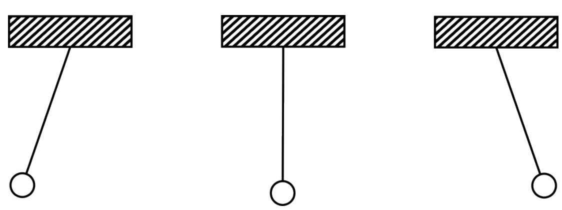
Chapter 9

Dynamical Systems

Mathematics can be used to model the real-world. A key property that many-real world systems exhibit is that they change over time. A dynamical system is any system where state of the system and perhaps even the rules that the system obeys (also called its *dynamics*) change as a function of time.

A pendulum at rest is not a dynamical system, as it does not change; however if you give it a push it will start swinging: the position and velocity of the pendulum will change over time. The pendulum is a dynamical system and the changes we observe are not random; they are governed by rules.



We can use dynamical systems to model complex, real-world systems that change over time and in response to a control signal. This is used in the design of aircraft, launching rockets into space, developing control software for robots and much more. In games, dynamical systems are used for physics simulations. The stock market, and the human brain are also dynamical systems. In addition, dynamical systems also gives us general tools to construct and explore systems with novel behaviour.

A dynamical system is defined by a function F. This function takes a state of our system x at a given time t and returns the state of the system at time $t + 1$.[1]

[1] In this lecture we are only going to be working with dynamical systems where our state x is a scalar, i.e. a number. Our state can also be represened by a vector of arbitrary size. Each element in our vector represents one dimension of our state, thus our state can encode as many values as we like. This allows us to build far more interesting dynamical systems, but mathematically they also get much more complex.

$$x_{t+1} = F(x_t)$$

We could also include many other things as parameters to this function if we wished. For example, F might depend on the current time t (i.e. the rules that govern the system change over time), or if we are actively interfering in the dynamical system (e.g. moving a robot's legs to stabalise its balance) we might include a control paramter.

This sort of equation is called a *difference equation*. This equation will (if we knew F) tell us the value of a given x_{t+1} in terms of the previous value of x_t. We will be assuming our dynamical system works in discrete time intervals (that is when we are working with timesteps of appreciable size). If we wanted to work with continuous time t, we would use *differential equations*:

$$\frac{dx}{dt} = F(x)$$

Here the $\frac{dx}{dt}$ is understood as meaning the change in x with respect to t for arbitrarily small increments of t. However, to work with this sort of equation we need to use calculus, which is not within the scope of this course.

Before we get to considering dynamical systems proper, we will consider sequences. Conceptually these are very similar.

9.1 Sequences

A sequence is an ordered list of numbers. Note that we write sequences between braces { }. Confusingly, this is the same as how we notate sets. However, in a sequence we *can* have repeted elements. For example, consider the following sequence:

$$a = \{4, 2, 2, 6, 8, 3, 5\}$$

Here the variable a equals this particular sequence. We often need to refer to elements of sequences. To refer to the first element in the sequence a we would write a_1. The second element is a_2, and the nth element is a_n.

Sequences can be finite (like the one above) or infinite. A sequence might represent a set of values that have been gathered from a real-world application – a particular collection of arbitrary numbers – or it might be purely theoretical and defined by a particular formula. For example, a sequence might contain:

- the closing price of a given stock for the last 365 days;
- the sum of the first n natural numbers, for increasing values of n;
- the average yearly rainfall for the last 20 years.

Each of these sequences is useful. A stock market trading algorithm might use the former in developing a mathematical model of the movements of the stock and thus work out the best time to buy or sell. The sum of the natural numbers is a formula that is needed in various places in mathematics, such as simplifying summations. A climate scientist might want to find a formula fitting the last sequence to predict rainfall for future years.

A sequence is *increasing* if the numbers always get bigger. That is if, for two successive values of a, a_k and $a_k + 1$ it is the case that $a_k < a_{k+1}$. A sequence is *non-decreasing* if the numbers never get smaller, thus for a_k and a_{k+1}, $a_k \leq a_{k+1}$. As you might expect, a sequence is *decreasing* if $a_k > a_{k+1}$ and it is *non-increasing* if $a_k \geq a_{k+1}$. When a sequence is either increasing, non-decreasing, non-increasing, or decreasing, it never changes direction. This property of never changing direction is called *monotonicity*. This such a sequence would be called *monotonic*.

