MIE 1613 Term Project

Interest Rate Modelling with Applications to Mortgage Markets

David Islip

In [1]:
```
import pandas as pd
from pandas.tseries.offsets import BDay
import numpy as np
import re

import matplotlib.pyplot as plt
from matplotlib import rc
import matplotlib as mpl
from mpl_toolkits.mplot3d import Axes3D

from arch.univariate import GARCH, Normal, ZeroMean, StudentsT
from itertools import product
from arch.bootstrap import MCS
from arch import arch_model
import statsmodels.formula.api as smf
import statsmodels.tsa.api as smt
import statsmodels.api as sm
import scipy.stats as scs

import quandl
import pickle

from sklearn.linear_model import LassoCV
from sklearn.linear_model import Lasso
from sklearn.model_selection import KFold
from sklearn.model_selection import GridSearchCV
from sklearn.metrics import r2_score

%matplotlib inline
rc('text', usetex=True)
```

In [2]:
```
#Read in the data
TD = pd.read_csv('IO Files\TD_Forecast.csv')
yield_curve = pd.read_csv('IO Files\yield_curves_update.csv')
yield_curve.Date = pd.to_datetime(yield_curve.Date)
yield_curve[yield_curve.Date <= '2020-04-01' ]
yield_curve.drop(' ', axis = 1, inplace = True)
```

In [3]:
```
benchmarks = quandl.get(["BOC/V39079", "BOC/V39065","BOC/V39051","BOC/V39053","BOC/V39055","BOC/V3905
6"],api_key='xY-nMTUQzLJNXPS6Nfyv').reindex()
benchmarks.dropna(inplace = True)
benchmarks.columns = [ 'Overnight Target Rate ', '3-mth T-Bill Rate ','2-yr Govt. Bond Yield ', '5-yr
 Govt. Bond Yield ','10-yr Govt. Bond Yield ', '30-yr Govt. Bond Yield ']
benchmarks.columns = [i.strip() for i in benchmarks.columns]
benchmarks.reset_index(inplace=True)
```

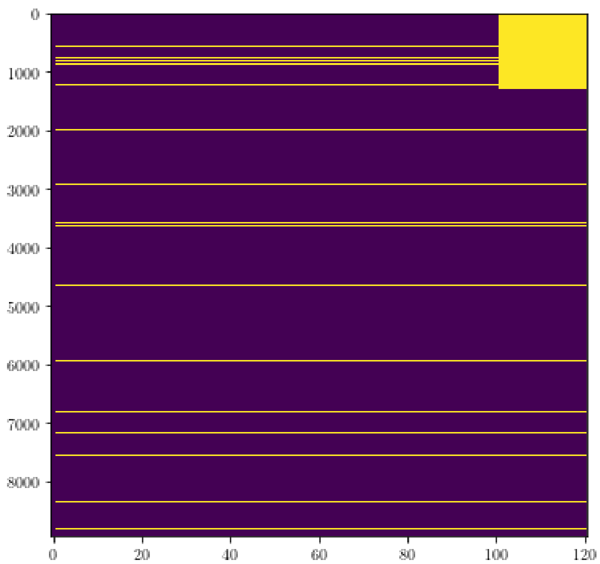
Data Cleaning

In [4]:
```
TD.columns = [i.strip() for i in TD.columns]
TD = TD.drop(['Overnight Target Rate'], axis = 1)
TD
```

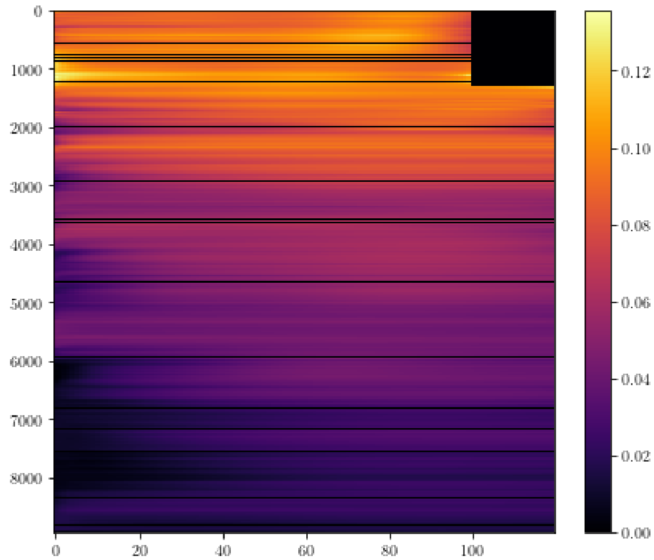
Out[4]:

	Year	Quarter	3-mth T-Bill Rate	2-yr Govt. Bond Yield	5-yr Govt. Bond Yield	10-yr Govt. Bond Yield	30-yr Govt. Bond Yield	10-yr-2-yr Govt Spread	Quarter End Date
0	2020	Q1	0.0040	0.0040	0.0050	0.0065	0.0100	0.0025	03/31/2020
1	2020	Q2	0.0015	0.0050	0.0060	0.0075	0.0105	0.0031	06/30/2020
2	2020	Q3	0.0015	0.0065	0.0075	0.0090	0.0115	0.0025	09/30/2020
3	2020	Q4	0.0015	0.0080	0.0090	0.0105	0.0130	0.0025	12/31/2020
4	2021	Q1	0.0015	0.0100	0.0110	0.0125	0.0150	0.0025	03/31/2021
5	2021	Q2	0.0028	0.0120	0.0130	0.0145	0.0170	0.0025	06/30/2021
6	2021	Q3	0.0053	0.0135	0.0145	0.0160	0.0185	0.0025	09/30/2021
7	2021	Q4	0.0078	0.0150	0.0160	0.0175	0.0200	0.0025	12/31/2021

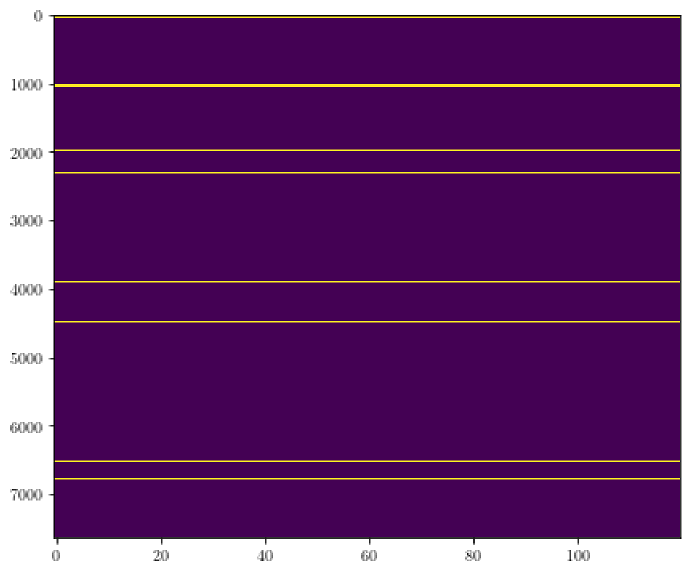
In [5]:
```
#### Investigate Missing Values
plt.figure(figsize = (6,6))
plt.imshow(yield_curve == 0,aspect='auto');
#yield_curve = yield_curve.replace(miss, 0.0);
```



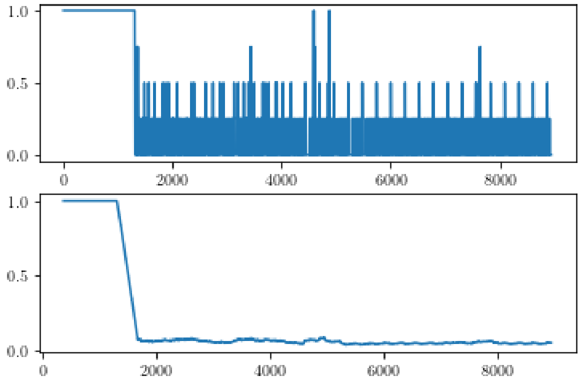
In [6]:
```
#Check if there are any columns that are not numeric
for i in range(1, len(yield_curve.columns)):
    if yield_curve.iloc[:,i].dtype != 'float64':
        print(yield_curve.iloc[:,i].dtype)
        break
plt.figure(figsize = (7,6))
plt.imshow(yield_curve.iloc[:,1:],aspect='auto',  cmap='inferno');
plt.colorbar();
```



In [7]:
```
yield_curve.iloc[0:1306];

plt.figure(figsize = (7,6))
plt.imshow(yield_curve.iloc[1306:, 1:]==0,aspect='auto');
#plt.colorbar()
```



In [8]:
```python
#4 day rolling average
fig, axs = plt.subplots(2)
(yield_curve.iloc[:,1:] == 0).any(axis = 1).astype(float).rolling(4).mean().plot(ax = axs[0]);
(yield_curve.iloc[:,1:] == 0).any(axis = 1).astype(float).rolling(365).mean().plot(ax = axs[1]);
```



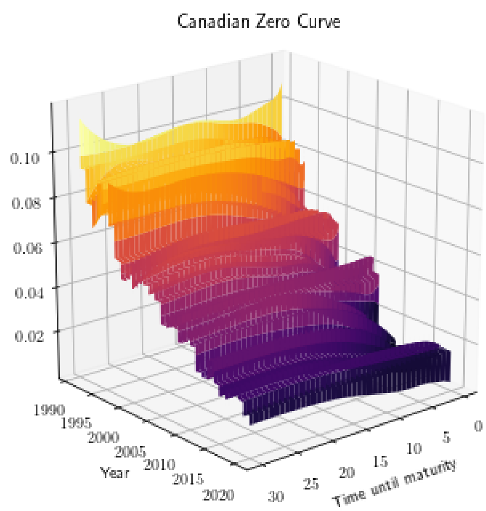
In [9]:
```python
#Map the columns to numeric durations
new_columns = {}
for col in yield_curve.columns[1:]:
    temp = re.findall(r'\d+', col)
    res = list(map(int, temp))
    new_columns[col] = res[0]/100
```

In [10]:
```python
#Convert Columns to numeric
yield_curve.rename(columns = new_columns, inplace = True)
```

