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A Stochastic Approach to Prepayment Modeling

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Abstract

A new type of prepayment model for use in the valuation of mortgage-backed securities is presented. The model is based on a simple axiomatic characterization of the prepayment decision by the individual in terms of a continuous time, discrete state stochastic process.

One advantage of the stochastic approach compared to a traditional regression model is that information on the variability of prepayments is retained. This information is shown to have a significant effect on the value of mortgage-backed derivative securities. Furthermore, the model explains important path dependent properties of prepayments such as seasoning and burnout in a natural way, which improves fit accuracy for mean prepayment rates. This is demonstrated by comparing the stochastic mean to a nonlinear regression model based on time and mortgage rate information for generic Ginnie Mae collateral.

Contents

Illustrations

Tables

Chapter 1

Introduction

Mortgage-backed securities (MBSs) constitute one of the largest segments of the U.S. debt market. Since their introduction in 1970, MBSs have grown to account for almost 15% of the total domestic debt. Ordinary fixed-income valuation techniques are inadequate for the valuation of mortgage-backed securities because of the unique and important effect of mortgage prepayments. Hence an understanding of prepayments and how they complicate fixed-income valuation techniques is essential to understanding and investing in MBSs.

Underlying any mortgage-backed security are individual mortgages. A mortgage is a pledge of property as collateral for a loan. The pledge is cancelled if the loan is repaid according to the terms of the agreement. Otherwise, the lender may choose to assume ownership of the collateral. The term *mortgage* may also refer to the legal document which represents such an agreement. More typically, however, *mortgage* is used to refer to the loan rather than the pledge of property. The kind of property usually pledged in a mortgage is land or buildings. Although other types of property may be mortgaged, such as automobiles and boats, loan agreements specifying such property as collateral are not commonly called mortgages.

Mortgage payments consist of two parts, principal and interest, which are paid monthly. The principal payments are made according to a schedule agreed upon at the outset of the contract, the main requirement for which is that the sum of the scheduled principal payments must equal the amount of money initially borrowed. The schedule may require periodic principal reduction, which is called amortization, or no payments at all until maturity, when the entire balance comes due. A large payment due at maturity is called a balloon payment, and a loan with a balloon payment is sometimes called a balloon mortgage.

Interest represents a fee for the use of the principal. The interest rate is also agreed upon at the outset, but may be either deterministic or stochastic in nature. The interest portion of each payment is computed by multiplying the interest rate by the remaining principal balance at the start of the month.

There are a wide variety of mortgage types available. The usual kind is the fixed-rate, level pay mortgage, which gets its name from the fact that the interest rate is constant and the principal repayment amounts are designed so that each monthly payment is exactly the same amount. In order for this to be the case, the initial payments are almost exclusively interest with very little devoted to principal reduction. The interest amount of each payment decreases rapidly towards the end of the loan period, so that the last payments are almost exclusively for the reduction of principal. The term of this or any other kind of mortgage may be 15, 30, or sometimes some other number of years, with 30 being the most common.

Another common kind of mortgage is the adjustable rate mortgage (ARM). The interest rate on an adjustable mortgage resets periodically according to the value of some time varying index. Adjustable-rate mortgages often specify a low fixed rate, or teaser, for an introductory period, after which the rate begins resetting off an index. Their popularity waxes and wanes with fluctuations in the interest rates because of the teaser period. When interest rates are low, borrowers wish to lock in low rates and hence prefer fixed-rate mortgages. When interest rates are high they do not wish to lock in the prevailing rate and hence avoid fixed-rate mortgages. By turning to ARMs they can get below market rates initially, and may get lower rates in the future as well provided that interest rates fall.

Mortgages may themselves be pooled and used as collateral for the issue of other debt securities. Any security whose cash flows depend on the payments from a pool of mortgages is a mortgage-backed security, but frequently the term MBS is used as in a narrow sense as a synonym for a particular kind of security called a pass-through, in which case other kinds of mortgage-backed securities are termed mortgage derivatives.

A pass-through is a certificate which entitles its holder to all of the interest and principal payments on an underlying collection or pool of mortgages, less a small servicing fee. The process of creating a pass-through begins with lending institutions such as banks and savings and loans, which issue mortgages to homeowners. After a lender originates a sufficient number of mortgages which meet certain criteria, the mortgages may be pooled and sold to an issuing agency. The issuer sets the mortgages aside and sells the right to the payments from the pool as a pass-through. The issuer promises to make up any unpaid principal balance to the owners of the pass-through in case any homeowners with mortgages in the pool default on their payments. Because the cash flows for most other types of mortgage-backed securities are generated from an underlying pass-through rather than directly from individual mortgages, pass-throughs are often referred to as collateral. Although both residential and commercial mortgages can be used to create a mortgage-backed security, almost all MBS collateral consists of residential mortgages.

The first mortgage-backed pass-through security was issued by the Government National Mortgage Association in February, 1970. Since then the mortgage-backed securities market has experienced tremendous growth. In 1980 over $110 billion in securitized mortgage debt was outstanding in the United States and by 1990 this amount had grown to approximately $1 trillion, almost 40% of all residential debt in the country.[1] In comparison, the total long-term debt in all U.S. bond markets was $8 trillion at that time.[2] By 1992 the value of outstanding residential debt had risen to $2.5 trillion[3], at least $1.4 trillion of which has been securitized.[4]

The explosive growth in the mortgage market has been accompanied by a proliferation of mortgage-backed securities. Instruments such as collateralized mortgage obligations, planned amortization classes, targeted amortization classes, and very accurately defined maturity bonds have evolved in recent years to meet investors' demands for the high yields offered by mortgage-backed securities without some of the particular risks these securities may involve. Other types of derivatives, such as interest only and principal only bonds, floaters, and inverse floaters, have evolved to offset or emphasize other types of risk.

The existence of these derivative securities has been facilitated to a great degree by the increased issuance of the most basic of mortgage-backed instruments, pass-throughs, which serve as the building blocks for more complicated structures. Pass-throughs are also an important investment instrument in their own right. Hence questions relating to the fair valuation and risk assessment of these securities are of prime concern to a large number of investors controlling a significant proportion of the total amount of domestic investment capital. Our analysis of this topic begins by addressing the evolution of the mortgage and mortgage-backed security.

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[1]Ames (1993), 5.

[2]Fabozzi (1991), 7.

[3]Gerber (1992), 111.

[4]Ames (1993), 5.

1.1 Background

The following summary of the early history of mortgages is from Homer and Sylla's *A History of Interest Rates* (1991). Mortgages have existed in various forms for thousands of years, at least since the time of the Babylonians and perhaps earlier. Some provisions for mortgages were made in the Code of Hammurabi, a written specification of Babylonian laws that was collected and recorded during the reign of Hammurabi, the sixth king of the first dynasty of Babylon, who ruled from 1792 to 1750 BC. The Code governed many aspects of Babylonian culture, including business and family customs as well as civil and criminal law. With respect to mortgages, the Code recognized loan contracts at interest and allowed for the pledge of property as collateral to protect the creditor. Any property could be pledged, including wife, concubine, land, house, credits, and so on. There were explicit and detailed regulations concerning the mortgage of land, farmland in particular. Temples were generally the center of banking activity.

The Romans and Greeks both made use of the mortgage as a financial contract. As in Babylonia, temples were the center of finance in Greece. Most loans made by temples were personal loans secured by land. They were usually made for five years, though some for only one. A few were for much longer, up to fifty-five years, although loan terms of such duration may have been due to delinquency rather than contract. Interest on these loans was typically due in annual installments.

Records of Roman financial activity are not as comprehensive as those of the Greeks, perhaps because the Romans left most commerce and banking matters to the endeavor of foreigners. What little evidence there is suggests that most financial dealings were limited to personal loans extended by foreign traders. These loans were secured by real estate and were usually made for short periods.

In the Middle Ages a credit instrument called a census became a popular means for creditors to invest in land and for debtors to obtain funds. There were various kinds of census which closely resembled modern mortgages and annuities. Farmers, nobles, and states with property could raise money by selling a census, which was typically secured by land but could also be secured by other assets such as tax receipts. The buyer paid cash for the contract which entitled him to annual interest payments. A census could be perpetual, in which case it would run forever and the principal would never be repaid, just like the British consols of the mid eighteenth century. Unlike the British consol, however, there is no medieval census which still pays interest.

A census could also be for the life of either the buyer or seller, like a modern life annuity, or it could run for a fixed number of years. This kind of census, which was called temporary, is like the modern mortgage. The census could pay either a fixed or variable rate of interest, and it might or might have an early prepayment option at the discretion of the buyer, the seller, or both.

The census continued to thrive through the Renaissance and into the modern period. By the eighteenth century, a temporary census or personal loan secured by real estate had become known as a mortgage. In England at this time, mortgages had become somewhat more standard: the contract rate was set to a variable level that was renegotiated yearly, and either party could terminate the loan agreement early, six months after notification. Two centuries later in the United States the concept was little changed. In the early part of this century American home mortgages were typically balloon loans of five or ten year terms, with both parties able to call for early repayment of principal.

Fabozzi and Modigliani's *Mortgage and Mortgage-Backed Securities Markets* provides an excellent discussion of the recent history of mortgages and the rise to prominence of the the modern level pay, fixed rate mortgage. The level pay, fixed rate mortgage was not introduced until the 1930s in response to the Great Depression. This type of mortgage is now termed "traditional", despite the fact that it has only existed for six decades, a comparatively short time considering the nearly 400 decades of recorded history of mortgages.

The callable balloon mortgage proved disastrous during the economic turmoil of the Depression. In order to raise cash to cover withdrawals, banks were forced to call their loans en masse. However, most borrowers, even those performing on their interest obligation, were unable to raise sufficient funds to repay the principal from their own resources and could not refinance the loans through other lenders. Thus there were massive numbers of defaults and foreclosures. The government responded to protect both borrowers and lenders by establishing several agencies. To provide credit facilities for commercial banks the government established the Federal Reserve, and for savings and loan institutions the Federal Home Loan Bank. To protect and foster the ability of borrowers to obtain funds for the purchase of a house, the Federal Housing Administration (FHA) was established in 1934 through the National Housing Act. Congress instructed the FHA to initiate programs to assist in the construction, acquisition, and rehabilitation of single and multifamily homes.

The FHA moved to fulfill its obligation by offering to insure mortgages against default. Thus investors would have a much smaller exposure to credit risk and would be more willing to provide funds for mortgages. At the same time, the FHA devised a new type of mortgage which would offer much greater protection to borrowers, the fixed rate, level pay mortgage. This new mortgage first of all eliminated the threat of early call on the part of the lender. Second, it provided for the amortization of the loan over its term, so that the borrower would not have to find a new source of financing in order to make the balloon payment due at maturity. To guarantee the success of the new design the FHA limited its insurance coverage for the lenders to this type of mortgage.

Success of the FHA in making the acquisition of homes for individuals easier met with only limited success, however, because the supply of funds available for mortgages was limited due to the illiquidity of such an investment. In response, Congress created the Federal National Mortgage Association (FNMA, or Fannie Mae) in 1938. Fannie Mae was charged with establishing a liquid secondary market for mortgages, primarily through the purchase and sale of FHA insured mortgages. Efforts to establish such a market proved somewhat ineffective until the establishment of two other government-related agencies similar to Fannie Mae, Ginnie Mae and Freddie Mac, and the introduction of the first mortgage-backed security in February, 1970. Since these three agencies mainly perform their function by issuing mortgage-backed securities they are now known collectively as agency issuers.

The reason for the initial lack of success in the development of a secondary market was related to the reliance upon depository institutions for mortgage funds. By 1950 banks and savings and loans held nearly 50% of outstanding mortgages in their portfolios, and by the mid 1970s this figure had grown to 64%. The problem with this arrangement was that these depository institutions were encouraged or required by legislation to confine many of their financial activities such as deposit seeking and lending to their local markets. This resulted in a poor allocation of capital resources, with some regions having a shortage of funds and others a surplus. In most areas there was usually a shortage due to a fundamental contradiction in the nature of depository institutions.

This fundamental contradiction arises for depository institutions that originate mortgages because their liabilities, in the form of ordinary savings deposits, are typically set at a floating rate, whereas their assets, in the form of mortgages, usually return a fixed rate. Thus if interest rates rise, the value of their assets falls while the value of their liabilities remains constant, creating a shortfall. Outside investors are reticent to purchase individual mortgages from originators first because there is a large amount of risk involved in the purchase of individual mortgages. Consequently the market is illiquid, making individual loans even less attractive.

To circumvent this problem the agencies have established programs to buy pools of mortgages meeting certain criteria from mortgage originators and resell the rights to the mortgage cash flows as a security. The enticement to the mortgage originator is that, without assuming any interest rate risk by holding the mortgages in a portfolio, the originator may retain a small portion of the interest payments as a servicing fee. The issuer generates revenue by withholding an additional small amount of the interest. Despite the fact that the interest rate paid to the holder of a pass-through, called the pass-through coupon or simply coupon, is less than the rate paid by the homeowner, called the mortgage rate or mortgage coupon, investors may nonetheless find these securities provide attractive yields compared to other types of fixed-income investments. Furthermore, pass-throughs are guaranteed by the issuer against default.

With the introduction of mortgage-backed securities the dependence upon banks and savings and loans was broken. Mortgage-backed securities opened the door for outside investors to supply funds without assuming the risk of owning individual mortgages. Thus it is the mortgage-backed security which has allowed the growth of the secondary mortgage-market and hence made it easier for individual Americans to obtain the credit necessary for the purchase of a home.

1.2 Prepayments

One important feature common to all home mortgages is the right of the borrower to prepay any portion of the outstanding principal at any time. Partial prepayment has the effect of shortening the stream of promised cash flows rather than reducing their amount, since as long as there is any principal outstanding the payment schedule must be maintained. In addition, most mortgages are not assumable, but are due on sale. This means that if a mortgaged property is sold, the remaining principal balance is due immediately. Together, the prepayment option and due-on-sale clause significantly alter the stream of cash expected from any collection of mortgages, and will be discussed in more detail later.

The ubiquitousness of the prepayment option for residential mortgages may on its face be somewhat surprising, since commercial loans do not necessarily contain this feature. There are a number of reasons why such is the case, however. Probably foremost is several hundred years of tradition. The prepayment option was already standard in England over two hundred years ago, and due to the close ethnic, social, and commercial ties between England and the United States it is not surprising that it became standard here as well. Another compelling reason is of course that homeowners may sell their houses at any time, and since it is imperative for the lender to ensure the creditworthiness of the borrower, it is in the lender's interest to allow and even require prepayment in some circumstances to prevent the debt from being passed to someone who is more likely to default. Although it is theoretically possible to prohibit prepayments under other conditions, some experts believe that mortgage with such prohibitions would not be successful in the marketplace since homeowners have become so used to having the prepayment option.[5]

Furthermore, the prepayment option can be exercised at no cost to the borrower. Many mortgages had prepayment penalties which were typically on the order of 50% to 100% of a full year of interest payments up until the late 1970's. An important court case in California in 1978, *Cynthia J. Wellenkamp v. Bank of America, et al.*, effectively killed such penalties for subsequent mortgages.[6] This case actually directly concerned the enforcement of the due-on-sale clause. The court ruled that lenders could not require repayment of a mortgage loan upon sale of the mortgaged property for the sole purpose of raising mortgage rates.[7] While the direct effect of this ruling was to transfer a particular claim from lenders to borrowers which was estimated by Dietrich *et al.* (1983) to be more than the entire net worth of all state-chartered California savings and loan institutions in 1981, the broader effect was to establish a precedent that prepayment penalties and other anti-consumer provisions would not be upheld by the courts. Today there are mortgage securities with prepayment prohibitions for a few years, but these are exclusively backed by non-single family residences, such as multi-family residential or commercial property.[8] Despite the fact that there are no penalties for early repayment of traditional loans, this does not translate into a lack of financial cost associated with refinancing, but it is important to note that such costs are exactly those involved in obtaining a new loan.

[5] Senft (1995), personal communication.

[6] *Ibid.*

[7] Green and Shoven (1986), 43.

[8] Senft (1995), personal communication.

The main difference between a mortgage-backed security and other types of fixed-income securities arises from the fact that a homeowner has the right to prepay any or all of the remaining principal balance at any time. While most other fixed-income securities pay cash flows according to a schedule which is known at the outset, the cash flows from a mortgage-backed security are stochastic. Furthermore, the way that homeowners exercise their prepayment option differs fundamentally from the way other financial options are exercised. Financial options are typically exercised to maximize the present value of expected future cash flows, whereas this is not the case for mortgages. Many homeowners prepay their mortgages as a result of unpredictable events which precipitate the sale of the mortgaged home, such as purchase of a new home as a result of increased financial means, a job-related move, or down-sizing in response to a lost job. Although some homeowners presumably do prepay their mortgages as the result of an economic analysis, it is unlikely that many of these decisions are optimal in terms of maximum expected present value, since such an analysis would require computer simulation or advanced knowledge of finance and partial differential equations. Furthermore, since the option to prepay a mortgage is neither a tradable nor a synthetically reproducible asset, sophisticated investors cannot drive inefficiencies out of the market, which renders standard financial option pricing theory useless. Thus prepayments make mortgage-backed securities fundamentally different from other fixed-income securities.

The importance of prepayments stems from the fact that the value of any cash flow is intimately tied to the date at which it is received. Consider a scenario in which interest rates are constant at 10% per year. An investment which returns \$1 one year hence is worth $\$1/(1+.10) = \$0.91$ today, while an investment which returns the same cash flow two years from now is worth only $\$1/(1+.10)^2 = \$0.83$ today. If there were no mortgage prepayments the present value of a pass-through could be computed as in this example by discounting the known cash flows back to the present using the appropriate discount rate. In reality, prepayments occur throughout the life of a pool in uncertain amounts. The uncertainty in the prepayment amount complicates the valuation process. Furthermore, it introduces three types of risk to the investor: price risk, reinvestment risk, and extension risk.

Price risk refers to the sensitivity of a pass-through's value to the dates at which the cash flows are received, as just illustrated. Since the entire outstanding principal balance or any portion of it can be prepaid at any time, any of the cash flows associated with a pass-through can depart from the scheduled amount, causing potentially dramatic changes in price. To see this, consider a pass-through with a face value of \$1 million selling at a price of \$1.1 million, which might happen if the pass-through paid a coupon higher than the discount rate. If all the mortgages in the pool paid off entirely tomorrow, the \$1.1 million investment would be returned as a single \$1 million principal payment, resulting in a sudden 9% loss.

Reinvestment risk refers to the fact that principal tends to be prepaid in declining interest rate environments, when borrowers can reduce their interest payments by refinancing their mortgages. Mortgage investors who receive the principal payments must then reinvest them at a lower rate that that at which their initial investment was made.

Extension risk refers to the tendency of prepayments to slow in high interest rate environments, causing the term of a pass-through to lengthen and its price to become much more volatile. In order to understand how the term of a security affects its price, consider another simple example involving two cash flows, the first a \$1.10 payment to be received in one year, the second a \$17.45 payment to be received in thirty years. If interest rates are a constant 10% then both cash flows have a present value of \$1. If interest rates are instead a constant 15%, then the present value of the first cash flow declines to \$0.96, a difference of only 4%, while the second cash flow has a present value of only \$0.26, a 74% decline. The present value or price of the cash flow with the longer term is more sensitive to changes in interest rates because the discount factor determining present value is applied to the cash flow through more periods. Thus pools with slow prepayments that continue to generate cash flows in the distant future have greater degree of extension risk than pools with high prepayment activity which return principal more quickly.

An alternate way of looking at extension is through its direct impact on price. The present value of the cash flows actually realized from a pass-through increases with extension if purchased at a discount, and decreases if purchased at a premium. An investor who purchases at a discount a pool that pays a coupon below the currently available rate may expect to receive an adequate yield, because the initial principal investment is made at a discount but returned at par. The investor realizes a gain whenever he receives principal payments, and the sooner the gains are made, the higher the yield is, which makes the investment more valuable. Thus an increase in prepayment rates drives the price of a discount pass-through upwards. Likewise, if an investor pays a premium for a pass-through with a high coupon which subsequently experiences high prepayments, he will earn little coupon income and the initial principal investment will be returned at par for a loss. Hence an increase in prepayment rates drives the price of a premium pass-through downwards.

Any investment intelligent financial decision must take into account not only present and expected future worth but also risk. Because of the complex nature of mortgage-backed securities, the risks in such an investment are many, and they are all integrally connected to the way in which prepayments occur. Thus it is imperative for investors to understand prepayments before making a decision that involves mortgage-backed securities.

1.3 Factors Affecting Prepayments

In order to predict the course of prepayments it is necessary to first recognize the factors that affect them. Foremost is perhaps the identity of the issuing agency. The issuer plays a role in the prepayment process because it dictates the specifications of the mortgages which may be included in pools and may make certain requirements of lenders who wish to securitize mortgages, which can have important consequences on the underlying borrower characteristics.

1.3.1 Issuers

While some pass-throughs are issued by private companies, most are issued by one of three government-related agencies: the Federal National Mortgage Association, the Government National Mortgage Association, and the Federal Home Loan Mortgage Association. The difference in the volume of securities issued between agencies and private companies primarily owes to a difference in credit quality, which makes the agency pass-throughs more desirable to investors. The three agencies were established to help provide Americans with easier access to housing by ensuring an abundance of mortgage credit. Each has an active program securitizing mortgages which encourages easy mortgage credit by providing creditors incentive to lend money. A complete discussion of the agency issuers requires mention of two other organizations, the Federal Housing Administration and the Department of Veterans Affairs. What follows is condensed from Lowell (1992) and Axelson and Smuckler (1993).

Federal Housing Administration

Congress created the Federal Housing Administration (FHA) in 1934 to provide mortgage lenders with insurance against loan default, making it easier for prospective homeowners to qualify to borrow funds for the purchase of a home. The FHA is a division of the Department of Housing and Urban Development (HUD) and operates various insurance concerns, including the Section 203(b) program, which provides coverage for mortgage loans collateralized by single-family homes.

HUD and the FHA impose various conditions on the lender, borrower, and loan. The 1993 HUD Appropriations Act requires that the maximum loan amount qualifying for Section 203(b) coverage be 95% of the median single family house price for the area, as determined by HUD. Until 1995 the loan to value ratio (LTV) may not exceed 95%. The borrower must make a minimum down payment of 3% of the first $25,000 of the appraised value of the house and 5% of any excess. In addition, the borrower is required to pay a mortgage insurance premium (MIP) for the insurance. For loans originated from 1991 on, the MIP is 0.5% of the outstanding loan balance each year for 5 years on loans with an LTV under 90%, and for 8 years on loans with an LTV over 90%. Beginning in 1995, the MIP is 0.5% of the loan balance each year for 11 years on loans with an LTV under 90%, 0.5% per year for 30 years on loans with an LTV between 90 and 95%, and 0.55% per year for 30 years on loans with an LTV over 95%.

Lenders must meet certain criteria in order to originate FHA insured loans. To obtain HUD approval, called an "Eagle," lenders must demonstrate and maintain a $100,000 net worth and a credit line of $250,000. They must also maintain adequate staff and facilities to originate and service loans. Lenders who have met these requirements may submit loans for FHA insurance before settlement. Lenders who meet the higher standards of the Direct Endorsement Program may submit loans after settlement. Direct Endorsement Program requirements include a minimum $250,000 net worth, 5 years of experience with at least 15 successful mortgage insurance applications, and completion of a HUD training course on underwriting requirements by all underwriting and technical staff. Lenders may only charge the borrower fees for such items as credit reports, surveys, title examinations, recording, appraisal, and taxes. There is no restriction on interest rates, which may be negotiated freely. HUD has established the Mortgage Review Board to ensure that lenders abide by its requirements. In the event of a violation the Board is empowered to take disciplinary action ranging from a letter of reprimand to suspension of HUD approval for up to 6 years, or longer for egregious violations.

Department of Veterans Affairs

The Department of Veterans Affairs (VA) has offered a service similar to that of the FHA for the exclusive benefit of veterans of the armed services since 1940. The VA does not provide full insurance, however, but instead guarantees the repayment of a percentage of the loan amount in the event of default.

Borrowers are required to pay a guarantee fee for the service the VA provides. This fee differs from a typical insurance premium in that it is a single up-front payment based on the size of the down payment, although no down payment is required. There is no limit on the size of a VA loan. The borrower may be required to pay fees for such items as credit reports, title examinations, surveys, hazard insurance, and taxes, and origination. The origination fee is limited to 1% of the loan amount. The lender may charge discount points as well, but the seller of the house must pay these rather than the borrower. The interest rate of the loan is set by the VA.

There are no special lender requirements to be able to submit loans for VA approval prior to settlement. However, those who meet certain requirements may obtain a special status which entitles all their loans to automatic guarantee without prior approval. Some of the requirements to qualify for automatic approval include having $50,000 in working capital, which is the excess of current assets over current liabilities, 3 years experience originating VA guaranteed loans, and at least one full time employee with 3 years experience devoted to making and reviewing VA loan underwriting decisions.

Federal National Mortgage Association

Congress established the Federal National Mortgage Association, commonly called Fannie Mae, in 1938 to provide a secondary market for FHA loans and later for VA loans. In 1968 Congress split the original association into two entities, one becoming a private company with the original name, the other remaining a government operated company named the Government National Mortgage Association. Although privately owned and managed, Fannie Mae is overseen by the Secretary of the Treasury and the Secretary of Housing and Urban Development. Its new mission is "to provide financial products and services that increase the availability and affordability of housing for low-, moderate-, and middle-income Americans,"[9] which it fulfills through an ex-

[9] Demaris (1993), 247

panded role in the secondary market. It now purchases a wide variety of non-FHA/VA loans, called conventional loans, as well as FHA/VA loans, although its role in the securitization of FHA/VA loans has diminished greatly. By securitizing and selling much of its mortgage portfolio, Fannie Mae provides a means of channeling money from the capital markets to lenders, thereby facilitating home purchases. Currently Fannie Mae is the largest provider of funds for mortgage loans in the country.

Fannie Mae purchases mortgages from nearly 3,000 primary lenders nationwide under a variety of programs. To be eligible to sell loans to Fannie Mae and service them, lenders must meet a number of criteria. They must obtain proper licenses to conduct business, maintain an acceptable net worth, demonstrate the ability to originate and service loans, and execute a contract that specifies origination, selling, and servicing guidelines. They must also demonstrate that each borrower is able and willing to repay and that the mortgaged property has sufficient value to cover the loan in the event of default. Required evidence includes credit reports, bank and deposit statements, employment verification, and property appraisal conducted according to strict regulations. Failure to comply with any requirements may result in suspension of eligibility, an order to repurchase loans, or other penalties as appropriate.

Eligible lenders may sell an array of mortgage products to Fannie Mae, including 10, 15, 20, and 30 year fixed-rate mortgages, 1 and 3 year ARMs indexed to treasuries or the 11th District Cost of Funds Index (COFI), and 7 year balloon mortgages with a refinancing option to name a few. Fannie Mae administers separate programs for conventional mortgages and FHA/VA loans. Mortgaged properties must be a one to four family residence, including units in certain condominiums, cooperatives, and related developments to be included in any program. With a few exceptions, Congress has limited the maximum original balance of a loan that Fannie Mae can purchase to $203,150 for single family homes in 1993, with higher limits for two to four family homes. Maximum LTVs vary, but generally lie between 70 and 95%. Fannie Mae requires that loans with an LTV above 80% be insured for one fourth of the property value. All Fannie Mae securities are fully modified, which means that investors receive scheduled principal and interest payments on time regardless of whether the mortgage payments have actually been made, in addition to any prepayments.

Pools are required to represent at least $1 million in original face value, although they may include aged in addition to new mortgages. More than one originator can combine mortgages within a single pool which may therefore represent various areas from around the country. Mortgage rates may vary by as much as 200 basis points (one one-hundredth of one percent), and servicing fees may range from 200 to 250 basis points. The actual payment delay is 25 days, which means that investors receive payments 25 days after they are due from the borrower, which is usually the first of the month. Since mortgage payments are made in arrears, an interest payment on a mortgage for the balance outstanding January first is not due until February first, and so the investor would not receive a payment for the January period until February 25th.

Federal Home Loan Mortgage Corporation

The Federal Home Loan Mortgage Corporation, or Freddie Mac, was created by Congress in 1970 to provide a secondary market for loans originated by federally insured savings banks and savings and loan associations. It is authorized to purchase FHA and VA backed mortgages and did so up until 1981, at which point it began to deal exclusively with conventional mortgages. Freddie Mac carries out its mission by purchasing mortgages, pooling them, and then selling the rights to the income generated by the pools as mortgage-backed securities. At its inception Freddie Mac was an entity within the Federal Home Loan Bank System, but the 1989 Financial Institutions Reform, Recovery, and Enforcement Act privatized Freddie Mac, giving it a corporate structure similar to that of Fannie Mae. Freddie Mac remains under the regulatory control of HUD.

Today it is no longer a requirement that lenders be federally insured thrifts in order to sell loans to Freddie Mac. Any institution may gain Freddie Mac approval by demonstrating good lending practices and maintaining minimum net worth requirements. After gaining such approval, lenders must meet other conditions governing underwriting, delivery, and servicing standards to remain in good standing. Freddie Mac expects that the mortgage loans it purchases are secured by property of sufficient value to recover the outstanding principal balance in the event of default, and that the lenders have evidence that the borrower can make timely payments according to the mortgage contract.

Freddie Mac has a wide assortment of programs for securitizing loans. The older programs issued partially modified pass-throughs, which guarantee the timely payment of interest and the eventual payment of principal. These programs had a payment delay of 45 days. Newer programs issue fully modified securities with shorter payment delays of only 15 days.

Freddie Mac allows the greatest heterogeneity of pools among the agencies. The original term for the mortgages in a single pool may vary anywhere from 15 to 30 years, and the mortgages may be seasoned. Thus the actual maturity of similar securities may vary substantially. Under one program, all mortgage rates in a single pool must at least as great as the pass-through coupon and may be as much as 250 basis points higher. In another program, mortgage rates may not be more than 200 basis points above the pass-through rate and may be below it, although the maximum spread of rates is limited to 100 basis points.

Government National Mortgage Association

The Government National Mortgage Association, or Ginnie Mae, was created in 1968 when Congress split the Federal National Mortgage Association into two parts, one of which remained Fannie Mae, the other of which became Ginnie Mae. While Fannie Mae became a privately owned corporation, Ginnie Mae is a government agency under the administration of the Department of Housing and Urban Development. Ginnie Mae was created to help provide home financing for families who have difficulty obtaining credit under other programs, and does so by maintaining an active securitization program for FHA and VA mortgages.

To gain approval to participate in Ginnie Mae securitization programs, lenders must first get an Eagle and show that they are in good standing with HUD. They must have prior experience with Fannie Mae servicing mortgages, or demonstrate in some other fashion the ability and experience to issue and service mortgage-backed securities. Lenders must also meet minimum net worth requirements.

Ginnie Mae pass-throughs are fully modified. Because Ginnie Mae is a government agency, its securities are backed by the full faith and credit of the United States and hence are regarded as having the lowest possible credit risk. Ginnie Mae has two different securitization programs, GNMA I, established in 1970, and GNMA II, established in 1983. Each of these are further divided by mortgage type and other pool distinctions, including original terms of 15 or 30 years, single family traditional mortgages, adjustable-rate mortgages, and other mortgage types. By far the largest program with the most liquid market is that for 30 year traditional mortgages on single family homes.

Of the MBS programs of the three agency issuers, Ginnie Mae's are the most homogeneous. All mortgages in a Ginnie Mae pass through must be no older than twelve months at issue, and 90 percent of the mortgages in any 30 year pool must have original terms of 20 years or more. For the GNMA I program, interest rates on all mortgages in a single pool must be the same, the servicer is allowed to deduct a 50 basis point servicing fee, and all mortgages must be originated by a single lender.

The GNMA II program permits greater flexibility. Mortgages in a single pool may be originated by different lenders, so the pools are usually more geographically dispersed. Originators are allowed to pool mortgage with different interest rates and retain all interest over the lowest rate in the pool as servicing, and servicing fees are allowed to range from 50 to 100 basis points. There are also differences in minimum pool size and payment delays. For the GNMA I program, the actual payment delay is 15 days, as compared to 20 days for GNMA II. The longer delay for GNMA II is due to the fact that all payments to MBS holders are routed from the servicers through a central paying agent, Chemical Bank, whereas in the older program the servicers remit payments directly to the pass-through owners. The minimum pool size for GNMA I is only \$1 million, whereas the minimum for GNMA II is \$7 million.

One last major difference between the agency programs deserves mention. Because most Freddie Mac and Fannie Mae pass-throughs are backed by conventional loans which are due on sale, whereas Ginnie Mae securities are backed by FHA and VA loans which are are assumable, prepayments on Ginnie Maes tend to be much lower than those on Fannie Maes or Freddie Macs. Furthermore, FHA and VA loans provide implicit subsidies for veterans and people of modest financial means, so the rates tend to be below market rates on conventional loans, which also tends to favor lower prepayment rates. Last, people with conventional loans are generally considered more mobile due to their superior financial position and hence more likely to prepay than those with FHA or VA loans. These differences help explain the discrepancies between prepayment rates for various MBS programs, which might vary by as much as a factor of two under otherwise similar conditions.

1.3.2 Other Determinants of Prepayments

Apart from a specification of a particular agency issuance program there are a number of explanatory variables which seem to affect prepayments to varying degrees. The variable with the single greatest influence appears to be the current market mortgage financing rate, which is closely related to the current coupon. The current market mortgage financing rate is the rate at which an ordinary borrower can obtain on a new mortgage at the current point in time. The current coupon is defined as the coupon rate that will make a pass-through collateralized by recently originated mortgages sell at par, a quantity that depends on the collateral type.

The current coupon for Fannie Mae and Ginnie Mae typically differ by a positive amount. This make sense because Fannie Mae as a private company does not have as good credit as the government corporation Ginnie Mae, so investors demand a premium for assuming the increased risk associated with Fannie Mae securities. There is also a difference between the current coupons for 15 and 30 year original term mortgages, with the 30 years commanding the higher coupon. This agrees with our expectation as well since the duration and hence price risk of the 30 year collateral is much greater than that of the 15 year collateral.

If there is a newly issued pass-through trading exactly at par then the current coupon can be observed directly. Otherwise it must be inferred from the coupons of securities trading nearest above and below par. Because the pass-through coupon differs from the mortgage financing rate by a fixed amount corresponding to servicing fees, the two move up and down together through time in lock step. There may be a minor discrepancy between the two rates owing to the fact that a slight degree of heterogeneity of coupon is permitted in most pools, so the average coupon rate within pools with a particular pass-through coupon may vary by a small amount.

To illustrate the relationship between mortgage rate and prepayments we have plotted prepayments versus relative coupon for all Ginnie Mae 30 year pass-throughs issued between 1984 and 1994 in Figure 1.1. The relative coupon at time t is defined as the current coupon when the security was issued minus the current coupon at t. Relative coupon is defined this way so that it corresponds to refinancing incentive, so that when mortgage rates fall, corresponding to a rise in relative coupon, there is stronger incentive to refinance. Thus we expect a positive relationship between prepayments and relative coupon. Figure 1.1 confirms this intuition. The vertical axis is represents single monthly mortality (SMM). The single monthly mortality at time t_i is the amount of principal which is prepaid between t_{i-1} and t_i divided by the outstanding principal at time t_{i-1} less scheduled principal payments for the interval. The strong relationship between prepayments and the relative coupon is usually referred to as the refinancing effect, because much of the prepayment activity is presumably due to people refinancing their mortgages when rates fall.

Behind the current coupon, the factor widely considered to have the next greatest impact on prepayments is seasoning, which refers to the gradual increase in prepay-

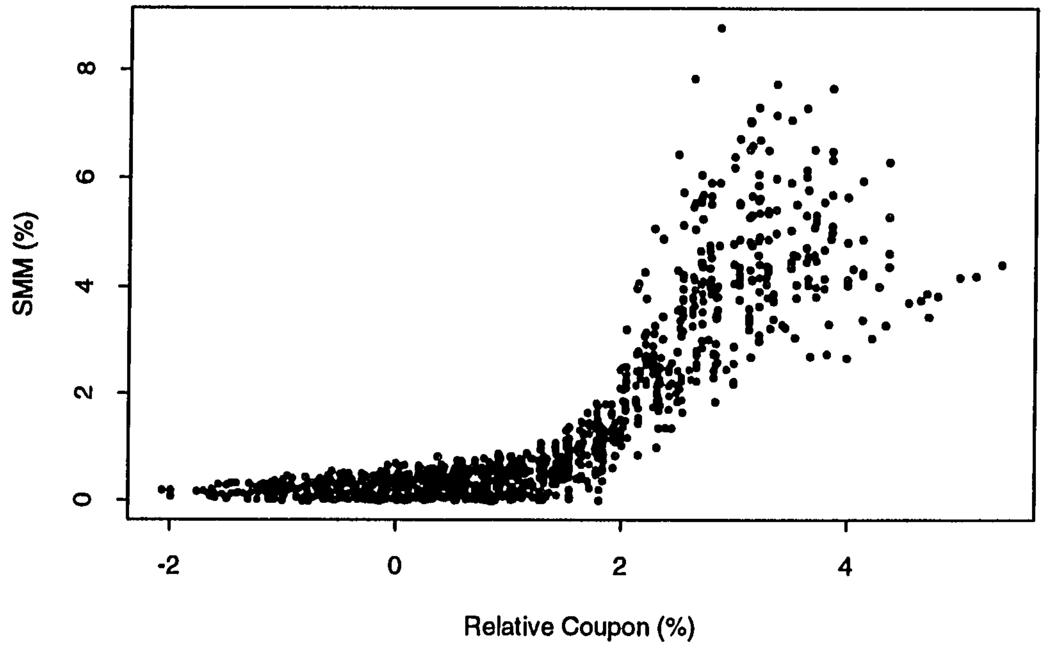

Figure 1.1 Prepayments for Ginnie Mae 30 Year Pass-Throughs.

ments for any pool of newly originated mortgages. For such a pool prepayments usually begin near zero and increase to some maximum level after a number of years. To illustrate, Figure 1.3 shows prepayments adjusted for the relative coupon affect versus time for all Ginnie Mae 30 year 8% coupon pools issued in 1987 (8s of 87). The time origin corresponds to February 1, 1990. The adjusted prepayments were computed by dividing actual prepayments by the value of a nonparametric fit of the data in Figure 1.1 evaluated at the appropriate value of the relative coupon.

Seasoning is usually explained in terms of aversion to relocation after the purchase of a new home. See for instance Schorin (1992). The traditional argument is that because new homeowners are not inclined to immediately move again, prepayments due to housing sales will initially be small. A few homeowners will encounter unforeseen circumstances that will require them to sell their houses and move right away, so this number will be positive even though close to zero. With the passage of time, however, circumstances or preferences may change somewhat so that the probability

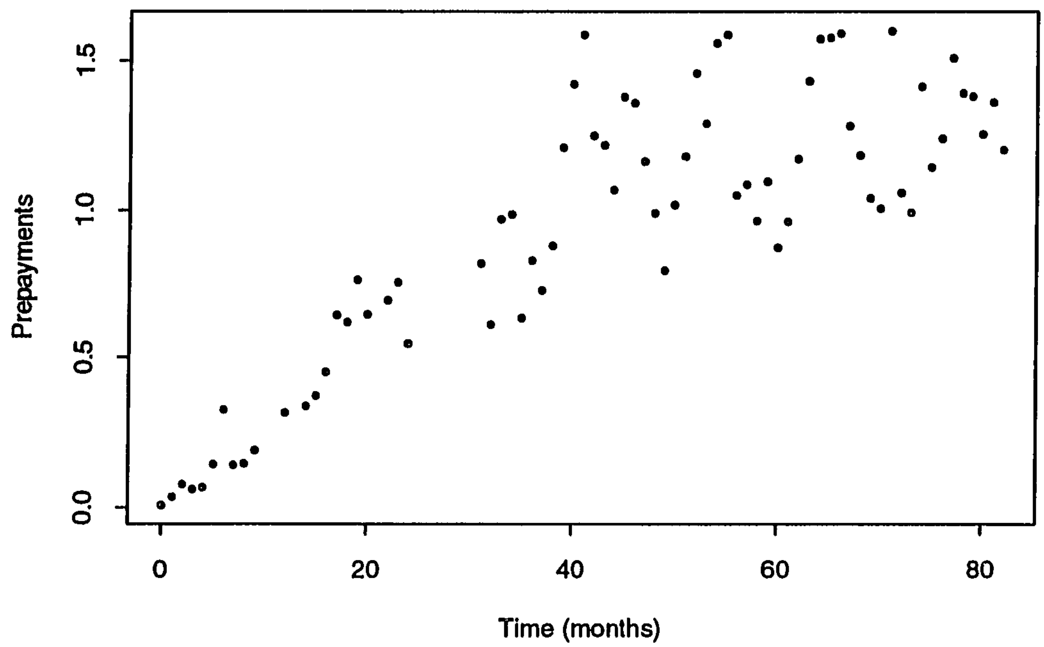

Figure 1.2 Refinancing-adjusted prepayments for the 8s of 87.

of the sale of property and termination of a mortgage will increase slowly to some steady state level.

Although this argument does not explain why such behavior should be evinced by the component of prepayments due to refinancing, it could easily be modified to do so. If we suppose there is an aversion to refinancing, due to inconvenience and time requirements, then it would be unlikely for most borrowers to undergo this process several times in succession. As time passes there would be more opportunity for rates to move sufficiently low to prompt refinancing-averse borrowers to do so in order to obtain lower interest obligations and hence save money.

The speed with which seasoning occurs is in part related to the relative coupon. Many authors, including Schorin (1992), Kang and Zenios (1992), and Richard and Roll (1989), have recognized that securities tend to season much more quickly in a high relative coupon environment than they do when the relative coupon is small or negative. This fact is evident in a comparison of Figure 1.2 and Figure 1.3, which shows refinancing-adjusted prepayments for the 11.5s of 87. The 8s in Figure 1.2

require about 40 months before coming close to a maximum mean value and continue to rise slowly through month 75 at least, possibly even longer, while the 11.5s are fully seasoned by month 20. The 3.5% difference in relative coupons shortens the time required for full seasoning roughly by a factor of four.

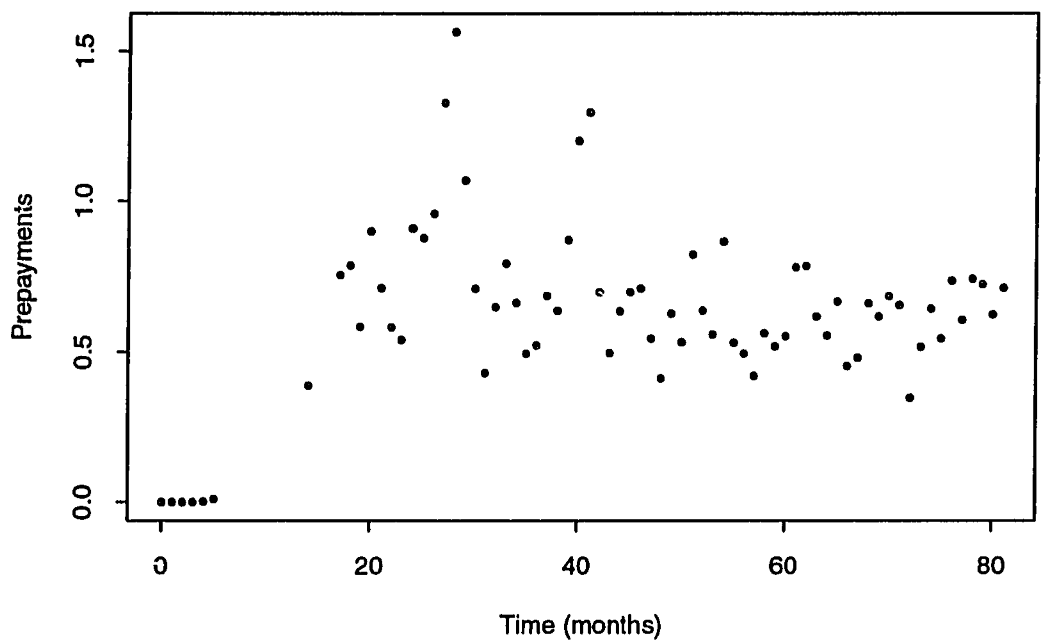

Figure 1.3 Refinancing-adjusted prepayments for the 11.5s of 87.