Sequences may (or may not) be *bounded*. Upper and lower bounds are values that are either above or below *all* values in the sequence. An upper bound is a value m such that $m \geq a_n$ for every n. Similarly, a lower bound is a value m such that $m \leq a_n$, for every n. For the sequence $\{2, 3, 4\}$ we can give an infinite number of upper and lower bounds. Upper bounds include 4, 10, and 972. Lower bounds include 2, 1, and -123. A sequence which has both upper and lower bounds is called *bounded*.

9.1.1 Notating Sequences

As with sets, we can take two approaches to notating sequences. We can write out each element, i.e. give an extensional definition. Obviously this only works when the number of elements is relatively small, e.g.

$$\{1, 2, 3, 4\}$$

Alternatively, we can write a formula for the sequence. This is necessary when dealing with infinite sequences. One notation for giving the formula for a sequence is as follows:

$$\{f(n)\}_{n=1}^{\infty}$$

The above formula has three parts. First, the value of a given term is given by the funtion $f(n)$. The second part defines the values of n. At the bottom, we see the lower bound of n. Here n starts at 1. At the top, we see the upper board. Here n continues to infinity, so this is an infinite sequence. We could define the squares of the natural numbers between $n = 2$ and $n = 4$ this way:

$$\{n^2\}_{n=2}^{6} = \{4, 9, 16, 25, 36\}$$

It might be helpful to visualise this with a table:

n	2	3	4	5	6
n^2	4	9	16	25	36

Recursive Definition

We could also define a sequence by giving its initial term and a rule to generate subsequent terms. This is called a *recursive definition.* This formula is said to be in *function iteration form.*

$$a_n = 2 \times a_{n-1} \qquad a_1 = 1$$

$$a = \{1, 2, 4, 8, 16, 32, \dots\}$$

Here we have really defined an infinite set of equations for all values of n. In each, the value of a_n is related to the value of a_{n-1}. Thus to work out the value of the sequence for any n, we would need to recursively substitute all of the formulas back to a_1. This is also called finding a numerical solution of this sequence. For long or complex sequences this can be slow.

$$a_2 = 2 \times a_1 \qquad a_3 = 2 \times a_2 \qquad a_4 = 2 \times a_3 \qquad a_5 = 2 \times a_4 \qquad \dots$$

$$a_5 = 2 \times 2 \times 2 \times 2 \times 1$$

Explicit Definition

Sometimes we can identify an explicit definition for a sequence. This is a formula for any value a_n which does not refer to any other values of a. Taking a sequence and finding such an explcit formula is also called an analytic solution. For example, the explicit definition for the above sequence a is:

$$a_n = 2^n$$

This formula is easier to work with as no recursion is required. It allows us to quickly find the value of the sequence for n. However, in practice it can be hard to find such a solution to a sequence. It is generally easier to work out how a system changes relative to its previous state, than find a solution for any state as an explicit formula.

9.1.2 Arithmetic Sequences

Some sequences with well-known characteristics have names. An *arithmetic sequence* is a particular kind of sequence completely defined by an initial value and a common difference. Say you have a bank account with a starting balance of £4 (initial value) and you deposit £2 each day (common difference). The money in the account at the end of each day would follow the sequence:

$$a = \{4, 6, 8, 10, \dots\}$$

The sequence $\{4, 6, 8, 10, \dots\}$ is given by the recursive definition below. The value of a at each step n is the previous value, a_{n-1}, plus 2. Note that we need to specify the value for a_1 as that is the first element in the sequence and it cannot be recursively defined in terms of anything else.