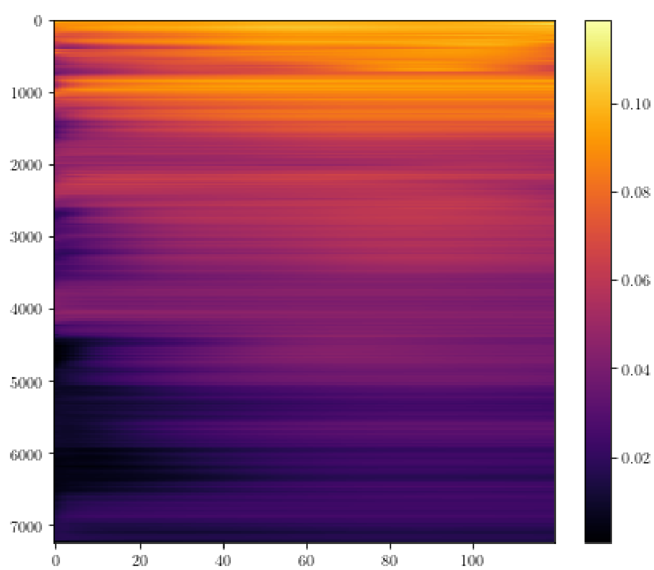
Remove the missing rows and the missing square and plot the yield curve

In [11]:
```python
yc = yield_curve[(yield_curve != 0).all(axis = 1)]
```

In [12]:
```
#plot the yield curve
x = yc.columns[1:]
y = yc.Date.dt.year
X,Y = np.meshgrid(x,y)
Z = yc.iloc[:,1:].values
fig = plt.figure(figsize = (6,6))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(X, Y, Z, cmap='inferno')
ax.view_init(20, 50)
plt.xlabel('Time until maturity');
plt.ylabel('Year');
plt.title('Canadian Zero Curve');
#plt.colorbar();
plt.savefig("p1.png", dpi = 350)
```



In [13]:
```
plt.figure(figsize = (7,6))
plt.imshow(yc.iloc[:,1:],aspect='auto',  cmap='inferno');
plt.colorbar();
plt.savefig("p2.png", dpi = 350)
```



Fixed Income: Government Bonds and Monetary Policy

What are bonds?

- Face value and Coupon
- An IOU
- Price and yield

Government Bonds:

- 5 year fixed:
- Government bonds are issued by governments to raise money to finance projects or day-to-day operations
- Are safe investments
- Traded in financial market

Monetary Policy: Overnight Rate:

- https://www.bankofcanada.ca/core-functions/monetary-policy/ (https://www.bankofcanada.ca/core-functions/monetary-policy/)

Fixed Income and Mortgages

- Funding Sources securitization NHA MBS, and covered bonds
- Overnight affects variable rate mortgage
- In both cases, there are spreads between the respective instruments that are dependent on consumer specific competition
- For example . . . ### Why Model Zero Coupon Bonds

Bolder "The most fundamental underlying driver of fixed-income securities' returns is the sovereign yield curve.'

Bolder: "Simply plotting the YTM for a selection of bonds would be misleading. Firstly, the YTM measure, which is a complicated average of zero-coupon rates, cannot be used to discount a single cash flow. In fact, the YTM cannot be used to price any set of bonds apart from the specific bond to which it refers. Secondly, the implicit reinvestment assumption and the coupon effect make the YTM measure extremely difficult to interpret as a yield curve."

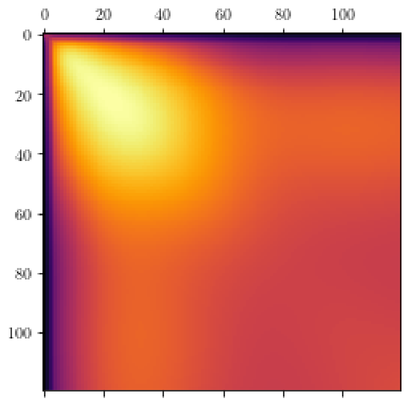
PCA analysis as per Meucci - Risk and Asset Allocation

Let Y_t^v represent the yield to maturity (i.e the yield curve) then changes in the yield to maturity $X_t^v = Y_t^v - Y_{t-1}^v$ form the set of random variables that are approximately iid.

In [14]:
```
N_start = 1465+int(5*365*5/7.47) #start index
N = 10 #number of pca factors to store
nn = 3 #number of pca factors to use for modelling
X = yc.iloc[:,1:].diff()
print(yc.Date.iloc[N_start])
X = X.iloc[N_start:,:]
```

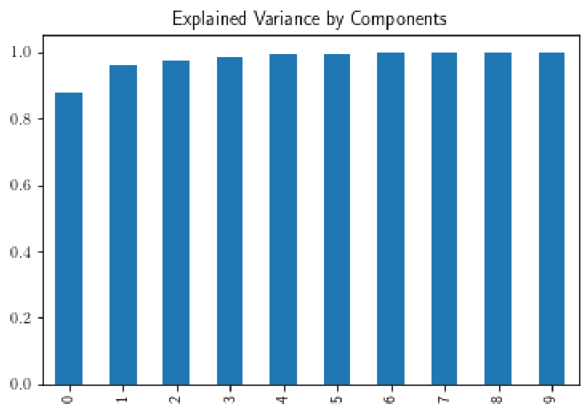
```
2001-12-24 00:00:00
```

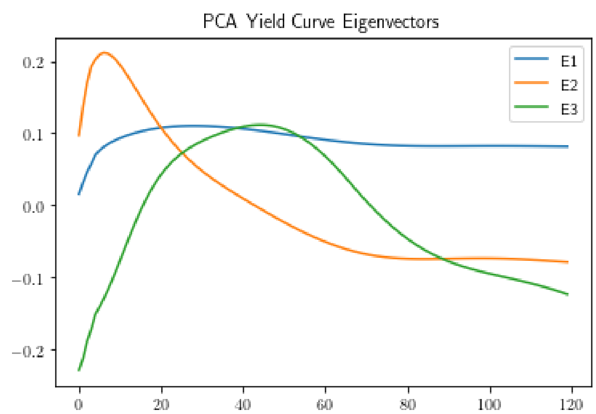
In [15]:
```
#estimate the covariance matrix
from sklearn.covariance import EmpiricalCovariance
covX_ = EmpiricalCovariance().fit(X)
plt.matshow(covX_.covariance_, aspect='auto',  cmap='inferno');
covX = covX_.covariance_
plt.savefig("p3.png", dpi = 350)
```

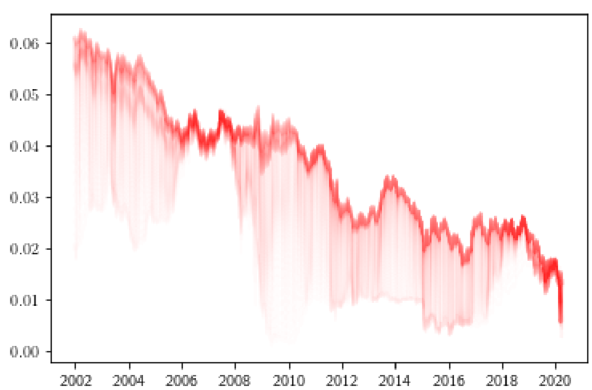


In [16]:
```
#PCA
# eigenvectors and eigenvalues for the from the covariance matrix
eig_val_cov, eig_vec_cov = np.linalg.eig(covX)

# Make a list of (eigenvalue, eigenvector) tuples
eig_pairs = [(np.abs(eig_val_cov[i]), eig_vec_cov[:,i]) for i in range(len(eig_val_cov))]

# Sort the (eigenvalue, eigenvector) tuples from high to low
eig_pairs.sort(key=lambda x: x[0], reverse=True)

# Visually confirm that the list is correctly sorted by decreasing eigenvalues
#for i in eig_pairs:
#    print(i[0])

components_ = np.vstack([eig_pairs[i][1] for i in range(N)])
eig_sort = np.vstack([eig_pairs[i][0] for i in range(len(eig_pairs))])

explained_variance_ratio_  =  eig_sort[:N].cumsum()/eig_sort.sum()
```

```
In [78]: #Variance explained by the first three components
explained_variance_ratio_[:3]
pd.DataFrame(explained_variance_ratio_).plot.bar(legend = False, title = "Explained Variance by Components");
plt.savefig("ExplainedVar.png", dpi = 300)
pd.DataFrame(components_[0:3], index = ["E1", "E2", "E3"]).transpose().plot(title = "PCA Yield Curve Eigenvectors");
plt.savefig("YieldCurveFactors.png", dpi = 300)
plt.figure();
plt.plot(yc.Date.iloc[N_start:], yc.iloc[:,1:].iloc[N_start:], color = 'red', alpha = 0.01);
```





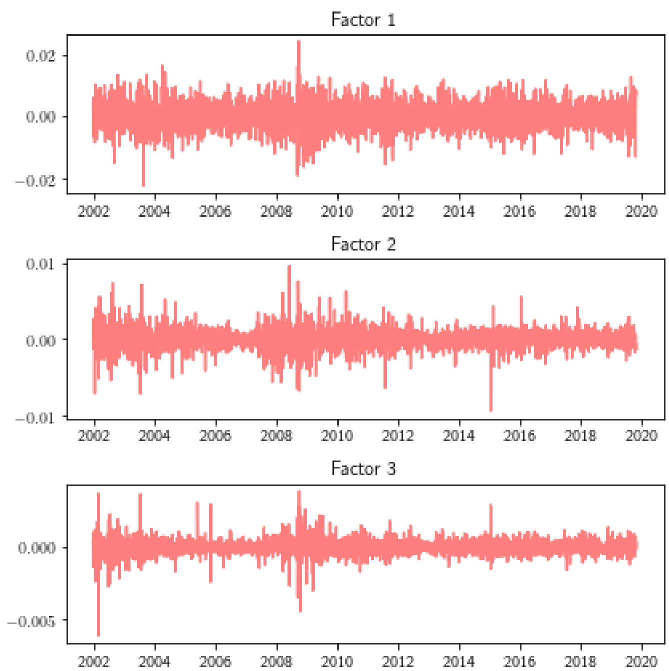


The matrix of eigen-values $E = (e^1, \ldots, e^n)$

The factors $F = (X - \mathbb{E}[X])E$

Then $X \approx FE^\intercal$

In [96]:
```
E = components_[0:3].transpose()
F = np.matmul((X - X.mean()).values, E)
fig, axs = plt.subplots(3, figsize = (6,6))
#Showing pre corona virus for visual clarity
axs[0].set_title('Factor 1')
axs[0].plot(yc.Date.iloc[N_start:-100], F[:-100,0], color = 'red', alpha = 0.5, label = "F1");
axs[1].set_title('Factor 2')
axs[1].plot(yc.Date.iloc[N_start:-100], F[:-100,1],color = 'red', alpha = 0.5, label = "F2");
axs[2].set_title('Factor 3')
axs[2].plot(yc.Date.iloc[N_start:-100], F[:-100,2],color = 'red', alpha = 0.5, label = "F3");
plt.tight_layout()
plt.savefig("Factors_time.png", dpi = 300)
```



