Asymptotic Power and Efficiency of Lepage-Type Tests for the Treatment of Combined Location-Scale Alternatives

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Abstract

For the two-sample location and scale problem Lepage (1971) constructed a test that is based on a combination of the Wilcoxon test statistic and the Ansari-Bradley test statistic. We replace both components by arbitrary linear rank tests and obtain so-called Lepage-type tests that were introduced by Büning and Thadewald (2000). In the present paper we compute their asymptotic efficacies.

The results of these calculations give rise to an idea how to construct adaptive tests based on the concept of Hogg (1974).

We also include asymmetric densities in our study. It turns out that, for moderately skew densities, a combination of linear rank test statistics designed for symmetric densities is sufficient. Therefore, in our proposed adaptive test occur only tests designed for symmetric densities. For extremely skew densities the application of the combination of Savage-scores tests is suggested.

A Monte Carlo study confirms the asymptotic results. Moreover, it shows that the adaptive test proposed is a serious competitor also for moderate sample sizes.

1 Introduction

Let $X_1, \ldots, X_m$ and $X_{m+1}, \ldots, X_{m+n}$ be two samples from absolutely continuous populations F_1 and F_2, respectively. We consider the Behrens-Fisher Problem,

$$F_2(x) = F_1\left(\frac{x-\theta}{\tau}\right)$$

where θ and $\tau = e^{\vartheta}$ are location and scale parameters. In the following we assume that $F := F_1$ is twice continuously differentiable on $(-\infty, \infty)$ except for a set of Lebesgue measure zero; f' denotes the derivative of the density f where it exists and it is defined to be zero, otherwise. The Fisher information is assumed to exist.

We test the hypothesis

$$H_0: \qquad \theta = 0 \quad \text{and} \quad \tau = 1$$

against the alternative

$$H_1: \qquad \theta \neq 0 \quad \text{or} \quad \tau \neq 1.$$

This problem was considered by Lepage (1975) and Büning and Thadewald (2000). In the present paper we compute the asymptotic efficacies and power functions. Moreover, we consider symmetric as well as asymmetric densities.

2 Linear rank tests

In this section we recall well-known results for linear rank tests for the two-sample location and scale problem, respectively.

Assumption 1 *The scores $a_N(i)$ and $b_N(i)$ are assumed to satisfy*

$$\lim_{N\to\infty}\int_0^1 (a_N(1+\lfloor uN\rfloor)-\phi(u))^2\,du=0$$

and

$$\lim_{N\to\infty}\int_0^1 (b_N(1+\lfloor uN\rfloor)-\psi(u))^2\,du=0$$

with square integrable score functions

$$\phi_1(u,g_1):=\phi_1(u) = -\frac{g_1'(G_1^{-1}(u))}{g_1(G_1^{-1}(u))} \quad \textit{and} \tag{1}$$

$$\phi_2(u,g_2):=\phi_2(u) = -1-G_2^{-1}(u)\frac{g_2'(G_2^{-1}(u))}{g_2(G_2^{-1}(u))}. \tag{2}$$

Moreover, we assume that

$$\begin{aligned} |\phi_j(u)| &\leq K_j\big(u(1-u)\big)^{-\frac{1}{2}+\delta} \quad \textit{and} \\ |\phi_j'(u)| &\leq K_j'\big(u(1-u)\big)^{-\frac{3}{2}+\delta} \end{aligned}$$

for some $\delta>0$, $j=1,2$. Define

$$d_L(f,g_1):=\int_0^1 \phi_1'(u,g_1)\cdot f(F^{-1}(u))\,du \quad \textit{and} \quad I_L(g_1):=\int_0^1 \phi_1^2(u,g_1)\,du,$$

as well as

$$d_S(f,g_2):=\int_0^1 \phi_2'(u,g_2)\cdot F^{-1}(u)f(F^{-1}(u))\,du \quad \textit{and} \quad I_S(g_2):=\int_0^1 \phi_2^2(u,g_2)\,du,$$

where $I_L(g_1)$ and $I_S(g_2)$ are the Fisher-informations of the density functions g_1 and g_2 defined by (1) and (2) concerning the location and scale problems, respectively, ϕ' and ψ' represent the derivatives of ϕ and ψ almost everywhere. It is assumed that $\int_0^1 \phi_1(u,g_1)\,du=\int_0^1 \phi_2(u,g_2)\,du=0$ and $0<I_L(g_1),I_S(g_2)<\infty$.

We use the notations

$$C_L(f,g) := d_L(f,g) \cdot I_L(g)^{-1/2} \qquad \textit{and} \qquad C_S(f,g) := d_S(f,g) \cdot I_S(g)^{-1/2}$$

Moreover, let

$$\begin{aligned} d_{12}(f,g_1) &:= \int_0^1 \phi_1'(u,g_1) F^{-1}(u) f(F^{-1}(u))\, du, \\ d_{21}(f,g_2) &:= \int_0^1 \phi_2'(u,g_2) f(F^{-1}(u))\, du, \end{aligned}$$

and

$$C_{12}(f,g_1) = d_{12}(f,g_1) \cdot I_L(g_1)^{-1/2}, \qquad C_{21}(f,g_2) = d_{21}(f,g_2) \cdot I_S(g_2)^{-1/2}.$$

Assumption 2 *We assume that the two score functions $\phi_1(u,g_1)$ and $\phi_2(u,g_2)$ are orthogonal in the Hilbert space of square integrable functions.*

Assumption 3 *Moreover, we assume that* $0 < d_L(f,g_1), d_S(f,g_2) < \infty$.

Let

$$T_1 = \sum_{i=1}^{m} a_N(R_i)$$

and

$$T_2 = \sum_{i=1}^{m} b_N(R_i)$$

with $N = m + n$, be linear rank statistics for the location problem and for the scale problem, respectively.

Proposition 1 (Hájek, $\check{S}$idák, and Sen, 1999, Ch.6) *Under H_0 the limiting distributions of T_1/σ_1 and T_2/σ_2 are standardnormal with*

$$\begin{aligned} \sigma_1^2 &= \frac{mn}{N} I_L(g_1) \quad \textit{and} \\ \sigma_2^2 &= \frac{mn}{N} I_S(g_2), \end{aligned}$$

respectively.

Let

$$T = \left(\frac{T_1}{\sigma_1}\right)^2 + \left(\frac{T_2}{\sigma_2}\right)^2$$

be the combined test statistic which we call Lepage-type statistic (cf. Büning and Thadewald, 2000).

Corollary 1 *Under H_0 the limiting distribution of T is χ^2 with two degrees of freedom.*

Therefore, asymptotic critical values for the test problem (H_0, H_1) are given by the quantile $\chi^2_{2,1-\alpha}$, and H_0 is rejected if $T > \chi^2_{2,1-\alpha}$.

Next we give some examples of Lepage-type tests together with their two components, the location and scale test.

Example 1 (Classical Lepage test, cf. Lepage, 1975) *Wilcoxon-test and Ansari-Bradley test,*

$$\begin{aligned} T_1 = WI &= \sum_{i=1}^{m} R_i \\ T_2 = AB &= \sum_{i=1}^{m} \left(\left| R_i - \frac{N+1}{2} \right| - \frac{N+1}{2} \right) \end{aligned}$$

The corresponding score functions are (cf. Hájek, $\check{S}$idák, and Sen, p. 15)

$$\begin{aligned} \phi_{1,LP}(u) &= 2u - 1 \qquad \textit{and} \\ \phi_{2,LP}(u) &= 2|2u-1| - 1. \end{aligned}$$

For the next examples we present only the score functions. The scores are gererated by setting $a_N(i) = \phi_1\left(\frac{i}{N+1}\right)$ and $b_N(i) = \phi_2\left(\frac{i}{N+1}\right)$, except for the Savage scores where we have $a_N(i) = \mathbf{E}(X_{(i)})$ ($X_{(i)}$ the ith order statatistic from an exponentially distributed random variable). For finite sample sizes we may modify the scores slightly to have $\overline{a}_N = \overline{b}_N = 0$.

