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QCD phase transition in cosmology

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Abstract

Nice abstract. Very nice indeed!!

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1 Introduction

According to the ΛCDM model, a cosmological phase transition between quark-gluon plasma and hadron phase took place at $t \approx 10^{-5}$ s with $T \sim 160 - 180$ MeV. This phase transition may have induced inhomogeneities in the baryon-to-photon ratio and in the ratio of protons and neutrons, opening the possibility to influence Big Bang Nucleosynthesis (BBN).

BBN is the process in which the lightest elements, such as ^{4}He, D, ^{3}He and ^{7}Li, were produced in the early Universe (from 10 s to 20 min at a temperature of $T \sim 0.1 - 1$ MeV). BBN is one of the main observational evidences for the ΛCDM model, the other ones being the Cosmic Microwave Background and the expansion of the Universe. The abundances of the lightest elements can be calculated with the Standard Big Bang Nucleosynthesis (SBBN) model, which has a single free parameter, the baryon-to-photon ratio η, and is based on the Standard Model physics alone.

Despite being the simplest model which is in good agreement with astronomical observations, SBBN poses a problem of fine-tuning: the predicted abundances can vary if the conditions or the physics at that time are changed. Also, this model has disagreements with other methods used to estimate the value of η [1]. The objective of nonstandard BBN models is to alleviate these problems. For instance, the Inhomogeneous Big Bang Nucleosynthesis model (IBBN) assumes the existence of regions with high and low baryon densities during the process of nucleosynthesis, consequently allowing η to be a function of position, and adding the separation of the high-density regions as another parameter for the model. It is in this context that the QCD phase transition could influence primordial nucleosynthesis, giving an explanation for such inhomogeneities.

If the QCD phase transition is of first order then, according to nucleation theory, it would have proceeded with the formation of bubbles of hadronic matter in the quark-gluon plasma. These bubbles continue to grow during the process and, near the end of the transition, the last shrinking regions would have been filled with a concentrated baryon number due to the difficulty for the baryons to diffuse through the phase boundary [1]. Therefore, the baryon density in the quark-gluon plasma is higher than in the hadron phase, giving a baryon density contrast that will originate the required inhomogeneities for IBBN, if they survive until $T \sim 0.1 - 1$ MeV [2].

In the high-density regions produced by the phase transition, the neutrons are diffused more than the protons because they do not interact electromagnetically with electrons. So in IBBN there is not only inhomogeneities in the baryon number but in the neutron-to-proton ratios: the high-density regions have a lower neutron-to-proton ratio while the low-density regions have a higher neutron-to-proton ratio [3].

In the model of IBBN there are four parameters which can affect nucleosynthesis: the baryon-to-photon ratio, the average density contrast, the average distance between inhomogeneities and the average volume fraction of the high-density regions. A wide set of values for these parameters are examined by Kurki-Suonio *et al* [3] together with observational constrains in the primordial abundances. Their results are summarized in a baryon-to-photon ratio ranging from 2.2×10^{-10} to 7×10^{-10} for reasonable values of the baryon density contrast (which has an upper bound of 100 if chemical equilibrium is attained during the phase transition) and for a mean separation of the high-density regions less than 150 m at 100 MeV after the phase transition, and less than 1 m at the time of the QCD phase transition. A more recent estimation of the IBBN parameters was done by Kainulainen *et al* [4]. They got similar values of η depending on the bounds for the abundances. Nevertheless, they concluded that the average separation of the high-density regions has to be much greater than 150 m at 1

MeV, which means that it is needed an average separation of inhomogeneities (at ~ 150 MeV) several orders of magnitude greater than the typical distance scale of the QCD phase transition calculated with lattice methods[1]. However, the uncertainties involved allow the theory to have this disagreement with the calculations, making IBBN a still-viable explanation for the formation of the lightest elements in the Universe.

In this paper we are going to... c=1=kb

Checar trabajos recientes en el tema

2 Cosmology

Modern cosmology studies the properties of the universe as a whole at scales of the order of few 100 Mpc, with the aim of understanding its origin and evolution. It tries to solve fundamental questions, such as the predominance of matter over antimatter, the origin of the chemical elements and how the structures in the Universe were made, just to name a few.

The standard cosmological model is based on the *cosmological principle*, which states that the Universe is homogeneous and isotropic, and on the assumption that our physical models are valid at all times throughout the Universe. The theory of General Relativity is used to describe the dynamics of the Universe; we need a metric $g_{\mu\nu}$ which describes the geometry of the Universe and an energy-momentum tensor $T_{\mu\nu}$ representing its physical content, together with the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ and R are respectively the Ricci tensor and Ricci scalar and G is Newton's gravitational constant.