Figure 1.3 shows another effect called burnout, which is the decay in prepayment rates which is observed in most MBSs after they have full seasoned. Most researchers believe that burnout is caused by the cumulative effect of prepayments on the part of those borrowers most sensitive to lower interest rates. As the individuals with the lowest refinancing costs or highest sensitivity to refinancing opportunities prepay, the proportion of individuals with low sensitivity remaining in a pool increases. This leads to a gradual decline in prepayment rates.

An idea closely related to burnout is that prepayments at time t are strongly dependent not only on mortgage financing rates at t, but also on the entire path rates have followed to get to their levels at t. To understand this consider a simple example. Suppose interest rates remain at a constant high level over some reference period and then fall precipitously. One expects a flurry of refinancing activity at the end of the period as homeowners seek to cash in on the advantageous move in interest rates. If instead interest rates begin at the same level, fall immediately by the same amount, and then hold steady for the remainder of the period, one anticipates an initial wave of refinancings. Eventually the proportion of those inclined to refinance declines, leaving only those individuals insensitive to refinancing opportunities still holding their original mortgages. Thus the overall level of refinancing diminishes over time. By the end of the period, refinancings occur relatively infrequently.

There are many other candidate explanatory variables, including seasonal indicators, geographical indicators, yield curve indicators, measures of aggregate housing activity, and so on. There is evidence that some of these other factors, in particular seasonality, have significance in explaining prepayment activity. However, it remains that the bulk of the explanatory power lies in the factors discussed previously, which will for the sake of convenience serve as the focus for subsequent inquiry. Furthermore, the techniques to be developed later on are easily extendable to include additional factors, so their omission is of little consequence.

1.4 Purpose

It is clear that the valuation of mortgage-backed securities requires an understanding of prepayments. There are a wealth of models that have been constructed in the last decade to explain prepayments, but those actually employed by practitioners for evaluating and forecasting prepayments and identifying the various risks associated with a mortgage-backed security have been exclusively regression models. The principal shortcoming of these regression models is that they ignore the stochastic nature of prepayments. By doing so their useful capabilities are limited to the determination of mean prepayment rates conditional upon the future path of interest rates or other covariates, and functions related to the conditional mean, such as unconditional mean prepayments or confidence bands for the conditional mean path. However, questions about quantities depending on actual prepayment or conditional prepayments cannot be answered. For instance, an investor with a view on the future course of interest rates could not use a regression model to obtain estimates as to the probabilities that the price of a particular mortgage derivative might decline to zero, remain at its current level, or increase by 10% over some period. These and similar questions are of great importance in real applications. Furthermore, it will be shown that some derivative securities are seriously misvalued if a deterministic function, i.e., a regression line, is used to predict prepayments due to the underspecification of prepayment variability.

Accordingly, the purpose of this work is to develop stochastic prepayment models which will address these concerns. In Chapter 2 we will discuss earlier models, address the general question of how they are used in valuation, and point out some of the limitations of the previous work. In Chapter 3 we shall propose various dynamic stochastic alternatives which will explain financially motivated prepayments along with seasoning and burnout. When practical, the likelihood functions will be derived. These models will be seen to incorporate the path dependent nature of prepayments in a natural way. Model selection and estimation issues will be analyzed in Chapter 4. We will also discuss violations of assumptions and some alternatives. In the last chapter we shall present forecasts and prediction confidence bands for out-of-sample data, give an example where the use of a regression model to forecast prepayments leads to inaccurate security valuation, compare the mean of the stochastic model to a regression line, and offer some suggestions as to possible improvements and directions for future work.

Chapter 2

Prepayment Models and Valuation Techniques

2.1 Valuation

Valuation methods for mortgage-backed securities may be grouped into three categories: deterministic cash flow analyses, partly stochastic cash flow analyses, and contingent claim methods. The most basic of these is the deterministic cash flow analysis.

Suppose one wishes to determine the present value of a derivative security collateralized by a pool of mortgages. Suppose furthermore that one has identified a prepayment function $\mathbf{f}(\tilde{\mathbf{r}})$, where $\mathbf{f}$ is a vector-valued function of the stochastic state variable $\tilde{\mathbf{r}}$. For simplicity, assume that $\tilde{\mathbf{r}}$ is the vector of future interest rates. For a particular realization $\mathbf{r}$ of $\tilde{\mathbf{r}}$, $\mathbf{p} = \mathbf{f}(\mathbf{r})$ is the prepayment stream for the underlying pool. A complete description of the security, which would be available to an investor, requires a statement of scheduled cash flows $\mathbf{s}$ as well as a rule $\mathbf{g}$ for distributing prepayments to bondholders over time. Thus $\mathbf{g}(\mathbf{p})$ is the vector of cash flows paid to the securities under consideration in excess of $\mathbf{s}$ due to the prepayments $\mathbf{p}$. Both $\mathbf{s}$ and $\mathbf{g}$ are vector-valued because several derivative securities may be collateralized by a single pool of mortgages, which also explains why $\mathbf{g}$ may be very complicated. Nonetheless, it is known at the outset.

The analysis proceeds in a straightforward manner, starting with a guess at the future path of interest rates $\mathbf{r}$. Assuming that $\tilde{\mathbf{r}}$ realizes the path $\mathbf{r}$, then prepayments are given by $\mathbf{p} = \mathbf{f}(\mathbf{r})$. The total cash flows to the bond are $\mathbf{cf} = \mathbf{s} + \mathbf{g}(\mathbf{p})$. The present value of the cash flows is found simply by discounting and summing them,

$$PV = \sum_{i=1}^{n} \frac{\mathbf{cf}_i}{(1 + \mathbf{d}_i)^i},$$

where a subscript denotes a component of a vector and $\mathbf{d}$ is the vector of discount factors.

The method of choosing the discount factors may vary as well, but it is usually an appropriately chosen spread over the forward curve or the realized interest path $\mathbf{r}$. The process may be repeated for other choices of $\mathbf{r}$.

The obvious failure of this approach is that it assumes away all of the randomness in the future of the security. Not only are prepayments assumed to be deterministic, but interest rates are as well. Presumably this method only attempts to provide rough information about the mean and range of possible present values, but it does not necessarily even provide that. Since the expected value of a function of a random variable is not always the function of the expected value of the variable, the estimate of the mean obtained in this fashion is liable to differ significantly from the true value. Likewise, one should not expect the extreme present values of the security to be given by discounting the cash flows corresponding to one's view of extreme interest rate paths, especially for complicated securities.

A second class of valuation techniques may be described as a partially stochastic cash flow analysis, and is one of two methods popular in the literature, where it is commonly referred to as an option-adjusted analysis. This method is nearly the same as the deterministic cash flow analysis, except that a large number of vector-valued state variables are simulated from a distribution, and the present value of the security is determined as the average of the present values for each state. The primary flaw in this approach is that while the state variable is assumed to be stochastic, prepayments are not. For a given realization of the state variable, the prepayment function is deterministic, but the prepayment function is usually identified by estimating the parameters of a regression model. Thus in the estimation stage the prepayment function is formulated as a stochastic function, but in the simulation or prediction stage it is computed by a deterministic function. It will be shown in Chapter 5 that ignoring the randomness in prepayments can have a drastic effect on the expected present value of a security.

The last type of valuation, which is also popular in the literature, is a contingent claim method. A mortgage-backed security may be may be viewed as a long position in a bond and a short position in an American call on the bond struck at par. This is because the bond owner receives the scheduled mortgage payments if there are no prepayments. However, if the homeowner prepays the mortgage, the bond owner receives par and loses all subsequent cash flows. Hence the bond owner is short a call. Since prepayment may occur at any time, the call is of the American type. The bond may be valued by ordinary fixed-income techniques since it is a series of deterministic cash flows, so the problem has been reduced to valuing a call.

The standard rational option pricing technique goes as follows: Begin by positing an underlying stochastic process on which the value of the mortgage-backed security depends. This could be a Brownian motion representation of the price of the underlying home, a Markov process for the short-term interest rate, or some other plausible process. Once this process has been identified, stochastic calculus is used to obtain a deterministic partial differential equation satisfied by the mortgage-backed security. The partial differential equation is then solved by finite differences.

For example, suppose one wishes to compute the value of a mortgage-backed security S. Suppose also that the source of randomness driving prepayment behavior is the short-term interest rate r, which follows the mean-reverting process

$$dr = (l - r)sdt + \sigma r dz. \tag{2.1}$$

Here l, r, and σ are constant parameters and dz is standard Wiener process. A standard Wiener process is a stochastic function of time with the following two properties:

1. For any times t_1 and t_2 such that $t_x < t_2$, $z(t_2) - z(t_1) \sim \epsilon\sqrt{t_2 - t_1}$ where ϵ is a standard normal random variable.

2. For any times t_1, t_2, t_3, and t_4 such that $t_1 < t_2 < t_3 < t_4$, $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are statistically independent.

The value B at time t of a bond maturing at time T in the near future is given by

$$B = e^{-r(T-t)},$$

so

$$dB = rBdt - (T - t)Bdr.$$

Substitution for dr yields

$$dB = (r - (T - t)(l - r)s)Bdt - (T - t)\sigma r B dz. \tag{2.2}$$

At this point it is necessary to introduce a result from stochastic calculus known as Ito's lemma, which says that if

$$dx = a(x, t)dt + b(x, t)dz \tag{2.3}$$

and $G = G(x, t)$, then

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right) dt + \frac{\partial G}{\partial x}bdz. \tag{2.4}$$

Applying Ito's lemma to the security S, we have from (2.1), (2.3), and (2.4) that

$$dS = \left(\frac{\partial S}{\partial r}(l - r)s + \frac{\partial S}{\partial t} + \frac{1}{2}\frac{\partial^2 S}{\partial r^2}\sigma^2 r^2\right) dt + \frac{\partial S}{\partial r}\sigma r dz. \tag{2.5}$$

Let portfolio Π = long $B(T-t)$ securities and long $\partial S/\partial r$ bonds. Then

$$d\Pi = B(T-t)dS + \frac{\partial S}{\partial r}dB.$$

Substitute for dS and dB from (2.5) and (2.2) to get

$$d\Pi = \left(\frac{1}{2}(T-t)B\sigma^2 r^2 \frac{\partial^2 S}{\partial r^2} + (T-t)B\frac{\partial S}{\partial t} + rB\frac{\partial S}{\partial r}\right) dr.$$

Since Π does not depend on dz it is riskless, and hence must earn the riskless return r. That is,

$$d\Pi = r\Pi dt,$$

which reduces to

$$\frac{1}{2}\sigma^2 r^2 \frac{\partial^2 S}{\partial r^2} + \frac{\partial S}{\partial t} = rS \tag{2.6}$$

after substituting the appropriate expressions for Π and $d\Pi$. Finite difference techniques can be used to solve (2.6) when appropriate boundary conditions are specified.

There are a number of important assumptions involved in this type of valuation:

1. The price of the mortgage-backed derivative security is contingent only upon the specified variables, which in this example were the short rate r and time.

2. The specified variables follow known stochastic processes. In the example, r followed a mean-reverting process.

3. Markets are frictionless. That is, borrowing and lending occur at the same rate, there are no transactions costs or taxes, securities are perfectly divisible, short selling with full use of proceeds is permitted, and trading is continuous and instantaneous.

4. There are no arbitrage opportunities.

5. Investors act optimally.

These first two assumptions are probably not satisfied in reality. In this example, the short interest rate r and time have been identified as the variables affecting the price of a mortgage derivative. Although some investors might believe the price of an MBS to be contingent on only these two variables, others believe the entire term structure of interest rates is important, or that the underlying home values are important. If there are other significant factors then the partial differential equation obtained in the analysis is missing important terms. It is also necessary that there exist an additional independent tradable contingent asset for each stochastic variable introduced, otherwise a riskless portfolio cannot be constructed and the system cannot be reduced to a deterministic partial differential equation.

The correctness of the specified stochastic processes must also be questioned. Some investors believe that short interest rates follow a mean-reverting process and some do not. Although a mean-reverting process is plausible, perhaps the true process does not have a pull toward a long term mean value. Different processes lead to different partial differential equations, and it is impossible to know for certain exactly what process the underlying variables satisfy.

Although the first two assumptions are evidently not satisfied exactly, one might believe they could be formulated to represent a close approximation to reality, so that inferences and conclusions drawn from the contingent claim model would provide insight about the behavior of real mortgage derivatives. However, the last three assumptions are not remotely satisfied. One cannot borrow and lend money at the same rate, trading decisions can require a considerable amount of time to implement, and in the mortgage derivative market the bid-offer spread can amount to a sizable fraction of the price of a security, up to 10% or more, effectively eliminating the possibility of continuous trading. Not only does this mean instantaneously riskless portfolios must be held longer than mere instants, but also that price differences may exist between portfolios with identical cash flows. Since the underlying asset S, the prepayment option, is neither tradable nor synthetically reproducible, the arbitrage condition does not exist in reality, another reason that the riskless portfolio may not be constructed as argued. Together these effects invalidate the deterministic partial differential equation obtained in the analysis.

To compound this problem, the equation itself is usually not solved correctly, owing to a misspecification of boundary conditions. Borrowers are typically assumed to act in a financially optimal fashion; that is, they act to maximize the present value of their portfolios. If so, then the value of the prepayment option is the greater of its values alive and exercised on any exercise date. Since the option is American, this condition applies at every point in time. However, it is well known that many borrowers will not exercise their prepayment option no matter how profitable, and others exercise the option when deeply out of the money. This fact is typically ignored in the contingent claim analysis.

2.2 Previous Research

Almost all published work on prepayments can be classified as either a contingent claim analysis or a regression model. There are a few minor exceptions, including some neural network models which were first examined by Yamamoto and Zenios (1993). The neural network models have not demonstrated any superiority to traditional, less computationally intensive methods.

The contingent claim approach has been very popular in the literature, and is preferred by many academics owing to the method's aesthetic appeal. The aesthetic draw is evident in the remarks of one proponent, who complains that empirical models "are somewhat unsatisfying to the financial economist in that borrower prepayment behavior is passively accepted without rational economically motivated explanations."[10] There is nothing ad-hoc about option pricing; contingent events occur if they maximize expected present value.

Kau, Keenan, Muller, and Epperson (1987, 1992), Titman and Torous (1989), or Schwartz and Torous (1992) offer simple, straightforward analyses. These models make all the incorrect assumptions previously outlined, and hence are of little practical value.

Others have made attempts to relax some of the assumptions, particularly the optimal exercise provision. This provision requires that the borrower exercise his call whenever it is in the money. One simple way to relax this assumption is explored by Leung and Sirmans (1990), who make the minor adjustment of allowing deterministic transactions costs (i.e. points and fees for refinancing), which is equivalent to shifting

[10] Archer and Ling, 1988

the strike price of the option from zero by an amount equal to the transaction cost. A subsequent work by Archer and Ling (1993) adopts a similar but slightly more general approach, in that transactions costs were allowed to vary according to some distribution. Murphy (1991) and McConnell and Singh (1994) specify different classes of borrowers with differing fixed transactions costs for each class.

Schwartz and Torous (1992) and Kau, Keenan, Muller, and Epperson (1993) take the more important step of incorporating empirical prepayment models into their analyses. In these models only a fraction of live calls exercised at any point in the state space is determined according to a prepayment function. It is unclear whether the incorporation of an empirical prepayment model is an improvement or not; it is fundamentally inconsistent to allow suboptimal exercise and still suppose that the valuation equation holds, which is based on perfect market and no arbitrage assumptions.

Although different contingent claims analyses may incorporate various incorrect assumptions, perfect markets and no arbitrage are fundamental conditions common to all. As explained earlier, these two assumption are not remotely satisfied in the case of mortgage-backed derivatives, which is why contingent claims methods are incorrect in this context, and why they are not used by practitioners who invest real money.

The earliest empirical method used to quantify mortgage prepayments for valuation purposes was a rule of thumb called the twelve year prepaid life assumption, which was based on the observation that most mortgages are not held to maturity. The rule was simply to assume that there would be no prepayments on a pool of mortgages for the first twelve years. At the end of this period, the pool mortgages would prepay in full.

Mortgage-backed securities traders soon recognized the deficiency of this rule and began turning to empirical data for guidance. The Federal Housing Administration (FHA) prepared tables of prepayment rates using actual data for FHA insured loans issued past a certain date, indexed by age of loan. The tables were typically used with the current prepayment information on a pool to estimate future prepayments in a simple fashion. The current prepayment rate on the pool would be compared to the table value for loans of similar age to obtain a multiplier. The multiplier would then be applied to the table values for older loans to see how the pool in question would prepay as it matured.

This procedure had a number of shortcomings. Besides neglecting the effect of various factors known to affect prepayments, such as interest rate levels, the FHA experience did not provide a set standard. The statistical techniques used to analyze the underlying data and even the method for selecting data changed from year to year. For instance, until the 1983 table was issued, all mortgages back to 1957 were included in the analysis, but subsequently only mortgages from 1970 and later were included. The tables had the additional potential to confuse expressed prepayment speeds, which were typically quoted as a multiple of a table value. If one quoted the prepayment rate of a security as 200% FHA, this figure could be based on tables from several different years and so imply different absolute speeds.

In order to simplify the analysis and comparison of mortgage-backed securities, the Public Securities Administration (PSA) developed the Standard Prepayment Model as an approximation to FHA experience. As with FHA experience, the sole variable in the PSA model is pool age. The PSA model estimates the conditional prepayment rate (CPR) of a pool, which is the fraction of outstanding principal prepaid during a given month, expressed at an annualized rate. According to the PSA model, prepayments begin at the initial level of 0.2% per year in the first month and increase 0.2% every month thereafter until reaching a maximum of 6.0% per year in month thirty. After that, prepayments level off and remain at 6.0% per year until maturity.

The primary use of the PSA model is as a comparative tool. Prepayment rates of various securities may be expressed easily and unambiguously in terms of PSA speed, facilitating comparison on an age-adjusted basis. For this purpose the PSA curve is still widely used. For instance, a 15 month old pool prepaying at 50% PSA has a CPR of 1.5%, while a 90 month old pool prepaying at 200% PSA has a CPR of 12%. In absolute terms, the older pool is prepaying eight times as fast as the younger pool, but only four times as fast on an age adjusted basis, all else begin equal. As a predictive tool, however, the PSA curve is no better than FHA experience, as it fails to account for any explanatory factor other than time.

The first serious attempt at modeling prepayments is due to Curley and Guttentag (1974). They point out the absurdity of the 12 year prepaid life assumption and argue that age, current interest rates, and current discount points all affect prepayments. Subsequent work by Peters (1979) identifies more than a dozen other explanatory variables that affect prepayments for various kinds of mortgages. Milonas and Lacey (1988) perform a series of linear regressions of prepayments on various explanatory variables, and find both interest rates and time to have a significant effect. In particular, they find that in declining interest rate environments there is a lagged relationship between interest rates and prepayment experience, suggesting that prepayment decisions are made months before the cash payment is received. They also find that the structural relationship between prepayments and interest rates differs for pools with different mortgage coupons.

Green and Shoven (1986) use a proportional hazards formulation to model the prepayments on nearly 4,000 mortgages held by two large California savings and loans. According to the authors, if P denotes the probability that a mortgage of age a will prepay given exogenous factors $x_1, \ldots, x_n$ at time t, then

$$P = \lambda(a)\pi(x_1, \ldots, x_n),$$

where $\lambda(a)$ is the proportion of the population that would prepay even under completely stationary, homogeneous conditions, and $\pi(x_1, \ldots, x_n)$ is a function which is greater or less than one depending on whether the exogenous factors $x_1, \ldots, x_n$ make prepayments more or less likely. The particular form they choose for π is

$$\pi(x_1, \ldots, x_n) = \exp \sum_{i=1}^{n} \beta_i x_i.$$

The two main assumptions they make are that the effect of the exogenous variables is the same multiplicative amount for any age mortgage, and that there is no dependence of prepayments today on past or anticipated future values of any exogenous factors. In the actual estimation of their model they choose $n = 1$ and use a measure of the difference between the mortgage coupon and current market rates for the variable x. The authors calculate point estimates of parameters and standard deviations of their estimates and estimate prepayments under various interest rate scenarios, but they do not present a measure of the goodness-of-fit of their model or compare predicted prepayments for out-of-sample points to actual data.

Some more recent work on prepayments is based on Green and Shoven's original proportional hazards model, such as Cooperstein and Meyers (1991) and Schwartz and Torous (1989, 1993), but these models do not explain the observed data well. In particular, Schwartz and Torous (1993) construct a baseline hazard function whose log is a cubic polynomial in time. Such a function has the unfortunate tendency of increasing rapidly in the later part of the mortgage term, in contrast to actual prepayment aging behavior, which tends to fall off.

A study by Navratil (1985) models prepayments for pools of mortgages using a linear regression based on a logistic transformation. The transformation is given by

$$Z_{it} = \log \frac{P_{it}}{100 - P_{it}},$$

where P_{it} is the prepayment rate for pool i at time t. He proceeds with a linear regression based on the model

$$Z_{it} = a + bD_{it} + cG_{it} + d(DG)_{it} + \epsilon_{it},$$

where D_{it} is the difference between the current mortgage refinancing rate and the mortgage coupon for the ith pool at time t, G_{it} is the indicator variable assuming the value one if D_{it} is nonnegative and zero otherwise, and ϵ_{it} is a random error term.

Richard and Roll (1989) published the first outline of a large scale dealer model. They model prepayments as a product of four factors, which they call refinancing, seasoning, seasonality, and burnout. The refinancing factor is a sigmoidal function of an interest rate variable, representing the tendency of homeowners to refinance when interest rates fall. Seasoning is an increasing function of age, with a slight dependence on the interest rate variable, and represents the tendency of refinancings for newly issued mortgages to begin at a low level, increasing gradually to a steady state. Seasoning occurs much more rapidly when interest rates are very low, which is why there is a dependence on the interest rate variable. The seasonal component depends on the month of the year and explains the pattern of higher levels of refinancing during the summer and fall and lower levels during the winter months. Burnout is a decreasing function of age. It represents the relative decline in refinancings for very old mortgages, which happens since the people who have not refinanced over time are apparently insensitive to such opportunities. Their data include over 100,000 observations on mortgage prepayments spanning 1979 to 1988. Their results are impressive: they obtain an R-squared of 94.6%. Most have greeted this result with scepticism, for they do not disclose the number of parameters they estimate or any other details of their model.

Perhaps the best prepayment model so far is due to Kang and Zenios (1992), and is based on the Richard and Roll model. It is the first and only large-scale, empirical model published with any great detail and appearing in an academic journal Kang and Zenios include three explanatory variables: the age t of the mortgage, the month m, and the relative coupon C/R, where C denotes the mortgage contract rate and

R denotes the current market rate for newly issued mortgages. They use the same multiplicative model for the prepayment function P as Richard and Roll, which may be written

$$P(\{C/R\}_t, t, m, C/R) = s\rho\sigma b\epsilon,$$

where $\{C/R\}_t$ denotes the sequence of observed values of C/R from origination to time t, $s(m, C/R)$ represents seasonality, $\rho(C/R)$ refinancing, $\sigma(\{C/R\}_t, t, C/R)$ seasoning, and $b(\{C/R\}_t, t, C/R)$ burnout. The term ϵ is a normal random error with unit mean. This is a poor choice for the error distribution, as it admits the possibility of negative prepayments. A lognormal distribution would avoid this problem and hence would be a better choice.

Each factor is fit using a piecewise linear approximation. The model is constructed in stages by minimizing the total squared error of the fit at each successive step. The data consist of monthly observations of t, m, C/R, the prepayment rate P, and w, the outstanding principal balance. The first stage is to fit the seasonal variations. To do this, all the CPR observations corresponding to securities within some age bracket $[t_0, t_1]$ are arranged in groups having approximately the same coupon. Twenty-one intervals of C/R in the range [0.5, 1.6] were used in all. For K securities, and n_j observations for the jth security, $j = 1, \ldots, K$, there would be $\sum_{j=1}^{K} n_j$ observations altogether. Letting $C(m, C/R)$ denote the set of observations corresponding to month m and relative coupon C/R, then the seasonal adjustment is estimated by

$$\hat{s}(m, C/R) = \frac{\displaystyle\sum_{i \in C(m,C/R)} w_i P_i}{\displaystyle\sum_{n=1}^{12} \sum_{i \in C(m,C/R)} w_i P_i}$$

The seasonality factors for each coupon class are examined to make sure they fall within two standard deviations of each other. If they do not, the age interval $[t_0, t_1]$ is reduced and the process repeats. This is done because the authors believe that while relative coupon should not affect seasonality, seasoning and burnout should. Since seasoning and burnout are only noticeable for very young and very old mortgages, the prepayments for medium age mortgages should display only pure seasonality effects. When suitable consistency of the seasonality factors has been achieved, the seasonally adjusted prepayments P_{si} are obtained from the original observations P_i by

$$P_{si} = P_i / \hat{s}(m_i, C/R_i).$$

The refinancing effect is found by fitting a piecewise linear function to the seasonally adjusted prepayments. Let $y_i, i = 0, \ldots, n$ be particular values which divide the range of C/R into n bins of width $\delta = (y_n - y_0)/N$. Typically, $y_0 = 0.5, y_1 = 1.6$, and $n = 21$. For $j = 1, \ldots, n$, define the basis functions g_j by

$$g_j(x) = \begin{cases} 0, & x < y_j, \\ (x - y_j)/\delta, & y_j \leq x < y_{j+1}, \\ 1, & x \geq y_{j+1} \end{cases}$$

and the refinancing function by

$$G(x, \alpha) = \alpha_0 + \sum_{j=1}^{n} \alpha_j g_j(x).$$

Then an estimate of the refinancing effect is $\hat{\rho}(C/R) = G(x, \alpha^*)$, where α^* is the argument which minimizes

$$\Phi(\alpha) = \sum_{i=1}^{N} w_i (P_{s\,i} - G(C/R_i, \alpha))^2$$

subject to the condition that $\alpha_i > 0, i = 1, \ldots, n$.

It is important to notice that the refinancing function attempts to explain some of the variability due to the age effects, which have not been explicitly modeled yet. In this stage of the estimation procedure, ideally one would only like to include observations where the underlying security has seasoned fully but not begun to burn out. Since burnout and seasoning reduce prepayment activity, the intervening period should have the highest prepayment rates. Thus the residuals of the refinancing effect

$$P_{sri} = \frac{P_i}{\hat{s}(m_i, C/R_i)\hat{\rho}(C/R)}$$

should be less than one in magnitude. The result of including all the observations is that the residuals tend to be greater than one in magnitude for medium age observations. The best refinancing estimation window may be identified by inspecting a time series plot of the residuals, and then $\hat{\rho}$ may be recomputed using only observations from this interval. Premium and discount bonds exhibit different aging patterns, with premiums seasoning and burning out much faster than discounts. The authors found that for premium pools an acceptable refinancing estimation window was 24 to 48 months, while 84 to 180 months worked well for discount pools.

The age effects are modeled with a similar basis function approach, and likewise depend on the history of prepayments. At this point the remaining effects are easily distinguished, since seasoning occurs early in the life of a security, while burnout occurs later on. Separate models are estimated for various classes of bonds, including high premiums, premiums, pars, discounts, and deep discounts. For a given class there is a maximum month T at which the pools are considered fully seasoned. For discount bonds which season slowly, 120 months works well, while for premium bonds 90 months is a suitable choice. For $j = 1, \ldots, T$, define a new set of basis functions g_j by

$$g_j(t) = \begin{cases} 0, & t < 0, \\ t/j, & 0 \leq t < j, \\ 1, & t \geq t_j \end{cases}$$

and the seasoning function by

$$G(t, \alpha) = \alpha_0 + \sum_{j=1}^{T} \alpha_j g_j(t).$$

An estimate of the seasoning effect for the set of securities S for which each member has belonged to class C/R over its entire history is given by $\tilde{\sigma}(t, C/R) = G(t; \alpha^*)$, where α^* is the argument which minimizes

$$\Phi(\alpha; C/R) = \sum_{i \in S} w_i (P_{s\,r\,i} - G(t_i, \alpha))^2$$

subject to the condition that $\sum_{j=1}^{T} \alpha_j = 1$ and $\alpha_j > 0$ for each j. It is unlikely, however, that a security belongs to one class over its entire history. Hence we can estimate the seasoning factor with

$$\hat{\sigma}(\{C/R\}_t, t, C/R) = \sum_{\tau=1}^{t} \tilde{\sigma}(t, C/R_\tau)/t.$$

Burnout is modeled similarly. Since the effect occurs later, however, more appropriate basis functions are

$$g_j(t) = \begin{cases} 1, & t < j, \\ \exp(-\beta(t - j)) & t \geq t_j \end{cases}$$

In all, the authors examine several hundred thousand observations. They present graphs of some of the fits they obtain for the four factors using data from various issuing agencies. Their values for R-squared range from .78 to .85, which is more credible than the figures reported by Richard and Roll. Furthermore, they present their model in enough detail so that one might attempt to replicate their results.

Others have published descriptions of similar large scale, nonlinear regression models, including Carron *et al.* (1988), Ben-Dov, Hayre, and Pica (1992), and Patruno (1994). These publications stem from the research departments of various investment banks; in fact, nearly every sizable dealer of mortgage securities has published a prepayment model. Such publications are produced as marketing tools, and hence are published in trade magazines rather than scholarly journals. Most details are omitted in order that competitors not benefit from any proprietary knowledge. The minimum amount of information is divulged such that the source firm is made to seem resourceful and on the cutting edge of development, which is always done without actually presenting substance. For this reason the Kang and Zenios model is an anomaly, the sole concrete representative of its class.

Chapter 3

Dynamic Prepayment Models

Despite the fact that the dealer models are widely acknowledged by practitioners as the best, they are just as widely acknowledged to be poor at predicting prepayments.[11] The models have been characterized by some as "inaccurate"[12], and others have gone so far as to say that they are no better at predicting prepayments than a constant linear extrapolation.[13]

Most empirical prepayment models do not account for the dynamic properties of the prepayment process, regarding it instead as a static affair. Regression models find patterns in past behavior, but historical trends are valid for predicting future behavior only if the generating mechanism has simple invariant characteristics. This has not been the case with mortgages and mortgage prepayments. According to one researcher at Goldman, Sachs, "Previous models have tended to provide seriously inaccurate predictions whenever the conditions influencing borrowers' prepayment decisions have changed to any significant degree."[14] In other words, regression models fail when something changes, or more accurately, fail because things change.

The fundamental problem with a regression approach to prepayment models is that a regression line is a fixed model, whereas the mortgage market is a dynamic system. New mortgages are continually originated whenever there is a refinancing or purchase of a new home, while simultaneously old mortgages are paid down for numerous reasons through prepayments and normal amortization. Thus it is natural to turn to a dynamic model. Since prepayments are obviously inherently random, we will be particularly interested in stochastic models, and it will be shown eventually that using deterministic functions to estimate stochastic prepayment levels can result in serious misvaluations of mortgage-backed derivative securities.

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[11] Moore (1994), personal communication

[12] Choi (1994), 57.

[13] Yamamoto and Zenios (1993), 87.

[14] Patruno (1994), 42.

3.0.1 General Deterministic Models

To model a deterministic system one can proceed in a straightforward manner by writing down the effect corresponding to each cause. This typically leads to a system of differential equations, which sometimes can be solved analytically and otherwise must be solved numerically. As an example we consider the classical predator-prey problem first analyzed by Lotka (1925) and Volterra (1926). The problem is based on an environment in which two species live, one of which preys upon the other such as wolves and sheep.

A simple model of the number of sheep as a function of time can be formulated by supposing that the increment in the sheep population over a short period is proportional to the population size and the length of the period. We can write

$$\Delta S = aS\Delta t, \tag{3.1}$$

where $S(t)$ is the number of sheep at time t and a is the growth rate. Taking the limit we obtain $S' = aS$, which has the solution

$$S(t) = S_0 e^{at}.$$

Of course we have ignored the discrete nature of the population and seasonal variation of the birth rate. More importantly, we have ignored the wolves.

We can incorporate the affect of the wolves by adding additional postulates. First, in the absence of sheep, the wolves will starve at a rate proportional to their number. Second, the interaction between the two species causes the number of wolves to increase and the number of sheep to decrease in direct proportion to the size of both populations. Consequently we can write

$$\begin{aligned} S' &= S(a - \alpha W), \\ W' &= W(-c + \gamma S), \end{aligned}$$

where W is the number of wolves, and c is their death rate, and α and γ are positive constants representing the interaction between the species. Equations (3.2) are commonly called the Lotka-Volterra equations.

Several possibilities arise concerning the possible fate of the wolves and sheep. It may be that the wolves will eat all the sheep and then perish by starvation. Perhaps the wolves will die out before all the sheep are eaten after which the sheep will multiply without bound. Alternatively, the two species may reach an equilibrium in which the number of each remains constant, or the population levels will never attain such an equilibrium but oscillate around it.

A simple analysis such as that given by Boyce and DiPrima (1977) reveals that the system specified by (3.2) has two critical points located at $(S, W) = (0, 0)$ and $(c/\gamma, a/\alpha)$. The point at $(0, 0)$ is a saddle point whose only entrance is along the line $S = 0$; all other trajectories recede from here. In fact, all other trajectories are closed curves about the other critical point. In the vicinity of $(c/\gamma, a/\alpha)$ the orbits are approximately elliptical and are given by

$$S(t) = \frac{c}{\gamma} + \frac{c}{\gamma} K \cos(\sqrt{ac}t + \phi),$$
$$W(t) = \frac{a}{\alpha} + \frac{a}{\alpha}\sqrt{\frac{c}{a}} K \cos(\sqrt{ac}t + \phi),$$

where the constants K and ϕ are determined by the initial conditions.

Under the hypotheses of the model there is no danger of either species dying out. The population levels vary sinusoidally about $(c/\gamma, a/\alpha)$, with the sheep cycle leading the wolf cycle by one quarter of a period. This behavior would be nearly impossible to predict using a regression model based on early empirical data. Such a model might easily forecast the extinction of one or both species, particularly if the data were from the beginning of the second quarter of the period of the cosine for the sheep. In this case the sheep would appear to be decreasing at a linear rate, while the wolves would maintain stable numbers early on. The only reasonable forecast for a regression model based on this data would be the extinction of the sheep, after which the wolves would necessarily starve.

With more data the cyclic nature of the populations would become evident, but there would be no way to anticipate the effect of some unforeseen shock to the system, such as the poaching of a number of the wolves by hunters. While this deterministic model is simple and presumably inadequate for describing most real environments, it should nonetheless illustrate the importance of dynamic models and the corresponding inadequacies of regression models for describing dynamic systems.

3.0.2 General Stochastic Models

For stochastic systems it is sometimes less clear how to proceed since each cause does not necessarily have a single, reliable effect. While at first it may be less clear how to model a stochastic dynamic system, we may in fact proceed in an analogous fashion and write down a system of stochastic difference or differential equations corresponding to a set of axioms.

One of the simplest examples of the axiomatic approach is the construction of the Poisson distribution based on a simple model for the occurrence of events. The events might correspond to the arrival of customers in a store. Suppose one wishes to analyze the total number of customers that might enter the store up to any point in time. This process might be described by the following simple set of axioms:

1. The probability that a customer enters the store in any short time interval is proportional to the duration of the interval.

2. The probability that two customers enter in any short interval is negligible provided the interval is small enough.

3. The numbers of customers entering the store over disjoint intervals are independent.

Letting $n(a,b)$ denote the number of arrivals in the interval (a,b), we can specify our axioms mathematically

1. $P(n(t,t+\Delta t)=1)=\theta\Delta t$.

2. $P(n(t,t+\Delta t)>1)=o(\Delta t)$.

3. $P(n(t,s)=k \text{ and } n(u,v)=l)=P(n(t,s)=k)P(n(u,v)=l)$ whenever $(t,s)\cap(u,v)=\emptyset$.

Conditioning on the number of customers to visit the store at a previous point in time, we derive a differential-difference equation for the likelihood function as follows:

$$
\begin{aligned}
&P(n(0,t+\Delta t)=k) \\
&= P(n(t,t+\Delta t)=0|n(0,t)=k)P(n(0,t)=k)+ \\
&\quad P(n(t,t+\Delta t)=1|n(0,t)=k-1)P(n(0,t)=k-1)+o(\Delta t) \\
&= (1-\theta\Delta t)P(n(0,t)=k)+\theta\Delta tP(n(0,t)=k-1)+o(\Delta t).
\end{aligned}
$$

Rearranging terms gives

$$
\begin{aligned}
&\frac{P(n(0,t+\Delta t)=k)-P(n(0,t)=k)}{\Delta t} \\
&= \theta[P(n(0,t)=k)-P(n(0,t)=k-1)]+\frac{o(\Delta t)}{\Delta t}.
\end{aligned}
$$

In the limit as Δt approaches zero,

$$\frac{dP(n(0,t)=k)}{dt} = \theta[P(n(0,t)=k) - P(n(0,t)=k-1)].$$

It can be easily verified that the solution to this equation is

$$P(n(0,t)=k) = \frac{e^{-\theta t}(\theta t)^k}{k!},$$

which is the discrete density function for the Poisson distribution.

More Complicated Stochastic Models

The Poisson example clearly illustrates the idea of the axiomatic approach. In such simple systems the likelihood function can be written down, which makes parameter estimation a simple task. However, in most cases the solution of the resulting differential-difference equation is not as easy to find. The axiomatic method is sometimes abandoned because construction of the likelihood function based on the differential-difference equations for the purpose of parameter estimation is frequently a daunting task, as it requires consideration of all possible paths to every possible outcome. At this point practitioners might despair and turn to regression models, but this is unnecessary. For more complicated systems it may be that there is a deterministic system with the same behavior for large populations. If not, then a useful technique developed by Atkinson *et al.* (1983) can be employed to obtain parameter estimates through simulation. The technique is called SIMEST, an acronym for simulation-based estimation.

Consider the following stochastic model to describe the onset and growth of secondary tumors in cancer patients due to Thompson et al. The primary motivation for their model was evidence that the hazard rate of discovery of secondary tumors after the removal of a primary tumor was constant in time. This was an indication that besides a well known metastatic process whose intensity was proportional to the size of the primary tumor, there was an additional systemic process with constant intensity involved. Using data on the time from the removal of the primary tumor to the first discovery of a secondary tumor, they employed the axiomatic approach to construct a model based on four postulates:

1. Each tumor originates from a single cell and grows exponentially at rate α.

2. The probability that a tumor of size $Y_j(t)$, not previously detected and removed by time t, is detectable in $(t, t + \Delta t)$ is $bY_j(t)\Delta t + o(\Delta t)$.

3. Until the removal of the primary the probability of metastasis in $(t, t + \Delta t)$ is $aY_0(t)\Delta t$.

4. The probability of systemic occurrence of a tumor in $(t, t + \Delta t)$ is $\lambda\Delta t + o(\Delta t)$, independent of the prior history of the patient.

Despite the apparent simplicity of their axioms, the resulting likelihood function is extremely complex. Just one of several terms in the expression for the likelihood is

$$\begin{aligned} P(T_1 = S', T_2 > S) = & \int\int p(t;1)p(S';e^{\alpha}u)[\lambda + ae^{\alpha(t-u)}] \\ & *H(v(S-S');S',e^{\alpha u})H(v(S-S')e^{\alpha S'};u,e^{\alpha(t-u)}) \\ & *\exp[w(S-S') - \lambda(t-u) - \frac{a}{\alpha}(e^{\alpha(t-u)} - 1)]dudt \\ + & \int\int p(t;1)p(S'-u;1)H(v(S-S');S'-u,1) \\ & *\lambda\exp[w(S-S') - \lambda u - \lambda t - \frac{a}{\alpha}(e^{\alpha t} - 1)]dudt \end{aligned}$$

where

$$\begin{aligned} H(s;t,z) &= \exp\{\frac{az}{\alpha}e^{\alpha t}(e^{s} - 1)\log[1 + e^{-s}(e^{-\alpha t} - 1)], \\ & \quad + \frac{\lambda s}{\alpha} - \frac{\lambda}{\alpha}\log[1 + e^{-\alpha t}(e^{s} - 1)]\} \\ p(t;z) &= bz\exp[\alpha t - \frac{bz}{\alpha}(e^{\alpha t} - 1)], \\ w(y) &= \lambda(\int_0^y e^{-v(u)}du - y), \end{aligned}$$

and $v(u)$ is determined by the relation

$$u = \int_0^v \frac{ds}{a + b + \alpha s - \alpha e^{-s}}.$$

Almost one and a half person-years of work were required to derive the full likelihood expression for this model and evaluating it is equivalent in complexity to four dimensional quadrature. Clearly this approach is not a practical, general method for dynamical model building; however, in the same article, Thompson et al. propose an alternative, the simulation based approach for parameter estimation and inference called SIMEST. The technique arises naturally from the axiomatic formulation of the model, which describes how events evolve forwards in time and thus lends itself to simulation.

If we adopt the notation

D time of detection of primary tumor,

M time of origin of first metastasis,

S time of origin of first systemic tumor,

R time of origin of first recurrent tumor,

d^* time from origin to detection of first recurrent,

D^* time from detection of primary to detection of first recurrent,

then it follows directly from the postulates that the tumor volume at time t is $ce^{\alpha t}$ where c is the volume of one cell, and the distribution functions of various intervals of interest are

$$\begin{aligned}
F_D(t) &= 1 - \exp(-\frac{bc}{1}e^{\alpha t}), \\
F_M(t) &= 1 - \exp(-\frac{ac}{\alpha}e^{\alpha t}), \\
F_S(t) &= 1 - \exp(-\lambda t), \\
F_{d^*} &= 1 - \exp(-\frac{bc}{\alpha}e^{\alpha t}).
\end{aligned}$$

These distributions can be used to simulate the occurrences of secondary tumors. By binning the observations and determining actual and simulated bin frequencies, a goodness of fit statistic can be calculated and used to optimize parameter values.

From a practical standpoint the SIMEST approach to the cancer model is immeasurably preferable to attempting to find the likelihood function because of the vast reduction of time and effort needed in the analysis. Furthermore, for the purpose of model selection and validation it would likely be desirable to add additional postulates or modify existing ones. Such changes could be made at little cost in the case of SIMEST, but would basically require derivation of an entirely new likelihood function from scratch.

The key to the complexity of the problem is whether or not the transition probabilities are independent of the history of the process. If so, the equations governing the likelihood function may typically be written down and either solved analytically or numerically. However, if any transition probabilities depend on the path to the current state then the individual particles must be tracked to find out how they came to be wherever they are at each point in time. This requirement invalidates the differential-difference equation approach, which only examines the gross population levels from moment to moment. In this case the simulation based procedure of Thompson et al. is useful.

3.1 Stochastic Prepayment Models

3.1.1 A Simple Model

The refinancing effect is the single greatest determinant of prepayments. Therefore, the simplest model of prepayment behavior considered here is a two state system which includes only age of mortgage and mortgage rate differential as explanatory variables. In this model a borrower begins in state one with the origination of his mortgage. As long as the borrower is in this state and has a positive outstanding balance, he is sensitive to refinancing opportunities. The second state may be reached from the first if there is a prepayment. We assume that prepayment only occurs in full; that is, there are no partial curtailments, hence the second state represents early mortgage termination. This assumption is justified since partial curtailments account for only about 0.15% of outstanding mortgage balances.[15] For an individual borrower in state 1, the probability of a transition to state 2 or prepayment in a short interval is given by

$$\begin{aligned} P(\text{transition in } (t, t+\Delta t)) &= \theta_{12}(t)\Delta t + o(\Delta t), \\ \theta_{12}(t) &= \gamma_0 + \gamma_1(r_0 - r(t-\gamma_2) - \gamma_3)^+, \end{aligned} \tag{3.2}$$

where t denotes calendar time, $r(t)$ is the mortgage refinancing rate at time t, and r_0 is the coupon rate being paid on the outstanding mortgage. It is usually convenient for us to let the time origin correspond to the mortgage origination date, so that t is the same as the age of the mortgage. If we let T denote the original term of the mortgage, then clearly the only allowable times are $t \in (0, T)$. Also, since r_0 is usually the market rate at some point near the closing date, we usually have $r_0 \simeq r(0)$. In the case when we are referring to multiple mortgages with possibly different origination

[15] Hayre and Lauterbach, 1991

times, we shall explicitly denote the origination date for the ith mortgage by t_{i0}, and the mortgage coupon by r_{i0}.

