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Risk parity portfolio optimization under a Markov regime-switching framework

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We formulate and solve a risk parity optimization problem under a Markov regime-switching framework to improve parameter estimation and to systematically mitigate the sensitivity of optimal portfolios to estimation error. A regime-switching factor model of returns is introduced to account for the abrupt changes in the behaviour of economic time series associated with financial cycles. This model incorporates market dynamics in an effort to improve parameter estimation. We proceed to use this model for risk parity optimization and also consider the construction of a robust version of the risk parity optimization by introducing uncertainty structures to the estimated market parameters. We test our model by constructing a regime-switching risk parity portfolio based on the Fama–French three-factor model. The out-of-sample computational results show that a regime-switching risk parity portfolio can consistently outperform its nominal counterpart, maintaining a similar *ex post* level of risk while delivering higher-than-nominal returns over a long-term investment horizon. Moreover, we present a dynamic portfolio rebalancing policy that further magnifies the benefits of a regime-switching portfolio.

Keywords: Risk parity; Asset allocation; Factor model; Markov regime switching; Robust optimization; Uncertainty

1. Introduction

The problem of optimal portfolio construction has been widely explored in academic literature since the introduction of Modern Portfolio Theory in Markowitz (1952). While a powerful tool with widespread use, the solutions yielded by a mean-variance analysis suffer from considerable drawbacks with regard to its practical implementation. The optimal mean-variance solution is significantly sensitive to the level of noise in the estimated input parameters, specifically the asset expected returns and covariance matrix. Best and Grauer (1991) explored and measured the sensitivity of optimal mean-variance portfolios to the changes in the asset expected returns, and suggest that the covariance estimates tend to be more stable (Merton 1980). Chopra and Ziemba (1993) found that estimation errors can significantly impact portfolio construction, often yielding overconcentrated (non-diversified) and unintuitive portfolios, arguing that estimation errors in expected returns can have an impact 10 times larger than errors in the covariance matrix during mean-variance optimization. This is followed by Broadie (1993), in which the effects of estimation error is not only studied but also suggests an alternative method for adjusting for the bias inherent in mean-variance analysis. This motivated a stream of academic research on the subject of parameter uncertainty and methods by which to mitigate its impact, such as resampling (Michaud and Michaud 2007). Other more recent publications have attempted to capture and mitigate estimation error through robust optimization methods, creating uncertainty sets around noisy parameters (Goldfarb and Iyengar 2003, Tütüncü and Koenig 2004, Delage and Ye 2010).

The risk parity portfolio, also referred to as equally weighted risk contribution (ERC) portfolio, enables the construction of diversified portfolios where resources are allocated solely based on the measure of risk in such a way that the risk contribution of every asset is the same (Maillard *et al.* 2010, Demey *et al.* 2010, Roncalli 2013). Consequently, these approaches do not require the estimation of expected returns. Avoiding the use of estimated returns is, within itself,

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a way of significantly reducing estimation error. Moreover, the resulting portfolios tend to be well diversified. However, the risk parity approach does require the estimation of a risk measure and therefore could still be subject to the adverse effects of estimation error.

The topic of risk parity portfolios has gained popularity in recent years, with research in both academia and industry. Bai *et al.* (2016) present an alternative and more computationally efficient formulation of the risk parity problem, and study scenarios with multiple budget constraints where the solution space may not be convex. Chaves *et al.* (2011) measure the empirical performance of risk parity portfolios relative to other typical asset allocation techniques, including mean-variance optimization. They proceed to show the dependency of risk parity on the selection of portfolio constituents, advising users of the unobserved risk arising from the selection of constituents and their corresponding asset classes. Lohre *et al.* (2014) present an alternative view on risk parity, where a diverse risk parity portfolio can be constructed by decomposing a portfolio based on its underlying risk measures via principal component analysis (PCA). Rather than seeking risk diversification from an individual asset level, they use PCA to diversify risk from uncorrelated underlying factors, which is particularly relevant when considering different asset classes. However, throughout this paper we will focus solely on the fundamental risk parity problem proposed in Maillard *et al.* (2010). The motivation for our choice is to study the effect that a regime-switching framework could have on the basis risk parity problem, and we wish to isolate this from additional constraints that may otherwise affect the assessment of our proposed framework. Nevertheless, adding these constraints and insights from the aforementioned publications may prove valuable to expand the regime-switching framework, which may be the subject of future work.

This paper considers methods aimed at improving parameter estimation in the context of risk parity optimization, where we focus primarily on improving the estimated asset covariances. First, we formulate a factor model of asset returns which explicitly incorporates a Markov regime-switching framework (Hamilton 1989, 2010). By incorporating a regime-switching factor model of returns we attempt to more faithfully reflect the market dynamics within the covariance matrix estimate. Second, we attempt to further minimize the effect of estimation error by constructing an associated robust optimization problem by introducing uncertainty structures around the estimated parameters as prescribed in Goldfarb and Iyengar (2003). We are able to show that a regime-switching ERC portfolio shows that it can yield a higher risk-adjusted rate of return when compared to a nominal† ERC portfolio. Below we present a description of each section in this paper.

In Section 2, we present the risk parity non-linear program (NLP) introduced in Maillard *et al.* (2010), where the optimization objective is to equalize the risk contributions from the different components of the portfolio. The proposed model is an ideological variant of the well-known '$1/n$' portfolio, and it benefits from not necessitating an estimate of the asset expected returns.

In Section 3, we follow as in Hamilton (1989, 2010) and present a regime-switching factor model of asset returns. As described by Hamilton, economic time series periodically exhibit dramatic breaks in their behaviour, usually as a result of financial crises or political turmoil. Traditional factor models rely on a longitudinal linear regression through an extended time series. However, under the assumption that the observed data are the result of alternating regimes, a classic linear regression would result in an oversimplification of the market dynamics, and would incorrectly yield an estimate blending the distinct market regimes. Thus, it is proposed that any model of asset returns should take into consideration the current state (i.e. regime) of the market. Our computational results show that a two-state model suffices to faithfully capture the market dynamics, where a 'bull' market is be represented by a model with lower volatility, while a 'bear' market has the opposite properties. The unobserved properties that control the dynamics of these sudden changes can be modelled by a Markov chain, which leads to the development of a Markov regime-switching model. The introduction of a regime-switching framework into a nominal factor model of returns allows us to 'switch' between the estimated parameters corresponding to a bull or bear market, with the intent of improving the accuracy of estimation. However, the authors note that this introduces another source of uncertainty due to the regime estimation error. In practice, the Markov regime-switching model is estimated by following the Baum–Welch algorithm (Baum *et al.* 1970). The computational implementation of the regime-switching model used for the experiment in this paper was derived from the algorithms presented in Kole (2010) and Kritzman *et al.* (2012).

In Section 4, we present a regime-switching ERC portfolio model. By using the asset covariance matrix derived from the regime-switching factor model as the input of the risk parity NLP, we are able to incorporate the market dynamics implicitly into the portfolio optimization. The resulting portfolio is, therefore, better aligned with the current market state. By design, the regime-switching risk parity NLP retains the same level of complexity as the nominal ERC portfolio, ensuring its computational tractability. Thus, in practice, this method can be easily scaled to efficiently solve large-scale problems.

In Section 5, we proceed to use robust optimization in an attempt to further reduce the effect of estimation error in regime parameters. A robust framework is proposed in Kapsos *et al.* (2017) as a natural extension of robustness stemming from mean-variance optimization, where the worst-case estimate of the covariance matrix is considered. We apply this same principle under the assumption that the source of uncertainty stems from the factor model itself. Thus, for our purpose, the robust framework described in Goldfarb and Iyengar (2003) allows us to create uncertainty sets from a factor model calibrated directly from raw market data, without the additional challenge of estimating any other scenario-based probabilities, such as the stochastic programming framework proposed in Mulvey *et al.* (1995). We create these uncertainty sets around the regression coefficients, specifically the

† The authors note the use of the word 'nominal' throughout this paper in the context of the Operations Research discipline, where it serves to differentiate a basis model from its *robust* counterpart. It should not be taken in the context of Economics, where it serves to differentiate between *real* and *nominal* values. Thus, the basic ERC model shall be referred to as the *nominal*.

factor loadings and the variance of the residual term. This model serves to illustrate the adaptability of the regime-switching framework, which can be easily tailored to suit other investment methods.

Section 6 presents the results of two separate computational experiments. The theory in this paper is developed to accommodate any general factor model. However, we select the well-known Fama–French three-factor model (Fama and French 1993) to illustrate the regime-switching framework. To further improve the validity of our results, we use 15 years of weekly asset and factor returns to prepare and calibrate the model, and a further 11 years to perform the out-of-sample test. We then proceed to explain the process of regime estimation, including our choice of number of regimes. The first experiment involves a typical fixed-term rebalancing approach, where the portfolios are rebalanced every six months. The second experiment explores the use of the Markov regime-switching model for the purpose of formulating a dynamic rebalancing policy, only allowing the portfolio to be rebalanced when a change in regime is perceived. This serves to better align the portfolio regime-switching parameters with the current state of the market. The computational results are measured relative to the nominal ERC portfolio model. The relative portfolio wealth, Sharpe ratio, and turnover rate are used to measure overall performance. The results show that a two-state regime-switching model is able to consistently outperform the nominal ERC model, yielding a higher risk-adjusted rate of return. The experimental results also highlight that a robust formulation of the ERC portfolio can be an exceedingly conservative investment strategy, diminishing its potential gains during a market upside.

By construction, the ERC portfolio maximizes the diversification of risk between its assets. Although diversification by itself can lead to improved portfolio returns (Booth and Fama 1992), the diversification process is fully reliant on estimated risk parameters and can only be performed on an ex ante basis. Thus, inaccuracies in estimation can result in an imperfect distribution of risk among the assets, which, in turn, could lead to excessive *ex post* exposure to individual asset risk. Achieving a higher risk-adjusted return can be attributed, fundamentally, to the fact that the regime-switching model is able to better capture the market dynamics, improving the quality of parameter estimation by reflecting the properties of the current financial cycle.

2. Risk parity portfolio

Risk parity portfolios, also referred to as ERC portfolios, solve the asset allocation problem by distributing the wealth in such a way that each asset has an equal contribution to the overall portfolio risk. The concept is similar to that of the '1/n' portfolio, with the distinction that it is the risk, and not wealth itself, that is evenly distributed between the assets.

Starting with the framework presented in Markowitz (1952), the portfolio expected return and variance are given by

$$\begin{aligned}\mu_p &= \mu^\top \phi,\\ \sigma_p^2 &= \phi^\top \Sigma \phi,\end{aligned} \tag{1}$$

where μ_p is the portfolio expected return, $\boldsymbol{\mu} \in \mathbb{R}^n$ is the vector of asset expected returns, $\boldsymbol{\phi} \in \mathbb{R}^n$ is the vector of asset weights (i.e. the proportion of wealth invested in each asset), σ_p^2 is the portfolio variance and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is the asset covariance matrix. The individual risk contribution of each asset can be derived from Equation (1) by an Euler decomposition of the portfolio standard deviation. As shown in Maillard *et al.* (2010), the risk contribution per asset is given by

$$\begin{aligned}\sigma_p &= \sqrt{\phi^\top \Sigma \phi} = \sum_{i=1}^{n} \sigma_i,\\ \sigma_i &= \phi_i \frac{\partial \sigma_p}{\partial \phi_i} = \phi_i \frac{(\Sigma\phi)_i}{\sqrt{\phi^\top \Sigma \phi}},\end{aligned} \tag{2}$$

where ϕ_i is the proportion of wealth allocated to asset i (i.e. the weight of asset i), $\partial\sigma_p/\partial\phi_i$ is the marginal risk contribution per asset and σ_i is the individual risk contribution of asset i. The only risk measure we consider in this paper is the volatility of the portfolio, i.e. the portfolio variance or standard deviation. However, any other risk measure that can be decomposed into its marginal risk contribution per asset can be used. The objective of the ERC method is to find a portfolio where $\sigma_i = \sigma_j \forall i, j$. By minimizing the squared differences in risk contribution, the optimization model can be written as

$$\begin{aligned}\text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\phi_i(\Sigma\phi)_i - \phi_j(\Sigma\phi)_j\right)^2\\ \text{s.t.} \quad & \mathbf{1}^\top \phi = 1\\ & \phi \geq 0,\end{aligned} \tag{3}$$

where the constraint $\mathbf{1}^\top \boldsymbol{\phi} = 1$ ensures that the entirety of the available portfolio wealth is distributed amongst the assets. This portfolio is not only conceptually similar to the '1/n' portfolio but its risk can also be shown to lie between the '1/n' and minimum variance portfolios. As proved in Maillard *et al.* (2010), the portfolio volatilities are ordered in the following way,

$$\sigma_{mv} \leq \sigma_{ERC} \leq \sigma_{1/n}.$$

In addition to the optimization problem shown in Equation (3), Maillard *et al.* (2010) presents an alternative model by which to compute ERC portfolios,

$$\begin{aligned}\min \quad & \psi^\top \Sigma \psi\\ \text{s.t.} \quad & \sum_{i=1}^{n} \ln \psi_i \geq c\\ & \psi \geq 0,\end{aligned} \tag{4}$$

where c is an arbitrary constant and $\boldsymbol{\psi}$ is an auxiliary weight allocation variable not restricted by the equality constraint given in Equation (3), i.e. the total sum of these auxiliary weights is unrestricted. The true asset weights corresponding to the optimal portfolio can be recovered by $\phi_i^* = \psi_i^* / \sum_{i=1}^{n} \psi_i^*$, where x^* implies optimality. The first ERC formulation is preferred from a numerical point of view, as the second problem incorporates a non-linear inequality constraint. However, the second problem may be solved using sequential quadratic programming .