It can be proven that the cosmological principle is satisfied by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - k\, r^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right), \tag{2}$$

where r, θ and ϕ are the usual spherical coordinates, t is the *cosmic time*, $a(t)$ is the *scale factor* and k is a constant representing the curvature of space. These coordinates are known as *comoving coordinates* and do not change with the expansion of the Universe. In a *comoving frame* the matter of the Universe which only moves with the expansion, is at rest, so it can be defined a *comoving volume* which remains constant with the expansion. In this volume the number densities of objects in rest with respect to the comoving frame, remain constant. For $k = 0$ we have a spatially-flat Universe (Euclidean space), for $k = 1$ we have a positively curved Universe (elliptical space) and for $k = -1$ we have a negatively curved Universe (hyperbolic space).

The assumptions of homogeneity and isotropy also set strong constraints on the energy-momentum tensor [5], which has to take the form of that of an ideal gas:

$$T_{\mu\nu} = [\epsilon(t) + P(t)] U_\mu U_\nu + P(t)\, g_{\mu\nu}\,, \tag{3}$$

where $\epsilon(t)$ is the energy density, $P(t)$ is the pressure of the fluid and U_μ is the four-velocity of the fluid, which can have more than one component or *species*. If these components are non-interacting, then we will have that $\epsilon(t) = \sum \epsilon_i(t)$ and $P(t) = \sum P_i(t)$.

[1]pongo valor? **Sí. En el peor de los casos, lo quitamos luego.**

The dynamics of the Universe can be established with the three equations above. The results are the *Friedmann equations*:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\epsilon\,, \tag{4a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\epsilon + 3P\right)\,. \tag{4b}$$

Combining equations (4a) and (4b), or equivalently, using the more general relation $T^{\mu\nu}{}_{;\nu} = 0$ from the Einstein field equations, we get a third equation:

$$\frac{d\epsilon}{dt} = -3\,\frac{1}{a}\frac{da}{dt}\left(\epsilon + P\right). \tag{5}$$

Nevertheless, there are three variables (a, ϵ, P) for just two independent equations. To complete the system, an *equation of state* is introduced, which relates the density with the pressure of each component of the fluid through the relation:

$$P_i(t) = \omega_i\,\epsilon_i(t), \tag{6}$$

where ω_i is a value that depends on the nature of the component of the fluid. Cosmological observations suggest the existence of three principal components of the Universe: relativistic particles ($\omega = 1/3$), non-relativistic particles ($\omega = 0$) and dark energy ($\omega = -1$).

3 Phase transitions

Boiling water or making ice in a fridge may be an everyday phenomenon but these phenomena have something in common with more exotic ones like superconductivity, superfluidity and the subject of this work. The common thing between these is that they go through a phase transition. A phase transition occurs when a thermodynamic system undergoes through a drastic change of its external properties, for example it may drop its resistance to zero (superconductivity) or it may go through a compressible object to an incompressible (condensation). In thermodynamics, the several types of phase transitions are classified according to the mathematical properties of the free energy $F = U - TS$, where U is the total energy, S is the entropy and T the temperature [6]. We say that the phase transition is of the n-th order when the n-derivative with respect to one of the thermodynamic variables of the free energy is discontinuous and the temperature at which this happens is called the critical temperature T_c. For example, the transition of water from liquid to gas is a first order transition and has the qualitative behaviour shown in Figure 1.

There are some transitions called crossovers where there is no discontinuity in any derivative of the free energy. This transitions may point that the indicator that the system is undergoing a phase transition are not the derivatives of the free energy. Landau realized that when a phase transition occurs there is a symmetry breaking in the system [6]. Furthermore, the symmetry breaking can be studied quantitatively through a quantity called the order parameter of the system. For example, a ferromagnet can be modeled as a 2-D cuadratic lattice of N atoms each carrying either spin $\frac{1}{2}$ or $-\frac{1}{2}$. If we start at a temperature $T < T_c$, the spins are aligned in a preferred direction and there is a non-zero magnetization M as shown in Figure 2(a) . If we start increasing the temperature, then at some critical temperature T_c, the spins will now point in a random way and the magnetization is drops to zero as shown in Figure 2(b). The

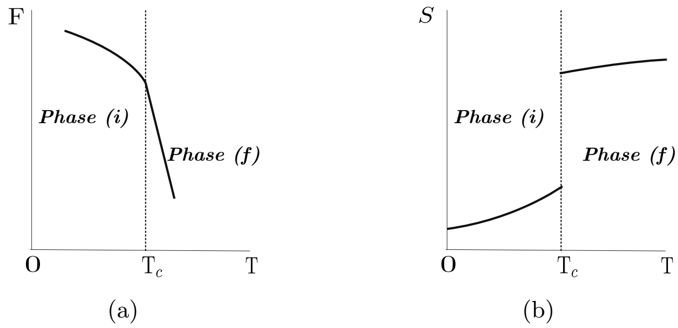
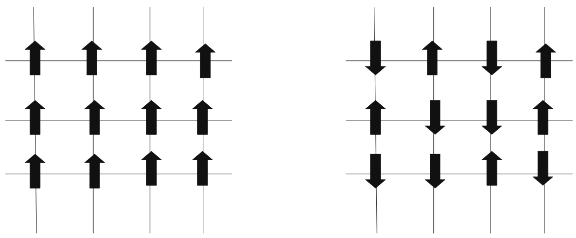