Example 2 (Gastwirth tests, cf. Büning and Thadewald, 2000)

$$\begin{aligned}\phi_{1,GA}(u) &= \phi_1(u, f_{U-L}) = \begin{cases} 4u-1 & \textit{if} \quad 0 < u \le \frac{1}{4} \\ 0 & \textit{if} \quad \frac{1}{4} < u < \frac{3}{4} \\ 4u-3 & \textit{if} \quad \frac{3}{4} \le u < 1 \end{cases} \qquad \textit{and} \\ \phi_{2,GA}(u) &= \begin{cases} 1-4u & \textit{if} \quad 0 < u \le \frac{1}{4} \\ 0 & \textit{if} \quad \frac{1}{4} < u < \frac{3}{4} \\ 4u-3 & \textit{if} \quad \frac{3}{4} \le u < 1. \end{cases} - \frac{1}{4}\end{aligned}$$

Note that $\phi_{1,GA}(u)$ is optimal score function for the location problem if the underlying density is uniform-logistic (0.75), cf. Büning and Kössler, 1999). The resulting Lepage-type test is called Gastwirth test, and it is abbreviated by LPGA.

Example 3 (van der Waerden test and Klotz test, cf. Büning and Thadewald, 2000)

$$\begin{aligned}\phi_{1,N}(u) &= \phi_1(u, f_{nor}) = \Phi^{-1}(u) \\ \phi_{2,N}(u) &= \phi_2(u, f_{nor}) = \left(\Phi^{-1}(u)\right)^2 - 1.\end{aligned}$$

The score functions $\phi_1(u, f_{nor})$ and $\phi_2(u, f_{nor})$ are optimal for the location and scale problems, respectively, if the underlying density is normal. The resulting Lepage-type test is called normal scores test, and it is abbreviated by LPN.

Example 4 (Long-tail test and Mood test, cf. Büning and Thadewald, 2000)

$$\begin{aligned}\phi_{1,LT}(u) &= \phi_1(u, f_{L-D}) = \begin{cases} -1 & \textit{if} \quad 0 < u \le \frac{1}{4} \\ 2(2u-1) & \textit{if} \quad \frac{1}{4} < u < \frac{3}{4} \\ 1 & \textit{if} \quad \frac{3}{4} \le u < 1 \end{cases} \qquad \textit{and} \\ \phi_{2,LT}(u) &= \phi_2(u, f_{t_2}) = 3(2u-1)^2 - 1.\end{aligned}$$

The score function $\phi_{1,LT}(u, f_{L-D})$ is optimal score function for the location problem if the underlying density is logistic-doubleexponential (0.75), cf. Büning and Kössler, 1999). The score function $\phi_{2,LT}(u, f_{t_2})$ ist optimal score function for the scale problem if the underlying density is t with two degrees of freedom. The resulting Lepage-type test is called Long-tail scores test, and it is abbreviated by LPLT.

Example 5 (Cauchy scores test)

$$\begin{aligned}\phi_1(u) &= \phi_1(u, f_{Cau}) = -\sin(2\pi u) \\ \phi_2(u) &= \phi_2(u, f_{Cau}) = \cos(2\pi u).\end{aligned}$$

These score functions are optimal for the location and scale problems, respectively, if the underlying density is Cauchy. The resulting Lepage-type test is called Cauchy-test and it is abbreviated by LPCA.

Example 6 (Logistic scores test)

$$\begin{aligned}\phi_1(u) &= \phi_1(u, f_{Log}) = 2u - 1 \\ \phi_2(u) &= \phi_2(u, f_{Log}) = -1 - (2u-1)\ln(\frac{1}{u} - 1).\end{aligned}$$

These score functions are optimal for the location and scale problems, respectively, if the underlying density is logistic. The resulting Lepage-type test is called Logistic scores test and it is abbreviated by LPlog.

Example 7 (Hogg-Fisher-Randles scores test)

$$\begin{aligned}\phi_{1,HFR}(u) &= \phi(u, f_{L-E}) = \begin{cases} u - \frac{3}{8} & \textit{if} \quad u \le \frac{1}{2} \\ \frac{1}{8} & \textit{if} \quad u > \frac{1}{2} \end{cases} \\ \phi_{2,HFR}(u) &= -\frac{1}{2} + \begin{cases} -\frac{3}{5}u + \frac{3}{5} & \textit{if} \quad u \le \frac{1}{2} \\ u - \frac{1}{5} & \textit{if} \quad u > \frac{1}{2}. \end{cases}\end{aligned}$$

The score function $\phi_{1,HFR}(u)$ is optimal for the location problem if the underlying density is logistic-exponential (0.75), cf. Büning and Kössler (1999). Note that the score functions $\phi_{1,HFR}(u)$ and $\phi_{2,HFR}(u)$ are orthogonal in the Hilbert space of square integrable functions. The resulting Lepage-type test is called HFR-test, and it is abbreviated by LPHFR.

This test is originally designed for right-skew densities. For left-skew densities we may use the scores $\phi_{1,-HFR}(u) = -\phi_{1,HFR}(1-u)$ and $\phi_{2,-HFR}(u) = \phi_{1,HFR}(1-u)$, respectively. We call the corresponding test antisymmetric HFR-test, and it is abbreviated by LP-HFR.

Example 8 (Savage scores tests)

$$\begin{aligned}\phi_{1,SA}(u) &= \phi(u, f_{nGu}) = -1 - \ln(1-u) \\ \phi_{2,SA}(u) &= 1 - 4u - \ln(1-u)\end{aligned}$$

The score function $\phi_{1,SA}(u)$ is optimal for the location problem if the underlying density is of "negative" Gumbel type, cf, Hájek, $\check{S}$idák, and Sen, p.15. Note that the score functions $\phi_{1,SA}(u)$ and $\phi_{2,SA}(u)$ are also orthogonal in the space of square integrable functions. The resulting Lepage-type test is called Savage test, and it is abbreviated by LPSA.

This test is originally designed for left-skew densities. For right-skew densities we may use the scores $\phi_{1,-SA}(u) = -\phi_{1,SA}(1-u)$ and $\phi_{2,-SA}(u) = \phi_{1,SA}(1-u)$, respectively. We call this test antisymmetric Savage test, and it is abbreviated by LP-SA.

Note that in all the examples the assumptions 1 and 2 are satisfied. Assumption 3 is also satisfied in most cases. Exceptions exist for Examples 3, 6 and 8 with some densities (uniform and exponential).

Curves of the score functions are given in Figure 1. The continuous line and the dashed line are for the location test and for the scale test, respectively.

Figure 1 here

3 Asymptotic efficacies of the Lepage type tests

The asymptotic efficacies and asymptotic power functions are computed unter the following assumption.