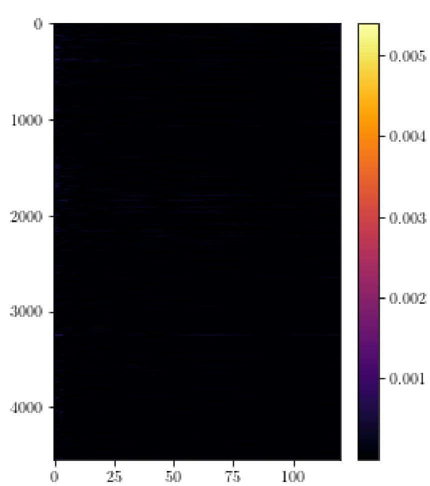
In [19]:
```
E.shape
```

Out[19]: (120, 3)

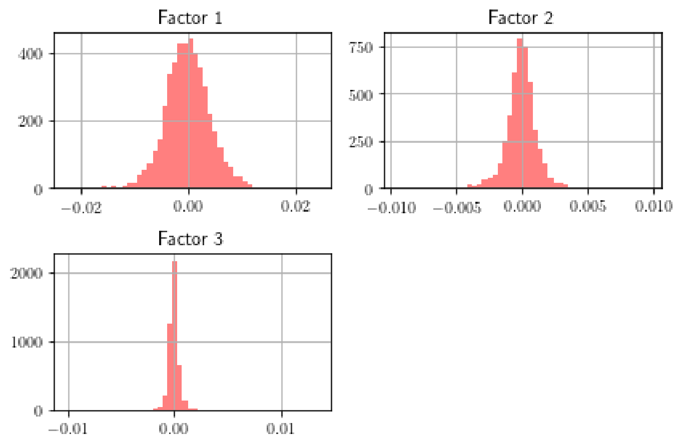
In [20]:
```
F.shape
```

Out[20]: (4547, 3)

In [21]:
```
X_ = np.matmul(F, E.transpose())
plt.figure(figsize = (4,5))
plt.imshow(np.abs(X.values - X_),aspect='auto',  cmap='inferno');
plt.colorbar();
```



In [98]:
```
pd.DataFrame(F, columns = ["Factor 1", "Factor 2", "Factor 3"]).hist(bins = 50, color = 'red', alpha =
0.5)
plt.title("Factor Distributions")
plt.tight_layout();
plt.savefig("Factor Distribution.png")
```



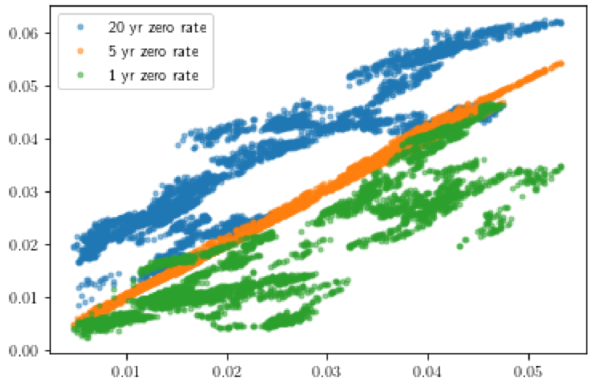
Correlation with Average Benchmark Rate

In [23]:
```
benchmarks.head()
```

Out[23]:

	Date	Overnight Target Rate	3-mth T-Bill Rate	2-yr Govt. Bond Yield	5-yr Govt. Bond Yield	10-yr Govt. Bond Yield	30-yr Govt. Bond Yield
0	2001-12-07	2.25	2.04	3.49	4.83	5.53	5.76
1	2001-12-10	2.25	2.02	3.34	4.73	5.48	5.73
2	2001-12-11	2.25	2.00	3.28	4.67	5.43	5.69
3	2001-12-12	2.25	2.00	3.26	4.65	5.40	5.65
4	2001-12-13	2.25	2.02	3.34	4.73	5.47	5.70

In [24]:
```
data = pd.merge(yc, benchmarks, left_on='Date', right_on = 'Date', how = 'inner');
zcb_cols = [i for i in data.columns if type(i) == float]
bond_cols = [i for i in data.columns if type(i) != float and i != 'Date']
data[bond_cols] = data[bond_cols]/100
plt.plot(data['5-yr Govt. Bond Yield'], data[20.0], '.', alpha = 0.5, label = "20 yr zero rate");
plt.plot(data['5-yr Govt. Bond Yield'], data[5.0], '.', alpha = 0.5, label = "5 yr zero rate");
plt.plot(data['5-yr Govt. Bond Yield'], data[1.0], '.', alpha = 0.5, label = "1 yr zero rate");
plt.legend();

zcb_cols = [i for i in data.columns if type(i) == float]
bond_cols = [i for i in data.columns if type(i) != float and i != 'Date']
```



In [25]:
```
bond_cols
```

Out[25]:
```
['Overnight Target Rate',
 '3-mth T-Bill Rate',
 '2-yr Govt. Bond Yield',
 '5-yr Govt. Bond Yield',
 '10-yr Govt. Bond Yield',
 '30-yr Govt. Bond Yield']
```

First derive some intuition as to why the benchmark yields are approximately a linear combination of zero rates

All government bonds can be expressed as a combination of zero coupon bonds. Since, the benchmark bonds are an aggregate pool of traded bonds and are agreed upon by the market to represent the benchmark it is assumed that the price of the bond is given by some combination of zero coupon bonds. Assume that the benchmark bond trading at price P_t has cashflows α_i at times $i = 1, \ldots, N$ in the future - then the yield r is the interest rate that makes the discounted cashflows equal to the price. Since the bond is a portfolio of zero coupon bonds:

$$\sum_i \alpha_i e^{-ir} = \sum_i \alpha_i e^{-iY^{(i)}}$$

Using the first order taylor approximation: $e^{-ir} \approx 1 - ir$, implies

$$r \sum_i i\alpha_i = \sum_i iY^{(i)}\alpha_i \implies r = \sum_i \beta_i Y^{(i)}$$

where $\beta_i = \frac{i\alpha_i}{\sum_i i\alpha_i}$

Therefore, the benchmark bond yields are approximately linear combinations of the zero yields.

Use the linear approximation/assumption to derive the yield changes in terms of the PCA factors

The benchmarks $R_t \in \mathbb{R}^b$ can be written as: $A^\intercal Y_t$ where $A \in \mathbb{R}^{v \times b}$, and $Y_t \in \mathbb{R}^v$ is the vector of zero yields

Therefore, $R_t - R_{t-1} = A^\intercal(Y_t - Y_{t-1}) = A^\intercal(X_t) = A^\intercal(\sum_{i=1}^k e_i F_t^i) = \sum_{i=1}^k (A^\intercal e_i) F_t^i$

Estimation using Least Squares

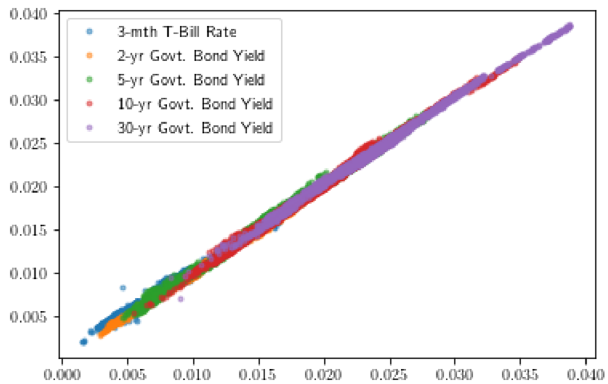
For a given benchmark k, column a_k can be estimated using least squares using lasso regularized regression

In [26]:
```python
model = False

def estimate_bond_zcb(col_name, zcb_cols, data, plt_flag = 0):
    'this function trains a lasso regression on the first half of the dataset'
    X =  100*data[zcb_cols]
    y =  100*data[col_name]
    hw = len(y)//2
    X_train = X.iloc[:hw,:]
    y_train = y.iloc[:hw]

    lasso = Lasso(random_state=0, max_iter=10000, fit_intercept=False)
    alphas = np.logspace(-4, -1, 10)

    tuned_parameters = [{'alpha': alphas}]
    n_folds = 5

    clf = GridSearchCV(lasso, tuned_parameters, cv=n_folds, refit=True, scoring = 'r2')
    clf.fit(X_train, y_train)
    scores = clf.cv_results_['mean_test_score']
    scores_std = clf.cv_results_['std_test_score']

    if plt_flag:
        plt.figure().set_size_inches(8, 6)
        plt.semilogx(alphas, scores)

        # plot error lines showing +/- std. errors of the scores
        std_error = scores_std / np.sqrt(n_folds)

        plt.semilogx(alphas, scores + std_error, 'b--')
        plt.semilogx(alphas, scores - std_error, 'b--')

        # alpha=0.2 controls the translucency of the fill color
        plt.fill_between(alphas, scores + std_error, scores - std_error, alpha=0.2)

        plt.ylabel('CV $R^2$ +/- std error')
        plt.xlabel('alpha')
        plt.axhline(np.max(scores), linestyle='--', color='.5')
        plt.xlim([alphas[0], alphas[-1]])

    return clf

if model:
    benchmark = {}
    for col in bond_cols:
        if col != 'Overnight Target Rate':
            benchmark[col] = estimate_bond_zcb(col, zcb_cols, data)
else:
    with open("benchmark_models.pkl", 'rb') as f:
        benchmark = pickle.load(f)
```

In [27]:
```python
import pickle
with open("benchmark_models.pkl", 'wb') as f:
    pickle.dump(benchmark, f)
```

```
In [28]: for col in bond_cols:
             if col != 'Overnight Target Rate':
                 y_pred = benchmark[col].best_estimator_.predict(data[zcb_cols])
                 hw = len(y_pred)//2
                 r2_oos = r2_score(data[col].iloc[hw:], y_pred[hw:])
                 plt.plot(y_pred[hw:], data[col].iloc[hw:],'.', label = col, alpha = 0.5)
                 print(col+ " Linear Model Out of Sample R2: ", r2_oos)
                 plt.legend();
```

```
3-mth T-Bill Rate Linear Model Out of Sample R2:  0.9747279544532022
2-yr Govt. Bond Yield Linear Model Out of Sample R2:  0.9982043142320964
5-yr Govt. Bond Yield Linear Model Out of Sample R2:  0.9919700045735047
10-yr Govt. Bond Yield Linear Model Out of Sample R2:  0.9920326096409087
30-yr Govt. Bond Yield Linear Model Out of Sample R2:  0.9919947123585225
```