Figure 1: The qualitive behaviour of a first order phase transition from phase (i) to phase (f) is shown. (a) The graph of the free energy F as a function of the temperature T. (b) The free energy F has a discontinuity at one of its first derivatives $S = -\frac{\partial F}{\partial T}$ [7].

key in all of this is that for temperatures higher than the critical temperature, a symmetry is present as the system remains unchanged by the transformation $S_i \to -S_i$. In the other hand, for temperatures below the critical temperature, the system no longer has the symmetry and we say the symmetry was spontaneously broken.



(a) $T < T_c$; $M \neq 0$.

(b) $T > T_c$; $M = 0$.

Figure 2: The second order phase transition that appears in ferromagnetic materials. (a) The ferromagnet has a non-zero magnetization $M \neq 0$ for $T < T_c$ and the symmetry is broken. (b) The ferromagnet has a zero magnetization $M = 0$ for $T > T_c$ and the symmetry is restored.

4 The QCD Phase Transition

As far as we know, there are four fundamental forces in nature: the weak and strong nuclear force, gravity and electromagnetism. The strong nuclear force is responsible for gluing the protons and neutrons inside the nucleus overcoming the electromagnetic repulsive force between them. The theory that describes this force is called quantum chromodynamics. The particles that mediate this force are called gluons and have influence on particles with a color charge. The particles that have a color charge are the quarks which are Dirac fermions. Quantum chromodynamics is a Yang-mills theory with gauge group $SU(3)$ which has 8 generators corresponding to the 8 gluons.

The main difference between quantum chromodynamics and electromagnetism is the fact that free quarks are not found in nature but joined in groups that make zero color charge such as baryons and mesons forming an hadron gas. Such phenomenon is called confinement. This occurs because the force between color-charged objects increases with the distance. This is called asymptotic freedom. Nevertheless, for high energies the strong force decreases and quarks and gluons can deconfine from the colorless objects and exist as free particles in a state named as quark-gluon plasma QGP. This can also be achieved with high temperatures T or high baryon chemical potential μ_B. Although quarks were free particles, they still strongly interact in the QGP. For this reason, it resembles relativistic fluid (cite).

Additionally, QCD underwent another phase transition called chiral phase transition. Roughly speaking, when the quark masses are approximately zero, if one transforms the right and left hand-side of the quark Dirac spinor independently the theory is unaffected and we have a chiral symmetry. In the present universe the chiral symmetry is broken by the non-zero masses, so we said it is an approximately symmetry due to the small quark masses. Nevertheless, this approximately symmetry was broken giving place to the chiral phase transition. Similar to the confinement-deconfinement phase, this phase transition occurs for high temperature or baryon chemical potentials.

According to Chapter 3, there may be an order parameter that indicates when these transitions occur. In the case of confinement, the order parameter is the trace of the Polyakov Loop. And in the case of the quiral phase transition, the order parameter is the condensate of quarks [8]. It is not the object of this work to give an extensive study of this properties but rather just give a qualitative overview just to make clear what the model that will be given is doing.

8¡In order to get the evolution of the cosmological QCD phase transition we need to know the densities and pressures of the confined and deconfined phases. Let us consider the QGP and the hadron gas as a weakly interacting gas of particles. Then, the energy density ρ for one of the phases is given by [**?**]

$$\rho = \frac{g}{(2\pi)^2} \int d^3p f(p) E(p), \tag{7}$$

where g is the number of degrees of freedom of the particles, $E(p) = \sqrt{p^2 + m^2}$ for relativistic particles and $f(p)$ is the Fermi-Dirac distribution for fermions and the Bose-Einstein