According to (3.2) the conditional probability of a prepayment over $(t, t + \Delta t)$ is proportional by factor γ_1 to the amount by which the mortgage rate differential exceeds some threshold level γ_3. The threshold γ_3 represents a minimum cost of refinancing, since refinancings will not occur until the benefit of lower monthly payments, which is determined in part by the mortgage differential, exceeds the points and other costs of refinancing, which also depend on the differential. Heterogeneity among both fixed and floating refinancing costs is implicitly accounted for by the sensitivity γ_1; as interest rates fall, the proportion of borrowers whose potential savings exceed total costs will increase whether the population is heterogeneous or not, although perhaps with varying degree. The prepayment incentive is not manifest immediately, however, but after a period γ_2 which reflects the time required to reach the decision to refinance given the current market conditions as well as the time required by the servicer to process the prepayment. This model also allows for the contribution of housing turnover to prepayments. The portion of turnover not related to mortgage rate movements is measured by the constant γ_0, while the part which is not is subsumed by the γ_1 term.

Letting $n(s,t)$ be the 0-1 valued variable representing the number of times a borrower has terminated a particular mortgage in the interval (s,t) and letting $F(t)$ denote the distribution of $n(0,t)$, we can find an expression for F in analogous fashion to the Poisson example. Conditioning on the distribution at time t yields

$$\begin{aligned} F(t+\Delta t) &= F(t) + P(n(t,t+\Delta t)|n(0,t)=0)(1-F(t)) \\ &= F(t) + \theta(t)(1-F(t))\Delta t + o(t), \end{aligned}$$

where $P(n(t,t+\Delta t)=1|n(0,t)=0) = \theta(t)\Delta t + o(t)$. Rearranging terms gives

$$\frac{F(t+\Delta t)-F(t)}{\Delta t} = \theta(t)(1-F(t)) + \frac{o(\Delta t)}{\Delta t}.$$

In the limit as Δt approaches zero we see

$$\frac{dF}{dt} = \theta(t)(1-F(t)),$$

which has the solution

$$F(t) = 1 - \exp\{-\int_0^t \theta(\tau)d\tau\}. \tag{3.3}$$

This is a general result for any θ which depends on t. Substituting the form postulated in (3.2) we get

$$F(t) = 1 - \exp\{-\gamma_0 t - \gamma_1 \int_0^t (r_0 - r(\tau - \gamma_2) - \gamma_3)^+ d\tau\}.$$

The corresponding density function is

$$f(t) = \{\gamma_0 + \gamma_1(r_0 - r(t - \gamma_2) - \gamma_3)\} \exp\{-\gamma_0 t - \gamma_1 \int_0^t (r_0 - r(\tau - \gamma_2) - \gamma_3)^+ d\tau\}.$$

If we assume that the actions of every borrower are independent, then we can write the joint density for the prepayment times of n individuals as

$$\begin{aligned} f_n(\mathbf{t}) = \prod_{i=1}^{n} &\{\gamma_0 + \gamma_1(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+\} \\ &\cdot \exp\{-\gamma_0 t_i - \gamma_1 \int_{t_{i0}}^{t_i} (r_{i0} - r(\tau - \gamma_2) - \gamma_3)^+ d\tau\} \end{aligned}$$

where $\mathbf{t} = (t_1, \ldots, t_n)'$. Thus the log likelihood is

$$\begin{aligned} l(\gamma) = &\sum_{i=1}^{n} \log\{\gamma_0 + \gamma_1(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+\} \\ &- \sum_{i=1}^{n}\{\gamma_0 t_i + \gamma_1 \int_{t_{i0}}^{t_i} (r_{i0} - r(\tau - \gamma_2) - \gamma_3)^+ d\tau\} \end{aligned}$$

where $\gamma = (\gamma_0, \ldots, \gamma_3)'$. The resulting necessary conditions for the maximum likelihood estimates are therefore

$$\begin{aligned} \frac{\partial l}{\partial \gamma_0} &= \sum_{i=1}^{n} \frac{1}{\gamma_0 + \gamma_1(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+} - \sum_{i=1}^{n} t_i \\ &= 0, \\ \frac{\partial l}{\partial \gamma_1} &= \sum_{i=1}^{n} \frac{(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+}{\gamma_0 + \gamma_1(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+} \\ &\quad - \sum_{i=1}^{n} \int_{t_{i0}}^{t_i} (r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+ d\tau \\ &= 0, \\ \frac{\partial l}{\partial \gamma_2} &= \sum_{i=1}^{n} \frac{\gamma_1 r'(t_i - \gamma_2)(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+}{\gamma_0 + \gamma_1(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+} \\ &\quad - \sum_{i=1}^{n} \gamma_1 \int_{t_{i0}}^{t_i} r_i'(t - \gamma_2)(r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+ d\tau \\ &= 0, \end{aligned}$$

$$\begin{aligned}
\frac{\partial l}{\partial \gamma_3} &= \sum_{i=1}^{n} \frac{-\gamma_1 (r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+}{\gamma_0 + \gamma_1 (r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+} \\
&\quad + \sum_{i=1}^{n} \gamma_1 \int_{t_{i0}}^{t_i} (r_{i0} - r(t_i - \gamma_2) - \gamma_3)^+ d\tau \\
&= 0,
\end{aligned}$$

which can be solved by an iterative approximation technique.

3.1.2 A Three State Model

The simple two state model described in the previous section only accounted for the single most basic and important effect on the prepayment decision, the mortgage rate differential. By means of a slight modification to the two state model we can easily provide a mechanism to explain another significant factor, seasoning. To motivate the modification we begin by considering why seasoning might occur.

One suspects that people who have just moved or refinanced are intrinsically more stable in some sense than those who have not and are hence less likely to repeat the action for some period of time. People who buy a new house are likely to be there for some time; people who refinance may try to time an interest rate trough and might therefore expect higher future rates in the near term. An alternative explanation lies in the observation that most homeowners probably do not follow the mortgage market closely, certainly not on a daily or weekly basis. Having recently closed a mortgage one could reasonably expect that it would take a period of many months before there would be any significant chance for another favorable refinancing opportunity to arise, justifying a period of inattentiveness to market conditions. One might also believe that the typical homeowner is somewhat averse to refinancing since the process can involve a significant commitment of time to collect documents, fill out forms, arrange and attend meetings and consultations with loan officers, and so on. It seems unlikely that an ordinary borrower, having just completed the process of refinancing or obtaining a new loan, would be immediately receptive to the idea of going through the routine again. With the passage of time, however, the individual might come again to the point where he would at least entertain the idea.

We can account for this period of aversion by adding an additional state to the model, state 0. An individual with a new mortgage begins in state 0, where he will not consider the refinancing decision. Over time he may transition to state 1, in which he is sensitive to refinancing opportunities and subject to housing turnover. Prepayment for either of these two reasons corresponds to the transition to state 2, in which the mortgage has been terminated. This model can be diagramed as in Figure 3.1.

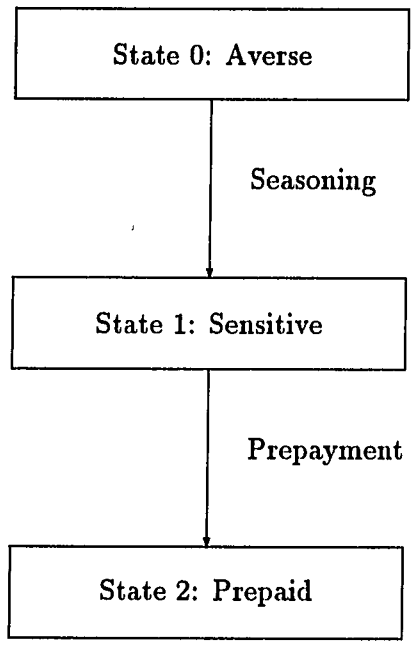

Figure 3.1 Three State Model

We can analyze the behavior of this model qualitatively to ascertain that it should capture the effect of seasoning. Let $n_0(t)$ denote the number of borrowers in state 0 at time t, $n_1(t)$ the number in state 1, and $n(t)$ the sum $n_1(t) + n_2(t)$. Suppose that the transition probabilities at time t are $\theta_{01}(t)$ for a borrower in state 0 and $\theta_{12}(t)$ for a borrower in state 1. The observed prepayment rate for all borrowers at time t is therefore $\theta_{12}(t)n_1(t)/n(t)$. Initially, since all individuals begin in state 1, $n_1(t)/n(t)$ is zero so prepayments start at zero. As time passes the proportion of unprepaid borrowers in state 1 increases, producing a corresponding increase in prepayments. Eventually, $n_0(t)$ becomes small so that $n_1(t)/n(t)$ approaches 1 and expected prepayments reach their steady state value $\theta_{12}(t)$.

It has been observed that seasoning occurs rapidly for discount mortgages and much more slowly for premiums, hence we would expect θ_0 to depend on the mortgage differential. With this in mind, we hypothesize that the probability that a refinancing-averse borrower becomes receptive in the interval $(t, t+\Delta t)$ is given by

$$P(\text{transition in } (t, t+\Delta t)) = \theta_{01}(t)\Delta t + o(\Delta t),$$

where

$$\theta_{01}(t) = \alpha_0 + \alpha_1(r_0 - r(t))^+.$$

The constant term is present because seasoning is observed even in premium mortgages, for which the interest rate differential is negative. The distribution of transition times from state 0 to state 1 is therefore, by (3.3),

$$F(t) = 1 - \exp\{-\alpha_0 t - \alpha_1 \int_0^t (r_0 - r(\tau))^+ d\tau\}. \tag{3.4}$$

We now derive a differential equation for the likelihood function of the total time until prepayment for a single individual in much the same fashion as we derived the Poisson distribution. Let $\pi_i(t)$ denote the probability that an individual is in state i at time t. We can write an expression involving $\pi_i(t)$ by conditioning on the states an individual could possibly be in a short period earlier that might result in him being in state i at time t. For state 0 this is

$$\begin{aligned}\pi_0(t+\Delta t) &= \pi_0(t)P(\text{no transition in } (t, t+\Delta t) | \text{in state 0 at time } t) \\ &= \pi_0(t)(1 - \theta_{01}(t)\Delta t) + o(\Delta t). \qquad (3.5)\end{aligned}$$

For state 1 we have

$$\begin{aligned}\pi_1(t+\Delta t) &= P(\text{in state 0 at time } t) \\ &\quad \cdot P(\text{transition from state 0 to 1 in } (t, t+\Delta t)) \\ &\quad + P(\text{in state 1 at time } t) \\ &\quad \cdot P(\text{no transition in } (t, t+\Delta t) | \text{in state 1 at time } t) \\ &= \pi_0(t)\theta_{01}(t)\Delta t + \pi_1(t)(1 - \theta_{12}(t)\Delta t) + o(\Delta t). \qquad (3.6)\end{aligned}$$

Similarly,

$$\pi_2(t+\Delta t) = \pi_1(t)\theta_{12}(t)\Delta t + \pi_2(t) + o(\Delta t). \tag{3.7}$$

Rearranging terms in (3.5)-(3.7) gives the system

$$
\begin{aligned}
\frac{\pi_0(t+\Delta t)-\pi_0(t)}{\Delta t} &= -\pi_0(t)\theta_{01}(t)+\frac{o(\Delta t)}{\Delta t}, \\
\frac{\pi_1(t+\Delta t)-\pi_1(t)}{\Delta t} &= \pi_0(t)\theta_{01}(t)-\pi_1(t)\theta_{12}(t)+\frac{o(\Delta t)}{\Delta t}, \\
\frac{\pi_2(t+\Delta t)-\pi_2(t)}{\Delta t} &= \pi_1(t)\theta_{12}(t)+\frac{o(\Delta t)}{\Delta t}. \qquad (3.8)
\end{aligned}
$$

In the limit as Δt approaches zero, (3.8) reduces to the system of first order linear differential equations with non-constant coefficients

$$
\begin{aligned}
\pi_0' &= -\theta_{01}(t)\pi_0, && (3.9)\\
\pi_1' &= \theta_{01}(t)\pi_0-\theta_{12}(t)\pi_1, && (3.10)\\
\pi_2' &= \theta_{12}(t)\pi_1. && (3.11)
\end{aligned}
$$

The initial conditions are $\pi_0(0)=1$ and $\pi_1(0)=\pi_2(0)=0$, signifying that at the outset every borrower is certain to begin in state 0.

By inspection we see that (3.9) has the solution

$$\pi_0(t) = e^{-\int_0^t \theta_{01}(\tau)d\tau}. \qquad (3.12)$$

Substituting (3.12) into (3.10) and rearranging yields

$$\pi_1' + \theta_{12}(t)\pi_1 = \theta_{01}(t)e^{-\int_0^t \theta_{01}(\tau)d\tau}, \qquad (3.13)$$

which is a nonhomogeneous first order linear differential equation. Multiplying each side of (3.13) by the integrating factor

$$\mu(t) = e^{\int_0^t \theta_{12}(\tau)d\tau}$$

yields

$$\left[\pi_1 e^{\int_0^t \theta_{12}(\tau)d\tau}\right]' = \theta_{01}(t)e^{-\int_0^t[\theta_{01}(\tau)-\theta_{12}(\tau)]d\tau}.$$

Therefore

$$\pi_1 e^{\int_0^t \theta_{12}(\tau)d\tau} = \int_0^t \theta_{01}(s)e^{-\int_0^s[\theta_{01}(\tau)-\theta_{12}(\tau)]d\tau}ds + c.$$

Imposing the condition $\pi_1(0)=0$, we see that

$$\pi_1(t) = e^{-\int_0^t \theta_{12}(\tau)d\tau}\int_0^t \theta_{01}(s)e^{-\int_0^s[\theta_{01}(\tau)-\theta_{12}(\tau)]d\tau}ds. \qquad (3.14)$$

We can now solve (3.11)

$$\begin{aligned}\pi_2(t) &= \int_0^t \theta_{12}(s)\pi_1(s)ds + c \\ &= \int_0^t \theta_{12}(u)e^{-\int_0^u \theta_{12}(\tau)d\tau}\int_0^u \theta_{01}(s)e^{-\int_0^s[\theta_{01}(\tau)-\theta_{12}(\tau)]d\tau}dsdu. \end{aligned} \tag{3.15}$$

We have defined $\pi_i(t)$ to be the probability that an individual occupies state i at time t. The observations and the quantities of interest are prepayment times, which in the model correspond to the time of first arrival in state two. For an arbitrary state i, $\pi_i(t)$ is not the probability $f_i(t)$ of first arrival in i by time t. In general, f_i is larger than π_i because an individual who has arrived in state i by any point in time has a chance of subsequently transitioning to a different state. For this reason we can also see that while f_i is monotonically increasing, in general π_i is not. State two is special because it is an absorbing state: once an individual has prepaid, he cannot subsequently make a transition to any other state and remains in state two indefinitely. Thus the probability an individual has arrived in state two by time t is the same as the probability he is in state two at time t, so $\pi_2(t) = f_2(t)$, which is the distribution function for prepayment times.

To obtain the likelihood function for prepayment times we could differentiate (3.15) with respect to t, but we can obtain the result more easily from the original system (3.9)-(3.11), from which we observe

$$\begin{aligned}\pi_2'(t) &= \theta_{12}(t)\pi_1(t) \\ &= \theta_{12}(t)e^{-\int_0^t \theta_{12}(\tau)d\tau}\int_0^t \theta_{01}(s)e^{-\int_0^s[\theta_{01}(\tau)-\theta_{12}(\tau)]d\tau}ds.\end{aligned}$$

in conjunction with (3.14). By independence the joint likelihood function for the prepayment times of N individuals is

$$\begin{aligned}L_N(\theta) &= \prod_{i=1}^N \theta_{12}(t_i)e^{-\int_{t_{i0}}^{t_i}\theta_{12}(\tau)d\tau}\int_{t_{i0}}^{t_i}\theta_{01}(s)e^{-\int_{t_{i0}}^{s}[\theta_{01}(\tau)-\theta_{12}(\tau)]d\tau}ds \\ &= \prod_{i=1}^N\{\gamma_0+\gamma_1(r_0-r(t_i-\gamma_2)-\gamma-3)^+\} \\ &\quad\cdot\exp\left(-\int_{t_{i0}}^{t_i}\{\gamma_0+\gamma_1(r_0-r(\tau-\gamma_2)-\gamma-3)^+\}d\tau\right) \\ &\quad\cdot\int_{t_{i0}}^{t_i}[\alpha_0+\alpha_1(r_0-r(s))^+]\exp\left(-\int_{t_{i0}}^{s}[\alpha_0+\alpha_1(r_0-r(\tau))^+\right. \\ &\quad\left.-\gamma_0-\gamma_1(r_0-r(\tau-\gamma_2)-\gamma_3)^+]d\tau\right)ds,\end{aligned}$$

where θ is the vector of parameters. For given $\{t_1, \ldots, t_N\}$ the MLEs can in theory be found from the preceding expression using a nonlinear optimization routine, but depending on the size of N this may be difficult in practice.

3.1.3 A Four State Model

We now modify the three state model to allow for burnout. We do this by adding a new state, state three, corresponding to burned-out mortgages, which may be reached from the sensitive state, one. The model is flowcharted in Figure 3.2.

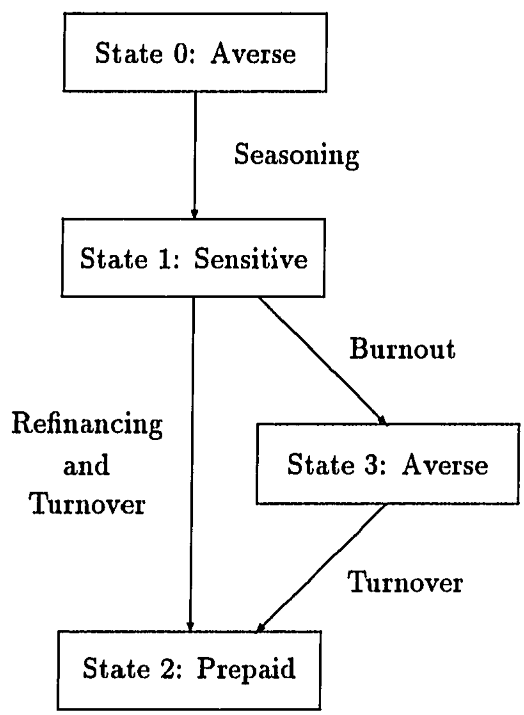

Figure 3.2 Four State Model

The easiest explanation of burnout is that as time progresses and the balance of a mortgage diminishes, the potential benefit of refinancing diminishes as well. Recall that for a fixed rate mortgage, the bulk of each payment in the early years consists of interest, while later payments consist primarily of principal. This means that the present value of the interest portion of the remaining payments is very small for old mortgages, and hence that there is little financial incentive for refinancing. Since some of the costs in the refinancing process are fixed, such as title searches, application fees, and so on, at some point the balance of the mortgage reaches a critical level so that no matter how low interest rates fall, it is never advantageous for the borrower to refinance. If one associates a cost with the time requirements for refinancing, then the critical balance is seen to be somewhat higher. The critical balance is not likely to be the same for all individuals due to differences in fixed costs, time requirements, costs associated with time expenditure, and level of sophistication of the analysis. Indeed, for many if not most homeowners, the decision to forego any future refinancing opportunities might not even be a conscious one, making it impossible to specify with precision the outstanding balance at the time the decision is made. The result of this heterogeneity and uncertainty in the critical balance is that we can at best assert that burnout occurs with increasing probability as balance decreases. Thus we add the postulate that the probability a homeowner in the receptive state burns out in the interval $(t, t+\Delta t)$ is given by

$$\begin{aligned} P(\text{burnout in } (t, t+\Delta t)) &= \theta_{13}(t)\Delta t + o(\Delta t), \\ \theta_{13}(t) &= \beta_0 \left(1 - \frac{b(t)}{b_0}\right), \end{aligned}$$

where b_0 is the initial mortgage balance and $b(t)$ is the remaining balance at time t.

Although homeowners who have burned out will not consider refinancing their mortgage, we suppose that they are subject to the same forces producing turnover as other homeowners. To substantiate this, we observe that even in heavily burned-out MBS pools prepayments continue at some positive residual level. Hence we postulate that the probability that a borrower who has burned out will prepay the balance of his mortgage in the interval $(t, t+\Delta t)$ is

$$P(\text{prepayment in } (t, t+\Delta t)) = \theta_{32}\Delta t + o(\Delta t),$$

where $\theta_{32}(t) = \gamma_0$. Here γ_0 is the same parameter as that representing the expected prepayment rate of receptive homeowners due to turnover. While the underlying driving mechanism is different for seasoning and burnout, the dynamics are the same. That is, in each case the effect is caused by shifting population proportions.

We now analyze the behavior of the model qualitatively to ascertain that it should capture the effect of burnout. Let $n_i(t)$ denote the number of borrowers in state i at time t and $n(t)$ denote the sum $n_1(t) + n_3(t)$. If we suppose that by time t_0 seasoning has occurred to its full extent and burnout has not yet begun to any significant extent, then for $t > t_0$ we have $n_0(t) = n_3(t_0) = 0$ and so $n(t)$ represents the total number of borrowers who have not yet prepaid. If we denote the probability of a transition from state i to state j in $(t, t + \Delta t)$ by $\theta_{ij}(t)$, then the observed overall prepayment rate at time t_0 is $\theta_{12}(t)$. This is because at t_0 all individuals are in state 1. As time passes the proportion of unprepaid borrowers in state 1 decreases, while the proportion in state 3 increases since $\theta_{13} > 0$. The prepayment rate at $t > t_0$ is

$$\frac{n_1(t)\theta_{12}(t) + n_3(t)\theta_{32}(t)}{n(t)}.$$

We observe that $\theta_{32} < \theta_{12}$ since prepayment in state 3 owes solely to turnover rather than turnover and refinancing, so we can conclude that the aggregate prepayment rate has declined from its value at t_0. Eventually, $n_1(t)$ becomes small in comparison to $n(t)$ and so expected prepayments approach their steady state value $\theta_{32}(t)$. This is the overall pattern evidenced by historical data so we conclude the model has the proper qualitative characteristics with regard to the burnout phenomenon.

$$\begin{aligned}
\pi_0' &= -\theta_{01}(t)\pi_0, &\quad (3.16)\\
\pi_1' &= \theta_{01}(t)\pi_0 - (\theta_{12}(t) + \theta_{13}(t))\pi_1, &\quad (3.17)\\
\pi_2' &= \theta_{12}(t)\pi_1 + \theta_{32}(t)\pi_3, &\quad (3.18)\\
\pi_3' &= \theta_{13}(t)\pi_1 - \theta_{32}(t)\pi_3. &\quad (3.19)
\end{aligned}$$

The appropriate initial conditions are $\pi_0(0) = 1$ and $\pi_1(0) = \pi_2(0) = \pi_3(0) = 0$.

We see immediately that (3.16) has the solution

$$\pi_0(t) = e^{-\int_0^t \theta_{01}(\tau)d\tau}.$$

If we substitute this expression into (3.17) we find

$$\pi_1' + (\theta_{12}(t) + \theta_{13}(t))\pi_1 = \theta_{01}(t)e^{-\int_0^t \theta_{01}(\tau)d\tau}. \tag{3.20}$$

Multiplying both sides of (3.20) by the integrating factor

$$\mu(t) = e^{\int_0^t (\theta_{12}(\tau)+\theta_{13}(\tau))d\tau}$$

we get

$$\left[\pi_1 e^{\int_0^t (\theta_{12}(\tau)+\theta_{13}(\tau))d\tau}\right]' = \theta_{01}(t)e^{-\int_0^t [\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}.$$

This leads to the solution

$$\pi_1(t) = e^{-\int_0^t (\theta_{12}(\tau)+\theta_{13}(\tau))d\tau} \int_0^t \theta_{01}(s)e^{-\int_0^s [\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau} ds. \tag{3.21}$$

We can save some labor by observing that the solution to the general first order linear equation

$$p' + p(t)p = g(t)$$

is

$$p = e^{-\int_0^t p(\tau)d\tau}\left[\int_0^t e^{\int_0^s p(\tau)d\tau} g(s)ds + c\right],$$

which can be easily verified. Letting $p(t) = \theta_{32}(t)$ and $g(t) = \theta_{13}(t)\pi_1(t)$, we see that the solution to (3.19) is given by

$$\begin{aligned}
\pi_3(t) &= e^{-\int_0^t \theta_{32}(\tau)d\tau}\left[\int_0^t e^{\int_0^s \theta_{32}(\tau)d\tau}\theta_{13}(s)\pi_1(s)ds + c\right] && (3.22)\\
&= e^{-\int_0^t \theta_{32}(\tau)d\tau} \int_0^t \theta_{13}(s)e^{-\int_0^s (\theta_{12}(\tau)+\theta_{13}(\tau)-\theta_{32}(\tau))d\tau}\\
&\quad \cdot \int_0^s \theta_{01}(u)e^{-\int_0^u [\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau} du ds. && (3.23)
\end{aligned}$$

The constant c in (3.22) must be zero to satisfy the condition $\pi_3(0) = 0$. We can now solve (3.18), getting

$$\begin{aligned}
\pi_2(t) &= \int_0^t [\theta_{12}(v)\pi_1(v) + \theta_{32}(v)\pi_3(v)]dv + c\\
&= \int_0^t [\theta_{12}(v)e^{-\int_0^v (\theta_{12}(\tau)+\theta_{13}(\tau))d\tau}\\
&\quad \int_0^v \theta_{01}(s)e^{-\int_0^s [\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau} ds\\
&\quad +\theta_{32}(v)e^{-\int_0^v \theta_{32}(\tau)d\tau} \int_0^v \theta_{13}(s)e^{-\int_0^s (\theta_{12}(\tau)+\theta_{13}(\tau)-\theta_{32}(\tau))d\tau}\\
&\quad \cdot \int_0^s \theta_{01}(u)e^{-\int_0^u [\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau} du ds]dv.
\end{aligned}$$

We can also write down the likelihood function for prepayment times by substituting the values for π_1 and π_3 given by (3.21) and (3.23) into (3.18). For one individual this yields

$$\begin{aligned}\pi_2' &= \theta_{12}(t)e^{-\int_0^t(\theta_{12}(\tau)+\theta_{13}(\tau))d\tau}\int_0^t \theta_{01}(s)e^{-\int_0^s[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}ds \\ &+\theta_{32}(t)e^{-\int_0^t \theta_{32}(\tau)d\tau}\int_0^t \theta_{13}(s)e^{-\int_0^s(\theta_{12}(\tau)+\theta_{13}(\tau)-\theta_{32}(\tau))d\tau} \\ &\cdot\int_0^s \theta_{01}(u)e^{-\int_0^u[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}duds.\end{aligned}$$

The joint likelihood function for N individuals who have prepaid at times $\{t_1,\ldots,t_N\}$ is thus

$$\begin{aligned}L_N(\theta) &= \prod_{i=1}^N \Big[\theta_{32}(t_i)e^{-\int_{t_{i0}}^{t_i}\theta_{32}(\tau)d\tau}\int_{t_{i0}}^{t_i}\theta_{13}(s)e^{-\int_{t_{i0}}^{s}(\theta_{12}(\tau)+\theta_{13}(\tau)-\theta_{32}(\tau))d\tau} \\ &\cdot\int_{t_{i0}}^{s}\theta_{01}(u)e^{-\int_{t_{i0}}^{u}[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}duds \\ &+\theta_{12}(t_i)e^{-\int_{t_{i0}}^{t_i}(\theta_{12}(\tau)+\theta_{13}(\tau))d\tau} \\ &\cdot\int_{t_{i0}}^{t_i}\theta_{01}(s)e^{-\int_{t_{i0}}^{s}[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}ds\Big],\end{aligned}$$

where **theta** is the vector of parameters.

3.1.4 An Alternative Four State Model

One might object to the previous model on the basis that there are probably few if any people who are completely insensitive to the interest rate incentive to prepay. Furthermore, there is probably more heterogeneity in the responsiveness of different borrowers to falling interest rates. We can allow for these possibilities by modifying the previous model somewhat. The alternative model is diagramed in Figure 3.3. We still have four states with the alternative specification, but now individuals in state three are no longer insensitive to refinancing opportunities but are merely less sensitive than than those in state one. This approach still allows seasoning, burnout, and prepayment, but now there are additional ways in which both burnout and seasoning can occur.

Seasoning can still occur as in the original model by the transition of borrowers from a refinancing-averse state to a refinancing-sensitive state, resulting in an increase in prepayments gradually rising from their initial value of zero. Furthermore, we can allow for the possibility that seasoning might occur over an extended period of time by supposing that individuals who become sensitive to interest levels may first experience a transitional phase in state three where they are only somewhat sensitive.

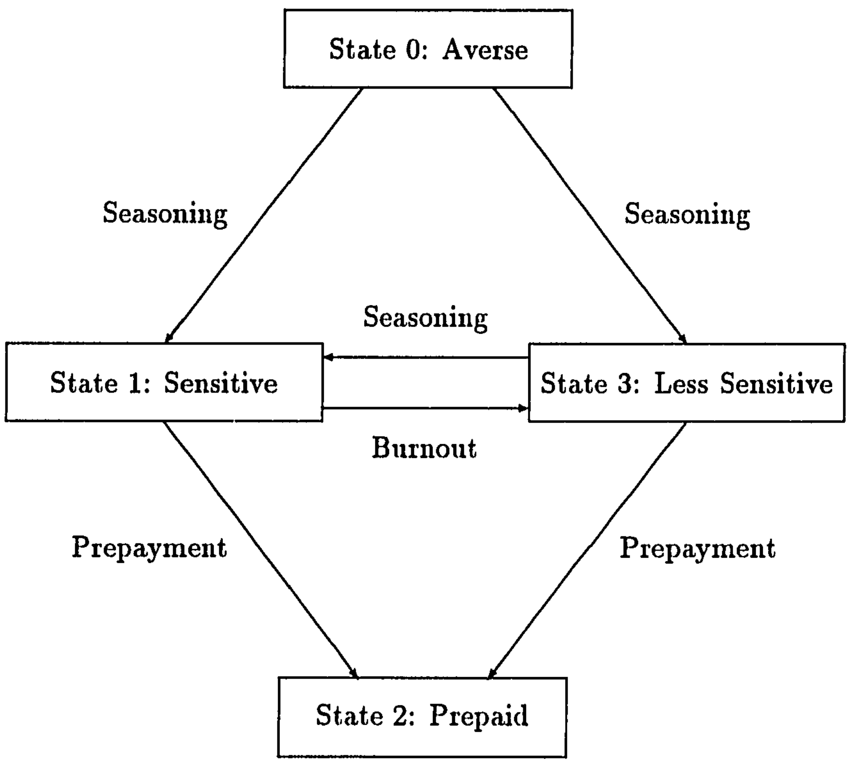

Figure 3.3 Alternative Four State Model

This might correspond to a borrower who winds up actively seeking to refinance at moderate interest levels for some reason, say because he needs to increase his current cash position, who would have refinanced earlier only if interest rates had fallen drastically and generated a wealth of news stories and perhaps calls from mortgage brokers.

Burnout still occurs in the same fashion, as sensitive borrowers with declining balances transition to state three for which prepayment rates are lower. The shifting proportion of individuals in states one and three produces the burnout effect. There is another way for this to occur in the new model, which is evident if we suppose there were no transitions between states one and three. Consider what happens to the aggregate prepayment rate in this case if seasoning has already occurred to its fullest extent. Because borrowers in state one prepay at a faster rate, the fraction of remaining borrowers in state one will decline with time. This results in a corresponding decline in the aggregate prepayment rate.

The transition probabilities for the new model are similar to those for the original four state model. For seasoning we have

$$\theta_{01}(t) = \theta_{03}(t) = \alpha_0 + \alpha_1[r_0 - r(t)]^+,$$

for prepayment

$$\theta_{12}(t) = \gamma_0 + \gamma_1[r_0 - r(t - \gamma_2) - \gamma_3]^+,$$
$$\theta_{32}(t) = \gamma_0 + \gamma_4[r_0 - r(t - \gamma_2) - \gamma_3]^+,$$

and for burnout

$$\theta_{13}(t) = \beta_0[1 - \frac{b(t)}{b_0}],$$

where all parameters are positive and $\gamma_1 > \gamma_4$. A bit more care must be taken to describe θ_{31} since we don't want to allow an individual who has already burned out to reseason. If we let $I_j(t, i)$ be the indicator of the event that borrower i has visited state j on or before time t, then we can write

$$\theta_{31}(t, i) = \alpha_0 + \alpha_1[r_0 - r(t)]^+[1 - I_1(t, i)].$$

It is also necessary to specify that $100\alpha_2$ percent of the population makes the initial transition to state one, with the remainder going to state three.

The transition probabilities in this model are history-dependent, so the likelihood function would be very difficult to derive. Thus it would be useful to employ SIMEST to obtain parameter estimates for this model.

3.2 Deterministic Analogs

There is a natural way to construct deterministic analogs to the stochastic models just described based on conditional expectations. Begin by considering a general continuous time, finite state stochastic prepayment model. Suppose there are K states and that the probability an individual in state i at time t will make a transition to state j in the interval $(t, t+\Delta)$ is given by

$$P_{ij}(t, t+\Delta t) = \theta_{ij}(t)\Delta t + o(\Delta t). \tag{3.24}$$

Let $n_i(t)$ denote the number of borrowers in state i at time t. For each i and $j \in \{1, \ldots, K\}$, $t > 0$, and $k \in 1, \ldots, n_i(t)$, let $\{\epsilon_{ijk}(t)\}$ be conditionally independent and identically distributed indicator functions such that given the history of the process up to time t, $\epsilon_{ijk}(t) = 1$ if the kth borrower in state i makes a transition to state j in $(t, t+\Delta)$, and 0 otherwise.

We define the deterministic analog of the stochastic model by setting

$$\bar{n}_i(t+\Delta t) = E(n_i(t+\Delta t) | n_i(t) = \bar{n}_i(t)) \tag{3.25}$$

for $i = 1, \ldots, K$. Since

$$n_i(t+\Delta t) = n_i(t) + \sum_{j \neq i} \sum_{k=1}^{n_j(t)} \epsilon_{jik}(t) - \sum_{j \neq i} \sum_{k=1}^{n_j(t)} \epsilon_{ijk}(t),$$

it follows from (3.25) that

$$\begin{aligned}
\bar{n}_i(t+\Delta t) &= E(n_i(t) + \sum_{j \neq i} \sum_{k=1}^{n_j(t)} \epsilon_{jik}(t) - \sum_{j \neq i} \sum_{k=1}^{n_j(t)} \epsilon_{ijk}(t) | n(t) = \bar{n}(t)) \\
&= \bar{n}_i(t) + \sum_{j \neq i} \bar{n}_j(t) p_{ji}(t) - \bar{n}_i(t) \sum_{j \neq i} p_{ij}(t)
\end{aligned}$$

Substituting the right side of (3.24) for $p(t)$ and rearranging we get

$$\frac{\bar{n}_i(t+\Delta t) - \bar{n}_i(t)}{\Delta t} = \sum_{j \neq i} \bar{n}_j(t)\theta_{ji}(t) - \bar{n}_i(t) \sum_{j \neq i} \theta_{ij}(t) + \frac{o(\Delta t)}{\Delta t}.$$

In the limit as Δt goes to zero we get the system of differential equations

$$\frac{d\bar{n}_i(t)}{dt} = \sum_{j \neq i} \bar{n}_j(t)\theta_{ji}(t) - \bar{n}_i(t) \sum_{j \neq i} \theta_{ij}(t), \quad \text{i} = 1, \ldots, \text{K}. \tag{3.26}$$

3.2.1 Relation to the Stochastic Mean

It is not difficult to show that the solution of the deterministic system (3.26) is precisely the mean population trajectory of the corresponding stochastic system.

We can rewrite (3.26) in vector notation as

$$\bar{\mathbf{n}}' = \mathbf{A}(t)\bar{\mathbf{n}} \tag{3.27}$$

where $\bar{\mathbf{n}} = (\bar{n}_1, \ldots, \bar{n}_K)'$ and $\mathbf{A}(t)$ is the matrix whose (i,j)th element is

$$A_{ij}(t) = \begin{cases} \theta_{ji}(t) & \text{if } i \neq j, \\ -\sum_{k \neq i} \theta_{ik}(t) & \text{else.} \end{cases}$$

Let $\Psi(t)$ be a fundamental matrix for (3.27), which from elementary differential equations theory is guaranteed to exist if the θ_{ij}s are continuous. Then for any set of initial conditions $\mathbf{n}_0$, the solution to (3.27) is given by

$$\bar{\mathbf{n}} \;=\; \Psi(t)\Psi^{-1}(0)\bar{\mathbf{n}}_0. \tag{3.28}$$

Invertibility of $\Psi(t)$ is assured since its columns are linearly independent.

The corresponding stochastic system has the same form as (3.26), but with the $\bar{n}_i$s replaced with p_is. We write this succinctly in analog to (3.27) as

$$\pi' = \mathbf{A}(t)\pi.$$

Letting $N = \sum_1^K n_i$ denote the total number of borrowers, $\epsilon_{ij}(t)$ be the indicator of the event that individual i occupies state j at time t, and $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{iK})'$, we have that

$$\begin{aligned} E(\mathbf{n}(t)) &= E(\sum_{i=1}^{N} \epsilon_i(t)) \\ &= \sum_{i=1}^{N} E(\epsilon_i(t)) \\ &= N\pi(t) \\ &= N\Psi(t)\Psi^{-1}(0)\pi_0 \\ &= \Psi(t)\Psi^{-1}(0)\mathbf{n}_0 \\ &= \bar{\mathbf{n}}(t) \end{aligned}$$

3.2.2 Two State Model

The differential equations describing the evolution of the deterministic system corresponding to the two state model are

$$\begin{aligned} n_0' &= -\theta_{01}(t)n_0 \\ n_1' &= -n_0 \end{aligned} \tag{3.29}$$

with initial conditions $n_0(0) = N$ and $n_1(0) = 0$. Here N is the total number of borrowers and

$$\theta_{01}(t) = \gamma_0 + \gamma_1[r_0 - r(t-\gamma_2) - \gamma - 3]^+.$$

By inspection we see that the solution to (3.29) is given by

$$\begin{aligned} n_0(t) &= Ne^{-\int_0^t \theta_{01}(\tau)d\tau}, \\ n_1(t) &= N(1 - e^{-\int_0^t \theta_{01}(\tau)d\tau}). \end{aligned}$$

3.2.3 Three State Model

The deterministic model corresponding to the stochastic three state model of Section 3.1.2 is defined by the following system:

$$\begin{aligned} n_0' &= -\theta_{01}(t)n_0, \\ n_1' &= \theta_{01}(t)n_0 - \theta_{12}(t)n_1(t), \\ n_2' &= \theta_{12}(t)n_1. \end{aligned} \tag{3.30}$$

The initial conditions are $n_0(0) = N$ and $n_1(0) = n_2(0) = 0$, where N is the total number of borrowers and the θ_{ij}s are the same as in Section 3.1.2. By the same analysis the solution to (3.30) is given by

$$\begin{aligned} n_0(t) &= e^{-\int_0^t \theta_{01}(\tau)d\tau}, \\ n_1(t) &= e^{-\int_0^t \theta_{12}(\tau)d\tau} \int_0^t \theta_{01}(s) e^{-\int_0^s [\theta_{01}(\tau) - \theta_{12}(\tau)]d\tau} ds, \\ n_2(t) &= \int_0^t \theta_{12}(u) e^{-\int_0^u \theta_{12}(\tau)d\tau} \int_0^u \theta_{01}(s) e^{-\int_0^s [\theta_{01}(\tau) - \theta_{12}(\tau)]d\tau} ds du. \end{aligned}$$

3.2.4 Four State Model

The equivalent deterministic formulation of the stochastic model presented in Section 3.1.3 is

$$n_0' = -\theta_{01}(t)n_0, \tag{3.31}$$

$$
\begin{aligned}
n_1' &= \theta_{01}(t)n_0 - (\theta_{12}(t) + \theta_{13}(t))n_1, & (3.32)\\
n_2' &= \theta_{12}(t)n_1 + \theta_{32}(t)n_3, & (3.33)\\
n_3' &= \theta_{13}(t)n_1 - \theta_{32}(t)n_3, & (3.34)
\end{aligned}
$$

with corresponding initial conditions $n_0(0) = N$ and $n_1(0) = n_2(0) = n_3(0) = 0$. By Section 3.1.3 the solution is given by

$$
\begin{aligned}
n_0(t) &= Ne^{-\int_0^t \theta_{01}(\tau)d\tau},\\
n_1(t) &= Ne^{-\int_0^t(\theta_{12}(\tau)+\theta_{13}(\tau))d\tau}\int_0^t \theta_{01}(s)e^{-\int_0^s[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}ds,\\
n_2(t) &= N\int_0^t[\theta_{12}(v)e^{-\int_0^v(\theta_{12}(\tau)+\theta_{13}(\tau))d\tau}\\
&\quad\cdot\int_0^v \theta_{01}(s)e^{-\int_0^s[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}ds\\
&\quad+\theta_{32}(v)e^{-\int_0^v \theta_{32}(\tau)d\tau}\int_0^v \theta_{13}(s)e^{-\int_0^s(\theta_{12}(\tau)+\theta_{13}(\tau)-\theta_{32}(\tau))d\tau}\\
&\quad\cdot\int_0^s \theta_{01}(u)e^{-\int_0^u[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}duds]dv,\\
n_3(t) &= Ne^{-\int_0^t \theta_{32}(\tau)d\tau}\int_0^t \theta_{13}(s)e^{-\int_0^s(\theta_{12}(\tau)+\theta_{13}(\tau)-\theta_{32}(\tau))d\tau}\\
&\quad\cdot\int_0^s \theta_{01}(u)e^{-\int_0^u[\theta_{01}(\tau)-(\theta_{12}(\tau)+\theta_{13}(\tau))]d\tau}duds.
\end{aligned}
$$

3.2.5 Alternative Four State Model

It is not possible to write down a system of differential equations for the deterministic analog of the model in Section 3.1.4. This is true because the flow in and out of each state depends not only on current population levels, but on the entire history of the flow as well. By the result of Section 3.2.1, the most straightforward technique for finding the deterministic population levels as a function of time would therefore be to simulate the stochastic model repeatedly and compute the mean path.

Chapter 4

Estimation and Model Selection

4.1 Data

The data used in this analysis consist of monthly observations for all outstanding GNMA collateral of thirty year original term. Observations are grouped by origination year and pass-through coupon, and consist of the date, weighted average coupon, weighted average loan age (WALA), outstanding principal balance, number of pools, single monthly mortality, and factor, which is the current principal as a percentage of the original principal. For later reference, each coupon-origination year combination is called a class; there are 114 different classes in all. The origination period coincides with the observation period, which spans from January 1984 to December 1993. The pass-though coupons range from 15% for mortgages originated in 1984 down to a low of 6% for mortgages originated in the 1990s, in one-half percent increments. For just 1984, coupons ranged from 8.5% to 15%, while for 1992 the range was much lower, extending from 6% to 10%. During this time the coupons for mortgage-backed securities declined in response to steadily falling interest rates.

The data were obtained from EJV Partners, a financial services company. EJV obtains information on specific pools directly from its partners, six major broker-dealers, and then aggregates the raw data so that a single observation represents prepayment activity for all pools of a particular class. Thus a single observation in the dataset might be an aggregate figure for thousands or even tens of thousands of underlying pools. The largest observations are based on more than 11,000 pools representing over $43.6 billion of original principal. Altogether the data track prepayments on mortgages whose original principal totals approximately one-half trillion dollars.

A number of problems were encountered in analyzing the data. Most important was the issue of missing and incomplete observations, which proved to be a great practical complication since the simplest computer algorithms for reading and processing data expect sequential observations and hence are inadequate in this case. Of much greater significance, however, is that absolute prepayment probabilities cannot be determined at any time following a missing observation. The reason for this is that the single monthly mortality represents an empirical estimate of conditional prepayment probability. As explained in Chapter 1, the single monthly mortality for month i is the amount of principal which is prepaid between the end of months $i-1$ and i divided by the outstanding principal at the end of month $i-1$ less scheduled principal payments for the interval. For equal-sized mortgages originated simultaneously and no partial curtailments this is equal to the empirical probability that a mortgage prepays in month and i conditional on it having not previously prepaid. To see this, suppose that we observe a pool of N mortgages over a period of $T-1$ months. Let N_i denote the number of mortgages that prepay in month i and b_i be the scheduled balance. Set

$$N_t = N - \sum_{i=1}^{T-1} N_i,$$

and denote the empirical unconditional prepayment probability for $i = 1, \ldots, T$ by $\hat{p}_i = N_i/N$. Then

$$\begin{aligned} SMM_i &= \frac{b_i N_i}{b_i \sum_{j=i}^{T} N_i} \\ &= \frac{\hat{p}_i}{1 - \sum_{j=1}^{i-1} \hat{p}_j}, \end{aligned} \tag{4.1}$$

so the $\hat{p}_i$ can be determined recursively by the relation

$$\hat{p}_i = SMM_i \left(1 - \sum_{j=1}^{i-1} \hat{p}_j\right). \tag{4.2}$$

From (4.2) it is evident that missing observations can be problematic for determining the unconditional probabilities that will be required to obtain maximum likelihood estimates of model parameters.