For the purpose of this paper, the alternative formulation presented in Equation (4) is preferred due to its compatibility with the robust optimization framework presented in Goldfarb and Iyengar (2003), which we will study in Section 5. For reference, throughout this paper we will refer to the optimization problem presented above as the nominal ERC model.

3. Regime-switching factor model

In this section, we develop a Markov regime-switching variant of a traditional multi-factor model of asset returns. We begin by presenting a traditional factor model of returns. The vector of asset returns is assumed to be a random variable and is denoted by $\boldsymbol{r} \in \mathbb{R}^n$. Suppose that the return vector is given by

$$r = \alpha + V^\top f + \epsilon, \tag{5}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ is the vector of regression intercepts, $\boldsymbol{f} \sim \mathcal{N}(\bar{\boldsymbol{f}}, \boldsymbol{F}) \in \mathbb{R}^m$ is the vector of factor returns, $\boldsymbol{V} \in \mathbb{R}^{m\times n}$ is the matrix of factor loadings for the n assets and $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{D}) \in \mathbb{R}^n$ is the vector of residual returns. As before, the matrix $\boldsymbol{F}$ denotes the factor covariance matrix, while the matrix $\boldsymbol{D}$ denotes the diagonal matrix of residual variance. The model assumes that the residual returns are independent of one another, i.e. $\mathrm{cov}(\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_j) = 0$ for $i \neq j$; and it also assumes that the residual returns are independent of the factor returns, i.e. $\mathrm{cov}(\boldsymbol{\epsilon}_i, \boldsymbol{f}_j) = 0 \forall i, j$. The authors note that this model does not assume nor require that the factors be independent of one another, i.e. the factor covariance matrix, $\boldsymbol{F}$, does not need to be a diagonal matrix. However, in practice we may need $\boldsymbol{F} \succeq 0$. The asset expected returns and covariance matrix can be expressed as

$$\begin{aligned} \mu &= \alpha + V^\top \bar{f}, \\ \Sigma &= V^\top F V + D, \end{aligned} \tag{6}$$

implying that $\boldsymbol{r} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The factor model is purposely designed to separate the expected factor returns from the intercept, $\boldsymbol{\alpha}$, i.e. we do not require that the factor expected returns, $\bar{\boldsymbol{f}}$, are equal to zero. The reason for this will become apparent as we develop the regime-switching model.

Next, we introduce a Markov regime-switching-based factor model of asset returns. Following the derivation presented in Hamilton (2010) and Ang and Timmermann (2012), we describe the behaviour of a variable, y_t, which undergoes a dramatic change at a given time, t_0. Suppose the observed behaviour of this variable for $t = 1, 2, \ldots, t_0$ can be modelled by a first-order autoregression,

$$y_t = c_1 + \beta_1 y_{t-1} + \sigma_1 \varepsilon_t,$$

with intercept c_1, autocorrelation β_1, volatility σ_1 and $\varepsilon_t \sim \mathcal{N}(0, 1)$. Now, suppose that at time t_0 the typical behaviour of the series undergoes a significant change, where we can describe it by

$$y_t = c_2 + \beta_2 y_{t-1} + \sigma_2 \varepsilon_t,$$

for $t = t_0 + 1, t_0 + 2, \ldots$. Instead of believing the two systems are independent, we assume that these two are governed by a larger model which encompasses both states,

$$y_t = c_{s_t} + \beta_{s_t} y_{t-1} + \sigma_{s_t} \varepsilon_t, \tag{7}$$

where s_t is a random variable governed by the same probability law that governs the state of the market. In the example above, s_t happened to assume the value $s_t = 1$ for $t = 1, 2, \ldots, t_0$ and $s_t = 2$ for $t = t_0 + 1, t_0 + 2, \ldots$. Certainly, this behaviour is akin to that of financial cycles, where the market is observed to periodically, albeit abruptly, alternate between bull and bear states.

It is possible to describe this model by means of a two-state Markov chain. We assume a discrete Markov chain where the probability distribution of the state s_t at time t is dependent only in the most recent state s_{t-1} and, therefore, does not depend on any further preceding states,

$$P\{s_t = j | s_{t-1} = i, s_{t-2} = k, \cdots\} = P\{s_t = j | s_{t-1} = i\}.$$

In this type of Markov model, the latent variable dictating the state is unobservable, but the state-dependent market parameters are visible. However, an estimated probability distribution of each state can be derived directly from raw market data by applying an expectation maximization (EM) algorithm. This method is an iterative technique by which to derive the maximum likelihood. In this paper, we use the Baum–Welch algorithm, which is a special case of EM (Baum *et al.* 1970). The two-state transition matrix describing the probability of switching from one regime to another is given by

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \tag{8}$$

where γ_{ij} gives the probability that state i will be followed by state j. Thus, by construction, every row must sum up to unity, $\sum_{j=1}^{2} \gamma_{ij} = 1$ for $i = 1, 2$.

The Markov model used in this paper does not apply any restrictions to the estimation of parameters to avoid making any a priori assumptions or imposing any subjective conditions. It is possible to employ restrictions if one has specific beliefs or assumptions regarding the regimes and the regime-dependent parameters. However, if restrictions are applied the resulting estimates may differ from what the raw data would otherwise suggest. For the interested reader, one possible approach for incorporating restrictions can be found in Binning and Maih (2015).

We proceed by applying Hamilton's model to describe the behaviour of the factors within our factor model of returns shown in Equation (5). As before, suppose that the historical returns of a given asset for time $t = 1, 2, \ldots, t_0$ could be described by

$$r_t = \alpha + V_1^\top f_{1t} + \epsilon$$

with long-term average equal to $\boldsymbol{\mu}_1 = \boldsymbol{\alpha} + \boldsymbol{V}_1^\top \bar{\boldsymbol{f}}_1$ and volatility $\boldsymbol{\Sigma}_1 = \boldsymbol{V}_1^\top \boldsymbol{F}_1 \boldsymbol{V}_1 + \boldsymbol{D}$. Now, suppose that at time t_0 there was a significant change in the average value of the time series, which could be described by

$$r_t = \alpha + V_2^\top f_{2t} + \epsilon$$

for subsequent time periods ($t = t_0 + 1, t_0 + 2, \ldots$), with a long-term average given by $\boldsymbol{\mu}_2 = \boldsymbol{\alpha} + \boldsymbol{V}_2^\top \bar{\boldsymbol{f}}_2$ and volatility $\boldsymbol{\Sigma}_2 = \boldsymbol{V}_2^\top \boldsymbol{F}_2 \boldsymbol{V}_2 + \boldsymbol{D}$. The transition from state 1 to state 2 is due to an imperfectly predictable event or force. Therefore, it cannot be precisely determined if the transition between states will happen at a given time t_0. It follows that the transition from one state to the other is governed by a single stochastic model encompassing both states,

$$r_t = \alpha + V_{s_t}^\top f_{s_t,t} + \epsilon, \tag{9}$$

where, as before, s_t is a random variable indicating the current market regime. The model is constructed under the assumption that both the factors and the assets follow a normal distribution. Thus, the factors are modeled as a simplified version of the Markov model presented in Equation (7), where the factors follow a regime-dependent Gaussian process, $\boldsymbol{f}_{s_t,t} \sim \mathcal{N}(\bar{\boldsymbol{f}}_{s_t}, \boldsymbol{F}_{s_t}) \in \mathbb{R}^m$. The regime-dependent joint distribution of the factors is estimated directly through the Baum–Welch algorithm. Therefore, this regime-switching factor model of returns assumes that the factors are non-stationary, with their behaviour governed by two distinct, alternating states.

We continue with the stochastic process presented in Equation (9), and provide a more convenient yet equivalent formulation of this two-state model,

$$r_t = \alpha + I_1(V_1^\top f_{1t}) + I_2(V_2^\top f_{2t}) + \epsilon, \tag{10}$$

where $I_i = 1$ when the current state $s_t = i$ and $I_i = 0$ otherwise, thereby explicitly presenting both states of the model in a single equation. The choice of the number of market states to be estimated is a difficult one. Introducing additional regimes should yield greater accuracy in the estimation of parameters, provided the regimes are representative of the actual number of regimes in existence. However, increasing the number of regimes requires the estimation of additional parameters, thereby increasing the corresponding estimation error. Our choice of a two-state model is studied in Section 6 and is justified by the experimental results that follow.

The formulation in Equation (10) allows us to derive the regime-switching expected returns and covariance matrix directly from this model. The expected returns are the probability-weighted average of the expected returns of the two regimes. The derivation of the covariance matrix, on the other hand, results in a collection of terms that arise naturally when calculating $\sigma_i^2 = \mathrm{var}(r_i)$ and $\sigma_{ij} = \mathrm{cov}(r_i, r_j)$. The two equations are given below,

$$\begin{aligned} \mu_{RS_i} &= \alpha + \gamma_{i1} V_1^\top \bar{f}_1 + \gamma_{i2} V_2^\top \bar{f}_2, \\ \Sigma_{RS_i} &= \gamma_{i1} V_1^\top F_1 V_1 + \gamma_{i2} V_2^\top F_2 V_2 + \gamma_{i1}(1-\gamma_{i1}) V_1^\top \bar{f}_1 \bar{f}_1^\top V_1 \\ &\quad + \gamma_{i2}(1-\gamma_{i2}) V_2^\top \bar{f}_2 \bar{f}_2^\top V_2 \\ &\quad - \gamma_{i1}\gamma_{i2} V_1^\top \bar{f}_1 \bar{f}_2^\top V_2 - \gamma_{i1}\gamma_{i2} V_2^\top \bar{f}_2 \bar{f}_1^\top V_1 + D, \end{aligned} \tag{11}$$

where $\boldsymbol{\mu}_{RS_i} \in \mathbb{R}^n$ is the vector of asset expected returns corresponding to current state i, $\boldsymbol{\Sigma}_{RS_i} \in \mathbb{R}^{n\times n}$ is the asset covariance matrix corresponding to current state i, and $\gamma_{ij} = \mathbb{E}[I_j | s_t = i]$, i.e. the probability of switching from state i to state j. These probabilities correspond to the two-state transition matrix given in Equation (8). If necessary, this model can be easily expanded to account for additional market regimes by modifying Equation (10) accordingly and proceeding with the derivation of the expected returns and covariance matrix as before. For brevity, however, this derivation is left up to the reader.

The authors note that the Markov model is used to estimate the transition probabilities and the regime-switching parameters by means of an in-sample analysis of available historical data during the current time period. The model calculates the current-time covariance matrix by identifying the current state i, and it does not attempt to forecast any future changes in regime. In other words, this is not a predictive model, it simply serves to identify the parameters corresponding to the present state of the market. Thus, this serves to position a regime-switching ERC portfolio in a favourable position given the present estimate of the market state. A potential drawback arises from the possible misalignment between the portfolio and the market if a sudden change of state is observed during the subsequent time period. Nevertheless, this is justified by the fact that we use a probability-weighted approach to construct the risk measure. Moreover, this potential misalignment can be mitigated by periodically rebalancing the portfolio.

4. Regime-switching risk parity portfolio

The Markov regime-switching factor model shown in Equation (10), like any other factor model of returns, can be used to prepare a regime-switching ERC optimization problem by constructing the corresponding asset covariance matrix. However, unlike the nominal ERC, the regime-switching model is able to capture the non-stationarity of the market factors when estimating the asset parameters.

The formulation of NLP for the regime-switching ERC portfolio is similar to the nominal NLP given in Equation (4), except we replace the covariance matrix with its regime-switching counterpart from Equation (11). The regime-switching framework is incorporated simply by replacing the nominal covariance matrix with its regime-switching counterpart,

$$\begin{aligned} \min \quad & \psi^\top \Sigma_{RS_i} \psi \\ \text{s.t.} \quad & \sum_{j=1}^{n} \ln \psi_j \geq c \\ & \psi \geq 0, \end{aligned} \tag{12}$$

where the true weights of the regime-switching portfolio can be derived from the optimal solution as before, $\phi_j^* = \psi_j^* / \sum_{j=1}^n \psi_j^*$. The Markov regime-switching properties are fully encompassed within the covariance matrix.