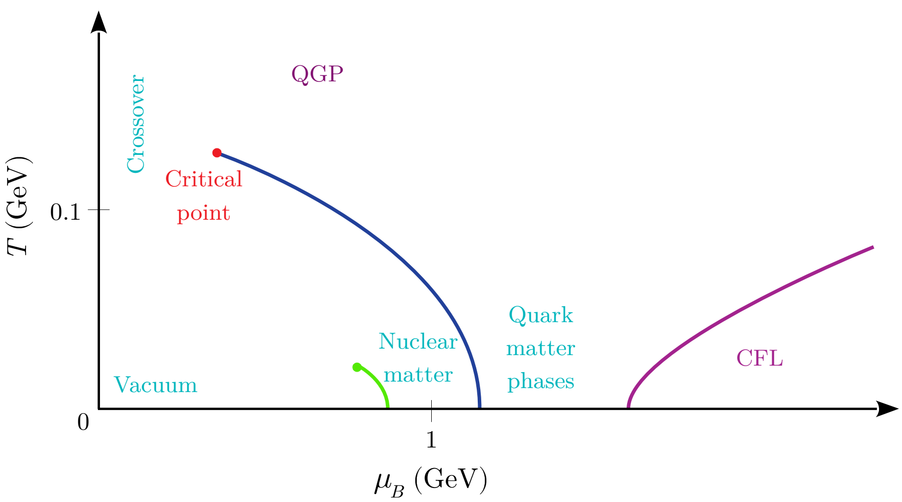


Figure 3: The QCD phase diagram for the chiral phase transition. The point where the transition changes from a crossover to a first order is called the critical point. CFL stands for color flavor locking and is another possibly state of matter for QCD. Nevertheless, it is irrelevant for this work. [9]

distribution for bosons

$$f(p) = \frac{1}{e^{\frac{E(p)-\mu}{T}} \pm 1}, \tag{8}$$

where $+$ is for the fermions and the $-$ for bosons. Using eq. (8) in (7) yields

$$\rho = \frac{\pi^2}{30} g T^3 \times \begin{cases} 1, & \text{bosons.} \\ \frac{7}{8}, & \text{fermions.} \end{cases} \tag{9}$$

Assuming both of the phases are perfect fluids, they must satisfy $P = w\rho$, where $w = \frac{1}{3}$ because we are considering in both phases that the particles are relativistic and then, they must behave like radiation. We have

$$\rho = \frac{\pi^2}{30} T^3 \times \begin{cases} g_{QGP}, & \text{QGP .} \\ g_h, & \text{hadron gas.} \end{cases} \quad \text{and} \quad P = \frac{\pi^2}{90} T^3 \times \begin{cases} g_{QGP}, & \text{QGP.} \\ g_h, & \text{hadron gas.} \end{cases}, \tag{10}$$

where $g_{QGP} = \frac{7}{8} g_{QGP,f} + g_{QGP,b}$ and $g_{QGP,f}$ are the fermionic degrees of freedom for the QGP and $g_{QGP,b}$ the bosonic degrees of freedom for the QGP. And similarly g_h for the hadron gas phase. Let us determine both g's.

In both phases we have photons, neutrinos, electons and muons which have a lower mass than the QCD phase transition temperature (which is $T_c = 160 MeV$ for this work). This common number of degrees of freedom for both phases is

$$g_{both} = \frac{7}{8} \left(\underbrace{6}_{neutrinos} + \underbrace{4}_{electrons} + \underbrace{4}_{muons} \right) + \underbrace{2}_{photons} = 14.25, \tag{11}$$

where we have counted the spin, the particle/antiparticle possibilities; and the fact that there are only left-handed neutrinos in the standard model.

In the case of the quark gluon-plasma, we have three quarks (up, down an strange) not yet annihilated by the fall of the temperature of the universe, 8 massless gluons. So we have

$$g_{QGP} = \frac{7}{8} (\underbrace{3 \times 3 \times 2 \times 2}_{quarks}) + (\underbrace{8 \times 2}_{gluons}) + g_{both} = 47.5 + 14.25 = 61.75, \tag{12}$$

where we have counted the possible spin, the particle/antiparticle degrees of freedom, the color variations for the quarks and the two possible polarizations of the gluons.

For the hadronic state we count only three pions which are the only relativistic hadrons below the critical temperature. So

$$g_h = \underbrace{3}_{pions} + g_{both} = 17.25 \tag{13}$$

One final ingredient for the ρ's and P's is missing. The main difference between the QGP and the hadronic phases is the fact that the quarks are confined in hadrons. One way to account for this is adding a negative presusre B (or bag constant) to the QGP pressure P_{QGP} that reflects the fact that hadrons are stopped from being free by an inward pressure; and sum the same bag constant B to the energy density ρ_{QGP} to account for the fact that we now have confined states in the hadronic phase. This is known as the MIT Bag Model. Therefore,

$$\rho = \begin{cases} \frac{\pi^2}{30} g T^4 (61.75) + B, & \text{QGP.} \\ \frac{\pi^2}{30} g T^4 (17.25), & \text{fermions.} \end{cases} \quad \text{and} \quad P = \begin{cases} \frac{\pi^2}{90} g T^4 (61.75) - B, & \text{QGP.} \\ \frac{\pi^2}{90} g T^4 (17.25), & \text{fermions.} \end{cases}. \tag{14}$$

5 Nucleation Theory

If we want to study the dynamics of the QCD phase transition assuming it was first order, we must use nucleation theory. Suppose we have a substance in a high temperature phase H and we start to cool it until we reach its critical temperature T_c where the phase transition begins. At this point bubbles of a lower temperature phase L will start to form in the substance and will start to grow until all the substance is now in the L phase. The main goal of nucleation theory is giving an expression for the nucleation rate or the probability of forming a bubble of critical radius r_* per unit time per unit volume. There are two types of nucleation theory that apply for different scenarios. Homogeneous nucleation theory is used when the system is pure and inhomogeneous nucleation theory when the formation of droplets and bubbles is caused by impurities. In the case of the cosmological QCD phase transition, homogeneous nucleation theory is more appropriate because this was the first QCD phase transition ever made. Therefore, it was impossible to have impurities (hadrons in this case) that could cause the bubble formations.

