Experimentally Determining the Relationship Between the Mass of an Object and its Kelvin Wake Angle

How does the mass of a moving object in water affect its wake angle and wave speed?

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IB Physics HL

May 2019

Personal Engagement:

Due to my interest in mathematics, I often find myself browsing different university's mathematics department websites. When browsing Cambridge Mathematics' webpage, I read a claim that stated that the angle a duck makes with deep water, also known as the angle of the wake of a body is a constant value of $2\sin^{-1}(\frac{1}{3}) \approx 38.9°$ (Undergraduate Mathematics). This at first seemed ludicrous to me, why such a specific angle? Would the velocity of the propagated wave be affected by this angle? I was walking around a lake in Bowen Island, B.C. with my father, when I happened to see ducks waddling in the water and noticed that their waves indeed extended at a specific angle. This curious observation prompted me to further research this topic.

Background information:

A wake is defined as the wave pattern emanating downstream of an object in a flow or produced by a moving object (e.g. a ship), caused by pressure differences of the fluids (*Wake*). The Kelvin wave angle is the phenomenon behind this observation. First studied by Lord Kelvin, the Kelvin wake angle consists of a V-shaped pattern made up of multiple individual waves (Rožman, 2009). This phenomenon states that the angle of the wake of an object is independent of its velocity and independent of its mass (Kai-kwong, 2013). Furthermore, according to T.E. Faber, a relationship exists between the wake angle, the velocity of the object making the waves, and the wave speed (Faber, 1995).

This relationship will be utilized throughout the data analysis section in order to obtain the wave speed. The purpose of my internal assessment is not to prove the kelvin wake angle, but rather to see if it holds in real life, by varying the masses of a given object in a body of water. The duck shown below left is a clear example of the kelvin wake problem showing the different waves emanating from it (Rožman) and the wake and wake angle (image modified by the author). The image shown below right represents the emanating wave as a mathematical model (Georgi, 1993).

Figure 1. A duck in deep water

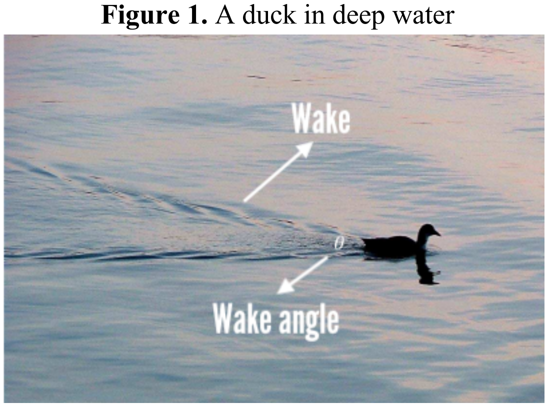
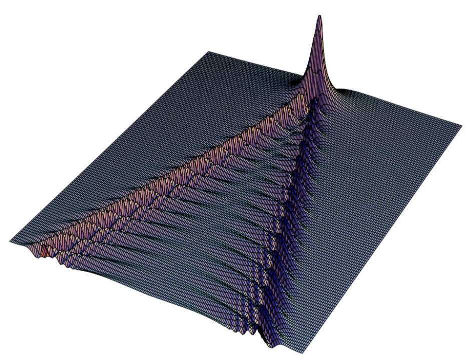


Figure 2. A mathematical model of the wake of an object.



Research Question:

The purpose of this experiment is to examine the angle of the wave emanating from an object as it travels in a body of water with a constant velocity, and subsequently the wave speed. This purpose develops into the research question: *How does the mass of a moving object in water affect its wake angle and wave speed?*

A wake angle is defined is the angle contained by the ends of the emanating wave, a body of water may vary in length ranging from shallow (depth less than 10 cm) to deep (depth greater than 50 cm), and the wave speed is measured in *m/s*.

Hypothesis:

If the mass of a moving object in water is changed whilst its velocity is kept constant, then it is hypothesized that the wake angle and wave speed will undergo no significant change.

