometric Series
  - Here *r* is the common ratio

$$\sum_{i=1}^{n} G_i = \frac{G_1(r^n - 1)}{r - 1}$$

Sum of first n squares

- It is also common to want to sum the first n square numbers
  - You can derive this using simultaneous equations
  - Or just look up the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Exercise 17:

What is the sum of the first 20 natural numbers?

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Arithmetic Properties of Summation

Arithmetic Properties of Summation

- When you cannot resolve a summation to a number, we often need to transform it
  - Factor constants out of a summation
  - Break up summation across a sum or difference
- By transforming it, we can often get it to a state where it is easier to work with
  - e.g. so we can substitute one of the formulas we've seen

Factor Constants out of Summation

- Due to the distributivity of multiplication over addition
  - n(a+b) = na + nb
- We can factor constants out of a series
  - 1c + 2c + 3c = c (1 + 2 + 3)
- This lets us transform summations into more manageable forms

$$\sum_{i=1}^{n} ci = c \sum_{i=1}^{n} i$$

Break up Summation across a Sum

- Due to the associativity of addition
  - (a + b) + c = a + (b + c)
- We can split summations across sums
  - (1 + a) + (2 + b) + (3 + c)

    = (1 + 2 + 3) + (a + b + c)
- This lets us transform summations into more manageable forms

$$\sum_{i=1}^{n}(a+b) = \sum_{i=1}^{n} a + \sum_{i=1}^{n} b$$

Exercise 18:

Simplify the following summations

1. $\sum_{i=1}^{n} 3ci$
2. $\sum_{i=1}^{n} (3i + 4)$
3. $\sum_{i=1}^{n} (2i^2 + 3i)$

Hint:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Product Notation

Product Notation

- To add a sequence we can use sigma notation and use summation
- What if we want to multiply a sequence? e.g.
  - $1 \times 2 \times 3 \times \ldots \times n\text{-}1 \times n$
- Product Notation
  - Uses the Greek capital letter pi, Π

Parts of a Product

- a - formula to multiply
- i - variable
- $=1$ - Lower limit (inclusive)
- n - Upper Limit (inclusive)

$$\prod_{i=1}^{n} a$$

Example Product

- Multiply the first 5 elements of the sequence a:
  - $a_1 \times \ldots \times a_5$
- Lower limit = 1
- Upper Limit = 5
- Variable = *k*
- Formula to multiply = a_k

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5 .$$

Product of a constant

- The product of *n* values of a constant *c* is given by the n^{th} power of *c*
- The product of, e.g.
  - $10 \times 10 \times 10 \times 10 = 10^4 = 10000$
  - $c \times c \times c = c^3$

$$\prod_{k=1}^{n} c = c^n$$

Exercise 19:

Compute the following products:

1. $\prod_{k=1}^{4} 2$
2. $\prod_{k=1}^{4} k$
3. $\prod_{k=1}^{3} \frac{k}{k+1}$
4. $\prod_{k=0}^{7} 2^k$

Summary

Summary

- Sigma Notation – Summation
- Product Notation
- Sequences
- Summation of a Series

Further reading

- Deriving the sum of squares formula: https://trans4mind.com/personal_development/mathematics/series/sumNaturalSquares.htm