To have a rough idea of what is (quantum) nucleation theory consider a quantum system with wavefunction ϕ and the potential shown in Figure 4. This potential $U(\phi)$ has two degenerate minima ϕ_L and ϕ_R. For $T > T_c$ the minimum ϕ_H corresponds to the high temperature phase because is the least energy configuration available. The other minimum ϕ_L is the vacuum corresponding to the low temperature phase. The basic idea of quantum nucleation theory is that if the system is in ϕ_H or near it, then it has a probability $I \sim e^{-S}$, where S is the action evaluated in a path that goes from ϕ_H to ϕ_L and back, to go to the other vacuum ϕ_L. As T approaches T_c, the path between both vacua shortens and the probability increases until $T < T_c$ and we have a similar potential like in Figure 4 but with ϕ_H higher than ϕ_L. This probability is basically the nucleation rate I.

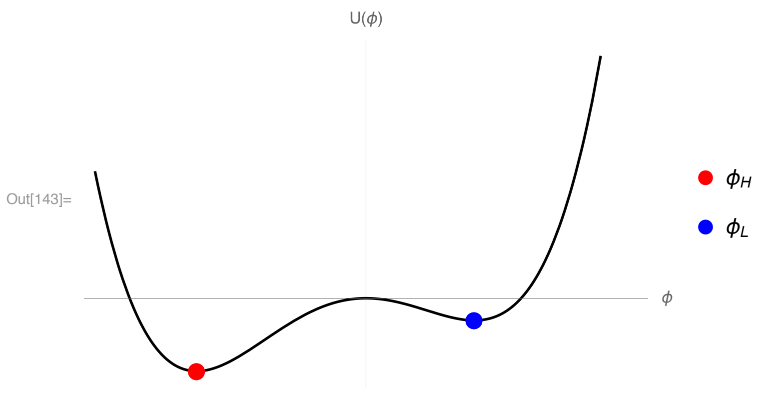


Figure 4: Potential for ϕ.

Nevertheless, the nucleation rate used in this work was computed by Csernai and Kapusta [10] using Langers formalism for classical systems [11] and relativistic hydrodynamics. The nucleation rate is

$$I = \frac{4}{\pi}\left(\frac{\sigma}{3T}\right)^{3/2}\frac{\sigma(3\zeta_A + 4\eta_A)r_*}{3(\Delta\omega)^2\xi^4}e^{-\Delta F_*/T}, \tag{15}$$

where σ is the surface tension of the bubbles, T is the temperature, ζ is the bulk viscosity of the phase inside the bubbles, η is the shear viscosity of the this phase, $\Delta\omega$ is the difference of enthalpy density $\omega = P + \rho$ between the phases and ξ is the correlation length of the bubble that measures the length at which the thickness of the bubble stops to be correlated between two points.

6 QCD phase transition equations

The modern approach to the QCD phase transition is to model it through nucleation theory, where the L-phase is made of *spherical* hadronic *bubbles* that are formed on the QGP.

These bubbles are mainly modeled using classical ideas from thermodynamics and deformable media.

The first thing to consider is the surface tension σ of a bubble, defined as the ratio of the work W to the bubble's surface change ΔA due to W, *ie.* σ is a factor of proportionality such that the work W required to expand a bubble by ΔA is simply $\sigma \Delta A$[2].

()Therefore, the work required to *create* a bubble *out of nowhere* has the contribution σS, with S the bubble's surface area.

The interior and exterior of the bubbles correspond to different phases, so in principle there may be a difference of pressure at the surface of the bubble. This tells us that to nucleate a bubble of volume V, another contribution to the work must be of the form $V\Delta P$. Furthermore, because the internal and external pressures are in opposition, we expect that the greater P_{ext} is compared to P_{int} the more work is required, so

$$\Delta P = P_{ext} - P_{int} = P_q - P_h$$

The total work required to create a bubble is then,

$$W = \sigma S + V\left(P_q - P_h\right)$$

Because at constant T the change in the Helmholtz free energy ΔF satisfies $\Delta F = W$, then for an *spherical* growing bubble

$$\Delta F = 4\pi r^2 \sigma + \frac{4\pi}{3} r^3 (P_{ext} - P_{int}).$$

At equilibrium ΔF reaches a minimum ΔF_* given by

$$r_* = \frac{2\sigma}{P_{int} - P_{ext}}, \tag{16}$$

that corresponds to a critical-size radius of the form

$$\Delta F_* = \frac{4\pi}{3} \sigma r_*^3 \,. \tag{17}$$

The relevance of equations (16) and (17) is that they allow us to compute the nucleation rate (15) by means of (14).