Method of investigation:

As my research question states, the relationship between the mass of an object and its wake angle, and subsequently its wave speed was researched. This was done by utilizing a plastic boat and attaching it to a fishing line and hook. A mass was placed on the plastic toy boat and the boat was reeled in at a constant velocity while filmed with a camera. The experiment was divided into two parts; the first part was performed on a shallow body of water on the roof of my garage, whereas the second part was performed on a deeper body of water in my jacuzzi. The variation in depth was not the dependent variable, but rather provided more bodies of water in which to conduct my experiment.

Each trial's respective wake angles were analyzed, and it was observed whether mass scales positively, negatively, or neutrally linear with the wake angle. In order to reduce random error, each sample was repeated for three to four trials before varying the mass. The number of trials was not further increased in order to limit the total time taken for the experiment.

Materials:

- Plastic Toy Boat
- Plastic Pyrex container
- Varying masses (See Table 2.)
- Fishing hook
- Fishing line
- Gloves
- Fishing reel
- Fishing rod
- Tripod
- Ladder
- A video camera
- Measuring tape
- Specific video analysis software (ImageJ)

Ethics and Safety:

Before conducting this experiment there are a number of ethical and safety concerns that must first be addressed.

If a small boat with a loud motor noise is launched in a naturally occurring pond or lake, it can negatively affect the local biotic life by decreasing their living space and drawing them away from their home. To mitigate this problem, a simple plastic toy was used, and experimentation was confined to within the bounds of private property.

Secondly, improper disposal of the plastic boat and plastic Pyrex jar can pollute living areas and cause negative health benefits to surrounding people and animals, thus these apparatus were reused instead of discarded. In regard to safety, conducting an experiment on the roof of a garage poses significant risk as falling off of the roof could lead to lasting physical damage. Secondly, the sharp hook used to attach the plastic boat to the fishing line is dangerous as it could potentially pierce the experimenter's skin, thus gloves were used whilst performing this task.

Variables:

Independent Variables:
• Angle of wake
• Wave Speed

Dependent Variable:
• Mass of object

Controlled Variables:
• Shape and size of object

Reason: Perhaps a differing shape of an object could lead to a change in the angle of the wake, as the surface area of the object will come into contact with the water at a different angle. Because this property was not observed throughout the experiments, it was controlled so as to limit its potential change on the wake angle.

• Velocity of object

Reason: According to Lord Kelvin the wake angle is hypothesized to be independent of the velocity of the object in water. An attempt was made to maintain the rate of reeling, and subsequently the velocity of the object in the water constant so as to bypass the verification of this assumption.

• Position of video camera

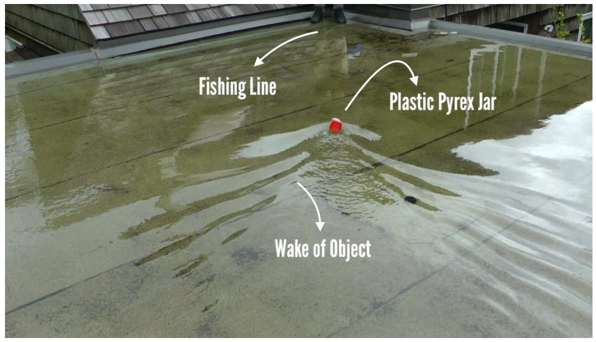
Reason: To have a comparable base for all samples, the position of the camera remained constant throughout every trial as a variation in camera position would affect the angle obtained.

Methodology:

Part 1: Shallow Water

1. Set up tripod on far side of roof and attach the video camera to the tripod.
2. Use protective gloves for the following two steps.
3. Place the fishing hook onto the end of the fishing line attached to the fishing rod.
4. Attach the fishing hook to the plastic Pyrex jar and extend fishing line to opposite end of the body of water.
5. Begin the video recording and reel in the Pyrex jar at a constant velocity.
6. Repeat steps 1-5 with an added weight inside of the plastic Pyrex jar.
7. Analyze the recorded footage through video analysis software to obtain the angle of the object's wake.

