Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: https://www.tandfonline.com/loi/rquf20

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Quantitative Finance, Vol. 10, No. 2, February 2010, 221–234

Routledge
Taylor & Francis Group

Analysis of the rebalancing frequency in log-optimal portfolio selection

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(*Received 22 November 2007; in final form 14 November 2008*)

In a dynamic investment situation, the right timing of portfolio revisions and adjustments is essential to sustain long-term growth. A high rebalancing frequency reduces the portfolio performance in the presence of transaction costs, whereas a low rebalancing frequency entails a static investment strategy that hardly reacts to changing market conditions. This article studies a family of portfolio problems in a Black–Scholes type economy which depend parametrically on the rebalancing frequency. As an objective criterion we use log-utility, which has strong theoretical appeal and represents a natural choice if the primary goal is long-term performance. We argue that continuous rebalancing only slightly outperforms discrete rebalancing if there are no transaction costs and if the rebalancing intervals are shorter than about one year. Our analysis also reveals that diversification has a dual effect on the mean and variance of the portfolio growth rate as well as on their sensitivities with respect to the rebalancing frequency.

Keywords: Portfolio selection; Log utility; Growth-optimal portfolio; Rebalancing frequency; Kelly criterion

1. Introduction

Since the pioneering work of Markowitz (1952), portfolio theory has constituted a favourite topic of finance researchers and practitioners. Its popularity has recently been boosted by the revolution in information technology, which makes it possible to solve large-scale portfolio problems in short time on an ordinary personal computer. As opposed to the traditional static Markowitz approach, the present article addresses a dynamic investment situation, in which an agent periodically rebalances a portfolio in order to maintain a long-term goal for asset allocation. The right choice of a suitable goal (or objective criterion) has been—and still is—a subject of considerable dispute. Under the premise that the agent has a tail preference, thus assessing an investment strategy only on the basis of its long-term performance, one can argue that the best policy is the one which maximizes the expected portfolio growth rate. This implies the use of a so-called log-criterion or log-utility, that is, the agent should maximize the expected logarithm of period wealth over the set of all admissible investment strategies. However, the choice of an adequate objective criterion is not the only critical decision a serious investor must make. Another important choice concerns the frequency of scheduled portfolio revisions and adjustments, the rebalancing frequency. Transaction costs, administrative expenses, taxes, and opportunity costs make frequent rebalancing highly unattractive. Conversely, very infrequent rebalancing may result in inferior portfolio performance, as too much flexibility to react to changes in economic circumstances is sacrificed. Finding the right compromise between the two extremes is a nontrivial problem faced by many finance practitioners, and it is also tied to a number of interesting theoretical questions: How accurate is the continuous-time approximation used in most theoretical work? In other words, can the optimal growth rate of a continuous-time model be reasonably approached by a real investor? Under what circumstances is it admissible to disregard transaction costs? What is the impact of the rebalancing frequency on the optimal portfolio composition and the statistical properties of the portfolio growth rate? In the present article we attempt to address these questions—and some others which arise on

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Quantitative Finance
ISSN 1469–7688 print/ISSN 1469–7696 online © 2010 Taylor & Francis
http://www.informaworld.com
DOI: 10.1080/14697680802629400

the way—for a log-utility investor in a Black–Scholes type economy, that is, under the assumption that asset prices follow geometric Brownian motions. Analytical formulae will be provided for the limiting cases of extremely high and low rebalancing frequencies. Numerical experiments suggest that the obtained first-order approximations are accurate over a relatively large range of rebalancing frequencies.

The log criterion was first suggested by Kelly (1956) in an information theoretical framework and further developed by Latané (1959) and Breiman (1961). The logarithm's superiority to other possible utility functions has also been advocated by Hakansson and Ziemba (1995), Thorp (1971, 2007), and Algoet and Cover (1988). More recent contributions to the theory of log-optimal portfolio selection are reported in Cover and Thomas (1991) and Luenberger (1998), while Dempster *et al.* (2007, 2008) demonstrate that using the log-criterion can—maybe surprisingly—result in positive portfolio growth even if all assets in the market destroy, rather than create, value. However, the log-criterion has also been a source of controversy. Merton and Samuelson, (1974) criticized the popular idea that any utility function of distant future wealth could be replaced by the logarithm (even if one was only interested in short-term decisions). It was shown in Luenberger (1993) that the expected log-criterion is justified if investment opportunities are evaluated only on the basis of long-term results. A recent survey of the theoretical and practical aspects of the log-utility approach as well as an extensive list of additional relevant references can be found in MacLean and Ziemba (2007).

Several attempts have been undertaken to solve Merton-type portfolio models which explicitly include transaction costs; see e.g. Cadenillas (2000) or Muthuraman and Kumar (2006) for a survey of recent developments in this field. Most results are limited to the case of only one risky asset. Davis and Norman (1990), for instance, consider proportional transaction costs, while Korn (1998) addresses situations in which the transaction costs consist of fixed *and* proportional components. There are also a few extensions to multi-stock problems. For example, Liu (2004) solves a portfolio model with proportional transaction costs in an economy of several stocks with independent returns, and Morton and Pliska, (1995) study a multi-asset portfolio problem where the transaction costs are of the management-fee type, i.e. being proportional to the investor's wealth. Moreover, Bielecki and Pliska (2000) elaborate a very general model with both fixed and proportional transaction costs and securities prices that depend on economic factors.

In spite of impressive theoretical advances in recent years, the problem of obtaining optimal rebalancing policies in the presence of transaction costs remains very difficult if the number of stocks rises to a range compatible with practical use. We therefore suggest an analysis of portfolio problems in which rebalancing is free of charge but restricted to certain discrete time points. Such problems are more tractable than those with transaction costs—especially if the underlying asset universe is large—and also reveal under what circumstances market frictions can safely be disregarded. The present article adopts the perspective of a log-utility investor in a frictionless Black–Scholes economy consisting of several assets with correlated Gaussian returns. The portfolio composition is adjusted at equally spaced time points whose spacing is denoted by τ (hence, the rebalancing frequency is given by τ^{-1}). We derive approximate formulae for the optimal investment strategy as well as the mean and variance of the portfolio growth rate, which are correct to first order in τ and which can easily be evaluated for an asset universe comprising several thousand stocks. Numerical experiments suggest that these formulae are very precise if τ is smaller than about a year. Subsequently, we determine the asymptotic properties of the log-optimal portfolio as τ tends to infinity. Interpolation of the two extreme solutions gives us a qualitative understanding of the log-optimal portfolio for all intermediate values of τ. An examination of several examples suggests that continuous-time rebalancing only marginally outperforms discrete-time rebalancing (in a frictionless market without transaction costs) given that the rebalancing intervals are no longer than about one year. As frequent rebalancing is not necessary to sustain portfolio growth in frictionless markets, we conclude that transaction costs often have a marginal effect on portfolio growth in frictional markets. Our analysis further reveals a dual effect of diversification: even though it hardly improves the portfolio's log mean under continuous rebalancing, diversification can virtually offset performance losses due to infrequent rebalancing. The log variance, in contrast, is affected by diversification in the exact opposite way.

Notice that this paper elaborates a theoretical result about an important aspect of portfolio theory, which is valid under the given assumptions. A real investor might face additional hurdles that are disregarded in our analysis. In particular, our conclusions may have to be revised if the geometric Brownian motion model of asset prices is dismissed or if the parameters of the asset price processes are no longer assumed to be deterministic and perfectly known.

The remainder of this article develops as follows. Section 2 introduces the basic notation and specifies a probabilistic model for the asset market to be considered. Subsequently, section 3 addresses the log-optimal portfolio problem under continuous rebalancing, which is formulated as a stochastic optimization problem in continuous time. We prove that this infinite-dimensional mathematical program is equivalent to a finite-dimensional single-stage problem. The latter can be solved by standard techniques. Our main results are presented in section 4, where a parametric family of portfolio problems in discrete time is investigated; the underlying parameter τ characterizes the length of the rebalancing intervals. Each of these multistage problems has an equivalent myopic reformulation as a convex one-stage stochastic program. We provide approximate analytical solutions in the limits of very frequent and infrequent rebalancing. When the length of the rebalancing intervals tends to zero, we recover the exact solution of the continuous-time problem. Section 5 provides intuitive consistency checks and outlines how our results can be used in practice. Simple analytical formulae for the portfolio weights as well as the mean and variance of the portfolio growth rate as functions of τ are obtained in important special cases: a two-asset economy with one risk-free and one risky asset and an n-asset economy with several identical stocks. Conclusions are presented in section 6.

2. Market model

All random quantities are defined as measurable mappings on an abstract probability space $(\Omega, \mathcal{F}, P)$, which is referred to as the sample space. As a notational convention, random objects will be represented in boldface, while their realizations will be denoted by the same symbols in normal face. The dependence of the random objects on the samples $\omega \in \Omega$ will be notationally suppressed most of the time.

Consider a market with $n+1$ assets. The price of asset i is denoted by $\boldsymbol{p}_i$, where i ranges from 0 to n. We assume that the assets are continuously traded, and their prices are modelled by geometric Brownian motions, that is,

$$\frac{\mathrm{d}\boldsymbol{p}_i}{\boldsymbol{p}_i} = \mu_i\,\mathrm{d}t + \mathrm{d}\boldsymbol{z}_i.$$

The constant parameter μ_i characterizes the asset's drift rate, and $\boldsymbol{z}_i$ denotes a Wiener process whose variance rate may be different from 1. Furthermore, we impose a time-invariant correlation structure,

$$\mathrm{cov}(\mathrm{d}\boldsymbol{z}_i, \mathrm{d}\boldsymbol{z}_j) = \mathrm{E}(\mathrm{d}\boldsymbol{z}_i \mathrm{d}\boldsymbol{z}_j) = \sigma_{ij}\,\mathrm{d}t,$$

and use the convention $\sigma_i = \sqrt{\sigma_{ii}}$. By applying Itô's lemma it can be verified that each asset has a lognormal distribution at time t,

$$\boldsymbol{p}_i(t) = \boldsymbol{p}_i(0)\exp(\nu_i t + \boldsymbol{z}_i(t)),$$

that is, the logarithm of $\boldsymbol{p}_i(t)$ has expected value $\nu_i t = (\mu_i - \frac{1}{2}\sigma_i^2)t$ and variance $\sigma_i^2 t$. The new parameter ν_i can conveniently be interpreted as the expected *logarithmic growth rate* or, in short, *growth rate* of asset i.