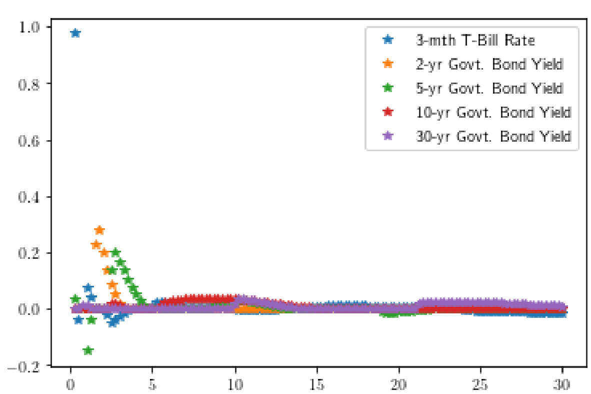
The linear regression approximation is good

```
In [29]: ## do they all sum to one?
         for col in bond_cols:
             if col != 'Overnight Target Rate':
                 print(np.sum(benchmark[col].best_estimator_.coef_))
```

```
0.9819173957211692
1.0019546425141475
1.0101152064845698
1.0080992909655437
1.0091064643521168
```

ZCB Weightings across tenors

```
In [30]: for col in bond_cols:
             if col != 'Overnight Target Rate':
                 plt.plot(zcb_cols, benchmark[col].best_estimator_.coef_,'*', label = col)
                 plt.legend();
```



In [31]:
```
benchmark[col].best_estimator_.coef_.shape
```

Out[31]:
```
(120,)
```

In [32]:
```
#Assemble the A matrix
A = np.vstack([benchmark[col].best_estimator_.coef_ for col in bond_cols if col != 'Overnight Target R
ate' ])
```

Finally: given the following equations

$R_t - R_{t-1} = \sum_{i=1}^{k}(A^{\intercal}e_i)F_t^i$

$Y_t - Y_{t-1} = \sum_{i=1}^{k} e_i F_t^i$

the key is to simulate the factors F using a stochastic process and project the distribution to some time horizon τ

Monte Carlo

Hagenbjörk, Meucci, and Litterman either show or assume a zero mean process for the PCA factors. I will do the same. How to select the distribution: maybe use the R package to do this?

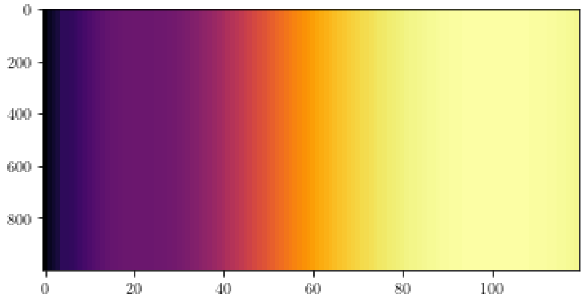
Model # 1: Random Walks

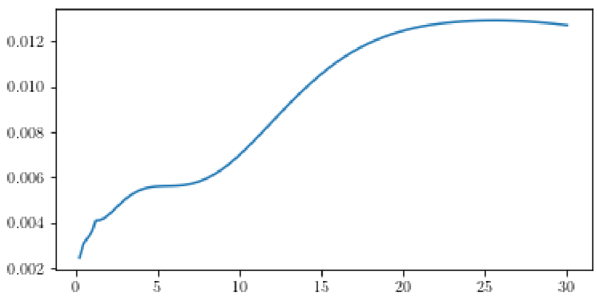
$F_t = \sigma\epsilon_t$

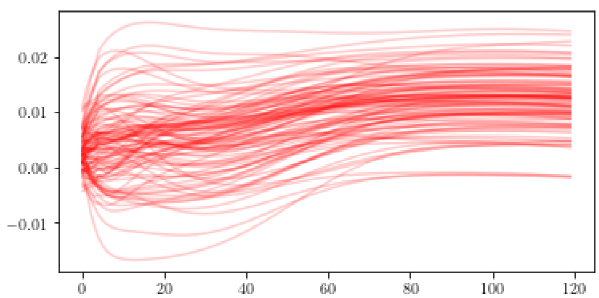
Model # 2: Volatility Clustering

$F_t = \sigma_t\epsilon_t$

In [33]:
```
#Model 1: Note the variance is the eigenvalues
tau = 252
N_rep = 1000
start_date = data.Date.iloc[-1]+BDay(1)
end_date = data.Date.iloc[-1]+BDay(tau)
date_grid = pd.bdate_range(start_date,end_date)
def RW(tau, N_rep, F, E, data, zcb_cols, bond_cols, plots = False):
        #estimation
    eigen_values = pd.DataFrame(F, columns = ["F1", "F2", "F3"], index = X.index).var()

    y_end = data[zcb_cols].iloc[-1,:].values*np.ones([N_rep, len(zcb_cols)])
    y_end = y_end.astype(float)
    benchmark_bond_cols = [x for x in bond_cols if x != 'Overnight Target Rate']
    r_end = data[benchmark_bond_cols].iloc[-1,:].values*np.ones([N_rep, len(benchmark_bond_cols)])
    r_end = r_end.astype(float)

    #sampling/distrubtion selection
    F_out = np.random.multivariate_normal(np.zeros(len(eigen_values)), np.diag(eigen_values), [tau, N_
rep]);
    X_out = np.matmul(F_out, E.transpose())
    X_out_R = np.matmul(X_out, A.transpose())

    Y_out = np.zeros(X_out.shape)
    R_out = np.zeros(X_out_R.shape)

    #projection via random walk
    for t in range(tau):
        if t == 0:
            Y_out[t,:,:] = X_out[t,:,:] + y_end
            R_out[t,:,:] = X_out_R[t,:,:] + r_end
        else:
            Y_out[t,:,:] = X_out[t,:,:] + Y_out[t-1,:,:]
            R_out[t,:,:] = X_out_R[t,:,:] + R_out[t-1,:,:]

    if plots:
        plt.figure(figsize = (6,3))
        plt.imshow(y_end,aspect='auto',  cmap='inferno');
        plt.figure(figsize = (6,3))
        plt.plot(yc.iloc[-1,1:]);
        plt.figure(figsize = (6,3))
        plt.plot(Y_out[tau-1,0:100,:].transpose(), alpha = 0.2, color = 'red');

    return Y_out, R_out

Y_out, R_out = RW(tau, N_rep, F, E, data, zcb_cols, bond_cols, plots = True);
```



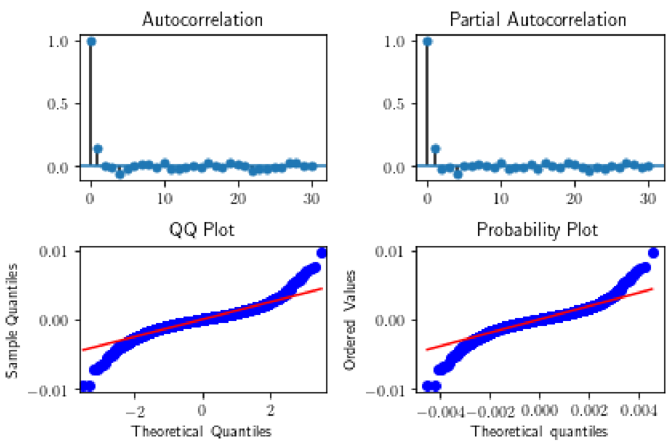




Time Series Exploration

In [68]:

```
#Model 2: Note the variance follows a garch model


def tsplot(y, name, lags=None, figsize=(6, 4)):
    if not isinstance(y, pd.Series):
        y = pd.Series(y)

    fig = plt.figure(figsize=figsize)
    #mpl.rcParams['font.family'] = 'Ubuntu Mono'
    layout = (2, 2)
    #ts_ax = plt.subplot2grid(layout, (0, 0), colspan=2)
    acf_ax = plt.subplot2grid(layout, (0, 0))
    pacf_ax = plt.subplot2grid(layout, (0, 1))
    qq_ax = plt.subplot2grid(layout, (1, 0))
    pp_ax = plt.subplot2grid(layout, (1, 1))

    #y.plot(ax=ts_ax)
    #ts_ax.set_title('Time Series Analysis Plots')
    smt.graphics.plot_acf(y, lags=lags, ax=acf_ax, alpha=0.5)
    smt.graphics.plot_pacf(y, lags=lags, ax=pacf_ax, alpha=0.5)
    sm.qqplot(y, line='s', ax=qq_ax)
    qq_ax.set_title('QQ Plot')
    scs.probplot(y, sparams=(y.mean(), y.std()), plot=pp_ax)
    plt.tight_layout()
    plt.savefig(name, dpi = 300)
    return

tsplot(F[:,1], 'NormalQQ.png',lags=30)
```



Garch Model Selection

$$
\begin{aligned}
F_t^k &= \epsilon_t \\
\epsilon_t &= \sigma_t e_t \\
\sigma_t^\lambda &= \omega + \sum_{i=1}^{p} \alpha_i |\epsilon_{t-i}|^\lambda \\
&\quad + \sum_{j=1}^{o} \gamma_j |\epsilon_{t-j}|^\lambda I\left[\epsilon_{t-j} < 0\right] + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^\lambda
\end{aligned}
$$

Reading the source code...

code below is the same as:

```
res.conditional_volatility
```

the code below:

```
sigma2 = np.zeros_like(res.resid)
from arch.utility.array import ensure1d

params = ensure1d(res.params, "params", False)
                    sigma2 = am.volatility.compute_variance(params,
                    resids = res.resid, sigma2 = sigma_2,
                    backcast=am.volatility.backcast(res.resid),
                    var_bounds=am.volatility.variance_bounds(res.resid))

sigma2**0.5
```

```
def cpt_losses(am, res):
    lls = -0.5 * (log(2 * np.pi) + log(sigma2) + resids ** 2.0 / sigma2)
```

Model Selection

For each independent factor - capture the intertemporal effects via Garch modelling

- test against student vs normal distribution
- and values of $(p, o, q) \in \{1, 2, 3, 4\} \times \{0, 1, 2, 3\} \times \{1, 2, 3, 4\}$