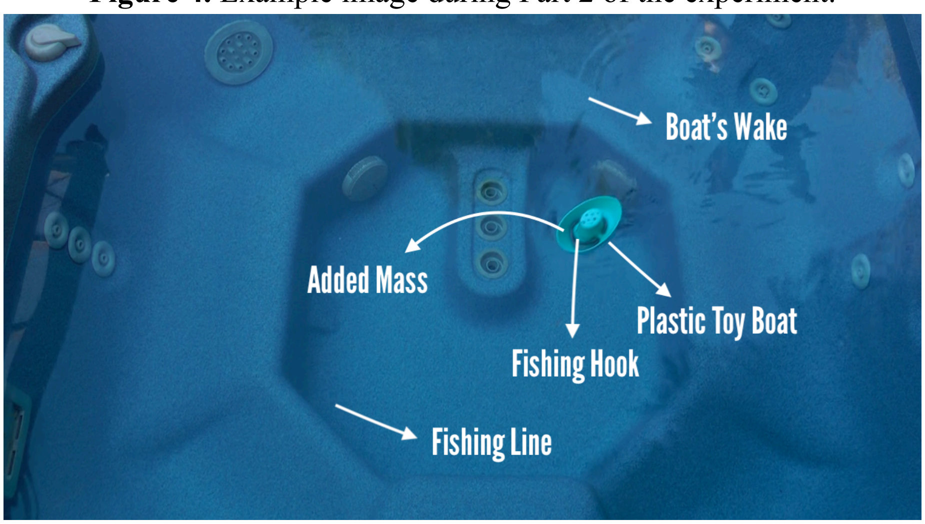
Figure 3. Example image during Part 1 of the experiment.



Part 2: Deep Water

1. Set up the ladder on the right of the jacuzzi.
2. Place the tripod with attached camera onto the third rung of the ladder so that the camera has a bird's eye view of the jacuzzi.
3. Use protective gloves for the following two steps.
4. Attach the fishing hook onto the end of the fishing line attached to the fishing rod.
5. Attach the fishing hook to the top of the plastic toy boat and extend fishing line to opposite end of the body of water.
6. Begin the video recording and reel in the plastic toy boat at a constant velocity.
7. Repeat steps 1-6 with an added weight inside of the plastic toy boat.
8. Analyze the recorded footage through video analysis software to obtain the angle of the object's wake.

Figure 4. Example image during Part 2 of the experiment.



<u>**Data Analysis:**</u>

Table 1. Initial Measurements of Trial Conditions.

Description	Quantitative measurements
Length of Fishing Line *	14.310 ± 0.003 *m*
Depth of Shallow Body of Water *	(1.35 ± 0.03) x 10^{-1} *m*
Width of Shallow Body of Water *	6.782 ± 0.003 *m*
Hypotenuse Width of Shallow Body of Water *	9.591 ± 0.005 *m*
Average Velocity for Object in Shallow Body of Water	(9.591 ± 0.005) x 10^{-1} ms^{-1}

Depth of Jacuzzi *	$1.041 \pm 0.003\ m$
Width of Jacuzzi *	$1.740 \pm 0.003\ m$
Hypotenuse Width of Jacuzzi	$2.461 \pm 0.005\ m$
Average Velocity for Object in Jacuzzi	$(6.15 \pm 0.02) \times 10^{-1}\ ms^{-1}$
Mass of Plastic Toy Boat	$(4.02 \pm 0.02) \times 10^{1}\ g$
Mass of Plastic Pyrex	$(1.53 \pm 0.02) \times 10^{1}\ g$

* Length measured using a measuring tape.

Sample calculation for calibrated measurement:

Depth of Jacuzzi = $41.00 \pm 0.04\ inches \times \frac{2.54\ cm}{inch} = 104.14 \pm 0.1016\ cm \times \frac{1m}{1.0 \times 10^2\ cm} = 1.0414\ \pm\ 0.001016\ m$

$(1.0414\ \pm\ 0.001016\ m) - (0.00 \pm\ 0.001016\ m) = 1.0414 \pm\ 0.002032\ m = 1.041\ \pm\ 0.003\ m$

After obtaining the recorded footage of each sample, a picture of each recordings was taken near when the object had covered half of its total distance as that moment in time displayed a more prominent wake. These pictures were then analyzed using the image analysis software ImageJ. The steps taken for the image analysis are as follows:

1. Obtain a picture of each recording near when the object has covered half of its total distance.
2. Open the image file with the ImageJ application.
3. Click on the angle tool.
4. Select three points in the following order:
   i. Select a point behind the object on the line created by the object's wake.
   ii. Select a point on the centre of the object.
   iii. Select a point behind the object on the line created by the object's wake, on the opposite side of the first point selected.
5. Select "Analyze" then "Measure" from the menu bar to obtain the angle of the object's wake.