In the remainder, asset 0 will be used as the numeraire, and we will frequently work with discounted asset prices

$$\frac{\boldsymbol{p}_i(t)}{\boldsymbol{p}_0(t)} = \frac{\boldsymbol{p}_i(0)}{\boldsymbol{p}_0(0)}\exp(\tilde{\nu}_i t + \tilde{\boldsymbol{z}}_i(t)). \tag{1}$$

Here, the constants $\tilde{\nu}_i = \nu_i - \nu_0$ denote the excess growth rates over the numeraire, and the Wiener processes $\tilde{\boldsymbol{z}}_i = \boldsymbol{z}_i - \boldsymbol{z}_0$ have correlation structure

$$\mathrm{cov}(\mathrm{d}\tilde{\boldsymbol{z}}_i, \mathrm{d}\tilde{\boldsymbol{z}}_j) = \tilde{\sigma}_{ij}\,\mathrm{d}t \quad \text{with} \quad \tilde{\sigma}_{ij} = \sigma_{ij} - \sigma_{i0} - \sigma_{0j} + \sigma_0^2.$$

As before, we will use the convention $\tilde{\sigma}_i = \sqrt{\tilde{\sigma}_{ii}}$. The stochastic differential equations governing the dynamics of the discounted prices can be represented as

$$\frac{\mathrm{d}(\boldsymbol{p}_i/\boldsymbol{p}_0)}{\boldsymbol{p}_i/\boldsymbol{p}_0} = \tilde{\mu}_i \mathrm{d}t + \mathrm{d}\tilde{\boldsymbol{z}}_i,$$

$$\text{where} \quad \tilde{\mu}_i = \tilde{\nu}_i + \frac{\tilde{\sigma}_i^2}{2} = \mu_i - \mu_0 - \sigma_{i0} + \sigma_0^2.$$

For the sake of transparency, we will frequently use matrix notation. Therefore, we introduce an n-vector $\tilde{\mu}$ with entries $\tilde{\mu}_i$ as well as an $n \times n$ matrix $\tilde{S}$ with entries $\tilde{\sigma}_{ij}$, where the indices i and j range from 1 to n. Moreover, we will often work with the n-dimensional Wiener process $\tilde{\boldsymbol{z}} = (\tilde{\boldsymbol{z}}_1, \ldots, \tilde{\boldsymbol{z}}_n)$.

Observe that the covariance matrix $\tilde{S}$ is positive definite if there is at most one risk-free asset and if the Wiener processes driving the risky assets are linearly independent; this will always be assumed henceforth. In addition, it should be emphasized that the numeraire can be chosen freely by permuting the set of available assets. Thus, the numeraire can (and frequently will) be risky. This flexibility becomes useful when addressing portfolio selection problems, below, as it always allows us to choose the numeraire from the portfolio constituents.

3. Continuous-time rebalancing

The information $\mathcal{F}_0^t$ available at time t by continuously observing price movements is conveniently expressed as the σ-algebra induced by the stochastic asset prices up to time t, that is,

$$\mathcal{F}_0^t = \sigma(\boldsymbol{p}_i(s) | i = 0, \ldots, n, s \in [0, t]).$$

We denote by $\mathbb{W}_0$ the space of all $\mathcal{F}_0^t$-progressively measurable stochastic processes taking values in the standard simplex $W = \{w \in \mathbb{R}_+^n \mid \sum_{i=1}^n w_i \leq 1\}$. Each process $\boldsymbol{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n) \in \mathbb{W}_0$ characterizes an investment strategy in the asset market under consideration. By convention $\boldsymbol{w}_i(t)$ specifies the percentage of wealth to be allocated to asset i at time t, where i ranges from 1 to n. It is tacitly assumed that the residual capital is invested in the numeraire. The specification of W implies that no asset (including the numeraire) may be sold short at any time.

Consider now a dynamically rebalanced portfolio corresponding to some trading strategy $\boldsymbol{w} \in \mathbb{W}_0$ and denote its value process by $\boldsymbol{\pi}$. By using (1), the real portfolio return (relative to the numeraire) over an infinitesimal time interval can be expressed as the weighted average of the real asset returns, i.e.

$$\frac{\mathrm{d}(\boldsymbol{\pi}/\boldsymbol{p}_0)}{\boldsymbol{\pi}/\boldsymbol{p}_0} = \sum_{i=1}^n \boldsymbol{w}_i \frac{\mathrm{d}(\boldsymbol{p}_i/\boldsymbol{p}_0)}{\boldsymbol{p}_i/\boldsymbol{p}_0} = \boldsymbol{w}^\top \tilde{\mu}\,\mathrm{d}t + \boldsymbol{w}^\top \mathrm{d}\tilde{\boldsymbol{z}}.$$

By the measurability and boundedness properties of $\boldsymbol{w} \in \mathbb{W}_0$, this stochastic differential equation has a straightforward solution.

$$\frac{\boldsymbol{\pi}(t)}{\boldsymbol{p}_0(t)} = \frac{\boldsymbol{\pi}(0)}{\boldsymbol{p}_0(0)} \exp\left(\int_0^t \boldsymbol{w}(s)^\top \tilde{\mu} - \frac{1}{2}\boldsymbol{w}(s)^\top \tilde{S}\boldsymbol{w}(s)\mathrm{d}s + \int_0^t \boldsymbol{w}(s)^\top \mathrm{d}\tilde{\boldsymbol{z}}\right). \tag{2}$$

An investor seeking to maximize the expected (annualized) growth rate of his or her portfolio thus faces the following optimization problem,

$$\underset{\boldsymbol{w}\in\mathbb{W}_0}{\text{maximize}}\ \mathrm{E}\ln\left(\frac{\boldsymbol{\pi}(1)}{\boldsymbol{\pi}(0)}\right). \qquad \mathcal{P}(0)$$

Using stationarity of the asset returns and the separability properties of the logarithmic utility function, we can reformulate problem $\mathcal{P}(0)$ as a one-stage maximization problem over a finite-dimensional space, that is,

$$\underset{w\in W}{\text{maximize}}\ \varphi_0(w). \qquad \mathcal{P}'(0)$$

The corresponding objective function is given by

$$\varphi_0(w)=\nu_0+w^\top\tilde{\mu}-\frac{1}{2}w^\top\tilde{S}w,$$

which is continuous and strictly concave as $\tilde{S}$ is positive definite. Compactness of the feasible set W thus ensures that problem $\mathcal{P}'(0)$ has a unique solution. The following proposition makes the relation between $\mathcal{P}(0)$ and $\mathcal{P}'(0)$ precise.

Proposition 3.1: *The maximization problems $\mathcal{P}(0)$ and $\mathcal{P}'(0)$ are equivalent in the following sense. First, the optimal values coincide,*

$$\max\mathcal{P}(0)=\max\mathcal{P}'(0).$$

Moreover, if w^* is a solution of $\mathcal{P}'(0)$, then $\boldsymbol{w}^*(t)\equiv w^*$, $t\in[0,1]$, solves $\mathcal{P}(0)$. Conversely, if $\boldsymbol{w}^*$ solves $\mathcal{P}(0)$, then there is a solution w^* of $\mathcal{P}'(0)$ such that $\boldsymbol{w}^*(t)\equiv w^*$ P-almost surely for Lebesgue almost all $t\in[0,1]$.

Proof: Plugging (2) into the objective of problem $\mathcal{P}(0)$, we find

$$\begin{aligned}\max\mathcal{P}(0)&=\max_{\boldsymbol{w}\in\mathbb{W}_0}\ \mathrm{E}\int_0^1\varphi_0(\boldsymbol{w}(t))\mathrm{d}t\\&\le\mathrm{E}\int_0^1\max_{w\in W}\varphi_0(w)\mathrm{d}t=\max\mathcal{P}'(0).\end{aligned}$$

The first equality follows from the definition of φ_0 and the fact that the expectation of an Itô-integral vanishes, while the inequality follows from relaxing the requirement that $\boldsymbol{w}$ must be progressively measurable. The second equality exploits the fact that the integrand is deterministic and time-independent. Thus, we have max $\mathcal{P}(0)\le$ max $\mathcal{P}'(0)$. By reducing the feasible set of problem $\mathcal{P}(0)$ to the space of time-independent and deterministic trading strategies, we can easily prove the converse inequality, max $\mathcal{P}(0)\le$ max $\mathcal{P}'(0)$. Thus, the optimal values of $\mathcal{P}(0)$ and $\mathcal{P}'(0)$ coincide. This reasoning also reveals that $\mathcal{P}(0)$ is solvable and that the maximum is attained by the deterministic strategy $\boldsymbol{w}^*(t)\equiv w^*$, $t\in[0,1]$, where w^* solves $\mathcal{P}'(0)$.† Next, we must show that every optimal strategy of $\mathcal{P}(0)$ is essentially of this form. To this end, define a random function $\boldsymbol{f}$ on the set of essentially bounded random variables $\boldsymbol{v}$, namely,

$$\boldsymbol{f}(\boldsymbol{v})=\varphi_0(\mathrm{E}[\boldsymbol{v}])+\nabla\varphi_0(\mathrm{E}[\boldsymbol{v}])^\top(\boldsymbol{v}-\mathrm{E}[\boldsymbol{v}])-\varphi_0(\boldsymbol{v}).$$

By construction, we have

$$\boldsymbol{f}(\boldsymbol{v})(\omega)\begin{cases}=0 & \text{for all } \omega\in\text{with } \boldsymbol{v}(\omega)=\ \mathrm{E}[\boldsymbol{v}]\ ,\\>0 & \text{otherwise,}\end{cases}$$

since the quadratic function φ_0 is strictly concave. Select now a nondeterministic investment strategy $\boldsymbol{w}\in\mathbb{W}_0$. By this we mean that the set of all t for which $P(\boldsymbol{w}(t)\ne\mathrm{E}[\boldsymbol{w}(t)])>0$ has a nonzero Lebesgue measure. A standard measure-theoretic result (Ash 1972, Theorem 1.6.6(b)) implies

$$\int_0^1\varphi_0(\mathrm{E}[\boldsymbol{w}(t)])-\mathrm{E}[\varphi_0(\boldsymbol{w}(t))]\mathrm{d}t=\int_0^1\mathrm{E}[\boldsymbol{f}(\boldsymbol{w}(t))]\mathrm{d}t>0,$$

that is, the nondeterministic strategy $\boldsymbol{w}\in\mathbb{W}_0$ is strictly outperformed by the deterministic strategy $\mathrm{E}[\boldsymbol{w}]\in\mathbb{W}_0$. Thus, if $\boldsymbol{w}^*$ solves $\mathcal{P}(0)$, $\boldsymbol{w}^*$ must be deterministic (up to almost sure equivalence), and $w^*=\mathrm{E}[\boldsymbol{w}^*_t]$ must be the unique solution of $\mathcal{P}'(0)$ for Lebesgue almost all $t\in[0,1]$. □

To solve problem $\mathcal{P}'(0)$, we assume without loss of generality that the solution lies in the interior of W, that is, it characterizes a portfolio in which all assets (including the numeraire) have strictly positive weight. Otherwise, we may pretend that those assets which do not enter the optimal portfolio are not available for purchase, and we may neglect them from the beginning. Under this assumption, the optimal solution of the quadratic program $\mathcal{P}'(0)$ is easily seen to be $w=\tilde{S}^{-1}\tilde{\mu}$. Plugging this allocation vector back into the objective function shows that the optimal value of $\mathcal{P}'(0)$ is $\nu_0+\frac{1}{2}\tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu}$. By Proposition 3.1, this solution of $\mathcal{P}'(0)$ easily translates to a solution for the original problem $\mathcal{P}(0)$. Note that such a solution was first obtained by Merton (1969) via methods of stochastic optimal control theory. The approach presented here, which reduces $\mathcal{P}(0)$ to a finite-dimensional deterministic equivalent problem $\mathcal{P}'(0)$, relies on less sophisticated techniques. Its main benefit is that it easily extends to the discrete-time case and facilitates the analysis of changing rebalancing frequencies.