$$a_n = a_{n-1} + 2 \qquad a_1 = 4$$

With our sequence so defined, we can find a numerical solution for the 5th value in this sequence. We do this by taking the formula above where $n = 5$ and substituing the formula for a_4, then a_3, then a_2, and finally a_1. This gives us the result below:

$$a_5 = 4 + 2 + 2 + 2 + 2 = 12$$

Alternatively, we can define the same sequence with an explicit formula. Here we start with an inital value of 4, and add on our common difference 2 a number of times equal to $n - 1$:

$$a_n = 4 + 2(n - 1)$$

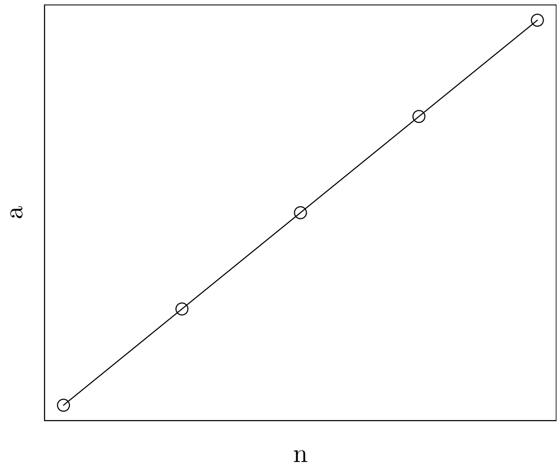
Using this formula, the fifth value of the sequence a can be found as a function of n, without recursion:

$$a_5 = 4 + 2(5 - 1) = 12$$

The difference between successive terms of an arithmetic sequence is constant. In the above sequence, the difference between each pair of terms $\Delta a_n = a_{n+1} - a_n = 2$. The symbol Δ (the Greek capital letter delta) is used to refer to a difference.

n	1	2	3	4	5
a_n	4	6	8	10	12
Δa_n	2	2	2	2	2

If we plot this sequence on a graph, you will see that it is increasing a constant rate. As the differences between each step are constant, the gradient of the graph is constant.



We can give both recursive and explicit formulas for any arithmetic sequence. Here a_n is the nth value of the sequence a, a_1 is the initial value, and d is the common difference. By substituting values of a_1 and d we can get the formulas for any arithmetic sequence.

$$a_n = a_{n-1} + d$$
$$a_n = a_1 + d(n-1)$$

9.1.3 Geometric Sequences

A *geometric sequence* on the other hand is defined by an initial value and a common ratio. Geometric sequences can be defined by can also give recursive or explicit formuas:

$$g_n = g_{n-1} \times r$$
$$g_n = g_1 \times r^{n-1}$$

For example, with an initial value of 2 and a common ratio of 3, we get the sequence that starts:

$$g = \{2, 6, 18, 54, 162, \dots\}$$

The recursive and explicit formulas for this sequence would be:

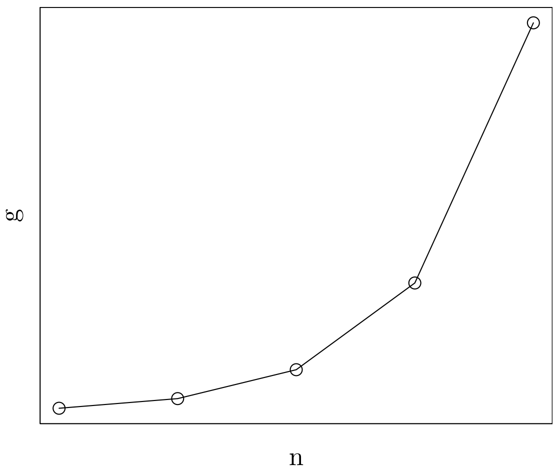
$$g_1 = 2$$
$$g_n = g_{n-1} \times 3$$
$$g_n = 2 \times 3^{n-1}$$

The difference between successive terms of a geometric sequence is not constant. In the above sequence, the difference between each pair of terms Δg_n

is also a geometric sequence! This means that the difference between pairs of differences $\Delta\Delta g_n$ is also geometric! All of these sequences are increasing exponentially.

n	1	2	3	4	5
g_n	2	6	18	54	162
Δg_n	4	12	36	108	
$\Delta\Delta g_n$	8	24	72		

We can see this exponential growth if we plot g_n on a graph:



This is the same pattern we observe for compound interest. If we put money in a bank account and earned %1 interest each day, the money at the end of each day would follow the geometric sequence $m_n = 4 \times 1.01^n$.

$$m = \{4, 4.04, 4.0804, 4.121204, 4.16241604, 4.2040402004, 4.246080602404, \ldots\}$$

Exercise 1 Give the first 4 values of the following sequences. What is the length of each sequence?