Figure 5. Example image analysis using ImageJ.

The angle obtained through the use of technology does not include uncertainty as it is taken to be exact, but a reasonable value for the uncertainty of the angle is 5°. This value was approximated by varying the wake angle slightly and examining the difference in angle obtained. Thus, the angles obtained are taken to two significant figures so as to match the digit of the uncertainty. To obtain the Wave speed, the following equation construction was utilized (French, 2013):

Figure 6. Construction of moving object
Image modified by author from (French).

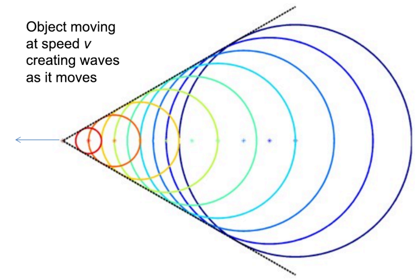
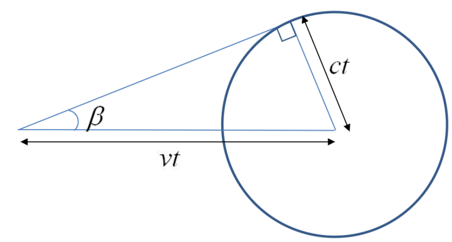


Figure 7. Geometric simplification of Moving Object.
Image obtained from (French).



$$vt \sin\beta = ct \qquad \text{(French)}$$
$$c = \frac{vt \sin\beta}{t}$$
$$c = v \sin\beta$$

If the angle β contains uncertainty k, we determine the value of c as follows:
$$c = v\sin(\beta \pm k) = v\,(\sin\beta \cos k \pm \sin k \cos\beta)$$

The results of the image analysis are outlined below:

Table 2. Wake Angle and Wave speed obtained from Samples in Part 1.

Trial Number	Sample	Mass of Object ± 0.2 *g*	Angle Obtained ± 5°	Average Trial Angle Observed	Wave Speed (ms^{-1})	Average Wave Speed (ms^{-1})
1	1	15.3	47	54 ± 13°	0.71 ± 0.06	0.8 ± 0.2
	2	15.3	51		0.74 ± 0.06	
	3	15.3	63		0.85 ± 0.04	
2	1	21.8	49	58 ± 16°	0.72 ± 0.06	0.8 ± 0.2
	2	21.8	53		0.76 ± 0.06	
	3	21.8	71		0.90 ± 0.03	
3	1	27.4	49	63 ± 17°	0.72 ± 0.06	0.8 ± 0.2
	2	27.4	72		0.91 ± 0.03	
	3	27.4	68		0.89 ± 0.04	
4	1	39.5	58	62 ± 12°	0.81 ± 0.05	0.8 ± 0.1
	2	39.5	71		0.90 ± 0.03	
	3	39.5	57		0.81 ± 0.05	

Sample Average Calculation for Average Angle:

Average = $\frac{\sum x_i}{n} \pm \frac{\max\{x_i\} - \min\{x_i\}}{2}$ where x_i represents every value and *n* represents the total number of values.