Another interesting quantity related to problem $\mathcal{P}(0)$ is the variance of the (annual) portfolio growth rate given that the portfolio is managed according to the log-optimal investment strategy. A straightforward calculation yields

$$\begin{aligned}\mathrm{Var}\ln\left(\frac{\boldsymbol{\pi}(1)}{\boldsymbol{\pi}(0)}\right)&=\mathrm{E}\left[\left(\int_0^1\tilde{\mu}^\top\tilde{S}^{-1}\mathrm{d}\tilde{\boldsymbol{z}}+\int_0^1\mathrm{d}\boldsymbol{z}_0\right)^2\right]\\&=\tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu}+2\tilde{\mu}^\top\tilde{S}^{-1}\varsigma+\sigma_0^2,\end{aligned}$$

where the n-vector ς has elements $\varsigma_i=\sigma_{i0}-\sigma_0^2$. Notice that both ς and σ_0^2 vanish if the numeraire is risk-free, in which case the formula for the variance of the portfolio growth rate simplifies to $\tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu}$.

†In addition, the use of the 'max'-operators in the proposition statement is justified.

4. Discrete-time rebalancing

The optimal solution of problem $\mathcal{P}(0)$ keeps the portfolio weights constant. Thus, at any time point the investor must sell (buy) assets that currently grow faster (slower) than his or her portfolio. High transaction costs and onerous administrative burdens that go along with each reallocation of assets, however, make frequent portfolio changes undesirable or even infeasible. Therefore, we now investigate the log-optimal portfolio problem under the additional premise that rebalancing is restricted to discrete time points $h\tau$, $h \in \mathbb{N}_0$; the constant $\tau > 0$ characterizes the length of a rebalancing interval. In this section we will derive analytical formulae for the sensitivity of the optimal portfolio weights as well as the expectation and the variance of the portfolio growth rate with respect to small changes of the parameter τ.

Assume that our investor observes the asset prices only at the start times of the rebalancing intervals. For notational convenience we define $\boldsymbol{p}_{i,h} = \boldsymbol{p}_i(h\tau)$ for every nonnegative integer h and for i between 0 and n. Then, the information available to the investor at the beginning of the hth rebalancing interval can conveniently be expressed as the σ-algebra

$$\mathcal{F}_\tau^h = \sigma(\boldsymbol{p}_{i,g} | i = 0, \ldots, n, g = 0, \ldots, h).$$

In analogy to the continuous-time case considered before, we denote by $\mathbb{W}_\tau$ the space of $\mathcal{F}_\tau^h$-adapted discrete-time stochastic processes valued in the closed simplex W. To every $\boldsymbol{w} \in \mathbb{W}_\tau$ we assign a portfolio value process $\boldsymbol{\pi}$. By convention, $\boldsymbol{w}_h$ and $\boldsymbol{\pi}_h$ stand for the portfolio weight vector and the portfolio value at the beginning of the hth rebalancing interval, respectively. The (discounted) portfolio value is determined recursively by means of the dynamic budget constraint

$$\frac{\boldsymbol{\pi}_{h+1}/\boldsymbol{p}_{0,h+1}}{\boldsymbol{\pi}_h/\boldsymbol{p}_{0,h}} = 1 + \sum_{i=1}^{n} \boldsymbol{w}_{i,h}\left(\frac{\boldsymbol{p}_{i,h+1}/\boldsymbol{p}_{0,h+1}}{\boldsymbol{p}_{i,h}/\boldsymbol{p}_{0,h}} - 1\right). \tag{3}$$

Here, $\boldsymbol{\pi}_0$ denotes initial wealth, which is a deterministic random variable. Let us assume that $\tau^{-1} = H \in \mathbb{N}$. Then, the problem of maximizing the portfolio's expected growth rate per unit time can be formulated as

$$\underset{\boldsymbol{w} \in \mathbb{W}_\tau}{\text{maximize}} \ \mathrm{E}\ln\left(\frac{\boldsymbol{\pi}_H}{\boldsymbol{\pi}_0}\right). \tag{$\mathcal{P}(\tau)$}$$

Short selling is precluded explicitly in the definition of the set W. However, in the discrete-time setting under consideration, this restriction is redundant since short selling involves the risk of losing more money than initially invested. In fact, if any asset is sold short, there is a nonzero probability of negative terminal wealth, which is penalized by an infinitely negative utility. The use of a logarithmic utility function in a discrete-time framework therefore impedes short selling.†

Going from continuous- to discrete-time rebalancing reduces the portfolio manager's flexibility. This transition is admittedly somewhat artificial in the absence of transaction costs, but its analysis can provide insights that are also valuable for investors in frictional markets. It is intuitively clear that decreasing the rebalancing frequency lowers the achievable portfolio growth rate. Even though this qualitative result seems obvious, its proof requires a subtle argument.

Proposition 4.1: *Continuous-time rebalancing outperforms discrete-time rebalancing, that is,* $\sup \mathcal{P}(0) \geq \sup \mathcal{P}(\tau)$ *for all* $\tau > 0$.

Proof: Let $\boldsymbol{w} \in \mathbb{W}_\tau$ be a discretely rebalanced strategy with rebalancing intervals of length τ. Moreover, denote by $\boldsymbol{\pi}$ the associated discrete-time wealth process, which is determined by (3). Since the asset prices are modelled as continuous-time stochastic processes, our portfolio can be assigned a unique value $\hat{\boldsymbol{\pi}}(t)$ at any time $t \in \mathbb{R}_+$. In fact, we have

$$\hat{\boldsymbol{\pi}}(t) = \boldsymbol{\pi}_h \frac{\boldsymbol{p}_0(t)}{\boldsymbol{p}_{0,h}}\left(1 + \sum_{i=1}^{n} \boldsymbol{w}_{i,h}\left(\frac{\boldsymbol{p}_i(t)/\boldsymbol{p}_0(t)}{\boldsymbol{p}_{i,h}/\boldsymbol{p}_{0,h}} - 1\right)\right),$$

where h is the largest integer smaller or equal to t/τ. Notice that the discrete-time process $\boldsymbol{\pi}$ and the continuous-time process $\hat{\boldsymbol{\pi}}$ are consistent in the sense that $\hat{\boldsymbol{\pi}}(t) = \boldsymbol{\pi}_h$ for $t = h\tau$. Analogously, the assets in our portfolio can be assigned weights $\hat{\boldsymbol{w}}(t)$ at all times $t \in \mathbb{R}_+$. Set

$$\hat{\boldsymbol{w}}(t) = (\hat{\boldsymbol{w}}_1(t), \ldots, \hat{\boldsymbol{w}}_n(t)), \quad \text{where } \hat{\boldsymbol{w}}_i(t) = \boldsymbol{w}_{i,h}\frac{\boldsymbol{\pi}_h \boldsymbol{p}_i(t)}{\boldsymbol{p}_{i,h}\hat{\boldsymbol{\pi}}(t)},$$

and h is the largest integer smaller or equal to t/τ. Again, consistency is guaranteed by the relations $\hat{\boldsymbol{w}}(t) = \boldsymbol{w}_h$ for $t = h\tau$. It can easily be checked that $\hat{\boldsymbol{w}}$ is contained in $\mathbb{W}_0$ and generates the wealth process $\hat{\boldsymbol{\pi}}$; thus it is feasible in $\mathcal{P}(0)$. By consistency of the discrete- and continuous-time processes, the objective value of $\hat{\boldsymbol{w}}$ in $\mathcal{P}(0)$ is the same as the objective value of $\boldsymbol{w}$ in $\mathcal{P}(\tau)$. As the choice of $\boldsymbol{w}$ was arbitrary, the optimum of $\mathcal{P}(0)$ is no smaller than the optimum of $\mathcal{P}(\tau)$. □

Since the asset prices are governed by geometric Brownian motions, the total asset returns are independent and identically distributed over all rebalancing periods. We may write

$$\frac{\boldsymbol{p}_{i,h+1}}{\boldsymbol{p}_{i,h}} = \mathrm{e}^{\nu_i\tau + \boldsymbol{\varepsilon}_{i,h}\sqrt{\tau}}, \quad \text{where} \quad \boldsymbol{\varepsilon}_{i,h} = \frac{\boldsymbol{z}_i((h+1)\tau) - \boldsymbol{z}_i(h\tau)}{\sqrt{\tau}}.$$

The random variables $\boldsymbol{\varepsilon}_{i,h}$ are jointly normally distributed with zero mean and covariances $\mathrm{cov}(\boldsymbol{\varepsilon}_{i,g}, \boldsymbol{\varepsilon}_{j,h}) = \sigma_{ij}\delta_{gh}$. When dealing with discounted asset prices, we will further need the related random variables $\tilde{\boldsymbol{\varepsilon}}_{i,h} = \boldsymbol{\varepsilon}_{i,h} - \boldsymbol{\varepsilon}_{0,h}$, which are also normally distributed with zero mean and covariances $\mathrm{cov}(\tilde{\boldsymbol{\varepsilon}}_{i,g}, \tilde{\boldsymbol{\varepsilon}}_{j,h}) = \tilde{\sigma}_{ij}\,\delta_{gh}$. Using stationarity of the asset returns, the absence of transaction costs, and the separability properties of the logarithmic utility function,

†Short selling is possible, however, if the rebalancing dates are not predetermined but may depend on the realized asset price paths.

we can reformulate problem $\mathcal{P}(\tau)$ as a finite-dimensional one-stage problem, that is,

$$\underset{w \in W}{\text{maximize}} \ \varphi_\tau(w). \qquad \mathcal{P}'(\tau)$$

The corresponding objective function is given by

$$\varphi_\tau(w) = \nu_0 + \frac{1}{\tau}\mathrm{E}\left\{ \ln\left(1 + \sum_{i=1}^{n} w_i\left[\mathrm{e}^{\tilde{\nu}_i\tau + \tilde{\boldsymbol{\varepsilon}}_i\sqrt{\tau}} - 1\right]\right)\right\},$$

where we use the convention $\tilde{\boldsymbol{\varepsilon}}_i = \tilde{\boldsymbol{\varepsilon}}_{i,0}$. It can be shown that φ_τ is finite, continuous, and strictly concave on W for each parameter $\tau > 0$ (technical details are provided in appendix A). Compactness of the feasible set thus ensures that problem $\mathcal{P}'(\tau)$ has a unique solution. The following result, which is an extension of Proposition 3.1, makes the relation between problems $\mathcal{P}(\tau)$ and $\mathcal{P}'(\tau)$ precise.

Proposition 4.2: *The maximization problems $\mathcal{P}(\tau)$ and $\mathcal{P}'(\tau)$ are equivalent in the following sense. First, the optimal values coincide,*

$$\max \mathcal{P}(\tau) = \max \mathcal{P}'(\tau).$$

Moreover, if w^* is a solution of $\mathcal{P}'(\tau)$, then $\boldsymbol{w}_h^* \equiv w^*$, $h = 0, \ldots, H-1$, solves $\mathcal{P}(\tau)$. Conversely, if $\boldsymbol{w}^*$ is a solution of $\mathcal{P}(\tau)$, then for each $h = 0, \ldots, H-1$ there is a solution w^* of $\mathcal{P}'(\tau)$ such that $\boldsymbol{w}_h^* \equiv w^*$ almost surely.