Another quantity of obvious interest is the fraction $h(t)$ of volume occupied by the hadronic phase, which has the form $h(t) = \frac{V_h(t)}{\Omega(t)}$, where $\Omega(t)$ is the total volume. From now on, we shall work on the comoving frame, so that Ω is constant.

We shall now derive an expression for the hadronic volume $V_h(t)$ that will automatically provide us with one for $h(t)$ itself.

[2] Explicar por qu es lineal?

The volume of a bubble at time t must also depend on its original size (determined by the critical radius), or equivalently on the time t' at which it was created, so that in general we have a function of the form $V(t,t')$ which considers the initial volume $V(t',t')$ and the expansion from t' to t.

A naive way to compute $V_h(t)$ would be to *add* the contributions of $V(t,t')$ at different times t'. However, this would not consider how many bubbles are created at t', or more precisely, on the interval $[t', t'+dt']$. This is where the nucleation rate comes into play, because it quantifies how many bubbles are successfully nucleated per unit time per unit volume, which implies that the number of bubbles of critical sized formed on $[t', t'+dt']$ is given by

$$dt' I(t') V_q(t') \tag{18}$$

with V_q the volume occupied by the QGP phase. The presence of this factor is due to the fact that bubbles can only be formed on the QGP phase (there is no sense on them forming on the hadronic phase).

Now, the fraction of volume occupied by the QGP at t' is by construction $1 - h(t')$ so

$$V_q(t') = (1 - h(t'))\Omega\,. \tag{19}$$

Considering the previous discussion, it must be that

$$h(t)\Omega := V_h(t) = \int_{t_c}^{t} dt' I(t') V_q(t') V(t,t') = \int_{t_c}^{t} dt' I(t')(1-h(t'))\Omega V(t,t')\,,$$

where t_c is the critical time at which the transition occurs[3]. From this equation it is evident that

$$h(t) \;=\; \int_{t_c}^{t} dt' I(t')(1-h(t'))V(t,t') \tag{20}$$

Eq. (20) is an integral equation for $h(t)$ that, when coupled to the cosmological evolution of the universe, provides with the aspects of the QCD phase transition that we are interested on. However, before doing the coupling, we need an explicit expression for $V(t,t')$, which can be obtained by rewriting V as

$$V(t,t') = \frac{4}{3}\pi(r(t') + \Delta r(t,t'))^3 = \frac{4}{3}\pi\left(r_*(t') + \Delta r(t,t')\right)^3$$

So that $\Delta r(t,t')$ accounts for the expansion of the bubble. Therefore, if $v(t'')$ is the speed of the expanding surface of the bubble, then $\Delta r\left(T(t')\right) = \int_{t'}^{t} dt'' v(t'')$.

Thus,

$$V(t,t') = \frac{4}{3}\pi\left(r_*\left(t'\right) + \int_{t'}^{t} dt'' v\left(T(t'')\right)\right)^3 \tag{21}$$

//Choro de $v(T)$ para explicar a $V(t',t)$ (José)

Another quantity that may be of interest is the number of nucleated bubbles at time t, $N(t)$; or its density $n(t)$. Their expressions may be derived similarly to that of $h(t)$, because equation (18) is the number of bubbles nucleated between t' and $t'+dt'$ and therefore, from equation (19) it follows that

$$\frac{N(t)}{\Omega} = \int_{t_c}^{t} dt' I(t')(1-h(t'))\,.$$

[3]**En algn momento hay que mencionar la normalizacin que usamos para R.**

Now a question arises: is $\frac{N(t)}{\Omega}$ to be interpreted as a number density $n(t)$, or should $n(t) = \frac{N(t)}{V_h(t)}$?

The latter option may naively make sense, specially noting that a quantity such as the mass density is the ratio of mass to volume occupied by that mass.

However, the information we require from $n(t)$ (or any X-density for that matter) is the average number of particles (quantity X) given an arbitrary portion of space or volume. This is illustrated on Figure 5, where the red region is arbitrary, but note that if we consider it to occupy the *total* volume, then when multiplying by ω we would get the total number of particles as we should.

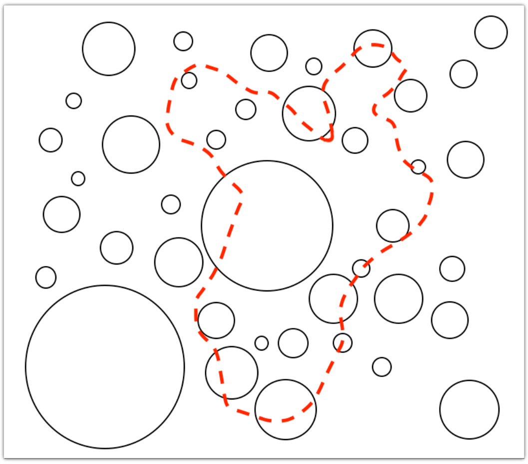


Figure 5: Cartoon of the phase transition. The number density must be obtained using the total volume Ω, as this allows us to know the average number of bubbles on an arbitrary sub-region (red).

