Lattice-Based Zero-Knowledge Proofs: New Techniques for Shorter and Faster Constructions and Applications

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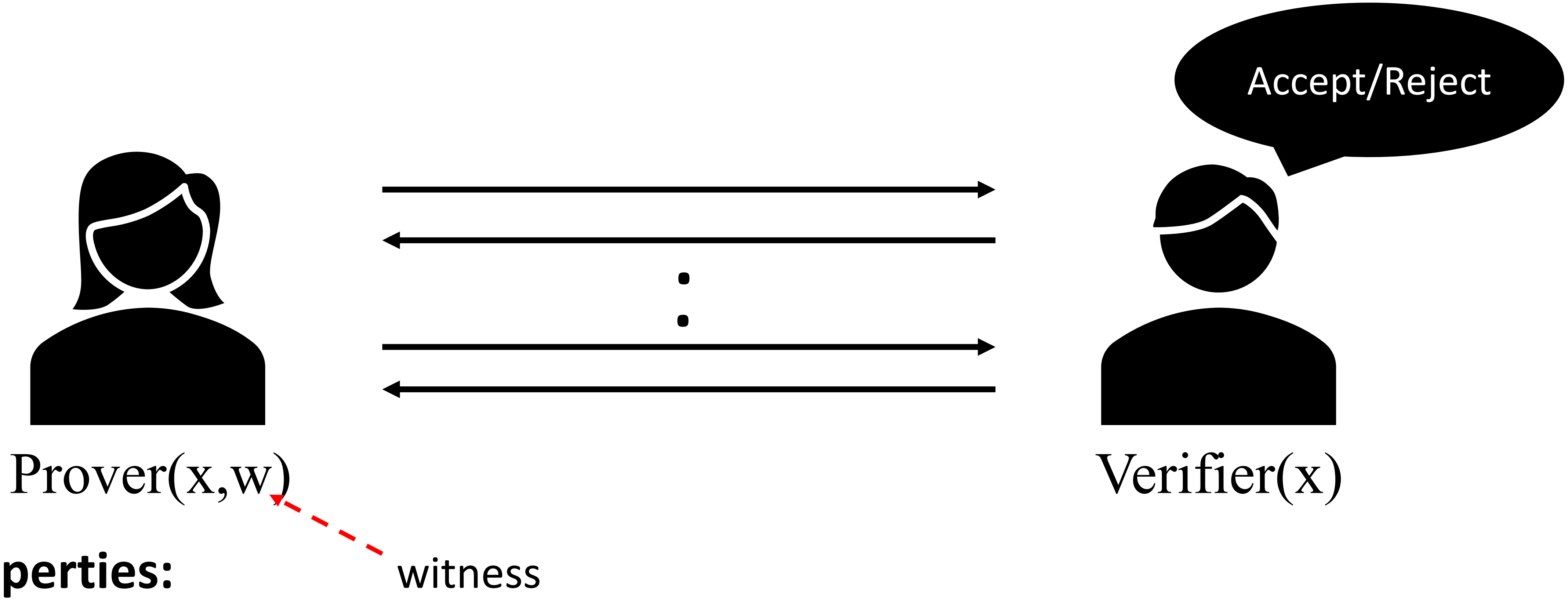
Acknowledgement: Some Slides courtesy of Muhammed. F. Esgin.

Outline

- **Background:** Efficient Zero-Knowledge Proofs (ZKPs) for linear relations
  - Schnorr proof ZKP of knowledge of discrete-log
  - Lattice analogue of DL: Module-RingSIS / Module-RingLWE
  - Difficulties and solutions in porting DL-based to lattice-based proof
  - Lyubahsevky proof of knowledge of Module-Ring-LWE witness [Lyu12]

- **Our new techniques:** Efficient Lattice-based ZKPs for `non-linear' relations of degree k > 1
  - Framework for ZKPs for non-linear relation of degree k > 1
    - Issues in porting DL-based to lattice-based proofs in non-linear setting
      - Our `one-shot' (short proof) soundness analysis technique: adjugate matrices
    - Application: Commitments of Bits Proofs
      - Speed-up technique 1: Extraction with large challenges and NTT-friendly rings
    - Application: One-of-Many Proofs
    - Application: anonymous authentication -- Ring Signatures
    - Application: Integer Range Proofs
      - Speed-up technique 2: CRT-packing technique supporting inter-slot operations
      - Improves run-time by packing factor s

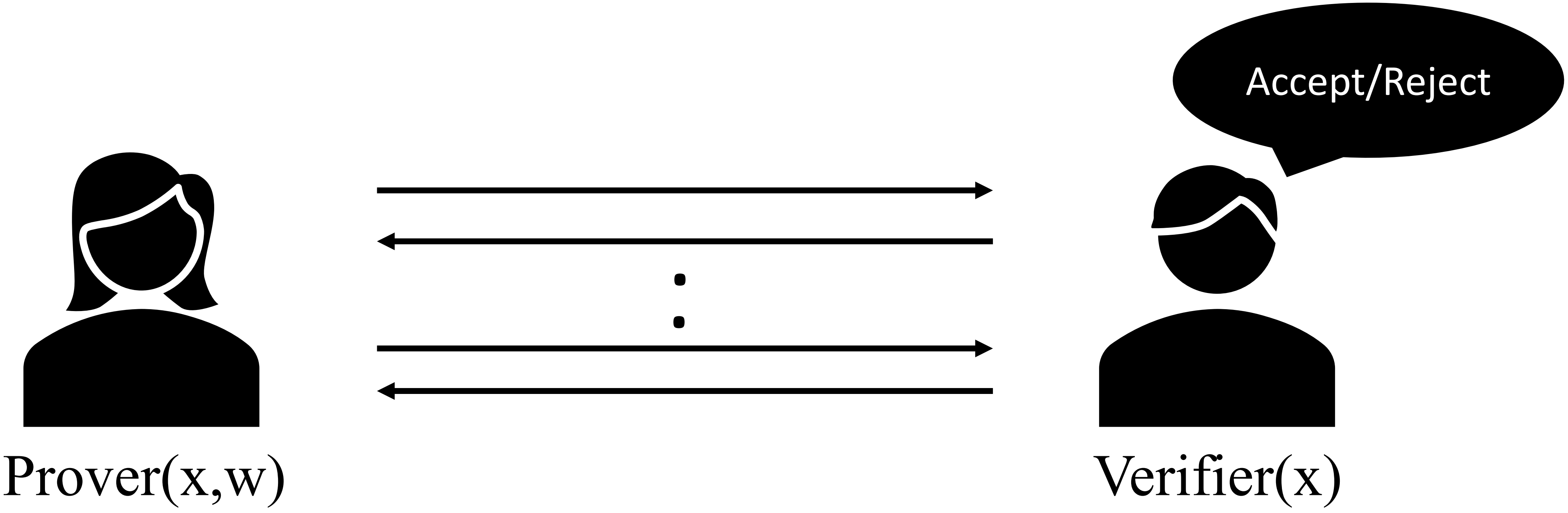
Zero-Knowledge Proofs [GMR85]



Properties:

1) Completeness
2) Soundness
3) Zero-Knowledge

Zero-Knowledge Proofs [GMR85]



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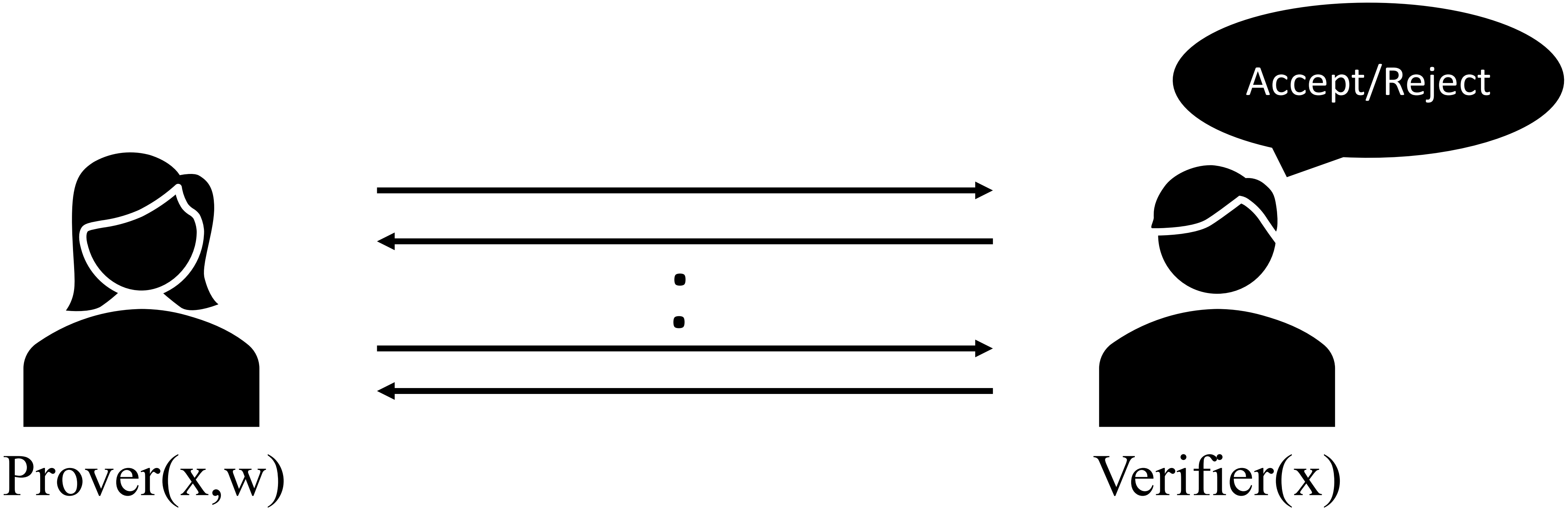