This problem was addressed in a simple manner by linearly interpolating missing values. Errors in determining the SMM have their greatest influence on early observations since there are a relatively large number of undetermined unconditional probabilities that can be affected. Furthermore, for young pools that have experienced little prepayment activity the factor

$$1 - \sum_{j=1}^{i-1} \hat{p}_j$$

is close to unity, so errors in SMM observation translate into errors in unconditional probability of similar magnitude. Fortunately, actual SMMs for young mortgages are usually very smooth and very small, under one-half of one percent, so linear interpolation works well in this case. Seasoned mortgages exhibit much larger, more volatile SMMs, typically reaching four to six percent in active refinancing periods with monthly variation of up to a percent or more. In this case linear interpolation does not work as well, but the errors are less important, both because there are fewer absolute probabilities which are affected and because they are much smaller than the SMMs in magnitude.

The most important reason that interpolation is successful, however, is due to the fact that the SMM for one month affects the magnitude of the unconditional prepayment probability for another month only to a slight degree. Suppose the SMM datum at month t is missing for collateral experiencing heavy refinancing of 5%, and the estimated SMM found through linear interpolation is 4%. If the attenuation of the outstanding principal balance up to that point is 50%, then from (4.2) the prepayment probability for month t will be be computed as 0.020 rather than 0.025, a relative error of 20%. However, the relative error in the next month's probability is less than 1%. Thus, even when the approximations for missing SMM values are poor, and the effects on the determination of the corresponding unconditional prepayment probabilities are large, the probability estimates for other months are substantially unaffected.

Alternatives to interpolation were explored and rejected due to another significant problem with the data concerning consistency. The dataset contains redundant information, in that ideally the sequences of observations of principal balance, SMM, and factor can each be derived from the other if the original balance is known. For instance, prepayments for month i can be inferred as the difference between the amortized amount of principal left over at month $i-1$ less the actual principal remaining at month i. By its definition, SMM can then computed as the ratio of prepayments over the the amortized principal from month $i-1$. However, the amortization factor is not the same for all mortgages in a pool since the pool may contain mortgages with a small range of coupon and a larger range of origination dates, so this computation can yield SMMs significantly different from the true values.

Imprecision in the amortization factor is aggravated by the fact that the set of pools upon which each observation is based changes dynamically through time. As interest rates fall, homeowners with high interest rate mortgages refinance into lower rate mortgages, causing the volume of lower rate pools to grow. The growth of original principal through time is confounded with prepayments in the principal balance data.

In addition, original principal growth can result in consecutive months of static WALA values or even in decreases in WALA with the passage of calendar time, which happens when origination for a particular month and coupon is large compared to the total previously originated amount for that coupon. In this case, there may be several distinct observations with the same WALA. The reasonable solution to this problem was to substitute a single observation with the weighted average balance, SMM, and current coupon for the multiple observations.

The conjunction of averaging, pool growth, and heterogeneity invariably led to instances of negative amortization and negative prepayment probabilities in all types of SMM inference strategies which relied on the redundancy of the data. In contrast, linear interpolation never has this problem, and so was selected as the best method for inferring missing SMMs for the computation of the unconditional probabilities.

4.2 Estimation

The models developed in the previous chapter are all descriptions of a continuous time process. The data, however, are discrete, representing the total number of prepayments for each collateral type summed over monthly intervals. To accommodate the data it is natural to consider multinomial approximations for the continuous time models. The components of the multinomial vector are the number of prepayments for each month, and the multinomial probabilities are the integrals over monthly periods of the continuous-time prepayment probability.

If the number of borrowers in class i who prepay during month j relative to the mortgage origination date is n_{ij}, and each borrower prepays in month j with probability p_{ij}, then assuming independence the joint likelihood for all borrowers in class i is proportional to

$$\prod_{j=1}^{T_i} p_{ij}^{n_{ij}},$$

where $T_i - 1$ is the index for the last observation month, and n_{i,T_i} is the number of borrowers who have not prepaid by the end of the observation period. The joint likelihood for the borrowers in all classes is proportional to

$$l = \prod_{i=1}^{C} \prod_{j=1}^{T_i} p_{ij}^{n_{ij}}$$

where C is the number of borrower classes. In order to get a rough idea of the magnitude l, consider that there are about 100 classes, each with approximately

60 months worth of data on an original par of \$5 billion. If the average mortgage size is \$100,000 and the monthly prepayment probabilities are 0.01, then about 500 mortgages should prepay every month. In this case we have

$$
\begin{aligned}
l &\approx \prod_{i=1}^{100} \prod_{j=1}^{50} 0.01^{500} \\
&= 10^{-5e6},
\end{aligned}
$$

a number too small for ordinary computers to evaluate without underflow. Thus it is more convenient to work with the negative log of the likelihood,

$$
\begin{aligned}
-\log l &= -\sum_{i=1}^{C} \sum_{j=1}^{T_i} n_{ij} \log p_{ij} \\
&\approx 1e7
\end{aligned}
$$

and find parameter estimates which minimize this function.

For a particular model the p_{ij}s can be determined by integrating the continuous time likelihood function derived in Chapter 3 for collateral of type i over month j. This was done using the IMSL package of FORTRAN subroutines for mathematical applications, in particular, the subroutine DIVPAG, which employs either an Adams-Moulton method for high accuracy or Gear's method for stiff systems of differential equations. Before the general neighborhood of the solution was identified stiffness was a serious problem for the models that included seasoning, because the optimization code would sometimes choose coordinates for which the seasoning rate was orders of magnitude different than that for the prepayment rate. As refinements were made to the initial search region, the problem of stiffness disappeared so that the Adams-Moulton method could be used reliably.

Another problem with numerical integration relates to the optimization stage of the estimation procedure. Despite the fact that the most precise integrator in the IMSL library was employed, the resulting numerical solutions contained enough noise to render derivative-based optimization routines useless. Accordingly, the robust multivariate Nelder-Mead method was selected as an optimizer. The Nelder-Mead algorithm has the advantage of performing well even for very noisy functions. Its significant drawback is that it typically converges on the optimum at a slow rate, particularly in higher dimensions.

In addition, Nelder-Mead is not particularly well suited for constrained problems, and in many instances model parameters are required to be positive in order to avoid nonsensical scenarios involving negative prepayments. This was not a difficulty however, as a simple exponential transformation of an unconstrained parameter value maps it into the positive reals. For the case of parameters constrained to lie in a finite range, as in population proportions, a logistic transformation

$$f(x) = \frac{1}{1+e^x}$$

suffices.

Starting coordinates for the algorithm simplex were obtained by sampling uniform random numbers from an initial coordinate range. In each case the search procedure was executed several times to reduce the chances that the search would stall in flat areas or local optima.

Maximum likelihood estimates obtained using the Nelder-Mead optimization algorithm and the IMSL integration routine DIVPAG are presented in table 4.1 for the two, three, and four state models. The estimates are not based on the entire data set: the last twelve observations were reserved in order to later analyze the model accuracy on out-of-sample data. This reduced the number of different collateral types from 114 to 106, owing to a short observation period for some of the newer classes. In addition, although interpolated SMMs were used to determine unconditional empirical prepayments, only the empirical numbers for non-missing months were used in the determination of the log likelihood. As indicated earlier, large errors can occur in the interpolated probabilities. To avoid such an error and possible bias introduced by the implied linear SMM, the prepayment predictions of the model being estimated were substituted for the missing months. This substitution clearly biases the negative log likelihood downwards, but because our interest lies in the performance of each model relative to the others, this seems like a reasonable course of action.

The mean traces based on the MLEs for the various models will shown in a succession of figures along with actual prepayments over a representative sample of the 106 classes. The sample, or test set, consists of six different collateral types with coupons ranging from 8% to 14% and origin years from 1984 to 1989. Included in the test set are the data for the Ginnie Mae 8s issued in 1989, the 9s of 88, the 11.5s of 87, the 10.5s of 85, the 13s of 85, and the 14s of 84. These collateral run the gamut of prepayment behavior from fast to slow, demonstrate a range of seasoning from partial to complete, and in the case of the higher coupon classes, exhibit burnout. Since it is impractical to display results for all 106 classes for each of the various models and

Parameter	2 State	3 State	4 State
$\log\alpha_0$	—	-4.82	-4.87
$\log\alpha_1$	—	-4.42	-4.39
$\log\beta$	—	—	-5.71
$\log\gamma_0$	-5.74	-3.63	-3.59
$\log\gamma_1$	-4.27	-2.85	-2.87
γ_2	2.73	3.25	3.05
γ_3	1.14	1.13	1.08
$-\log l$	6.255e6	6.164e6	6.164e6

Table 4.1 Maximum likelihood parameter estimates for various prepayment models.

procedures that will be examined later on, this test set will serve as a useful standard for visual comparison.

The test set was selected with emphasis on having as wide a range of coupon and origination year and hence prepayment behavior as possible, rather than according to largest balance. In fact, smaller balances were preferable from the standpoint of facilitating simulation studies. The greatest proportion of the data corresponds to low coupon unseasoned mortgages, such as the 8s of 89 and 9s of 88. It is expected that most of the models will fit this type of collateral well, as it does not display particularly complicated prepayment behavior. The next greatest proportion of the data is the medium coupon, moderately seasoned mortgages such as the 11.5s of 87 and 10.5s of 85, for which it is expected the models will not fit quite as well. Due to the complicated nature of burnout, the worst fits are expected to occur with the high coupon, highly seasoned collateral, which represents a small minority of the total number of mortgages in the dataset. Typical total original principal balances for this type of collateral are around \$100 million, two orders of magnitude smaller than the \$10 to \$40 billion common for younger collateral. Thus the fit of mean prepayment activity to historical data will typically be even better than the test set may indicate. Nonetheless, the importance of prepayment responses to rapidly falling rates for seasoned collateral should not be underestimated. Similar conditions will doubtless recur eventually, and the impact of very low rates on the valuation of mortgage-backed securities is tremendous. Thus it is important that the prepayment model be able to adequately explain all collateral types represented by the testset.

Figures 4.1 and 4.2 illustrate the fit for the two state model, which is clearly inadequate, accounting for neither seasoning nor burnout. In the first two panels of Figure 4.1 the model values are biased significantly upwards from the observed prepayment rates for the early months of the mortgage. This can be interpreted as a failure to explain the seasoning process, since the model predicts that prepayments start off at their long term rate which is much higher than the initial rate exhibited by the real data. In the last two panels of the next figure, prepayments react to early changes in the mortgage refinancing rate much more vigorously than in reality, again due to the absence of seasoning. In the last panel of Figure 4.1 the mean response to interest fluctuations is about right early on but significantly too high in the later months, while in the first panel of Figure 4.2 the mean response is too low throughout. This casts doubt on whether a single response function is adequate to model the entire population's response to changes in mortgage rates. The last two panels of 4.2 capture the mean prepayment response in the early months, but fail to account for the burnout evident in later months, which is particularly well illustrated by the 13s of 85. In the first peak of the relative coupon at 5% near month 25, prepayments rise to a high of nearly 6%. At the end of the observation period the relative coupon rises once again above 5%, but this time prepayments remain for the most part in the 2% to 3% range.

Figures 4.3 and 4.4 show actual data for the test set and the fit obtained using the maximum likelihood estimates in the three state model. The three state model allows for a significant improvement by accounting for seasoning. Although the negative log likelihood function improves by only 1.5%, from 6.255e6 to 6.164e6, this amounts to a very large change in the likelihood itself. The improvement can be seen by comparing the 8s of 89 and the 9s of 88 of the three state model in Figure 4.3 to the two state model in Figure 4.1. In the three state model, prepayments rise smoothly from zero up to their steady state value. Because the bulk of the data represent young collateral, the addition of seasoning results in a significant improvement of the likelihood. An additional benefit can be noted in the middle part of the data for the 10.5s of 85 shown in Figure 4.4. This improvement is due to better estimation of the steady-state prepayment rate in the three state model. The two state model tends to underestimate this level in order to reduce the error for early prepayment rates.

Improvement is not evident in all the panels, however. The inclusion of seasoning actually impairs the fit for the 11.5s of 87, which season very rapidly and seem to achieve steady-state prepayment rates of nearly 1% by the seventh month. This

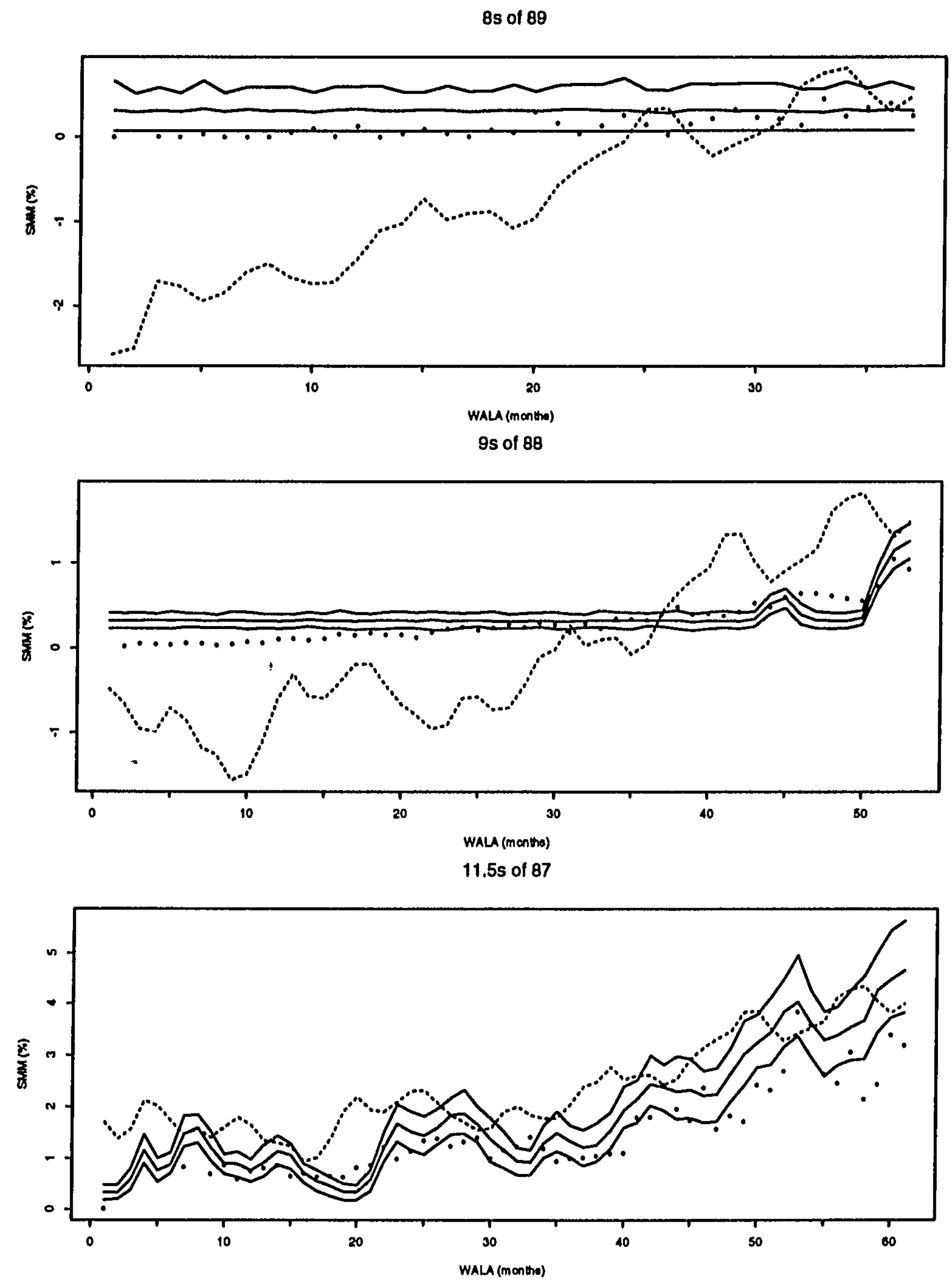

Figure 4.1 Two state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

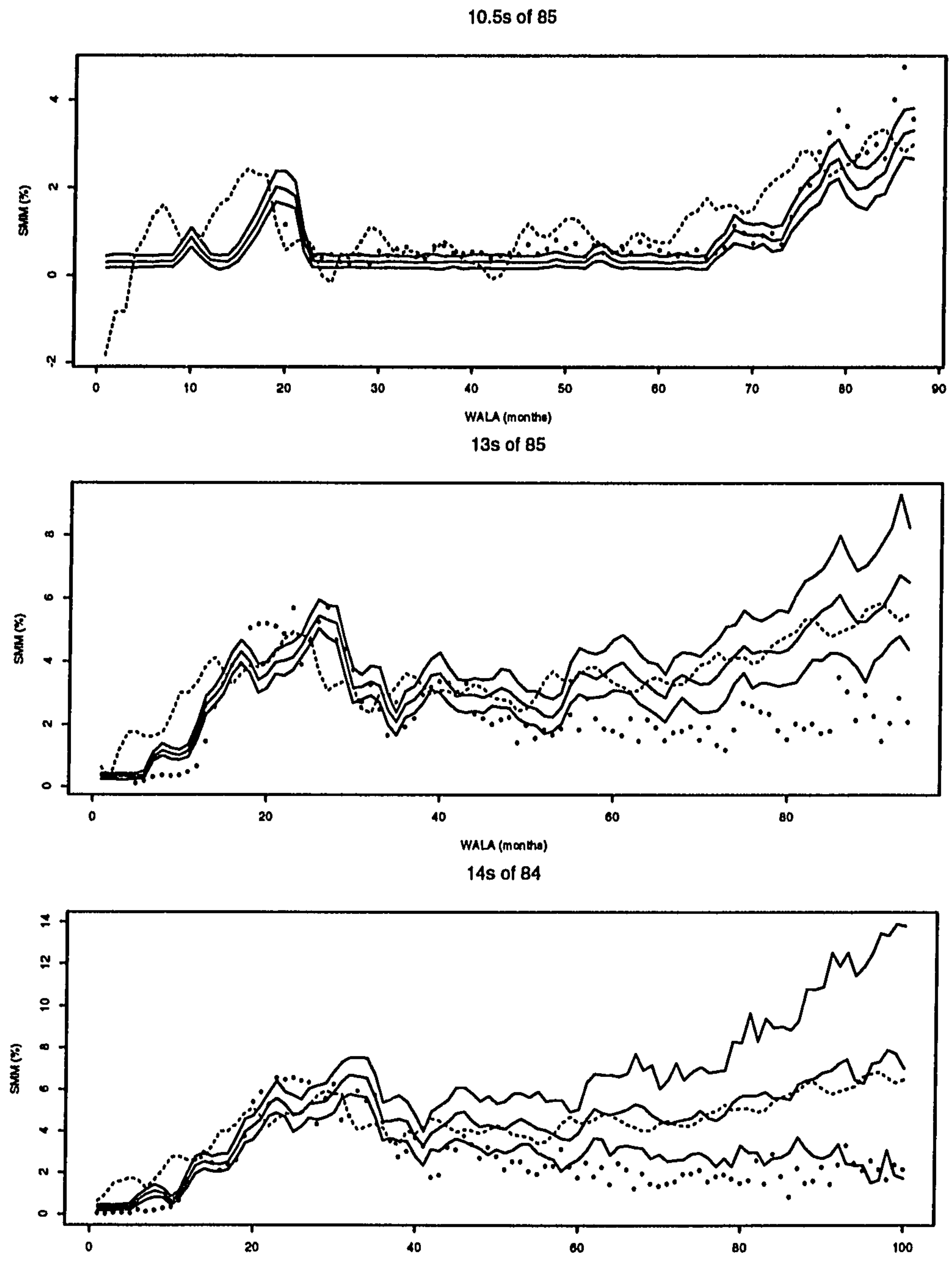

Figure 4.2 Two state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

pattern is not typical for the remainder of the data, however. As with the two state model, the failure of the three state model to account for burnout can be seen for the older high coupon examples in Figure 4.4. The maximum likelihood fit in the three state model settles on a lower interest rate sensitivity, however, so the early peak in prepayments is significantly underestimated, while the later rise in relative coupon does not lead to as large overestimates.

Fits for the four state model are shown in Figures 4.5 and 4.6. The four state model was unsuccessful in explaining burnout. Inclusion of the burnout state failed to improve the likelihood function by so much as one percent, and visually the fits seem unchanged as well. The sensitivity to interest rates is largely unchanged: $\log \gamma_1$ changes only from -2.85 to -2.87 with the additional state, so the model is not allowing for a subpopulation of mortgage holders with a strong sensitivity to refinancing opportunities. Although such a subpopulation might explain the initial peak in prepayments for the older high coupon examples such as the 13s of 85 and 14s of 84, there are too many pools with behavior similar to that of the 11.5s of 87. An examination of Figures 4.5 and 4.6 reveals that medium coupon moderately seasoned collateral already demonstrates lower sensitivity in reality than the model predicts, so increasing the relative coupon sensitivity would worsen the overall fit.

Overall, the three state model provides the best tradeoff between accuracy of fit and dimensionality. The increase in the number of parameters from four to six provides a noticable visual improvement in fit, as well as an 11% improvement in the likelihood function, so the three state model is preferred to the two state model. Adding a seventh parameter to create the fourth state provides an insignificant gain in accuracy, so the four state model is rejected. Of the models presented so far, the three state model is deemed to be the best. Yet on the basis of the fit accuracy it seems there is significant room for improvement.

4.3 Alternative Models

An examination of Figures 4.2, 4.4, and 4.6 indicates that a single prepayment sensitivity is insufficient to explain the burnout evident in highly seasoned collateral. Not even the four state model, despite allowing for two separate prepayment rates among subpopulations of mortgage holders, was able to explain this phenomenon. Perhaps this failure is due to the fact that there really are two subpopulations, but the four state specification was too restrictive in requiring one population to be wholly averse

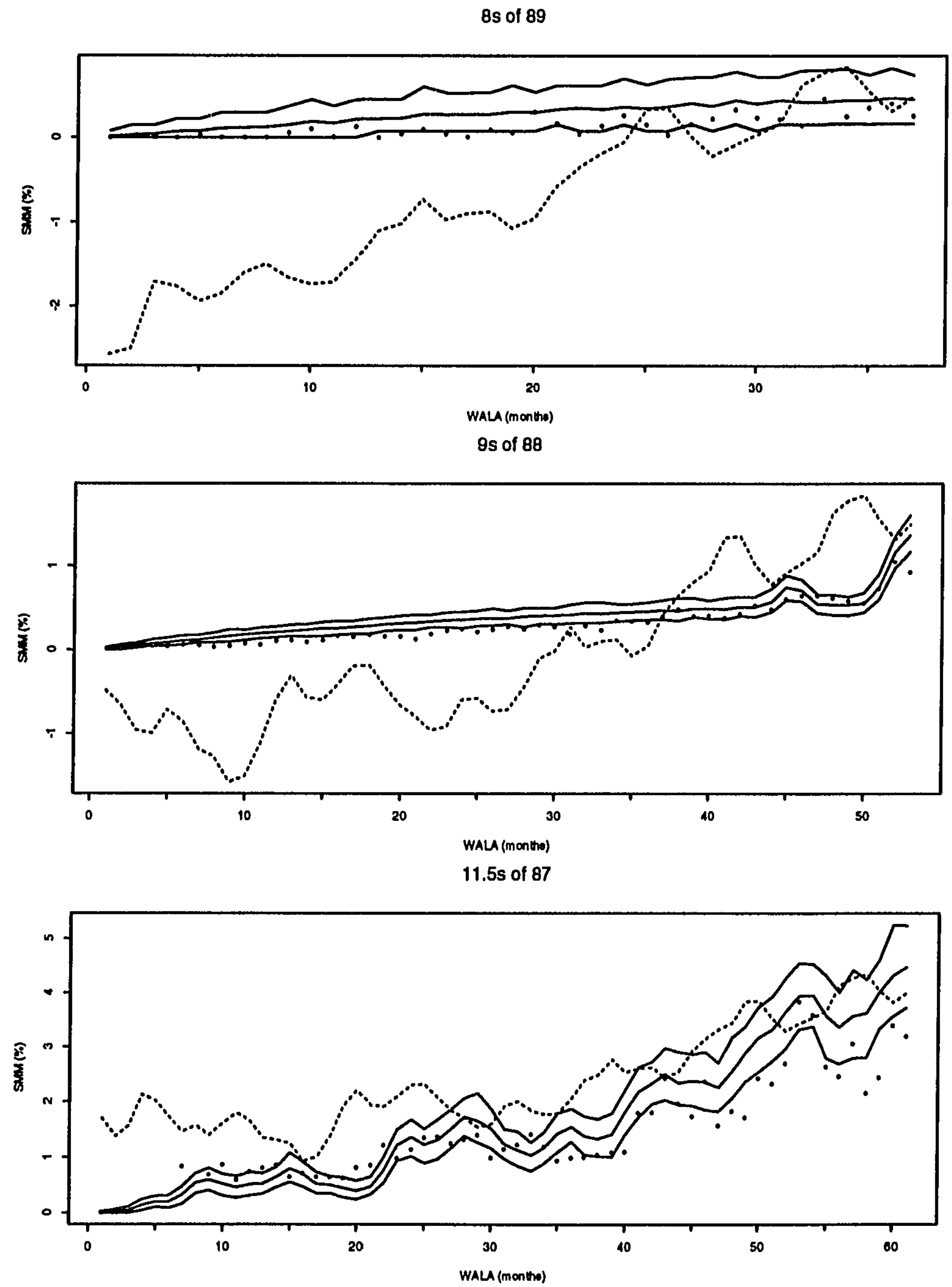

Figure 4.3 Three state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

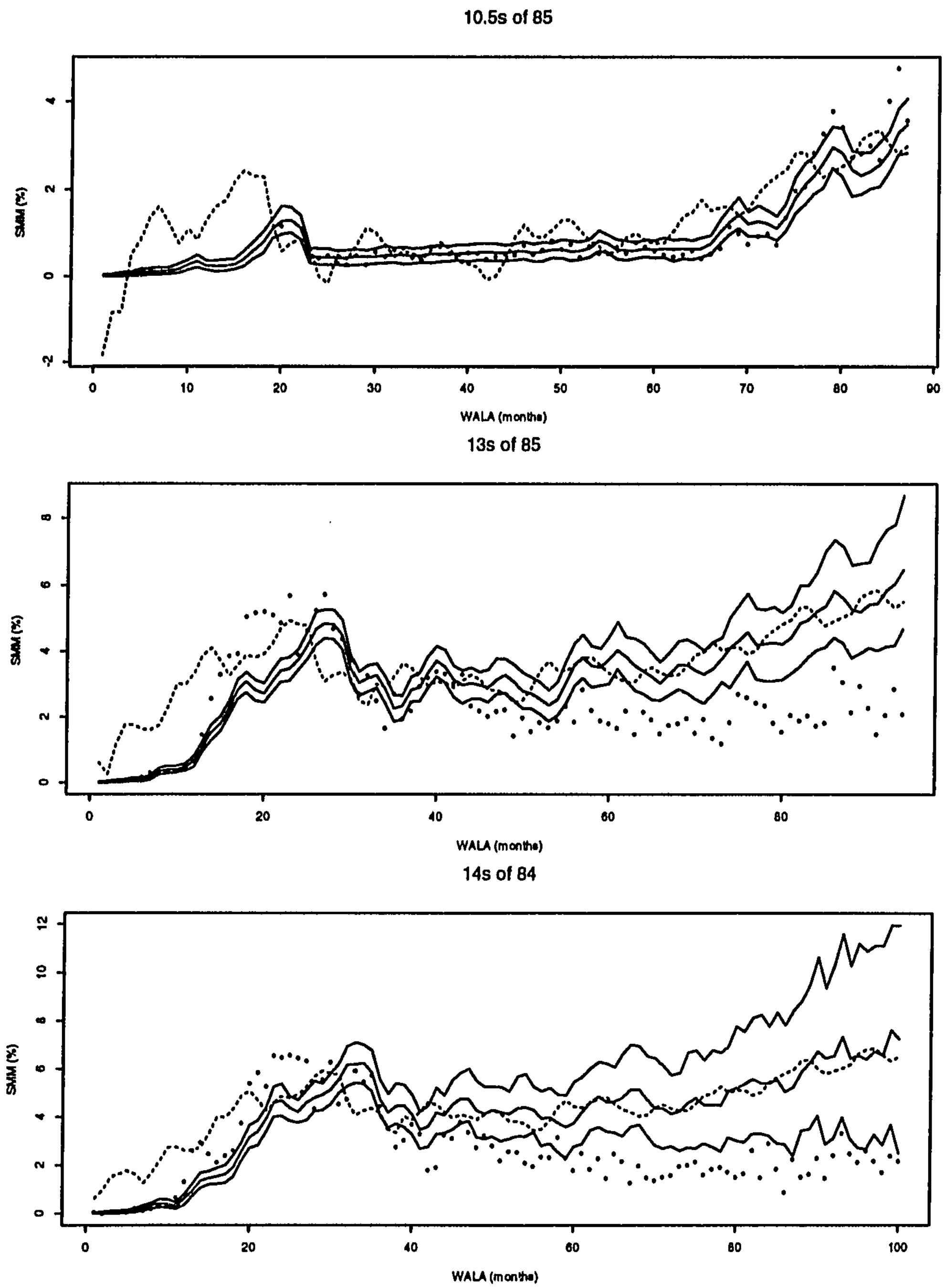

Figure 4.4 Three state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

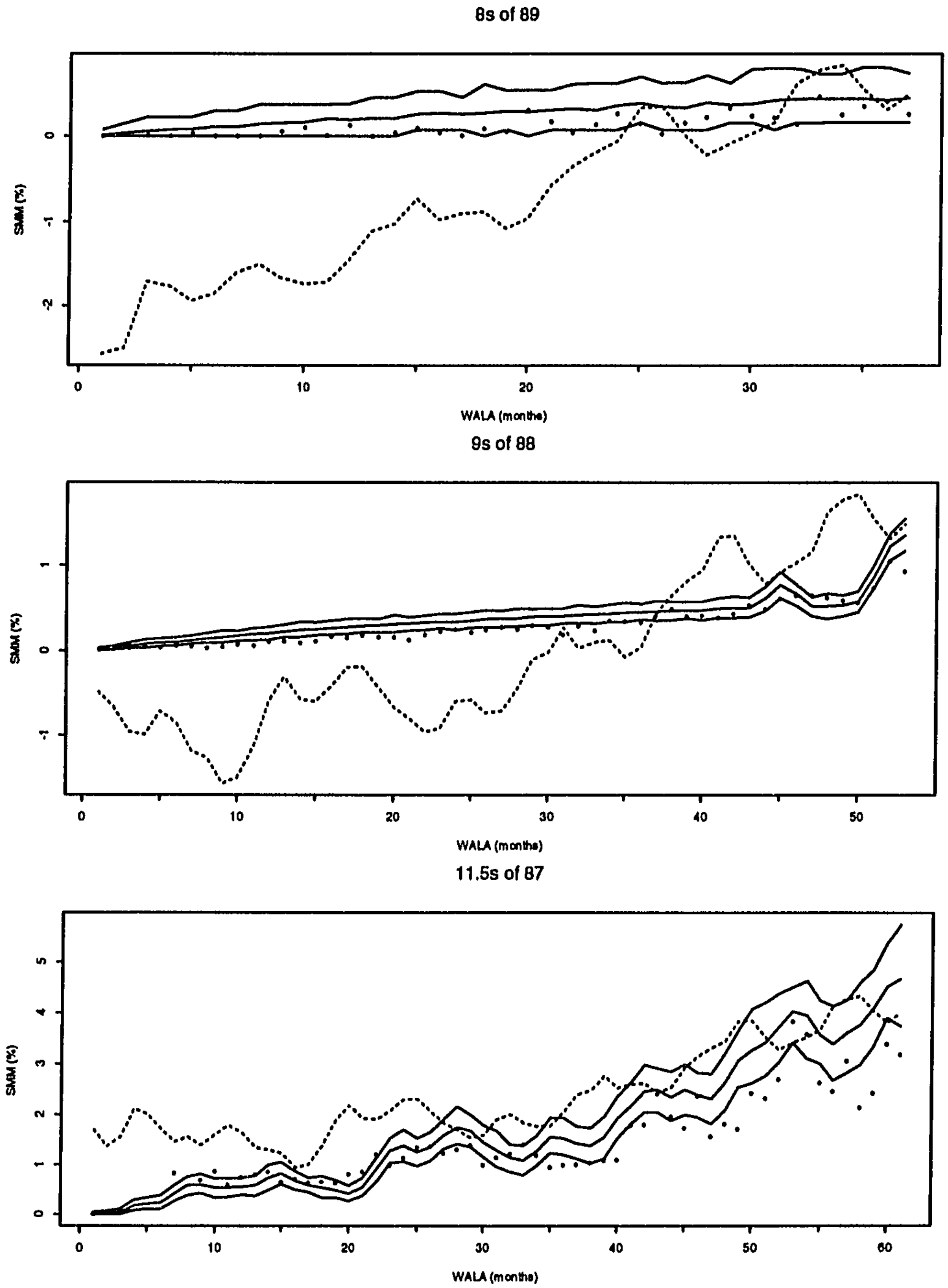

Figure 4.5 Four state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

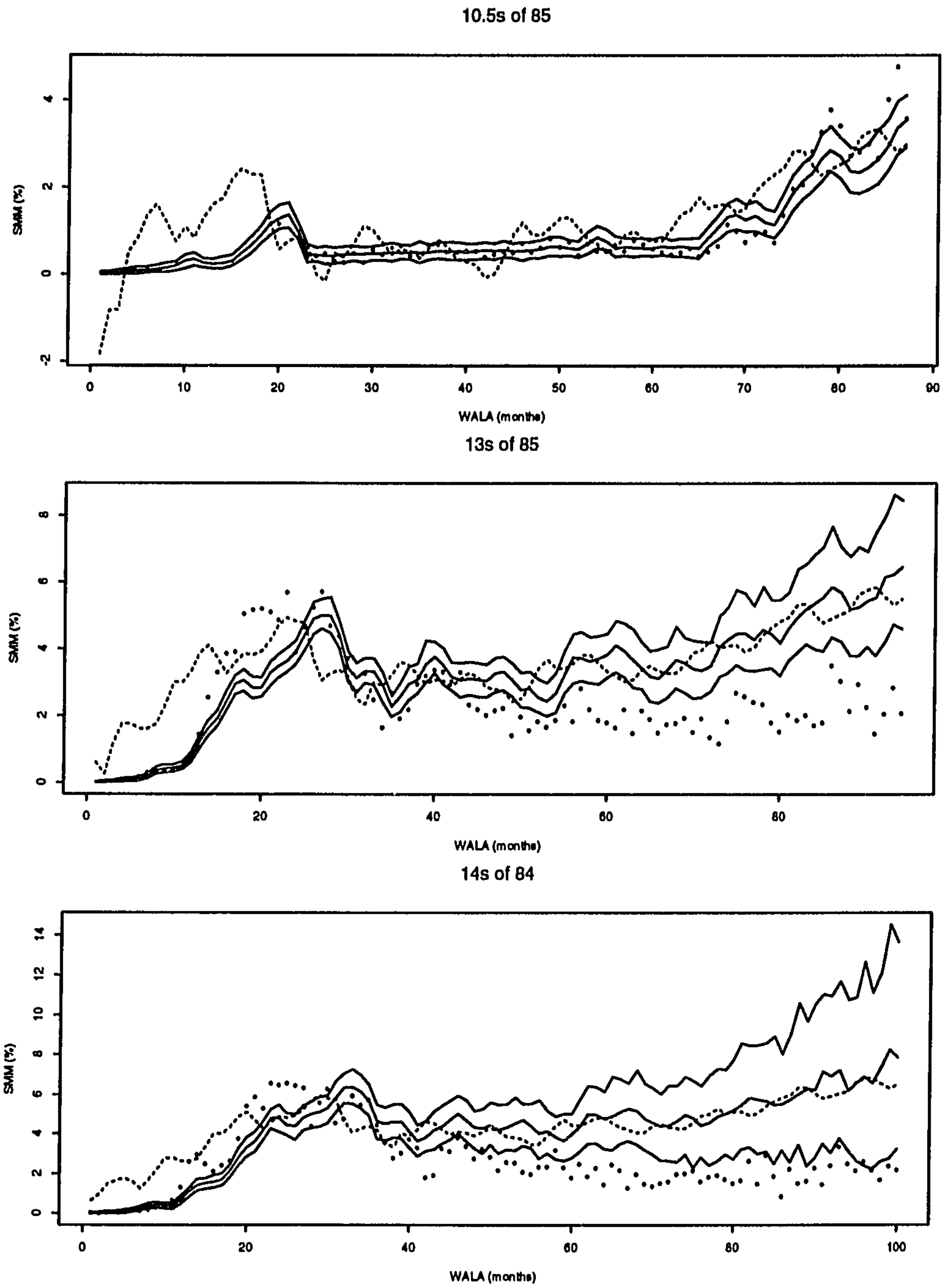

Figure 4.6 Four state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

to financially induced prepayments. Hence we shall relax this restriction and allow the insensitive prepayers of the four state model to be sensitive to mortgage rates, although to a lesser degree than the other prepayer group. To do this it is convenient to introduce an additional state. The new model can be described by the five state system shown in Figure 4.7. In the five state model, every borrower belongs to exactly one group, and each group follows the three state specification independently.

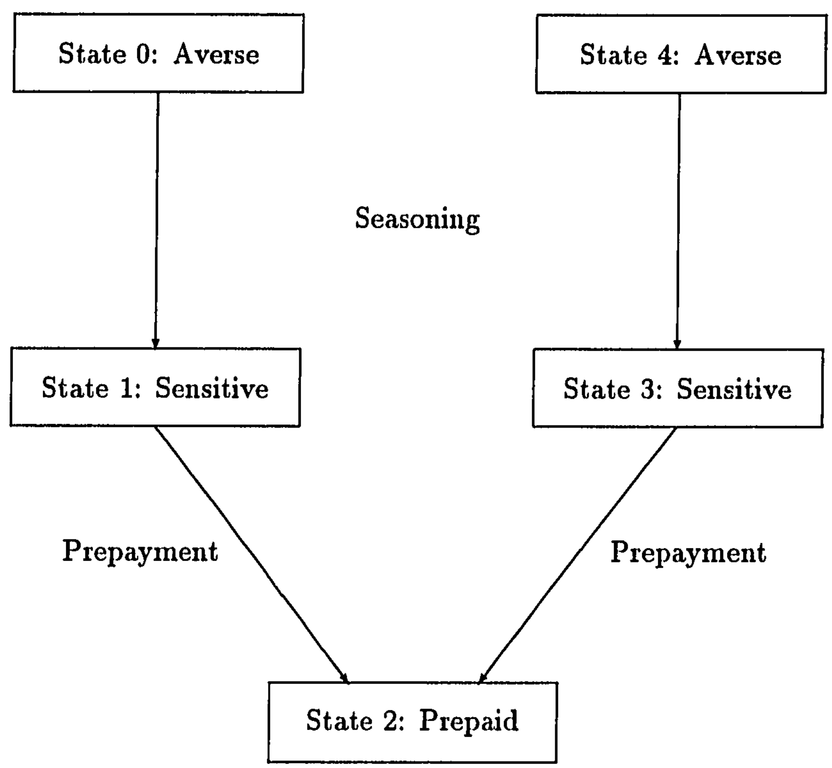

Figure 4.7 Five State, Two Population Model

In order to limit the increase in the number of model parameters, some of the population characteristics can be required to be shared between groups. It makes sense to require the prepayment delay to be the same for each subpopulation, since this is primarily due to institutional processing requirements which are common to all prepayments. It also seems reasonable to suppose that the mortgage rate threshold is the same for each group because it is presumably related to points, fees and other charges which would be the same or similar for all borrowers. The most important difference between borrowers is the prepayment sensitivity to relative coupon. Although of less obvious significance, a difference between seasoning rates for the different groups will be allowed. Accordingly, we define the transition probabilities for the five state model in a similar fashion to the three state model:

$$p(\text{transition from } i \text{ to } j \text{ in } [t, t+\Delta t]) = \theta_{ij}(t)\Delta t,$$

where the θ_{ij}s are defined by

$$\begin{aligned}
\theta_{01}(t) &= \alpha_0 + \alpha_1(r_0 - r(t))^+, \\
\theta_{43}(t) &= \alpha_0 + \alpha_2(r_0 - r(t))^+, \\
\theta_{12}(t) &= \gamma_0 + \gamma_1(r_0 - r(t-\gamma_2) - \gamma_3)^+, \\
\theta_{32}(t) &= \gamma_0 + \gamma_4(r_0 - r(t-\gamma_2) - \gamma_3)^+, \\
\theta_{ij}(t) &= 0, \text{ otherwise.}
\end{aligned}$$

An additional parameter β is required to determine the initial conditions. If there are N borrowers altogether, then βN begin in state zero and $(1-\beta)N$ begin in state five. This completely specifies the two population model.

Maximum likelihood estimates for the two population model are presented in Table 4.2 alongside the results for the three state model, which are included again for comparative purposes. According to the table, borrowers who begin in state zero have a prepayment sensitivity given by $\log \gamma_1 = -3.12$, compared to $\log \gamma_4 = -5.09$ for the borrowers beginning in state 5. Thus the people starting off in state five may be characterized as slow prepayers and those in state zero as fast, with the fast prepayers $\gamma_1/\gamma_4 = 7.2$ times as likely to prepay for a given relative coupon.

The fast prepayers season more quickly as well if interest rates show favorable movement. By hypothesis both groups season at equal rates when there is no economic prepayment incentive. Empirically this rate is found to be $\alpha_0 = 0.8\%$ per month, which implies a half-life for the initial state of 89 months. If the relative coupon rises above the threshold $\gamma_3 = 1.16\%$, however, monthly seasoning rates immediately rise by $\alpha_2 = 0.6\%$ per point for the slow group, and a much larger $\alpha_1 = 4.3\%$ per point for the fast group. Thus falling mortgage rates lead to a flood of seasoning by the fast prepayers and a corresponding rise in prepayments. Since the initial population proportions are skewed towards a predominance of the fast prepayers,

Parameter	3 State	5 State
$\log \alpha_0$	-4.82	-4.85
$\log \alpha_1$	-4.42	-3.15
$\log \alpha_2$	—	-5.10
$\log \beta$	—	-.34
$\log \gamma_0$	-3.63	-3.64
$\log \gamma_1$	-2.85	-3.12
γ_2	3.25	2.88
γ_3	1.13	1.16
$\log \gamma_4$	—	-5.09
$-\log l$	6.164e6	6.155e6

Table 4.2 Maximum likelihood estimates for the three and five state models.

with $1/(1 + \exp \beta) = 58\%$ in state zero and the remaining 42% in state five, high prepayment rates can be sustained for some time but not indefinitely. As the fast group prepays, the slow group quickly comes to predominate and overall prepayment rates fall, resulting in the burnout that is seen in the 13s of 85 and 14s of 84.

The relative improvement in the log likelihood function obtained due to the division of borrowers into two classes is only 0.13%, yet once again this represents a large relative gain in the likelihood. The improvement in the fit is visually evident as well, and can be seen in Figures 4.8 and 4.9, which show the mean traces and confidence bands for the fitted two population model over the test set data.

From Figure 4.9 it is clear that the five population model explains the burnout phenomenon. Fit for both the 13s of 85 and 14s of 84 are much better, both in the early and later portions of the plots. The improvement is due to the fact that both the high prepayment sensitivity demonstrated in the early period and the low sensitivity demonstrated in the later period can be identified through the additional prepayment response in the five state model, whereas the three state model identifies a single sensitivity that is somewhere in the middle, causing it to underestimate the peak and overestimate the tail in the last two panels of Figure 4.4.