Proof: The proof is widely parallel to that of Proposition 3.1. The only difference is that the time integral is replaced by a sum, while the concave quadratic function φ_0 is replaced by the strictly concave function φ_τ. Further details are omitted for brevity. □

Proposition 4.3: *The unique solution w^* of problem $\mathcal{P}'(\tau)$, $\tau \geq 0$, satisfies the following necessary and sufficient optimality condition:*

$$\nabla\varphi_\tau(w^*)^\top(w - w^*) \leq 0 \quad \forall w \in W. \qquad (4)$$

Proof: See Rockafellar and Wets (1998, Theorem 6.12). Notice that $\nabla\varphi_\tau(w^*)^\top\,(w - w^*)$ is the directional derivative of φ_τ at w^* for $w - w^*$, and its existence can be proved by means of the dominated convergence theorem. □

Next, we introduce a set $\mathcal{S}(\tau) \subset \{0, 1, \ldots, n\}$ for each $\tau \geq 0$ which contains the indices of the (strictly) positively weighted assets in the optimal portfolio corresponding to problem $\mathcal{P}'(\tau)$. We call problem $\mathcal{P}'(\tau)$ *nondegenerate* if its solution w^* assigns strictly positive weight to the numeraire, $0 \in \mathcal{S}(\tau)$, and if the partial derivatives $\partial\varphi_\tau(w^*)/\partial w_i$ are strictly negative for all $i \notin \mathcal{S}(\tau)$. Requiring the numeraire to have positive weight is nonrestrictive as it can be chosen freely, and since at least one asset must have nonzero weight. With the numeraire having strictly positive weight, the optimality condition (4) reduces to

$$\partial\varphi_\tau(w^*)/\partial w_i = 0 \quad i \in \mathcal{S}(\tau), \qquad (5a)$$

$$\partial\varphi_\tau(w^*)/\partial w_i \leq 0 \quad i \notin \mathcal{S}(\tau). \qquad (5b)$$

If problem $\mathcal{P}'(\tau)$ is nondegenerate, then the inequalities in (5b) are strict. Notice that nondegeneracy holds generically, whereas degeneracy can always be removed by slightly perturbing the problem data. Without much loss of generality, we may thus assume that the continuously rebalanced reference problem $\mathcal{P}'(0)$ is nondegenerate. Proposition A.3 in the appendix then implies that the optimal portfolio associated with problem $\mathcal{P}'(\tau)$ comprises the same assets for all small values of τ, that is, $\mathcal{S}(\tau)$ is locally constant at 0. We may therefore pretend that the assets in the complement of $\mathcal{S}(0)$ are not available for purchase, and we may neglect them in the entire analysis.

For notational convenience, we introduce two $n \times n$ matrices Q and M with entries $Q_{ij} = \tilde{\sigma}_{ij}^2$ and $M_{ij} = \tilde{\mu}_i\,\delta_{ij}$ respectively. Thereby, the indices i and j range from 1 to n, and δ_{ij} stands for the Kronecker delta. We also recall that ς was defined earlier as the n-vector with entries $\varsigma_i = \sigma_{i0} - \sigma_0^2$. With these conventions, we are now ready to state our main result.

Theorem 4.4: *Suppose that problem $\mathcal{P}'(0)$ is nondegenerate and—after a suitable reduction of the asset universe—that $\mathcal{S}(0)$ comprises all available assets. Then, we can derive the following estimates.*

(i) *The unique optimal solution of problem $\mathcal{P}'(\tau)$ is representable as*

$$w^*(\tau) = w^{(0)} - w^{(1)}\tau + o(\tau) \quad \text{for } \tau \downarrow 0,$$

where $w^{(0)} = \tilde{S}^{-1}\tilde{\mu}$ coincides with the optimal portfolio allocation under continuous rebalancing, and

$$w^{(1)} = \tilde{S}^{-1}\left(\frac{1}{2}Q - \frac{1}{2}M\tilde{S} - \tilde{S}M + \tilde{\mu}\tilde{\mu}^\top\right)\tilde{S}^{-1}\tilde{\mu}.$$

(ii) *The maximal value of problem $\mathcal{P}'(\tau)$ is representable as*

$$g^*(\tau) = g^{(0)} - g^{(1)}\tau + o(\tau) \quad \text{for } \tau \downarrow 0,$$

where $g^{(0)} = \nu_0 + \frac{1}{2}\tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu}$ coincides with the maximal expected portfolio growth rate under continuous rebalancing, and

$$g^{(1)} = \frac{1}{4}\tilde{\mu}^\top\tilde{S}^{-1}(Q - M\tilde{S} - \tilde{S}M + \tilde{\mu}\tilde{\mu}^\top)\tilde{S}^{-1}\tilde{\mu}.$$

(iii) *The variance of the growth rate of the optimal portfolio in problem $\mathcal{P}(\tau)$ is representable as†*

$$v^*(\tau) = v^{(0)} - v^{(1)}\tau + o(\tau) \quad \text{for } \tau \downarrow 0,$$

where $v^{(0)} = \tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu} + 2\tilde{\mu}^\top\tilde{S}^{-1}\varsigma + \sigma_0^2$ coincides with the variance of the growth rate

†Notice that the maximization problems $\mathcal{P}(\tau)$ and $\mathcal{P}'(\tau)$, $\tau \geq 0$, have the same optimal value and (essentially) the same solution. However, the variance of the optimal portfolio's growth rate over unit time can only be calculated from the objective function of problem $\mathcal{P}(\tau)$.

of the optimal portfolio under continuous rebalancing, and

$$v^{(1)} = \tilde{\mu}^\top \tilde{S}^{-1}\left(\frac{1}{2}Q - M\tilde{S} - \tilde{S}M + \frac{3}{2}\tilde{\mu}\tilde{\mu}^\top\right)\tilde{S}^{-1}\tilde{\mu} + \tilde{\mu}^\top \tilde{S}^{-1}\left(Q - M\tilde{S} - \tilde{S}M + 2\tilde{\mu}\tilde{\mu}^\top\right)\tilde{S}^{-1}\varsigma.$$

The proof of Theorem 4.4 is purely technical and thus deferred to appendix B. We first observe that Theorem 4.4 is consistent with our findings in section 3, that is, the portfolio weights as well as the mean and variance of the portfolio growth rate converge to Merton's continuous-time values for $\tau \downarrow 0$. Moreover, the functions $w(\tau)$, $g(\tau)$ and $v(\tau)$ are differentiable at the origin, and the (negative) derivatives $w^{(1)}$, $g^{(1)}$, and $v^{(1)}$ can be expressed in closed form. Notice that $g^{(1)}$ must be nonnegative since the expected growth rate of the log-optimal portfolio is monotonically decreasing in τ, see Proposition 4.1. As a consistency check, one can directly prove nonnegativity of $g^{(1)}$ by only manipulating its closed form representation. Technical details are provided in appendix A. Unlike $g^{(1)}$, the sensitivities $w^{(1)}$ and $v^{(1)}$ can be either positive or negative, as will be exemplified in section 5.1. From the proof of Theorem 4.4(i) in the appendix one sees that all terms depending on $w^{(1)}$ cancel out in the formula for $g^{(1)}$. Thus, up to second order in τ, misusing the optimal continuous-time allocation for discrete rebalancing is not worse than using the optimal discrete-time allocation. The formulae for $v^{(0)}$ and $v^{(1)}$ look cumbersome, but they become significantly simpler if the numeraire is riskless, which implies that σ_0^2 and ς vanish. The magnitudes of all Taylor coefficients introduced in Theorem 4.4 will be analysed more carefully in section 5 in a number of interesting special cases.

Remark 1: Analytical treatment of problem $\mathcal{P}'(\tau)$ is not only possible in the limit $\tau \downarrow 0$ but also for $\tau \uparrow \infty$, that is, if rebalancing takes place very infrequently. In the latter case, the maximum achievable growth rate over all portfolios coincides with the maximum growth rate over all individual assets. Accordingly, in the limit $\tau \uparrow \infty$ it is optimal to invest all money in the asset with the highest growth rate. Mathematical details are omitted for brevity of exposition.

5. Examples

In order to make the main results of section 4 more comprehensive and plausible, we present a series of analytical and numerical examples. Emphasis is put on consistency checks and the development of an intuition for the qualitative effects of discrete-time rebalancing.

5.1. One risk-free and one risky asset

In the case of a two-asset economy with a risk-free numeraire ($\mu_0 = r, \sigma_0 = 0$) and one risky asset ($\mu_1 = \mu$, $\sigma_1 = \sigma$) we find

$$\tilde{\mu} = \mu - r, \quad \tilde{S} = \sigma^2, \quad Q = \sigma^4, \quad M = \mu - r, \quad \varsigma = 0.$$

Thus, the optimal weight of the risky asset under continuous rebalancing is $w^{(0)} = \tilde{S}^{-1}\tilde{\mu} = (\mu - r)/\sigma^2$. We require $0 \le \mu - r \le \sigma^2$, which ensures that neither the risky nor the risk-free asset will be shorted. If the inequalities are strict, the portfolio problem corresponding to $\tau = 0$ is nondegenerate, and Theorem 4.4(i) applies. Thus, the portfolio weight of the risky asset changes at rate

$$w^{(1)} = \frac{1}{\sigma^2}\left(\frac{\sigma^4}{2} - \frac{3\sigma^2(\mu - r)}{2} + (\mu - r)^2\right)\frac{\mu - r}{\sigma^2} = \frac{\mu - r}{2} - \frac{3(\mu - r)^2}{2\sigma^2} + \frac{(\mu - r)^3}{\sigma^4}.$$

If $w^{(1)}$ is negative (positive), then the amount of money invested in the risky asset is increased (decreased) as rebalancing becomes less frequent. As easily can be checked, $w^{(1)}$ is negative for $\sigma^2/2 \le \mu - r \le \sigma^2$. The second inequality is redundant since the reference problem for $\tau = 0$ is assumed to be nondegenerate; the first inequality translates to $\nu \ge r$, where ν is the risky asset's growth rate. Hence, the weight of the risky asset increases with τ if its growth rate exceeds that of the numeraire. This result is plausible in light of Remark 1, which asserts that the weight of the fastest growing asset converges to 1 as τ tends to infinity. By using Theorem 4.4(ii) we next obtain $g^{(0)} = r + (\mu - r)^2/(2\sigma^2)$ and

$$g^{(1)} = \frac{\mu - r}{4\sigma^2}\left(\sigma^4 - 2\sigma^2(\mu - r) + (\mu - r)^2\right)\frac{\mu - r}{\sigma^2} = \frac{(\mu - r)^2}{4}\left(1 - \frac{\mu - r}{\sigma^2}\right)^2.$$

This representation manifests nonnegativity of $g^{(1)}$, which means that the portfolio growth rate decreases as the the parameter τ is increased, and it is thus consistent with Proposition A.4 in the appendix. Finally, Theorem 4.4(iii) yields the coefficients of the variance expansion, i.e. $v^{(0)} = (\mu - r)^2/\sigma^2$ and

$$v^{(1)} = \frac{\mu - r}{\sigma^2}\left(\frac{\sigma^4}{2} - 2\sigma^2(\mu - r) + \frac{3(\mu - r)^2}{2}\right)\frac{\mu - r}{\sigma^2} = (\mu - r)^2\left(\frac{1}{2} - \frac{2(\mu - r)}{\sigma^2} + \frac{3(\mu - r)^2}{2\sigma^4}\right).$$

The sensitivity $v^{(1)}$ is negative if $\sigma^2/3 \le (\mu - r) \le \sigma^2$ and positive otherwise. As before, the second inequality is redundant by nondegeneracy of the reference problem for $\tau = 0$.