The formulation of the regime-switching NLP retains the same level of complexity as its nominal counterpart. This, in turn, allows the NLP to be scaled upwards in size to efficiently solve it for a large number of assets. In addition, by incorporating the dynamics of the market within the input parameters, we are able to more accurately assess both the individual asset volatilities and their co-movements. This provides a more faithful estimate of the risk parameters, maximizing the ex ante diversification of risk while reducing unintended *ex post* exposure to individual asset risk compared to a non-regime-switching factor modelling approach.

We later show through a computational experiment how the inability of the nominal ERC portfolio to capture the directional information latent within the market hinders its performance. Under the thesis that the market is governed by two distinct regimes, the nominal factor model is, in effect, smoothing both states while calibrating its factor loadings. On the other hand, the regime-switching factor model is designed to yield a probability-weighted average of the two latent regimes, emphasizing the regime which currently governs the market. This information is then passed directly to the NLP formulation, which ensures the market dynamics are considered during the optimization process.

5. Robust risk parity portfolio

In this section, we follow as in Goldfarb and Iyengar (2003) by employing a robust optimization approach to both the nominal and regime-switching ERC portfolios in an attempt to mitigate the estimation error that might arise from estimation of the regime-switching risk model.

Robust risk parity models, as proposed in Kapsos *et al.* (2017), are a natural extension of robust mean-variance optimization, where the worst-case estimate of the covariance matrix is assumed while enforcing the diversification of risk. By design, our covariance matrix is derived from a factor model of asset returns. Applying the same principle as in Kapsos *et al.* (2017), an intuitive robust framework for our covariance matrix can be attained by introducing an uncertainty structure arising from the regression coefficients, as proposed in Goldfarb and Iyengar (2003). Thus, our robust risk parity model assumes the worst-case portfolio variance arising from noisy regression coefficients subject to the risk parity constraint. A detailed formulation of the robust model follows.

We begin by introducing uncertainty sets around the estimated factor loadings matrix, $\boldsymbol{V}$, and the residual variance matrix, $\boldsymbol{D}$, as follows,

$$\begin{aligned} S_d &= \left\{D : D = \operatorname{diag}(d), d_i \in [\underline{d}_i, \bar{d}_i], i = 1, \dots, n\right\}, \\ S_v &= \left\{V : V = V_0 + W, \|W_i\|_g \le \rho_i, i = 1, \dots, n\right\}, \end{aligned} \tag{13}$$

where d_i is the idiosyncratic risk of asset i, $\boldsymbol{W}_i$ is the ith column of strength matrix $\boldsymbol{W}$ around $\boldsymbol{V}_0$ and $\|\boldsymbol{W}_i\|_g = \sqrt{\boldsymbol{W}_i^\top \boldsymbol{G} \boldsymbol{W}_i}$ denotes an elliptic norm with respect to a symmetric, positive definite matrix $\boldsymbol{G}$. It should be noted that the uncertainty set for the factor loadings matrix $\boldsymbol{V}$ belongs is ellipsoidal .

Goldfarb and Iyengar (2003) present a robust optimization model tailored towards a minimum variance portfolio. Although our ERC portfolio differs from theirs, the objective function in Equation (4) is, in fact, exactly the same. Thus, the robust counterpart of the ERC portfolio can be derived in a similar fashion. The robust version of the objective function in Equation (4) is then

$$\min_{\psi} \max_{V \in S_v, D \in S_d} \psi^\top \Sigma \psi \iff \min_{\psi} \max_{V \in S_v, D \in S_d} \psi^\top V^\top F V \psi + \psi^\top D \psi. \tag{14}$$

The bounds $\underline{\boldsymbol{d}}_i \le \boldsymbol{d} \le \bar{\boldsymbol{d}}_i$ shown in Equation (13) imply that the residual portfolio variance has an upper bound, i.e. $\boldsymbol{\psi}^\top \boldsymbol{D} \boldsymbol{\psi} \le \boldsymbol{\psi}^\top \bar{\boldsymbol{D}} \boldsymbol{\psi}$, where $\bar{\boldsymbol{D}} = \operatorname{diag}(\bar{\boldsymbol{d}})$. This simplifies the previous expression to

$$\begin{aligned} \min_{\psi} \quad \max_{V \in S_v} \quad & \psi^\top V^\top F V \psi + \delta \\ \text{s.t.} \quad & \left\| \begin{bmatrix} 2\bar{D}^{1/2}\psi \\ 1 - \delta \end{bmatrix} \right\| \le 1 + \delta. \end{aligned}$$

In Goldfarb and Iyengar (2003) it is shown that the remaining part of the objective function be reformulated into a collection of linear and second-order conic constraints through Lemma 1, which can be found in Appendix 1 for the reader's convenience.

Using part (ii) of Lemma 1, the nominal objective function is reformulated into its robust counterpart as follows,

$$\begin{aligned} \min \quad & \nu + \delta \\ \text{s.t.} \quad & \left\| \begin{bmatrix} 2\bar{D}^{1/2}\psi \\ 1 - \delta \end{bmatrix} \right\| \le 1 + \delta, \\ & \left\| \begin{bmatrix} 2\rho^\top \psi \\ \tau - \nu + \mathbf{1}^\top s \end{bmatrix} \right\| \le \tau + \nu - \mathbf{1}^\top s, \\ & \left\| \begin{bmatrix} 2u_k \\ 1 - \tau\theta_k - s_k \end{bmatrix} \right\| \le 1 - \tau\theta_k + s_k, \quad k = 1, \dots, m, \\ & \tau - \frac{1}{\lambda_{\max}(K)} \le 0, \\ & \tau \ge 0. \end{aligned} \tag{15}$$

Finally, the complete version of the robust ERC portfolio is given by adding the remaining constraints to Equation (15) above, one of which is the non-linear constraint to enforce risk parity,

$$\begin{aligned} \min \quad & \nu + \delta \\ \text{s.t.} \quad & \left\| \begin{bmatrix} 2\bar{D}^{1/2}\psi \\ 1 - \delta \end{bmatrix} \right\| \le 1 + \delta, \\ & \left\| \begin{bmatrix} 2\rho^\top \psi \\ \tau - \nu + \mathbf{1}^\top s \end{bmatrix} \right\| \le \tau + \nu - \mathbf{1}^\top s, \\ & \left\| \begin{bmatrix} 2u_k \\ 1 - \tau\theta_k - s_k \end{bmatrix} \right\| \le 1 - \tau\theta_k + s_k, \quad k = 1, \dots, m, \\ & \tau - \frac{1}{\lambda_{\max}(K)} \le 0, \\ & \sum_{j=1}^{n} \ln \psi_j \ge c, \\ & \psi, \tau \ge 0, \end{aligned} \tag{16}$$

where the dimensions of the different variables are $\boldsymbol{\psi} \in \mathbb{R}^n, \nu, \delta, \tau \in \mathbb{R}, \boldsymbol{u}, \boldsymbol{s} \in \mathbb{R}^m$, and the weights of the ERC portfolio can be derived as before,

$$\phi_i^* = \frac{\psi_i^*}{\sum_{i=1}^n \psi_i^*}.$$

5.1. *Robust regime-switching risk parity portfolio*

We proceed to create a robust counterpart of the regime-switching NLP by introducing uncertainty sets for the factor loadings matrix, $\boldsymbol{F}$, and the residual covariance matrix, $\boldsymbol{D}$. However, unlike the sets presented by Equation (13), each regime imposes its own uncertainty set around the matrices of factor loadings. For a two-regime model, the uncertainty sets are

$$\begin{aligned} S_d &= \left\{D : D = \mathrm{diag}(d), d_i \in [\underline{d}_i, \bar{d}_i], i = 1, \ldots, n\right\}, \\ S_{v_1} &= \left\{V_1 : V_1 = V_{1_0} + W_1, ||W_{1_i}||_g \le \rho_{1_i}, i = 1, \ldots, n\right\}, \\ S_{v_2} &= \left\{V_2 : V_2 = V_{2_0} + W_2, ||W_{2_i}||_g \le \rho_{2_i}, i = 1, \ldots, n\right\}, \end{aligned} \tag{17}$$

where $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are the factor loadings matrices corresponding to regime 1 and regime 2, respectively.

The formulation of the robust regime-switching model continues in the same manner as before. Although the regime-switching covariance matrix given in Equation (11) is more convoluted than its nominal counterpart can be segmented and treated in the same fashion as Equation (14),

$$\begin{aligned} &\min_{\psi} \max_{V \in S_v, D \in S_d} \psi^\top \Sigma_{RS_i} \psi \iff \\ &\min_{\psi} \max_{V \in S_v, D \in S_d} \gamma_{i1} \psi^\top V_1^\top F_1 V_1 \psi + \gamma_{i2} \psi^\top V_2^\top F_2 V_2 \psi \\ &\qquad + \gamma_{i1}(1-\gamma_{i1}) \psi^\top V_1^\top \bar{f}_1 \bar{f}_1^\top V_1 \psi \\ &\qquad + \gamma_{i2}(1-\gamma_{i2}) \psi^\top V_2^\top \bar{f}_2 \bar{f}_2^\top V_2 \psi \\ &\qquad - \gamma_{i1}\gamma_{i2} \psi^\top V_1^\top \bar{f}_1 \bar{f}_2^\top V_2 \psi \\ &\qquad - \gamma_{i1}\gamma_{i2} \psi^\top V_2^\top \bar{f}_2 \bar{f}_1^\top V_1 \psi + \psi^\top D \psi, \end{aligned} \tag{18}$$

where, even under this more convoluted formulation, the uncertainty sets apply only to $\boldsymbol{D}$, $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$. Again, the bounds $\underline{\boldsymbol{d}}_i \le \boldsymbol{d} \le \bar{\boldsymbol{d}}_i$ imply that the residual portfolio variance has an upper bound, i.e. $\boldsymbol{\psi}^\top \boldsymbol{D} \boldsymbol{\psi} \le \boldsymbol{\psi}^\top \bar{\boldsymbol{D}} \boldsymbol{\psi}$. This simplifies the previous expression to

$$\begin{aligned} \min_{\psi} \max_{V \in S_v} \quad & \gamma_{i1} \psi^\top V_1^\top F_1 V_1 \psi + \gamma_{i2} \psi^\top V_2^\top F_2 V_2 \psi \\ & + \gamma_{i1}(1-\gamma_{i1}) \psi^\top V_1^\top \bar{f}_1 \bar{f}_1^\top V_1 \psi \\ & + \gamma_{i2}(1-\gamma_{i2}) \psi^\top V_2^\top \bar{f}_2 \bar{f}_2^\top V_2 \psi \\ & - \gamma_{i1}\gamma_{i2} \psi^\top V_1^\top \bar{f}_1 \bar{f}_2^\top V_2 \psi \\ & - \gamma_{i1}\gamma_{i2} \psi^\top V_2^\top \bar{f}_2 \bar{f}_1^\top V_1 \psi + \delta \\ \text{s.t.} \quad & \left\| \begin{bmatrix} 2\bar{D}^{1/2}\psi \\ 1-\delta \end{bmatrix} \right\| \le 1 + \delta. \end{aligned}$$