The models are estimated on the first half of the observations and predictions are made for the second half. Finally, losses are constructed from these predictions.

using the model selection approach pioneered by Hansen https://onlinelibrary.wiley.com/doi/full/10.1002/jae.800 (https://onlinelibrary.wiley.com/doi/full/10.1002/jae.800)

In [51]:
```python
def compute_garch_model(split, scaledF, p, o, q, dist, plots = False):

    am = arch_model(scaledF)
    am = ZeroMean(scaledF)
    am.volatility = GARCH(p,o,q)
    am.distribution = dist

    res = am.fit(update_freq=5, disp='off', first_obs=0, last_obs=split)
    #print(res.summary())
    #res.plot();
    loss = 0;
    if split < len(scaledF):
        f = res.forecast(start=split+1, horizon=1)
        realized = (scaledF[split+1:]**2)**0.5
        forecasted = f.variance.values**0.5
        forecasted = forecasted[~np.isnan(forecasted)]
        #loss = np.log(forecasted**2) + (realized**2)/(forecasted**2)
        loss = np.abs(forecasted - realized)
    if plots:
        plt.plot(realized,'-');
        plt.plot(forecasted)
        plt.figure()
        print(res.summary())
        res.plot();
        tsplot(res.resid, lags=30)
    return loss, res, am

def compute_constant_loss(scaledF, split):
    realized = (scaledF[split+1:]**2)**0.5
    forecasted = (scaledF[:split]**2)**0.5
    loss = np.abs(forecasted - realized)
    return loss

def generateMCS(scaledF, distributions, ps, os, qs, split, printMCS = False, plotFinalModel = False):

    model_num = 0
    losses = []
    Model_lookup = {}

    for dist, p, o, q in product(distributions, ps, os, qs):

        loss, res, am = compute_garch_model(split, scaledF, p, o, q, dist, plots = False)
        Model_lookup[model_num] = [dist, p, o, q, dist.name, res.bic]
        losses.append(loss)
        model_num += 1
    lss = np.array(losses).transpose()
    #Adding the constant model in there for good measure
    Model_lookup[model_num] = ['RandomWalk', 0, 0, 0, 'rw', 0]
    loss = compute_constant_loss(scaledF, split)
    losses.append(loss)

    lss = np.array(losses).transpose()

    #perform the model confidence set procedure
    alpha = 0.05
    mcs = MCS(lss, size=alpha)
    mcs.compute()
    included = mcs.included
    excluded = mcs.excluded
    if printMCS:
        print('MCS P-values')
        print(mcs.pvalues)
        print('Included')
        print([Model_lookup[model][1:-1] for model in included])
        print('Excluded')
        print([Model_lookup[model][1:-1] for model in excluded])

    #select the minimum BIC model for use
    BICs = [Model_lookup[model][-1] for model_num in included]
    min_model = BICs.index(min(BICs))
    #get the lowest BIC model info
```

```python
    [dist, p, o, q] = Model_lookup[min_model][0:4]
    #train the lowest BIC model on the entire dataset
    loss, res, am = compute_garch_model(len(scaledF), scaledF, p, o, q, dist, plots = plotFinalModel)
    return res, am
```

In [52]:
```python
def generate_simulated_factor(F_i, printsMCS = False, printFinalModel = False):

    scaledF= (10**4)*F_i
    split = int(len(F)/2)
    distributions = [Normal(), StudentsT()]
    ps, os, qs = [[1,2], [0,1], [1,2]]

    res, am = generateMCS(scaledF, distributions, ps, os, qs, split, printsMCS, printFinalModel)

    Factor_Sims = np.zeros([N_rep, tau])
    for i in range(N_rep):
        garch_sim = am.simulate(res.params, tau, burn = 0)
        #Factor_Sims[i,:] = ((garch_sim.volatility)**0.5)*garch_sim.errors
        Factor_Sims[i,:] = garch_sim.errors
    return [Factor_Sims.T/(10**4), res, am]
```

In [53]:
```python
def Garch_Factors(tau, N_rep, F, E, data, zcb_cols, bond_cols, MonteCarloplots = False, printsMCS = Fa
lse, printFinalModel = False):
        #estimation
    eigen_values = pd.DataFrame(F, columns = ["F1", "F2", "F3"], index = X.index).var()

    y_end = data[zcb_cols].iloc[-1,:].values*np.ones([N_rep, len(zcb_cols)])
    y_end = y_end.astype(float)
    benchmark_bond_cols = [x for x in bond_cols if x != 'Overnight Target Rate']
    r_end = data[benchmark_bond_cols].iloc[-1,:].values*np.ones([N_rep, len(benchmark_bond_cols)])
    r_end = r_end.astype(float)

    results = []
    models = []
    fs = []
    for i in range(len(eigen_values)):
        [sims, res, am] = generate_simulated_factor(F[:,i], printsMCS, printFinalModel)
        results.append(res)
        models.append(am)
        fs.append(sims)
    F_out = np.moveaxis(np.array(fs), 0, -1)

    X_out = np.matmul(F_out, E.transpose())
    X_out_R = np.matmul(X_out, A.transpose())

    Y_out = np.zeros(X_out.shape)
    R_out = np.zeros(X_out_R.shape)

    #projection via random walk
    for t in range(tau):
        if t == 0:
            Y_out[t,:,:] = X_out[t,:,:] + y_end
            R_out[t,:,:] = X_out_R[t,:,:] + r_end
        else:
            Y_out[t,:,:] = X_out[t,:,:] + Y_out[t-1,:,:]
            R_out[t,:,:] = X_out_R[t,:,:] + R_out[t-1,:,:]

    if MonteCarloplots:
        plt.figure(figsize = (6,3))
        plt.imshow(y_end,aspect='auto',  cmap='inferno');
        plt.figure(figsize = (6,3))
        plt.plot(yc.iloc[-1,1:]);
        plt.figure(figsize = (6,3))
        plt.plot(Y_out[tau-1,0:100,:].transpose(), alpha = 0.2, color = 'red');

    return Y_out, R_out, results, models
```

Print out the Model Confidence Results

In [54]:
```
Y_out, R_out, GarchResults, GarchModels= Garch_Factors(tau, N_rep, F, E, data, zcb_cols, bond_cols,  M
onteCarloplots = False, printsMCS = True, printFinalModel = False)
```

```
MCS P-values
             Pvalue
Model index
4             0.000
0             0.000
12            0.000
8             0.000
13            0.000
9             0.000
14            0.000
10            0.000
2             0.000
6             0.000
15            0.000
11            0.000
16            0.000
7             0.001
1             0.258
5             0.258
3             1.000
Included
[[1, 0, 2, 'Normal'], [1, 1, 2, 'Normal'], [2, 0, 2, 'Normal']]
Excluded
[[1, 0, 1, 'Normal'], [1, 1, 1, 'Normal'], [2, 0, 1, 'Normal'], [2, 1, 1, 'Normal'], [2, 1, 2, 'Norma
l'], [1, 0, 1, "Standardized Student's t"], [1, 0, 2, "Standardized Student's t"], [1, 1, 1, "Standar
dized Student's t"], [1, 1, 2, "Standardized Student's t"], [2, 0, 1, "Standardized Student's t"],
[2, 0, 2, "Standardized Student's t"], [2, 1, 1, "Standardized Student's t"], [2, 1, 2, "Standardized
Student's t"], [0, 0, 0, 'rw']]
MCS P-values
             Pvalue
Model index
10            0.000
13            0.000
7             0.000
8             0.000
12            0.000
4             0.000
0             0.000
14            0.000
6             0.000
2             0.000
11            0.000
5             0.000
16            0.000
3             0.004
15            0.105
1             0.565
9             1.000
Included
[[1, 0, 2, 'Normal'], [1, 0, 2, "Standardized Student's t"], [2, 1, 2, "Standardized Student's t"]]
Excluded
[[1, 0, 1, 'Normal'], [1, 1, 1, 'Normal'], [1, 1, 2, 'Normal'], [2, 0, 1, 'Normal'], [2, 0, 2, 'Norma
l'], [2, 1, 1, 'Normal'], [2, 1, 2, 'Normal'], [1, 0, 1, "Standardized Student's t"], [1, 1, 1, "Stan
dardized Student's t"], [1, 1, 2, "Standardized Student's t"], [2, 0, 1, "Standardized Student's t"],
[2, 0, 2, "Standardized Student's t"], [2, 1, 1, "Standardized Student's t"], [0, 0, 0, 'rw']]
MCS P-values
             Pvalue
Model index
9             0.000
12            0.000
8             0.000
14            0.000
10            0.000
2             0.000
6             0.000
0             0.000
4             0.000
16            0.000
15            0.000
3             0.281
5             0.457
```

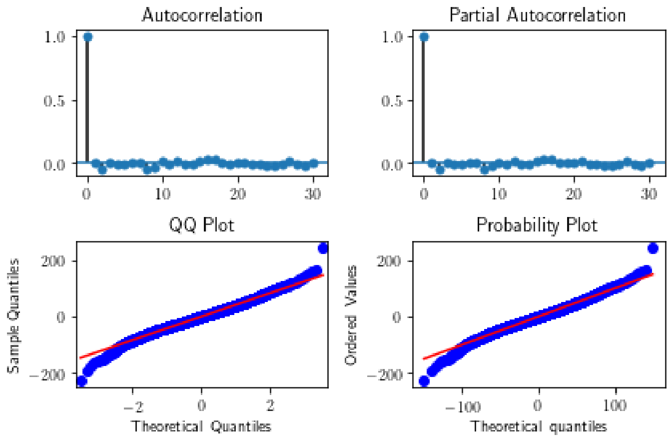
```
7            0.541
1            0.869
13           0.869
11           1.000
Included
[[1, 0, 2, 'Normal'], [1, 1, 2, 'Normal'], [2, 0, 2, 'Normal'], [2, 1, 2, 'Normal'], [1, 1, 2, "Stand
ardized Student's t"], [2, 0, 2, "Standardized Student's t"]]
Excluded
[[1, 0, 1, 'Normal'], [1, 1, 1, 'Normal'], [2, 0, 1, 'Normal'], [2, 1, 1, 'Normal'], [1, 0, 1, "Stand
ardized Student's t"], [1, 0, 2, "Standardized Student's t"], [1, 1, 1, "Standardized Student's t"],
[2, 0, 1, "Standardized Student's t"], [2, 1, 1, "Standardized Student's t"], [2, 1, 2, "Standardized
Student's t"], [0, 0, 0, 'rw']]
```