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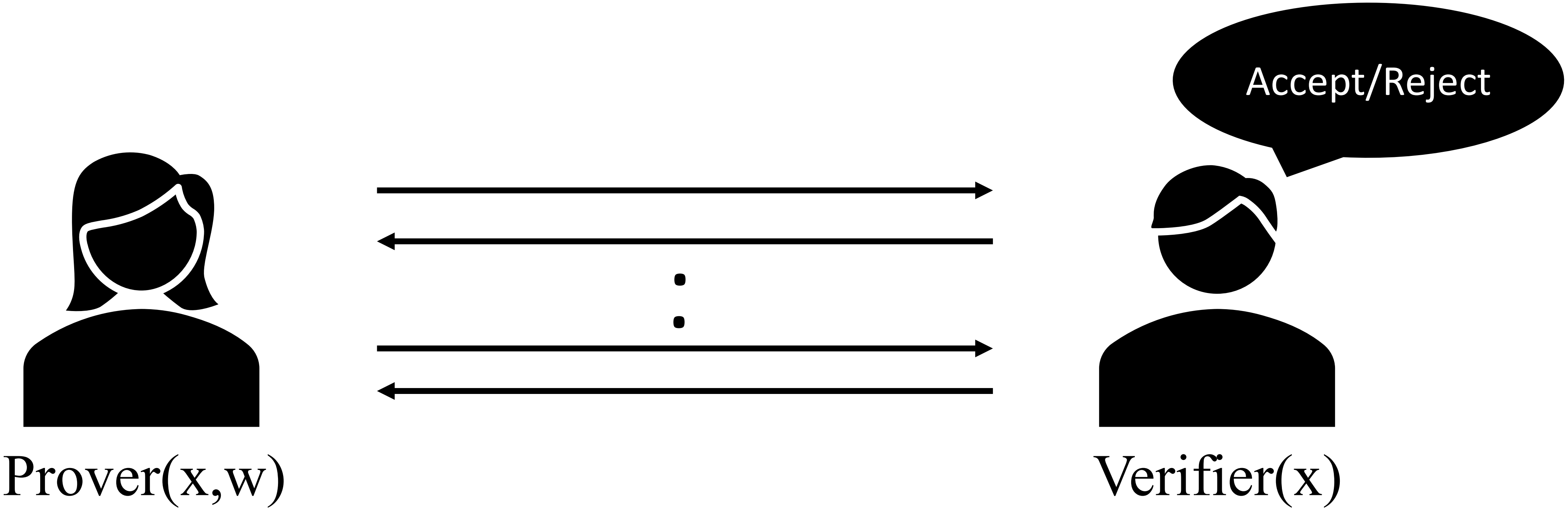

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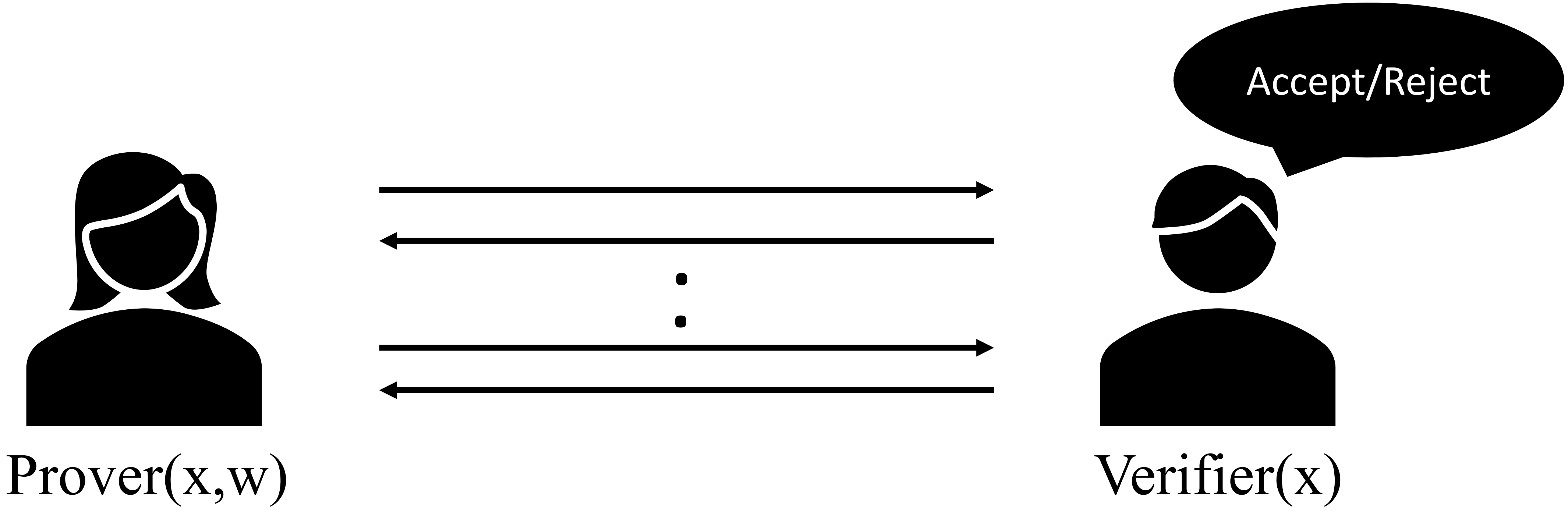


Properties:

1) Completeness
2) Soundness
3) Zero-Knowledge

We work in particular with Sigma protocols.
Easily made non-interactive using Fiat-Shamir heuristic.

Background: Types of ZKPs in Lattice-Based Crypto

- Two main types of ZK Proofs investigated in lattice-based crypto:
  - "Combinatorial" type (aka `Stern-type' [St96] ZK proofs) :
    - Verifier challenge chosen from a very small set (of size typically **3**)
    - Different prover response algorithm explicitly specified for each possible challenge
    - Pro: Very powerful – can be adapted to prove complex relations (e.g. [BLNW18])
    - Con: long/slow proofs: Many protocol repeats needed for high soundness level
  - "Algebraic" type (aka `Schnorr-type' [Sch89] ZK proofs) :
    - Verifier challenge can be chosen from a huge set (of size $> 2^{\lambda}$ for security parameter λ)
    - Prover response algorithm is an algebraic function of the verifier's challenge
    - Pro: can achieve short/fast proofs: `one-shot' challenge may be possible
    - Cons:
      - More limited in types of proofs so far achievable efficiently
      - May prove "approximate" (relaxed) relations rather than exact relations

Our focus in this talk

Classical ZKP 1: Schnorr proof ZKP of knowledge of discrete-log

Setup of Schnorr's ZKP of Knowledge of Discrete Log [Sch89]:

- Works in a cyclic multiplicative group $G = \langle g \rangle = \{1, g^1, g^2, \ldots, g^{q-1}\}$
  - where Discrete-Logarithm (DL) problem is hard
- Fixed public generator $g \in G$ for G
- Denote order (size) of G by q (assumed prime).
- Prover's Discrete-Log private-key (witness): $s \leftarrow U(Z_q)$.
- Prover's public-key (common input): $h = g^s \in G$.
- Write $h = Com(s)$.
  - Com is homomorphic from Z_q to G: $Com(s + t) = Com(s) \cdot Com(t)$

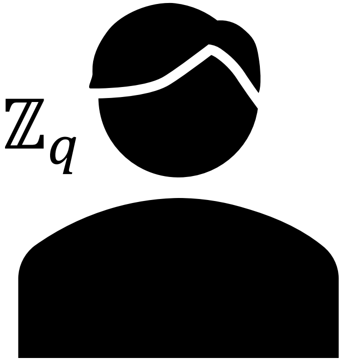
Classical ZKP 1: Schnorr proof ZKP of knowledge of discrete-log

$s \leftarrow \text{RandSet} = \mathbb{Z}_q$ $\quad u \leftarrow \mathbb{Z}_q$ $\qquad$ $A_1 = \text{Com}(s)$

$A_0 = \text{Com}(u)$ $\longrightarrow$

$x \leftarrow \text{ChallengeSet} = \mathbb{Z}_q$ $\longleftarrow$

$\boldsymbol{f} = u + x \cdot s$ $\quad \boldsymbol{f}$ $\longrightarrow$



Prover $\qquad\qquad$ Verifier

Correctness: homomorphic property of Com

Com(f) = Com(u+ x · s) = Com(u) · Com(s)^x

Soundness (2-special soundness): prover succeeds with prob > 1/|ChSet| → prover knows a **valid** opening (DL) of A_1