Next, we apply Lemma 1 to each of the factor loadings matrices to summarize the objective function as a quadratic function with a collection of linear and second-order conic constraints. For clarity, we also include the risk parity non-linear constraint to allow us to present the complete robust regime-switching model below,

$$\begin{aligned} \min \quad & \gamma_{i1}\nu_1 + \gamma_{i2}\nu_2 + \gamma_{i1}(1-\gamma_{i1}) \psi^\top V_1^\top \bar{f}_1 \bar{f}_1^\top V_1 \psi \\ & + \gamma_{i2}(1-\gamma_{i2}) \psi^\top V_2^\top \bar{f}_2 \bar{f}_2^\top V_2 \psi \\ & - \gamma_{i1}\gamma_{i2} \psi^\top V_1^\top \bar{f}_1 \bar{f}_2^\top V_2 \psi - \gamma_{i1}\gamma_{i2} \psi^\top V_2^\top \bar{f}_2 \bar{f}_1^\top V_1 \psi + \delta, \\ \text{s.t.} \quad & \left\| \begin{bmatrix} 2\bar{D}^{1/2}\psi \\ 1-\delta \end{bmatrix} \right\| \le 1 + \delta \\ & \left\| \begin{bmatrix} 2\rho_1^\top \psi \\ \tau_1 - \nu_1 + \mathbf{1}^\top s_1 \end{bmatrix} \right\| \le \tau_1 + \nu_1 - \mathbf{1}^\top s_1, \\ & \left\| \begin{bmatrix} 2\rho_2^\top \psi \\ \tau_2 - \nu_2 + \mathbf{1}^\top s_2 \end{bmatrix} \right\| \le \tau_2 + \nu_2 - \mathbf{1}^\top s_2, \\ & \left\| \begin{bmatrix} 2u_{1_k} \\ 1 - \tau_1\theta_{1_k} - s_{1_k} \end{bmatrix} \right\| \le 1 - \tau_1\theta_{1_k} + s_{1_k}, \quad k = 1, \ldots, m, \\ & \left\| \begin{bmatrix} 2u_{2_k} \\ 1 - \tau_2\theta_{2_k} - s_{2_k} \end{bmatrix} \right\| \le 1 - \tau_2\theta_{2_k} + s_{2_k}, \quad k = 1, \ldots, m, \\ & \tau_1 - \frac{1}{\lambda_{\max}(K_1)} \le 0, \\ & \tau_2 - \frac{1}{\lambda_{\max}(K_2)} \le 0, \\ & \sum_{j=1}^n \ln \psi_j \ge c, \\ & \psi, \tau_1, \tau_2 \ge 0, \end{aligned} \tag{19}$$

where the probabilities γ_{ij} are conditional on the current state i, the dimensions of the different variables are $\boldsymbol{\psi} \in \mathbb{R}^n, \nu_1, \nu_2, \delta, \tau_1, \tau_2 \in \mathbb{R}, \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{s_1}, \boldsymbol{s_2} \in \mathbb{R}^m$, and the weights of the ERC portfolio can be recovered as before,

$$\phi_j^* = \frac{\psi_j^*}{\sum_{i=j}^n \psi_j^*}.$$

6. Computational experiment

In this section we present the results of two separate computational experiments designed to demonstrate the performance of the regime-switching ERC portfolio. The Fama–French three-factor model was selected as the basis of the experiments to show how a well-known multi-factor model would behave under the proposed regime-switching framework (Fama and French 1993). We remind the reader that the model presented in this paper is applicable to any general single or multiple factor model. The Fama–French model attempts to explain the excess returns of a given asset based on the observation that small capitalization stocks (size premium) and value stocks (with high book-to-price ratios) have historically tended to outperform the market as a whole. It should be noted that both the size premium and value premium factors are subsets of the market factor, implying that the three factors are correlated. This is not an impediment to

Table 1. List of assets.

Consumer discretionary			*Energy*
F	Ford Motor	CVX	Chevron Corp.
GT	Goodyear Tire	MRO	Marathon Oil Corp.
FL	Foot Locker	OXY	Occidental Petroleum
DIS	The Walt Disney Comp.	XOM	Exxon Mobil Corp.
MCD	McDonald's Corp.	HAL	The Halliburton Company
Consumer staples			*Financials*
KR	The Kroger Company	AON	Aon Plc.
MO	Altria Group	BK	The Bank of New York Mellon Corp.
CL	Colgate-Palmolive	AXP	American Express Company
KO	The Coca Cola Company	C	Citigroup Inc.
PEP	Pepsico, Inc.	WFC	Wells Fargo & Company
Healthcare			*Industrials*
MRK	Merck & Co., Inc.	BA	Boeing Company
PFE	Pfizer Inc.	LMT	Lockheed Martin Corp.
LLY	Eli Lilly and Company	MMM	3M Company
BMY	Bristol-Myers Squibb	CAT	Caterpillar Inc.
JNJ	Johnson & Johnson	DE	Deere & Company
Information technology			*Materials*
IBM	Int'l Business Machines Corp.	PPG	PPG Industries
MSI	Motorola Solutions, Inc.	AA	Alcoa Inc.
HPQ	HP Inc.	DOW	Dow Chemical
TXN	Texas Instruments Inc.	DD	Du Pont (E. I.)
XRX	Xerox Corp.	IP	International Paper
Utilities			*Telecommunications*
ED	Consolidated Edinson Inc.	T	AT&T Inc.
ETR	Entergy Corp.	VZ	Verizon Communications Inc.
DTE	DTE Energy Company	S	Sprint Corp.
CNP	CenterPoint Energy, Inc.		
AEP	American Electric Power Co., Inc.		

our formulation, since, by construction, the model allows for a non-diagonal factor covariance matrix, $\boldsymbol{F}$.

For the experiments, we considered a diversified portfolio of U.S. stocks belonging to the S&P 500 index, with 5 stocks from each of the 10 *Global Industry Classification Standard* sectors,† which yielded a total of 48 stocks in our portfolio ($n = 48$). A list of the selected assets is presented in Table 1. The historical stock prices were obtained from *Yahoo! Finance*. The factor returns were obtained from Professor Kenneth R. French's website (French 2016). The length of the time series required for the experiments introduces a survivorship bias to our results since we only select stocks with sufficient historical data. However, we expect the bias to have a similar effect on all the portfolios tested, and it is the relative performance of the different risk parity portfolio strategies that are of main interest.

All experiments were performed on an Apple MacBook Pro computer (2.4 GHz Intel Core i5, 8 GB 1600 MHz DDR3 RAM) running OS X 'Sierra'. All computations were performed using the Julia programming language (version 0.6.0) using the optimization modeling language JuMP (Dunning *et al.* 2017) with IPOPT (version 3.12.6) as the optimization solver.

The estimation of regime switches requires a sufficiently long time period to properly estimate the parameters governing each market regime. Moreover, the question of how many regimes are present, or what is the optimal number of regimes to estimate, is discussed in this section by testing both a two-state and a three-state regime-switching model. We used a minimum of 15 years of weekly historical prices for the initial regime estimation prior to the beginning of the first investment period. We then performed the out-of-sample test for an additional 11 years, from 7 January 2000 to 31 December 2010 inclusive. Thus, 26 years of historical data were used during the experiments, from 18 January 1985 to 31 December 2010.

An overview of the two experimental procedures is presented here. During the first experiment, the portfolios were rebalanced periodically, with each investment period consisting of six months for a total of 22 investment periods ($t = 0, \ldots, 21$). At the beginning of each investment period, the previous 15 years of historical weekly prices were used to prepare the Fama–French three-factor model by means of an ordinary least-squares linear regression, which was then used to estimate the asset covariance matrix. This same 15-year window was used to estimate the regime-switching parameters and the regime-switching asset covariance matrix. Once the parameters were estimated and the current regime was identified, the portfolios were rebalanced immediately. The portfolios were then held static for the duration of the investment period, before repeating the process for the next period. For the case of the robust portfolios, we assumed the factors were stable and only constructed uncertainty sets around the matrix of factor loadings, $\boldsymbol{V}$, and the matrix of residual variances, $\boldsymbol{D}$ (i.e. we did not construct uncertainty sets around $\boldsymbol{F}$ or $\bar{\boldsymbol{f}}$).

† An exception was made for the Telecommunications sector, as only three stocks had sufficient historical data available for our test.

The second experiment relied on a more intuitive approach, where the portfolios were only rebalanced once a change in the current market regime was perceived. This allowed the portfolios to effectively react to current market conditions. Once a change in regime was identified, all the portfolios were rebalanced, including all the non-regime-switching portfolios. In the same fashion as the first experiment, the previous 15-year window of historical pricing data was used to prepare both regime-dependent and non-regime-dependent parameters. The portfolios were then held static until another change in regime was perceived.

The different investment strategies discussed in this paper were used to construct the portfolios in both experiments, including two-state and three-state versions of the regime-switching models. For the sake of comparison, an additional ERC portfolio where the covariance matrix is constructed by means of Bayesian Shrinkage is included in the experiments. The portfolio $\boldsymbol{\phi}_q^t$ was computed by solving the optimization problem for each of the following methods ($q = i, ii, \ldots, vii$): (i) the nominal ERC portfolio presented in Equation (4), (ii) the robust ERC portfolio presented in Equation (16), (iii) the two-state regime-switching portfolio presented in (12), (iv) the robust two-state regime-switching portfolio presented in (19), (v) the three-state regime-switching portfolio, (vi) the robust version of the three-state regime-switching portfolio and (vii) the Bayesian Shrinkage ERC portfolio. For convenience, we will refer to these portfolios as (i) the nominal portfolio, (ii) the robust portfolio, (iii) the two-state RS portfolio, (iv) the robust two-state RS portfolio, (v) the three-state RS portfolio, (vi) the robust three-state RS portfolio and (vii) the shrinkage portfolio, respectively.

The Bayesian shrinkage model is based on the methodology prescribed in Ledoit and Wolf (2003), where shrinkage is used to construct a covariance matrix which blends the factor-based assset covariances with the sample† asset covariances. The three-state regime-switching model is constructed as a natural extension of the two-state model described in Equation (10). However, the three-state model and its robust version are not included in the second experiment, where the rebalancing policy is dictated by a two-state regime-switching model.

The portfolios were evaluated by comparing their relative wealth, Sharpe ratio and turnover rate. The robust portfolios were assigned a joint confidence level of $\omega = 0.95$, in an effort to present significantly robust portfolios. Additionally, we proceed as in Goldfarb and Iyengar (2003) and set the upper bound on the variance of the residual returns to $\bar{d}_i = s_i^2$, where s_i^2 is the unbiased estimate of the variance of the residual return of the ith asset and is given by Equation (A3) in Appendix 1.

As we previously discussed, ERC models yield well-diversified portfolios and do not require the estimation of asset returns. Thus, by construction, they reduce the risk associated with concentrated portfolios and partially mitigate estimation error. The reader should be aware that errors in the estimated expected returns account for the majority of the overall estimation error (Chopra and Ziemba 1993). Therefore, the experiment is expected to yield fundamentally similar and stable portfolios, with the the experiment focusing on relative results between the portfolios. Nevertheless, as we will see in the subsequent results, the two-state regime-switching portfolio is able to significantly differentiate itself by maintaining a consistent lead over the accumulated wealth of its nominal counterpart. This is particularly true during the second experiment, where the regime-switching portfolio is allowed to dictate its rebalancing policy to react to current market conditions.

† The sample covariance matrix is based purely on the observed asset returns, directly computing the covariance of asset i against asset $j \forall i, j$ from their respective observed historical returns.

6.1. Estimation of regime switches

6.1.1. Determining the number of factors used for estimation. The regime-switching model is based on the Baum–Welch method (Baum *et al.* 1970), with a practical implementation derived from Kole (2010) and Kritzman *et al.* (2012). The assumption that a factor model is subject to the market dynamics is valid only if at least one of the factors is market-dependent, since only these factors will convey the latent market regime. In our experiment, all three of the Fama–French factors are market-dependent. In practice, however, performing the Baum–Welch method over a multivariate model can be computationally expensive and may also lead to considerable estimation errors in the form of noise. Thus, although theoretically more faithful, the inclusion of all market-dependent factors in the estimation process may not necessarily yield a better quality estimate.

In an effort to determine which factors best describe the latent state of the market, the regimes were estimated for the 15-year period leading up to the start of our experiment (1985–1999). Upon inspection of historical factor returns, we observe that the excess market returns and the ‘high-minus-low’ factors closely mimic each other, displaying high volatility concurrently. On the other hand, the volatility of the ‘small-minus-big’ factor lags behind these two. We proceed to estimate the regime switches for the following three scenarios: (i) considering only the excess market returns factor, (ii) considering both the excess market returns and the ‘high-minus-low’ factors and (iii) considering all three factors. The resulting smoothed probabilities are presented in Figure 1.

The estimation of regime switches will usually yield some random noise, as it is an empirical estimate based on observed factor returns and not determined by intuition or knowledge of past events. Nevertheless, the two-factor estimate shown above not only provides the cleanest plot, it also correctly identifies the ‘Black Monday’ market crash of 1987; the early 1990s recession; a brief ripple effect of the Asian crisis of 1997 and the Russian crisis of 1998; as well as the beginning of the ‘dot-com’ bubble.