Results for the factors

In [55]:
```python
k=0
for res in GarchResults:
    print("Factor: " +str(k+1))
    print(res)
    k+=1
```

```
Factor: 1
                       Zero Mean - GARCH Model Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.000
Mean Model:                 Zero Mean   Adj. R-squared:                  0.000
Vol Model:                      GARCH   Log-Likelihood:               -23217.9
Distribution:                  Normal   AIC:                           46441.7
Method:            Maximum Likelihood   BIC:                           46461.0
                                        No. Observations:                 4547
Date:                  Sun, Apr 19 2020  Df Residuals:                    4544
Time:                        21:50:57   Df Model:                            3
                              Volatility Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
omega         23.3359     10.754      2.170  3.000e-02    [  2.259, 44.413]
alpha[1]       0.0427  9.357e-03      4.568  4.913e-06 [2.441e-02,6.109e-02]
beta[1]        0.9439  1.460e-02     64.639      0.000    [  0.915,  0.973]
============================================================================

Covariance estimator: robust
Factor: 2
                       Zero Mean - GARCH Model Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.000
Mean Model:                 Zero Mean   Adj. R-squared:                  0.000
Vol Model:                      GARCH   Log-Likelihood:               -17425.7
Distribution:                  Normal   AIC:                           34857.4
Method:            Maximum Likelihood   BIC:                           34876.7
                                        No. Observations:                 4547
Date:                  Sun, Apr 19 2020  Df Residuals:                    4544
Time:                        21:51:00   Df Model:                            3
                              Volatility Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
omega          1.2608      0.680      1.854  6.381e-02  [-7.241e-02,  2.594]
alpha[1]       0.0526  1.458e-02      3.606  3.113e-04 [2.400e-02,8.115e-02]
beta[1]        0.9409  1.722e-02     54.633      0.000    [  0.907,  0.975]
============================================================================

Covariance estimator: robust
Factor: 3
                       Zero Mean - GARCH Model Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.000
Mean Model:                 Zero Mean   Adj. R-squared:                  0.000
Vol Model:                      GARCH   Log-Likelihood:               -13193.9
Distribution:                  Normal   AIC:                           26393.9
Method:            Maximum Likelihood   BIC:                           26413.1
                                        No. Observations:                 4547
Date:                  Sun, Apr 19 2020  Df Residuals:                    4544
Time:                        21:51:03   Df Model:                            3
                             Volatility Model
==========================================================================
                 coef    std err          t      P>|t|    95.0% Conf. Int.
--------------------------------------------------------------------------
omega          1.3619      0.674      2.019  4.345e-02 [4.004e-02,  2.684]
alpha[1]       0.1314  3.542e-02      3.710  2.074e-04 [6.199e-02,  0.201]
beta[1]        0.8145  5.554e-02     14.664  1.088e-48  [  0.706,  0.923]
==========================================================================

Covariance estimator: robust
```

In [67]:
```
tsplot(GarchResults[0].resid, 'Garch.png',lags=30)
```



TD Forecasts

In [57]:
```
TD #(forecasts as of march) and are end of period
```

Out[57]:

	Year	Quarter	3-mth T-Bill Rate	2-yr Govt. Bond Yield	5-yr Govt. Bond Yield	10-yr Govt. Bond Yield	30-yr Govt. Bond Yield	10-yr-2-yr Govt Spread	Quarter End Date
0	2020	Q1	0.0040	0.0040	0.0050	0.0065	0.0100	0.0025	03/31/2020
1	2020	Q2	0.0015	0.0050	0.0060	0.0075	0.0105	0.0031	06/30/2020
2	2020	Q3	0.0015	0.0065	0.0075	0.0090	0.0115	0.0025	09/30/2020
3	2020	Q4	0.0015	0.0080	0.0090	0.0105	0.0130	0.0025	12/31/2020
4	2021	Q1	0.0015	0.0100	0.0110	0.0125	0.0150	0.0025	03/31/2021
5	2021	Q2	0.0028	0.0120	0.0130	0.0145	0.0170	0.0025	06/30/2021
6	2021	Q3	0.0053	0.0135	0.0145	0.0160	0.0185	0.0025	09/30/2021
7	2021	Q4	0.0078	0.0150	0.0160	0.0175	0.0200	0.0025	12/31/2021

Relative Entropy Calibration using Covid 19 Forecasts

Given the simulations over τ for the benchmark rates, the question is how can one incorporte expertise into the simulations? One approach that has been used is to assign probabilities to each simulation and minimally distort them to meet the constraints imposed by expertise. The measure of probability distortion proposed by Avellenada has its roots in information theory and allows for a tractable solution for the twisted measure via duality.

That is, for a given initial distribution $\mathbb{P}_0 \in \mathcal{P}$ the problem is:

$$\min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\log(\frac{d\mathbb{P}}{d\mathbb{P}_0})]$$
$$\text{such that} \mathbb{E}^{\mathbb{P}}[G_i] = C_i, \quad i = 1, 2, \ldots, N$$

Where the $\int G_i(X)f(X)dX = C_i$ corresponds to the moment constraints imposed by prices or in our case expertise. Avellenada shows by introducing lagrange parameters that the above optimization is equivalent to:

$$\min_{\lambda} \log(Z(\lambda)) - \lambda^{\intercal} C$$

where $Z(\lambda) = \mathbb{E}^{\mathbb{P}_0}[e^{\lambda^{\intercal} G}]$ and the optimal density for a given λ is $\mathbb{P}_{\lambda} = \frac{1}{Z(\lambda)}\mathbb{P}_0(\omega)e^{\lambda^{\intercal} G(w)}$

First some notation: Denoting $R_t^{(n)}$ as the benchmark rates for replication n, p_n to be the discrete probabilities of rate path n, expressing the economist views on rate type k at time t as $\mathbb{E}^{\mathbb{P}}[R_{kt}] = \bar{r}_{kt}$. Therefore to calibrate the n simulated interest rates the dual problem takes the form:

...

In [58]:
```
benchmark_bond_cols = [x for x in bond_cols if x != 'Overnight Target Rate']
data[benchmark_bond_cols]
```

Out[58]:

	3-mth T-Bill Rate	2-yr Govt. Bond Yield	5-yr Govt. Bond Yield	10-yr Govt. Bond Yield	30-yr Govt. Bond Yield
0	0.0204	0.0349	0.0483	0.0553	0.0576
1	0.0202	0.0334	0.0473	0.0548	0.0573
2	0.0200	0.0328	0.0467	0.0543	0.0569
3	0.0200	0.0326	0.0465	0.0540	0.0565
4	0.0202	0.0334	0.0473	0.0547	0.0570
...	...	...	...	...	...
4553	0.0045	0.0059	0.0075	0.0084	0.0134
4554	0.0021	0.0044	0.0062	0.0072	0.0128
4555	0.0022	0.0044	0.0062	0.0075	0.0132
4556	0.0021	0.0042	0.0060	0.0071	0.0132
4557	0.0020	0.0040	0.0055	0.0063	0.0121

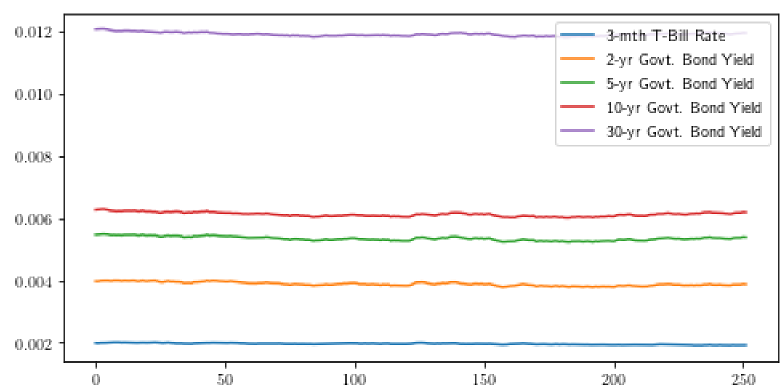
4558 rows × 5 columns

In [59]:
```
Simulated_Benchmarks = pd.DataFrame(R_out.mean(axis=1), index = date_grid, columns = benchmark_bond_co
ls)
plt.figure(figsize = (8,4))
plt.plot(R_out.mean(axis=1));
plt.legend(benchmark_bond_cols);
Mean_Rates = Simulated_Benchmarks.reset_index().melt(id_vars=['index'])
Mean_Rates
```

Out[59]:

	index	variable	value
0	2020-04-02	3-mth T-Bill Rate	0.001992
1	2020-04-03	3-mth T-Bill Rate	0.001987
2	2020-04-06	3-mth T-Bill Rate	0.001996
3	2020-04-07	3-mth T-Bill Rate	0.001995
4	2020-04-08	3-mth T-Bill Rate	0.001999
...	...	...	...
1255	2021-03-15	30-yr Govt. Bond Yield	0.011943
1256	2021-03-16	30-yr Govt. Bond Yield	0.011945
1257	2021-03-17	30-yr Govt. Bond Yield	0.011952
1258	2021-03-18	30-yr Govt. Bond Yield	0.011964
1259	2021-03-19	30-yr Govt. Bond Yield	0.011962

1260 rows × 3 columns



In [61]:
```python
def nearest(items, pivot):
    return min(items, key=lambda x: abs(x - pivot))
```

In [62]:
```python
## Stack TD forcasts in C
C = TD.melt(id_vars=['Year', 'Quarter', 'Quarter End Date'])
C['Quarter End Date'] = C['Quarter End Date'].astype('datetime64[ns]')

#C = C[C['variable'] != '3-mth T-Bill Rate']
C = C[C['Quarter End Date'] != '2020-03-31']
C = C[C['Quarter End Date'] <= date_grid.max()]
C['Nearest_Buisness_Day'] = C['Quarter End Date'].apply(lambda x: nearest(date_grid, x))

Constraints = pd.merge(C,Mean_Rates, how = 'inner',  left_on =['Nearest_Buisness_Day', 'variable'], ri
ght_on = ['index', 'variable'])

#get indices for the mean computation
Constrainted_Rates = R_out[date_grid.isin(Constraints['Nearest_Buisness_Day']),:,:]
Constrainted_Rates= np.moveaxis(Constrainted_Rates, 2,0)
Constrainted_Rates = np.reshape(Constrainted_Rates, [Constrainted_Rates.shape[0]*Constrainted_Rates.sh
ape[1],Constrainted_Rates.shape[2]] )
Constrainted_Rates.mean(axis=1)
```