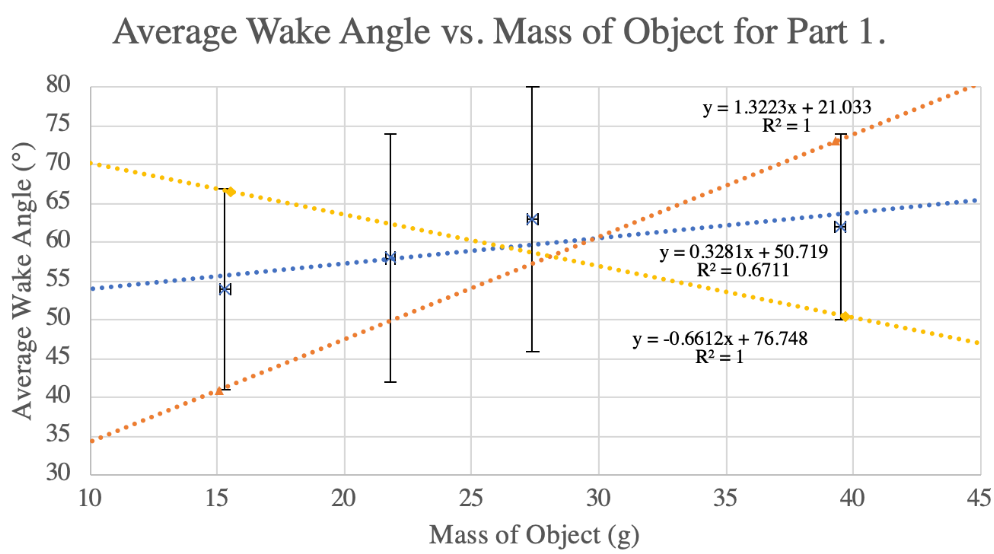
Thus, the Average = $\frac{47° + 51° + 63° + 49°}{4} \pm \frac{68° - 42°}{2} = 54.5 \pm 13° = 54 \pm 13°$

Sample Average Calculation for Average Wave Speed:

$$c = v\sin(\beta \pm k) = v\,(\sin\beta\cos k \pm \sin k\cos\beta)$$
$$c = \left((9.591\ \pm\ 0.005) \text{ x } 10^{-1}\ ms^{-1}\right) \text{ x } \sin(54° \pm 13°)$$
$$c = (0.9591\ \pm\ 0.0005\ ms^{-1}) \text{ x } (\sin(54°)\cos(13°) \pm \sin(13°)\cos(54°))$$
$$c = (0.9591\ \pm\ 0.0005\ ms^{-1}) \text{ x } (\sin(54°)\cos(13°) \pm \sin(13°)\cos(54°))$$
$$c = (0.9591\ \pm\ 0.0005\ ms^{-1}) \text{ x } (0.78828 \pm 0.13222)$$
$$c = (0.75604\ ms^{-1}) \pm (0.75604\ ms^{-1}) \text{ x } (\frac{0.0005}{0.9591} + \frac{0.13222}{0.78828})$$
$$c = (0.75604\ ms^{-1}) \pm (0.75604\ \ ms^{-1}) \text{x } (\frac{0.0005}{0.9591} + \frac{0.13222}{0.78828})$$
$$c = (0.75604\ ms^{-1}) \pm (0.75604\ ms^{-1}) \text{ x } (0.17295)$$
$$c = (0.75604\ ms^{-1}) \pm (0.13076\ ms^{-1})$$
$$c = 0.8 \pm 0.2\ ms^{-1}$$

The average wake angles compared to the masses of the objects for Part 1 are summarized in the following graph:

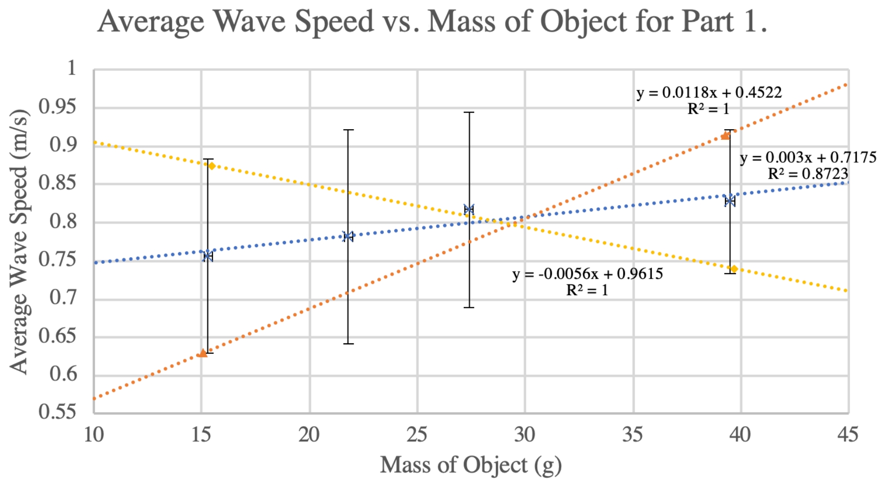
Figure 8. Graph of Average Wake Angle vs. Mass of Object for Part 1.