The gain in accuracy for the high coupon collateral is offset somewhat by a slight tradeoff for some of the middle coupons. Although the fit for the 10.5s of 85 is largely unchanged, the three state model fits the 11.5s of 87 a little better in the later months. The fast prepayers identified by the five state model do not appear to prepay as fast in

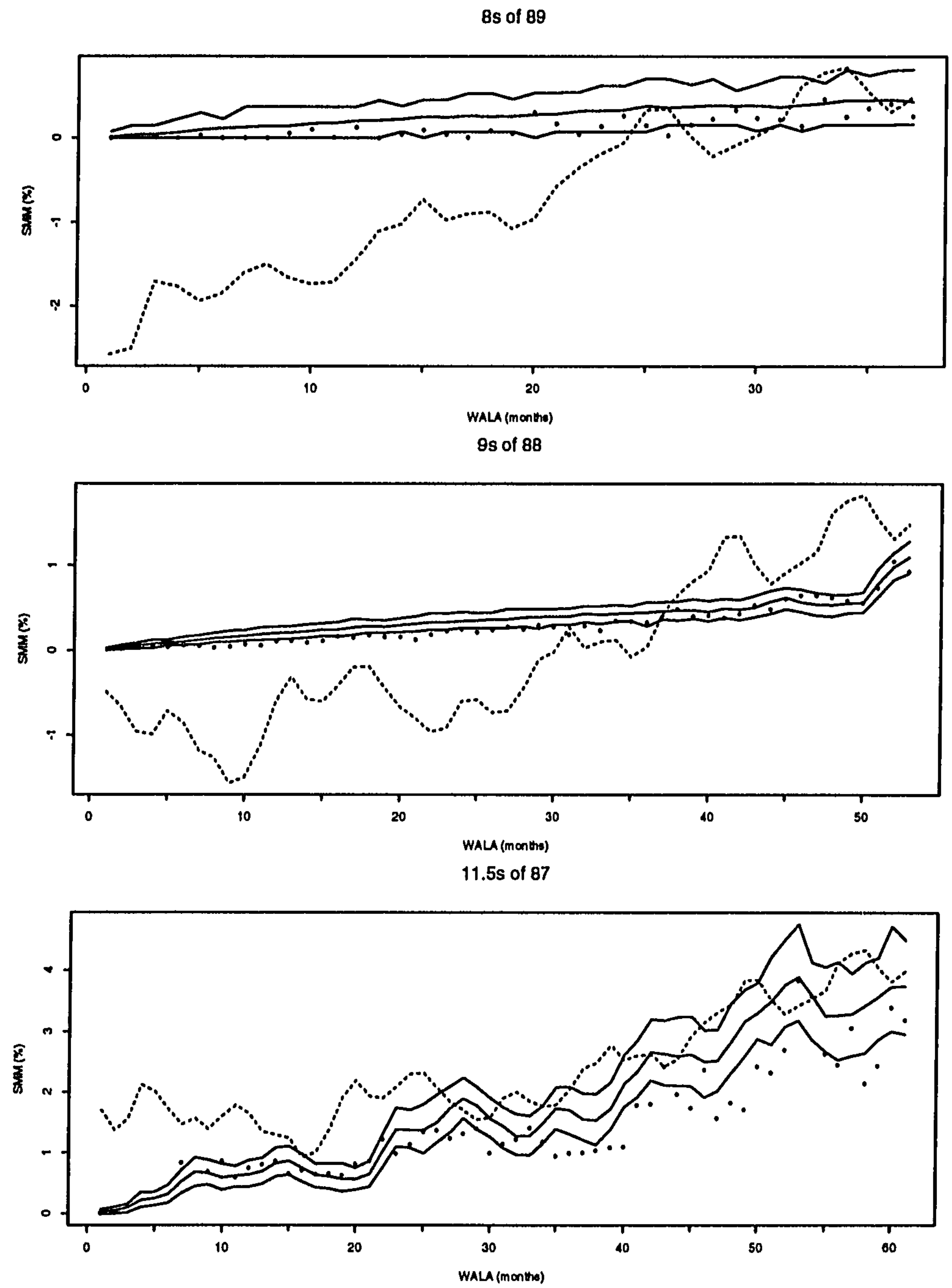

Figure 4.8 Five state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

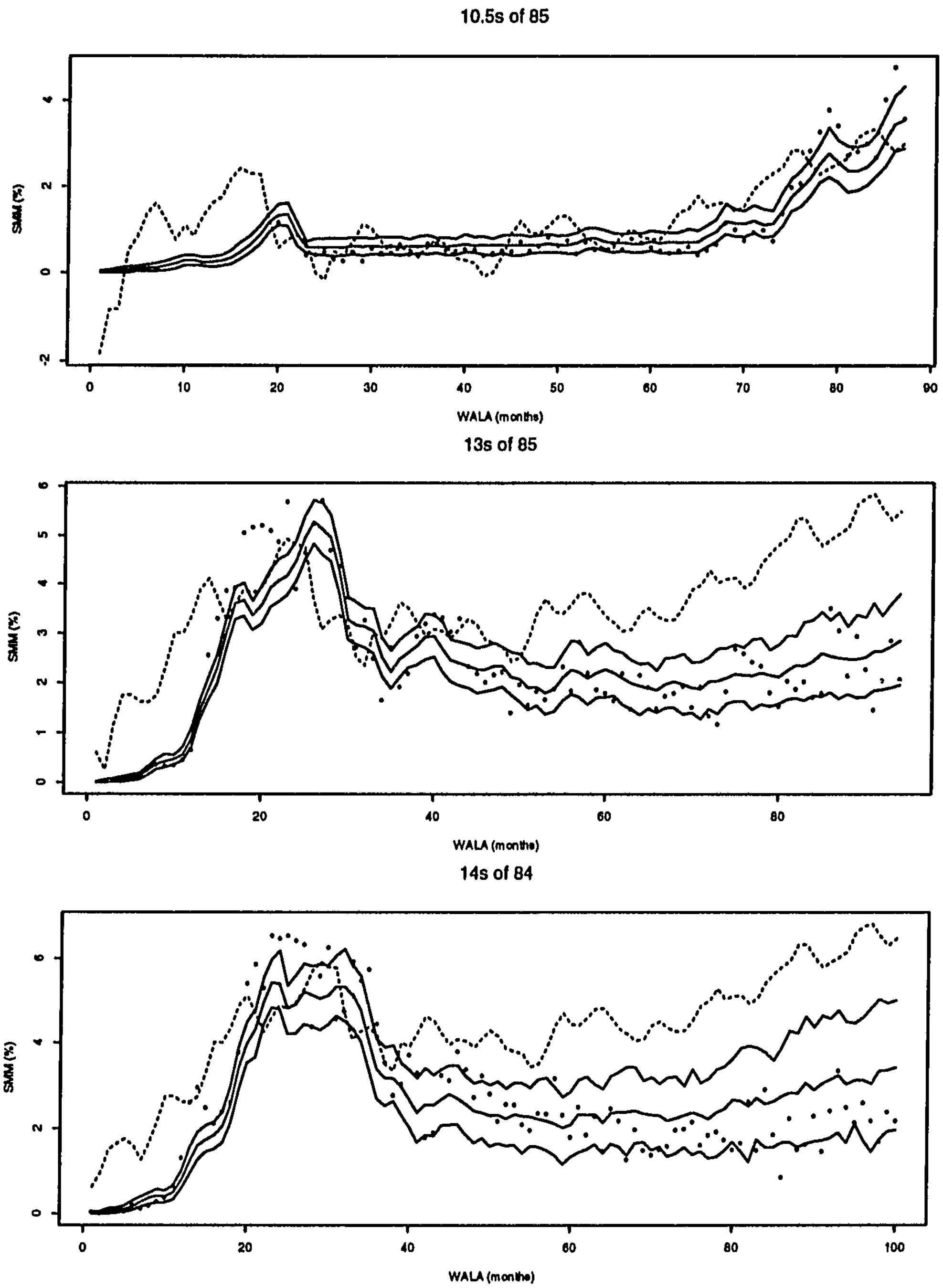

Figure 4.9 Five state model. Points represent prepayment observations, solid lines the simulated mean and 95% confidence bands. The dotted line is the relative coupon.

reality across all coupon classes. The fits for the lower coupon collateral represented by the 8s of 89, 9s of 88 for the two models are almost identical, but this discrepancy is not so great as to cause alarm.

The basic prepayment effects, including seasoning and burnout, seem to be adequately addressed by the model. Weighing the slight deterioration of the middle coupon fits and the increased number of parameters against the dramatic improvement for the high coupons, we conclude that the five state model is preferred to any of the others. Overall, the five state model seems to explain mean prepayment behavior very well.

4.4 Alternative Assumptions

Despite the visually appealing fit of the mean prepayment traces, on the basis of the confidence intervals we can easily reject the notion that the five state model or any other accurately represents the true prepayment process. Within the test set alone the majority of points fall outside the 95% confidence bands, which are based on simulations involving around \$100 million to \$1 billion in original principal. While the fit is visually appealing for much of the remaining data, as just mentioned, many of the other classes represent between 10 and 400 times as many mortgages. Since the multinomial distribution is asymptotically normal, we can surmise that the 95% confidence bands for other data may be roughly 1/3 to 1/20th as wide as those shown in Figures 4.1 through 4.6, which would make them far too narrow to contain more than a very small fraction of the actual prepayment observations. Thus we now examine some alternatives to previously made assumptions in the hope of explaining the variability of prepayments to a greater extent..

4.4.1 Unequal Original Balances

The first modification is to relax the assumption that all the mortgages have the same original principal amounts of \$100,000. Instead, we shall assume that the initial par value of the mortgage is a random variable

$$p \sim \$100,000\epsilon, \tag{4.3}$$

where $\log \epsilon \sim N(0, \sigma^2)$, so that

$$\log p \sim N(11.5, \sigma^2). \tag{4.4}$$

The choice of \$100,000 for the median initial par value in (4.3) is not important; varying this value affects the number of mortgages in each pool and hence has the modest effect of narrowing or widening the confidence bands by a constant factor not too much different from unity. However, we seek a larger, more fundamental change in prepayment variability.

The effect of varying σ is not as clear as the effect of varying the median initial balance. Figure 4.10 shows the results of a simulation study on the effect of varying σ from 0 to 0.4. The values for σ were selected because the interquartile range for $\sigma = 0.2$ is generously wide, covering a \$27,000 interval, and the upper and lower .01 quantiles vary from \$63,000 up to \$159,000, which is slightly above the stutory limit on the maximum size of a Ginnie Mae loan. Thus 0.2 is a realistic value for σ, and twice this should be a generous upper bound, even if the median loan size is smaller than \$100,000. For $\sigma = 0.4$ the interquartile range grows to cover a \$55,000 interval, with upper and lower .01 quantiles are \$40,000 and \$254,000.

Figure 4.10 clearly illustrates that the effects of varying σ in (4.4) are insignificant in affecting the distribution of prepayments over time. Neither the mean prepayment trace nor the confidence bands are influenced to any noticeable degree. In each case the number of simulations performed was equal to 5,865, the number of mortgages constituting the 10.5s of 85 data under the $p = \$100,000$ assumption.

4.4.2 Correlation

Perhaps the most important assumption, aside from the specification of prepayment response, is that of independence. Although it is not implausible to assume that conditional on the path of mortgage rates individual homeowners make their refinancing decisions independently, we must conclude on the basis of the consistent tendency of the data points to fall outside the confidence bands in the previous fits that this is not the case. All of the figures demonstrated so far have very narrow bands initially that spread out rapidly as the number of unprepaid mortgages diminishes, whereas the actual data do not show such a strong relationship between number of unprepaid mortgages and noise. Especially for active refinancing periods, prepayments seem to exhibit large variability even when the mortgage population has not been greatly attenuated, as for example in the case of the 14s of 84. The 14s of 84 exhibit SMM drops of nearly 2% not once but twice within a four month span, between months 27 and 31, despite a falling mortgage rate environment, showing that the variability

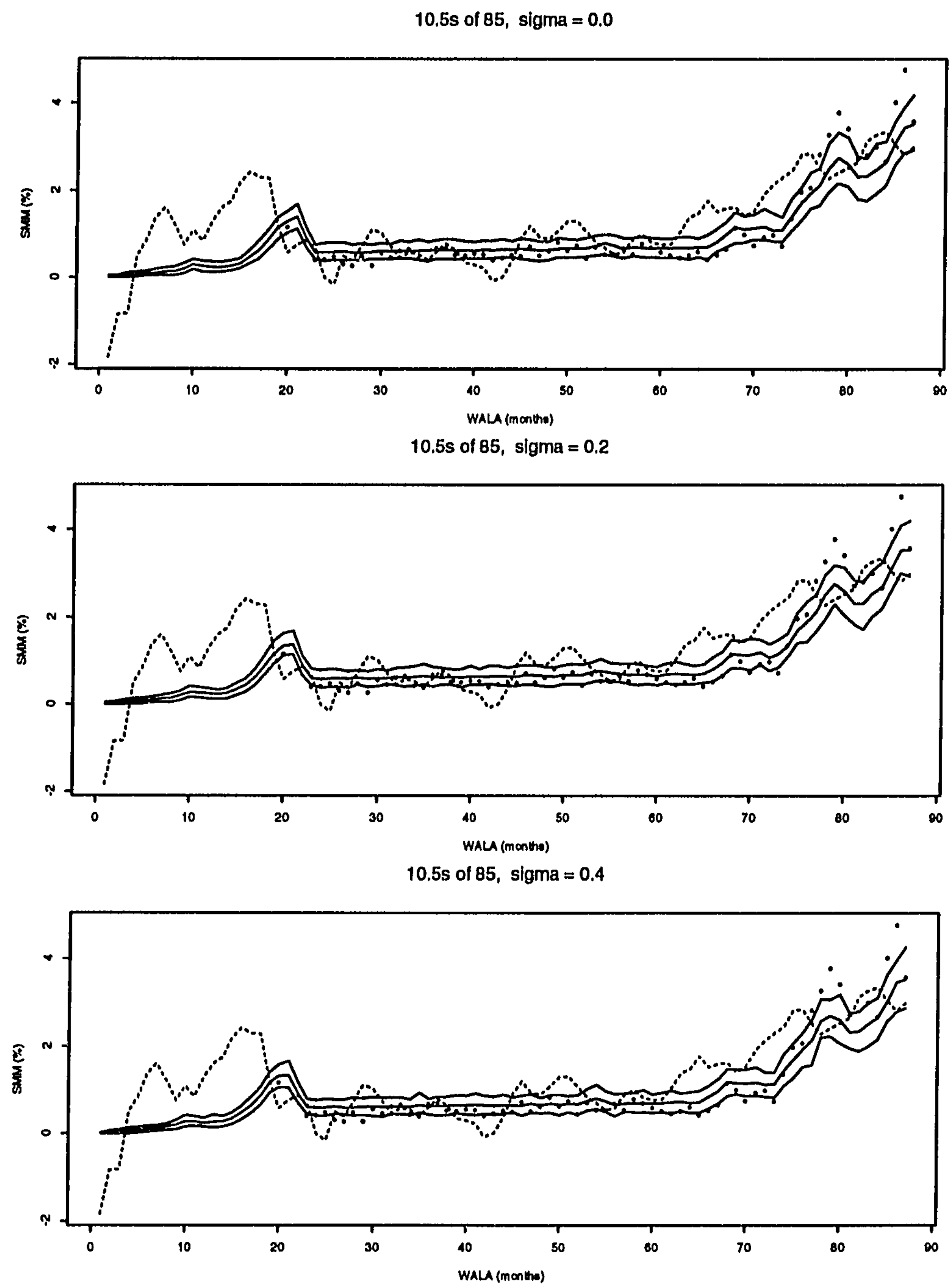

Figure 4.10 Five state model with lognormally distributed initial mortgage balances. There is no visible effect from changing the lognormal variance.

within the single actual realization exceeds the variability under the model across all possible realizations.

Furthermore, under the independence assumption the variance of the multinomial prepayments goes to zero as the number of mortgages increases. The data do not demonstrate this property either, as Figure 4.11 indicates. The first panel shows the 10s of 86, which do not experience any significant refinancing until age 65. The second shows the 8.5s of 88, which experience no significant refinancing throughout the entire observation period ending at age 54. The variability of the 10s of 86 is by a small margin the larger of the two, despite the fact that its initial principal balance is greater by two orders of magnitude. The 10s of 86 represent over $16 billion of principal and have an error standard deviation of 0.097, whereas the 8.5s of 88 account for less than $500 million and have an error standard deviation of only 0.091.

One simple way to model correlation between different borrowers is to introduce an unobserved noise process which multiplies the state variable. Prepayments would still be conditionally independent of one another given the history of the noise process, but since this is unobserved, prepayments would appear to be correlated. If we denote the noise process by $\{\eta_t\}$, then individuals respond not to the observed vector $\{r_t\}$, but instead to $\{r_t\eta_t\}$. It is convenient to suppose that $\log \eta_i$ is a simple ARMA process.

To motivate a suitable choice for the noise process, we consider an example involving the 11.5s of 87. Figure (4.12) shows a few simple time series diagnostic plots for the log of the residuals of the deterministic five state model fit to the series. The residual for month i is defined as

$$\frac{SMM_i}{h_i}, \tag{4.5}$$

where h_i is the discrete hazard function for the multinomial approximation to the fitted five state model; that is,

$$h_i = \frac{p_i(\hat{\theta})}{1 - \sum_{j=1}^{i-1} p_j(\hat{\theta})}$$

We shall denote the log of the ith fit residual by x_i.

Because the autocorrelation coefficients decay slowly to zero and the partial correlation coefficients falls immediately to zero after lag 1, we are justified in fitting an AR(1) model to the series $\{x_t\}$. The maximum likelihood estimate for the autoregressive coefficient was found using the statistical computing package S-Plus to be 0.69, and the variance of the corresponding white noise series was estimated as 0.081.

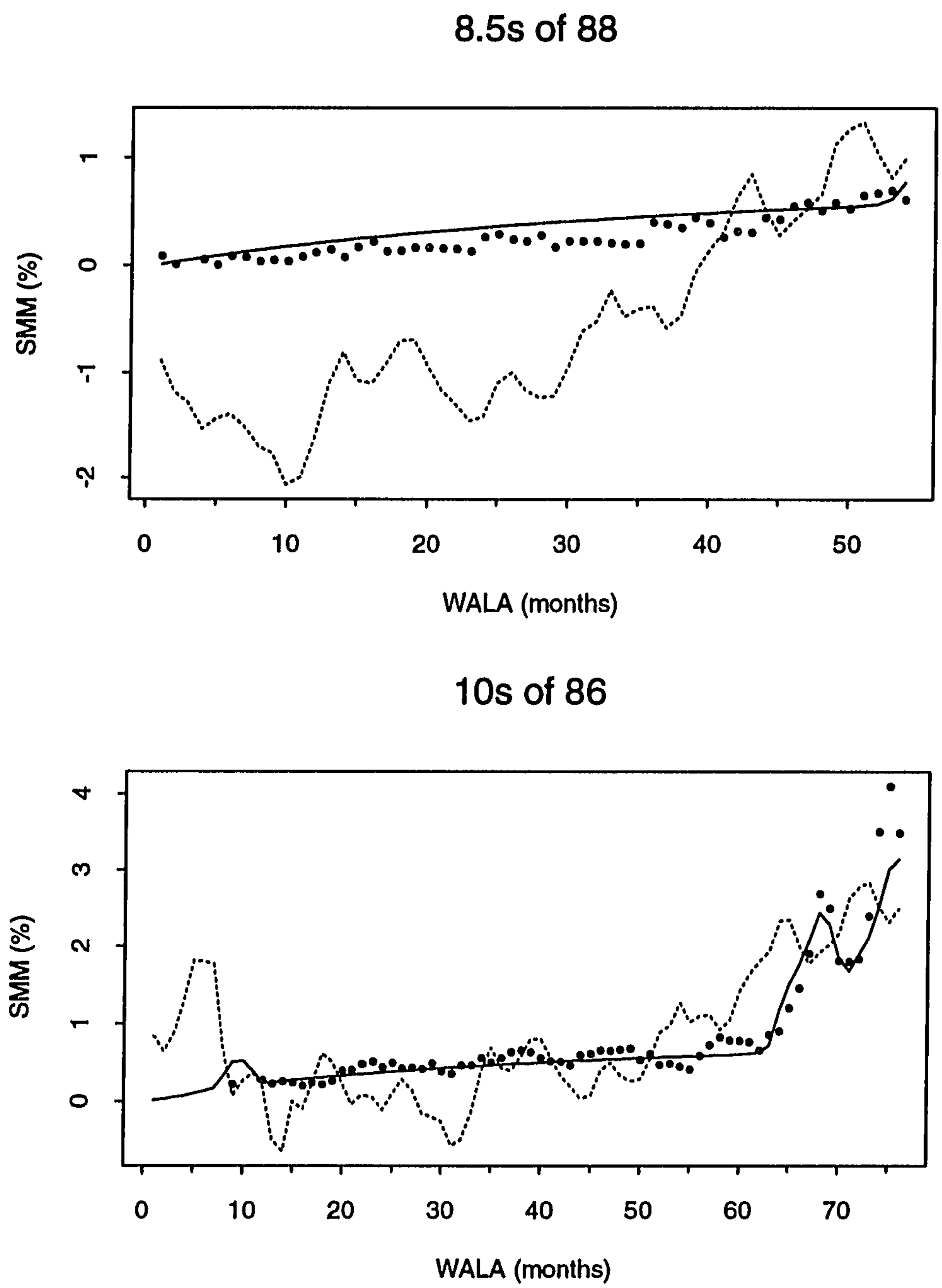

Figure 4.11 Prepayment variance does not approach zero as the number of mortgages grows. The 10s of 86 show greater prepayment variability despite original principal balance two orders of magnitude greater than the 8.5s of 88.

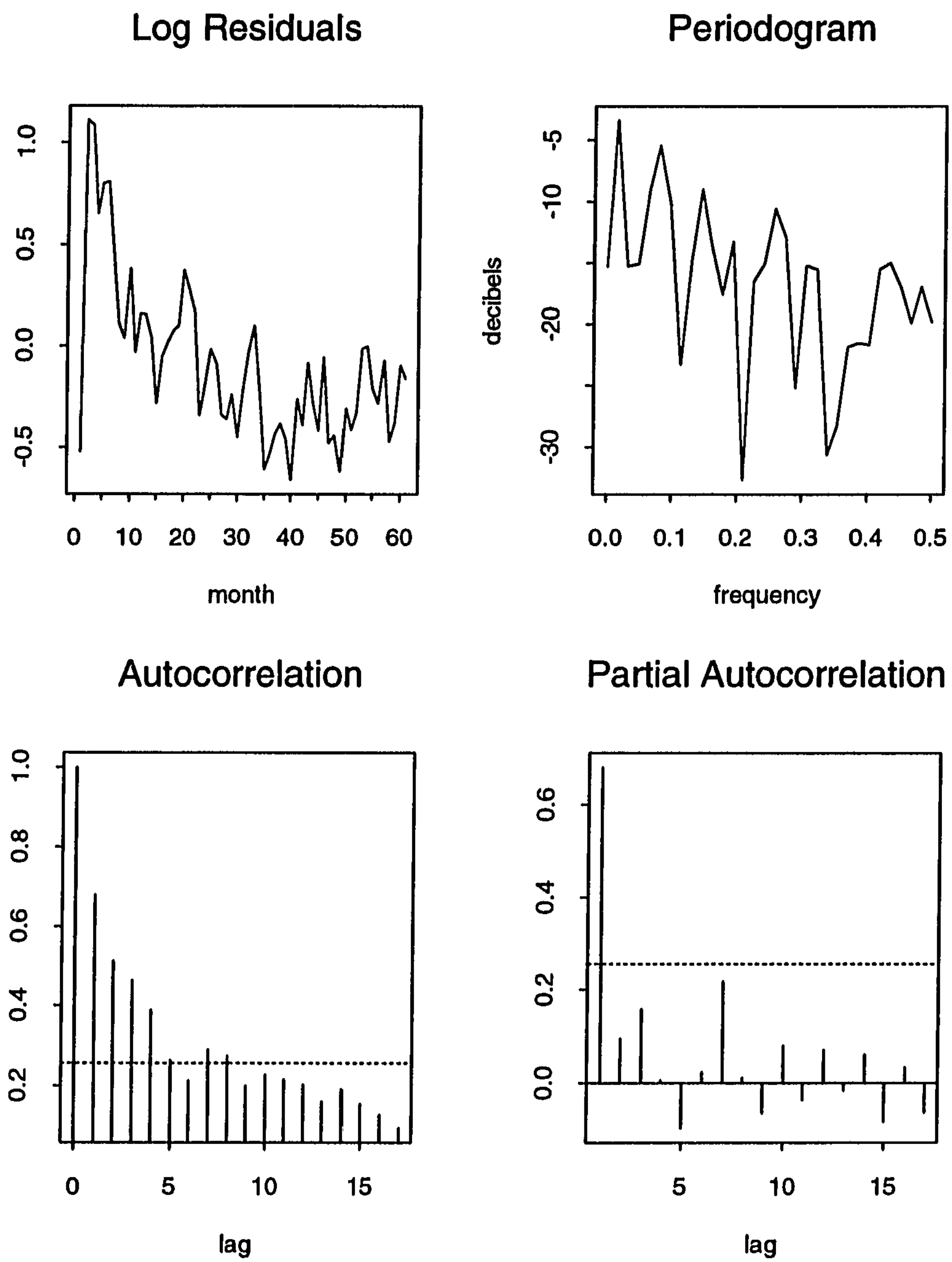

Figure 4.12 Time series plots for the log of the fit residuals for the 11.5s of 87.

The diagnostic plots in Figure 4.13 indicate that this model fits the time series very well. Thus x follows the process given by

$$x_i + .69x_{i-1} = \epsilon_i, \tag{4.6}$$

where

$$\epsilon_i \sim N(0, 0.0807).$$

Unfortunately, the relationship between $\log \eta_i$ and the fit residuals is unclear. If they are closely related then the AR model for $\{x_t\}$ might suffice to describe $\{\log \eta_t\}$. To see if this is true, we simulate the mean prepayment trace and 95% confidence bands for the 11.5s of 87 under the assumption that $\log \eta$ obeys (4.6). The result of the simulation is shown in the middle panel of Figure 4.14. The confidence bands have widened dramatically and now contain approximately 95% of the points. They appear neither excessively smooth nor rough, expanding and contracting in a fashion that seems more or less consistent with extreme moves in prepayments. Evidently there is a close enough relationship between the $\{x_t\}$ series and the $\{\log \eta_t\}$ series we have postulated to explain noise in prepayments and correlation in borrower prepayment decisions that both may be modeled by the same AR process.

To better understand how the parameter values in the noise process affect the prepayment confidence intervals, we perform two simulation studies, varying first the AR coefficient and then the error variance. First we fix the AR coefficient at 0.75 and allow the error standard deviation to range over the values 0.01, 0.044, and 0.09. The resulting mean trace and 95% confidence bands for the 11.5s of 87 are presented in Figure 4.14. As expected, the confidence bands widen with increasing σ. The change in width appears to be more or less by a multiplicative constant over the entire time line for each panel.

Figure 4.15 shows the effects of holding the error variance fixed at 0.044 and letting the AR coefficient range from 0.1 up to 0.9. We see in the figure that the variance of SMMs increases as the AR coefficient approaches unity, in much the same fashion as if the error variance of the AR process were increasing. Recall that for an AR(1) process, as the AR coefficient approaches 0 the process approaches white noise. As the AR coefficient approaches 1 the process approaches a random walk, which is characterized by arbitrarily long excursions away from the mean level due to the fact that the process variance is linearly related to time. Since large deviations from the mean prepayment level significantly alter the entire future path of prepayments due to the complicated interplay between the levels of the various states and observed SMMs,

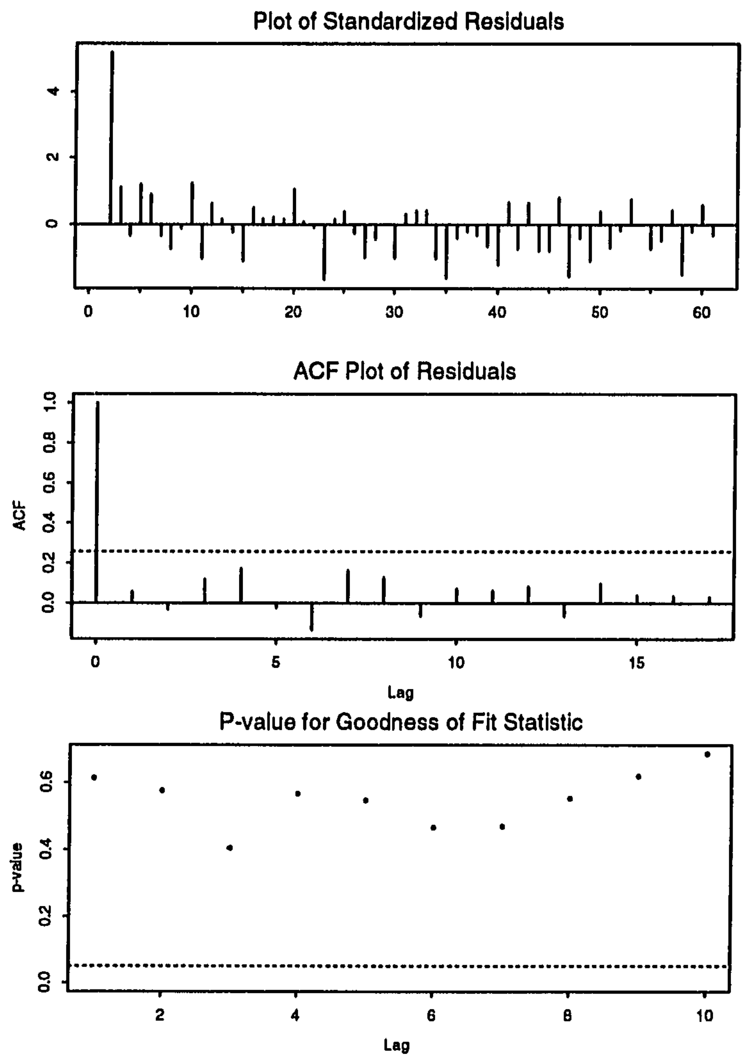

Figure 4.13 AR diagnostics for $\{x_t\}$ for the 11s of 87.

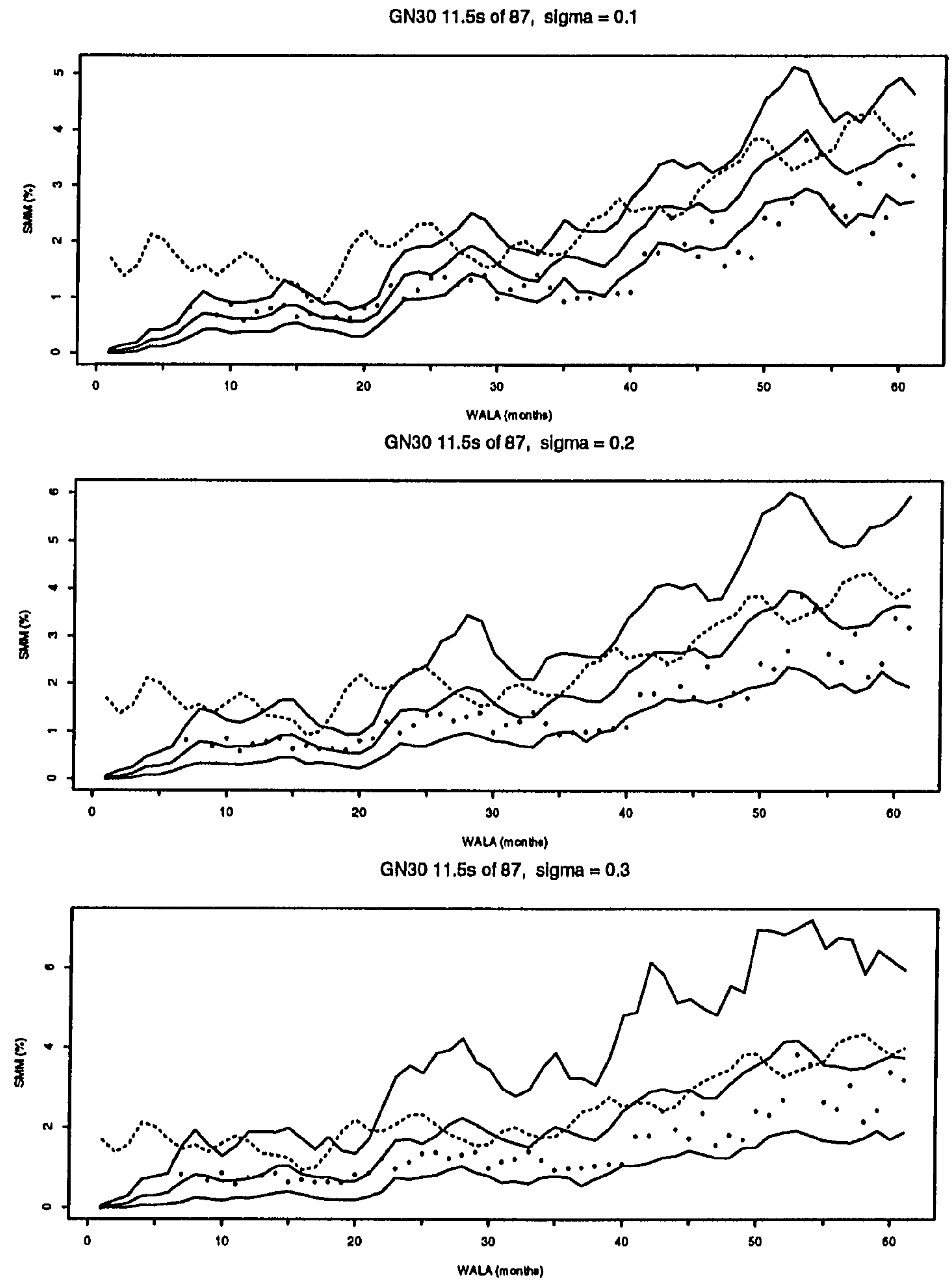

Figure 4.14 Five state model with AR(1) noise process for different error variances.

the increased prepayment variance corresponding to the AR process with coefficient nearest unity is probably due to the random walk-like behavior of this process. This is further evidence that the residual series $\{x_t\}$ is closely related to the log of the error process $\{\eta_t\}$

We would like to have a procedure for determining both the AR coefficient and the error variance for a single process that could be used to model the correlation structure over a larger set of collateral. Although one might argue that aside from interest rates, other unobserved factors affecting prepayments are intrinsic to particular collateral classes due to geographic concentration and exposure to associated local phenomena, like regional employment levels and housing activity. However, due to the aggregate nature of the data, this idea does not seem plausible. More likely is that all collateral classes are affected by a single error process representing factors such as national employment averages, housing activity, production, and so on due to geographic dispersion. Another problem with modeling the correlation structure on case by case basis is that for newly originated collateral there is no data on which to base estimates. Thus it seems appropriate to try to find a single noise process which explains the prepayment variability for all the data.

The approach employed here is to select a representative set of collateral, for example the test set, use it to determine AR parameter estimates, and posit that the estimates so obtained are valid for describing the noise process over the larger set of collateral. Supposing that an AR(1) process will suffice for this purpose, our strategy will be to fix one of the parameters in a more or less ad-hoc fashion, and then choose the other so that simulated 95% confidence intervals contain 95% of the data.

Accordingly, the residual series $\{x_t\}$ of the five state model fit was computed for each type of collateral in the test set according to (4.5). For each series $\{x_t\}$ obtained in this manner, the MLEs for an AR(1) process were computed. The results of these series of analyses are summarized in Table 4.3.

Although not shown here, plots corresponding to Figures 4.12 and 4.13 were examined for each of the models defined by Table 4.3. Without exception, the AR(1) model provides a good fit. Unfortunately, a single set of parameters does not seem to work for all classes. There is modest similarity between the AR coefficients, estimates for which range from 0.3858 for the 14s of 84 to 0.8370 for the 10.5s of 85. Most of the values seem to cluster near 0.70, however, which is close to the median 0.68. There is much less similarity between the values of the innovation variance, however.

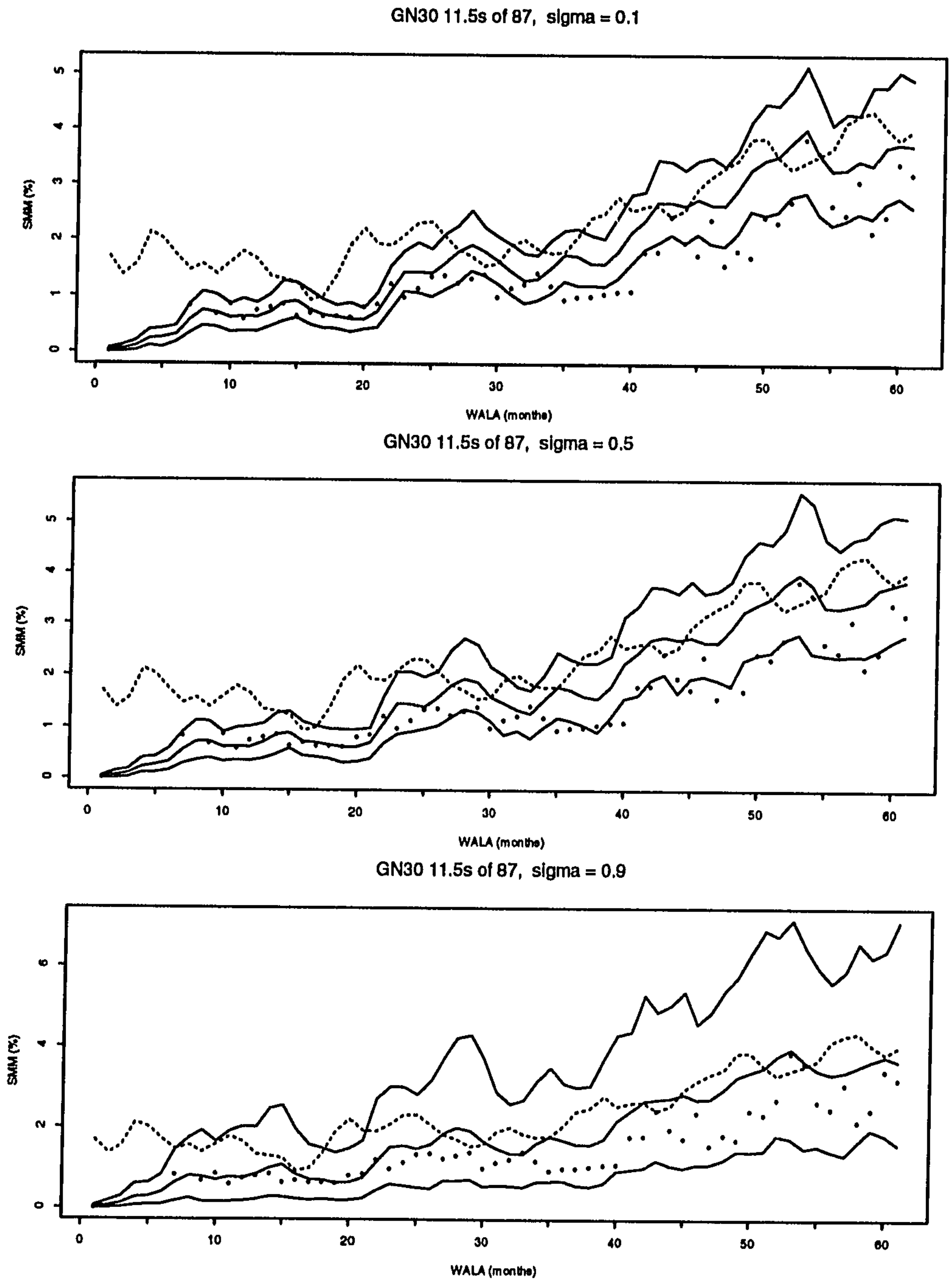

Figure 4.15 Five state model with AR(1) noise process for different AR coefficients.

Collateral	AR Coefficient	Coefficient Variance	Error Variance
8s of 89	0.6609	1.565e-2	1.6634
9s of 88	0.7912	0.719e-2	0.1012
11.5s of 87	0.6915	0.870e-2	0.0806
10.5s of 85	0.8370	0.348e-2	0.0555
13s of 85	0.4781	0.829e-2	0.0411
14s of 84	0.3858	0.860e-2	0.1026

Table 4.3 Maximum likelihood estimates for an AR(1) model of the log SMM residuals from the five state model.

Estimates vary by two orders of magnitude, from a low of 0.0411 for the 13s of 85 to a high of 1.6634 for the 8s of 89.

To resolve this problem in a sensible fashion, we select the median value for the AR coefficient, since the variation in this estimate is smaller. According to our strategy we determine the appropriate error variance σ^2 by holding the AR coefficient constant any varying σ until the 95% confidence bands of the prepayment simulations contain 95% of the actual SMMs. Table 4.4 shows the proportion of data contained in the 95% confidence intervals for various choices of σ, with the AR coefficient fixed at the median value 0.68.

Error Standard Deviation	Fraction in 95% CI
0.00	0.612
0.20	0.892
0.27	0.942
0.28	0.949
0.29	0.957
0.30	0.967
0.40	0.975

Table 4.4 Proportion of test set data falling within 95% confidence bands for the five state model with AR(1) noise. The AR coefficient is fixed at 0.68 as the error variances changes.

From the table it is evident that the appropriate value of σ is 0.28. It is interesting to note that is the same value as the maximum likelihood estimate of the error standard deviation for the $\{x_t\}$ series for the 11.5s of 87, to two significant figures.

Thus a suitable model of the noise process which can be used to establish correlation in the five state model is

$$\log \eta_i + .68 \log \eta_{i-1} = \epsilon_i,$$

where

$$\epsilon_i \sim N(0, 0.078).$$

Figures 4.16 and 4.17 show the 95% confidence bands obtained for the test set using this specification for noise in the stochastic five state model. Based on these figures we conclude that the five state model with AR noise adequately describes both mean prepayments and prepayment variability.

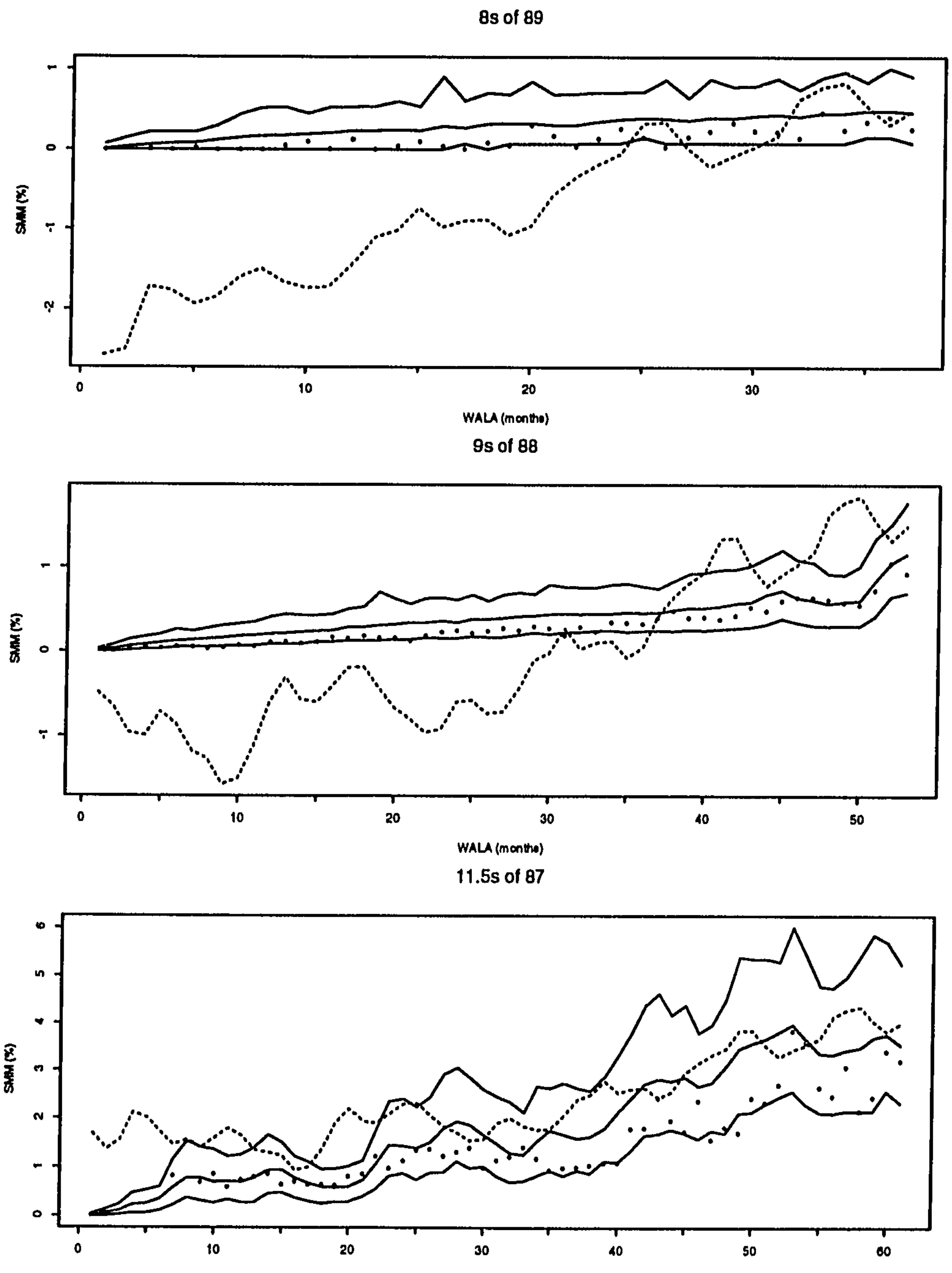

Figure 4.16 Five state model with AR(1) noise process.