5.2. Two no-growth stocks

Consider again the two-asset economy of the previous section, and assume additionally that $r = 0$ and $\mu = \sigma^2/2$. Thus, both assets have zero expected growth rate. If the rebalancing frequency amounts to τ^{-1}, the optimal portfolio growth rate can be approximated by

$$g^{(0)} - g^{(1)}\tau, \quad \text{where } g^{(0)} = \frac{\sigma^2}{8} \text{ and } g^{(1)} = \frac{\sigma^4}{64}.$$

This simple calculation shows that growth can be achieved by combining two no-growth stocks. Moreover, for reasonable volatility coefficients the loss incurred by infrequent rebalancing is only marginal. As a numerical example, let us assume that $\sigma = \ln 2$. Then, the return of the risky asset has the same mean and variance as the return of a fictitious 'digital' stock whose value in each year either doubles or reduces by one-half, each with a probability of 50%. With yearly rebalancing, our portfolio growth rate becomes $g^{(0)} - g^{(1)} \approx 5.6\%$. Substituting the risky asset by the fictitious digital stock, one gets a slightly higher expected growth rate of 5.9%, see Luenberger (1998, Example 15.2).

5.3. Several identical assets (independent case)

Consider a market with $n+1$ independent assets, all of which have the same drift rate μ and the same volatility σ. By definition, the parameters of the discounted price processes are

$$\tilde{\mu}_i = \sigma^2 \quad \text{and} \quad \tilde{\sigma}_{ij} = \begin{cases} 2\sigma^2 & i = j \\ \sigma^2 & i \neq j \end{cases} \quad \text{for} \quad i,j = 1,\dots,n.$$

In order to simplify notation, we denote by e the n-vector with identical entries $e_i = 1$, $i = 1,\dots,n$. Moreover, we set $E = ee^\top$, and let I be the n-dimensional identity matrix. Then, we have

$$\tilde{\mu} = \sigma^2 e, \quad \tilde{S} = \sigma^2(I+E), \quad Q = \sigma^4(3I+E),$$
$$M = \sigma^2 I, \quad \varsigma = -\sigma^2 e.$$

Let us first determine the composition of the log-optimal portfolio. By using the explicit formulae of Theorem 4.4(i) we find

$$w^{(0)} = (I+E)^{-1}e = \frac{1}{n+1}e$$

and

$$w^{(1)} = (I+E)^{-1}\left(\frac{\sigma^2}{2}(3I+E) - \frac{3\sigma^2}{2}(I+E) + \sigma^2 E\right)(I+E)^{-1}e = 0.$$

Thus, the optimal solution allocates the same share of wealth to each asset, no matter what the rebalancing frequency is. This result merely manifests the permutation symmetry of the available assets and confirms what we would have expected in the first place. Theorem 4.4(ii) implies that the maximum expected growth rate under continuous rebalancing amounts to

$$g^{(0)} = \nu_0 + \frac{\sigma^2}{2}e^\top(I+E)^{-1}e = \nu_0 + \frac{\sigma^2 n}{2(n+1)}.$$

As expected, diversification (i.e. letting n become large) increases the magnitude of the variance term, thereby increasing the portfolio growth rate to a maximum of $\nu_0 + \sigma^2/2$. See also the discussion of volatility pumping in Luenberger (1998, Chap. 15). The sensitivity of the portfolio growth rate with respect to τ can be written as

$$\begin{aligned} g^{(1)} &= \frac{\sigma^4}{4}e^\top(I+E)^{-1}(3I+E-2I-2E+E)(I+E)^{-1}e \\ &= \frac{\sigma^4 n}{4(n+1)^2}, \end{aligned}$$

which is very small for reasonable values of σ. Furthermore, by using Theorem 4.4(iii) and the fact that in the current setting $\varsigma = -\tilde{\mu}$, we obtain the coefficients of the variance expansion, i.e.

$$v^{(0)} = -\sigma^2 e^\top(I+E)^{-1}e + \sigma^2 = \frac{\sigma^2}{n+1}$$

and

$$\begin{aligned} v^{(1)} &= e^\top(I+E)^{-1}\left(-\sigma^4\frac{3I+2E}{2}\right)(I+E)^{-1}e \\ &= -\frac{\sigma^4 n(2n+3)}{2(n+1)^2}. \end{aligned}$$

These results suggest that diversification reduces the magnitude of the constant coefficient $v^{(0)}$, whereas the linear term $v^{(1)}$ is generally small but fairly insensitive to n. In conclusion, we have discovered a duality between growth and volatility with respect to diversification. Diversification does little to improve growth after $n \approx 10$, but it always lowers $g^{(1)}$ by about $1/n$. Conversely, diversification lowers variance, but it fails to improve $v^{(1)}$.

We conclude this example with a numerical calculation that shows how ineffective frequent rebalancing is at boosting the portfolio growth rate. For five independent assets ($n=4$) with identical volatilities $\sigma = 0.88$ we obtain $g^{(0)} = \nu_0 + 0.31$ and $g^{(1)} = 0.024$. Hence, the excess growth rate over the numeraire amounts to 31% if rebalancing is done continuously. This extraordinary growth rate is lowered by as little as 2.4% if the portfolio is rebalanced only once per year. Assume now that transaction costs are 0.1% of the transaction amount. Then, rebalancing once a year can degrade the portfolio performance at most by 10 basis points. In this situation, transaction costs have a negligible effect on portfolio growth and can safely be disregarded when designing portfolio strategies.

5.4. Several identical assets (dependent case)

Consider again $n+1$ assets with the same drift rate μ and the same volatility coefficient σ. In contrast to the previous section, however, assume that the assets are correlated, that is, $\sigma_{ij} = \varrho\sigma^2$ for all $0 \le i \neq j \le n$. For the covariance matrix of these $n+1$ assets to be positive definite, we must require $-n^{-1} \le \varrho \le 1$. Then, we obtain

$$\tilde{\mu}_i = \sigma^2(1-\varrho) \quad \text{and}$$
$$\tilde{\sigma}_{ij} = \begin{cases} 2\sigma^2(1-\varrho) & i=j \\ \sigma^2(1-\varrho) & i \neq j \end{cases} \quad \text{for } i,j = 1,\dots,n,$$

which implies that the results of the previous section carry over to the present setting if we replace σ^2 by $\sigma^2(1-\varrho)$ in the final formulae.

5.5. *A simple computational example*

If the market model exhibits no symmetries at all, we can tackle the reference problem $\mathcal{P}'(0)$ numerically by using a quadratic programming algorithm. This calculation reveals the set of active assets that have strictly positive weight in the optimal portfolio. Moreover, it allows us to check whether the reference problem is nondegenerate. In the unlikely case of a degenerate reference problem, however, we can recover nondegeneracy by slightly perturbing the parameters of the price processes. After reducing the asset universe to the set of active assets, Theorem 4.4 becomes applicable. Calculation of the sensitivities $w^{(1)}$, $g^{(1)}$, and $v^{(1)}$ is based on simple matrix manipulations, which can conveniently be carried out in Matlab, say, for an asset universe comprising several thousand titles.

In a market with very few independent assets, however, one may attempt to directly solve the nonlinear one-stage stochastic programs $\mathcal{P}'(\tau)$, $\tau \geq 0$, without making reference to the Taylor approximations derived in section 4. This approach requires discretization of the joint return distribution, e.g. by means of Monte Carlo sampling (MC). In addition, it requires the availability of a powerful nonlinear programming solver.

Let us now compare the direct MC approach with the semi-analytical approach based on Taylor approximation. For the sake of transparency and in order to keep the MC sample size manageable, we consider a market with only four assets. The numeraire with index 0 is chosen to be the asset with the largest growth rate. The relevant parameters of the market model are listed in table 1. Our parameter choice guarantees that all available assets are active, and that the results of our test calculations allow for a neat graphical representation. The MC approach is implemented as follows. For each τ we draw 300 000 samples from the joint return distribution. Next, the expectation in the objective function φ_τ is replaced by the sample average. The resulting approximate portfolio problem is solved by means of the sequential quadratic programming algorithm SNOPT (Gill *et al.* 2002); observe that this portfolio problem is nonquadratic. Since the optimal portfolio weights as well as the mean and variance of the portfolio growth rate turn out to be very insensitive to changes in τ, we solve the MC problem only for those values of τ which are multiples of 50 days and smaller than 10 years.

Figure 1 displays the dependence of the optimal portfolio weights on the parameter τ. The MC solution is virtually exact in this low-dimensional example. Observe that the Taylor approximation based on Theorem 4.4(i) hardly deviates from the MC solution as long as the rebalancing periods are smaller than half a year. We thus expect the Taylor approach to be sufficiently precise in many practical applications. Notice that the numeraire is the only asset whose weight increases with τ. This observation is consistent with Remark 1 which states that all the money will eventually (for $\tau \uparrow \infty$) be invested in the asset with the highest growth rate.

Table 1. Parameters of the asset price processes.

	σ_{ij}					
σ_{ij}	0	1	2	3	μ_i	ν_i
0	0.04000	0.10000	0.00005	0.00000	0.036	0.0160
1	0.10000	1.00000	0.00002	0.00000	0.240	−0.2600
2	0.00005	0.00002	0.04500	−0.03000	0.020	−0.0025
3	0.00000	0.00000	−0.03000	0.04000	0.014	−0.0060

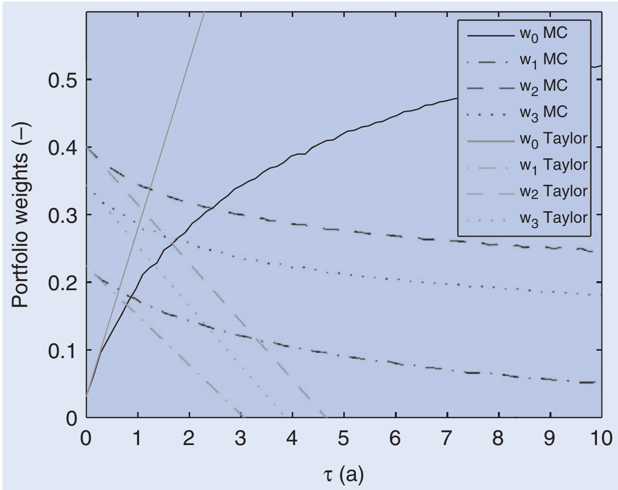

Figure 1. Optimal portfolio weights depending on the length of the rebalancing intervals.

Figure 2 visualizes the optimal portfolio growth rate as calculated with the MC and Taylor approaches. Again, the Taylor approximation is very accurate for short rebalancing intervals ($\tau < 0.5$ years). In accordance with Proposition A.4 and Remark 1, the portfolio growth rate is a monotonically decreasing function of τ which asymptotically approaches the value 1.6%, i.e. the expected growth rate of the numeraire.† Notice also that the Taylor approximation globally underestimates the achievable portfolio growth rate and thus represents a conservative approximation.

Finally, figure 3 shows the variance of the portfolio growth rate. As before, the Taylor approximation coincides with the MC solution for small rebalancing intervals. In the current parameterization, the log variance decreases at $\tau = 0$. Further numerical experiments have shown that the log variance is not a monotonic function of τ; it decreases until $\tau \approx 50$

†In fact, the portfolio growth rate does not saturate before $\tau \approx 200$ years. The saturation regime is outside the range of figure 2 as rebalancing periods longer than a few years are of minor interest.