The plotted smoothed probabilities suggest that, from a practical standpoint, the regime-switching estimates should only be based on a subset of two factors. However, the reader should recognize that this does not imply there is a change in the number of factors in the factor model of returns. This subset of two factors is only used to estimate the changes in regime, which serves two purposes. First, it allows us to divide the historic time series of returns into bullish and bearish market periods for all three Fama–French factors,

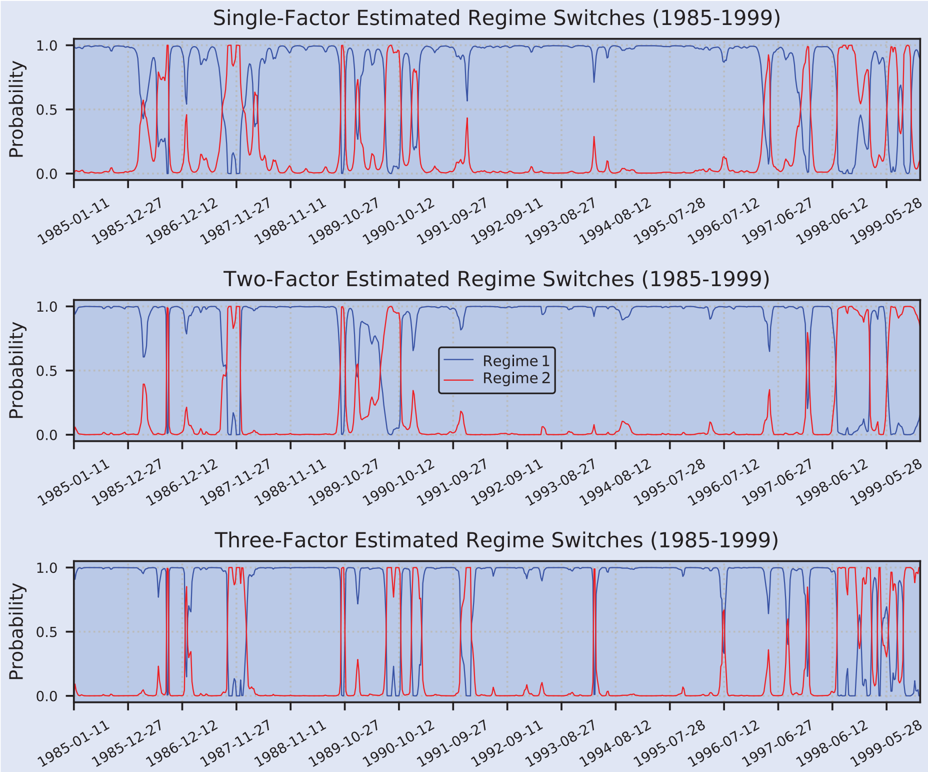

Figure 1. Estimated regime switches for the period of 1985–1999. The probability of having a bullish (bearish) market is shown in blue (red). Top: estimated switches based on the excess market return factor. Centre: estimated switches based on the excess market return and the ‘high-minus-low’ factors. Bottom: estimated switches based on the three Fama–French factors.

allowing us to calculate the regime-specific factor covariance matrix and factor loadings. Second, it serves to identify the ‘current’ market regime by inspecting the most recent entry of the resulting smoothed probabilities.

6.1.2. Determining the number of regimes. Another important consideration when working with regime-switching models is the question of how many regimes exist, and how many are we able to accurately estimate. Increasing the number of regimes requires the estimation of additional parameters, which, in turn, increases the estimation error of each parameter. On the other hand, if the model uses fewer regimes than there exist, then the model will not be representative of the true underlying regime-specific behaviour.

For clarity, we state the difference between a k-factor and a k-state model: the ‘k-factor’ corresponds to the number of factors used to estimate the changes in regime; a k-state model indicates that we assume there are only k regimes present, so the model will be fitted to k regimes accordingly.

We proceed to compare our two-state model against a three-state model. As discussed in the previous subsection, we use a two-factor model based on the excess market returns and the ‘high-minus-low’ Fama–French factors. The smoothed probabilities are presented in Figure 2.

A comparison of these two models suggests that a two-state model suffices to capture the market dynamics latent within the model factors. The first regime within the three-state model is well aligned with its counterpart on the two-state model. However, it seems the second and third regime of the three-state model are partitions of the two-state model’s second regime. In particular, the third regime in the three-state model is always short-lived and erratic, which could easily be attributed to estimation noise. However, for the sake of comparison, the three-state regime-switching model is included in the first of the two portfolio optimization experiments.

6.1.3. Estimation of regimes and Regime-Dependent Parameters. A sufficiently long time period of time is needed to properly estimate the asset parameters, which is why a 15-year rolling window was deemed adequate to perform the linear regressions when calculating the factor loadings and factor covariance matrix. A rolling window, where older data are discarded as new data becomes available with time, was preferred to an expanding window as to allow the model to provide a more contemporary view of the market.

Differing from the asset parameters, the changes in regime were estimated by means of an expanding window, beginning 15 years before the start of the experiment and increasing in length with time. An expanding window is preferable to provide continuity to the estimated changes in regime. While increasing the data set at one end, a rolling window has the disadvantage of trimming the dataset at the other end. As new data are added with time, periods that were once identified as volatile may no longer be considered so, which introduces another source of noise to the regime estimation. This problem is only magnified if earlier observations are discarded, justifying the choice to use an expanding window in lieu of a rolling window.

6.2. Fixed-term rebalancing experiment

The first computational experiment required the portfolios to be rebalanced every six months. Aside from the shared

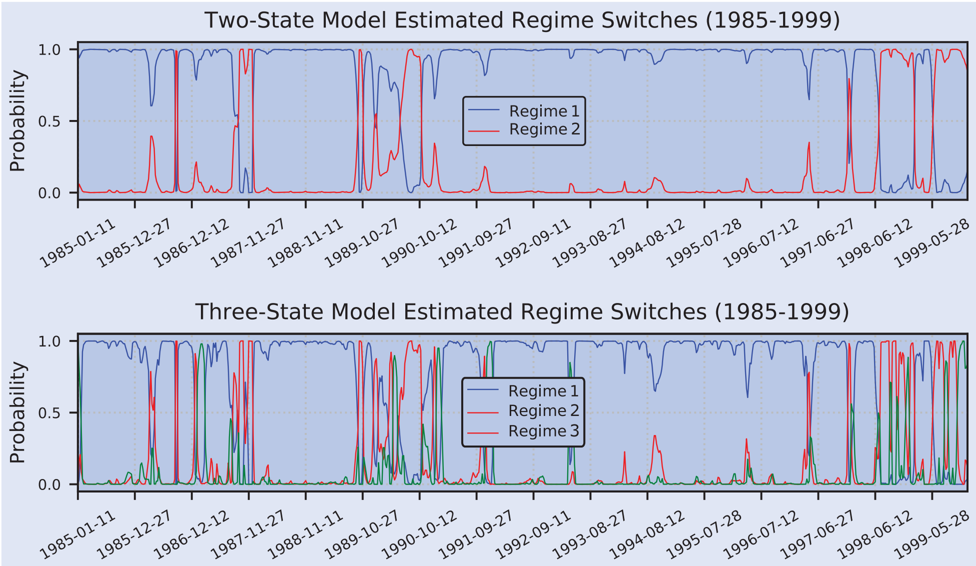

Figure 2. Estimated regime switches for the period of 1985–1999. Top: two-state model. Bottom: three-state model.

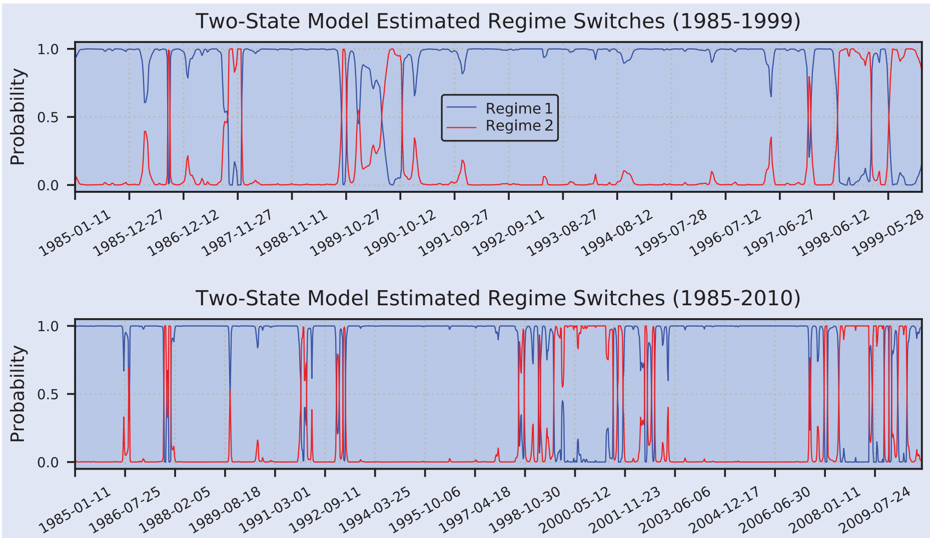

Figure 3. Estimated regime switches for test period 2000–2010. The probability of having a bullish (bearish) market is shown in blue (red). Top: regime switches estimated before the first investment period. Bottom: regime switches estimated after the last investment period.

benefits to all strategies from rebalancing, the regime-switching portfolios are expected to have the additional benefit of re-estimating the regime-dependent parameters at the beginning of each investment period. Correctly identifying a change in regime at the beginning of the current investment period would prompt the portfolio to swiftly shift its estimate of the covariance matrix to reflect the current market conditions before proceeding to re-optimize the portfolio. In theory, having a higher rebalancing frequency would allow the regime-switching portfolio to better mimic the market dynamics and to respond faster to a change in regime. However, in practice, portfolios tend to be reassessed periodically to minimize both transaction costs and associated labour. Thus, having six-month investment periods was considered to yield a more realistic experiment.

The current market regime is re-estimated at the beginning of each new investment period. If a change in regime is identified, the probability-weighted asset covariance matrix of the regime-switching portfolios would change to reflect the properties corresponding to the current regime. Such a sudden shift usually results in higher-than-nominal portfolio turnover rates. However, experimental results show that the increased accuracy of the covariance estimate is able to better reflect the market dynamics, resulting in higher risk-adjusted returns.

A sample of the estimated regime switches from this experiment is presented in Figure 3, which plots the smoothed probabilities of a two-state model prior to the start of the first investment period (top) and the last investment period (bottom).

The evolution of wealth of the seven portfolios is shown in Figure 4. The top plot shows the absolute wealth of the nominal ERC portfolio with an initial investment of 100 arbitrary units. The middle and bottom plots show the wealth of the remaining six other portfolios relative to the nominal, i.e. they show $100 \times ((w_q/w_1) - 1)$ for $q = 2, 3, \ldots, 7$, where w denotes the wealth of the portfolio as a function of time. For the sake of clarity, the remaining portfolios are separated into the middle and bottom sub-plots. The two- and three-state

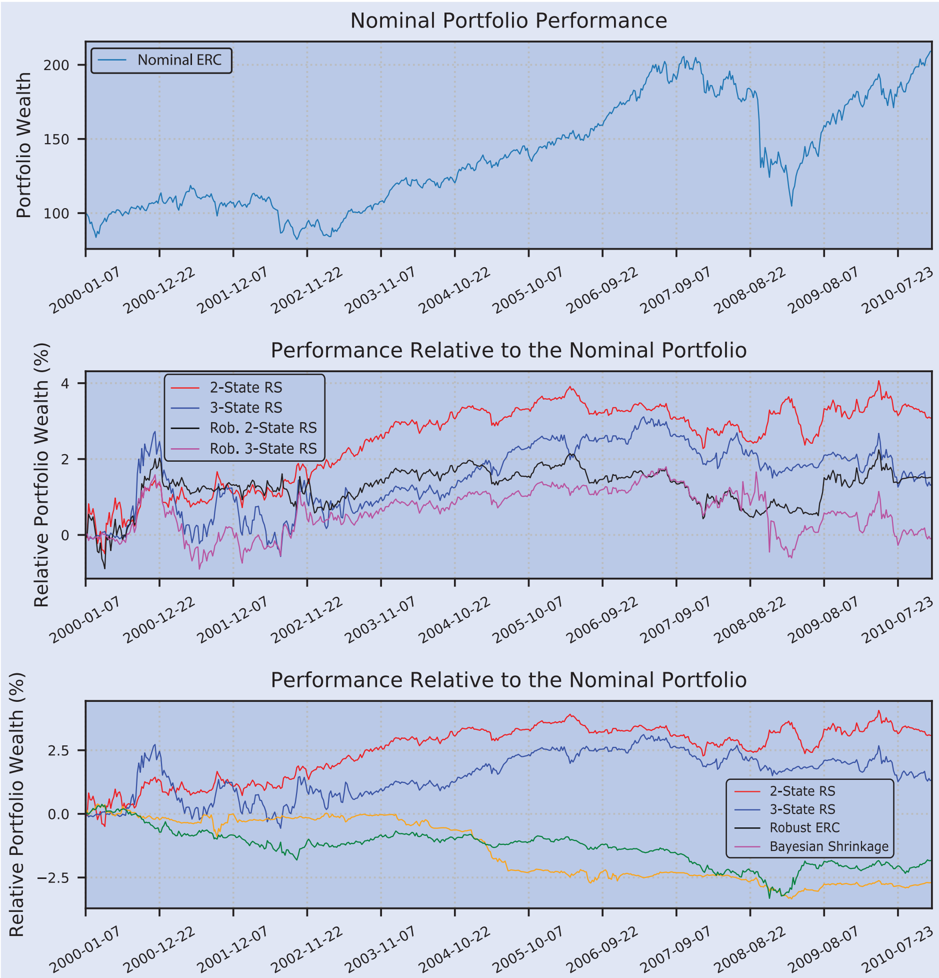

Figure 4. Evolution of relative wealth for test period 2000–2010. Top: evolution of absolute wealth of nominal portfolio. Middle: evolution of wealth relative to the nominal portfolio ($w_q/w_1 - 1$ for $q = 3,4,5,6$). Bottom: evolution of wealth relative to the nominal portfolio ($w_q/w_1 - 1$ for $q = 2,3,5,7$).