- Given commitment A_0, from **two** distinct successful challenge response pair pairs (x,f), (x',f'), extract witness s'

$$\boxed{A_0 \cdot A_1^x \stackrel{?}{=} \text{Com}(\boldsymbol{f})}$$

$$A_0 \cdot A_1^x = \text{Com}(\boldsymbol{f})$$
$$A_0 \cdot A_1^{x'} = \text{Com}(\boldsymbol{f}')$$
$$\Rightarrow A_1 = \text{Com}\left(\frac{f - f'}{x - x'}\right)$$

s'

Classical ZKP 1: Schnorr proof ZKP of knowledge of discrete-log

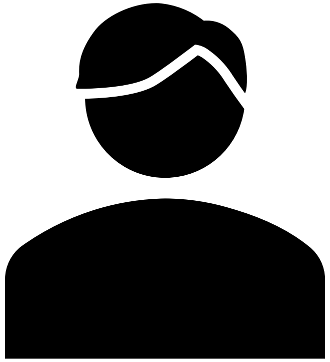
$s \leftarrow$ RSet = $\mathbb{Z}_q$

$h = \mathrm{Com}(s)$

$u \leftarrow \mathbb{Z}_n$ $\quad A_0 = \mathrm{Com}(u) \longrightarrow$

$\longleftarrow x \leftarrow$ CSet = $\mathbb{Z}_q$

$\boldsymbol{f} = u + x \cdot s \quad \boldsymbol{f} \longrightarrow$



Prover

Verifier

Honest-Verifier Zero-Knowledge (HVZK): An honest verified can efficiently simulate a proof transcript without the prover's witness!

$$\boxed{A_0 \cdot A_1^x \stackrel{?}{=} \mathrm{Com}(\boldsymbol{f})}$$

Transcript Simulator, given A_1:

- $x \leftarrow$ CSet = $\mathbb{Z}_q$
- $f \leftarrow \mathbb{Z}_q$
- $A_0 \cdot \stackrel{?}{=} \mathrm{Com}(\boldsymbol{f}) \cdot A_1^{-x}$

Application 1: Digital Signatures [Sch91]

- Fiat-Shamir Transformation: Generic conversion of an interactive ZK Sigma (3-move) proof to a non-interactive digital signature
  - Idea:
    - Prover uses a cryptographic one-way hash function H to generate challenge by hashing his protocol commitment A_0 and the signed message m
      - $x = H(A_0, m)$
- → Schnorr digital signature (similar to Digital Signature Standard, DSS):
    - **KG:** sk = s, $A_1 = \mathrm{Com}(s)$
    - **Sign**(s, m) = (x,f)
      - $A_0 = \mathrm{Com}(u)$
      - $x = H(A_0,m)$
      - $f = r + x \cdot s$
    - **Ver**(m,(x,f),pk):
      - $A_0 = \mathrm{Com}(\boldsymbol{f}) \cdot A_1^{-x}$
      - $x \stackrel{?}{=} H(A_0,m)$

Lattice analogue of DL Problem: Module-RingSIS / Module-RingLWE Problems

Structured lattice Setup:

- Work over a polynomial ring $R_q = \mathbb{Z}_q[x]/(x^d + 1)$ for integer q
- Fixed public uniformly random matrix $A \in {R_q}^{n \times m}$
- Conjectured-Hard Lattice problems
- Module−Ring−SIS$_{n,m,q,\beta}$ Problem:
  - Given $A \in {R_q}^{n \times m}$, find \`short' $v \in R_q^m$ ($\|v\| \leq \beta$) s.t. $A \cdot v = 0$
- Module−Ring−LWE$_{n,m,q,\alpha}$ Problem:
  - Given $A \in {R_q}^{n \times m}$, and $t = A \cdot s \in R_q^n$ for a \`short' $s \in R_q^m$ ($\|s\| \leq \alpha\, q \sqrt{m}$), find s (search-LWE) or distinguish t from uniform in R_q^n (decision-LWE)

→Typical Prover's private-key (witness): \`short' $s \leftarrow \mathrm{U}([-B, B]^{n \times m})$ = RandSet

→Typical Prover's public-key (common input): $t = A \cdot s \in R_q^n$

- Write $t = Com(s)$
  - Com is homomorphic from Dom$_s$ to R_q^n : Com(s + t) = Com(s) + Com(t)

Best known attacks take time 2^{λ} if

- $dn \geq \Omega(\lambda \cdot \frac{\log^2 \beta}{\log q})$, $\beta < q$ (SIS)
- $d(m-n) \geq \Omega(\lambda \cdot \frac{\log^2 \alpha^{-1}}{\log q})$, $\alpha^{-1} > 1$ (LWE)

→ Balanced with $m = 2n$, $\beta = \alpha^{-1}$

Hardness decreases with β → aim to minimize extracted witness norm in ZKPs!

Many SIS solutions /
Unique LWE solution with
$\mathrm{m} = 2\mathrm{n}, \beta = \alpha^{-1} \geq \sqrt{dm} \cdot q^{1/2}$

Lattice ZKP 1 Lattice-analogue of Schnorr ZKP
`Attempt 1'

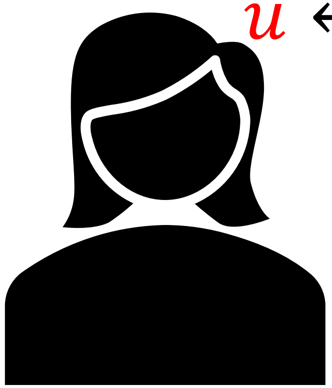
$s \leftarrow \text{SSet} = [-B, B]^{dm}$

$u \leftarrow RSet = [-B', B']^{dm}$ $A_0 = \text{Com}(u)$

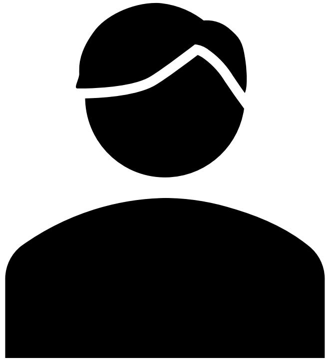
$A_1 = \text{Com}(s)$

$x \leftarrow \text{CSet} \subseteq R_q$

$\boldsymbol{f} = u + x \cdot s$ $\boldsymbol{f}$



Prover



Verifier

Correctness: homomorphic property of Com

Com(f) = Com(u+ x · s) = Com(u) · Com(s)^x

Soundness (2-special soundness): prover succeeds with prob > 1/|ChSet| → prover knows a **valid** opening of A_1

- Given commitment A_0, from **two** distinct successful challenge response pair pairs (x,f), (x',f'), extract witness s'

$A_0 + x \cdot A_1 \stackrel{?}{=} \text{Com}(\boldsymbol{f})$

$\|f\| \; ?< B' + \max_{x,s} \|x \cdot s\|_\infty$

$A_0 + x \cdot A_1 = \text{Com}(\boldsymbol{f})$

$A_0 + x' \cdot A_1 = \text{Com}(\boldsymbol{f}')$

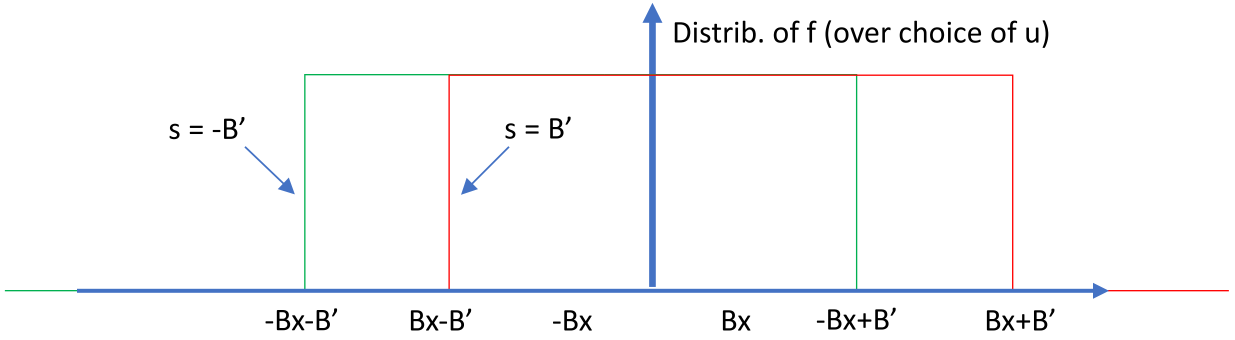
$\Rightarrow A_1 = \text{Com}\left(\frac{f - f'}{x - x'}\right)$ s'?