Assumption 4 *Let be $\Delta_1 \neq 0, \Delta_2 \neq 0$ and (θ_N, ϑ_N) a sequence of "near" alternatives with $\theta_N = N^{-1/2} \cdot \theta$ and $\vartheta_N = N^{-1/2} \cdot \vartheta$. Let be $\min(m,n) \to \infty$, $m/N \to \lambda$, $0 < \lambda > 1$.*

Theorem 1 *Under assumptions 1, 2 and 4 the Lepage-type tests are asymptotically noncentrally χ^2 distributed with two degrees of freedom and noncentrallity parameter*

$$\Delta^2 = \lambda(1-\lambda)\left(\frac{(\theta d_L(f,g_1) + \vartheta d_{12}(f,g_1))^2}{I_L(g_1)} + \frac{(\theta d_{21}(f,g_2) + \vartheta d_S(f,g_2))^2}{I_S(g_2)}\right) \quad (3)$$

Proof: 1. Let be $\boldsymbol{\theta} = (\theta, \vartheta)$. Since the cdf. F is twice continuously differentiable we have the expansion

$$\begin{aligned} F(x;\theta,\vartheta) &= F\big(\frac{x-\theta}{e^{\vartheta}}\big) \\ &= F(x;0,0) + \frac{\partial}{\partial\theta}F(x;\theta,\vartheta)\big|_{\boldsymbol{\theta}=\mathbf{0}}\theta + \frac{\partial}{\partial\vartheta}F(x;\theta,\vartheta)\big|_{\boldsymbol{\theta}=\mathbf{0}}\vartheta + \mathcal{O}(||\boldsymbol{\theta}||^2) \\ &= F(x) - f(x)\theta - xf(x)\vartheta + \mathcal{O}(||\boldsymbol{\theta}||^2) \end{aligned}$$

2. From the Chernoff-Savage theorem (cf. e.g. Puri and Sen (1971, Section 3.6)) we have that T_j, $j = 1, 2$, are asymptotically normal, and the expectations of T_j, $j = 1, 2$, are given by

$$\begin{aligned} \mathbf{E}T_j &= m\int_{-\infty}^{\infty}\phi_j\Big(\frac{m}{N}F(x) + \frac{n}{N}F\big(\frac{x-\theta_N}{e^{\vartheta_N}}\big)\Big)\,dF(x) \\ &\sim m\Big(\int_0^1\phi_j(u)\,du - \frac{n}{N}\int_{-\infty}^{\infty}\phi_j'(F(x))f(x)(\theta_N + \vartheta_N x)\,dF(x)\Big) \\ &= -\frac{mn}{N}\int_0^1\phi_j'(u)f(F^{-1}(u))(\theta_N + \vartheta_N F^{-1}(u))\,du \\ &= -\frac{mn}{N}\begin{cases}\theta_N d_L(f,g_1) + \vartheta_N d_{12}(f,g_1) & \text{if} \quad j=1 \\ \theta_N d_{21}(f,g_2) + \vartheta_N d_S(f,g_2) & \text{if} \quad j=2.\end{cases} \end{aligned}$$

The asymptotic variances are given in Proposition 1.

3. Therefore the asymptotic expectations Δ_j of T_j/σ_j are given by

$$\Delta_j = -\sqrt{\lambda(1-\lambda)}\begin{cases}\frac{\theta d_L(f,g_1)+\vartheta d_{12}(f,g_1)}{\sqrt{I_L(g_1)}} & \text{if} \quad j=1 \\ \frac{\theta d_{21}(f,g_2)+\vartheta d_S(f,g_2)}{\sqrt{I_S(g_2)}} & \text{if} \quad j=2.\end{cases}$$

4. Since the score functions are orthogonal the statistics T_1 and T_2 are, asymptotically, independent.

Therefore the asymptotic distribution of T is χ_2^2 and the noncentrality parameter is given by (3). ∎

Corollary 2 *The asymptotic power function of the Lepage type test is given by*

$$\beta(\theta,\vartheta) = 1 - F_{\chi_2^2,\Delta^2}(\chi^2_{2,1-\alpha}),$$

where F_{χ^2,Δ^2} is the cdf. of the noncentral χ^2 distribution with two degrees of freedom and noncentrality parameter Δ^2 which is given by (3).

Corollary 3 *If f is symmetric and if g_1 and g_2 are symmetric as in some examples above we obtain for the noncentrality parameter*

$$\Delta^2 = \lambda(1-\lambda)\Bigg(\theta^2 C_L^2(f,g_1) + \vartheta^2 C_S^2(f,g_2)\Bigg)$$

Corollary 4 *In the location problem ($\vartheta = 0$) we have*

$$\Delta^2 = \lambda(1-\lambda)\theta^2\Bigg(C_L^2(f,g_1) + C_{21}^2(f,g_2)\Bigg).$$

Corollary 5 *In the scale problem ($\theta = 0$) we have*

$$\Delta^2 = \lambda(1-\lambda)\vartheta^2\Bigg(C_S^2(f,g_2) + C_{12}^2(f,g_1)\Bigg).$$

For the examples presented above some values of the factors $C_L(f,g_1)$ and $C_S(f,g_2)$ are given in Table 1. The densities are uniform (abbreviated by Uni) normal (No), logistic (Lo), Cauchy (Cau), 'negative' Gumbel (nGu), Gumbel (Gu), Exponential (Exp), and two lognormals, $f_{lognormal}(x) = \frac{1}{x\tau\sqrt{2\pi}}exp(-\frac{\ln^2 x}{2\tau^2})$ with $x > 0$ and with parameters $\tau = 1$ and $\tau = 2$ (LN(1), LN(2)). Moreover, we considered four variants of the contaminated normal,

$$f_{CN}(x) = \frac{(1-\epsilon)}{\sigma_1}\phi(\frac{x-\mu_1}{\sigma_1}) + \frac{\epsilon}{\sigma_2}\phi(\frac{x-\mu_2}{\sigma_2})$$

with parameters $(\mu_1, \sigma_1, \mu_2, \sigma_2, \epsilon)$ = (0,1,0,3,0.9) (CN1), (1,2,-1,1,0.5) (CN2), (2,4,-1,1,0.5) (CN3) and (4,4,0,1,0.8) (CN4). Entries that are not given do not exist.

Values of the factors $C_{12}(f,g_1)$ and $C_{21}(f,g_2)$ are given in Table 2. The factor $C_{12}(f,g_1)$ can be considered as the effect of the scale alternative on the location test. On the other hand, the factor $C_{21}(f,g_2)$ can be considered as the effect of the location alternative on the scale test. Both effects may be positive or negative or zero. For symmetric densities and for tests designed for symmetric densities these values are zero. Again, for the uniform and for the exponential densities, some values do not exist.

Tables 1 and 2 here

In Figure 2 we present some univariate asymptotic power functions, $\gamma_{loc}(t)$ (location alternative, left column), $\gamma_{sca}(t)$ (scale alternative, right column) and $\gamma_{mix}(t)$

(alternative direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, middle column), where

$$\begin{aligned}
\gamma_{loc}(t) &:= \beta(\frac{t\sigma_F}{\sqrt{\lambda(1-\lambda)}}, 0) \\
\gamma_{sca}(t) &:= \beta(0, \frac{t\sigma_F}{\sqrt{\lambda(1-\lambda)}}) \\
\gamma_{mix}(t) &:= \beta(\frac{t\sigma_F}{\sqrt{2}\sqrt{\lambda(1-\lambda)}}, \frac{t\sigma_F}{\sqrt{2}\sqrt{\lambda(1-\lambda)}}).
\end{aligned}$$

For the mixed alternative, the factor $1/\sqrt{2}$ is introduced to have, for the direction vector, the same norm one as for the location and for scale alternative.