1. $\{n^2 + 3\}_{n=1}^{\infty}$
2. $\{\frac{n}{n+1}\}_{n=1}^{\infty}$
3. $\{n + 3\}_{n=4}^{17}$
4. $\{k + \frac{n}{2}\}_{n=k}^{k+4}$

Exercise 2 For the following sequences:

- state whether they are increasing, non-decreasing, non-increasing, or decreasing
- state whether they are monotonic and/or bounded
- give an upper and lower bound for each
- state if they are arithmetic sequences, and if so give their initial term and common difference
- state if they are geometric sequences, and if so give their initial term and common ratio
- where possible, give both recursive and explicit formulas for each

1. $\{1, 2, 5, 7, 8, 9\}$
2. $\{10, 7, 4, 4, 2\}$
3. $\{n, 2n, 3n\}$
4. $\{3, 4, 4, 4, 8\}$
5. $\{\frac{n}{n+1}\}_{n=1}^{\infty}$
6. The sequence that starts with 5, has a common difference of 0.25 and has 10 terms.
7. The sequence that starts with 1, has a common ratio of 2 and has 8 terms.

9.2 Dynamical Systems

A dynamical system is a system (i.e. a thing that can be in multiple states) that changes its state as a function of time. Discrete dynamical systems can be described by systems of difference equations. These equations represents some variable or state that changes between discrete time steps $t = 0$, $t = 1$, etc. For example:

$$a_{t+1} = a_t + 1$$

The above dynamical system defines an infinite set of equations that relate the value of a at time $t + 1$ as a function of the value of a at time t. In this example, the value of a always increases by 1 each time step. As t can keep increasing indefinitely, this set of equations is infinite.

$$a_1 = a_0 + 1 \qquad a_2 = a_2 + 1 \qquad a_3 = a_3 + 1 \quad \ldots$$

You may have noticed that the value of a follows an arithmetic sequence, just like we have seen above. Each successive value in the sequence is the state of our dynamical system at each subsequent time step. The nth value of the sequence is the state of the dynamical system at time t. The only difference is that we usually take the initial state in a dynamical system to be $t = 0$ rather than $n = 1$.

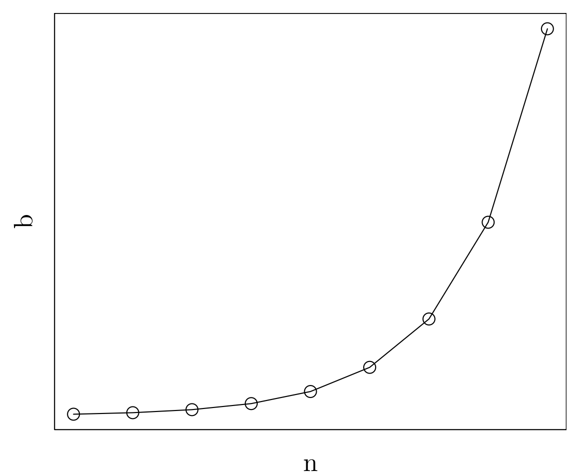
9.2.1 Example of Bacteria Growth

Imagine we were modelling the number of bacteria in a petri dish. At time 0 there is 1 bacteria in the dish, that is $b_0 = 1$. At each time step, the value of b (the number of bacteria in the dish) doubles. The number of bacteria in the dish can be modelled by the dynamical system:

$$b_0 = 1$$

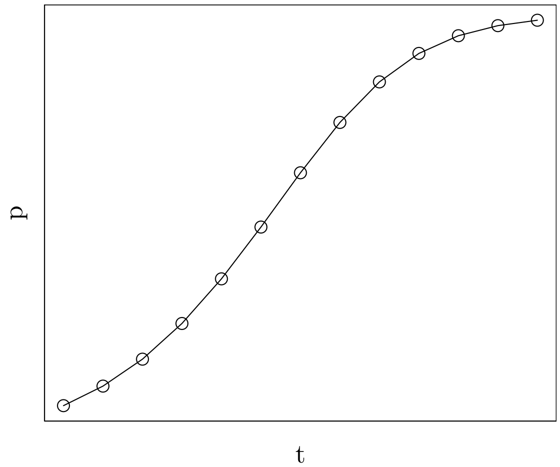
$$b_{t+1} = 2b_t$$

This dynamical system defines the geometric sequence $\{1, 2, 4, 8, 16, \dots\}$ where the starting index of the sequence is 0, thus $b_0 = 1$. If we plot this on a graph we see a familiar exponential growth:



Will this system really keep increasing exponentially forever? A better model might be given by p below. Again we have defined p_{t+1} in terms of p_t.

$$p_{t+1} = p_t + k(c - p_t)p_t \qquad k = 0.05 \qquad c = 10$$



This equation is a more complicated model of the dynamical system. However the behaviour of this model are perhaps more realistic. Here c is the carrying capacity. The number of bacteria in the dish increases but slows as it approaches the limit of the number of bacteria the food source can support.

9.2.2 Numerical Solutions

If we have a dynamical system with a state variable x and dynamics defined by function F, we can find soutions for a given x_t by repeatedly applying F.

$$\begin{aligned} x_{t+1} &= F(x_t) \\ x_1 &= F(x_0) \\ x_2 &= F(F(x_0)) \\ x_3 &= F(F(F(x_0))) \end{aligned}$$

Take for example the model of bacteria growth bounded by a carrying capacity given above:

$$F_p(x) = x + k(c - x)x \qquad k = 0.05 \qquad c = 10$$

We can work out a numerical solution for the number of bacteria in the dish after 5 time steps have passed by repeatedly applying this function to its own output:

$$\begin{aligned} F(1) &= 1.45 \\ F(1.45) &= 2.069875 \\ F(2.069875) &= 2.89059337421875 \\ F(2.89059337421875) &= 3.91811355857426 \\ F(3.91811355857426) &= 5.10958964496722 \end{aligned}$$

Exercise 3 In a user interface the opacity of a button after it is clicked changes from 1 to 0 over 20 frames. The first attempt at this animation used the following model:

$$o_{t+1} = o_t - \frac{1}{20}$$

Using a spreadhseet, work out the values of the opacity o for frames 0 (the value before the animation), frame 1 (the first frame of animation) to 20 (the final frame of animation). Then using your spreadsheet package, produce a graph of this sequence.

This animation does not look particularly good. Often animations look a lot better if they are smoothed so that the rate of change speeds up and then slows down. This smooths the step between the two end values. Given this, a second attempt at the animation used this updated model.

$$o_{t+1} = o_t + 0.3(o_t - 1.2)o_t$$

Again, work out the values of opacity for frames 0 to 20 using a spreadsheet and produce a graph of it.

Exercise 4 The temperature of a room after a number of minutes t is modelled by the following dynamical system. The starting temperature of the room is given by z_0.

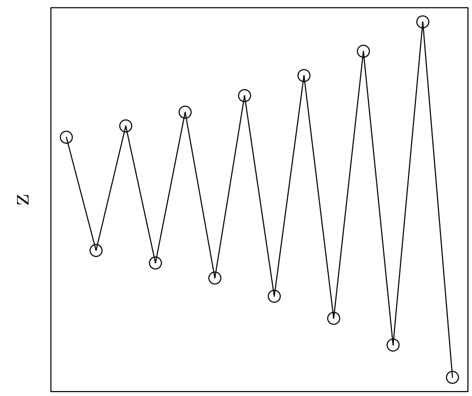
$$z_{t+1} = 0.8z_t + 4$$

Create a spreadsheet to calculate the values of z_t where $z_0 = 15$ up to $t = 16$ and plot these on a graph. After approximately how many minutes does the temperature reach 19?

An *equilibrium value* is a value of a dynamical system where the system does not change. Explore different starting values of z_0 (i.e. different starting temperatures) for this system by substituting them in your spreadsheet. Use this to find the equilibrium value for this system.