Difficulties & Solutions in porting DL-based to lattice-based ZK Proof

But, `Attempt 1' does not quite work…

Issues with `Attempt 1':

1. Zero-Knowledge Property is not satisfied:
   - Domain SSet and RSet for secrets s and u is `**short**' interval [-B,B] (< q)
     - Needed for hardness of the LWE/SIS lattice problems
     - Challenges x in ChallSet have to be `**short**' for same reason
   - Prover's response value $\boldsymbol{f} = u + x \cdot s$ leaks info. on secret s : $\mathbb{E}[\boldsymbol{f}] = x \cdot s$

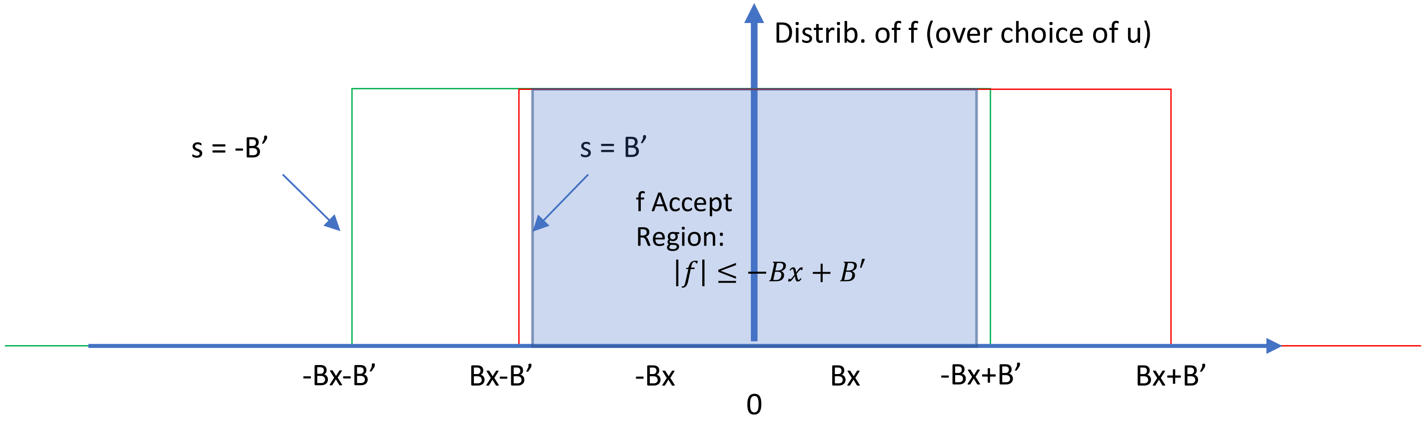


Difficulties & Solutions in porting DL-based to lattice-based ZK Proof

Main Issues with `Attempt 1':

1. Zero-Knowledge Property is not satisfied:

- Solution ([Lyu09,Lyu12]): Rejection sampling
  - Restart protocol with fresh u (and x) until f is independent of s, $\mathbb{E}[\boldsymbol{f}] = 0$



Acceptance probability
$p = (1 - \frac{|Bx|}{B'})^{md} = \Omega(1)$ if
$\frac{B'}{|Bx|} = O(md)$
Masking size linear in dimension.

Using discrete Gaussian (instead of uniform) distribution for u can reduce masking size [Lyu12].

Porting DL-based to lattice-based ZK Proof

Main Issues with `Attempt 1':

$$A_0 + x \cdot A_1 = \mathrm{Com}(\boldsymbol{f})$$
$$A_0 + x' \cdot A_1 = \mathrm{Com}(\boldsymbol{f}')$$
$$\Rightarrow A_1 = \mathrm{Com}\left(\frac{f-f'}{x-x'}\right)$$

2. Soundness Property is not satisfied

- Problem: extracted witness $s' = \frac{f-f'}{x-x'} \in R_q$
- s' may not be **`short' (<<q) → not in valid (secure) `short' Com domain**
  - **Issue:** $(x-x')^{-1}$ in R_q is usually not short in when $x-x'$ is short

- Solutions
  - Solution 1 (special challenges - efficiency compromise) [L+14,L+19]:
    - Use a special challenge space CSet $\subseteq R_q$ such that $(x-x')^{-1}$ is `short' for all x ≠ x' in CSet
    - But, largest such challenge space known is small (size 2*d* = *O*(λ))
      - *Low efficiency:* Many protocol repeats needed for high soundness level
  - Solution 2 (approximate relations – functionality compromise) [Lyu09,Lyu12]:
    - Prove knowledge of witness (c',s') to **approximate** relation $c' \cdot A_1 = \mathrm{Com}(s')$
    - c' is the `approximation' factor (must be `short' but not 1 as in exact relation)
    - ZK proof application must work securely with approximate proof

$$\Rightarrow (x - x') \cdot A_1 = \mathrm{Com}(f - f')$$

Lattice ZKP 1 Lattice-analogue of Schnorr ZKP
`Fixed Proof' idea (a-la [Lyu12])

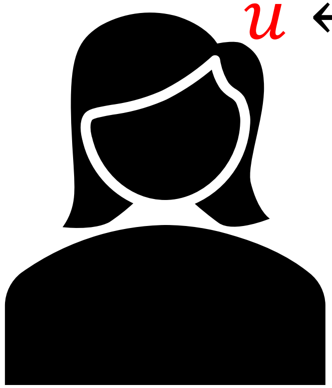
$s \leftarrow$ SSet $=[-B,B]^{dm}$ $\qquad A_1 = \mathrm{Com}(s)$

$u \leftarrow RSet = [-B',B']^{dm}$ $\quad A_0 = \mathrm{Com}(u)$ $\longrightarrow$

$x \leftarrow$ Cset=$\{0,1\}^d \subseteq R_q$ $\longleftarrow$

$\boldsymbol{f} = u + x \cdot s$ $\qquad \boldsymbol{f}$ $\longrightarrow$

Restart if $\|\boldsymbol{f}\|_\infty > \boldsymbol{B' - max_{x,s}\|x \cdot s\|_\infty}$



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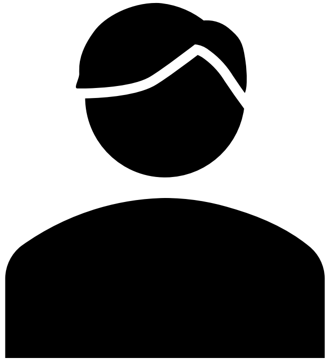

Verifier

$\boxed{A_0 + x \cdot A_1 \stackrel{?}{=} \mathrm{Com}(\boldsymbol{f})}$

$\boxed{\|f\| \;?< B' - max_{x,s}\|x \cdot} \; s\|_\infty$

Correctness: homomorphic property of Com

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- Given commitment A_0, from **two** distinct successful challenge response pair pairs (x,f), (x',f'), extract witness s'

$$A_0 + x \cdot A_1 = \mathrm{Com}(\boldsymbol{f})$$
$$A_0 + x' \cdot A_1 = \mathrm{Com}(\boldsymbol{f}')$$

s'

$$(\boldsymbol{x} - \boldsymbol{x}') \cdot \boldsymbol{A_1} = \mathbf{Com}(\boldsymbol{f} - \boldsymbol{f}')$$

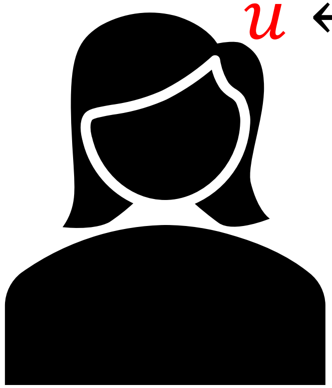
Relaxation factor

Lattice ZKP 1 Lattice-analogue of Schnorr ZKP
`Fixed Proof' idea (a-la [Lyu12])

$s \leftarrow$ SSet $=[-B, B]^{dm}$

$A_1 = \mathrm{Com}(s)$

$u \leftarrow RSet = [-B', B']^{dm}$ $A_0 = \mathrm{Com}(u)$

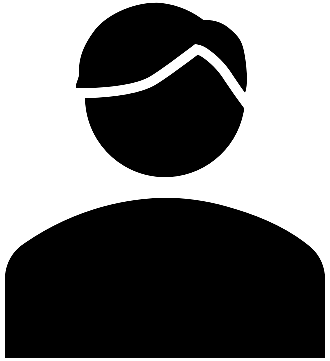


$x \leftarrow$ CSet $\subseteq R_q$

$\boldsymbol{f} = u + x \cdot s$ $\boldsymbol{f}$

Restart if $\|\boldsymbol{f}\|_\infty > B' - \boldsymbol{max_{x,s}}\|\boldsymbol{x} \cdot \boldsymbol{s}\|_\infty$

Prover



Verifier

$$\boxed{A_0 + x \cdot A_1 \stackrel{?}{=} \mathrm{Com}(\boldsymbol{f})}$$

Honest-Verifier Zero-Knowledge (HVZK): An honest verified can efficiently simulate a proof transcript without the prover's witness!