From the test examples above we choose the tests LPGA (continuous line, red), LPlog (long-dashed line, green), LPLT (dotted line, blue) and LP-SA (dash-dash-dot line, black). The first column presents the location alternative, the third column the scale alternative, and the second column presents the mixed alternative. Thirteen densities are considered, the normal, logistic, doubleexponential, Cauchy, Gumbel, uniform, exponential (the latter two for the tests LPGA, LPLT and LPHFR), the two lognormals and the four contaminated normals. They represent symmetric densities with short, medium, long and very long tails as well as skew densities. The factor t in the formula for the asymptotic power function is multiplied by the standard deviation σ_F of the underlying density if it exists (for the Cauchy we set $\sigma_F = \sigma_{Cau} = F^{-1}(\Phi(1)) = 1.8373$). This way we have similar power values for the various densities.

Figure 2 here

Especially we see that the test LPGA is the best for the uniform, for the exponential and for the normal (the latter together with the test LPlog). The test LPlog is the best for the normal, logistic, Gumbel, and the contaminated normals CN1-CN4. The test LPLT is the best for the long-tail densities DE and Cauchy, and the test LP-SA for the extremely skew lognormal densities.

Remark 1 *Note that the original Lepage test also was considered but there is almost no density for which it is among the best. Nearly the best test it is only for the Cauchy-density and for CN4 (for some alternative directions, scale or mixed alternatives (Cauchy), location and mixed alternatives (CN4)).*

This fact, however, is not surprising, because the components of the Lepage test are very different, the Wilcoxon test is a location test for moderately tailed *densities, whereas the Ansari-Bradley test is a scale test for heavily tailed densities.*

Remark 2 *For moderately asymmetric densities a combination of linear rank tests designed for symmetric densities is better than that designed for skew densities. An explanation for this fact may be that, for skew densities, changes in locations may result in changes of scales too, and vice versa.*

4 An adaptive test

One concept of adaptive tests is proposed by Hogg (1974). It is based on the independence of rank and order statistics (cf. Randles and Wolfe, 1979, p.388). The density is classified by order statistics, then a rank test is applied. It is quite common to classify the underlying distribution with respect to measures of tailweight and skewness.

There exist many measures of integral type or of quantile type (cf. e.g. Büning, 1991, Handl 1986, Hogg and Lenth, 1984). We choose the measures

$$\begin{aligned} t_{0.05,0.15}(F) &= \frac{F^{-1}(0.95) - F^{-1}(0.05)}{F^{-1}(0.85) - F^{-1}(0.15)} \\ s_{0.05}(F) &= \frac{F^{-1}(0.95) + F^{-1}(0.05) - 2F^{-1}(0.5)}{F^{-1}(0.95) - F^{-1}(0.05)} \end{aligned}$$

for tailweight and skewness, respectively. These measures are introduced by Groeneveld and Meeden (1984). Some examples are given in Table 3. The table shows that these measures are in accordance with our idea of tailweight and skewness.

Table 3 here

Replacing the quantile function $F^{-1}(.)$ by an estimate $\hat{Q}(.)$ we obtain estimates $\hat{t}$ and $\hat{s}$ of tailweight and skewness. To estimate the quantiles we use the "classical" estimate

$$\hat{Q}(u) = \begin{cases} X_{(1)} - (1-\epsilon)(X_{(2)} - X_{(1)}) & \text{if} \quad u < 1/(2 \cdot L) \\ (1-\epsilon) \cdot X_{(j)} + \epsilon \cdot X_{(j+1)} & \text{if} \quad 1/(2 \cdot L) \leq u \leq (2 \cdot L - 1)(2 \cdot L) \\ X_{(L)} + \epsilon(X_{(L)} - X_{(L-1)}) & \text{if} \quad u > (2 \cdot L - 1)/(2 \cdot L), \end{cases}$$

where $\epsilon = L \cdot u + 1/2 - j$, $j = \lfloor L \cdot u + 1/2 \rfloor$, and $X_{(i)}$ is the i-th order statistic of a sample of size L.

The results of the last section motivate such an adaptive test, where the LPGA-test is applied for short tails, the LPlog-test for medium tails, and the LPLT-test for densities with longer tails. More precisely, let

$$\hat{t} \;=\; \frac{\hat{Q}(0.95) - \hat{Q}(0.05)}{\hat{Q}(0.85) - \hat{Q}(0.15)}$$

be the the selector statistic. Then the Adaptive test $LPA(\hat{t})$ is defined by

$$LPA(\hat{t}) = \begin{cases} LPGA & \text{if} \quad \hat{t} \le 1.55 \\ LPlog & \text{if} \quad 1.55 < \hat{t} \le 1.8 \\ LPLT & \text{if} \quad \hat{t} > 1.8. \end{cases}$$

Remark 3 *We do not include a linear rank test designed for skew densities (e.g. the Savage scores test) in our adaptive test since only for extremely skew densities (as the lognormal) such a test is (asymptotically) better than the tests LPGA or LPlog. Moreover, for very skew densities and for finite sample sizes we have to expect large misclassification probabilities.*

However, for extremely skew densities the tests LP-SA or LPSA can be applied.

Remark 4 *For other adaptive tests based on linear rank tests we refer to Beier and Büning (1997), Büning and Kössler (1998), Hill, Padmanabhan and Puri (1988), Sun (1997) and Büning and Thadewald (2000).*

5 Simulation study

To find out whether the asymptotic theory can be applied for moderate or small sample sizes we performed a simulation study (10,000 replications). We chose the following six distributions: uniform (short tails), normal (medium tails), double-exponential (longer tails), Cauchy (very long tails), Gumbel (skew) and the exponential (very skew).

The powers of the following Lepage-type tests are compared: LPGA, LPlog, LPLT and the Adaptive test $LPA(\hat{t})$. For all the tests the asymptotic critical values are used. We restrict to the balanced cases with sample sizes $n_i = 25$, $n_i = 50$ and $n_i = 100$, $i = 1, 2$.

The alternatives are given by the parameters $(\theta_N, \vartheta_N) = N^{-1/2}(\theta \cdot 2\sigma_F, \vartheta \cdot 2\sigma_F)$, where we use the following (θ, ϑ)-combinations: $(t, 0), (0, t), (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})$, each variant with $t = 0, 1, 2, 3, 4$. The factor 2 stands for $(\sqrt{\lambda(1-\lambda)})^{-1}$ and the value σ_F is, as above, the standard deviation of the underlying distribution. if it exists. (Recall that we set $\sigma_F = 1.8373$ for the Cauchy.)

Figure 3 gives an impression of the finite and asymptotic power values. For each density that Lepage type test is chosen that is asymptotically selected by the Adaptive test. The dot-dashed (red) line is for N=50, the dashed (green) line for N=100, the dotted (blue) line for N=200, and the continuous (black) line for the asymptotic power. Again, the left, right and middle columns stay for the location, scale and mixed alternative, respectively. Tables of finite power values can be obtained from the author on request.

Figure 3 here

We summarize the results as follows:

1. For all tests considered the estimated power approaches the asymptotic power with increasing sample size. The convergence is relatively fast for location alternatives (except for the uniform and for the exponential) but considerably slower for scale and mixed alternatives. The result for the exponential is similar as in other studies (cf. Büning and Kössler, 1999).