Exercise 5 Make a copy of your spreadsheet for Exercise 4. In this copy, replace the difference equation for z with a difference equation of your own devising. Your should be able to get the spreadsheet to update the calculated values and re-generate the graph to let you rapidly visualise the behaviour of this system.

Your difference equation z_{t+1} can be any function of z_t you like. See what kinds of behaviour you can get it to exhibit and make a note of anything interesting you find. For instance, can you get it to oscilate between pairs of values as shown in the graph below?



9.3 Dynamical Systems with Multiple Dimensions

The dynamical systems we have seen involve only a single scalar value, such as opacity or temperature. Many real-world dynamical systems involve more than a single variable. For instance, to model a ball rolling down a slope we need to model both its position and its velocity.

To do this mathematically we need to replace our variable x with a vector $\underline{x}$. Vectors contain multiple values. This significantly complicates the definition of F as we would need to use vector and matrix arithmetic.

$$\underline{x}_{t+1} = F(\underline{x}_t)$$

However, we can extend our spreadsheets to model systems with multiple variables without worrying about vector and matrix math. Instead of having a single column of values of x_t at increasing values of t, we have multiple variables, each with their own column. A simple way to express this mathematically as a pair of difference equations that reference each other.

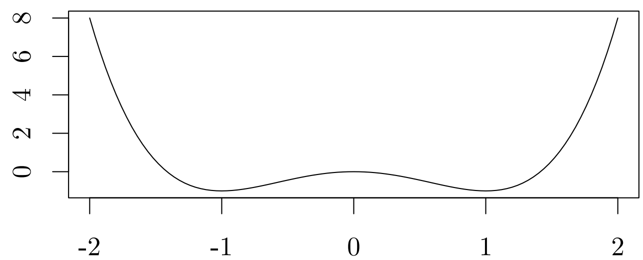
$$a_{t+1} = F_1(a_t, b_t)$$
$$b_{t+1} = F_2(b_t, a_t)$$

For example, using a dynamical system, we could model the movement of a bead strung on a wire. Imagine we had a wire whose position through two-dimensional space is given by the function[2].

$$y = x^4 - 2x^2$$

[2] To explore functions like this more fully, you might wish to use a graphing calculator such as www.desmos.com/calculator/

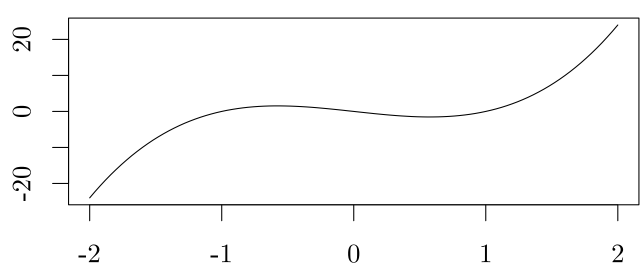
To make it easier to understand, let's plot this function on a graph:



The bead is going to move downhill from its starting position. By differentiating the above function, we get the function for the gradient of the wire at any point. Again, differentiation is part of calculus, which is beyond the scope of this course, so trust me when I say the gradient of the above function at any point x is given by the function:

$$\frac{dy}{dt} = 4x^3 - 4x$$

Which looks like this:



If the bead is going to behaive as it would in the real world, it will accelerate proportionally to this gradient, i.e. its velocity will increase (well, decrease, as it is going down hill) acording to the function above. The displacement u of the bead in the x dimension is just the last position plus the velocity. The displacement is given by the difference equation:

$$u_{t+1} = u_t + v_t$$

Where the velocity v of the bead is given by the difference equation:

$$v_{t+1} = (v_t + T(-(4u_t^3 - 4u_t))) \times F$$

which incorporates the formula for $\frac{dy}{dt}$ above. T is a small number representing the time scale of each step in our discrete dynamical system. F is a fractional value that represents friction.

Exercise 6 Construct a numerical solution to this dynamical system using a spreadsheet, for the values of T and F shown:

$$u_{t+1} = u_t + v_t$$

$$v_{t+1} = (v_t + 0.01(-(4u_t^3 - 4u_t))) \times 0.9 \qquad v_0 = 0$$

Explore different starting values of displacement u_0, i.e. different starting positions for the bead on this wire. Where does the bead need to start to observe the behaviour shown below?