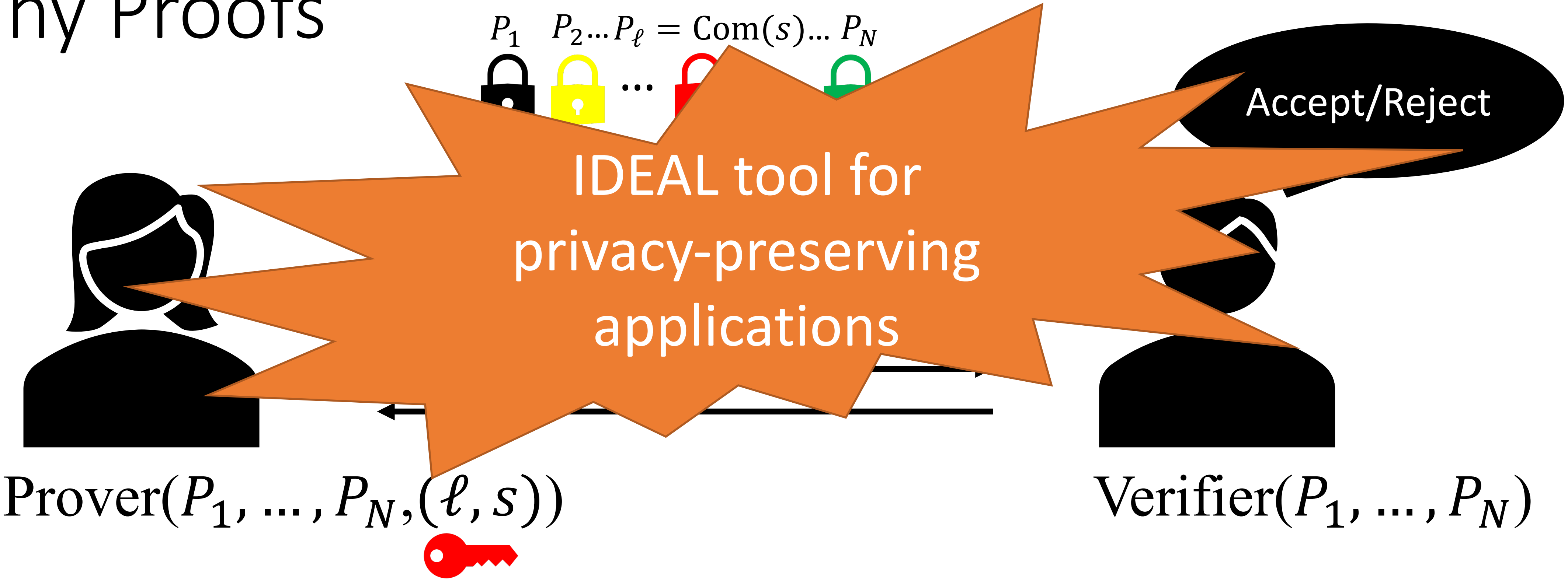
Accepted Transcript Simulator, given A_1:

- $x \leftarrow$ CSet $\subseteq R_q$
- $f \leftarrow$ AccSet = $[-(B' - \boldsymbol{max_{x,s}}\|\boldsymbol{x} \cdot \boldsymbol{s}\|_\infty), (B' - \boldsymbol{max_{x,s}}\|\boldsymbol{x} \cdot \boldsymbol{s}\|_\infty)]^{dm}$
- $A_0 = \mathrm{Com}(\boldsymbol{f}) - x \cdot A_1$

Application 1: Digital Signatures [Lyu12,L17+]

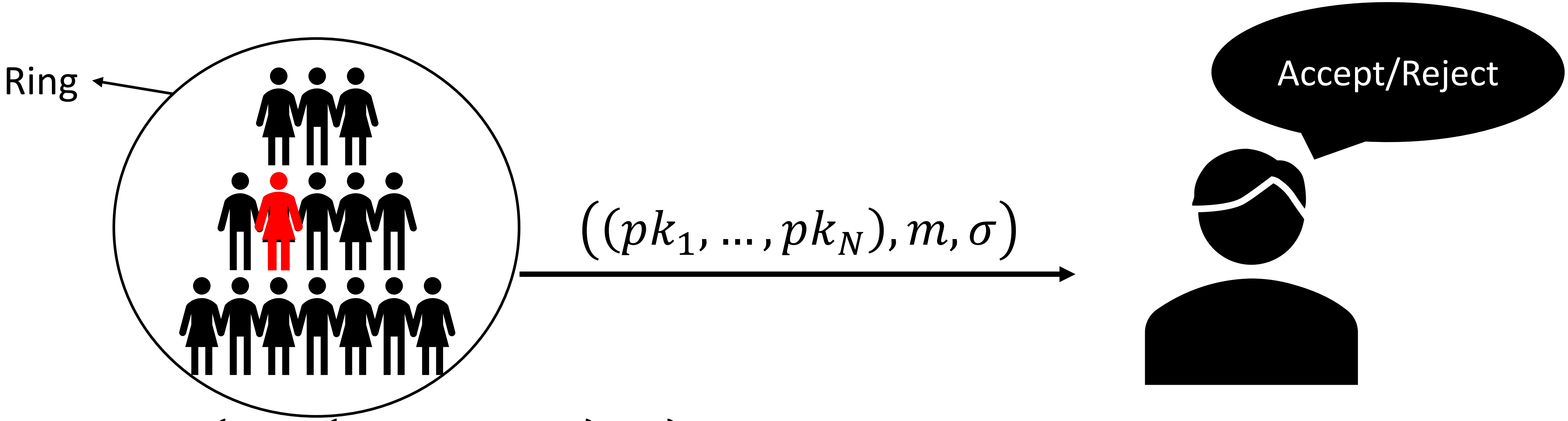
- Lyubashevsky digital signature idea [variant of Lyu12]
  - **KG:** sk = s, A_1 = Com(s)
  - **Sign**(s, m) = (x,f)
    - A_0 = Com(u)
    - x = H(A_0,m) ∈ $\{0,1\}^d$
    - f = u + x·s **Restart if $\|f\|_\infty > B' - max_{x,s}\|x \cdot s\|_\infty$.**
  - **Ver**(m,(x,f),pk):
    - $A_0 = \mathrm{Com}(f) - x \cdot A_1$
    - $x \stackrel{?}{=}$ H(A_0,m)
    - $\|f\| \, ?< B' - max_{x,s}\|x \cdot s\|_\infty$
- Unforgeability proof ideas:
  - ZK simulator → simulate obs signatures by programming H, without secret key s
  - Approx. relation soundness → forging alg. can be used to extract s' = $f - f'$ s.t
  - $(x - x') \cdot A_1 = \mathrm{Com}(f - f')$ → solve Module-RingSIS : Com($(x - x') \cdot s - (f - f')$) = 0
  - Hardness of decision Module-RingLWE → non-trivial solution for Module-RingSIS
- Optimised signature variants of above in NIST PQC second round:
  - Dilithium, Tesla

ZKPs for non-linear relations: One-out-of-Many Proofs



Goal: Prove knowledge of a secret associated to one of the public values without revealing the secret and the index of the public value

Ring Signatures [RST01, BKM09]



$$\sigma \leftarrow \mathrm{Sign}(sk_\ell, (pk_1, \dots, pk_N), m)$$

Properties:

1) Correctness
2) Unforgeability
3) Anonymity

1-out-of-N proof $\rightarrow$ Ring Signature

- Users commit to their secret keys to form their public keys:

$$pk_i = \mathrm{Com}(sk_i)$$

- Signer generates a non-interactive 1-out-of-N proof to prove knowledge of an opening of one of pk_i's
  - i.e., proving knowledge of sk_ℓ without revealing ℓ

1-out-of-N ZKP		**Ring Signature**
Completeness	$\Longrightarrow$	Correctness
Soundness	$\Longrightarrow$	Unforgeability
Zero-Knowledge	$\Longrightarrow$	Anonymity

The transition may not go so smoothly in the lattice setting!

Applications and Our Focus

- Set membership proofs, group signatures, …
- Privacy-aware cryptocurrencies, e.g., RingCT protocol in Monero
- e-voting systems
- …

- **We want:** short (sublinear-sized) and
  "post-quantum" one-out-of-many proofs
  with no trusted setup

Advanced Zero-Knowledge Proofs

Discrete Log.

- ZK proofs run smoothly
  - No protocol repetitions (negligible soundness error in single execution)
  - Exact soundness
  - **Any** commitment opening is valid
- Very short and scalable 1-out-of-N proofs due to Groth and Kohlweiss [GK15] and Bootle et al. [BCC+15]
  - Proof length: $O(\log N)$
  - Short **in practice** as well
  - Only a few KB even for $N = 10^9$

Lattice

- If you care about **efficiency**, then you have to make compromises
  - Relaxed soundness:
    prove knowledge of $(\gamma, \vec{s})$ s.t
    $\gamma \cdot C = \mathrm{Com}(\vec{s})$
- Only **short** openings are valid
  - $||\vec{s}|| \leq T$ for some $T < q$
- You may have to work
  - with a **small** set of challenges
  - over a **ring**, not a field
- Log-sized ring signature due to Libert et al. [LLNW15]
  - **NOT short** in practice
  - 75 MB for $N = 1000$

Our Results: Summary

- New technical tools for algebraic lattice-based protocols
  - Handling approximate ZK protocols for non-linear (degree $k > 1$) relations in lattice setting
    - **Many special sound protocols: Generalization of Lyubashevsky 2-sound protocol to $k > 1$ - non-linear relations**
      - Bounds on length of extracted witnesses and approximation factors
    - Speed-up Techniques: CRT message packing in commitment and adapting NTT-friendly rings
- Short one-out-of-many proofs from lattices
  - One shot challenges
  - Short both asymptotically and in practice
- Short ring signature from standard lattice assumptions
  - Based on Module-LWE and Module-SIS
  - No trusted setup
  - New ideas for soundness $\Longrightarrow$ unforgeability in a constraint (lattice) setting
- Variant proofs for range and set membership proofs
- Exploiting module variants of standard lattice assumptions for efficiency purposes [see the papers for details]

Lattice-Based Commitment schemes

- To hide low-entropy messages, need a randomised (hiding) commitment scheme Com(m; r)
- For remainder of this talk, Com will denote one of the two lattice-based (Module-LWE, Module-SIS) randomised commitment schemes [B+18]:
  - Hashed Message Commitment (HMC):

$$\mathrm{Com}(m, r) = \begin{pmatrix} G_r & G_m \end{pmatrix} \begin{pmatrix} r \\ m \end{pmatrix}$$

  - Unbounded-Message Commitment (UMC):

$$\mathrm{Com}(m, r) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} r + \begin{pmatrix} 0 \\ m \end{pmatrix}$$

Framework: ZKPs for non-linear relations

Prover

$$\xrightarrow{A_0, \ldots, A_k}$$

$$\xleftarrow{x \leftarrow \text{ChallengeSet}}$$

$$\xrightarrow{\boldsymbol{f}, \boldsymbol{r}}$$

<u>Witness Extraction</u>

How to extract **useful** secret information given a set of **accepting** protocol transcripts with the same initial message for a lattice-based commitment scheme Com?