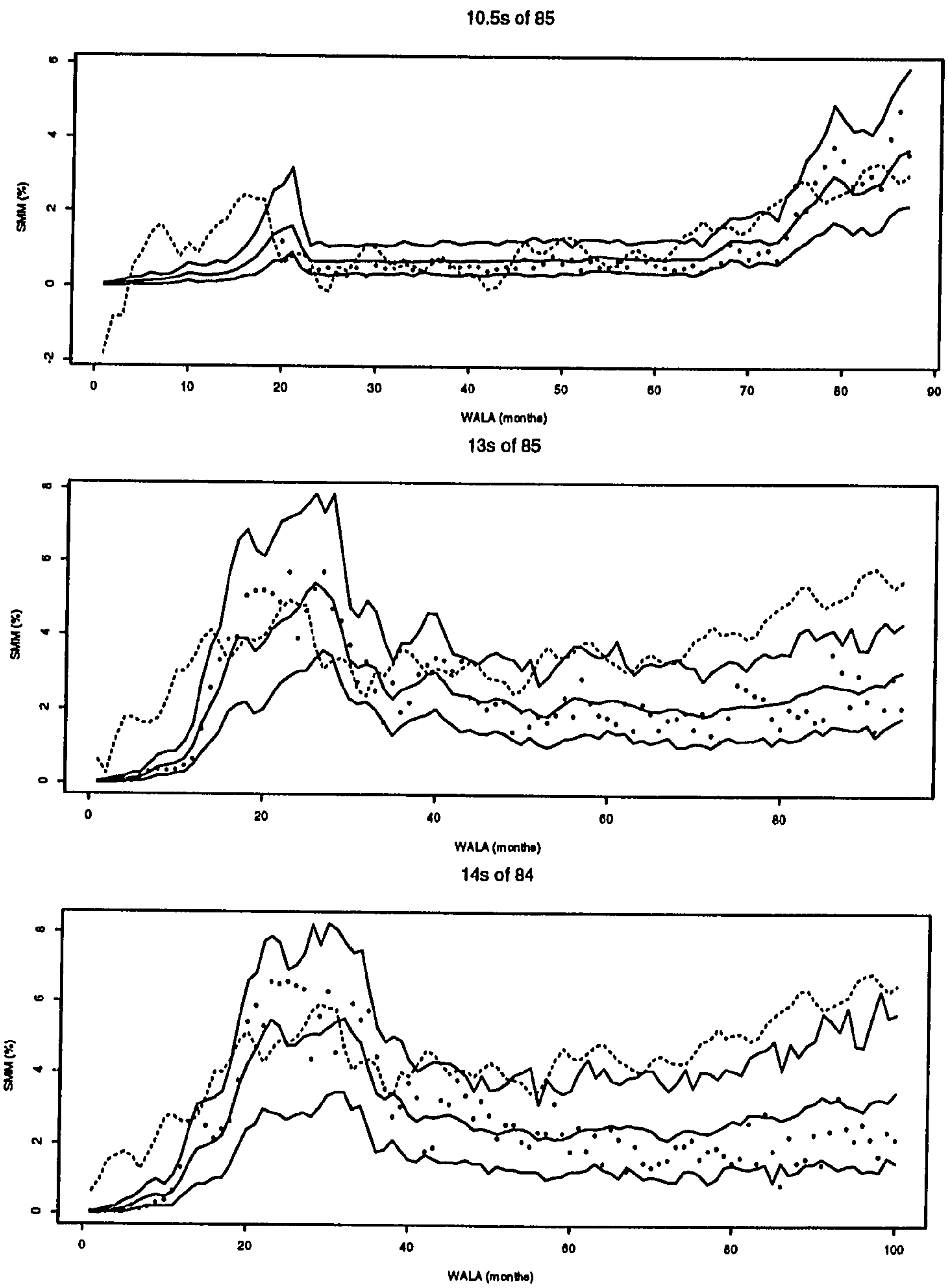

Figure 4.17 Five state model with AR(1) noise process.

Chapter 5

Model Evaluation

5.1 Prediction

In the previous chapter a number of fits of the model to historical data were presented. It is not too surprising that good agreement was generally obtained between the model values and actual prepayment rates, since the parameters of the model were chosen to optimize the fit using the likelihood as the optimality criterion. Of more interest would be an examination of the accuracy of the prepayment model on out-of-sample data.

Accordingly, the final twelve month period of observation was reserved from the analysis performed in the preceding chapter. The reserved observations spanned the period from January to December 1993. To illustrate the effectiveness of the five population model as a predictive tool, the fits from Chapter 4 were extended to the out-of-sample points. These are shown in Figures 5.1 and 5.2.

The predictive accuracy for the well-seasoned, high coupons is very good. All of the out-of-sample observations for both the 13s of 85 and 14s of 84 fall within the 95% confidence envelope. While the mean response of the 14s of 84 consistently exceeds the actual prepayment rate by about one percent, the mean for the 13s of 85 is very close to the actual prepayment data. The next most highly seasoned collateral, the 11.5s of 87, also yield very good predictive accuracy. For the most part actual prepayments fall within the confidence limits and close to the mean, this in spite of the fact that entering the prediction period the model mean was much higher than actual prepayments.

Accuracy for the less seasoned middle and low coupons was not as good, especially for the 9s of 88 and 10.5s of 85. Both collateral types demonstrate much higher prepayments than the model predicts. The 8s of 89 are a little better, probably due to the fact that while the prepayment response of the model is inaccurate, the relative coupon is very close to the refinancing threshold, so the number of borrowers

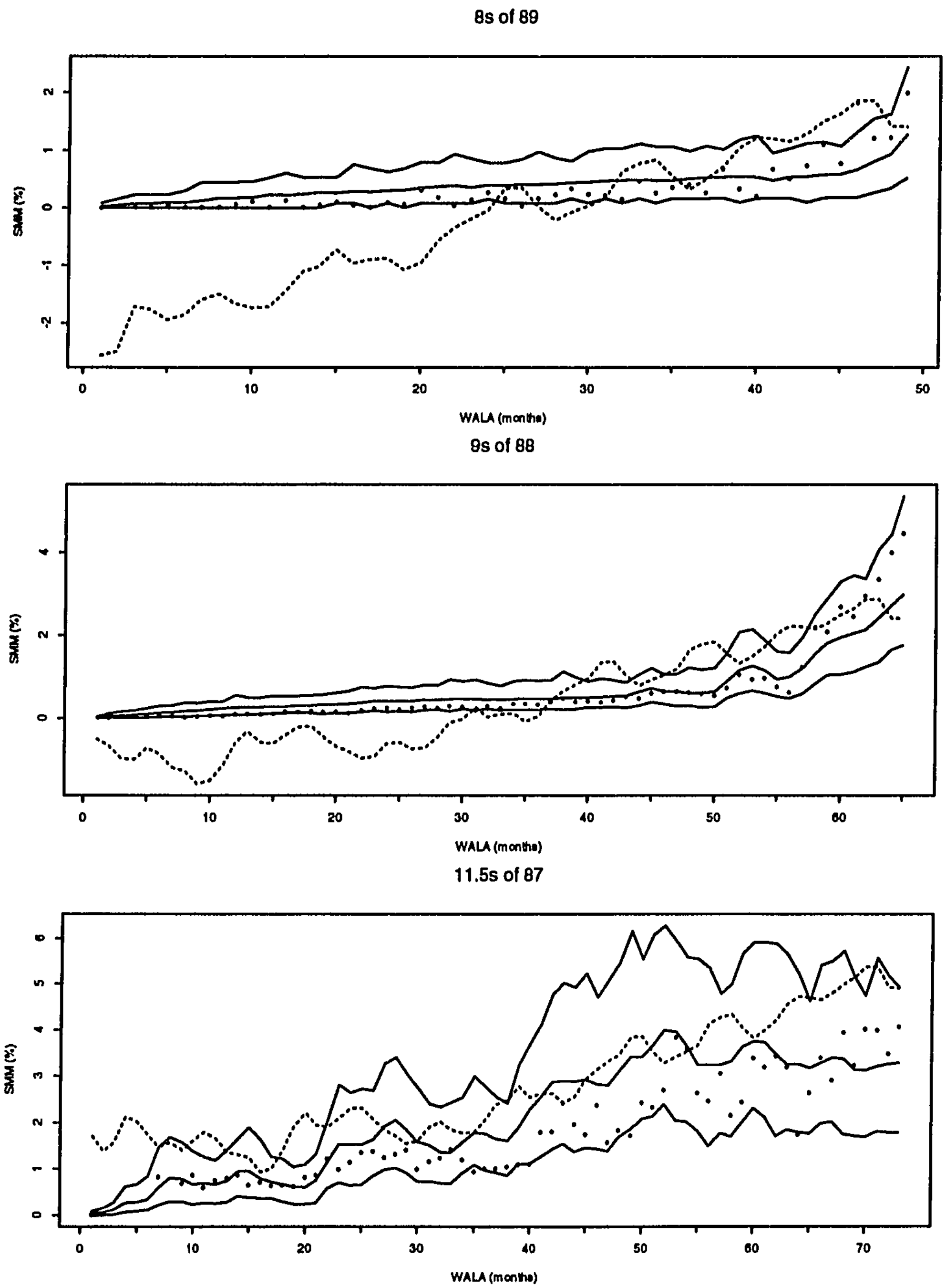

Figure 5.1 Five state model mean and 95% confidence bands with relative coupon. The last twelve points are out-of-sample observations.

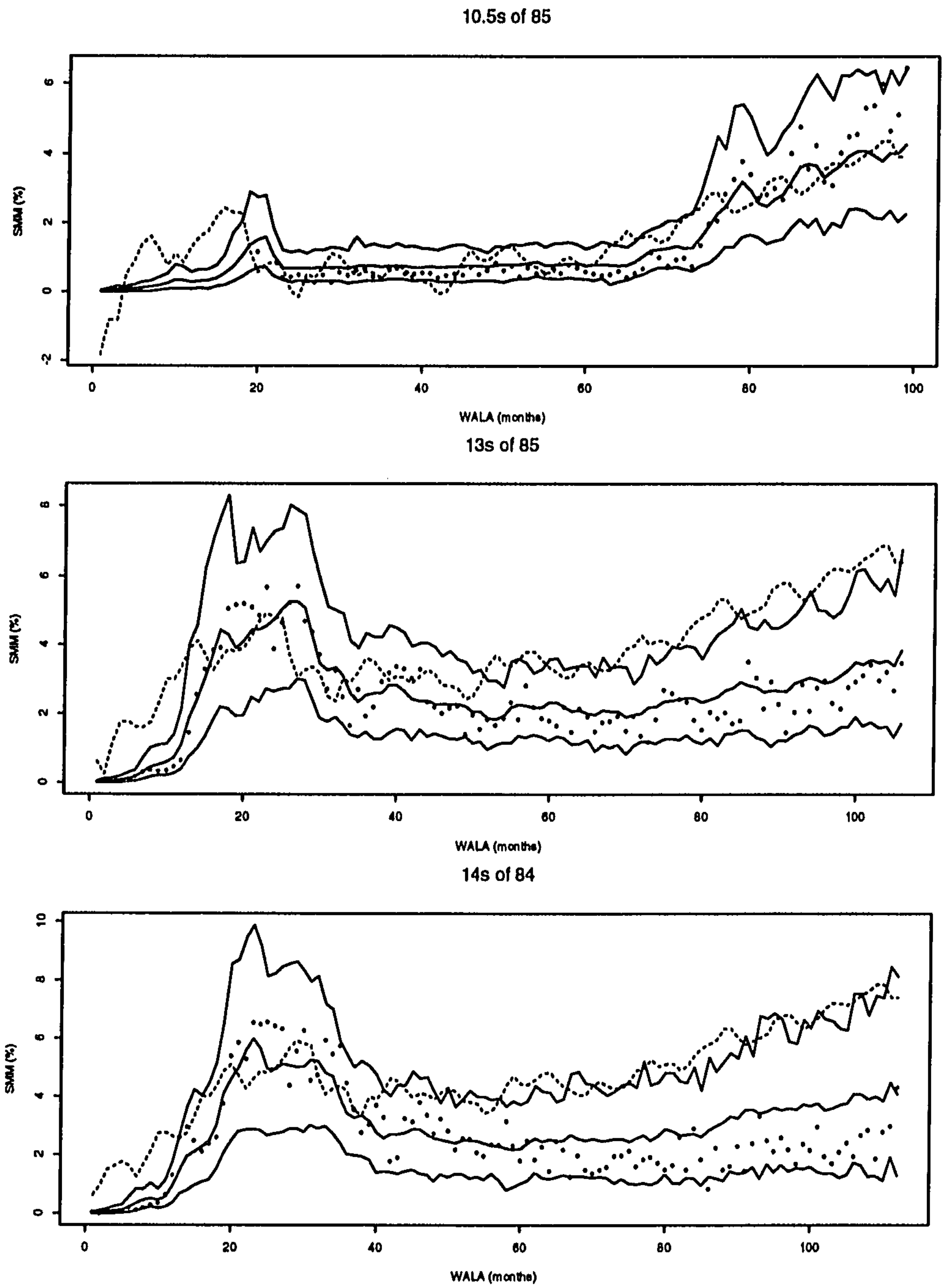

Figure 5.2 Five state model mean and 95% confidence bands with relative coupon. The last twelve points are out-of-sample observations.

for which prepayment is beneficial remains small. In other words, the absolute error is small because the absolute rates are small, whereas the relative error is larger.

Underestimation for the out-of-sample period is not surprising. Common consensus among many researchers and investors, such as Mattu (1995), Nabar (1995), Moore (1994), and Carron (1994), is that prepayment characteristics changed in the 1990s due to a fundamental change in the primary mortgage market. The change in the primary market was in turn brought about by falling mortgage rates, which consistently broke record lows for the modern era until hitting bottom in late 1993. In response to increased competition for the origination of mortgage loans in the still developing secondary market and the record low interest rates, originators initiated a much more aggressive campaign to market refinancing opportunities. Not only did fees and commissions fall, but brokers took the unprecedented step of establishing direct telephone and mail solicitation programs, which naturally resulted in an increase in prepayment response over historical average levels.

In order to examine the stability of the model parameters, new estimates based on the entire data set including 1993 are presented in Table 5.1, along with the original estimates based on the pre-1993 data only. Over the larger data sample the negative log likelihood function is larger as well owing to an increase in the number of multinomial bins. In order to facilitate a comparison, the likelihood for the pre-1993 estimates was recomputed using the entire data set.

Most of the parameter estimates display remarkable stability. All but one of the values in Table 5.1 differ across columns by less than six percent. The parameters for the slow prepayers are essentially the same, but the parameters for the fast prepayers change somewhat. There is a marked difference in the $\log \beta$ row, which changes from -0.34 to -0.78, corresponding to a change in the initial fast prepay population proportion from 58% to 67%. A modest change occurs in the seasoning interest rate sensitivity for the fast group as well The difference of 5.7% in the $\log \alpha_1$ row amounts to a change from $\alpha_1 = 4.3$ to 5.1% per month per point, a relative increase of 19%. Together these shifts allow the model to better fit the much higher prepayments characteristic of the 1993 data.

The fit for the test set using the re-estimated parameters and the five state model is shown in Figures 5.3 and 5.4. Despite the inclusion of the 1993 data, which demonstrates higher than normal behavior for the given levels of relative coupon, most of the fits agree well with the actual data. The only evidence of poor fit is in the middle period for the 11.5s of 87, in which the fit values exceed actual prepayments by a con-

Parameter	Pre 1993 Data	All Data
$\log \alpha_0$	-4.85	-4.90
$\log \alpha_1$	-3.15	-2.97
$\log \alpha_2$	-5.10	-5.06
$\log \beta$	-0.34	-.78
$\log \gamma_0$	-3.64	-3.66
$\log \gamma_1$	-3.12	-3.16
γ_2	2.88	2.72
γ_3	1.16	1.10
$\log \gamma_4$	-5.09	-5.08
$-\log l$	1.0586e7	1.0546e7

Table 5.1 Maximum likelihood estimates for the five state model over pre 1993 data and the entire dataset.

siderable amount. In fact, the fits for the other collateral types appear to improve. This is surprising, since the model employs static prepayment sensitivities, whereas true prepayment sensitivities have apparently experienced a significant shift.

5.2 Valuation

The fundamental innovation in the models presented here is their stochastic nature. For a regression based method, the predicted stream of prepayments is fixed for any particular path that the predictor variable follows so variability in the future value of any mortgage-backed security is due solely to the predictor. In contrast, the stochastic models presented here retain an element of uncertainty in the prepayment stream from a particular interest rate path, which adds to the uncertainty in the future value of the mortgage cash flows and could significantly affect the value of a derivative security. For example, a deep out of the money call on a mortgage would be very sensitive to an increase in prepayment volatility if a stochastic model were used in place of a deterministic one. Of course, not all derivatives would be significantly affected. One would expect PAC bonds to be relatively insensitive to the added variability the stochastic model introduces due to the stable payment schedule such bonds have even in the face of widely varying prepayments.

To illustrate these effects we shall value several mortgage-backed securities using the deterministic model and the stochastic model. It will be shown that sometimes

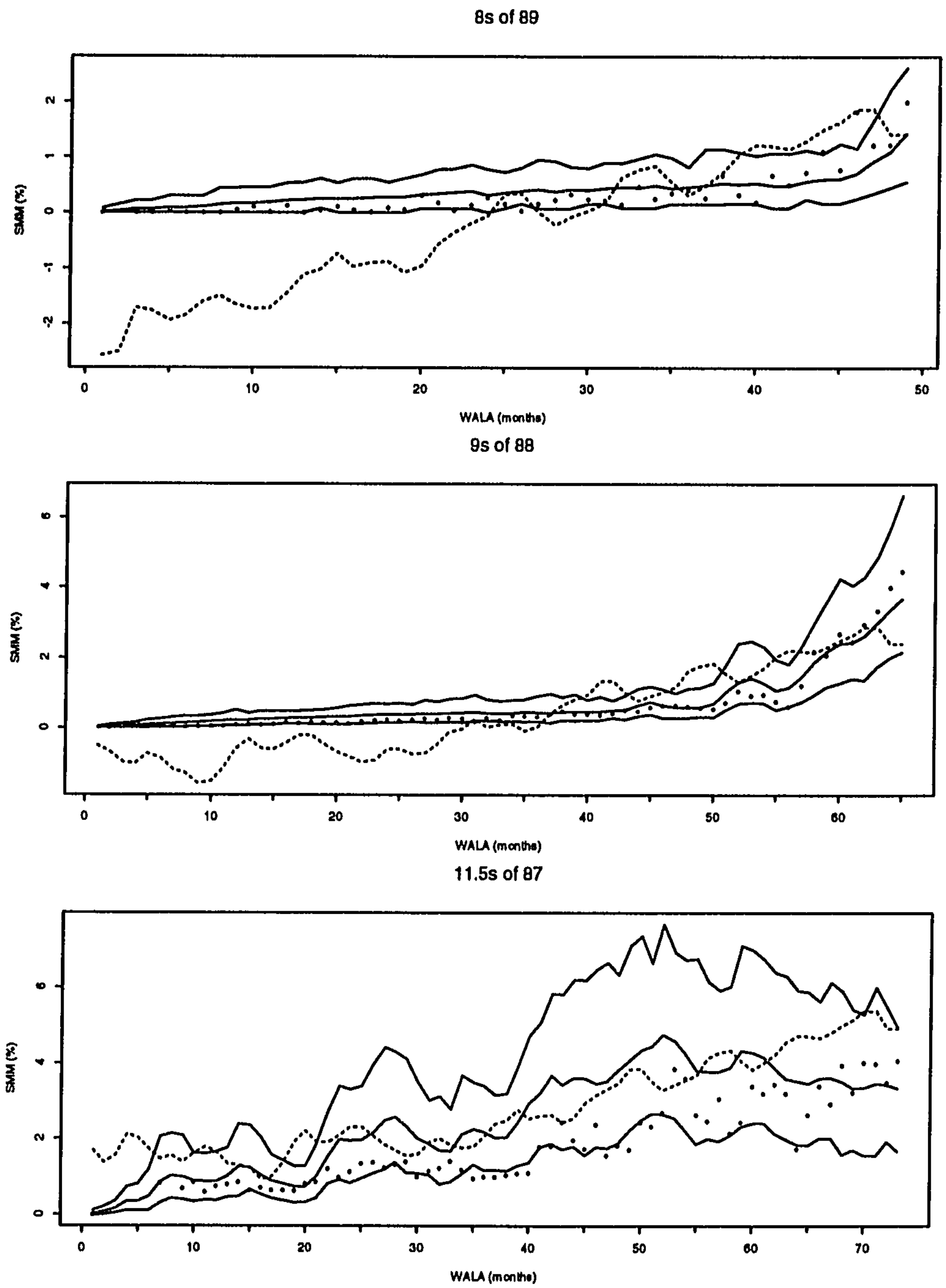

Figure 5.3 Five state model mean and 95% confidence bands with relative coupon. The fit covers the entire dataset.

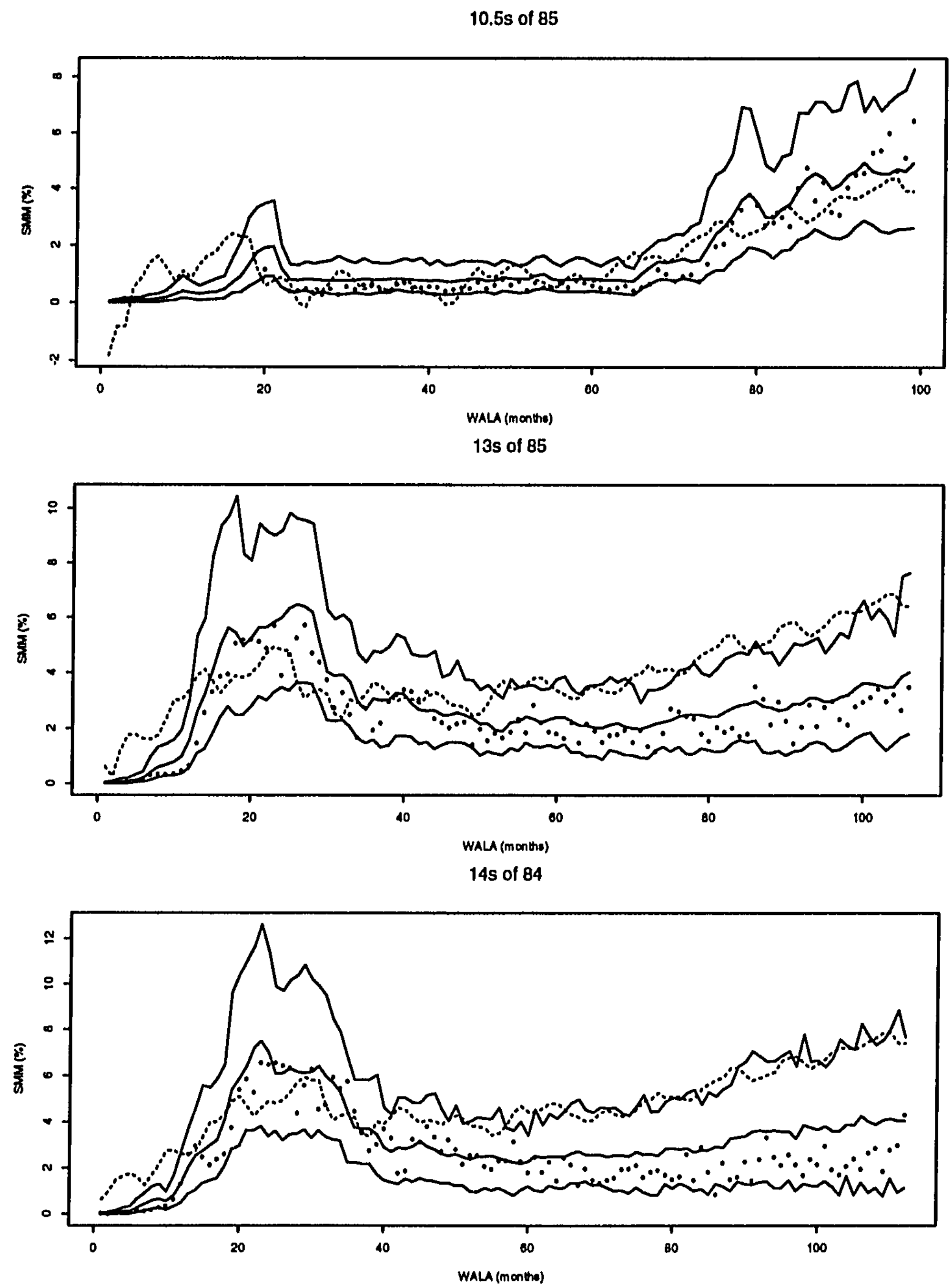

Figure 5.4 Five state model mean and 95% confidence bands with relative coupon. The fit covers the entire dataset.

the deterministic model significantly misprices a security owing to its underestimation of prepayment volatility. A single structure involving two primary classes of bonds will suffice for the illustration.

The illustration is based on a simple PAC structure with only two tranches, collateralized by a pool of newly originated loans with a par value of \$20 million. The first tranche is a PAC, the second a companion. The PAC receives all scheduled principal and interest payments from the pool until it is retired. In addition, while the companion remains alive, the PAC receives just those prepayments in excess of 500% PSA. If the companion is retired before the PAC, then all prepayments are assigned to the PAC. The companion bond receives all cash flows not distributed to the PAC. The PAC serves as collateral for two derivative securities, an interest-only (IO) and a principal-only (PO) strip. The IO receives all the interest payments to the PAC, while the PO receives all the principal.

We expect the values of the PAC and companion to be close to par for each model, since the term structure of interest rates should be nearly flat at 10%. In this case, the present value of an interest bearing principal obligation is par regardless of the timing of the principal payment. However, the present value of a principal obligation that does not bear interest would be affected by the timing. In this case the obligation is a pure discount bond whose present value decreases exponentially in the maturity date. Since the PAC is protected against prepayment up to 500% PSA until the paydown of the companion, the PO will only receive early cash flows if prepayments are extremely high, which is more likely under the stochastic model. Hence the value of the PO under the stochastic model should be greater. There would be a corresponding dollar for dollar decrease in the price of the IO since the sum of the IO and PO equals the PAC.

The stochastic valuation methodology outlined in Chapter 2 is used here. We shall assume that the appropriate one month discount factor for mortgage-backed cash flows is the Ginnie Mae current coupon r. Thus it is necessary to select a model with which to simulate r, which of course will also be used in the prepayment model.

There are a number of different processes for the riskless short-term interest rate r_s commonly found in the literature, the most common of which is due to Cox, Ingersoll, and Ross (1985). In the Cox, Ingersoll, and Ross model, r_s follows the process

$$dr_s = \kappa(\theta - r_s)dt + \sigma\sqrt{r_s}dz, \tag{5.1}$$

where t is time, dz a simple Wiener process, and κ, θ, and σ constants representing the mean reversion speed, the mean interest rate level, and the volatility of interest rates. We shall adapt this process to model the Ginnie Mae current coupon r.

Based on the work of Bunce, MacRae, and Szymanoski (1988) and Schwartz and Torous (1992), a suitable choice for the constants in (5.1) is given by $\kappa = 0.10$, $\theta = 0.065$, and $\sigma = 0.075$. We shall assume that the process followed by r is the same as for r_s, with the same value for the mean reversion speed and different values for the mean level and volatility. The mean and standard deviation of the actual monthly values of r from January 1982 to December 1993 are $\bar{r} = 0.010$ and $s = 0.026$. We set the mean rate level at $\bar{r}$, and set σ at the value which equates s with sample standard deviations of simulations of $\{r_t\}$. Doing so we determine that r follows the process

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz, \tag{5.2}$$

where $\kappa = 0.10$, $\theta = 0.10$, and $\sigma = .12$.

The valuation results for the deterministic and stochastic models are based on 10,000 simulations each and are summarized in Table 5.2. As expected, the present values of the PAC and companion are largely unchanged, differing by only 0.2% of par. It is interesting to note that for this structure, with a companion class relatively small in proportion to the class which is protected against prepayment variability, the PAC actually demonstrates greater variability in present value than the companion under the deterministic model. The reason for this is that since there are only two classes, the companion can be thought of as being a class which is protected against prepayments above 500% PSA. There is a complicated interplay between the simulated term structure of interest rates and the timing of cash flows which results in different classes having greater present value stability under the different prepayment models.

	Stochastic Model				Determistic Model			
Estimate	PAC	IO	PO	Companion	PAC	IO	PO	Companion
$\bar{PV}$	9.96	3.32	6.64	10.00	9.98	5.64	4.34	9.98
$\hat{\sigma}_{PV}$	0.37	0.88	1.14	0.48	0.49	0.81	1.15	0.37
$\hat{\sigma}_{\bar{PV}}$	0.004	0.009	0.012	0.005	0.005	0.008	0.012	0.004

Table 5.2 Valuation results for a two tranche, stripped PAC. Units are \$1 million.

The most significant observation is that both derivative securities are substantially affected by the increase in prepayment volatility associated with the stochastic model. The value of the PO increases from \$4.34 million under the deterministic model to \$6.64 million under the stochastic model, a relative difference of 153%. The standard error of each value is approximately \$0.01 million. Similarly, the price of the IO falls from \$5.64 million to \$3.32 million, a decrease of 42%. Thus we conclude that the prepayment variability of the stochastic model can have an important contribution on the value of a mortgage-backed derivative.

5.3 Comparison to a Regression Model

Due to the applied nature of the prepayment modeling problem, it is important that a new model be not only innovative, but useful as well. Both the deterministic and stochastic models presented so far involve new strategies for examining prepayments. Furthermore, it has been demonstrated that the five state model produces both good fit and prediction accuracy. It remains to be shown how the dynamicism of these models represents an improvement over traditional methods. Therefore the model of Kang and Zenios (1992) was selected in order to serve as a standard of comparison, since their model is probably the best in the literature which has been described in enough detail to implement.

The Kang and Zenios model is explained in detail in Chapter 2. It can be summarized as a multiplicative regression model, nonlinear in the parameters even under a log transformation. Although the full model involves four factors, seasonality, refinancing, seasoning, and burnout, seasonality will be omitted here to facilitate comparison with the five state model. For the same reason month can be eliminated from the set of explanatory variables. The reduced model, with refinancing, seasoning, and burnout, will be fit to the pre-1993 data, using age and the relative coupon ratio C/R as predictors.

Prepayments for the pre-1993 data are plotted against C/R in Figure 5.5. The first part of the regression is to model the refinancing effect, which is done in an iterative fashion. Since the age effect has not yet been removed, a first approximation to the refinancing function, $\hat{\rho}_0(C/R)$, is found by regressing prepayments against the relative coupon ratio C/R for all observations with WALAs between 48 and 180

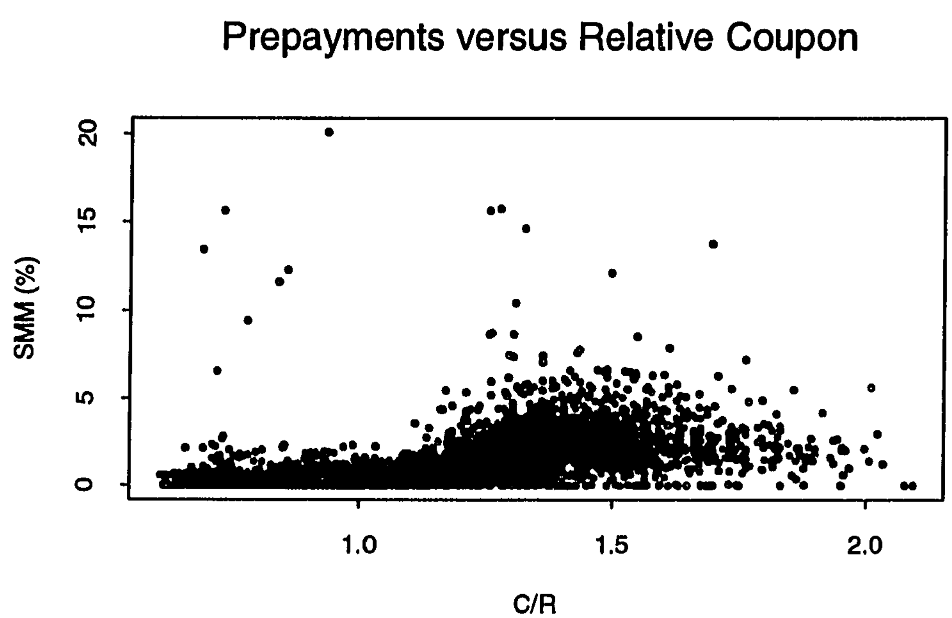

Figure 5.5 Prepayments for the pre-1993 data versus the relative coupon ratio.

months. The refinancing residuals correspond to the seasoning or aging effect

$$A_t = \frac{SMM(C/R,t)}{\hat{\rho}_0(C/R)}.$$

Figure 5.6 shows the initial fit and smooth of the age effect for both premium and discount bonds.

The age effect for the discount bonds does not reach a peak over the observation set, so it is impossible to exactly follow the algorithm in this case. According to Kang and Zenios, discount bonds do not season fully until after 100 months, but there are no observations for discount bonds older than 80 months. Thus we are faced with no alternative but to go ahead and use the preliminary discount refinancing function $\hat{\rho}_d(C/R)$ in our final estimate. There is probably little harm in using the discount age function obtained in this manner for in-sample fitting since $\hat{\rho}_d(C/R)$ attains a maximum value only slightly above 1.0. For extrapolation, however, one would expect the age function to produce estimates well below the true value.

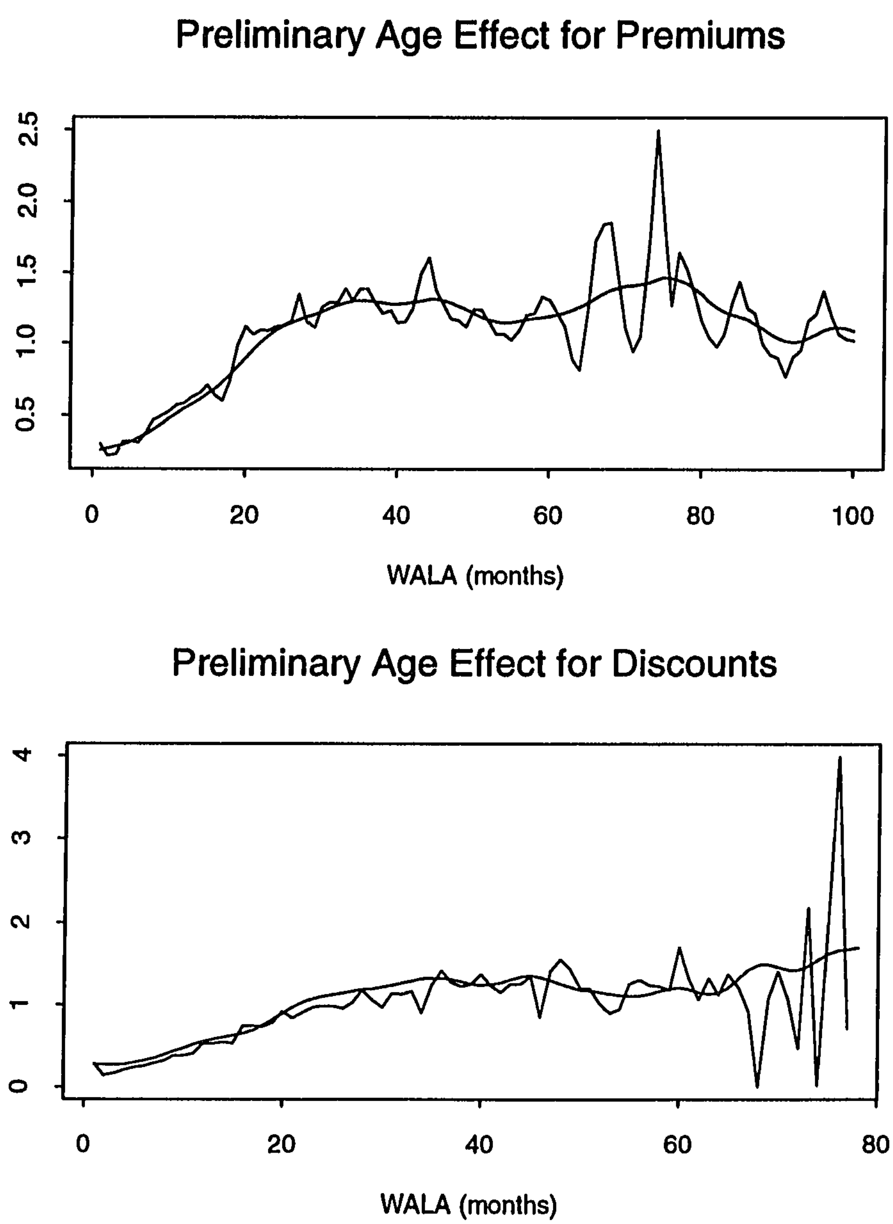

Figure 5.6 Fit and smooth fit of the age effect for premium and discount bonds based on the preliminary estimate of refinancing.

For the premium bonds, the age effect exceeds 1.0 in magnitude for ages 21 on. Kang and Zenios report that the premium age function exceeds unity only in the range from 24 to 48 months, which corresponds to the period when the premium bonds have fully seasoned but not yet begun to burn out. According to this reasoning, the premium bonds in our data set never burn out, which is certainly untrue. There is a slight dip in the smoothed fit at age 54, which we shall use as suitable justification for taking the interval from 21 to 54 months as the appropriate period for recomputing the premium refinancing effect $\hat{\rho}_p(C/R)$. The final estimate is calculated as

$$\hat{\rho}_p(C/R) = \begin{cases} \hat{\rho}_d(C/R) & \text{if} \quad C/R < 1, \\ \hat{\rho}_p(C/R) & \text{if} \quad C/R \geq 1. \end{cases}$$

which is plotted in Figure 5.7.

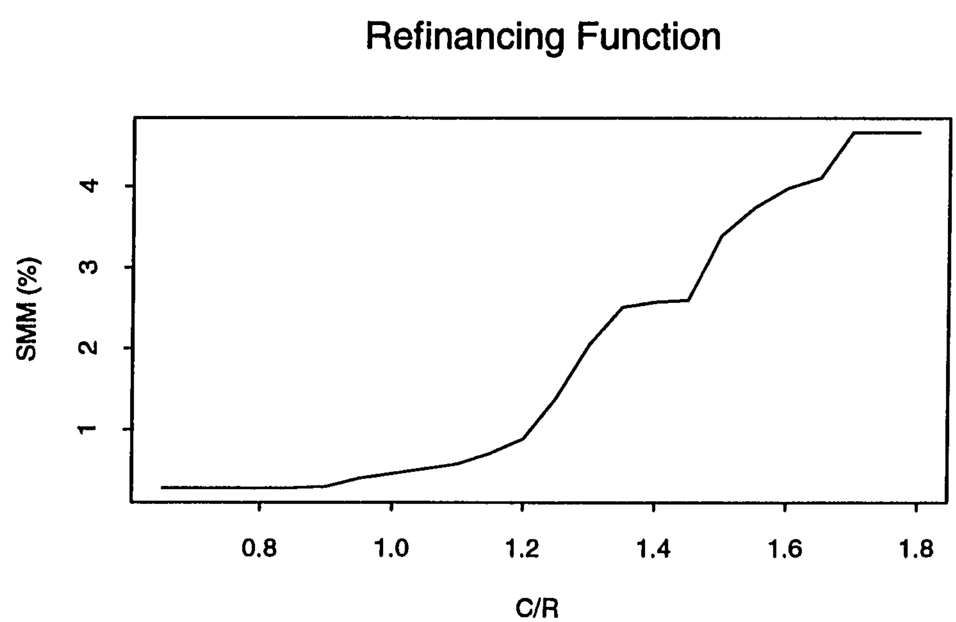

Figure 5.7 Final estimate of the refinancing function.

Kang and Zenios obtain final seasoning estimates are constructed by dividing the refinancing residuals into five classes corresponding to high premium, premium, par, discount, and deep discount observations, and determining a separate seasoning function for each class. We shall divide the observations into five similar classes, but because our data tend to higher relative coupons, we shall instead define classes as indicated in Table 5.3.

Class	C/R Range	% of Data
Discount	below 0.95	15
Par	0.95 — 1.05	28
Low Premium	1.05 — 1.15	25
Mid Premium	1.15 — 1.25	17
High Premium	above 1.25	15

Table 5.3 Data classes used for the determination of seasoning and burnout.

Due to the prevailing downward trend in interest rates over the observation period, most of the data fall in the premium category. Thus we have provided more subdivisions for the premium bonds. This is a benefit to the regression model since more parameters are used to describe the complicated behavior associated with burnout in the high coupon bonds.

The age function for selected classes is shown in Figure 5.8, and are equivalent to the product of the seasoning and burnout factors. Thus the age function is constrained to be piecewise linear and increasing until full seasoning has occurred, whereas it becomes piecewise decaying exponential after burnout begins. The only class which exhibits an aging function that attains a maximum value of unity followed by burnout corresponds to the high premium bonds, which season fully in about 30 months and begin to burnout after about 85 months. The par bonds appear to season fully near month 75, but there is no evidence of burnout by the end of the observation period, so they may in fact continue to season even past month 90. The discount bonds clearly continue to season over the entire period.

Both the discount and par age functions exceed unity, indicating that there is some confounding of aging and refinancing. In order that the fit accuracy not suffer because of this confounding, the fitted values were not constrained to be less that unity as in the Kang and Zenios prescription.