The best line fit for the sample data set shows a slight upward trend in the wake angle, which suggests that as the mass of the object increases, there is a directly proportional positive correlation between the average wake angle and the mass of the object. This observation contradicts the assertions made by Lord Kelvin. However, as the R^2 coefficient of correlation is relatively low, and the uncertainty in the average wake angle is quite high, there remains uncertainty as to this observation. Furthermore, this correlation suggests that for every gram of mass added, the average wake angle increases by 0.3281°, or that 3.05 g must be added to an object for its wake angle to increase by 1°.

The average wave speeds compared to the masses of the objects for Part 1 are summarized in the following graph:

Figure 9. Graph of Average Wave Speed vs. Mass of Object for Part 1.



The best line fit for the sample data set shows a very miniscule upward trend in the wave speed, which suggests that as the mass of the object increases, there is a directly proportional positive correlation between the average wave speed and the mass of the object. This correlation further suggests that for every gram of mass added, the wave speed increases by 0.003°, or that 30 *g* must be added to an object for its wave speed to increase by 0.1 ms^{-1}. The relatively large value of the R^2 coefficient of correlation supports this assertion.

An analysis of the video recordings of Part 2 follows.

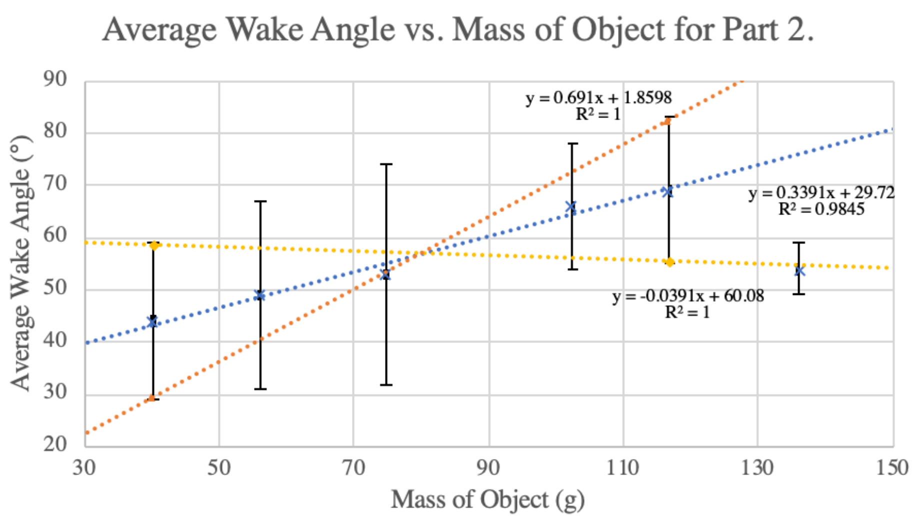
Table 3. Wake Angle and Wave speed obtained from Samples in Part 1.

Trial Number	Sample	Mass of Object ± 0.2 g	Angle Obtained ± 5°	Average Trial Angle Observed	Wave Speed (ms^{-1})	Average Wave Speed (ms^{-1})
1	1	40.2 g	58	44 ± 15°	0.52 ± 0.04	0.4 ± 0.2
	2	40.2 g	38		0.38 ± 0.05	
	3	40.2 g	39		0.39 ± 0.05	
	4	40.2 g	43		0.42 ± 0.05	
2	1	56.1 g	47	49 ± 18°	0.45 ± 0.04	0.4 ± 0.2
	2	56.1 g	34		0.35 ± 0.05	
	3	56.1 g	55		0.51 ± 0.04	
	4	56.1 g	59		0.53 ± 0.03	
3	1	74.8 g	47	53 ± 21°	0.45 ± 0.04	0.5 ± 0.2
	2	74.8 g	71		0.58 ± 0.02	
	3	74.8 g	42		0.41 ± 0.05	
	1	102.5 g	71		0.58 ± 0.02	