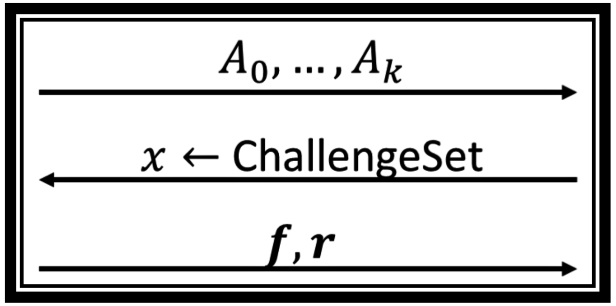
$$\boxed{A_0 + xA_1 + \cdots + x^k A_k \stackrel{?}{=} \mathrm{Com}(\boldsymbol{f}; \boldsymbol{r})}$$

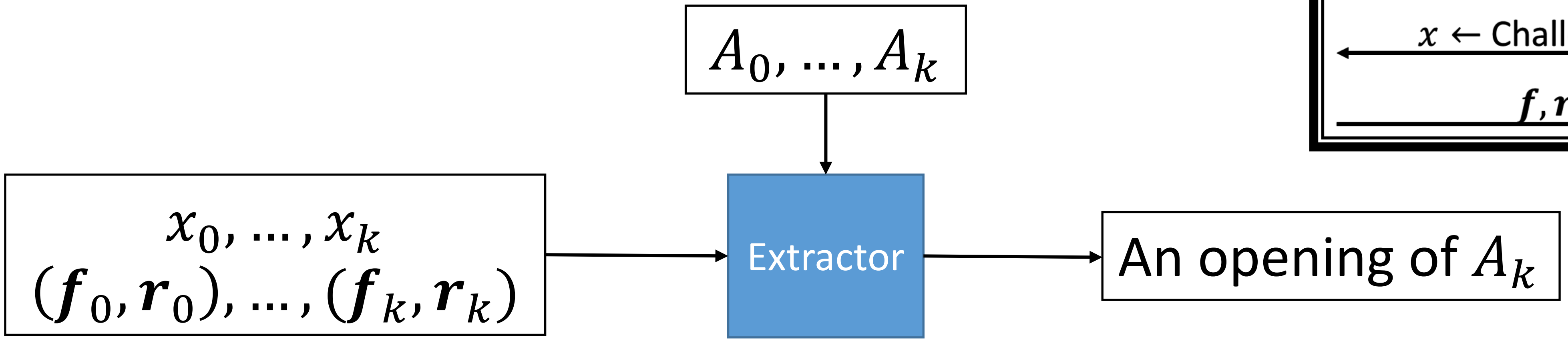
Efficient proof systems from [GK15] and [BCC+15] have this structure!

We need to 1) prove a degree-k relation for $k \geq 1$

2) extract a **valid** opening of A_k

Witness Extraction ($(k+1)$-special soundness)





s.t. $A_0 + x_i A_1 + \cdots + x_i^k A_k \stackrel{?}{=} \mathrm{Com}(\boldsymbol{f}_i; \boldsymbol{r}_i)$ for $i = 0, \ldots, k$

Proves a soundness error $\leq \frac{k}{|\mathrm{ChSet}|}$
(a cheating prover's max. success probability)

Witness Extraction

- We know that $A_0 + x_i A_1 + \cdots + x_i^k A_k \stackrel{?}{=} \mathrm{Com}(\boldsymbol{f}_i; \boldsymbol{r}_i)$ for $i = 0, \ldots, k$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^k \\ 1 & x_1 & x_1^2 & \cdots & x_1^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^k \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_k \end{pmatrix} = \begin{pmatrix} \mathrm{Com}(\boldsymbol{f}_0; \boldsymbol{r}_0) \\ \mathrm{Com}(\boldsymbol{f}_1; \boldsymbol{r}_1) \\ \vdots \\ \mathrm{Com}(\boldsymbol{f}_k; \boldsymbol{r}_k) \end{pmatrix}$$

over a ring $\mathfrak{R}$

$\boldsymbol{V}$, Vandermonde Matrix

- **Goal:** Recover an opening of A_k

For our lattice-based commitment, $(\vec{m}, \vec{r})$ is a valid opening of C if $C = \mathrm{Com}(\vec{m}; \vec{r})$ AND $(\vec{m}, \vec{r})$ is short!

Witness Extraction

We have $\boldsymbol{V} \cdot \boldsymbol{a} = \boldsymbol{c}$, and we want to eliminate $\boldsymbol{V}$

[Turner66]

$$\boldsymbol{V}^{-1} = \begin{pmatrix} \frac{*}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_k)} & \frac{*}{(x_0-x_1)(x_1-x_2)\cdots(x_1-x_k)} & \cdots & \frac{*}{(x_0-x_k)(x_1-x_k)\cdots(x_{k-1}-x_k)} \\ \frac{*}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_k)} & \frac{*}{(x_0-x_1)(x_1-x_2)\cdots(x_1-x_k)} & \cdots & \frac{*}{(x_0-x_k)(x_1-x_k)\cdots(x_{k-1}-x_k)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_k)} & \frac{-1}{(x_0-x_1)(x_1-x_2)\cdots(x_1-x_k)} & \cdots & \frac{(-1)^k}{(x_0-x_k)(x_1-x_k)\cdots(x_{k-1}-x_k)} \end{pmatrix}$$

Two approaches:

- Approach 1 [E+19a]: Use special challenge space so that challenge differences
  1) are invertible, and
  2) have a \`short' inverse!
  - Drawback: Small challenge space → multiple repetitions needed for high soundness security → Long proofs, length = $\tilde{O}(\lambda^2)$
- Approach 2 [This work]: Clear the denominators by multiplying by det(V) and find good bounds on det(V) for a set of \`short' challenges
  - Advantage: can support large challenge space (\`\`one-shot') → short proofs, length = $\tilde{O}(\lambda)$

Our approach: adjugate matrices

- Instead of multiplying by V^{-1} , we multiply by adj(V):
  - We have $\boldsymbol{V} \cdot \boldsymbol{a} = \boldsymbol{c} \rightarrow \boldsymbol{det}(\boldsymbol{V}) \cdot \boldsymbol{a} = \boldsymbol{adj}(\boldsymbol{V}) \cdot \boldsymbol{c}$
  - Relaxation factor: $\det(V) = \prod_{0 \leq i < j \leq k} (x_i - x_j)$

$$adj(V) = \begin{pmatrix} \frac{*}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_k)} & \frac{*}{(x_0-x_1)(x_1-x_2)\cdots(x_1-x_k)} & \cdots & \frac{*}{(x_0-x_k)(x_1-x_k)\cdots(x_{k-1}-x_k)} \\ \frac{*}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_k)} & \frac{*}{(x_0-x_1)(x_1-x_2)\cdots(x_1-x_k)} & \cdots & \frac{*}{(x_0-x_k)(x_1-x_k)\cdots(x_{k-1}-x_k)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\det(V)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_k)} & \frac{-\det(V)}{(x_0-x_1)(x_1-x_2)\cdots(x_1-x_k)} & \cdots & \frac{(-1)^k \det(V)}{(x_0-x_k)(x_1-x_k)\cdots(x_{k-1}-x_k)} \end{pmatrix}$$

  - Extracted witness for last commitment:

$$\det(V) \cdot A_k = \sum_{i=0}^{k} \Gamma_i \cdot Com(f_i; r_i) = Com(\underbrace{\sum_{i=0}^{k} \Gamma_i \cdot f_i}_{\hat{m}_k}; \underbrace{\sum_{i=0}^{k} \Gamma_i \cdot r_i}_{\hat{r}_k})$$

where $\Gamma_i = (-1)^{i+k} \prod_{0 \leq l < j \leq k \; l,j \neq i} (x_j - x_l)$

Our approach: adjugate matrices

- In particular, our adjugate matrix analysis approach allows large challenge spaces of the form

$$\mathcal{C}^d_{w,p} = \{ x \in \mathbb{Z}[X] \, : \, \deg(x) = d - 1 \wedge \mathsf{HW}(x) = w \wedge \|x\|_\infty = p \}.$$