Table 2. Average yearly returns, average *ex post* Sharpe ratio and average turnover rate for a fixed-term rebalancing policy.

	Fixed-term rebalancing: experimental results						
	Nom. ERC	Rob. ERC	2-State RS	3-State RS	Rob. 2-State	Rob. 3-State	Shrinkage
Yearly return	0.0695	0.0668	0.0724	0.0708	0.0709	0.0694	0.0677
Sharpe ratio	0.0511	0.0488	0.0531	0.0516	0.0514	0.0501	0.0492
Turnover	0.0222	0.0826	0.0722	0.1866	0.058	0.1494	0.02

regime-switching portfolios are shown in both of these for comparison. The middle and bottom plots serves to identify if any investment strategy has consistently outperformed the nominal ERC portfolio. The results demonstrate that the two-state regime-switching portfolio fared significantly better than the rest, even during bear market periods. The three-state regime-switching portfolio was also able to maintain a consistent lead over the nominal, but aside from a brief initial surge, it was not able to surpass its two-state counterpart. On the other hand, none of the robust portfolios were able to replicate this success. The robust regime-switching portfolios mimic their non-robust counterparts, peaking and dropping steeply at the same points in time, but they appear to be excessively conservative during bull market periods. Thus, any relative gains made during periods of distress are eroded once the market stabilizes. The robust ERC portfolio also suffers from having a conservative outlook during bull market periods. However, it also fails to capitalize during periods of market distress.

The average yearly returns are shown in Table 2, further reinforcing the results presented in Figure 4. The returns of the two-state regime-switching portfolio are shown to be consistently higher throughout this long-term investment horizon. Traditional robust strategies primarily focus on minimizing loss during periods of market distress, where parameter estimation is unstable. With that said, these same portfolios tend to underperform during bull market periods. Nevertheless, our regime-switching strategy does not suffer from this disadvantage due to its dynamic formulation, allowing it to maintain its higher rate of return through time. The frequency of rebalancing is also important for this type of portfolio. If rebalancing

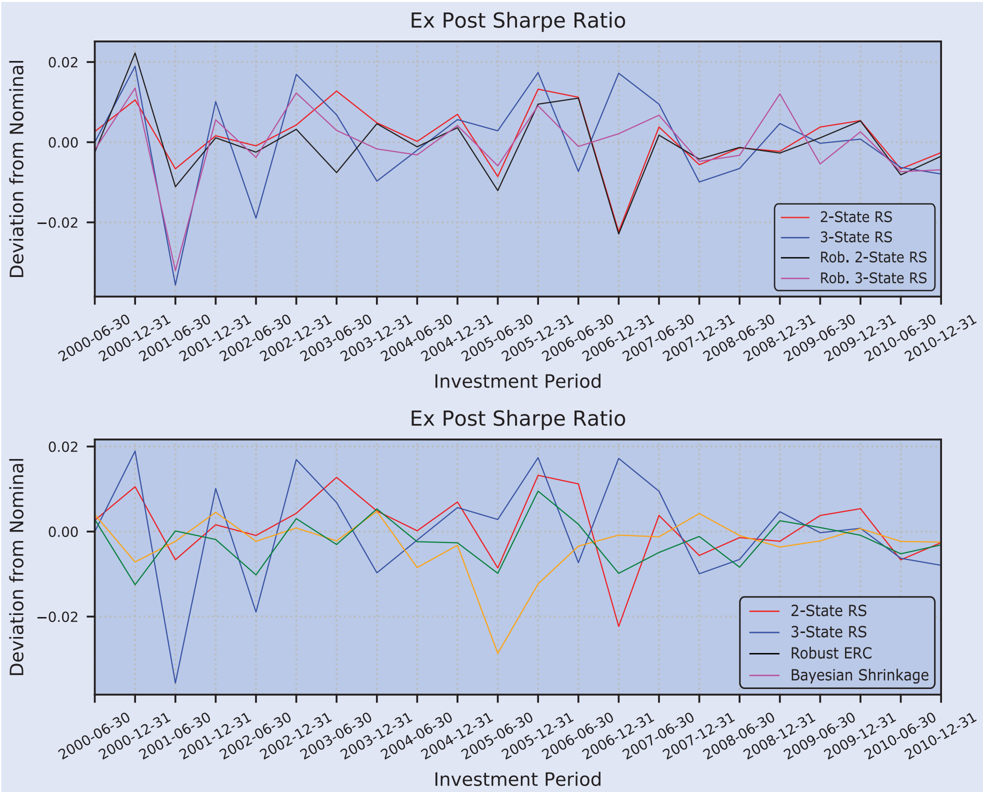

Figure 5. Evolution of *ex post* Sharpe ratio for test period 2000–2010. Values are shown as deviation from the nominal portfolio. Top: $S_q^t - S_1^t$ for $q = 3,4,5,6$. Bottom: $S_q^t - S_1^t$ for $q = 2,3,5,7$.

is infrequent, any changes in regime whilst in the middle of an investment period could hinder the portfolio performance. The same is true for a higher rebalancing frequency if a regime is estimated during turbulent periods with a high level of noise, potentially leading to large estimation errors.

We proceed to evaluate the risk-adjusted returns by calculating the *ex post* Sharpe ratio for each strategy (Sharpe 1994). The *ex post* Sharpe ratio for each portfolio q is defined as

$$\mathrm{S_q} = \frac{\mu^*_{p_q} - r_f}{s_{p_q}}, \quad q = 1, 2, \ldots, 7,$$

where μ^*_p is the realized return of the portfolio, r_f is the risk-free rate based on the one-month U.S. Treasury bill (French 2016) and σ^*_p is the observed variability of the realized returns (i.e. the observed standard deviation of the weekly portfolio returns).

Figure 5 shows the deviation from the nominal portfolio per investment period, i.e. $S_q^t - S_1^t$ for $q = 2, 3, \ldots, 7$ and $t = 1, \ldots, 22$. As before, the figure is divided into two sub-plots for clarity, with the two- and three-state regime-switching portfolios shown in both sub-plots for comparison. A deviation greater than zero indicate a better-than-nominal Sharpe ratio, while the opposite is true for values lower than zero. The average *ex post* Sharpe ratio shown in Table 2 was calculated by computing the average weekly returns corresponding to the entire investment horizon, and dividing this value by the corresponding standard deviation. The two-state regime-switching portfolio had the strongest risk-adjusted rate of return, in particular during periods of market distress. The three-state regime-switching portfolio also outperformed the nominal, but still trailed behind its two-state counterpart. Both the robust and non-robust three-state models appear to be quite volatile in general, eroding their Sharpe ratios.

The last aspect we consider is the cost of implementing these four strategies, especially considering that high transaction costs could erode any benefits gained from the different variants of the ERC portfolio. In general, trading fees consist of three categories: brokerage fees, market price impact, and the bid-ask spread. In reality, brokerage fees are usually the lowest of the three and, for large institutional investors, can be considered negligible. Assessing the remaining fees becomes increasingly difficult as these are dependent on the size of the transaction and the liquidity of the assets. However, large capitalization stocks, such as the constituents of the S&P 500, tend to have low fees due to their high liquidity and large traded volumes.

Transaction costs are better described by the turnover rate , i.e. $||\boldsymbol{\phi}_l^t - \boldsymbol{\phi}_l^{t-1}||$. As with the Sharpe ratio, Figure 6 shows the turnover rate of each portfolio as a deviation from the nominal.

Although not explicitly shown in this figure, it should be noted that the nominal ERC portfolio is very stable, with an exceptionally low turnover rate. On the other hand, we can see that the regime-switching portfolios are much more volatile, in particular during the two most prominent bear market periods during this test. This is expected since the estimated covariance matrix changes dramatically whenever a change in regime is perceived. The portfolio based on the Bayesian shrinkage method appears to be the most stable, having an average turnover rate lower than the nominal itself. For reference, Table 2 presents the average turnover rate per

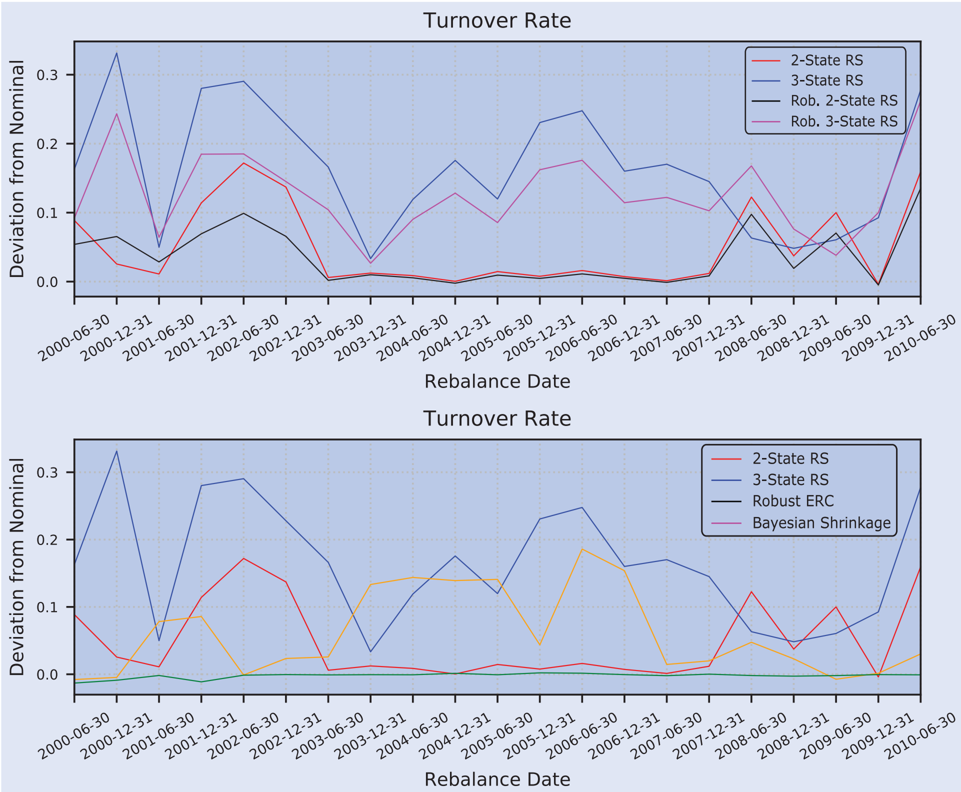

Figure 6. Portfolio turnover rate for test period 2000–2010. Values are shown as the absolute deviation from the nominal portfolio. Top: $||\phi_q^t - \phi_q^{t-1}|| - ||\phi_1^t - \phi_1^{t-1}||$ for $q = 3,4,5,6$. Bottom: $||\phi_q^t - \phi_q^{t-1}|| - ||\phi_1^t - \phi_1^{t-1}||$ for $q = 2,3,5,7$.

rebalancing period. The high turnover rate for the regime-switching portfolios should serve to understand the cost of improving the quality of estimation. Keeping a high fidelity to the market dynamics will have a similar effect on the portfolio turnover rate. Nevertheless, it can be clearly seen that the two-state regime-switching portfolio has a lower turnover rate than the three-state model, while also maintaining a better risk-adjusted performance, as previously shown.

6.3. Dynamic rebalancing experiment

A more intuitive approach to the regime-switching model was performed in a second experiment. Instead of having a series of fixed six-month investment periods, the portfolios were allowed to react to the market dynamics and were only rebalanced when a change in regime was perceived. This rebalancing policy is based solely on a two-state model. This is justified by the added noise observed in the three-state regime estimation shown previously in Figure 2, as well as by the underwhelming performance of the three-state portfolios during the fixed-term rebalancing experiment. Thus, the three-state regime-switching portfolios are dropped from the second experiment.