4	2	102.5 g	69	66 ± 12°	0.58 ± 0.03	0.55 ± 0.06
	3	102.5 g	58		0.52 ± 0.04	
5	1	116.9 g	74	69 ± 14°	0.59 ± 0.02	0.56 ± 0.06
	2	116.9 g	76		0.60 ± 0.02	
	3	116.9 g	58		0.52 ± 0.04	
6	1	136.0 g	54	54 ± 13°	0.50 ± 0.04	0.50 ± 0.04
	2	136.0 g	54		0.50 ± 0.04	
	3	136.0 g	54		0.50 ± 0.04	

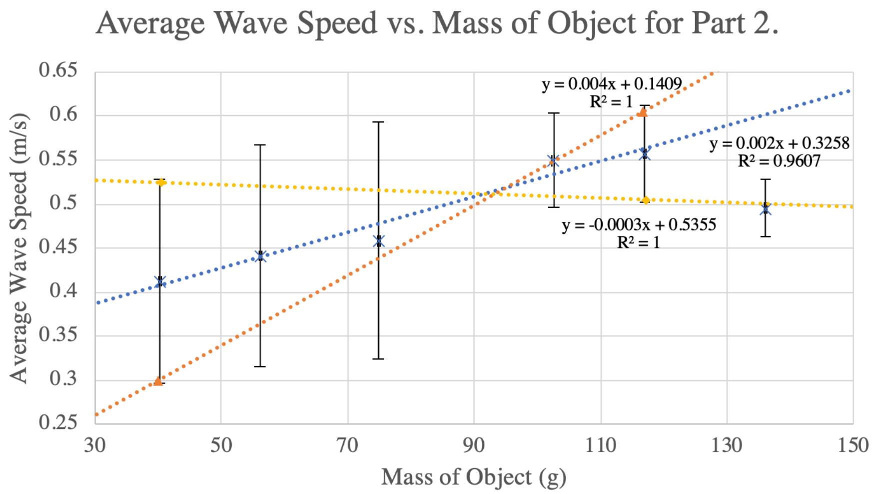
The average wake angle compared to the mass of the object for Part 2 is summarized in the following graph:

Figure 10. Graph of Average Wake Angle vs. Mass of Object for Part 2.



Yet again, the best line fit for the sample data set shows a slight upward trend in the wake angle, which suggests that as the mass of the object increases, there is a directly proportional positive correlation between the average wake angle and the mass of the object. The heaviest sample object was taken as an outlier since the velocity while reeling was considerably lower than the other samples' velocities, which could have skewed the results of the wave angle. The high value of the R^2 coefficient of correlation shows the certainty in this claim, although the large uncertainty in the wake angle shows how this increase in angle per unit mass may be a falsehood due to uncertainties propagated by errors. Furthermore, this correlation suggests that for every gram of mass added, the average wake angle increases by 0.339°, or that 2.95 *g* must be added to an object for its wake angle to increase by 1°. The mass required for a unit change in degrees is 96.7% of the value found in Part 1. The average wave speeds compared to the masses of the objects for Part 2 are summarized in the following graph:

Figure 11. Graph of Average Wave Speed vs. Mass of Object for Part 2.



The best line fit for the sample data set shows an upward trend in the wave speed. This increase suggests that as the mass of the object increases, there is a directly proportional positive correlation between the average wave speed and the mass of the object. The heaviest sample object was taken as an outlier since the velocity while reeling was considerably lower than the other samples' velocities, which could have skewed the results of the wave speed. The relatively large value of the R^2 coefficient of correlation supports this claim. This correlation further suggests that for every gram of mass added, the wave speed increases by 0.002 ms^{-1}, or that 50 g must be added to an object for its wave speed to increase by 0.1 ms^{-1}, a mass that is 167% of the value found in Part 1.

<u>Conclusion:</u>

In reference to the research question "*How does the mass of a moving object in water affect its wake angle and wave speed?*" it was determined that the mass of a moving object scales positively linearly with both the wake angle and the wave speed, thus nullifying our hypothesis and providing evidence that refutes Lord Kelvin's claim.