  - \`One shot' possible with \`short' challenges
    - e.g. size of $\mathcal{C}^d_{w,p}$ > 2^{256} if $(d, w, p) = (256, 60, 1)$
  - No invertibility condition on challenge space needed (V can even be singular)
    → no special condition on ring modulus q needed
    → can use \`NTT-friendly' *q*
  - Moderately short bounds on relaxation factor / witness size for small *k:*
    - *Relaxation factor:* $\det(V) \ \leq (2p)^{k(k+1)/2} \cdot w^{k(k+1)/2-1}$
    - *Extracted witness norm:*

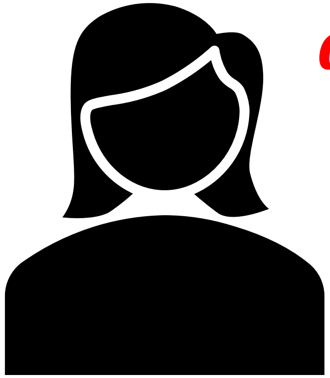
$$\|\widehat{m}_k\| \leq (k+1) \cdot d \cdot (2p)^{k(k-1)/2} \cdot w^{k(k-1)/2-1} \cdot max_i \|f_i\|$$
$$\|\hat{r}_k\| \leq (k+1) \cdot d \cdot (2p)^{k(k-1)/2} \cdot w^{k(k-1)/2-1} \cdot max_i \|r_i\|$$

Application: Commitments of Bits **Relaxed** ZKP

- One-shot variant of multi-shot lattice ZKP [E+19a], DL ZKP in [GK15]
  - Prover witness $\boldsymbol{b} \in \{\mathbf{0}, \mathbf{1}\}^{s}, \boldsymbol{r}$ ← Sset (`short')
  - Verifier input: $B = \mathrm{Com}(\boldsymbol{b}\ ; \boldsymbol{r})$
  - Original Goal: prove that b is a vector of **bits**
  - **Relaxed Goal: prove that b = y b' for vector of bits b' and `short' relaxation factor y**
  - ZKP Idea – encode binary requirement as a quadratic relation:
    - $b_i \in \{0,1\} \leftarrow (over\ a\ field) \quad b_i \cdot (1 - b_i) = 0$
    - Usual basic setting:
      - Prover sends commitment of masking randomness $A = Com(\boldsymbol{a}\ ; \boldsymbol{r_a})$
      - Verifier sends challenge x
      - Prover sends response encodings $f_i = a_i + \mathrm{x} \cdot b_i$
    - To verify binary requirement, verifier computes quadratic function of x over encodings:
      - $g_i(x) = f_i \cdot (x - f_i) = \left[-a_i^2\right] + [a_i(1 - 2b_i)] \cdot x + [b_i(1 - b_i)] \cdot x^2$
      - And checks that x^2 coefficient is zero, by checking
      - $Com(g_i(x))$ =? Com($\left[-a_i^2\right]$) + Com($[a_i(1 - 2b_i)]$)*x
      - To allow verifier to do this, prover also sends in first step commitments to the non-zero coefficients

Application: Commitment to bits ZKP (basic idea)

$\boldsymbol{b} \in \{\mathbf{0}, \mathbf{1}\}^{s}, \boldsymbol{r} \leftarrow \text{SSet}$



Prover

$A = Com(\boldsymbol{a}\,; \boldsymbol{r_a})$

$C = Com([a_i(1-2b_i)]; \boldsymbol{r_c})$

$D = Com([-a_i^2]; \boldsymbol{r_d})$

$\boldsymbol{a}, \boldsymbol{r_a} \longrightarrow$

$x \leftarrow \text{CSet} \subseteq R_q$

$\longleftarrow$

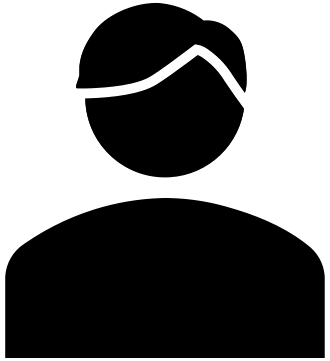
$\boldsymbol{f_i} = a_i + x \cdot b_i \qquad (\boldsymbol{f}, \boldsymbol{z_b}, \boldsymbol{z_c}) \longrightarrow$

$\boldsymbol{z_b} = \boldsymbol{r_a} + x \cdot \boldsymbol{r}$

$\boldsymbol{z_c} = \boldsymbol{r_d} + x \cdot \boldsymbol{r_c}$

Restart if $\|(\boldsymbol{f}, \boldsymbol{z_b}, \boldsymbol{z_c})\|_\infty > \boldsymbol{B'}$

$B = \text{Com}(\boldsymbol{b}\,; \boldsymbol{r})$



Verifier

$\boxed{A + x \cdot B \stackrel{?}{=} \text{Com}(\boldsymbol{f}; z_b)}$

$\boxed{D + x \cdot C \stackrel{?}{=} \text{Com}(\boldsymbol{g}; \boldsymbol{z_c})}$

$\|(\boldsymbol{f}, \boldsymbol{z_b}, \boldsymbol{z_c})\| \,?< B'$

$\boldsymbol{g} = [\boldsymbol{g_i}(\boldsymbol{x})] = [\boldsymbol{f_i} \cdot (\boldsymbol{x} - \boldsymbol{f_i})]$

Application: Commitment to bits ZKP (basic idea)

- Commit to bits ZKP Soundness argument sketch:
  - Using **three** rewindings of a prover on distinct challenges: x_1, x_2, x_3 (same commitments, but different responses $f_{i,j}$ ($j = 1,2,3$)
  - -> Get 3 relaxed openings $(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \widehat{\boldsymbol{c}}, \widehat{\boldsymbol{d}})$ of A,B,C,D
    - with relaxation factor $y = x_1 - x_2$
    - Must be same openings by binding of Com, hence:
    - $y \cdot f_{i,j} = x_j \cdot \hat{b}_i + \hat{a}_i$ (j=1,2,3)
    - $y \cdot f_{i,j} \cdot (x_j - f_{i,j}) = x_j \cdot \hat{c}_i + \hat{d}_i$ (j=1,2,3)
    - → Combine above pairs of relations to get a Vandermonde linear system over R_q:

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} \cdot \begin{pmatrix} -\hat{a}_i^2 - y\hat{d}_i \\ \hat{a}_i(y - 2\hat{b}_i) - y\hat{c}_i \\ \hat{b}_i(y - \hat{b}_i) \end{pmatrix} = 0$$

- Our adjugate technique implies $\det(V)\, \hat{b}_i(y - \hat{b}_i) = 0$ in R_q

Application: Commitment to bits ZKP (basic idea)

- Commit to bits ZKP Soundness argument sketch (cont.):
- Our adjugate technique implies $\det(V)\,\hat{b}_i\left(y-\hat{b}_i\right)$ = 0 in R_q, where
- $det(V) = (x_1 - x_2)\,(x_1 - x_3)(x_2 - x_3)$
- Want to use `NTT-friendly' rings and `large' challenges
  - Cannot assume det(V) is invertible in R_q
- But, still want to "cancel" det(V) factor
- → Speed-up Lemma 1:

Lemma 7. *Let* $f_1, \dots, f_n \in R$ *for some* $n \geq 1$. *If* $\prod_{i=1}^{n} f_i = 0$ *in* R_q *and* $q/2 > \|f_1\|_\infty \cdot \prod_{i=2}^{n} \|f_i\|_1$, *then there exists* $1 \leq j \leq n$ *such that* $f_j = 0$.

- -> We choose q large enough s.t. Lemma 7 applies: $q/2 > \det(V)\,\hat{b}_i\left(y-\hat{b}_i\right)$ → can cancel det(V) to conclude
  - $\hat{b}_i\left(y-\hat{b}_i\right)$ = 0 → "relaxed" soundness holds: $\hat{b}_i = y \cdot b_i'$ with $b_i' \in \{0,1\}$

Application: One-of-N ZKP

- One-shot variant of multi-shot lattice ZKP [E+19a], DL ZKP in [GK15]
  - Prover witness $\boldsymbol{\ell} \in [\boldsymbol{N}], \boldsymbol{r} \leftarrow$ Sset (\`short')
  - Verifier input: $(P_1, \dots, P_N)$
  - Original Goal [GK15]: prove that $P_\ell = \mathrm{Com}(\mathbf{0}\,;\boldsymbol{r})$
  - **Relaxed Goal (Our protocol): prove that $\boldsymbol{y}' \cdot P_\ell = \boldsymbol{Com}(\mathbf{0}\,;\hat{\boldsymbol{r}})$ for \`short' y' and $\hat{\boldsymbol{r}}$**
  - ZKP Idea – encode requirement as a polynomial relation:
    - Decompose $\ell = \sum_{j=0}^{k-1} \ell_j \boldsymbol{\beta}^j$ and $\boldsymbol{i} = \sum_{j=0}^{k-1} \boldsymbol{i}_j \boldsymbol{\beta}^j \in [\boldsymbol{N}]$ into $k = O(\log N)$ base-β digits
    - Write each digit ℓ_j in unary: $\boldsymbol{\delta}_j = (\delta_{\ell_j,0}, \dots, \delta_{\ell_j,\beta-1})$ is a bit vector with 1 in ℓ_j'th pos. and 0 else.
    - Then $P_\ell = \mathrm{Com}(\mathbf{0}\,;\boldsymbol{r})$ is equiv. to $\sum_{i\in[N]} \left(\prod_{j\in[k]} \delta_{\ell_j,i_j}\right) \cdot P_i = Com(0\,;r)$ (*)
    - Prover commits to $\boldsymbol{\delta}_j$'s and uses \`Commit to Bits' Protocol variant to prove $\boldsymbol{\delta}_j$'s are well formed
      - Prover sends commitments of masking randomness $A = Com(\boldsymbol{a}\,;\boldsymbol{r_a})$ (and C, D)
      - Verifier sends challenge x
      - Prover sends response encodings $f_{j,i_j} = a_{j,i_j} + \mathrm{x} \cdot \delta_{\ell_j,i_j}$
    - To verify 1-of-N relation (*), verifier computes degree k function of x over encodings:
      - $P(x) = \sum_{i\in[N]} p_i(x) \cdot P_i = \sum_{i\in[N]} \left(\prod_{j\in[k]} f_{j,i_j}\right) \cdot P_i = \sum_{i\in[N]} \left([e_{i,0}] + [e_{i,1}] \cdot x + \cdots + \left[\prod_{j\in[k]} \delta_{\ell_j,i_j}\right] \cdot x^k\right) \cdot P_i$
      - And checks that x^k coefficient is a commitment zero, by checking
      - $P(x) - ([\sum e_{i,0} P_i] + [\sum e_{i,1} P_i] \cdot x + \cdots + [\sum e_{i,k-1} P_i] \cdot x^{k-1})$ = Com(0,z) for a z sent by the prover
      - To allow verifier to do this, prover also sends in first step commitments in the coefficients of x^j (j < k)