In theory, this dynamic rebalancing policy consists of using the Markov model to regularly estimate the current market regime as time goes by. The portfolios are rebalanced only when a change in regime is identified. In practice, the 'current' regime was re-estimated every four weeks, and the portfolios would only be rebalanced if there was a consistent change in regime. We defined 'consistent' as having two consecutive four-week periods of the alternate regime in order to soften the impact of estimation error arising from random noise. The portfolio constituents and the investment horizon were the same as before. The estimated changes in regime are the same as those shown in Figure 3 corresponding to the previous experiment, which is not surprising since the investment horizon is the same.

The evolution of wealth is shown in Figure 7. The top plot shows the absolute wealth of the nominal ERC portfolio, while the bottom plot shows the wealth of the remaining portfolios as the percent deviation from the nominal portfolio. The rebalance dates are also plotted to show that regime changes are most prominent during volatile periods.

As before, the two-state regime-switching portfolio had the highest rate of return. Moreover, the results are significantly better than in the previous experiment. Not only did the nominal portfolio attain a higher return when compared to itself during the fixed-term experiment, but the regime-switching portfolio also fared better in both relative and absolute terms. Allowing the regime-switching portfolio to react to current market conditions enables it to swiftly update its estimated covariance matrix from an optimistic outlook to a pessimistic one, and vice versa. This is also reflected in the performance of the robust regime-switching portfolio. The reason for this outstanding rate of return is that the portfolios are allowed to rebalance as soon as a consistent change in the market is perceived, without having to endure prolonged periods of time under a misaligned covariance matrix while in-between fixed-term rebalance periods. The exceptional performance of the two-state regime-switching portfolio is emphasized by looking at the the average yearly returns provided in Table 3.

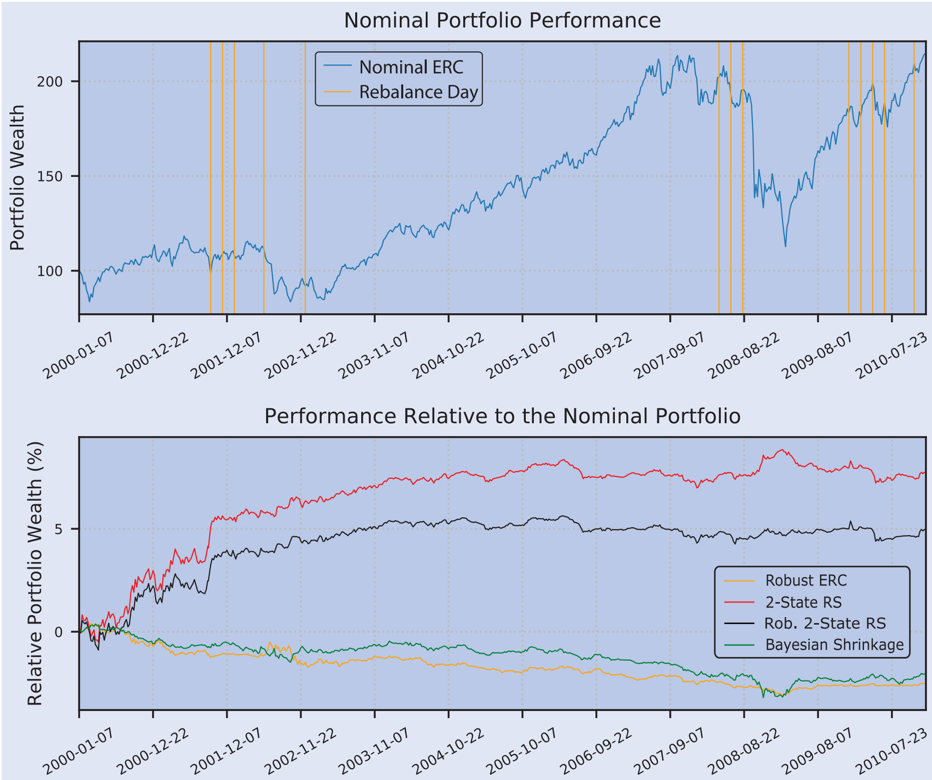

Figure 7. Evolution of relative wealth for test period 2000–2010. Top: evolution of absolute wealth of nominal portfolio. Bottom: evolution of wealth relative to the nominal portfolio ($w_q/w_1 - 1$ for $q = 2,3,4,7$).

Table 3. Average yearly returns, average *ex post* Sharpe ratio and average turnover rate for a dynamic rebalancing policy.

	Dynamic rebalancing: experimental results				
	Nom. ERC	Rob. ERC	2-State RS	Rob. 2-State RS	Shrinkage
Yearly return	0.0719	0.0694	0.0792	0.0766	0.0698
Sharpe ratio	0.0528	0.0508	0.0579	0.0557	0.0509
Turnover	0.0274	0.0516	0.1699	0.1314	0.0246

In line with the increased portfolio returns, the *ex post* Sharpe ratio was also superior in this experiment. However, we note that the results shown in Figure 8 may be biased due to the short time span of certain investment periods. During a volatile market state, the investment periods are short lived, often providing an unfair comparison point between the two experiments. A fair comparison of the *ex post* Sharpe ratio is achieved by comparing the average Sharpe ratio between both experiments (see Tables 2 and 3). This average is calculated in the same fashion as before. The two-state regime-switching Sharpe ratio is approximately 10% higher than its nominal counterpart, which is similar to the margin the average rate of return the regime-switching portfolio enjoyed over the nominal portfolio. Since the risk-free rate during this time period is considerably low, the similar proportions observed between the average returns and the average Sharpe ratios can only be attributed to having a similar level of risk. This, in turn, demonstrates that the regime-switching portfolio is able to attain a higher rate of return, even after adjusting for risk.

Finally, the turnover rate of the portfolios is shown in Figure 9. It should be noted that, as before, the Bayesian shrinkage portfolio still attains a lower turnover rate than the nominal. However, this has been the only advantage gained from using Bayesian shrinkage throughout this experiment. Since the portfolios were only rebalanced when a change in regime was identified, it is not surprising that every single rebalance by both the robust and non-robust regime-switching portfolios was significant, leading to a consistently large turnover rate relative to the nominal portfolio. This is emphasized by the average turnover rates shown in Table 3. Although these values are quite high, the reader should realize that the total number of rebalancing periods is considerably lower than before, with the portfolios only being rebalanced thirteen times.

6.4. *Analysis of results*

In general, portfolio optimization can only be performed on an ex ante basis. For the case of the ERC portfolio, this means that the even distribution of wealth from a risk perspective is entirely dependent on the estimated covariance matrix. We underline that estimation error will directly affect the allocation of wealth, resulting in an unequally weighted

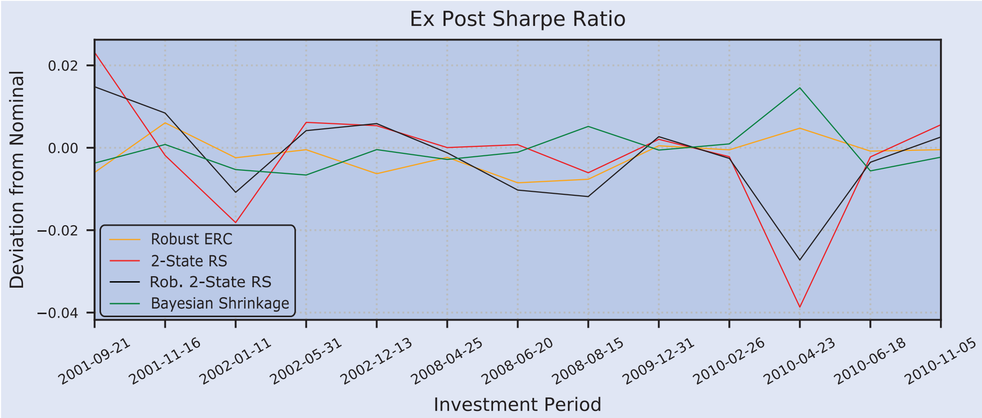

Figure 8. Evolution of *ex post* Sharpe ratio for test period 2000–2010. Values are shown as the absolute deviation from the nominal portfolio.

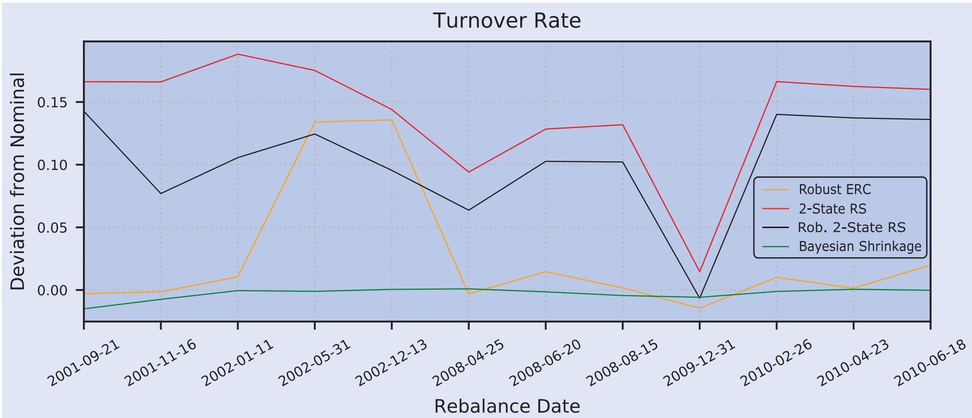

Figure 9. Portfolio turnover rate for test period 2000–2010. Values are shown as the absolute deviation from the nominal portfolio.

risk contribution per asset and increasing our exposure to individual asset risk. Thus, one can recognize how this would explicitly influence the *ex post* portfolio risk-adjusted rate of return.

The experiments were performed over a period of relatively continuous bull market states, briefly interrupted by the ‘ dot-com’ bubble, the early 2000s recession, and the recent 2008 financial crisis. Under the thesis that the market is governed by two distinct states, our results demonstrate that the nominal ERC portfolio is hindered by its static nature. The single-regime multi-factor model results in the smoothing of the two market states into a single set of estimated parameters, which in turn inhibits the model from benefiting from the directional insight gained by correctly identifying the current market regime and rebalancing the portfolio accordingly. Thus, a more detailed examination of the experimental results show that a prolonged bull market state enables the two-state regime-switching portfolio to attain a higher rate of return while still maintaining a similar risk level when compared to its nominal counterpart.

Moreover, deciding how many estimated states are present in the market is a challenge of its own. Our experiments have shown that increasing the number of states beyond two significantly increases estimation error, thereby hindering the performance of the three-state regime-switching portfolios and reinforcing the thesis that a two-state model is sufficiently adequate for our purpose.

The experiments only displayed the behaviour of the regime-switching ERC portfolios during prolonged bull market periods with sparse and short-lived bear market periods. However, we are only able to hypothesize about a broader set of scenarios, such as what the performance would be during a reverse scenario with long-lived bear market states. The experimental observations during the short-lived bear market states suggest a two-state regime-switching portfolio would be characterized by having a more conservative outlook, shifting its covariance matrix estimate to reflect this. This is exemplified on the first experiment by the surges in the regime-switching portfolio’s *ex post* Sharpe ratio observed during bear market periods. Thus, while the regime-switching portfolio performance is characterized by higher-than-nominal returns during bull market periods, the bear market performance is better described by a more conservative stance with a lower-than-nominal risk level.