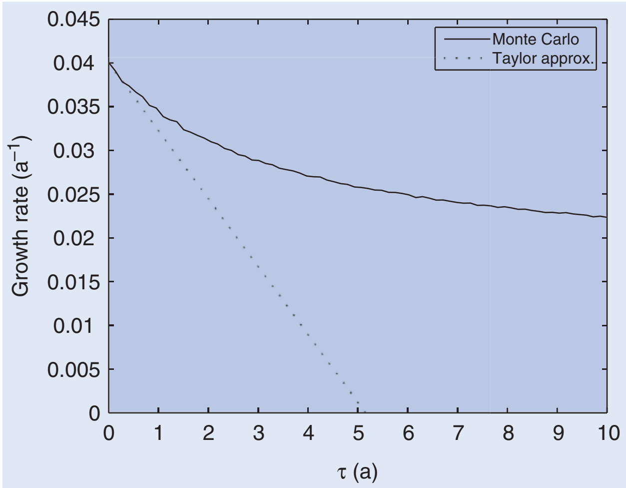

Figure 2. Optimal portfolio growth rate depending on the length of the rebalancing intervals.

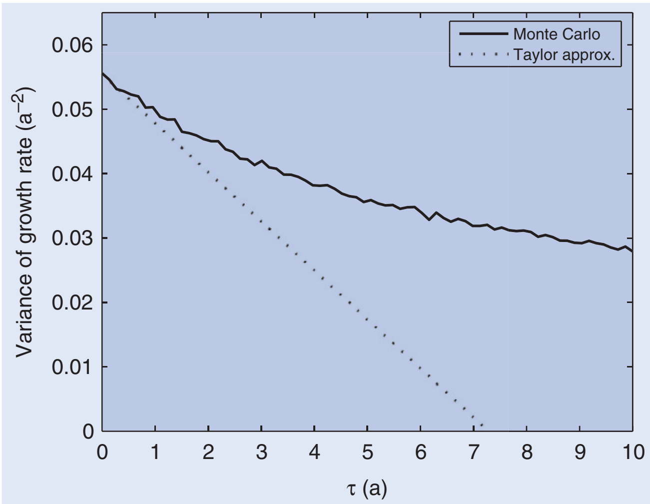

Figure 3. Variance of the optimal portfolio's growth rate depending on the length of the rebalancing intervals.

years, increases again, and eventually saturates at 4%, which represents the variance of the numeraire.

6. Conclusions

In a log-utility setting, we have studied the influence of the rebalancing frequency on the portfolio weights and the statistical properties of the portfolio growth rate. As part of this analysis, we solved the log-optimal portfolio problem to first order in τ which represents the length of the rebalancing intervals. Based on our numerical experiments we conjecture that the obtained approximate solution is very accurate if τ is of the order of one year. The approximation can quickly be evaluated, even if the underlying asset universe comprises several thousand titles. In contrast, a purely numerical approach based on Monte Carlo sampling, for instance, is time-consuming and can only cope with relatively few risk factors. We have shown in several examples that the loss incurred by infrequent rebalancing is surprisingly small. In a prototypical market of $n+1$ independent identical assets with drift rate μ and volatility σ, the expected portfolio growth rate (or log mean) is of the order $O(\mu)$, and its sensitivity with respect to τ is of the order $O(\sigma^4/n)$. Thus, although diversification does hardly improve portfolio growth for $\tau=0$, it can virtually offset the negative effects of infrequent rebalancing. Furthermore, the variance of the portfolio growth rate (or log variance) is of the order $O(\sigma^2/n)$, and its sensitivity with respect to τ is of the order $O(\sigma^4)$. Unlike in the case of the log mean, diversification improves the log variance for $\tau=0$, but does hardly mitigate the effects of infrequent rebalancing.

The results of this article can be extended to the important class of power utility functions $U(\pi_H)=(1/\gamma)\pi_H^{\gamma}$, $\gamma\in\mathbb{R}$, where $\pi_H\geq 0$ denotes final wealth. Like the logarithm, the limiting case for $\gamma\to 0$, all functions within this class exhibit convenient separation properties. Alternative approaches to portfolio rebalancing should also be investigated in the future. Instead of predetermined time points, one might want to rebalance the portfolio only when a significant mismatch between the actual and target states is detected. In such a framework, τ becomes a randomized stopping time. Furthermore, one could think of more realistic market models in which drift rates and covariances of the asset price processes are stochastic and/or unobservable. Then, our conclusion that frequent rebalancing is often ineffective would have to be carefully reconsidered.

Acknowledgement

Daniel Kuhn thanks the Swiss National Science Foundation for financial support.

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Appendix A: Technical background results

Proposition A.1: *Let $g(w,\lambda)=\varphi_{\lambda^2}(w)$ be the objective function of $\mathcal{P}'(\lambda^2)$. Then, g is continuous on $W\times\mathbb{R}$. Moreover, all higher-order partial derivatives of g are well-defined and continuous on the interior of $W\times\mathbb{R}$ and have a continuous extension to $W\times\mathbb{R}$.*

Proof: Consider the auxiliary functional

$$F: W\times\mathbb{R}\to\mathbb{R},$$

$$F(w,\lambda)=\mathrm{E}\left[\ln\left(1+\sum_{i=1}^{n}w_i\left(\mathrm{e}^{\tilde{\nu}_i\lambda^2+\tilde{\varepsilon}_i\lambda}-1\right)\right)\right].$$

By means of the dominated convergence theorem it can be shown that F has infinitely many continuous partial derivatives on the interior of $W\times\mathbb{R}$, all of which have a continuous extension to the entire domain $W\times\mathbb{R}$. A simple calculation shows that $F(w,0)=0$. Moreover, since the $\tilde{\varepsilon}_i$ are normally distributed with zero mean, F is symmetric around $\lambda=0$. This implies that all odd-order partial derivatives of F with respect to λ must vanish at $\lambda=0$. In conclusion, the objective function $g(w,\lambda)$ of problem $\mathcal{P}'(\lambda^2)$, which can be expressed as $\nu_0+F(w,\lambda)/\lambda^2$ for $\lambda\neq 0$ and as $\nu_0+\frac{1}{2}\partial^2F(w,0)/\partial\lambda^2$ for $\lambda=0$, is continuous on $W\times\mathbb{R}$ and has continuous partial derivatives of all orders on the interior of $W\times\mathbb{R}$ with continuous extensions to all boundary points. In particular, note that the division by λ^2 does not produce a pole. □

Proposition A.2: *The Hessian of φ_{λ^2} at $w\in W$ is negative definite and invertible for all parameters $\lambda\in\mathbb{R}$.*

Proof: Consider again the function F introduced in the proof of Proposition A.1. The Hessian matrix of F with respect to the first argument w is given by

$$H_w(F)(w,\lambda)=\nabla_w\nabla_w^\top F(w,\lambda)=-E\left[\frac{r(\tilde{\varepsilon},\lambda)r(\tilde{\varepsilon},\lambda)^\top}{(1+w^\top r(\tilde{\varepsilon},\lambda))^2}\right],$$

where $r(\tilde{\varepsilon},\lambda)$ is an n-vector whose ith entry $\mathrm{e}^{\tilde{\nu}_i\lambda^2+\tilde{\varepsilon}_i\lambda}-1$. We will argue that $H_w(F)$ is negative definite and invertible for all $w\in W$ and $\lambda\neq 0$. To this end, choose an arbitrary vector $\xi\neq 0$ in $\mathbb{R}^n$. Then, we find

$$\xi^\top H_w(F)(w,\lambda)\xi=-E\left[\frac{(\xi^\top r(\tilde{\varepsilon},\lambda))^2}{(1+w^\top r(\tilde{\varepsilon},\lambda))^2}\right]<0.$$

The last inequality follows from nonnegativity and continuity of the integrand and the fact that $\tilde{\varepsilon}$ has a strictly positive probability density function (this is equivalent to the covariance matrix $\tilde{\sigma}$ having full rank). Moreover, we use that $\xi^\top r(\tilde{\varepsilon},\lambda)$ cannot be zero for all $\tilde{\varepsilon}\in\mathbb{R}^n$ since the set $\{r(\tilde{\varepsilon},\lambda)|\tilde{\varepsilon}\in\mathbb{R}^n\}$ has dimension n for all $\lambda\neq 0$. As $\xi\neq 0$ was arbitrary, $H_w(F)$ is negative definite for all $\lambda\neq 0$. Fixing $w\in W$, the Hessian matrix of the objective function φ_{λ^2} is given by $H_w(F)/\lambda^2$ for $\lambda\neq 0$ and by $-\tilde{S}$ for $\lambda=0$. It is negative definite in any case and hence invertible. This observation completes the proof. □

Proposition A.2 implies that $\mathcal{P}'(\lambda^2)$ has a unique solution for each $\lambda\in\mathbb{R}$.

Proposition A.3: *If problem $\mathcal{P}'(\lambda_0^2)$ is nondegenerate for some $\lambda_0\in\mathbb{R}$, then the optimal value function $\lambda\mapsto\max\mathcal{P}'(\lambda^2)$ and the single-valued optimizer mapping $\lambda\mapsto\arg\max P'(\lambda^2)$ are infinitely often differentiable on a neighbourhood of λ_0. Moreover, $\mathcal{S}(\lambda^2)$ is locally constant at λ_0.*

Proof: By permutation symmetry, we may assume that $\mathcal{S}(\lambda_0^2)$ contains all nonnegative integers smaller or equal to $\hat{n}$, where $0\leq\hat{n}\leq n$. In order to keep notation simple, we set

$$\hat{w}=(w_1,\ldots,w_{\hat{n}})\quad\text{and}\quad\check{w}=(w_{\hat{n}+1},\ldots,w_n),$$

which implies that $w = (\hat{w}, \check{w})$. Moreover, we define an auxiliary function

$$\hat{g}(\hat{w}, \lambda) = \varphi_{\lambda^2}(w)\big|_{\check{w}=0}.$$

Notice that $\hat{g}$ and all its higher-order partial derivatives are continuous on the interior of $\hat{W} \times \mathbb{R}$, where $\hat{W} = \{\hat{w} \in \mathbb{R}^{\hat{n}}_{+} \,|\, \sum_{i=1}^{\hat{n}} \hat{w}_i \le 1\}$ is the compact standard simplex in $\mathbb{R}^{\hat{n}}$; for details see Proposition A.1. Denote by $w_0^* = (\hat{w}_0^*, \check{w}_0^*)$ the solution of the reference problem $\mathcal{P}'(\lambda_0^2)$. By construction, $\hat{w}_0^*$ lies in the interior of the simplex $\hat{W}$, while $\check{w}_0^*$ vanishes. Next, for some suitable neighbourhood U of λ_0 let $\hat{w}^* : U \to \mathbb{R}^{\hat{n}}$ be an infinitely often differentiable mapping with $\hat{w}^*(\lambda_0) = \hat{w}_0^*$ such that

$$\nabla_{\hat{w}} \hat{g}(\hat{w}^*(\lambda), \lambda) = 0 \quad \text{for all } \lambda \in \mathrm{U}.$$

The existence of $\hat{w}^*$ is ensured by the implicit function theorem (Munkres 1991), which applies since the Hessian of $\hat{g}$ with respect to its first argument is negative definite and invertible at $(\hat{w}_0^*, \lambda_0)$. Next, introduce a constant mapping $\check{w}^* : U \to \mathbb{R}^{n-\hat{n}}$ which vanishes on its whole domain, and define the product mapping $w^* = (\hat{w}^*, \check{w}^*)$. By construction of w^* and nondegeneracy of problem $\mathcal{P}'(\lambda_0^2)$, there is a neighbourhood $V \subset U$ of λ_0 such that $w^*(\lambda) \in W$ and

$$\left.\begin{array}{lll} w_i^*(\lambda) > 0, & \partial\varphi_{\lambda^2}(w^*(\lambda))/\partial w_i = 0 & i \in \mathcal{S}(\lambda_0^2) \\ w_i^*(\lambda) = 0, & \partial\varphi_{\lambda^2}(w^*(\lambda))/\partial w_i < 0 & i \notin \mathcal{S}(\lambda_0^2) \end{array}\right\}$$

for all $\lambda \in$ V.