2. For $N \geq 100$ the Lepage-type tests essentially maintain the level of significance. For N=50 they are sometimes slightly conservative. This fact holds especially for the Gastwirth scores and for the Wilcoxon scores.

3. For the considered tests we have: for relatively small up to large sample sizes the LPGA-test is the best for the uniform and for the exponential (location alternative, mixed alternative)

The LPLT-test is the best for the Cauchy and for the double exponential (location and mixed alternatives, the latter together with the test LPlog).

The LPlog-test is the best for the normal, Gumbel, exponential (scale alternative), and for the DE (scale and mixed alternatives, the latter together with the LPLT-test).

4. Comparing all the tests LPGA, LPlog, LPLT and $LPA(\hat{t})$ for moderate up to large sample sizes (N $\geq$ 100) we see that the Adaptive test $LPA(\hat{t})$ is, except for the exponential, always at least the second best. This fact is not surprising since the Adaptive test $LPA(\hat{t})$ is constructed in such a way that it is worse than the best test (for a given density, among the tests considered) but better than most of the other tests.

For relatively small sample sizes (N=50) we have the same tendency. Because of the higher misclassification probabilities the Adaptive test is somewhat worse than in the large sample case.

5. As a universal single test only the test LPlog can be considered. However, it is much worse than the Adaptive test for extreme distributions such as the uniform or the Cauchy.

6 Conclusions

In this paper we considered the location-scale alternative in the two-sample problem. We studied the Lepage-type tests indroduced by Büning and Thadewald (2000) and computed their asymptotic power functions. For various tests and densities we obtained efficacy values. These values are used for the design of an adaptive test.

Of course, there is no test which is the best for all densities, but on the whole the adaptive test proposed is the best one for this braod class of alternatives.

We also included tests designed for asymmetric densities, the Savage scores test, and the HFR-scores test. However, it turns out that tests designed for symmetric densies (LPGA and LPlog) are better also for skew densities. This findings are in accordance with that of Büning and Thadewald (2000). Only for extremely skew densities, such as for the lognormals, the Savage scores tests (LPSA and LP-SA, for left and right-skew densties, respectively) are better. For the other skew densities the tests LPGA (for short tails) and LPlog are better. This result is also in accordance to that of Büning and Thadewald (2000), who considered the Gastwirth and normal scores tests (there called LP1 and LP2). The normal scores test behaves similar as the test LPlog, the former (latter) is better for densities with slightly lighter (heavier) tails.

As an universe test the test LPlog can be recommended. The original Lepage test behaves worse.

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Table 1: The factors $C_L(f, g_1)$ (left columns) and $C_S(f, g_2)$ (right columns) for some Lepage type tests and for some densities

density	Gastwirth		normal scores		logistic scores		long-tail		Cauchy	
	Loc	Sca	VW	Klotz	WI	Sca	LT	Mood	Loc	Sca
Uni	4.90	2.32			3.46		2.45	2.24	0.00	1.41
No	.940	1.39	1.00	1.41	.977	1.40	.912	1.23	.656	1.05
Lo	.510	1.18	.564	1.18	.577	1.19	.561	1.18	.450	.988
CN1	.308	1.15	.352	1.33	.352	1.31	.344	1.14	.367	.967
CN5	.782	1.12	.836	1.04	.873	1.07	.844	1.11	.673	1.00
DE	.612	.975	.798	.988	.866	.998	.919	.933	.900	.834
Cau	.283	.628	.464	.526	.551	.584	.638	.680	.707	.707
nGu	.856	1.13	.903	1.29	.866	1.28	.796	1.11	.549	.938
Gu	.856	1.13	.903	1.29	.866	1.28	.796	1.11	.549	.938
Exp	2.45	.378		.421	1.73	.418	1.22	.373	.000	.319
CN2	.551	1.33	.587	1.48	.574	1.47	.536	1.31	.388	1.11
CN3	.360	1.17	.413	1.23	.434	1.24	.434	1.17	.381	1.04
CN4	.548	.881	.645	.996	.702	.989	.719	.911	.657	.831
LN(1)	1.60	.000	1.65	.000	1.25	.000	.974	.000	.282	.000
LN(2)	2.58	.000	3.69	.000	1.33	.000	.589	.000	-1.03	.000

density	Lepage		HFR		antis. HFR		Savage		antis. Savage	
	WI	AB	Loc	Sca	Loc	Sca	Loc	Sca	Loc	Sca
Uni	3.46	1.00	3.10	2.32	3.10	4.24				
No	.977	1.10	.874	.789	.874	.789	.903	1.03	.903	1.03
Lo	.577	1.02	.516	.732	.516	.732	.500	.866	.500	.866
CN1	.352	1.02	.315	.732	.315	.732	.315	.970	.315	.970
CN5	.873	.983	.781	.732	.781	.732	.731	.772	.731	.772
DE	.866	.866	.775	.620	.775	.620	.693	.724	.693	.724
Cau	.551	.702	.493	.502	.493	.502	.388	.391	.388	.391
nGu	.866	.988	.625	.623	.924	.791	1.00	1.20	.645	.683
Gu	.866	.988	.924	.791	.625	.623	.645	.683	1.00	1.20
Exp	1.73	.335	2.32	.549	.775	-.07	1.00	.000		.615
CN2	.574	1.17	.628	.752	.396	.928	.448	1.20	.613	.961
CN3	.434	1.08	.526	.691	.250	.854	.291	1.14	.431	.656
CN4	.702	.863	.748	.637	.508	.598	.460	.664	.685	.736
LN(1)	1.25	.000	1.71	.350	.538	-.350	.684	-.390	2.62	.392
LN(2)	1.33	.000	2.19	.175	.187	-.174	.526	-.196	8.88	.196

Table 2: The factors $C_{12}(f,g)$ (left columns) and $C_{21}(f,g)$ (right columns) for some Lepage type tests and for some densities

density	Gastwirth		normal scores		logistic scores		long-tail		Cauchy	
	C_{12}	C_{21}	C_{12}	C_{21}	C_{12}	C_{21}	C_{12}	C_{21}	C_{12}	C_{21}
nGu	-.121	.378	-.117	.596	-.234	.418	-.270	.372	-.303	.319
Gu	.121	-.378	.117	-.596	.234	-.418	.270	-.372	.303	-.319
Exp	.856	-2.32			.866		.797	-2.24	.549	-1.41
CN2	-.279	-.236	-.247	-.191	-.246	-.214	-.209	-.252	-.112	-.260
CN3	-.187	-.236	-.166	-.160	-.228	-.195	-.229	-.022	-.207	-.323
CN4	-.107	-.329	.008	.000	.055	-.286	.131	-.308	.261	-.265
LN(1)	.940	-1.59	1.00	-2.33	.977	-2.14	.912	-1.54	.656	-1.19
LN(2)	.471	-3.18	.500	-10.5	.489	-8.13	.456	-3.01	.325	-1.75
density	Lepage		HFR		antis. HFR		Savage		antis. Savage	
	C_{12}	C_{21}	C_{12}	C_{21}	C_{12}	C_{21}	C_{12}	C_{21}	C_{12}	C_{21}
Uni	.000	.000	.775	1.55	-.775	-1.55				
No	.000	.000	-.493	.350	.493	-.350	.596	-.390	-.596	.390
Lo	.000	.000	-.458	.207	.458	-.207	.500	-.289	-.500	.289
CN1	.000	.000	-.458	.126	.457	-.126	.560	-.159	-.560	.159
CN5	.000	.000	-.458	.312	.458	-.312	.421	-.474	-.446	.481
DE	.000	.000	-.387	.310	.387	-.310	.418	-.531	-.418	.531
Cau	.000	.000	-.314	.197	.314	-.197	.226	-.431	-.226	.431
nGu	-.234	.334	-.651	.549	.232	-.070	.423	.000	-.665	.615
Gu	.234	-.334	-.232	.070	.651	-.549	.665	-.615	-.423	.000
Exp	.866	-1.73	.625	-.620	.924	-1.86	1.00	-1.73	.645	
CN2	-.246	-.257	-.745	.021	.304	-.389	.406	-.371	-.839	.086
CN3	-.228	-.308	-.686	-.065	.279	-.376	.394	-.360	-.645	.121
CN4	.055	-.269	-.334	.059	.435	-.444	.405	-.602	-.390	.159
LN(1)	.977	-1.31	.874	-.486	.874	-1.38	.903	-1.32	.902	-2.03
LN(2)	.488	-4.48	.437	-1.13	.437	-2.08	.452	-1.40	.450	-12.7