The aging for a particular bond depends on the entire path of interest rates up to the present rather than just on the most recent value. However, the age estimates just obtained are based on the assumption that a bond in one class remained in that class at every moment through the course of time, which is not a likely scenario for most

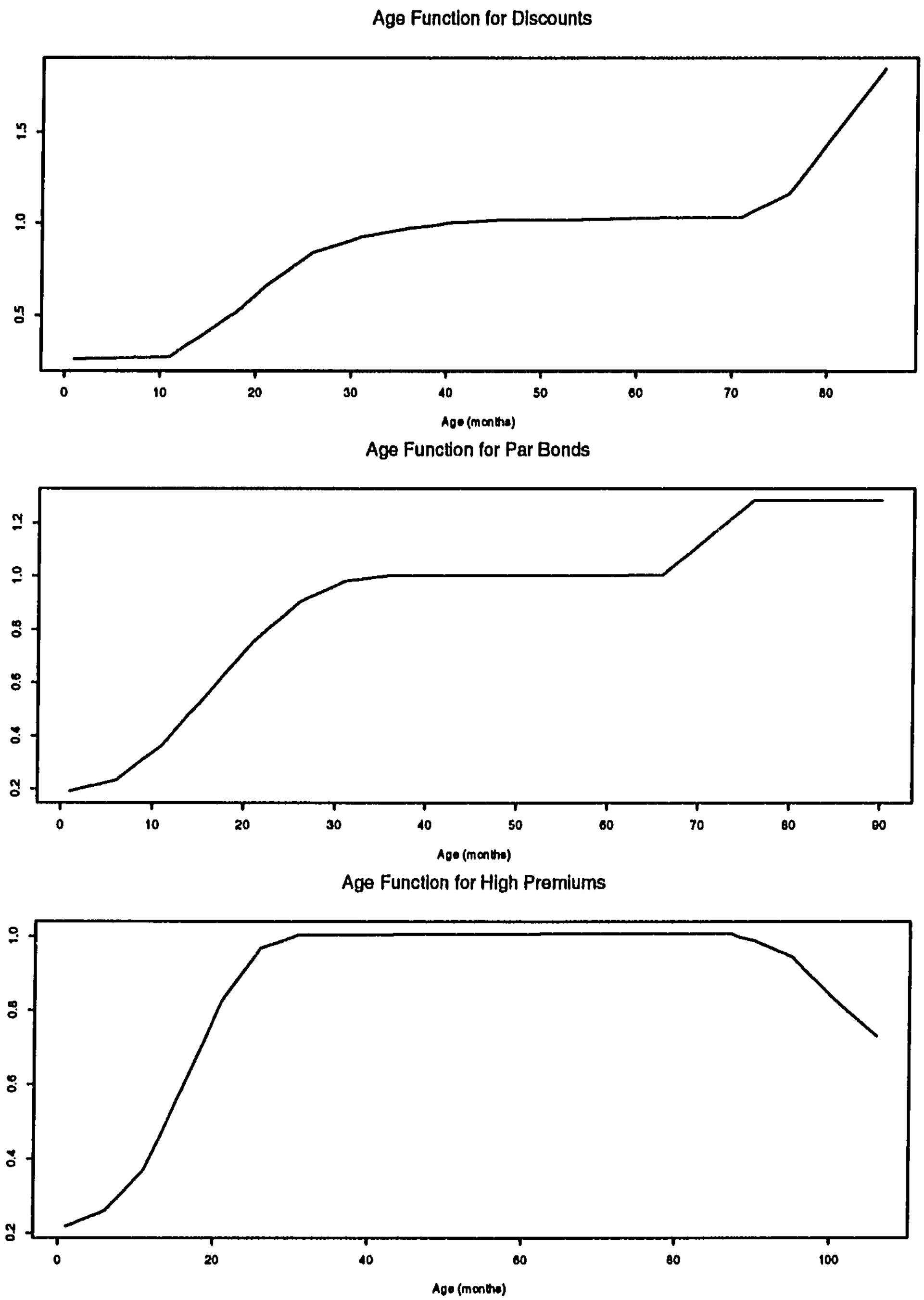

Figure 5.8 Age functions for selected bond classes.

bonds. Let $\{C/R\}_t$ denote the entire history of C/R up to time t for a particular bond and $\hat{a}(t, C/R|\{C/R\}_t = C/R)$ denote the age estimate based on the static C/R class assumption. Then Kang and Zenios propose

$$\hat{a}(t, C/R, \{C/R\}_t) = \sum_{\tau=1}^{t} \hat{a}(t, C/R|\{C/R\}_t = C/R)$$

as the final estimate of aging, which depends on the entire history of relative coupons.

Thus the regression function is given by

$$S\hat{M}M(t, C/R, \{C/R\}_t) = \hat{\rho}(C/R)\hat{a}(t, C/R, \{C/R\}_t).$$

Figures 5.9 and 5.10 compare the regression line with the deterministic five state model fit over the test set, while Table 5.4 compares the weighted total squared error, with pool balances as weights. For the low coupons the fits are comparable: for the 8s of 89, the regression model is slightly better, while for the 11.5s of 87 the dynamic model is slightly better. For the 9s of 88, the regression model is a little closer to the actual data for most of the observation period, but the mistimed peak at the end results in a total squared error almost twice as large as that of the dynamic model. The peak is mistimed because the regression model as implemented by Kang and Zenios fails to account for the lagged effect mortgage rate fluctuations have on prepayments. This is a minor deficiency which could be easily remedied. If it were, it seems the regression model would have a slight advantage for the low coupons. It is important to note that the difference in predicted values for the lowest coupons, the 8s and the 9s, are about .1 to .2%, which is a slight discrepancy indeed.

Collateral	Regression Model	5 State Model
8s of 89	39	117
9s of 88	1,152	669
11.5s of 87	6,790	6,222
10.5s of 85	7,721	1,744
13s of 85	132,785	11,525
14s of 84	79,098	8,264

Table 5.4 Weighted sum squared error for the 5 state model and regression fits to the test set data. Units are $ million - percent.

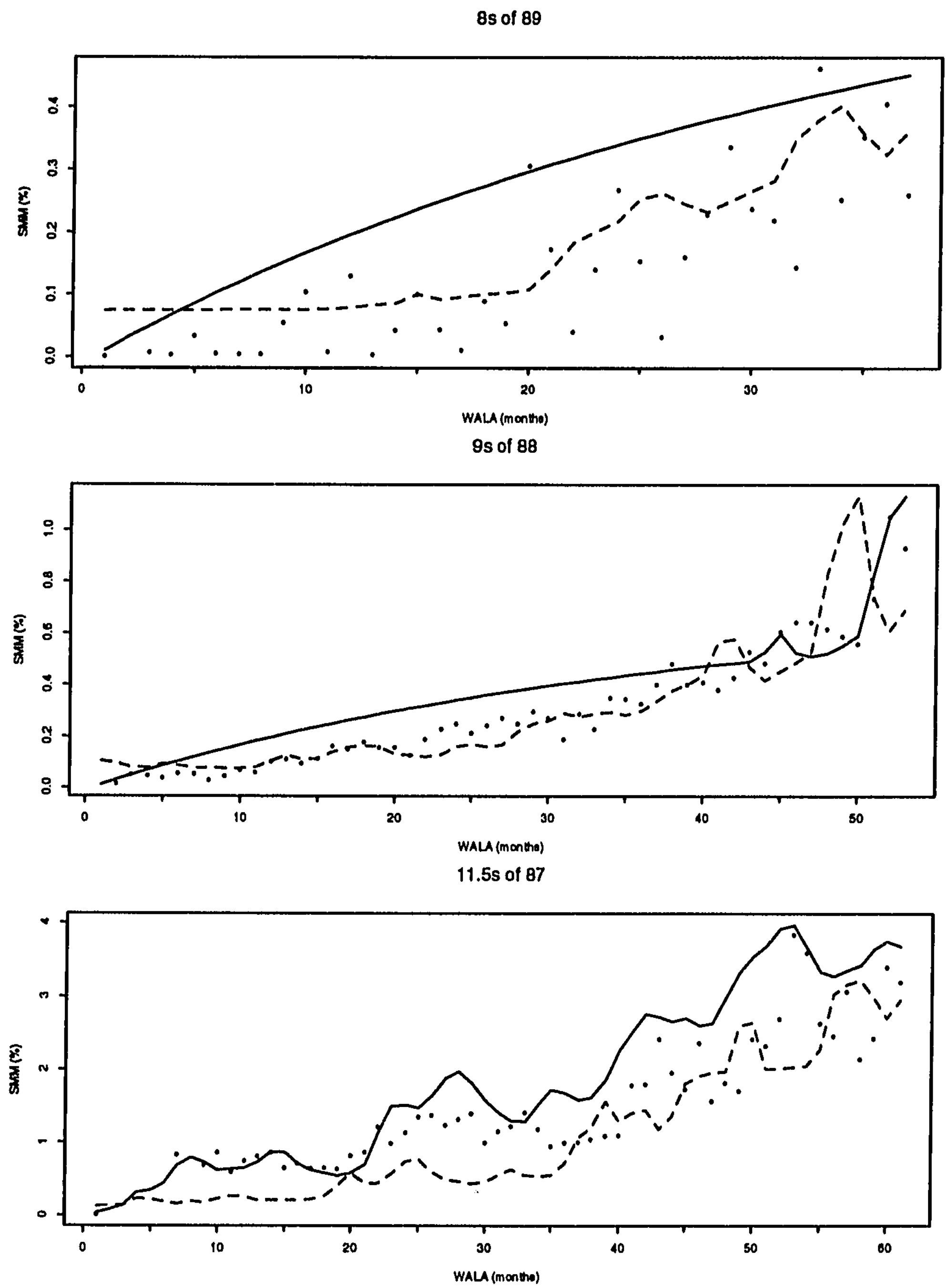

Figure 5.9 Comparison of the five state model (solid line) and regression model (dotted line).

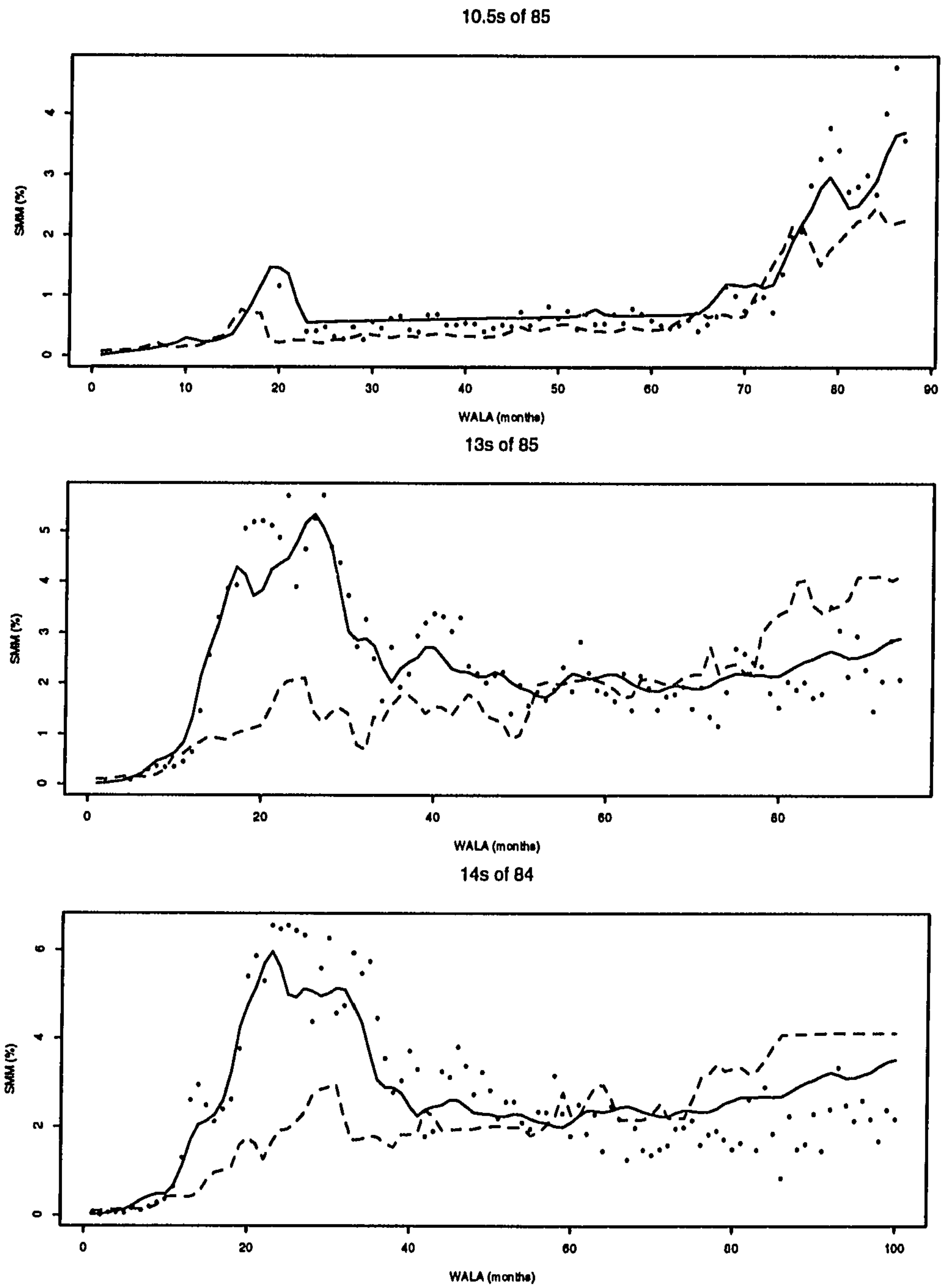

Figure 5.10 Comparison of the five state model (solid line) and regression model (dotted line).

The higher coupons reveal a huge difference in fit quality, and from Table 5.4 we see that the total squared error for the two models differs by an order of magnitude. For the 13s of 85, the regression total squared error is 11.5 times larger, and for the 14s of 84, 9.6 times as great. Lack of a delay term in the regression model does not account for this difference. In each panel of Figure 5.10 the regression model underestimates true SMMs by about 4% near month 25, and overestimates by almost 2% near month 90. Yet it under-fits the later prepayments for the 10.5s of 85 by 2%. In contrast, the dynamic model only underestimates by between 1 and 2% near month 25, and overestimates by at most 1% near month 90 for the two highest coupon classes, and underestimates later prepayments for the 10.5s of 85 by at most 1%.

Over the entire data set, the weighted total squared error of the regression model was 4.72e12 dollar-percent compared to only 1.65e12 dollar-percent for the five state model, a relative difference of about 300%. Appendix A compares the five state model and regression fits for every type of collateral.

5.4 Summary

Clearly the deterministic five state model outperforms the regression model, which fails to explain either burnout or aging in active refinancing periods. There is no way it can reconcile such high early prepayments and low later prepayments for a high coupon bond issued in one year with one issued in a later year, despite the "path dependent" way in which the age function was calculated. This is because the age function describes the burnout process with a static function of time, and so fails to account for population dynamics. Averaging is no substitute for a truely dynamic model.

The principal problem with the averaging scheme in the computation of the age function is that it lacks any sort of causal justification, unlike the specification of the refinancing function. It is intuitively appealing to suppose that there is an increasing relation between relative coupon and prepayments, since people can save money if they refinance or purchase a new home at lower interest rates, and the further rates fall, the more money can be saved. A piecewise linear function is a reasonable choice to model such a relationship. However, there is no such natural justification for supposing that there is a relation between prepayments and time that depends on the average value of various other increasing functions of time. It seems that such ad hoc descriptive measures are doomed to failure since they do not relate causes to effects.

We do not presume to suggest that a regression model cannot be made to describe such a system, but it seems apparent that to achieve a similar quality of description would require great increases in complexity. A regression model with a different static age function for each type of collateral would presumably provide an excellent fit, but would also be useless for prediction. One researcher concludes that "a new generation of prepayment models must be built to allow for constantly changing and unprecedented market conditions,"[16] but we instead suggest that the mortgage market is a dynamic system more appropriately and better described by a dynamic model.

We have also examined the particular importance of the stochastic model for valuation. The expected value of a very simple derivative security was shown to shift dramatically if the random component of prepayments was ignored in simulations. In addition to its impact on pricing, the stochastic variation is important for accurately assessing risk. For all investors there is a tradeoff between risk and expected return. Whereas a pension manager might prefer investments with smaller expected return and low risk, the manager of a hedge fund would require higher expected returns and the commensurate increase in risk. Any deterministic function, whether it result from a regression or dynamic model, eliminates the uncertainty in prepayments and hence is not suitable for accurately assessing risk.

In some applications, risk information may be entirely lacking if a deterministic model is employed. In a scenario analysis where a particular mortgage rate path is assumed and a mortgage-backed derivative security price is calculated conditional on the assumed path, a deterministic model will report a value with no variance. The stochastic model will however yield useful information on how the value of the security might vary even if the investor's view on the future path of interest rates is realized exactly.

It is unclear whether the five state model can be improved significantly by analyzing other possible explanatory variables or by using more complicated axiomatic descriptions, as in the alternative four state model described in Chapter 3. One possible additional explanatory variable is the month of the year. Since it is well known that homeowners are less mobile during the winter and early spring, with a resulting

[16] Choi (1994), 57.

decline in prepayments, perhaps a month indicator could be used to explain seasonal variations in the data.

Of more immediate concern, however, is probably a more thorough analysis of ways of introducing correlation and variability into the model. It seems natural that prepayments respond to relative coupon modified by noise, but the noise model selection procedure clearly needs refinement. Alternatively, perhaps the prepayment response parameters are random variables which vary from borrower to borrower. These possibilities merit further investigation.

Other avenues for exploration include extending the five state model to explain prepayments on generic collateral issued by Fannie Mae or Freddie Mac. This would probably require few if any modifications to the axioms, although certainly the parameters would need to be re-estimated, as both Fannie Mae and Freddie Mac loans demonstrate greater refinancing propensity than those of Ginnie Mae. Based on the addition of new variables or axioms involving more detailed local geographic, economic, or political factors, we could also extend the model to particular mortgage pools. It might also be interesting as well as useful to apply the axiomatic modeling approach to adjustable-rate mortgages.

In conclusion we note that the stochastic models and their deterministic counterparts analyzed here are very simple, involving only two variables and a handful of parameters, and yet they accurately describe the complicated prepayment patterns exhibited in over one hundred different classes of generic Ginnie Mae collateral. Path dependent features such as burnout and aging can be explained easily in terms of shifting population proportions, without having to rely on artificial, ad hoc features. Perhaps most important, stochastic models retain prepayment variability, which provides important information affecting not only a mortgage-backed derivative security's risk, but also its present value.

Appendix A

Figures A.1 through A.19 compare the five state model and regression model fit over every different collateral type in the data set. The regression line fits slightly better over the early months for the discount collateral, when prepayments are small in magnitude. The five state model is clearly better in refinancing environments, as it predicts both early prepayment peaks and subsequent burnout for the high coupons and late prepayment peaks for the medium coupons.

The weighted total squared error of the regression model is 4.72e12 dollar-percent compared to only 1.65e12 dollar-percent for the five state model, a relative difference of almost 300%.

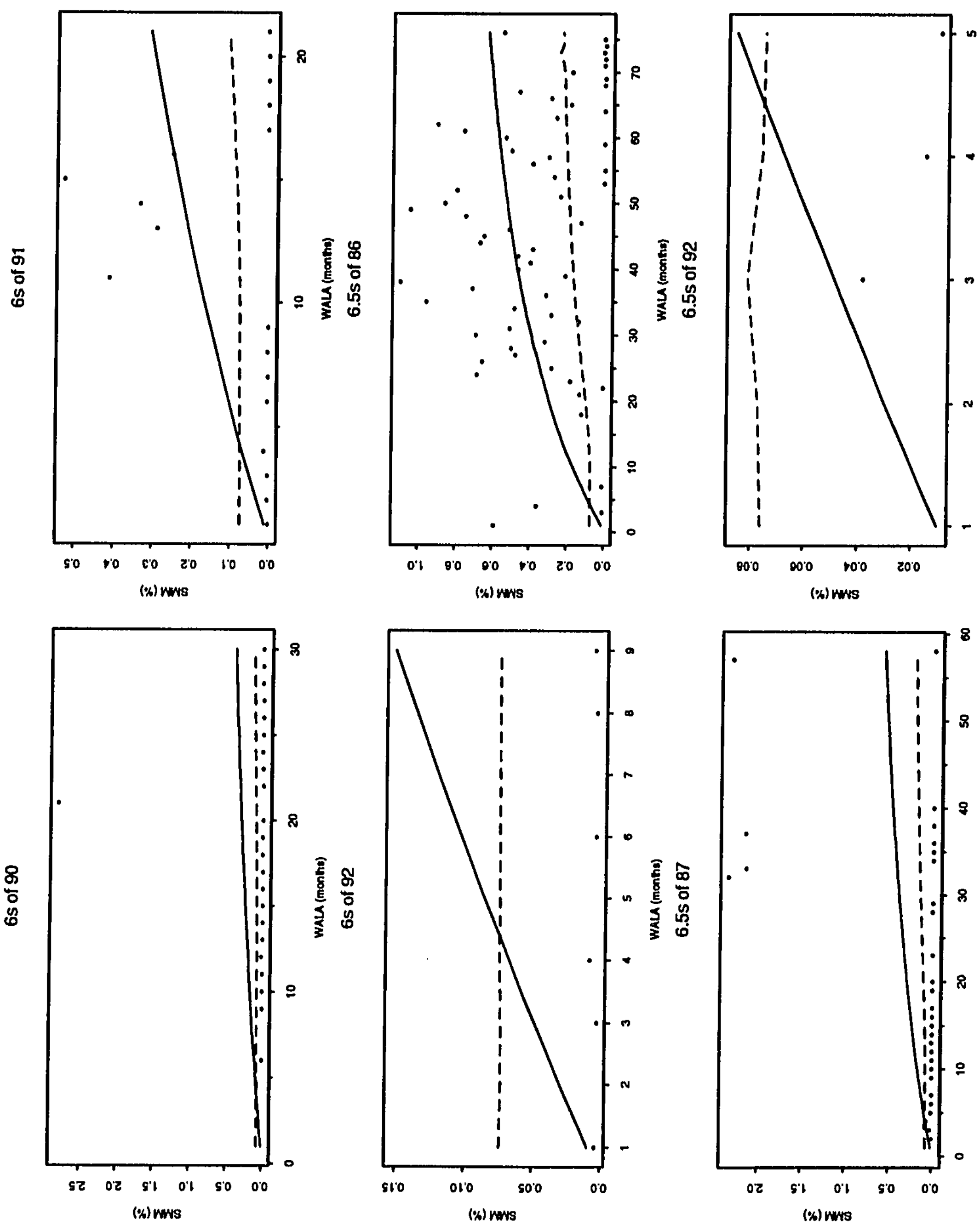

Figure A.1 Comparison of the five state model (solid line) and regression model (dotted line).

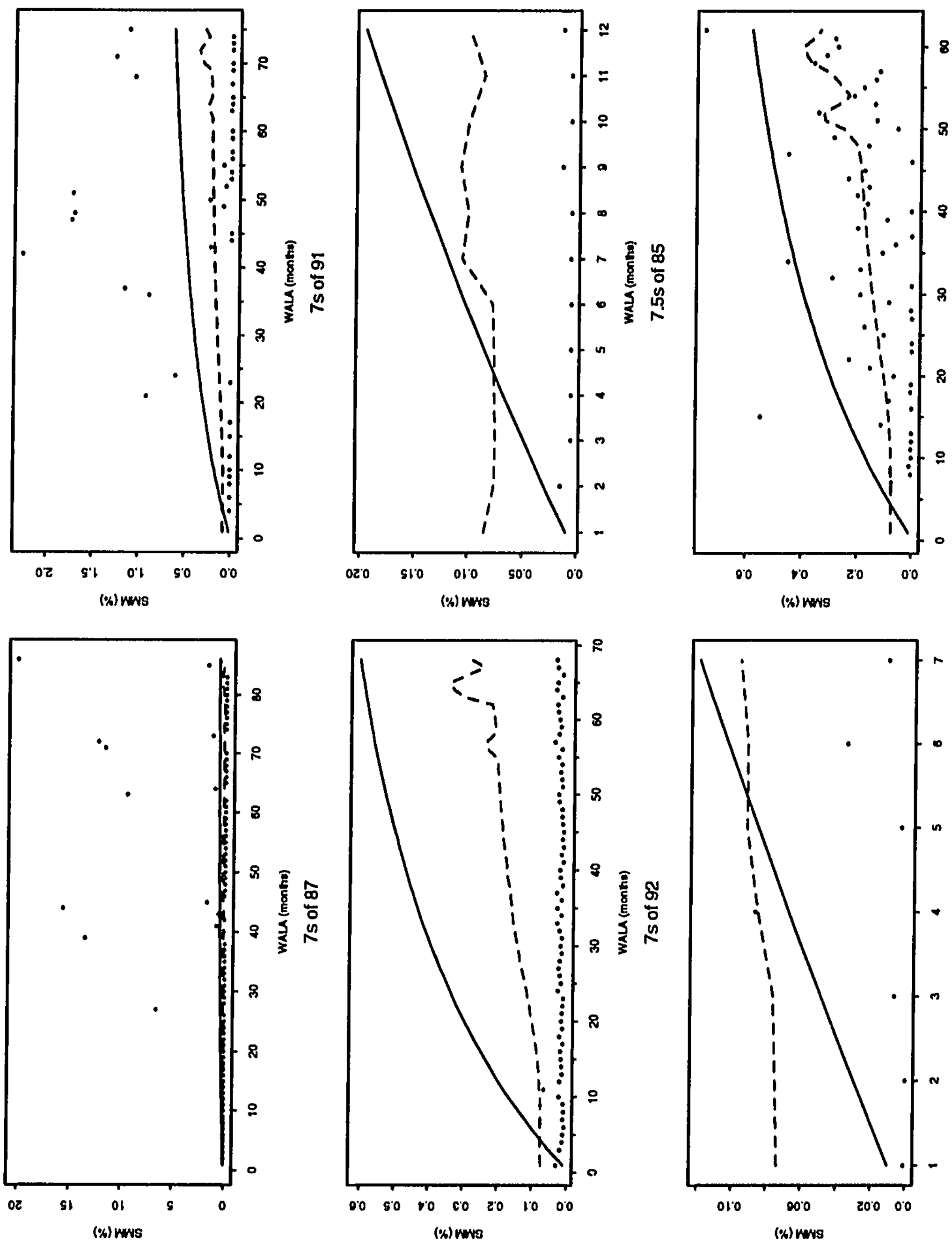

Figure A.2 Comparison of the five state model (solid line) and regression model (dotted line).

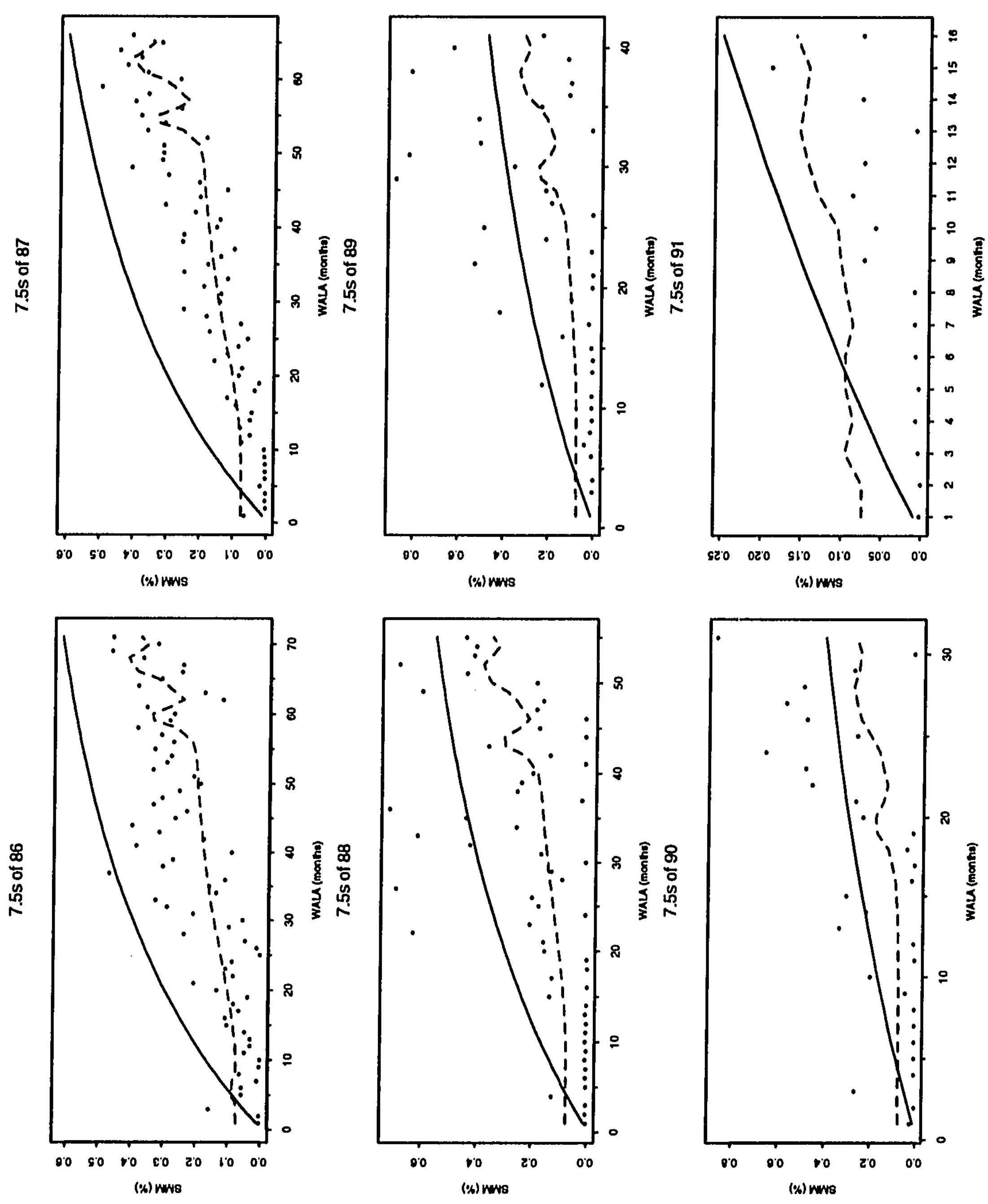

Figure A.3 Comparison of the five state model (solid line) and regression model (dotted line).

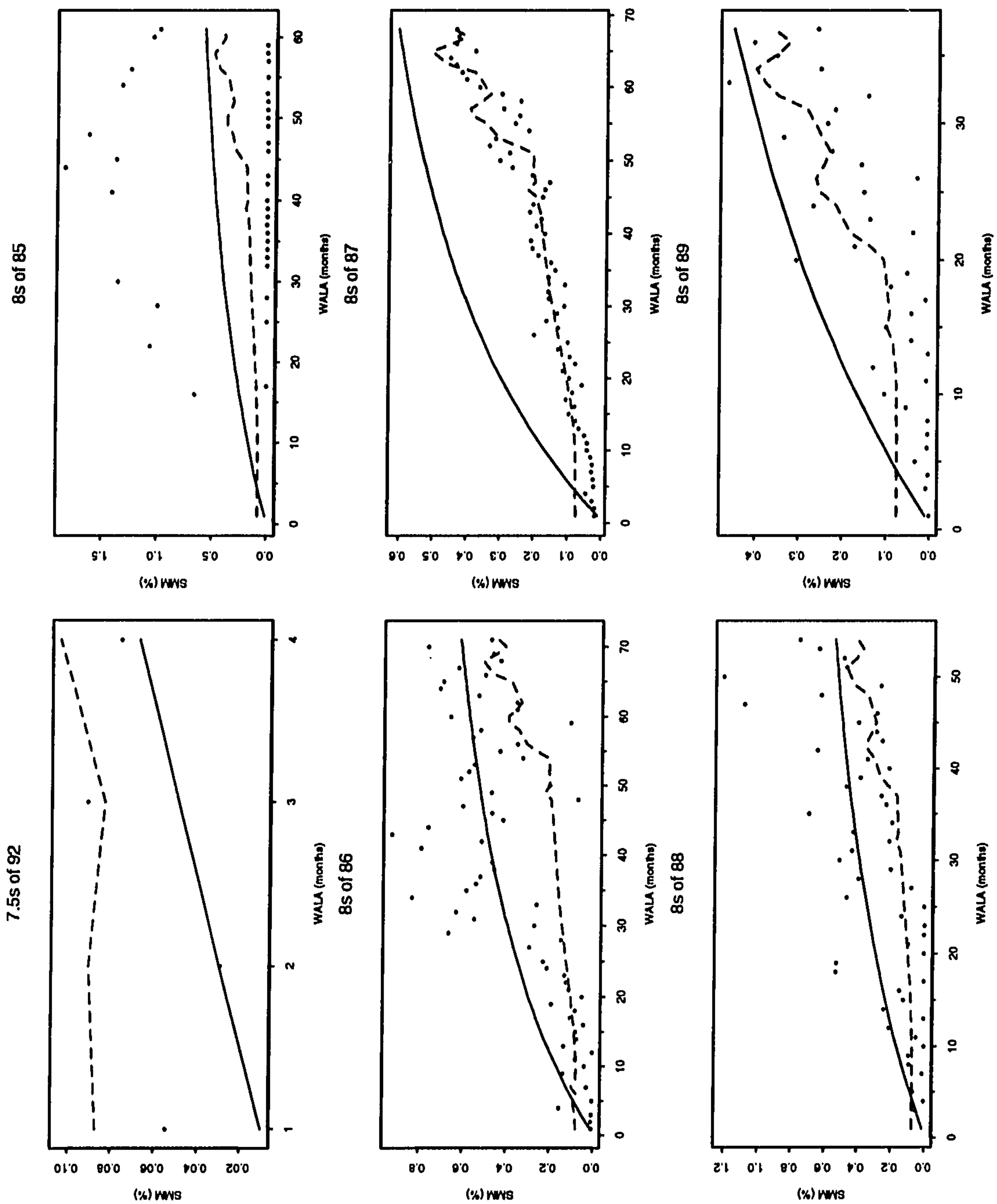

Figure A.4 Comparison of the five state model (solid line) and regression model (dotted line).

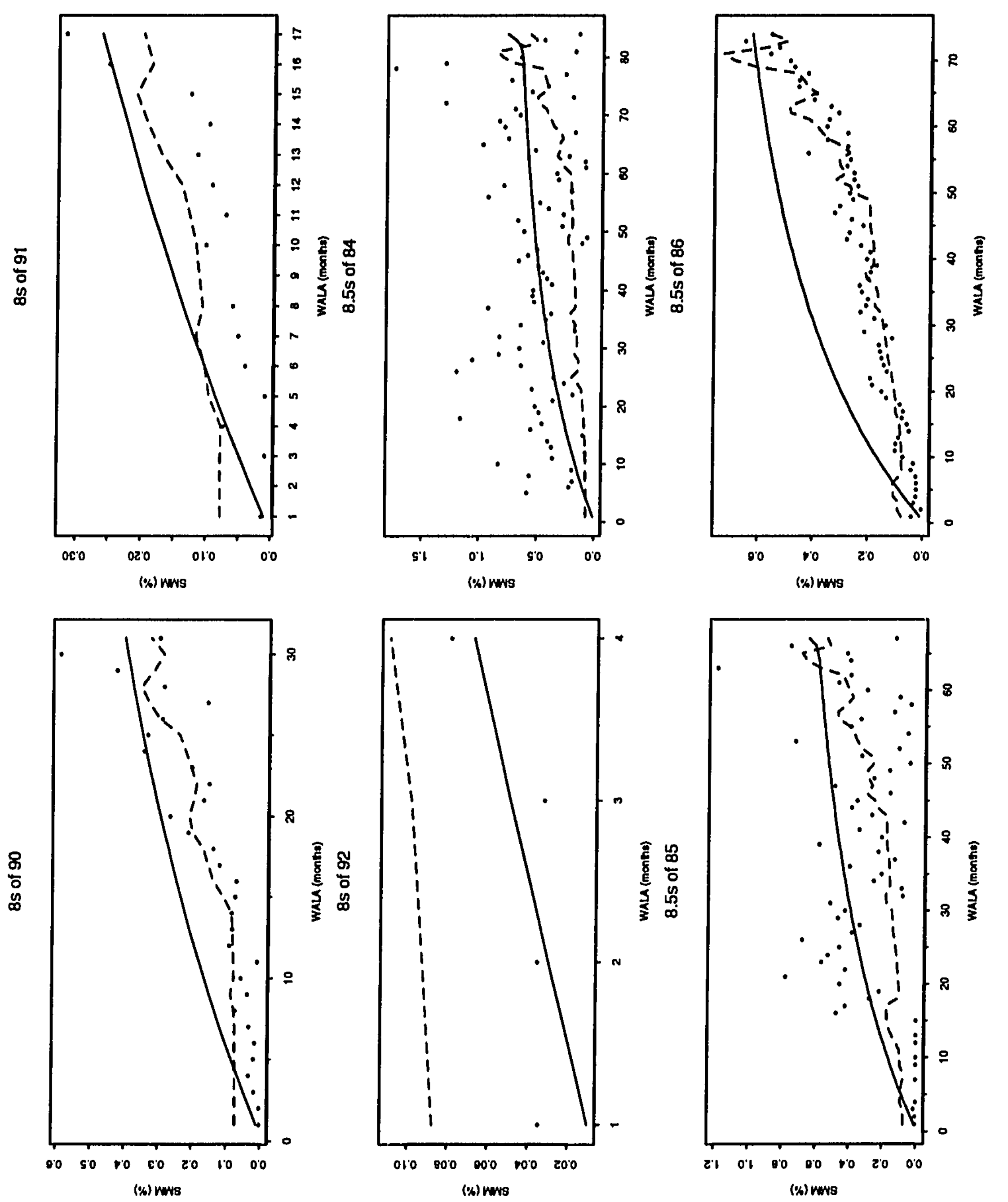

Figure A.5 Comparison of the five state model (solid line) and regression model (dotted line).

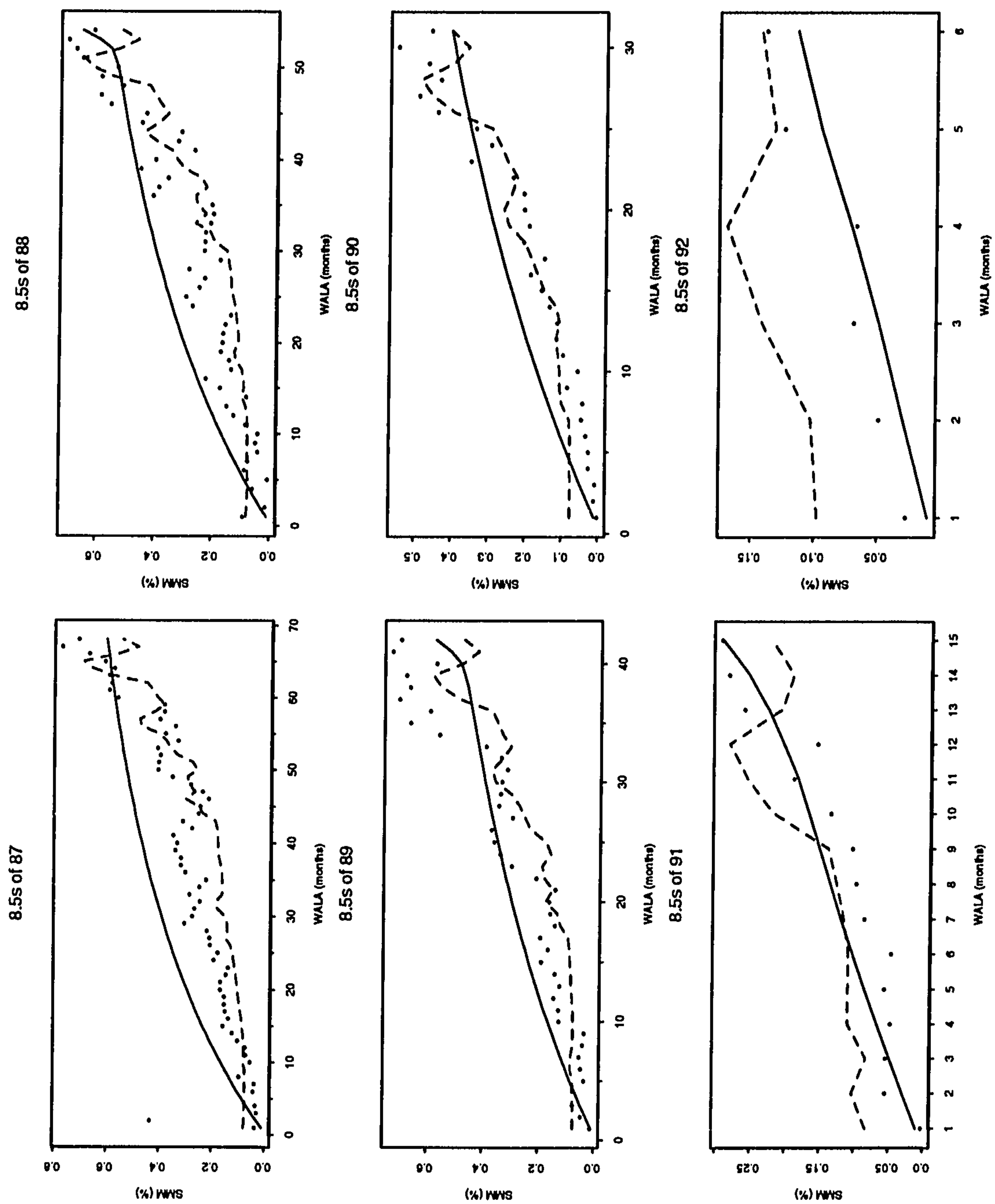

Figure A.6 Comparison of the five state model (solid line) and regression model (dotted line).

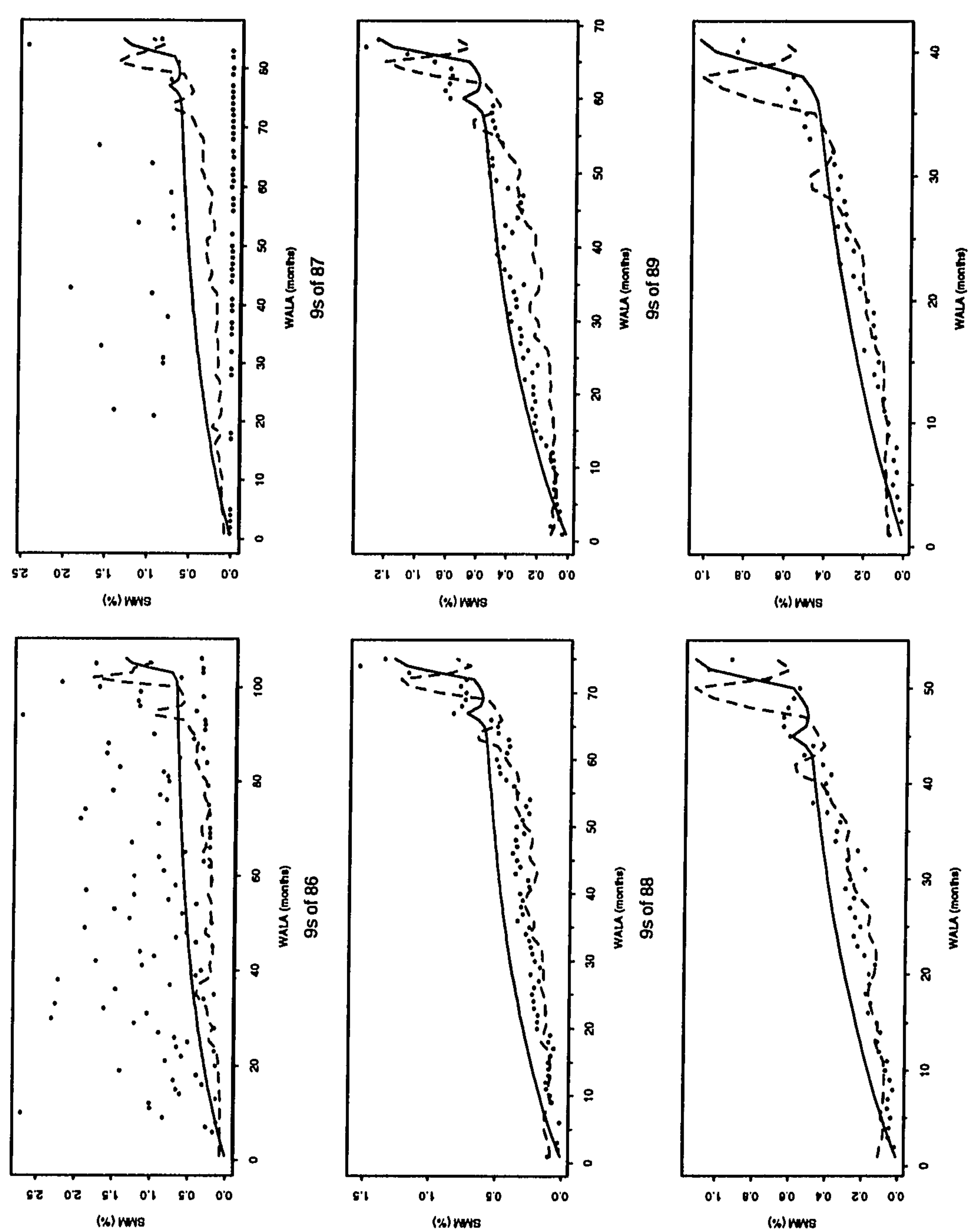

Figure A.7 Comparison of the five state model (solid line) and regression model (dotted line).