The discussion above implies that $n(t)$ should be computed using Ω and therefore, it satisfies equation (22)

$$n(t) = \int_{t_c}^{t} dt' I(t')(1 - h(t')) \,. \tag{22}$$

Hitherto we have not coupled the time dependence of $h(t)$ to the dynamics of space-time, this can be achieved by using equations (4a) and (5).

First we note that equation (5) implies that

$$c^2 d\rho = -3\frac{da}{a}(c^2\rho + P) \,.$$

If we work in units such that $c = 1$, then in the comoving frame $c^2\rho$ is simply the energy density ϵ and the equation above implies that

$$\frac{d\epsilon}{da} = -\frac{3\omega}{a} \,, \tag{23}$$

where $\omega = \epsilon + P$ is the enthalpy density.
Note that unlike the $\Delta\omega$ on (15), ω is the enthalpy density of the hole system and the analogous is true for ϵ. Given the intensive nature of this quantities, they must be given by the sum of their parts (bubbles and QGP) weighted by their respective volume fraction[4], i.e.

$$\begin{aligned} \epsilon(t) &= h(t)e_h(t) + (1 - h(t))e_q(t) \\ \omega(t) &= h(t)\omega_h(t) + (1 - h(t))\omega_q(t) \end{aligned} \tag{24}$$

[4]If not clear, multiply by Ω.

Equation (23) couples our model to a through (24). However, due to the time integration on (20) it is still necessary to relate a with t. This is precisely what equation (4a) does.
As discussed in 2, the k on equation (4a) relates to the curvature of spacetime. Because this transition happened after inflation, we can consider k to be zero and therefore equation (4a) reduces to

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}\epsilon}\,. \tag{25}$$

7 Method and Algorithm

Eqs. (20) and (23) define a system of integro-differential equations that determine the cosmological evolution of the phase transition. We can now proceed to solve numerically this system by recurring to some of the intermediate expressions presented on this section and 4.

The other elements needed to compute the interaction rate I of eq. (15) are particular for each type of first order phase-transition. The enthalpy density is $\omega = \epsilon + p$, where p is the pressure and ϵ is the energy density. Therefore, we need the energy density and the pressure density for both the hadronic phase h and the quark-gluon plasma phase q.

7.1 Algorithm

Equations (20) and (25) provide us with a set of coupled integro-differential equations, which must be solved numerically. These equation take into account bubble nucleation and growth, energy conservation, and Einsteins equations.

First, we need to perform a change of variable in eq. 20 with help of eq. 23 and expand eq. 25, which leads to

$$h(a) = \int_{a_c}^{a} da' \frac{1}{a'}\sqrt{\frac{3}{8\pi G\epsilon(a')}}\; I(T(a'))(1-h(a'))V(a,a'), \tag{26}$$

$$\frac{dT}{da} = -\frac{\frac{dh}{da}\left(\frac{g_h\pi^2}{30}T^4(a) - \frac{g_q\pi^2}{30}T^4(a) - B\right) + \frac{3}{a}\left[\frac{2g_h\pi^2}{45}h(a)T^4(a) + (1-h(a))\left(\frac{2g_q\pi^2}{45}T^4(a)\right)\right]}{\frac{4g_h\pi^2}{30}h(a)T^3(a) + (1-h(a))\frac{4g_q\pi^2}{30}T^3(a)}. \tag{27}$$

Now, we can approximate the integrals with a left Riemann sum. So let $P = \{[a_0,a_1],[a_1,a_2],\dots[a_{N-1},a_N]\}$ be a uniform partition of $[a_c,a]$, with $a_0 = a_c$, $a_N = a$ and $a_k = a_c + k\Delta a$, then

$$\begin{aligned} h(a_k) &= \int_{a_c}^{a_k} da' \frac{1}{a'}\sqrt{\frac{3}{8\pi G\epsilon(a')}}\; I(T(a'))(1-h(a'))V(a,a') \\ &\approx \sum_{i=0}^{k-1} \Delta a \frac{1}{a_i}\sqrt{\frac{3}{8\pi G\epsilon(a_i)}}\; I(T_i)(1-h_i)V(a_k,a_i), \end{aligned}$$

where $T_i = T(a_i)$ and $h_i = h(a_i)$. Now, in order to calculate $V(a_k,a_i)$ we have to do a change of variable in eq. (21) with with help of eq. (23), which gives us

$$V(a_k,a_i) = \frac{4\pi}{3}\left(r_*(T_i) + 3\int_{a_i}^{a_k} ds \frac{1}{a'}\sqrt{\frac{3}{8\pi G\epsilon(a')}}\; \left(1 - \frac{T(a')}{T_c}\right)^{3/2}\right)^3$$

$$\approx \frac{4\pi}{3}\left(r_*(T_i) + 3\sum_{j=i}^{k-1} \Delta a \, \frac{1}{a_j}\sqrt{\frac{3}{8\pi G\epsilon(T_j)}}\left(1 - \frac{T_j}{T_c}\right)^{3/2}\right)^3 .$$