As it satisfies the necessary and sufficient optimality conditions, $w^*(\lambda)$ is the unique solution of problem $\mathcal{P}'(\lambda^2)$. By construction, the optimizer mapping $w^*(\lambda)$ and the optimal value function $\varphi_{\lambda^2}(w^*(\lambda), \lambda)$ of problem $\mathcal{P}'(\lambda^2)$ are infinitely often differentiable on V. Moreover, we have $\mathcal{S}(\lambda^2) = \mathcal{S}(\lambda_0^2)$ on V, that is, the set of assets in the optimal portfolio is locally constant. □

Proposition A.4: *The sensitivity $g^{(1)}$ derived in Theorem 4.4(ii) is nonnegative.*

Proof: As usual, define $w^{(0)} = \tilde{S}^{-1}\tilde{\mu}$ as the vector of optimal portfolio weights in the continuous-time limit. Since, by assumption, all assets enter the optimal continuously rebalanced portfolio, each component of $w^{(0)}$ is strictly positive. We first reexpress $g^{(1)}$ in terms of $\tilde{S}$ and $w^{(0)}$,

$$\begin{aligned} 4g^{(1)} &= \sum_{i,j=1}^{n} w_i^{(0)} \tilde{\sigma}_{ij}^2 w_j^{(0)} - 2 \sum_{i,j,k=1}^{n} w_i^{(0)} \tilde{\sigma}_{ij} w_j^{(0)} \tilde{\sigma}_{ik} w_k^{(0)} \\ &\quad + \sum_{i,j,k,l=1}^{n} w_i^{(0)} \tilde{\sigma}_{ij} w_j^{(0)} w_k^{(0)} \tilde{\sigma}_{kl} w_l^{(0)} \\ &= \sum_{i,j,k,l=1}^{n} \tilde{\sigma}_{ij} \tilde{\sigma}_{kl} (w_i^{(0)} w_j^{(0)} \delta_{ik} \delta_{jl} - w_i^{(0)} w_j^{(0)} w_k^{(0)} \delta_{jl} \\ &\quad - w_i^{(0)} w_j^{(0)} w_l^{(0)} \delta_{ik} + w_i^{(0)} w_j^{(0)} w_k^{(0)} w_l^{(0)}). \end{aligned}$$

Symmetry of the covariance matrix $\tilde{S}$ is used to rearrange terms in the second line. Next, we introduce an $n \times n$ matrix A with elements $A_{ij} = w_i^{(0)} \tilde{\sigma}_{ij} w_j^{(0)}$. Notice that A inherits positivity of the covariance matrix $\tilde{S}$. Furthermore, we define an n-vector b with elements $b_i = 1/w_i^{(0)}$, all of whose entries are strictly positive. Using this new notation, we find

$$\begin{aligned} 4g^1 &= \sum_{i,j,k,l=1}^{n} A_{ij} A_{kl} (b_k b_l \delta_{ik} \delta_{jl} - b_l \delta_{jl} - b_k \delta_{ik} + 1) \\ &= \sum_{i,j,k,l=1}^{n} A_{ij} A_{kl} (1 - b_k \delta_{ik})(1 - b_l \delta_{jl}). \end{aligned}$$

Now, let C be the upper triangular Choleski decomposition matrix corresponding to A, that is, $A = C^\top C$, and define D as the symmetric matrix with entries $D_{ij} = 1 - b_i \delta_{ij}$. With these conventions, we can rewrite $g^{(1)}$ as

$$\begin{aligned} 4g^{(1)} &= \mathrm{Tr}(ADAD) \\ &= \mathrm{Tr}(C^\top C D C^\top C D) \\ &= \mathrm{Tr}(C D C^\top C D C^\top) \ge 0. \end{aligned}$$

The third equality follows from the fact that the trace of a product of square matrices is invariant under cyclic permutations, and the last inequality follows from symmetry of CDC$^\top$, which implies positivity of CDC$^\top$ CDC$^\top$. As positive matrices have nonnegative trace, the claim is established. □

Appendix B: Proof of Theorem 4.4

Proof of Theorem 4.4(i): Denote by $w^*(\lambda)$ the unique solution of problem $\mathcal{P}'(\lambda^2)$, $\lambda \in \mathbb{R}$. Since the reference problem for $\lambda = 0$ is nondegenerate, the mapping w^* is infinitely often differentiable on a neighbourhood of $\lambda = 0$; see Proposition A.3. Thus, by Taylor's theorem, w^* can be expanded in powers of λ, that is,

$$\arg\max \mathcal{P}'(\lambda^2) = w^*(\lambda) = w^{(0)} - w^{(1)} \lambda^2 + o(\lambda^2). \qquad \text{(B1)}$$

Symmetry with respect to the origin forbids odd powers of λ in the above expansion. Furthermore, Proposition A.3 and the assumption that $\mathcal{S}(0)$ contains all available assets imply that w^* fulfills the optimality conditions

$$\nabla \varphi_{\lambda^2}(w^*(\lambda)) = 0, \qquad \text{(B2)}$$

which determine the coefficients of the expansion (B1). To see this, we first introduce a random function

$$\psi(w, \lambda) = \ln\left(1 + \sum_{i=1}^{n} w_i \left(\mathrm{e}^{\tilde{\nu}_i \lambda^2 + \tilde{\varepsilon}_i \lambda} - 1\right)\right).$$

Next, we multiply (B2) by λ^2, differentiate k times with respect to λ, $k \in \mathbb{N}_0$, and express the result in terms of ψ. This yields

$$\begin{aligned} 0 \equiv \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \lambda^2 \frac{\partial}{\partial w_i} \varphi_{\lambda^2}(w^*(\lambda)) &= \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \frac{\partial}{\partial w_i} \mathrm{E}[\psi(w^*(\lambda), \lambda)] \\ &= \mathrm{E}\left[\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \frac{\partial}{\partial w_i} \psi(w^*(\lambda), \lambda)\right]. \end{aligned}$$

Interchanging the differentiation and the expectation operators is allowed by the dominated convergence theorem. Although the above identity holds for all $\lambda \in \mathbb{R}$, it is sufficient to consider the point $\lambda = 0$. For brevity of notation, we introduce random variables

$$\psi_{i,k} = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\frac{\partial}{\partial w_i}\psi(w^*(\lambda),\lambda)\bigg|_{\lambda=0}.$$

At optimality, the mean values $E\psi_{i,k}$ must vanish for all $i = 1, \ldots, n$ and for all nonnegative integers k. When calculating the expectations, we use the fact that odd monomials of the $\tilde{\varepsilon}_i$ have zero expectation. Moreover, we use the relations

$$E(\tilde{\varepsilon}_i\tilde{\varepsilon}_j) = \tilde{\sigma}_{ij} \quad \text{and} \quad E(\tilde{\varepsilon}_i\tilde{\varepsilon}_j\tilde{\varepsilon}_k\tilde{\varepsilon}_l) = \tilde{\sigma}_{ij}\tilde{\sigma}_{kl} + \tilde{\sigma}_{ik}\tilde{\sigma}_{jl} + \tilde{\sigma}_{il}\tilde{\sigma}_{jk}.$$

The last identity implies that all fourth-order moments of a Gaussian random vector can be expressed easily in terms of second-order moments; this useful property will substantially simplify our calculations below. To begin with, we find that $\psi_{i,0} = 0$ and $\psi_{i,1} = \tilde{\varepsilon}_i$, both of which have zero mean.† This is consistent with the underlying optimality conditions. The first nontrivial case is for $k = 2$, where

$$\psi_{i,2} = \tilde{\nu}_i + \frac{\tilde{\varepsilon}_i^2}{2} - \sum_{j=1}^n \tilde{\varepsilon}_i\tilde{\varepsilon}_j w_j^{(0)} \Rightarrow E\psi_{i,2} = \tilde{\mu}_i - \sum_{j=1}^n \tilde{\sigma}_{ij}w_j^{(0)}.$$

Consequently, the requirement that $E\psi_{i,2}$ must vanish implies $w^{(0)} = \tilde{S}^{-1}\tilde{\mu}$. The random variables $\psi_{i,3}$ are representable as odd polynomials in the $\tilde{\varepsilon}_i$, and no further calculation is necessary to see that they have zero expectation. Hence, we may directly proceed to the case $k = 4$. A tedious algebraic calculation yields

$$\begin{aligned}
\psi_{i,4} = {} & \frac{\tilde{\nu}_i^2}{2} + \frac{\tilde{\nu}_i\tilde{\varepsilon}_i^2}{2} + \frac{\tilde{\varepsilon}_i^4}{24} - \sum_{j=1}^n w_j^{(0)}\tilde{\varepsilon}_j\left(\tilde{\varepsilon}_i\tilde{\nu}_i + \frac{\tilde{\varepsilon}_i^3}{6}\right) \\
& - \sum_{j=1}^n w_j^{(0)}\left(\tilde{\nu}_j + \frac{\tilde{\varepsilon}_j^2}{2}\right)\left(\tilde{\nu}_i + \frac{\tilde{\varepsilon}_i^2}{2}\right) \\
& + \sum_{j,k=1}^n w_j^{(0)}\tilde{\varepsilon}_j\tilde{\varepsilon}_k w_k^{(0)}\left(\tilde{\nu}_i + \frac{\tilde{\varepsilon}_i^2}{2}\right) \\
& - \sum_{j=1}^n w_j^{(0)}\left(\tilde{\varepsilon}_j\tilde{\nu}_j + \frac{\tilde{\varepsilon}_j^3}{6}\right)\tilde{\varepsilon}_i + \sum_{j=1}^n w_j^{(1)}\tilde{\varepsilon}_j\tilde{\varepsilon}_i \\
& - \left(\sum_{j=1}^n w_j^{(0)}\tilde{\varepsilon}_j\right)^3\tilde{\varepsilon}_i \\
& + 2\left(\sum_{j=1}^n w_j^{(0)}\tilde{\varepsilon}_j\right)\left(\sum_{k=1}^n w_k^{(0)}(\tilde{\nu}_k + \frac{\tilde{\varepsilon}_k^2}{2})\right)\tilde{\varepsilon}_i.
\end{aligned}$$

Taking the expected value, substituting the explicit formula for $w^{(0)}$, and rearranging terms we find

$$\begin{aligned}
E\psi_{i,4} = {} & \frac{\tilde{\mu}_i^2}{2} + \sum_{j=1}^n \tilde{\sigma}_{ij}w_j^{(1)} \\
& + \sum_{j,k=1}^n\left(\tilde{\sigma}_{ij}\tilde{\mu}_j\tilde{\sigma}_{jk}^{-1}\tilde{\mu}_k - \tilde{\mu}_i\tilde{\mu}_j\tilde{\sigma}_{jk}^{-1}\tilde{\mu}_k - \frac{\tilde{\sigma}_{ij}^2}{2}\tilde{\sigma}_{jk}^{-1}\tilde{\mu}_k\right).
\end{aligned}$$