The linear relationship in Part 1 and Part 2 between the average wake angle and the mass of the object outputted an R^2 coefficient of correlation of 0.6711 and 0.9845 respectively as shown in Figure 8 and Figure 10. The low R^2 value for Part 1 coupled with a large uncertainty in the raw data and subsequently the processed data demonstrates how the certainty of the positively linear relationship is low. However, the high R^2 value for Part 2 increases the certainty of the positive linear relationship, although the uncertainty remains high.

The linear relationship in Part 1 and Part 2 between the average wave speed for six trials containing three or more samples each outputted an R^2 coefficient of correlation of 0.8723 and 0.9607 respectively as shown in Figure 9 and Figure 11. The high R^2 value for Parts 1 and 2 increases the certainty of the positive linear relationship, although the high uncertainty in the raw data and subsequently the processed data brings doubt upon high the relationship.

Connecting these conclusions to the real world may suggest that heavier ships have more acute wake angles or slower wave speeds than lighter ships, which could affect the speed of marine ships or influence the surrounding aquatic life. However, there are a number of limitations in generalizing the results of this experiment.

Errors and Limitation:

Throughout the experiment, the velocity of the object in the water remained constant as the experimenter reeled the fishing rod at a constant rate. However, imprecision resulting in the limitations of the exactness of human motions may have created a fluctuating velocity amongst different trials, which could have subsequently skewed the results.

A potential solution to this limitation is to implement a motor that could reel in the fishing rod at a more exact rate or utilize a remote-controlled toy boat whose velocity could be controlled more precisely. However, the utilization of a motor introduces an ethical dilemma which I have considered in the Ethics and Safety section of this investigation.

As shown throughout the Data Analysis section of this investigation, the raw and processed data contained a large amount of uncertainty, which was due to the high random error.

This large random error could be mitigated by repeating the experiment with a greater number of trials. However, as aforementioned, three to four trials per sample mass were examined in order to reduce random error but also prevent the experiment's time from becoming excessive.

While the object was being reeled in both Part 1 and Part 2, the surface area being exposed to the surface of the water was changing. This changing surface area likely caused a change to the object's wake and this change could have skewed the results.

As aforementioned, the introduction of a motor-powered object could reduce the fluctuations in the experimental procedure and yield more accurate results.

The positioning of the camera throughout each trial could have introduced an element of perspective which could shift the observed angle slightly.

To mitigate this problem, a camera could be placed directly above of the experiment so as to obtain the exact wake angle without perspective's skew. However, since placing a camera directly above the experiment poses a significant challenge, the constant position of the camera throughout this experiment seems suitable.

Future Work:

Firstly, future work includes the use of a motorized object to reduce random or systematic errors, and the analysis of more sample trials in order to reduce the random error.

Secondly, it would be beneficial to study the effect of the object's shape or velocity on the average wake angle and wave speed as opposed to focusing solely on their mass.

Thirdly, the depth of the water could be a factor that affects the results of the observations. However, the varying water depths in Part 1 and Part 2 of this investigation yielded similar results. The effect of changing this variable could be studied more in depth.

Lastly, as the analyzed data in Part 2 may suggest the presence of a quadratic relationship between variables, it may be useful to utilize models other than solely linear ones.

Works Cited

Faber, T. E. *Fluid Dynamics for Physicists*. 1995, pp. 188-94.

French, Andrew. “The shape of ship waves: The Kelvin wedge”. August 2013. Powerpoint presentation.

Georgi, H. *The Physics of Waves*. 1993.

Kai-kwong, H. *In the Wake of a Duck*. Hong Kong Observatory, 1 June 2013, http://www.hko.gov.hk/education/edu06nature/ele_kelvinwake_e.htm.

Rožman, S. *Wake Pattern of a Boat*. University of Ljubljana Faculty of Mathematics and Physics Department of Physics, 13 May 2009.

Undergraduate Mathematics. 23 Sept. 2015, https://www.maths.cam.ac.uk/undergrad.

Wake - Wikipedia. Wikipedia, 14 Dec. 2018, https://en.wikipedia.org/wiki/Wake.