Out[62]:
```
array([0.00197304, 0.00197635, 0.00193583, 0.00391892, 0.00390156,
       0.00380629, 0.00538514, 0.00535389, 0.00526549, 0.00615752,
       0.00612183, 0.00605871, 0.01193456, 0.0119082 , 0.0118658 ])
```

In [63]:
```python
Constraints
```

Out[63]:

	Year	Quarter	Quarter End Date	variable	value_x	Nearest_Buisness_Day	index	value_y
0	2020	Q2	2020-06-30	3-mth T-Bill Rate	0.0015	2020-06-30	2020-06-30	0.001973
1	2020	Q3	2020-09-30	3-mth T-Bill Rate	0.0015	2020-09-30	2020-09-30	0.001976
2	2020	Q4	2020-12-31	3-mth T-Bill Rate	0.0015	2020-12-31	2020-12-31	0.001936
3	2020	Q2	2020-06-30	2-yr Govt. Bond Yield	0.0050	2020-06-30	2020-06-30	0.003919
4	2020	Q3	2020-09-30	2-yr Govt. Bond Yield	0.0065	2020-09-30	2020-09-30	0.003902
5	2020	Q4	2020-12-31	2-yr Govt. Bond Yield	0.0080	2020-12-31	2020-12-31	0.003806
6	2020	Q2	2020-06-30	5-yr Govt. Bond Yield	0.0060	2020-06-30	2020-06-30	0.005385
7	2020	Q3	2020-09-30	5-yr Govt. Bond Yield	0.0075	2020-09-30	2020-09-30	0.005354
8	2020	Q4	2020-12-31	5-yr Govt. Bond Yield	0.0090	2020-12-31	2020-12-31	0.005265
9	2020	Q2	2020-06-30	10-yr Govt. Bond Yield	0.0075	2020-06-30	2020-06-30	0.006158
10	2020	Q3	2020-09-30	10-yr Govt. Bond Yield	0.0090	2020-09-30	2020-09-30	0.006122
11	2020	Q4	2020-12-31	10-yr Govt. Bond Yield	0.0105	2020-12-31	2020-12-31	0.006059
12	2020	Q2	2020-06-30	30-yr Govt. Bond Yield	0.0105	2020-06-30	2020-06-30	0.011935
13	2020	Q3	2020-09-30	30-yr Govt. Bond Yield	0.0115	2020-09-30	2020-09-30	0.011908
14	2020	Q4	2020-12-31	30-yr Govt. Bond Yield	0.0130	2020-12-31	2020-12-31	0.011866

In [64]:
```
(Constraints.value_y - Constrainted_Rates.mean(axis=1)) ##Proper alignment
```

Out[64]:
```
0     0.000000e+00
1     8.673617e-19
2    -2.168404e-19
3    -4.336809e-18
4     8.673617e-19
5     4.336809e-19
6    -3.469447e-18
7    -4.336809e-18
8     8.673617e-19
9     0.000000e+00
10    8.673617e-19
11    2.602085e-18
12    5.204170e-18
13    5.204170e-18
14    8.673617e-18
Name: value_y, dtype: float64
```

In [66]:
```
plt.title("Dsitribution of Interest Rates Against the Forecasts")
plt.plot(Constrainted_Rates, 'red', alpha = 0.05);
plt.plot(Constraints.value_x, 'blue');
plt.xlabel("Constraint Number")
plt.ylabel("Yield")
plt.savefig("constraints.png", dpi = 300)
```

In [91]:
```python
import cvxpy as cp
def min_rel_entropy_primal(R, C):

    (m, N) = R.shape
    p = cp.Variable(N)
    epsi = 0.125/100
    #objective = cp.Minimize(cp.square(cp.norm(cp.matmul(cov_half, x),2) + ((delta)**0.5)*cp.norm(x,
1)))
    constraints = [p >= 0,
                   cp.sum(p) == 1,
                    R@p <= C+epsi, R@p >= C-epsi ]
    objective = cp.Minimize(cp.sum(cp.kl_div(p, (1/N)*np.ones(N))))
    prob = cp.Problem(objective, constraints)
    prob.solve(verbose=True,solver=cp.SCS, max_iters=2000 )
    if p.value is None:
        print("Problem Infeasible")
    return p
p = min_rel_entropy_primal(Constrainted_Rates, Constraints.value_x)
```

```
----------------------------------------------------------------------------
	SCS v2.1.1 - Splitting Conic Solver
	(c) Brendan O'Donoghue, Stanford University, 2012
----------------------------------------------------------------------------
Lin-sys: sparse-direct, nnz in A = 34000
eps = 1.00e-04, alpha = 1.50, max_iters = 2000, normalize = 1, scale = 1.00
acceleration_lookback = 0, rho_x = 1.00e-03
Variables n = 2000, constraints m = 5031
Cones:	primal zero / dual free vars: 1
	linear vars: 2030
	exp vars: 3000, dual exp vars: 0
WARN: aa_init returned NULL, no acceleration applied.
Setup time: 1.09e-02s
----------------------------------------------------------------------------
 Iter | pri res | dua res | rel gap | pri obj | dua obj | kap/tau | time (s)
----------------------------------------------------------------------------
     0| 8.21e-01  3.24e+00  9.78e-01 -4.15e+01  2.06e+00  2.31e-14  1.77e-02
   100| 3.78e-04  2.20e-02  6.98e-03 -7.64e-01 -7.82e-01  1.67e-15  3.99e-01
   200| 6.59e-05  3.59e-03  2.07e-03 -7.53e-01 -7.48e-01  1.68e-15  7.99e-01
   300| 1.68e-05  1.39e-03  7.22e-04 -7.53e-01 -7.54e-01  6.23e-15  1.23e+00
   400| 5.68e-06  1.89e-04  9.25e-05 -7.53e-01 -7.52e-01  1.68e-15  1.68e+00
   460| 2.81e-06  6.01e-05  3.47e-05 -7.53e-01 -7.53e-01  9.59e-15  1.92e+00
----------------------------------------------------------------------------
Status: Solved
Timing: Solve time: 1.92e+00s
	Lin-sys: nnz in L factor: 41496, avg solve time: 8.82e-05s
	Cones: avg projection time: 3.98e-03s
	Acceleration: avg step time: 1.39e-08s
----------------------------------------------------------------------------
Error metrics:
dist(s, K) = 6.9765e-18, dist(y, K*) = 0.0000e+00, s'y/|s||y| = -3.3941e-12
primal res: |Ax + s - b|_2 / (1 + |b|_2) = 2.8128e-06
dual res:   |A'y + c|_2 / (1 + |c|_2) = 6.0101e-05
rel gap:    |c'x + b'y| / (1 + |c'x| + |b'y|) = 3.4749e-05
----------------------------------------------------------------------------
c'x = -0.7526, -b'y = -0.7526
============================================================================
```

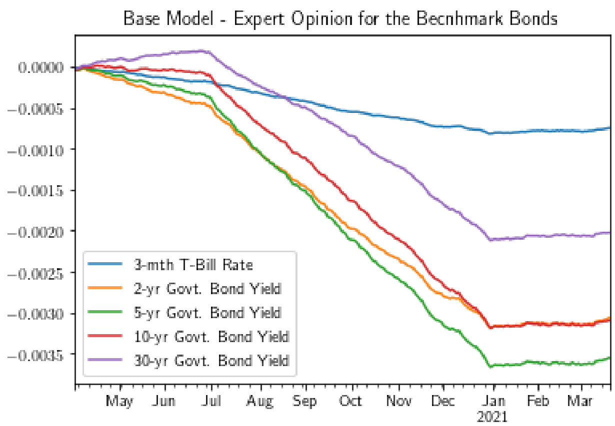
In [92]:
```
Simulated_Benchmarks
```

Out[92]:

	3-mth T-Bill Rate	2-yr Govt. Bond Yield	5-yr Govt. Bond Yield	10-yr Govt. Bond Yield	30-yr Govt. Bond Yield
2020-04-02	0.001992	0.003977	0.005475	0.006278	0.012082
2020-04-03	0.001987	0.003975	0.005479	0.006286	0.012089
2020-04-06	0.001996	0.003990	0.005490	0.006292	0.012093
2020-04-07	0.001995	0.003996	0.005498	0.006300	0.012097
2020-04-08	0.001999	0.003995	0.005491	0.006287	0.012083
...	...	...	...	...	...
2021-03-15	0.001923	0.003869	0.005375	0.006178	0.011943
2021-03-16	0.001924	0.003871	0.005376	0.006179	0.011945
2021-03-17	0.001925	0.003871	0.005378	0.006183	0.011952
2021-03-18	0.001928	0.003889	0.005399	0.006202	0.011964
2021-03-19	0.001925	0.003882	0.005393	0.006198	0.011962

252 rows × 5 columns

In [93]:
```
#Simulated_Benchmarks.plot()
Forecasted_Benchmarks = pd.DataFrame(np.matmul(np.moveaxis(R_out, 1,2), p.value), index = date_grid, c
olumns = benchmark_bond_cols)

(Simulated_Benchmarks- Forecasted_Benchmarks).plot(title = "Base Model - Expert Opinion for the Becnhm
ark Bonds");
```



Applications

Evaluation of the Difference between the short and long rates

Quantile Estimation of Term Shift (a critical quantity in determining the variable vs fixed decision)

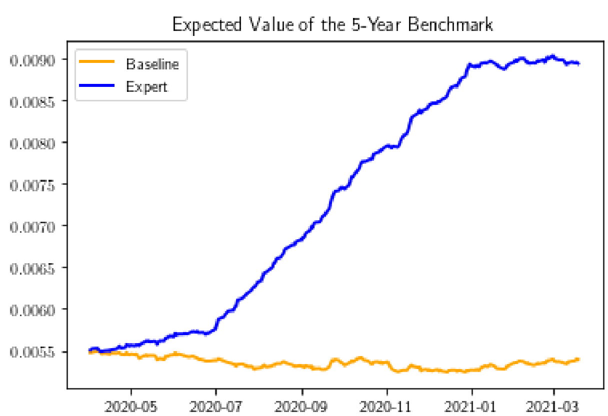
Probability that over the next y months 5 year t bill spread exceeds x?