The second experiment serves to highlight a secondary advantage of a two-state regime-switching ERC portfolio. Aside from estimating the transition probabilities and identifying the present state of the market, we can likewise use the Markov model to dictate the rebalancing policy of the portfolio. Intuitively, this allows us to re-estimate the portfolio parameters and rebalance whenever a change in regime is perceived. The advantage of this is threefold. First, this enables the regime-switching portfolio to swiftly adjust its covariance matrix to match the current market state and minimize its

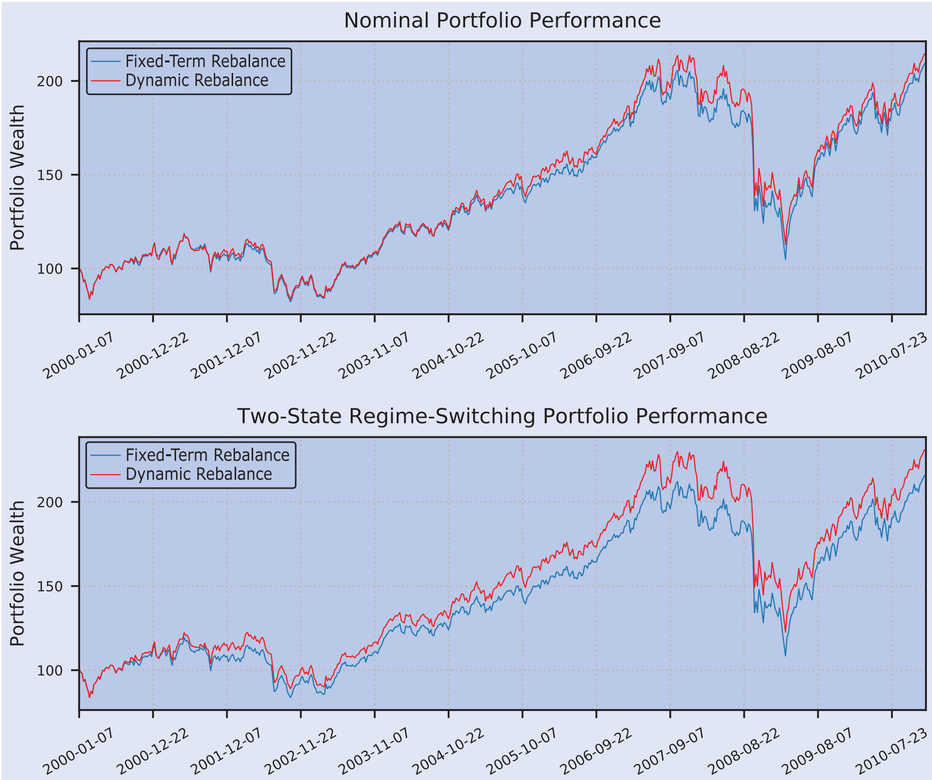

Figure 10. Evolution of wealth comparison for a fixed-term rebalancing policy and a dynamic rebalancing policy. Top: performance of the nominal ERC portfolios. Bottom: performance of the two-state regime-switching ERC portfolios.

exposure to individual asset risk. Second, it minimizes the time duration of potential misalignment between the model parameters and the current regime, ensuring the portfolio is consistently aligned with the market. Finally, it prevents the unnecessary rebalancing of the portfolio, thereby reducing associated transaction costs. Figure 10 compares of the portfolio evolution of wealth between the two rebalancing policies. Not only does this plot serve to emphasize the advantages of a two-state regime-switching portfolio over a nominal ERC portfolio but also to demonstrate the benefits of having a regime-dependent dynamic rebalancing policy.

Finally, the two-state regime-switching ERC portfolio was also compared against its robust counterpart. While the robust portfolio closely mimicked its changes in wealth, the experimental results show that it was never able to attain the same rate of return. A robust formulation of the regime-switching model had a counterproductive effect on the portfolio, demonstrating that any potential gains from the directional insight were eroded by its conservative predisposition.

7. Conclusion

This paper presented a risk parity portfolio optimization problem under a Markov regime-switching framework. The model was developed to accommodate any general multi-factor model of returns, and the Fama–French three-factor model was used as an example during the computational experiments. The regime-switching optimization problem proved to be computationally tractable, and did not show signs of deterioration for portfolios of increased size. Although the NLP incorporates the non-linear constraint $\sum_{j=1}^{n} \ln \psi_j \geq c$, the problem is still convex and can be easily adapted to incorporate other practical constraints (such as cardinality constraints), giving increased flexibility to further enhance and modify this formulation.

The computational results show that applying a regime-switching framework can be beneficial even in a scenario where the underlying model has low estimation error and is already well diversified. Allowing the NLP to ‘switch’ its parameters depending on the current market regime allows it to more faithfully render the non-stationarity of the market. However, unlike traditional robust strategies, our model does not penalize the portfolio by being unnecessarily conservative during bull market periods. Furthermore, the results showed that a two-state regime-switching portfolio can consistently provide higher-than-nominal risk-adjusted returns over long investment horizons. Thus, allowing the estimated parameters to closely mimic the oscillating market dynamics yields more accurate results.

The improved portfolio performance can be attributed to the better quality estimates of the asset covariance matrix. As the experiment shows, prolonged bull market periods results in a regime-switching portfolio that is able to keep a similar level of risk relative to its nominal counterpart while still delivering a higher rate of return. On the other hand, we can also observe that during a bear market the portfolio will instead have a more conservative stance, reducing its level of risk relative to the nominal. Although the ERC portfolio does not explicitly emphasize the minimization of risk or the maximization of returns, a higher risk-adjusted rate of return can be achieved by reducing exposure to concentrated risk. In other words, allowing the portfolio to capture the directional information latent within the market results in a more faithful diversification of the present level of risk, which is the overarching objective of the ERC portfolio.

Furthermore, the advantages of a regime-switching portfolio can be compounded by having a dynamic rebalancing policy, where the portfolio is allowed to numerically determine when should rebalancing take place. This magnifies the aforementioned benefits of the regime-switching ERC portfolio by promptly mitigating any potential misalignment of the portfolio parameters with the present state of the market, as well as preventing unnecessary rebalancing of the portfolio.

The two-state regime-switching model presented in this paper is constructed by means of an intuitive approach, where the regime-dependent asset returns are described by using indicator functions. This, in turn, allows for a natural derivation of the covariance matrix that intrinsically incorporates the market dynamics. However, by construction, this is a parametric model that requires the estimation of additional parameters as the number of estimated regimes is increased. A non-parametric model would reduce the number of parameters to be estimated, thereby increasing tractability and accuracy. Reformulating this problem into a non-parametric model could be the subject of future research.

Moreover, the use of regime switching for other risk parity models can be considered where factor models are used directly to estimate risk. For example, Roncalli and Weisang (2016) expose the relationship between risk factors and portfolio allocations and formulate an optimization model that achieves diversification based on this relationship. Incorporating the regime-switching approach for this model would be an interesting line of future research where the risk contributions of the assets would be influenced by multiple possible regimes of risk.

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Appendix 1. Lemma 1 from Goldfarb and Iyengar (2003)

Lemma 1 was introduced in Goldfarb and Iyengar (2003) to reformulate the 'worst-case variance' constraint as a collection of linear equalities, linear inequalities and restricted hyperbolic constraints. For the convenience of the reader, this lemma is presented below. The worst-case variance of a portfolio can be said to be less than ν if and only if

$$\max_{y:||y||_g \leq r} ||y_0 + y||_f^2 \leq \nu,$$

where $\boldsymbol{y}_0 = \boldsymbol{V}_0$ and $r = \boldsymbol{\rho}^\top|\boldsymbol{\phi}|$. This inequality constraint can be converted into a more computationally tractable formulation. This new version of the worst-case variance constraint is presented below.

Let $r, \nu > 0, \boldsymbol{y_0} = \boldsymbol{V}_0\boldsymbol{\phi}, \boldsymbol{y_0}, \boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{F}, \boldsymbol{G} \in \mathbb{R}^{m\times m}$ be positive definite matrices. Then the constraint

$$\max_{y:||y||_g \leq r} ||y_0 + y||_f^2 \leq \nu,$$

where $||\boldsymbol{x}||_f : \boldsymbol{x} \to \sqrt{\boldsymbol{x}^\top \boldsymbol{F}\boldsymbol{x}}$, is equivalent to either of the following:

(i) there exist $\tau, \sigma \geq 0$, and $t \in \mathbb{R}^m_+$ that satisfy

$$\nu \geq \tau + \mathbf{1}^\top t,$$
$$\sigma \leq \frac{1}{\lambda_{\max}(H)},$$
$$r^2 \leq \sigma\tau,$$
$$w_k^2 \quad (1 - \sigma\lambda_k)t_k, \quad k = 1, \ldots, m,$$

where $\boldsymbol{Q\Lambda Q}^\top$ is the spectral decomposition of $\boldsymbol{H} = \boldsymbol{G}^{-1/2}\boldsymbol{F}\boldsymbol{G}^{-1/2}$, $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_i)$, $\boldsymbol{w} = \boldsymbol{Q}^\top \boldsymbol{H}^{1/2}\boldsymbol{G}^{1/2}\boldsymbol{y}_0$ and $\lambda_{\max}$ $(\boldsymbol{H})$ is the largest eigenvalue of $\boldsymbol{H}$.

(ii) there exist $\tau \geq 0$, and $s \in \mathbb{R}^m_+$ that satisfy

$$r^2 \leq \tau(\nu - \mathbf{1}^\top s),$$
$$u_k^2 \leq (1 - \tau\theta_k)s_k, \quad k = 1, \ldots, m,$$
$$\tau \leq \frac{1}{\lambda_{\max}(K)},$$

where $\boldsymbol{P\Theta P}^\top$ is the spectral decomposition of $\boldsymbol{K} = \boldsymbol{F}^{1/2}\boldsymbol{G}^{-1}\boldsymbol{F}^{1/2}$, $\boldsymbol{\Theta} = \mathrm{diag}(\theta_i)$, $\boldsymbol{u} = \boldsymbol{P}^\top \boldsymbol{F}^{1/2}\boldsymbol{y}_0$ and $\lambda_{\max}(\boldsymbol{K})$ is the largest eigenvalue of $\boldsymbol{K}$.

Lemma 1 is proved by using the $\mathbb{S}$-procedure described in Boyd *et al.* (1994) †. The size of the uncertainty sets given in Equation (13) are controlled by the joint confidence level ω, which is incorporated into the computation of $\boldsymbol{\rho}$ in Lemma 1. The reader can find the equation relating $\boldsymbol{\rho}$ to ω in Equation (A4) in Appendix 2. A high joint confidence level implies we have a large uncertainty set, i.e. one demands robustness with respect to a very large set of parameter values. The typical choices of ω lie in the range 0.95–0.99.

† For a detailed proof of Lemma 1 see Goldfarb and Iyengar (2003).

Appendix 2. Parameter generation for the robust optimization model

The parameters used throughout this paper in the two robust optimization models arise from the procedure described in Goldfarb and Iyengar (2003). Let $\boldsymbol{r}_i \in \mathbb{R}^p$ be the vector of historical returns of asset i and let $\boldsymbol{B} = [\boldsymbol{f}^1\boldsymbol{f}^2 \ldots \boldsymbol{f}^p] \in \mathbb{R}^m \times p$ be the matrix of factor returns over all the periods $t = 1, \ldots, p$. We present the vector of returns in a different but equivalent form as the one given in Equation (5),

$$y_i = Ax_i + \epsilon_i, \tag{A1}$$

where $\boldsymbol{y}_i = [r_i^1 r_i^2 \ldots r_i^p]^\top$, $\boldsymbol{A} = [1B^\top]$, $\boldsymbol{x}_i = [\mu_i V_{1i} V_{2i} \ldots V_{mi}]^\top$ and $\boldsymbol{\epsilon}_i = [e_i^1 e_i^2 \ldots e_i^p]^\top$. The least square estimate $\bar{\boldsymbol{x}}_i$ of the true parameter $\boldsymbol{x}_i$ is given by the solution to the equation

$$\bar{\boldsymbol{x}}_i = (A^\top A)^{-1}A^\top y_i. \tag{A2}$$

Substituting Equation (A1) into Equation (A2) yields

$$\bar{x}_i - x_i = (A^\top A)^{-1}A^\top \epsilon_i = \mathcal{N}(0, \Sigma),$$

where $\Sigma = \sigma_i^2(A^\top A)^{-1}$. Although the true variance σ_i^2 is not known, in practice we replace σ_i^2 by $(m+1)s_i^2$, where s_i^2 is the unbiased estimate of σ_i^2. The unbiased estimate is given by

$$s_i^2 = \frac{||y_i - A\bar{x}_i||}{p - m - 1}. \tag{A3}$$

The resulting random variable

$$\mathcal{Y} = \frac{1}{\sigma_i^2}(\bar{x}_i - x_i)^\top (A^\top A)(\bar{x}_i - x_i),$$

follows the F-distribution with $(m+1)$ degrees of freedom in the numerator and $(p-m-1)$ degrees of freedom in the denominator. By setting the ω-confidence region for set $S_v(\omega)$, we obtain

$$S_v = \left\{V : V = V_0 + W, ||W_i||_g \leq \rho_i, i = 1, \ldots, n\right\},$$

where

$$V_0 = \bar{V},$$
$$G = (Q(A^\top A)^{-1}Q^\top)^{-1} = BB^\top - \frac{1}{t}(B\mathbf{1})(B\mathbf{1})^\top,$$
$$\rho_i = \sqrt{(m+1)c_{m+1}(\omega)s_i^2}, \quad i = 1, \ldots, n, \tag{A4}$$

where $c_J(\omega)$ is the ω-critical value. The upper bound of the variance of the residual is set to $\bar{\boldsymbol{D}} = \mathrm{diag}(s_i^2)$. Thus, all the necessary parameters to formulate a robust optimization problem can be extracted directly from raw market data. The same procedure is applied to generate the robust parameters corresponding to the regime-switching sets $S_{v_1}(\omega)$ and $S_{v_2}(\omega)$, with the only difference being that the $\boldsymbol{r}_i$ vectors of asset returns and the $\boldsymbol{B}$ matrix of factor returns are partitioned into $\boldsymbol{r}_{1_i}$, $\boldsymbol{r}_{2_i}$, $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$, corresponding to the time periods belonging to each regime.