Table 3: Measures for tailweight $t_{0.05,0.15}(F)$ and skewness $s_{0.05}(F)$ for some distributions.

Symmetric distributions		Skew distributions		
Density	Tailweight	Density	Tailweight	Skewness
Uniform	1.286	Exponential	1.697	0.564
U-L	1.474	Gumbel	1.655	0.280
Normal	1.587	negGumbel	1.655	-0.280
Logistic	1.697	L-E	1.799	0.349
CN1	1.697	CN2	1.592	0.277
L-D	1.864	CN3	1.707	0.439
DoubleExp	1.912	CN4	2.714	0.542
Cauchy	3.217	LN1	2.024	0.676
		LN2	3.426	0.928

Figure Captions

Figure 1: The score functions for the various Lepage type tests.

Figure 2: The asymptotic power functions for various Lepage type tests and various densities.

Figure 3: The asymptotic and finite power functions for three alternative configurations and various densities.

Figure 1: The score functions for the various Lepage type tests

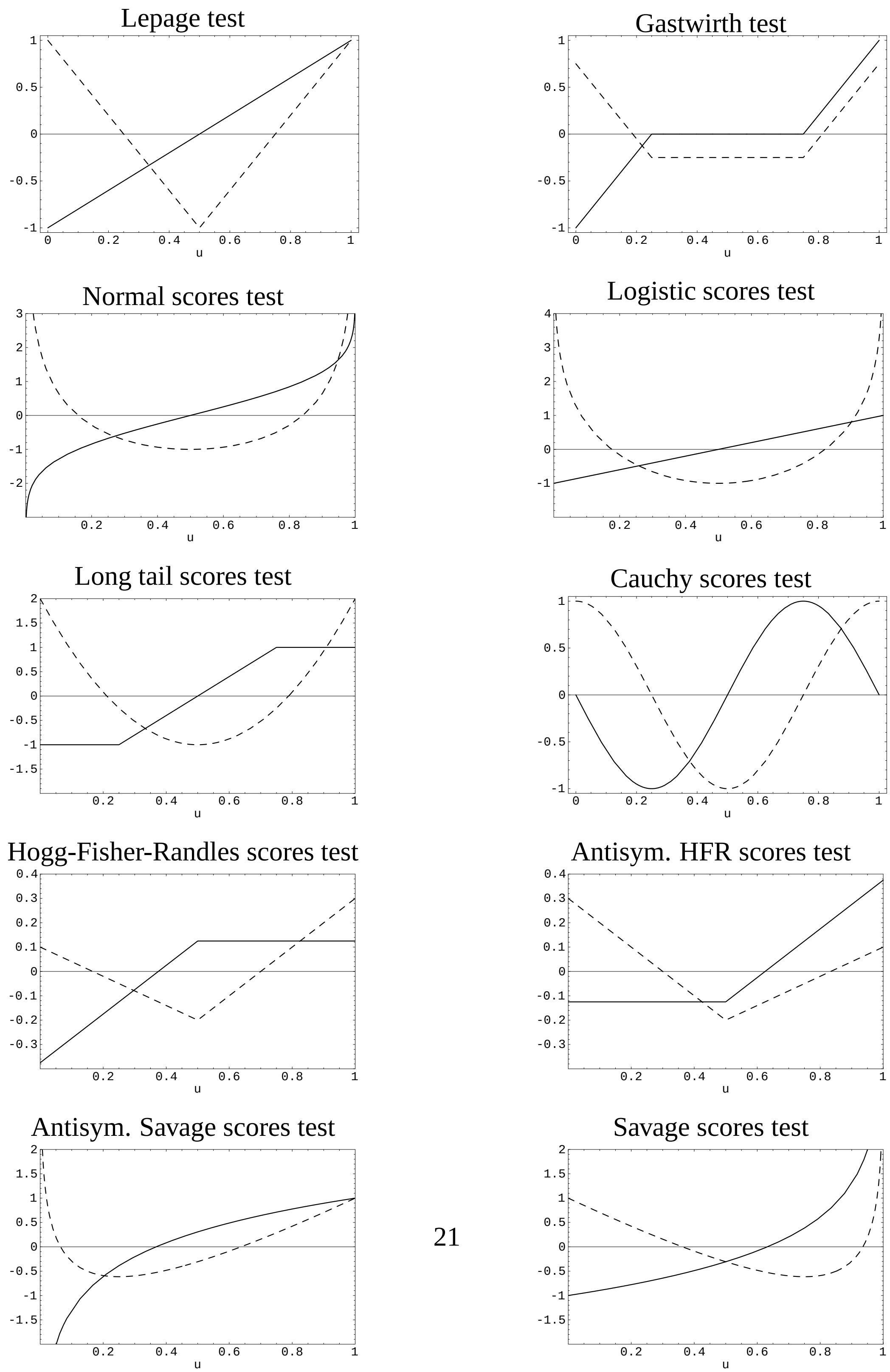
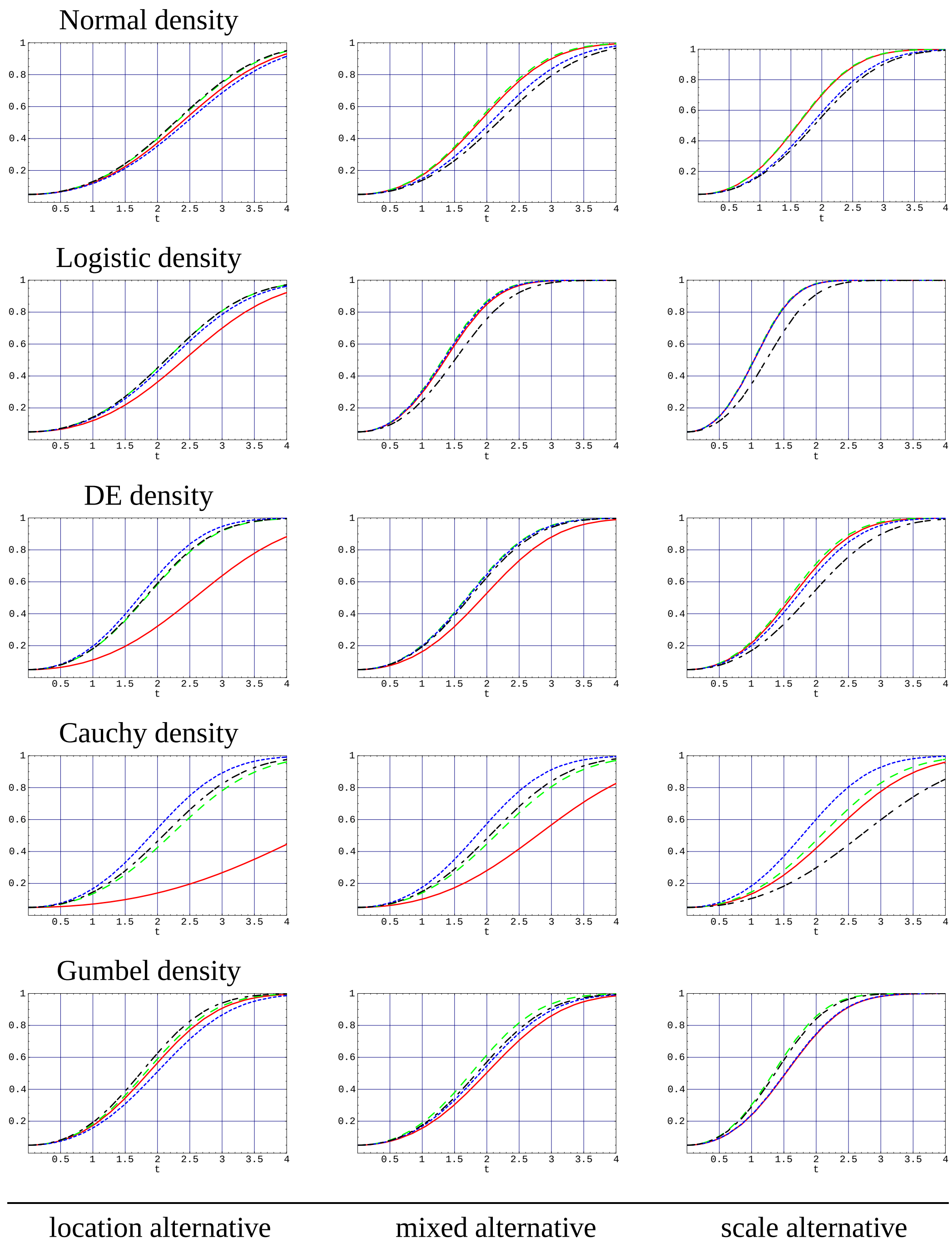
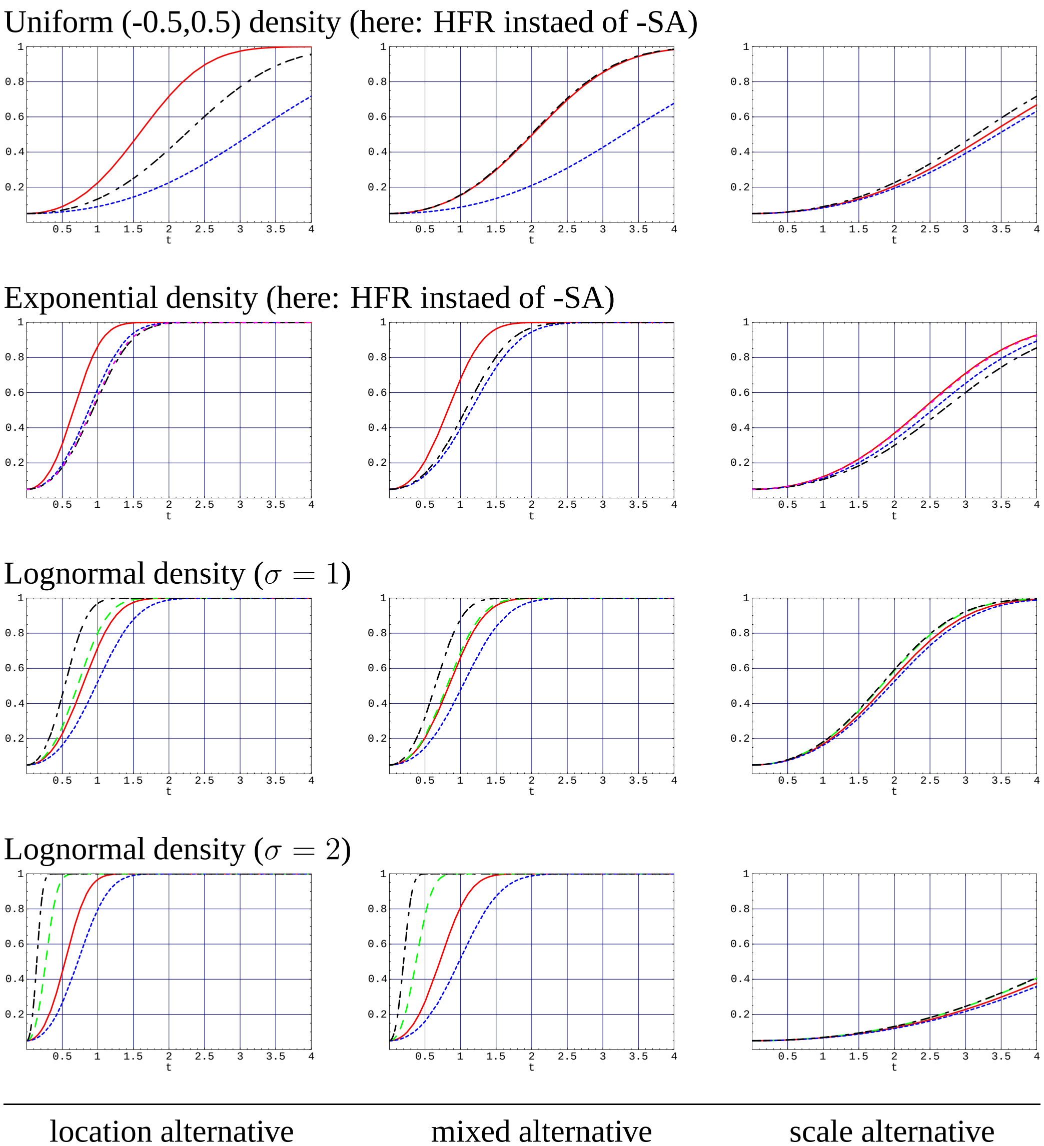


Figure 2: The asymptotic power functions for various Lepage type tests and various densities





Uniform (-0.5,0.5) density (here: HFR instaed of -SA)
Exponential density (here: HFR instaed of -SA)
Lognormal density ($\sigma = 1$)
Lognormal density ($\sigma = 2$)
0.2
0.4
0.6
0.8
1
0.5
1
1.5
2
2.5
3
3.5
4
t
location alternative
mixed alternative
scale alternative