Now, we can solve eq. (27) with Euler's method, that is,

$$T_i = T_{i-1} + \Delta a f(a_{i-1}, T_{i-1}, h_{i-1}),$$

where $f(a_{i-1}, T_{i-1}, h_{i-1})$ is the right side of equation (27). In order to calculate $f(a_{i-1}, T_{i-1}, h_{i-1})$ we need $\frac{dh(a_{i-1})}{da}$, which we can calculate with a backward difference

$$\frac{dh(a_{i-1})}{da} \approx \frac{h_{i-1} - h_{i-2}}{\Delta a}.$$

These approximations are in terms of the previous steps, so we have an explicit method, and therefore the algorithm builds the solution in each step using all the previous steps. The error in the Riemann sum is $\mathcal{O}(\Delta a)$, and the error in Euler's method is also $\mathcal{O}(\Delta a)$, so the total error in the algorithm is $\mathcal{O}(\Delta a)$.

Algorithm to solve a system of coupled integro-differential equations

---

Data: Δa, a_{final}

Result: Two arrays with the numerical solutions for T and h

Set $T = [T_c]$ and $h = [0]$, which are the arrays with the initial conditions

Set $a = a_c$ and $i = 1$

while $a \leq a_{final}$ **do**

  if $i = 1$ **then**

    $\frac{dh}{da} = 0$

  else

    $\frac{dh}{da} = \frac{h_{i-1} - h_{i-2}}{da}$

  $T_i = T_{i-1} + \Delta a f(a_{i-1}, T_{i-1}, h_{i-1})$, where f is the right side of eq. 27

  $$h_i = \sum_{j=0}^{i-1} \Delta a \frac{1}{a_j}\sqrt{\frac{3}{8\pi G\epsilon(T_j)}}\; I(T_j)(1 - h_j)V(a_i, a_j)$$

  $a_i = a_{i-1} + \Delta a$

  if $h > 1$ **then**

    break

  Append T_i to array T and h_i to array h

  $i = i + 1$

---

8 Results

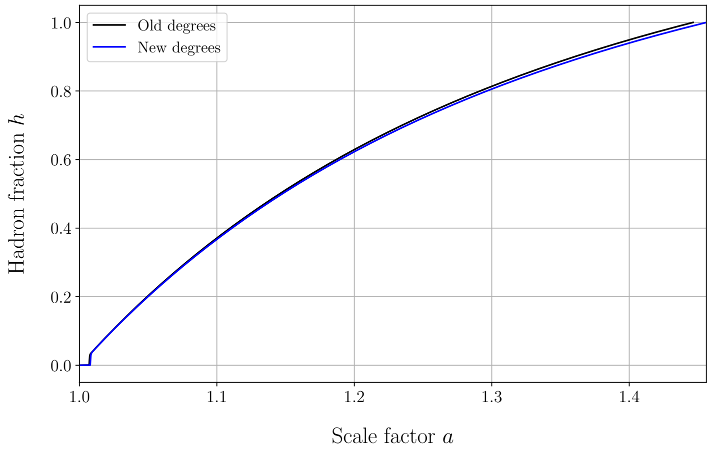


Figure 6: Hadron fraction

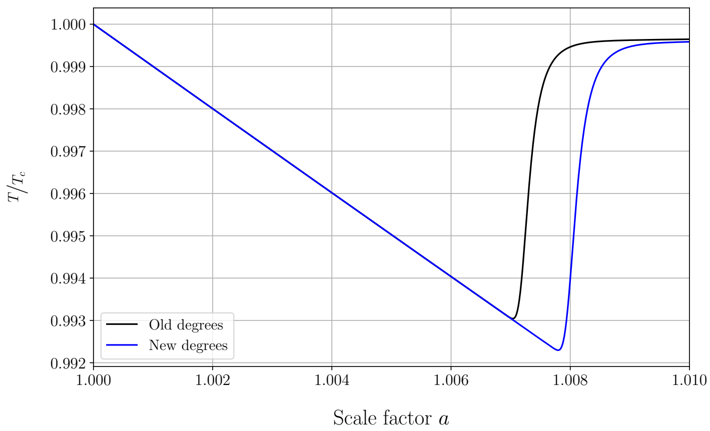


Figure 7: Temperature

9 Final remarks and outlook

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