As $E\psi_{i,4}$ must vanish for all $i = 1, \ldots, n$, we obtain

$$w^{(1)} = \tilde{S}^{-1}\left(\frac{1}{2}Q - \frac{1}{2}M\tilde{S} - \tilde{S}M + \tilde{\mu}\tilde{\mu}^\top\right)\tilde{S}^{-1}\tilde{\mu}.$$

The proof is completed by plugging the explicit formulae for the coefficients $w^{(0)}$ and $w^{(1)}$ into (B1) and replacing λ^2 by τ. □

Proof of Theorem 4.4(ii): We use the same notation as in the proof of Theorem 4.4(i). Since the reference problem for $\lambda = 0$ is nondegenerate, the mapping w^* is infinitely often differentiable on a neighbourhood of $\lambda = 0$, and $w^*(0)$ lies in the interior of W. Moreover, the parametric objective function $(w, \lambda) \mapsto \varphi_{\lambda^2}(w)$ is infinitely often differentiable on the interior of $W \times \mathbb{R}$; see Proposition A.1. Thus, the composed mapping

$$\lambda \mapsto \lambda^2(\varphi_{\lambda^2}(w^*(\lambda)) - \nu_0) = E[\psi(w^*(\lambda), \lambda)]$$

is locally smooth at the origin, and we may use Taylor's theorem to write

$$\begin{aligned}
\max \mathcal{P}'(\lambda^2) = {} & \varphi_{\lambda^2}(w^*(\lambda)) \\
= {} & \nu_0 + \frac{1}{\lambda^2}E[\psi(w^*(\lambda),\lambda)] \\
= {} & \nu_0 + \frac{1}{\lambda^2}\left(\sum_{k=0}^4 E[\psi_k]\lambda^k + o(\lambda^4)\right),
\end{aligned} \tag{B3}$$

where

$$\psi_k = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\psi(w^*(\lambda),\lambda)\bigg|_{\lambda=0}.$$

Note that the derivation of (B3) uses commutativity of the differentiation and expectation operators, which follows from the dominated convergence theorem. Using the Taylor approximation (B1) from the proof of Theorem 4.4(i), it is easily seen that $\psi_0 = 0$ and $\psi_1 = \sum_{i=1}^n w_i^{(0)}\tilde{\varepsilon}_i$, both of which have zero expectation. The first random variable with nonzero mean is ψ_2. It can be expressed as

$$\psi_2 = \sum_{i=1}^n w_i^{(0)}\left(\tilde{\nu}_i + \frac{\tilde{\varepsilon}_i^2}{2}\right) - \frac{1}{2}\sum_{i,j=1}^n w_i^{(0)}\tilde{\varepsilon}_i\tilde{\varepsilon}_j w_j^{(0)}.$$

Taking the expected value and substituting the explicit formula for $w^{(0)}$ yields $E\psi_2 = \frac{1}{2}\tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu}$. The random variable ψ_3, in contrast, is representable as an odd polynomial in the $\tilde{\varepsilon}_i$ and thus has zero expectation.

$$\begin{aligned}
\psi_3 = {} & \sum_{i=1}^n w_i^{(0)}\left(\tilde{\varepsilon}_i\tilde{\nu}_i + \frac{\tilde{\varepsilon}_i^3}{6}\right) - \sum_{i,j=1}^n w_i^{(0)}\tilde{\varepsilon}_i w_j^{(0)}\left(\tilde{\nu}_j + \frac{\tilde{\varepsilon}_j^2}{2}\right) \\
& + \frac{1}{3}\left(\sum_{i=1}^n w_i^{(0)}\tilde{\varepsilon}_i\right)^3 + \sum_{i=1}^n w_i^{(1)}\tilde{\varepsilon}_i.
\end{aligned}$$

†For small values of k the $\psi_{i,k}$ are found by expanding $\partial_{w_i}\psi(w^*(\lambda), \lambda)$ in powers of λ.

Although the contribution of ψ_3 to the expansion (B3) vanishes, it will be needed in the proof of Theorem 4.4(iii), below. Next, we evaluate ψ_4.

$$\begin{aligned}\psi_4 = &\sum_{i=1}^{n} w_i^{(0)}\left(\frac{\tilde{\nu}_i^2}{2}+\frac{\tilde{\nu}_i\tilde{\varepsilon}_i^2}{2}+\frac{\tilde{\varepsilon}_i^4}{24}\right)-\sum_{i=1}^{n} w_i^{(1)}\left(\tilde{\nu}_i+\frac{\tilde{\varepsilon}_i^2}{2}\right)\\ &-\frac{1}{2}\sum_{i,j=1}^{n} w_i^{(0)}w_j^{(0)}\left(\frac{\tilde{\varepsilon}_i^2\tilde{\varepsilon}_j^2}{4}+\tilde{\nu}_i\tilde{\nu}_j+\frac{\tilde{\nu}_i\tilde{\varepsilon}_j^2+\tilde{\nu}_j\tilde{\varepsilon}_i^2}{2}\right)\\ &+\sum_{i,l,k=1}^{n} w_i^{(0)}\Bigg(\tilde{\varepsilon}_i\tilde{\varepsilon}_j w_j^{(0)}w_k^{(0)}(\tilde{\nu}_k+\frac{\tilde{\varepsilon}_k^2}{2})+\tilde{\varepsilon}_i\tilde{\varepsilon}_j w_j^{(1)}\\ &-\tilde{\varepsilon}_i\tilde{\varepsilon}_j w_j^{(0)}\tilde{\nu}_j-\frac{w_j^{(0)}\tilde{\varepsilon}_i\tilde{\varepsilon}_j^3}{6}\Bigg)-\frac{1}{4}\left(\sum_{i=1}^{n} w_i^{(0)}\tilde{\varepsilon}_i\right)^4.\end{aligned}$$

It is worthwhile to remark that both terms involving $w^{(1)}$ cancel out after taking the expected value and substituting $w^{(0)}=\tilde{S}^{-1}\tilde{\mu}$. This considerably simplifies the evaluation of $E\psi_4$. A somewhat lengthy but conceptually simple calculation shows that

$$\begin{aligned}E\psi_4 &= \frac{1}{2}\sum_{i,j=1}^{n}\tilde{\mu}_i\tilde{\sigma}_{ij}^{-1}\tilde{\mu}_j^2-\frac{1}{4}\left(\sum_{i,j=1}^{n}\tilde{\mu}_i\tilde{\sigma}_{ij}^{-1}\tilde{\mu}_j\right)^2\\ &\quad-\frac{1}{4}\sum_{i,j,k,l=1}^{n}\tilde{\mu}_i\tilde{\sigma}_{ij}^{-1}\tilde{\sigma}_{jk}^2\tilde{\sigma}_{kl}^{-1}\tilde{\mu}_l\\ &= \frac{1}{4}\tilde{\mu}^\top\tilde{S}^{-1}(Q-M\tilde{S}-\tilde{S}M+\tilde{\mu}\tilde{\mu}^\top)\tilde{S}^{-1}\tilde{\mu}.\end{aligned}$$

The claim finally follows by plugging the formulae for the $E\psi_k$ into (B3) and replacing λ^2 by τ. □

Proof of Theorem 4.4(iii): Consider problem $\mathcal{P}(\tau)$, which is well-defined for $H=\tau^{-1}\in\mathbb{N}$. First, we express the variance of the optimal portfolio's growth rate as

$$v(\tau)=\operatorname{Var}\ln\left(\frac{\pi_H}{\pi_0}\right)=\sum_{h=0}^{H-1}\operatorname{Var}\ln\left(\frac{\pi_{h+1}}{\pi_h}\right)=H\operatorname{Var}\ln\left(\frac{\pi_1}{\pi_0}\right).$$

This is possible since the random variables π_{h+1}/π_h, $h\in\mathbb{N}_0$, are independent and identically distributed. Independence follows from the fact that the optimal strategy, which controls the wealth process π, is (essentially) deterministic, and the increments of Wiener processes are independent. Using the same notation as in the proof of Theorem 4.4(i), and setting $\tau=\lambda^2$, we find

$$\begin{aligned}v(\lambda^2)&=\frac{1}{\lambda^2}\operatorname{Var}\big[\psi(w^*(\lambda),\lambda)+\nu_0\lambda^2+\varepsilon_0\lambda\big]\\ &=\frac{1}{\lambda^2}E\big[\big(\psi(w^*(\lambda),\lambda)+\varepsilon_0\lambda\big)^2\big]-\frac{1}{\lambda^2}\big(E\big[\psi(w^*(\lambda),\lambda)\big]\big)^2.\end{aligned}\tag{B4}$$

The second term has already been analysed in Theorem 4.4(i). For present purposes, a second-order expansion in λ is sufficient, that is,

$$E\big[\psi(w^*(\lambda),\lambda)\big]=\frac{1}{2}\tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu}\lambda^2+o(\lambda^2).$$

To evaluate the first term in (B4), we introduce a new random function

$$\chi(\lambda)=\big(\psi(w^*(\lambda),\lambda)+\varepsilon_0\lambda\big)^2,\tag{B5}$$

whose expectation is smooth in λ. By using Taylor's theorem, we may thus write

$$E[\chi(\lambda)]=\sum_{k=0}^{4}E[\chi_k]\lambda^k+o(\lambda^4),\quad\text{where}\quad \chi_k=\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\chi(\lambda)\bigg|_{\lambda=0}.$$

As usual, interchangeability of the differentiation and expectation operators is ensured by the dominated convergence theorem. The (pointwise) expansion of the random function $\chi(\lambda)$ around the origin is conveniently obtained by plugging the well-known expansion of $\psi(w^*(\lambda),\lambda)$ into (B5). As ψ_0 vanishes we may conclude that χ_0 and χ_1 are zero, as well. The first nontrivial contribution comes from the second-order coefficient, which is given by $\chi_2=\psi_1^2+2\psi_1\varepsilon_0+\varepsilon_0^2$. Using our knowledge of ψ_1 from the proof of Theorem 4.4(ii), we find

$$E\chi_2=\tilde{\mu}^\top\tilde{S}^{-1}\tilde{\mu}+2\tilde{\mu}^\top\tilde{S}^{-1}\varsigma+\sigma_0^2.$$

The next coefficient $\chi_3=2\psi_1\psi_2+2\psi_2\varepsilon_0$ has zero expectation, again, as it is representable as an odd polynomial in the ε_i. Finally, the last relevant coefficient in our expansion amounts to $\chi_4=\psi_2^2+2\psi_1\psi_3+2\psi_3\varepsilon_0$. After a lengthy but straightforward calculation, which uses our knowledge of ψ_1, ψ_2, and ψ_3 from the proof of Theorem 4.4(ii), we obtain

$$\begin{aligned}E\chi_4&=\tilde{\mu}^\top\tilde{S}^{-1}\left(\frac{1}{2}Q-M\tilde{S}-\tilde{S}M+\frac{7}{4}\tilde{\mu}\tilde{\mu}^\top\right)\tilde{S}^{-1}\tilde{\mu}\\ &\quad+\tilde{\mu}^\top\tilde{S}^{-1}\left(Q-M\tilde{S}-\tilde{S}M+2\tilde{\mu}\tilde{\mu}^\top\right)\tilde{S}^{-1}\varsigma.\end{aligned}$$

Replacing the expectations in (B4) by their Taylor approximations, and substituting τ for λ^2, the claim follows. □