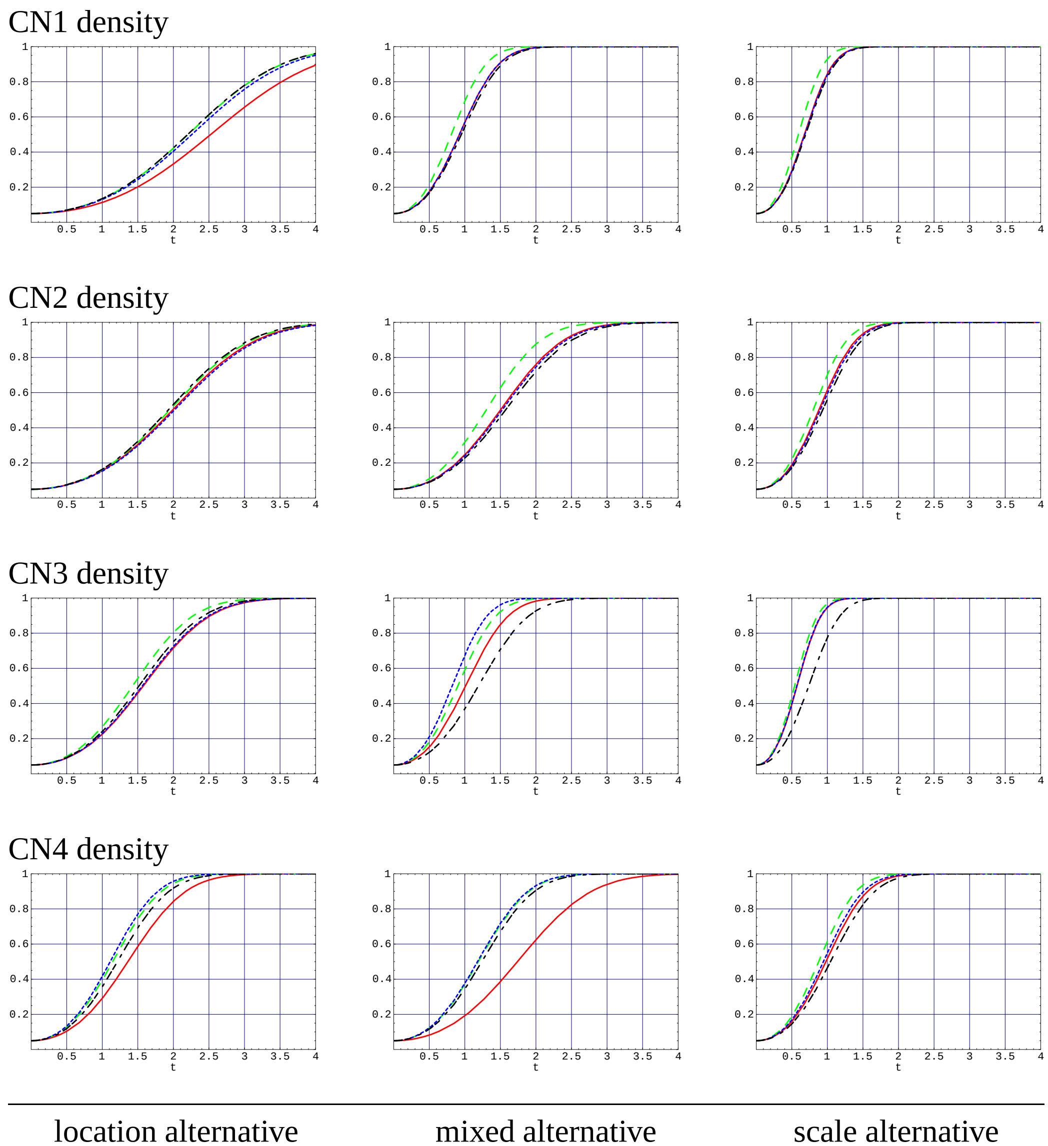
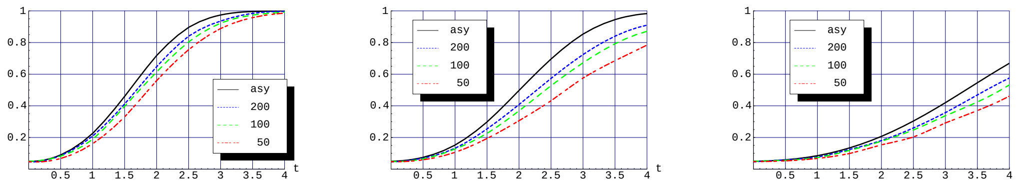
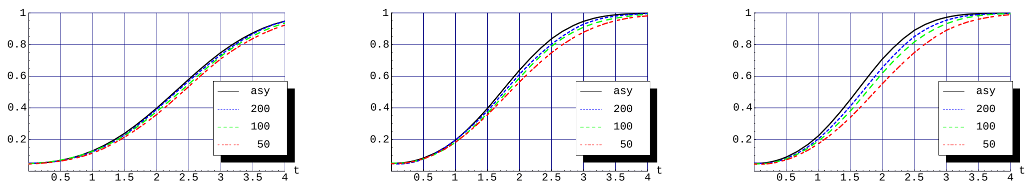

CN1 density
CN2 density
CN3 density
CN4 density
location alternative
mixed alternative
scale alternative

Figure 3: The asymptotic and finite power functions for three alternative configurations and various densities

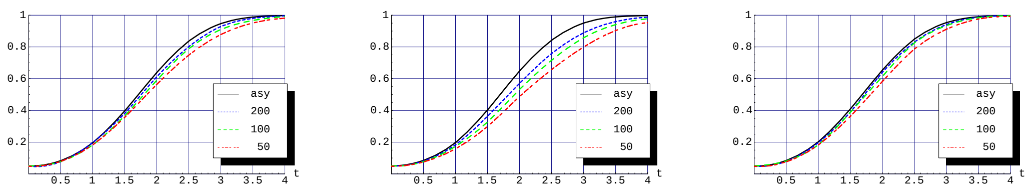
Uniform (-0.5,0.5) density (selected test LPGA)



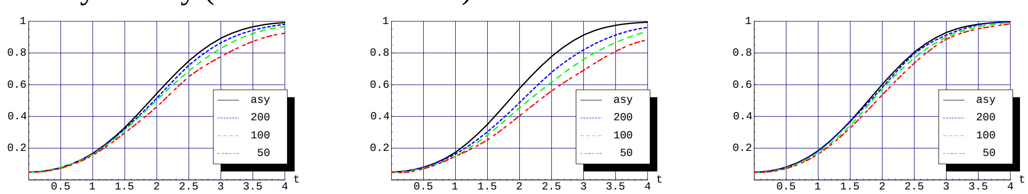
Normal density (selected test: LPlog)



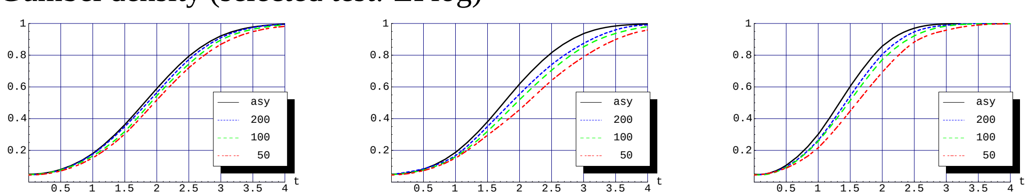
DE density (selected test: LPLT)



Cauchy density (selected test: LPLT)

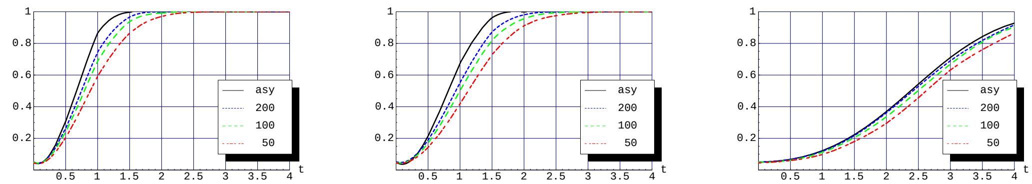


Gumbel density (selected test: LPlog)



location alternative | mixed alternative | scale alternative

Exponential density (selected test LPGA)



location alternative | mixed alternative | scale alternative