Probability of duration shift

Caveats

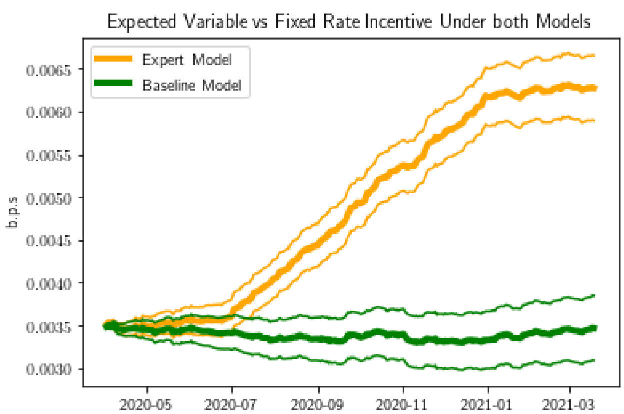
In [179]:
```
def t_mean_confidence_interval(data,alpha, std_err = 0):
    #If the user specifies a standard error
    #then the confidence interval will be defined
    #appropriately
    a = 1.0*np.array(data)
    n = len(a)
    if std_err == 0:
        m, se = np.mean(a), np.std(a,ddof=1)
        h = stats.t.ppf(1-alpha/2, n-1)*se/np.sqrt(n)
    else:
        m = np.mean(a)
        h = stats.t.ppf(1-alpha/2, n-1)*std_err
    return m, "+/-", h

def weighted_avg_and_std(values, weights, alpha):
    """
    Return the weighted average and standard deviation.

    values, weights -- Numpy ndarrays with the same shape.
    """
    average = np.average(values, weights=weights, axis = 1)
    # Fast and numerically precise:
    variance = np.average(((values.T-average).T)**2, weights=weights, axis = 1)

    std = variance**0.5
    n = values.shape[1]
    std_error = std/np.sqrt(n)

    h = stats.t.ppf(1-alpha/2, n-1)*std_error

    return (average, std, h)
    #return values, average
### Variable vs Fixed Decision
BR = np.moveaxis(R_out, 1,2)
```

In [181]:
```
## plotting some paths
import scipy.stats as stats
plt.title("Expected Value of the 5-Year Benchmark")
plt.plot(date_grid,BR[:,2,:].mean(axis=1), alpha = 1, color = 'orange', label = 'Baseline', linewidth
= 2);
plt.plot(date_grid, np.matmul(BR[:,2,:], p.value), alpha = 1, color = 'blue', label = "Expert", linewi
dth = 2);
plt.legend();
plt.savefig("Comparison.png")
```

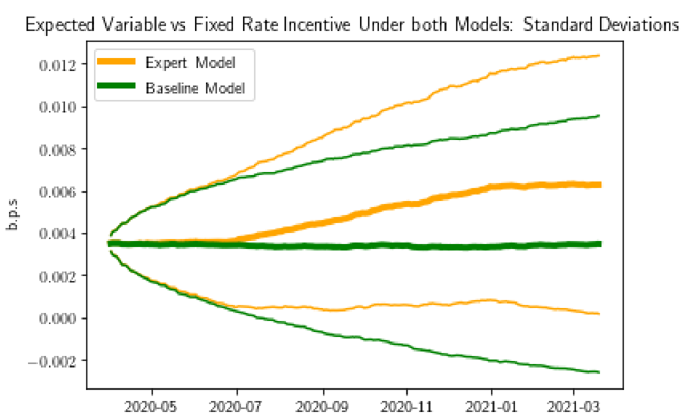


Expected Value of the Variable vs Fixed Incentive

In [200]:
```python
average, std, h = weighted_avg_and_std(BR[:,2,:] - BR[:,0,:], p.value, 0.05)
average_base, std_base, h_base = weighted_avg_and_std(BR[:,2,:] - BR[:,0,:], (1/N_rep)*np.ones_like(p.
value), 0.05)

plt.title("Expected Variable vs Fixed Rate Incentive Under both Models")
plt.plot(date_grid, average, color = 'orange', linewidth = 4, label = "Expert Model")
plt.plot(date_grid, average - h, color = 'orange');
plt.plot(date_grid, average + h, color = 'orange');

plt.plot(date_grid, average_base, color = 'green', linewidth = 4, label = "Baseline Model")
plt.plot(date_grid, average_base - h_base, color = 'green');
plt.plot(date_grid, average_base + h_base, color = 'green');
plt.legend();
plt.ylabel("b.p.s")
plt.savefig("FixedvsVariable.png")
```



In [203]:
```python
plt.title("Expected Variable vs Fixed Rate Incentive Under both Models: Standard Deviations")
plt.plot(date_grid, average, color = 'orange', linewidth = 4, label = "Expert Model")
plt.plot(date_grid, average - std, color = 'orange');
plt.plot(date_grid, average + std, color = 'orange');

plt.plot(date_grid, average_base, color = 'green', linewidth = 4, label = "Baseline Model")
plt.plot(date_grid, average_base - std_base, color = 'green');
plt.plot(date_grid, average_base + std_base, color = 'green');
plt.legend();
plt.ylabel("b.p.s")
plt.savefig("FixedvsVariable_Standard Deviations.png")
```



Expected Value of the Refinancing Incentive (3 year Window)

In [214]:
```
data.Date.isin(date_grid -  pd.DateOffset(years=3))
```

Out[214]:
```
0       False
1       False
2       False
3       False
4       False
        ...
4553    False
4554    False
4555    False
4556    False
4557    False
Name: Date, Length: 4558, dtype: bool
```

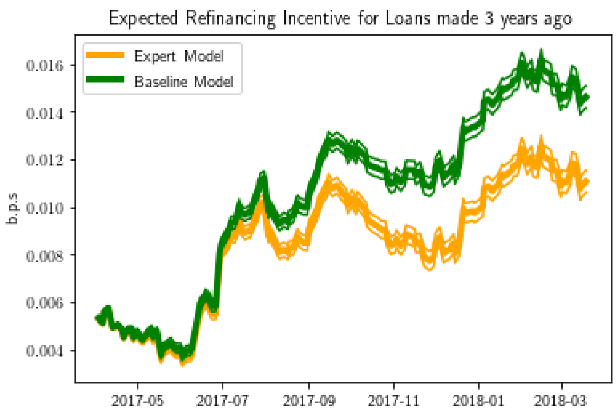
In [221]:
```
BR[date_grid.isin(five_yr_three_years_ago['LaterDate']),2,:]
```

Out[221]:
```
array([[ 0.00632813,  0.00474096,  0.00530862, ...,  0.00772153,
         0.00525143,  0.00518466],
       [ 0.00609271,  0.00417249,  0.00492143, ...,  0.00811551,
         0.00471519,  0.00483338],
       [ 0.00623164,  0.00394734,  0.00480783, ...,  0.00789044,
         0.00410222,  0.00525498],
       ...,
       [ 0.00848946,  0.01394503,  0.00221021, ...,  0.00241012,
         0.00814645, -0.00402648],
       [ 0.00810175,  0.01389651,  0.00148851, ...,  0.00324752,
         0.00839807, -0.00432433],
       [ 0.00819888,  0.01318787,  0.00057309, ...,  0.00357817,
         0.00746091, -0.00460875]])
```

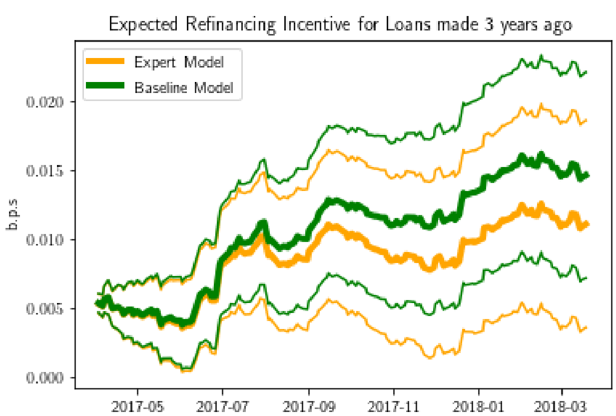
In [232]:
```
five_yr_three_years_ago = data[data.Date.isin(date_grid -  pd.DateOffset(years=3))][['Date', '5-yr Gov
t. Bond Yield']]
five_yr_three_years_ago['LaterDate'] = five_yr_three_years_ago['Date'] +  pd.DateOffset(years=3)

indic = date_grid.isin(five_yr_three_years_ago['LaterDate'])

corresponding_BR = BR[indic,2,:]

Refi = -1*(corresponding_BR - np.expand_dims(five_yr_three_years_ago['5-yr Govt. Bond Yield'].values,
axis = 1))
Refi.shape
```

Out[232]:
```
(141, 1000)
```

In [234]:
```
average, std, h = weighted_avg_and_std(Refi , p.value, 0.05)
average_base, std_base, h_base = weighted_avg_and_std(Refi , (1/N_rep)*np.ones_like(p.value), 0.05)

plt.title("Expected Refinancing Incentive for Loans made 3 years ago")
plt.plot(five_yr_three_years_ago['Date'], average, color = 'orange', linewidth = 4, label = "Expert Mo
del")
plt.plot(five_yr_three_years_ago['Date'], average - h, color = 'orange');
plt.plot(five_yr_three_years_ago['Date'], average + h, color = 'orange');

plt.plot(five_yr_three_years_ago['Date'], average_base, color = 'green', linewidth = 4, label = "Basel
ine Model")
plt.plot(five_yr_three_years_ago['Date'], average_base - h_base, color = 'green');
plt.plot(five_yr_three_years_ago['Date'], average_base + h_base, color = 'green');
plt.legend();
plt.ylabel("b.p.s")
plt.savefig("REFI.png")
```



In [235]:
```
average, std, h = weighted_avg_and_std(Refi , p.value, 0.05)
average_base, std_base, h_base = weighted_avg_and_std(Refi , (1/N_rep)*np.ones_like(p.value), 0.05)

plt.title("Expected Refinancing Incentive for Loans made 3 years ago")
plt.plot(five_yr_three_years_ago['Date'], average, color = 'orange', linewidth = 4, label = "Expert Mo
del")
plt.plot(five_yr_three_years_ago['Date'], average - std, color = 'orange');
plt.plot(five_yr_three_years_ago['Date'], average + std, color = 'orange');

plt.plot(five_yr_three_years_ago['Date'], average_base, color = 'green', linewidth = 4, label = "Basel
ine Model")
plt.plot(five_yr_three_years_ago['Date'], average_base - std_base, color = 'green');
plt.plot(five_yr_three_years_ago['Date'], average_base + std_base, color = 'green');
plt.legend();
plt.ylabel("b.p.s")
plt.savefig("REFI.png")
```



Conclusion

Nobody really knows what will happen during these perilous times. I surely do not know but maybe the expert's have a better idea.