Application: One-of-N ZKP

- Commit to bits ZKP Soundness argument sketch:
  - Using the extractor of our Relaxed `Commit to Bits' protocol with relaxation factor y = $x_1 - x_2$, we extract an opening $\hat{\ell}$ and $\widehat{p_i}$
  - Using **k+1** rewindings of a prover on distinct challenges: $x_1, \ldots, x_{k+1}$
  - → get a (k+1)'th order Vandermonde linear system with matrix V over R_q
  - → By our adjugate technique, extract a relaxed decommitment of the form
  - $\det(V)\, y^k P_{\hat{\ell}} = Com(0, \sum_{i \in [N]} \Gamma_i\, y^k \boldsymbol{z}_i)$
  - To reduce the relaxation factor to $\det(V)\, y$, we apply another observation:

Lemma 6. *Let $f, g \in R = \mathbb{Z}[X]/(X^d+1)$. If $f \cdot g^k = 0$ in $R_q = \mathbb{Z}_q[X]/(X^d+1)$ for some $k \in \mathbb{Z}^+$, then $f \cdot g = 0$ in R_q.*

  - We apply our bounds on det(V) and y to bound the extracted witness norm.
  - Moderately practical since k+1 = O(log N) is small –
    - in practice for N up to millions, usually optimal to use k a small constant k < 3

Application: Ring Signature Length Comparison

Ring Size	2^6	2^{10}	2^{16}	2^{20}	2^{30}
[LLNW15]	47000	75000	118000	146000	217000
[ESSLL19]	774	1021	1487	1862	3006
[**E**SLL19]	57	89	154	241	541

eprint.iacr.org/2018/773 – ”multi-shot” proofs (ACNS’19)

eprint.iacr.org/2019/445 – Advanced “one-shot” proofs

(to appear in CRYPTO’19)

Signature lengths are in KB.
Security level ≈ 128 bits

Application: Integer Range ZKP

- Integer range Proofs:
- Prover witness: $\ell \in [0, 2^k - 1]$, r `$short'$
- Verifier input: $(P \quad)$
- Original Goal: prove that $P = Com(\ell \,; r\,)$ with $\ell \in [0, 2^k - 1]$
- **Relaxed Goal (Ours): prove that $\boldsymbol{y' \cdot P \quad = Com(y' \cdot \ell \,; \hat{r})}$ for `short' y' and $\boldsymbol{\hat{r}}$**
- Basic ZKP idea:
  - Decompose $\ell = \boldsymbol{\sum_{i=0}^{k-1} \ell_i 2^i}$ in binary, $\ell_i \in \{0,1\}$
  - Prover commits to bits $B = Com(\ell_0, \ldots, \ell_{k-1};\ )$
  - Use `Commit to Bits' protocol to prove $\ell_i \in \{0,1\}$
    - Prover sends commitment of masking randomness $A = Com(\boldsymbol{a} \,; \boldsymbol{r_a})$
    - Verifier sends challenge x
    - Prover sends response encodings $f_i = Enc_x(\ell_i) = a_i + \mathrm{x} \cdot \ell_i$
  - Verifier checks `Commit to Bits' Proof and also checks that bits decompose ℓ
    - Inter-bit homomorphic encoding operation on encodings:
      - Verifier computes encoding v = $Enc_x\left(\sum_{i=0}^{k-1} 2^i \ell_i\right)$ from encodings of ℓ_i
      - $\sum_{i=0}^{k-1} 2^i \cdot Enc_x(\ell_i) = \sum_{i=0}^{k-1} 2^i \cdot a_i + x \ \cdot \sum_{i=0}^{k-1} 2^i \ell_i$
      - Checks that Com(v) and x*P are commitments to same ℓ

Speed-up technique 2: CRT-packing technique supporting inter-slot operations

- Efficiency problem:
  - Each bit ℓ_i consumes a whole ring element in the B commitment (UMC)

$$\mathrm{Com}(\ell, r) = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} r + \begin{bmatrix} 0 \\ \ell \end{bmatrix}$$

  - →k additional ring elements in commitment output
  - →Can maintain commitment length (set ring dimension d → d/k)
  - →But Com eval run-time still goes up by factor k (G_2 has $\geq k^2$ Ring elements)

- Our Speedup Technique 2: Use CRT-packing (a-la FHE) to pack k bits into 1 ring element

Speed-up technique 2: CRT-packing technique supporting inter-slot operations

- CRT message packing of k bits into 1 ring element:
  - Use R_q such that $z^d + 1$ splits into k irreducible factors $P_i(z)$ mod q (each of degree d/k:
    - $R_q \simeq R_q^{(1)} \times \cdots \times R_q^{(k)}$
    - $m \to CRT(m) = (m_1, \ldots, m_k) = (m \, mod \, P_1, \ldots, m \, mod \, P_k)$
- Packed Encoding is now:
  - $f = Enc_x(\ell_1, \ldots, \ell_k) = CRT^{-1}(a_1, \ldots, a_k) + \mathrm{x} \cdot CRT^{-1}(\ell_1, \ldots, \ell_k)$
  - Can extract from f encodings of individual slots:
    - $f_i = Enc_{x \, mod \, P_i}(\ell_i) = a_i + x \, mod \, P_i \cdot \ell_i$
  - But to support interslot homomrphic property of Enc, need all extracted encodings with respect to same x → need $x \, mod \, P_i = x$ for all i
    - $\sum_{i=0}^{k-1} 2^i \cdot Enc_x(\ell_i) = \sum_{i=0}^{k-1} 2^i \cdot a_i + x \cdot \sum_{i=0}^{k-1} 2^i \, \ell_i$
  - Our solution: choose challenge x of degree < d/k → $x \, mod \, P_i = x$ for all i

Speed-up technique 2: CRT-packing technique supporting inter-slot operations

Table 2: The (minimal) asymptotic time and space complexities of lattice-based protocols involving commitment to $k = O(\log q)$ messages. β_{SIS}: M-SIS solution norm, q: modulus, κ: the number of protocol repetitions, n: module rank for M-SIS, v: message vector dimension in a commitment, d: polynomial ring dimension, m: randomness vector dimension in a commitment. Assume: $\log q < \log^2 \beta_{\mathrm{SIS}}/2$ and degree-d polynomial multiplication costs $\widetilde{O}(d)$. To optimize both costs, one would set $n = v$ in all cases.

	Formula	**Multi-shot**[26, 19] $\kappa = \widetilde{O}(\lambda), v = k$	**One-shot** $\kappa = 1, v = k$	**One-shot + CRT** $\kappa = 1, v = O(1)$
Space UMC	$\kappa(n+v)d\log q$	$\widetilde{O}(\lambda^2 \log^2 \beta_{\mathrm{SIS}})$	$\widetilde{O}(\lambda \log^2 \beta_{\mathrm{SIS}})$	$\widetilde{O}(\lambda \log^2 \beta_{\mathrm{SIS}})$
Time UMC	$\kappa(n+v)md$	$\widetilde{O}(\lambda^2 \log^2 \beta_{\mathrm{SIS}})$	$\widetilde{O}(\lambda \log^2 \beta_{\mathrm{SIS}})$	$\widetilde{O}(\lambda \log^2 \beta_{\mathrm{SIS}}/\log q)$
Space HMC	$\kappa n d \log q$	$\widetilde{O}(\lambda^2 \log^2 \beta_{\mathrm{SIS}})$	$\widetilde{O}(\lambda \log^2 \beta_{\mathrm{SIS}})$	N/A
Time HMC	$\kappa n(m+v)d$	$\widetilde{O}(\lambda^2 \log^2 \beta_{\mathrm{SIS}})$	$\widetilde{O}(\lambda \log^2 \beta_{\mathrm{SIS}})$	N/A

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THANK YOU