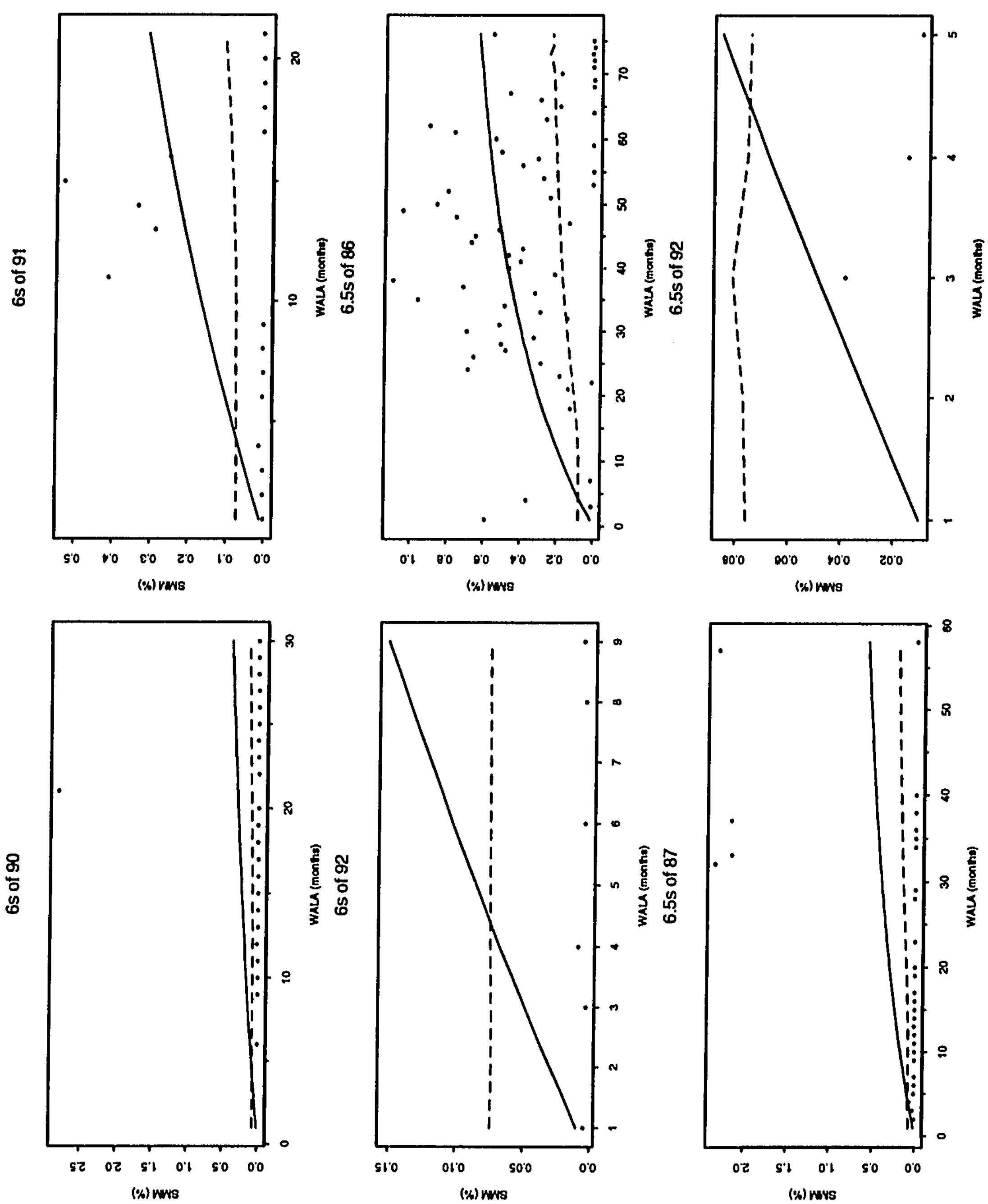

Figure A.8 Comparison of the five state model (solid line) and regression model (dotted line).

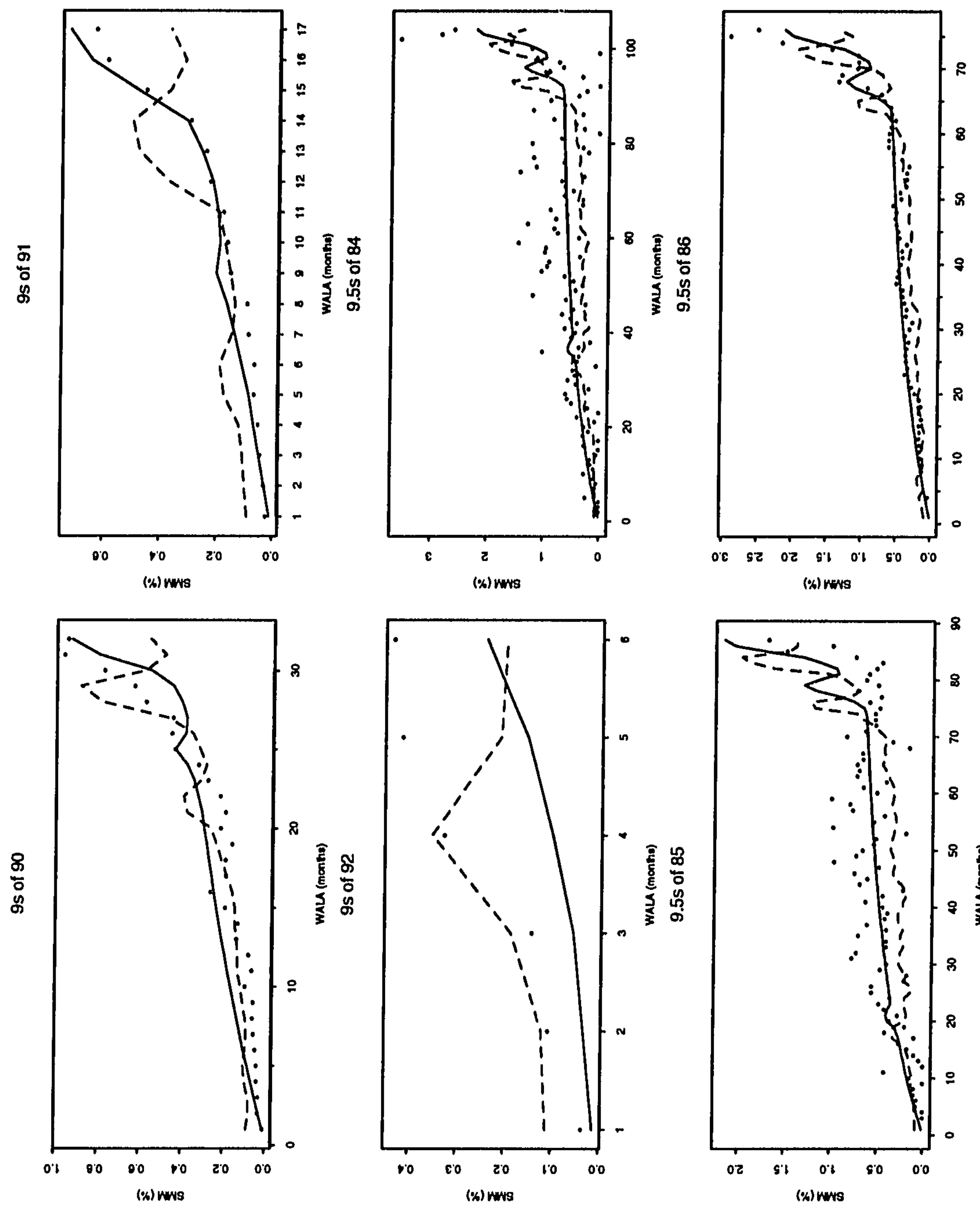

Figure A.9 Comparison of the five state model (solid line) and regression model (dotted line).

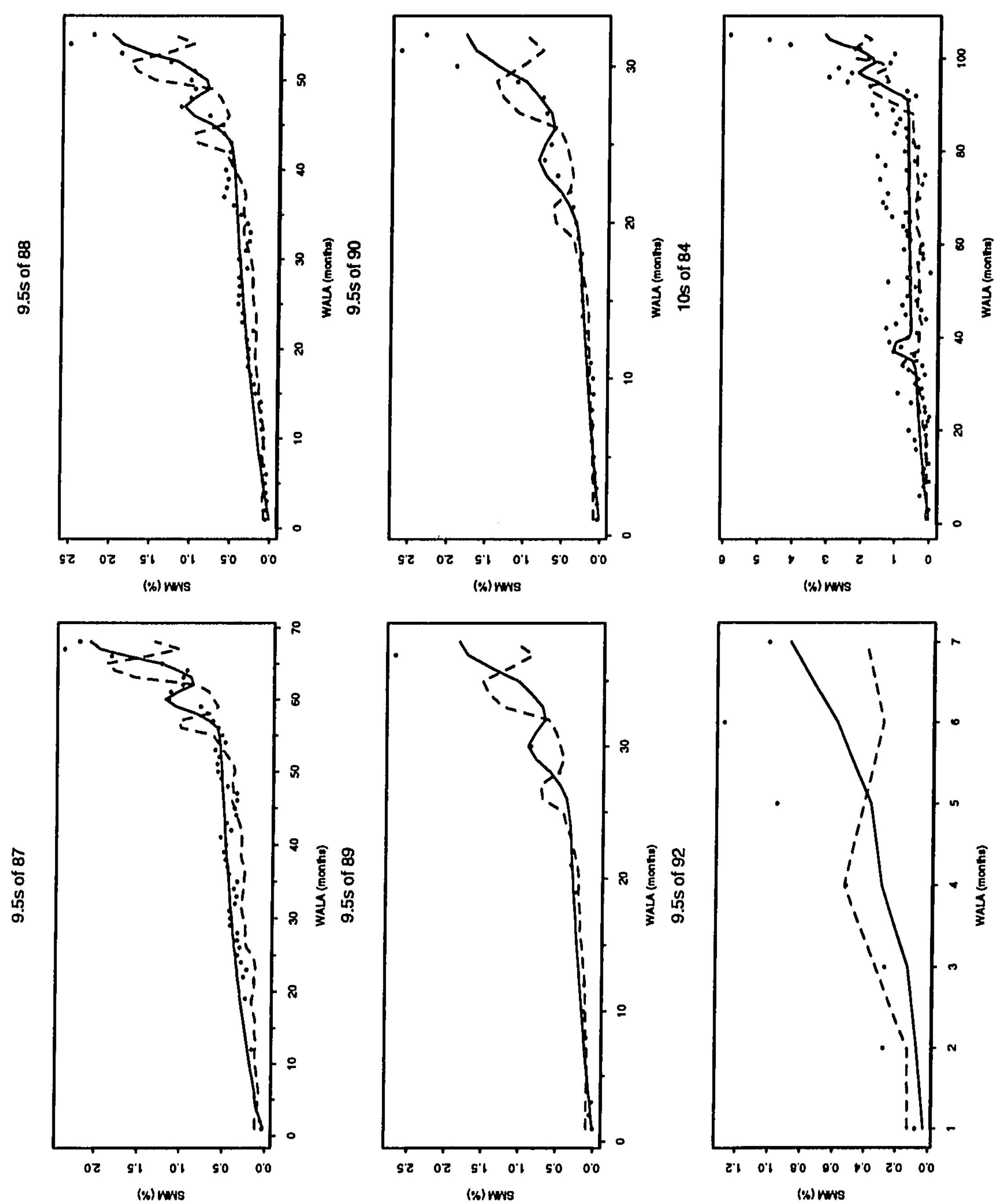

Figure A.10 Comparison of the five state model (solid line) and regression model (dotted line).

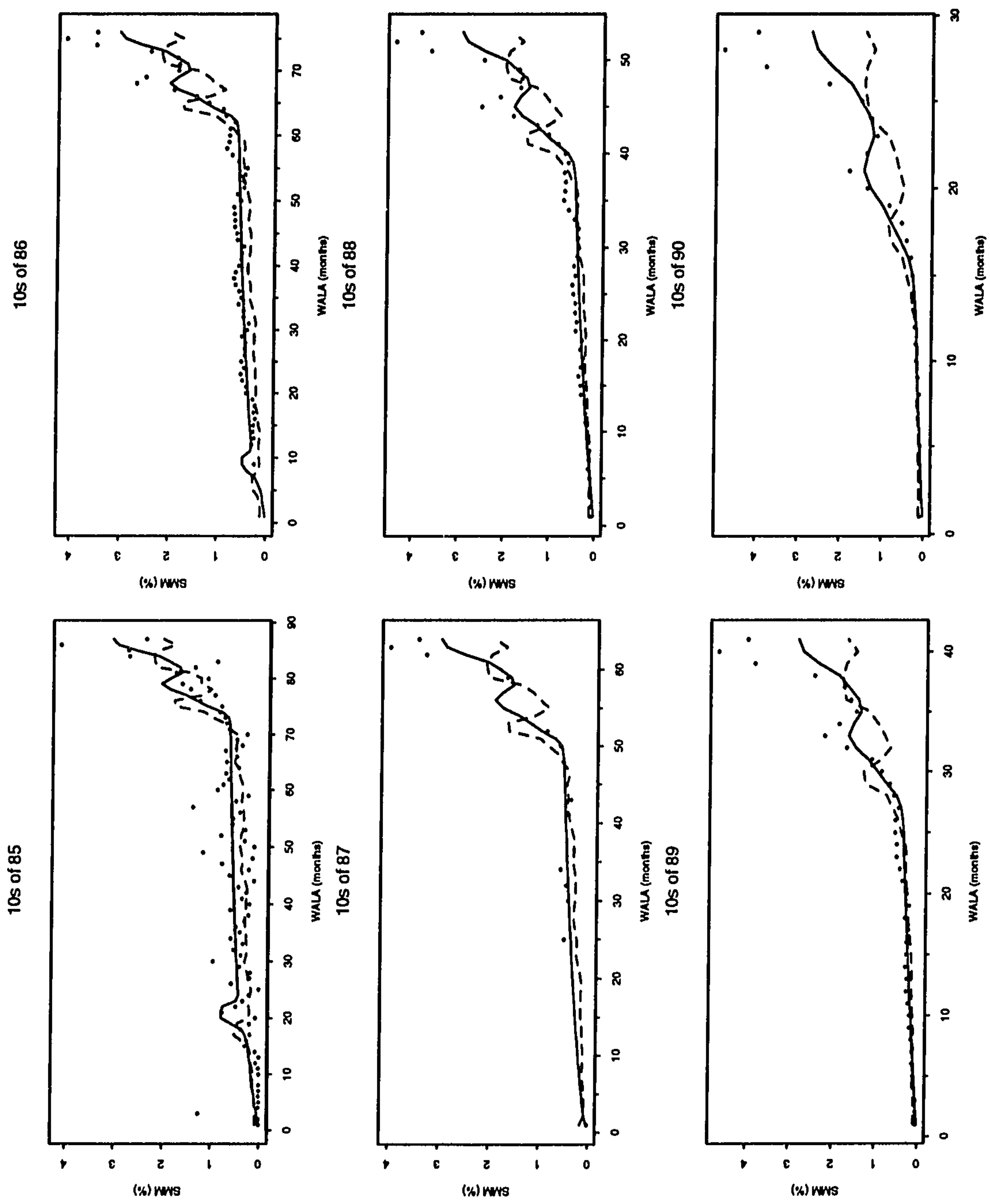

Figure A.11 Comparison of the five state model (solid line) and regression model (dotted line).

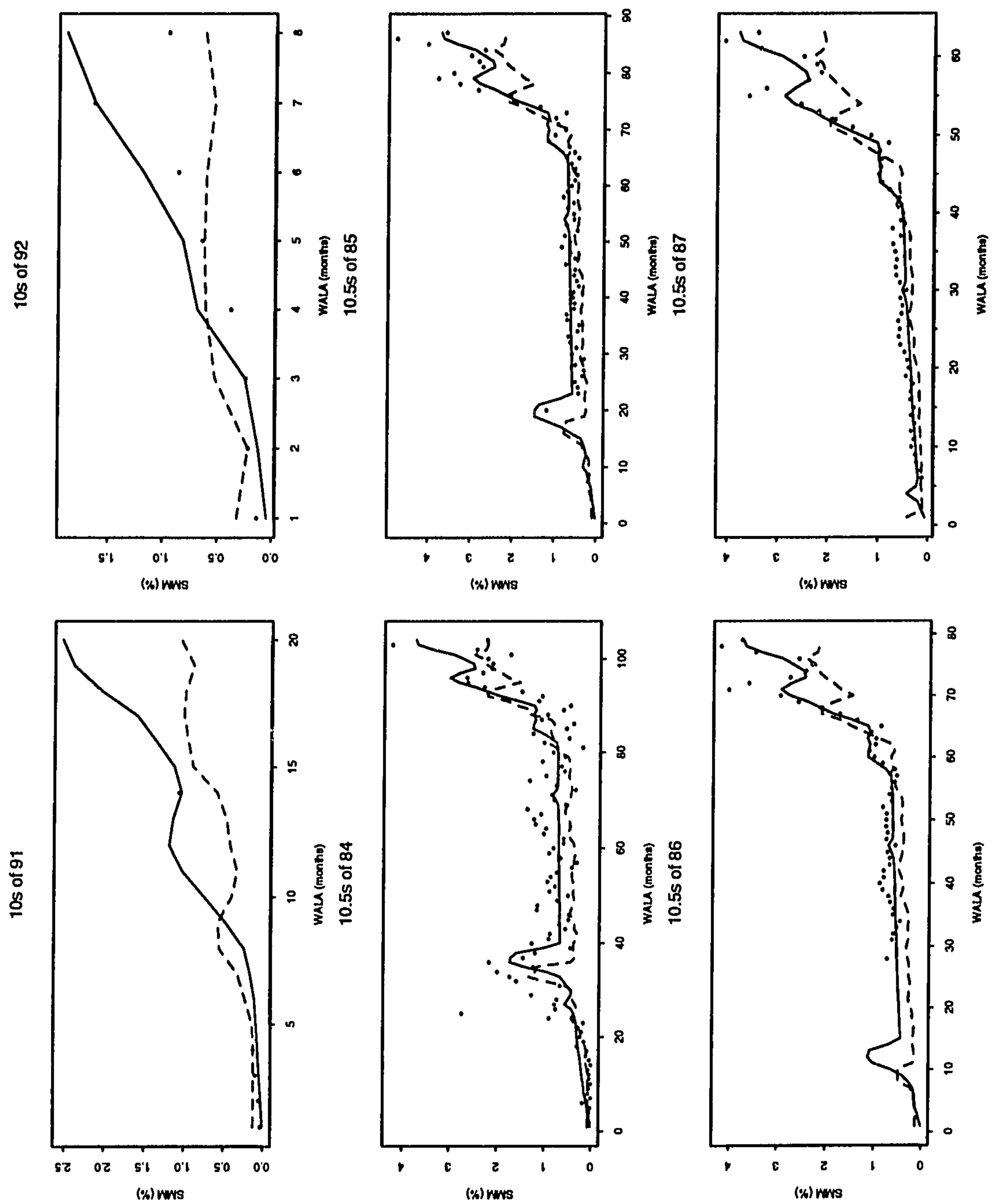

Figure A.12 Comparison of the five state model (solid line) and regression model (dotted line).

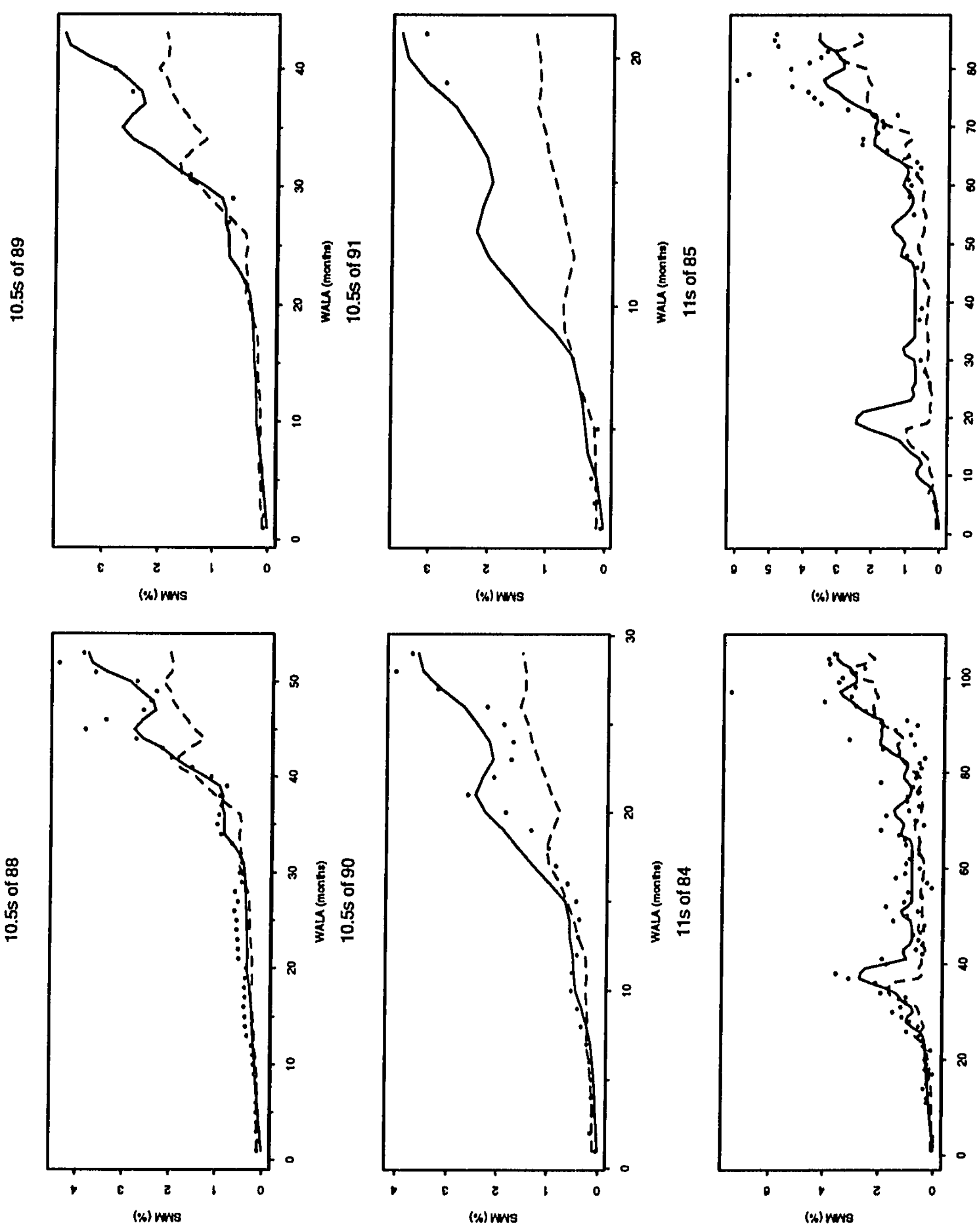

Figure A.13 Comparison of the five state model (solid line) and regression model (dotted line).

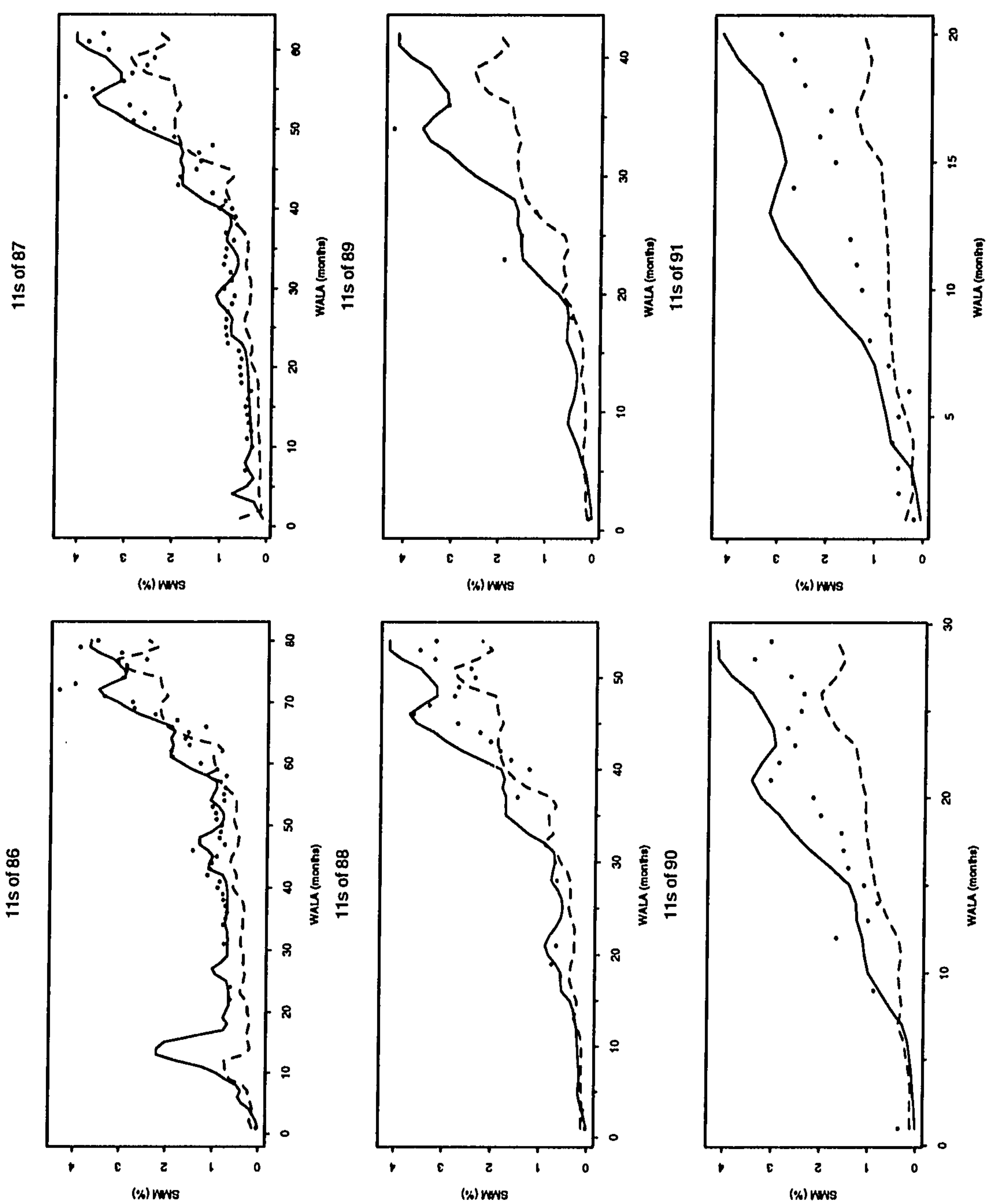

Figure A.14 Comparison of the five state model (solid line) and regression model (dotted line).

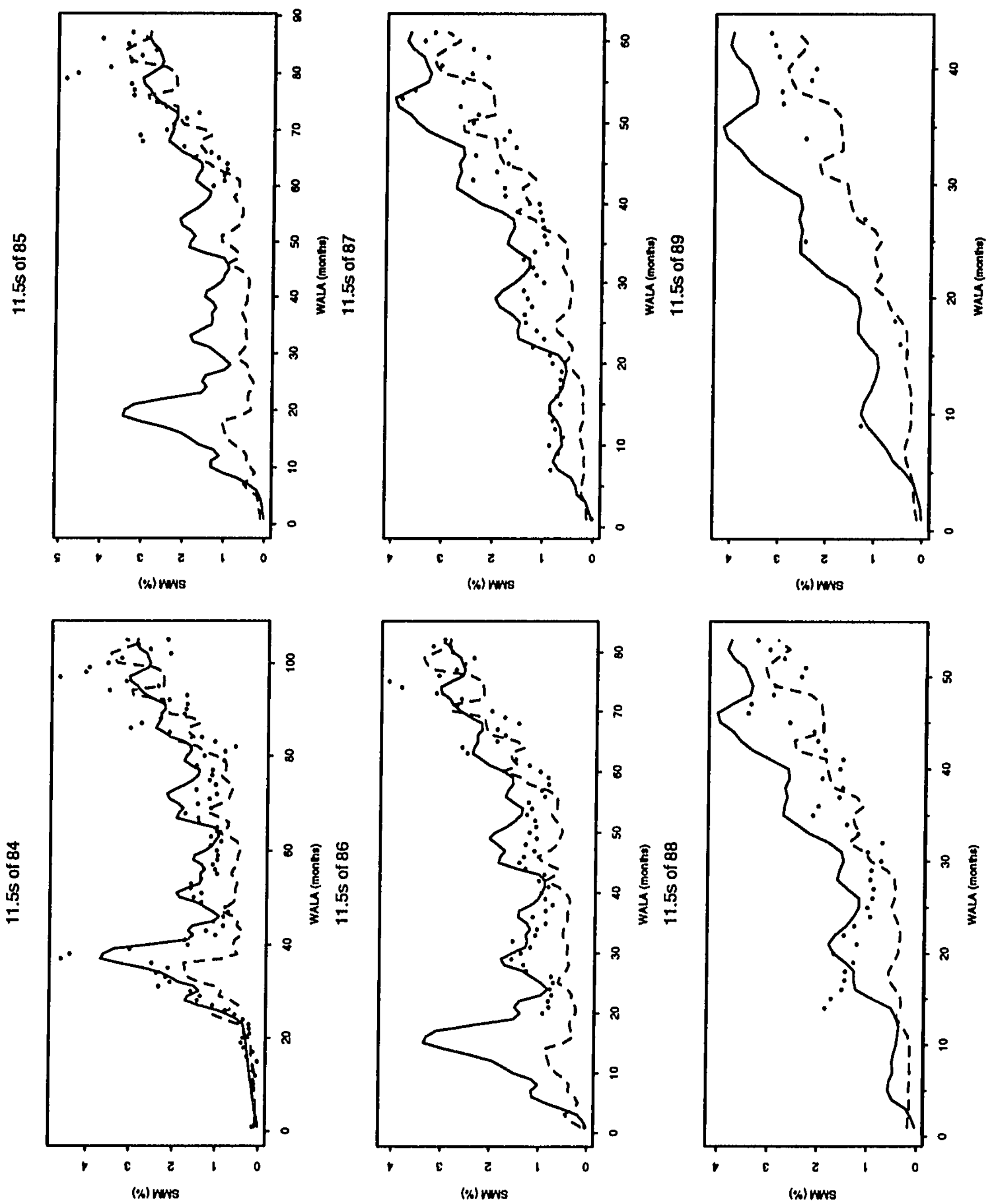

Figure A.15 Comparison of the five state model (solid line) and regression model (dotted line).

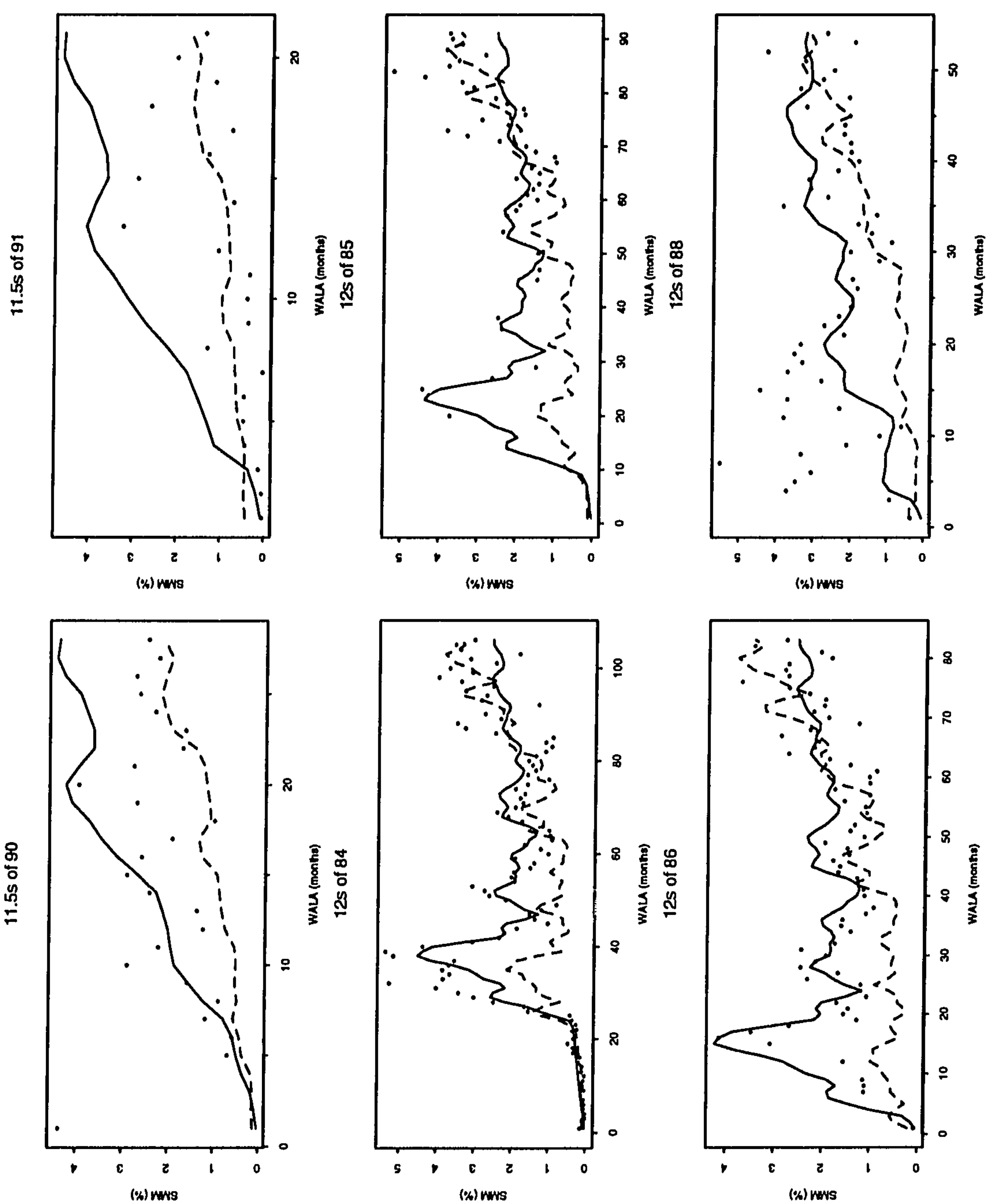

Figure A.16 Comparison of the five state model (solid line) and regression model (dotted line).

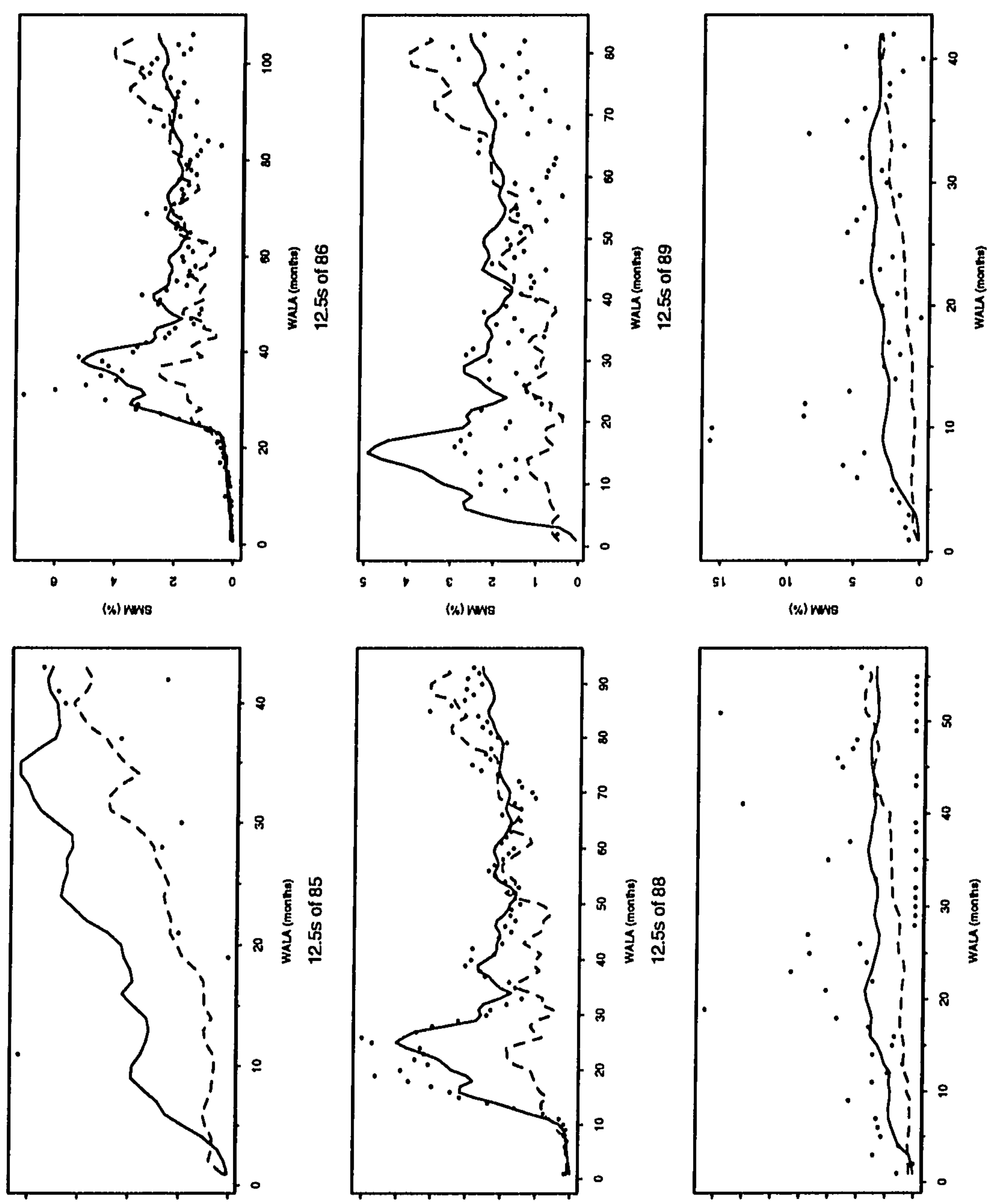

Figure A.17 Comparison of the five state model (solid line) and regression model (dotted line).

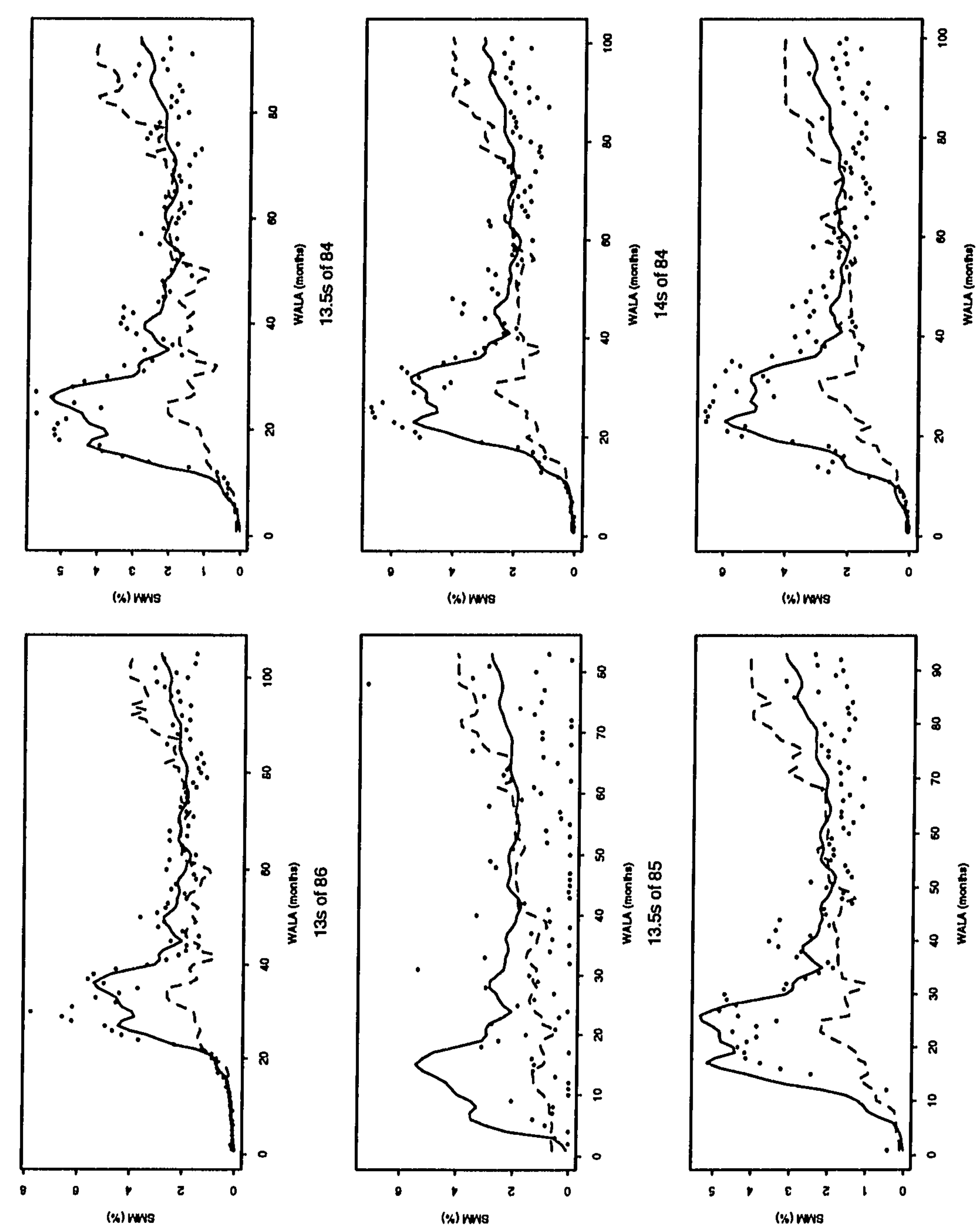

Figure A.18 Comparison of the five state model (solid line) and regression model (dotted line).

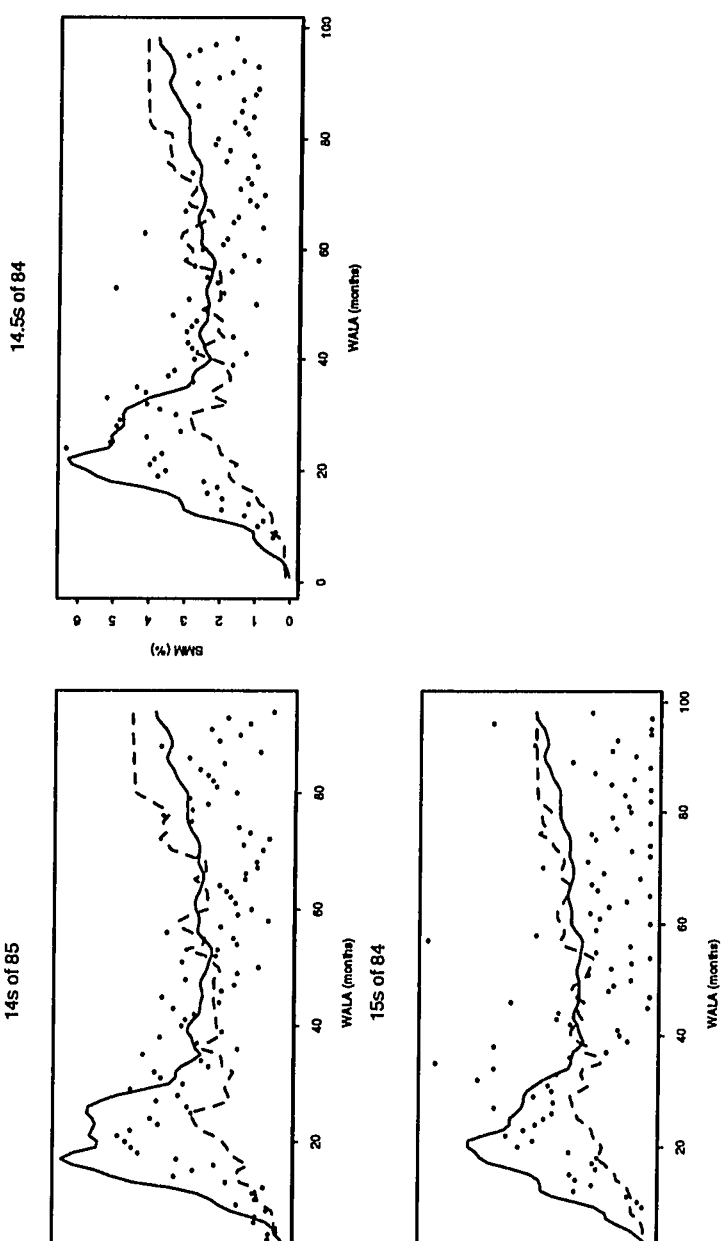

Figure A.19 Comparison of the five state model (solid line) and regression model (dotted line).

Appendix B

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