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Securitization of Financial Assets: Approximation in Theory and Practice

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Abstract. Asset-Backed Securitization (ABS) is an emerging sector of today banks' business. It represents an effective tool to turn unrated assets, such as commercial papers or lease contracts, into marketable financial products through the issuance of special notes, namely the asset-backed securities.

In this paper we analyze the problem of optimally selecting the assets to be converted into notes with respect to scenarios motivated by real-world problems. In particular, we study the case in which the assets amortization rule is characterized by constant periodic principal installments instead of the more classical amortization rule based on constant general (principal plus interests) installments. We show the computational advantages and the practical implications of this choice. The particular shape of the outstanding principal for the case of constant principal installments is exploited in the solution of a general model which selects assets at different dates.

Four approximation algorithms, based on LP-relaxation and on the implicit knapsack structure of the problem, are proposed for this general model. From a theoretical point of view we analyze the exact worst-case behavior of these algorithms compared to the optimal solution. Computational experiments are performed for a practical scenario suggested by a leasing bank. The results show that the proposed approximation algorithms are, on average, highly efficient and effective.

Keywords: asset-backed securitization, approximation algorithm, knapsack problem

1. Introducing ABS and amortization variants

Asset-Backed Securitization (ABS) is a financial tool which allows financial institutions (usually commercial banks) to move unmarketable assets (e.g. *lease assets*, *mortgage assets* or *commercial papers*) from their balance sheets in exchange for a long term loan which can be ploughed back into more profitable investments.

More precisely, the financial assets are converted into bonds (so called notes) and the proceeds of their market issuance become a long term loan for the assets owner (*the originator*). We will look at the ABS operation mainly from the point of view of this financial institution.

Our analysis will concentrate on the critical phase of the ABS operation avoiding to describe in detail the role of some of the participating operators, such as banks and insurance companies, which provide the credit protection (risk hedging) of the operation. It should be noted that the issue of credit protection is an interesting research topic in itself. However, the corresponding features such as credit guarantees and cash flow riskiness are beyond the scope of this paper.

In an ABS, the assets are sold by the originator to a *special purpose vehicle* (SPV), an institution created solely for that purpose. The SPV funds the purchase through issuing debt securities—the notes—which are collateralized by the assets. Note that the assets transfer is a true sale. Thus, if the originator becomes insolvent or is involved in bankruptcy the transferred financial assets will not be part of the bankruptcy assets. This makes the notes an interesting investment opportunity. In a *pass through* payment scheme the final investors who buy these notes receive periodic inflows (interests on their investments). These are directly related to the periodic installments paid by the holders of the assets (e.g. lessees or mortgage holders) to the originator (e.g. the lessor). Using the ABS structure the originator bypasses the problem of an impossible outright sale of its assets and thus reduces its overall exposure to them. For instance, lease or mortgage contracts which tie up the capital of leasing companies can be moved into notes. This replacement of illiquid assets improves the return on equity (ROE).

From the point of view of the originator, an ABS allows the achievement of three main financial objectives:

1. Replacement of the assets in the balance sheet, thereby improving ROE and allowing (if the originator is a bank) a more flexible keeping of the asset/liability composition constraints imposed by the control authorities (i.e. the Central Bank).
2. Diversification of fund sources. Although the originator may be low rated, its notes usually get a higher rating (e.g. AAA) due to the presence of banks and insurance companies which guarantee the whole operation. This implies that such notes can be dealt on the main financial markets allowing the originator to reach markets which would otherwise be unaccessible for him since attended only by more established companies.
3. Higher rated notes are more reliable investments and thus are allowed to pay lower interest rates to holders. If the cost to get a higher rating is lower than the saving obtained by issuing notes with higher rating, then the global cost to acquire funds decreases. Let us assume that an institution with a BB rating can get money at a rate such as Libor (London interbank offering rate) plus 150 basis points. Such an institution, as originator, may decide to pay an additional 100 basis points to get credit warranties[1] and be able to issue notes with rating AAA at the cost of Libor plus 10 basis points. In this case an ABS will produce a saving on interest rates of 40 basis points. This situation applies in practice, since there is no efficient market for the underlying assets.

The interest in this financial operation drastically increased in the last years all over Europe. In Italy, one of the most recent and relevant ABS has been performed by the public institution in charge of the management of the social security system, i.e. the Istituto Nazionale della Previdenza Sociale (INPS). This operation has allowed INPS to move delinquent contributions from its balance sheet. Other transactions of this type took place in the area of public housing agencies.

Many papers dealing with ABS from a modelling point of view have appeared in the last few years. Since an extensive review is beyond the scope of this paper we will only mention the papers by Kang and Zenios [6, 7] and by Mansini and Speranza [12, 13] and refer to the references given therein. For a better insight in the complex problem of securitization we suggest the textbooks [3, 5, 15].

In particular, motivated by the analysis of a real-world case, Mansini in [11] and then Mansini and Speranza in [12] have studied the problem of optimally selecting the assets to refund the loan. In their case only *lease assets* are considered, although many other types of assets have the same basic characteristics. In their paper the outstanding principal of the assets is computed based on constant general installments (the so called *French amortization*). The resulting problem of selecting assets *at a unique date* can be modelled as a d*-dimensional knapsack problem*, which is hardly tractable by exact algorithms but is typically solved by constructive heuristics (see e.g. [1, 16]) or metaheuristics (see e.g. [2, 4]). The authors also show that in the special case where all lease assets share the same financial characteristics (amortization rule, internal interest rate and term) all but one constraint turn out to be redundant and hence the model reduces to a classical 0–1 *knapsack problem* (KP), which is relatively easy to handle (cf. [8, 9, 14]). See [10] for a general introduction to knapsack problems. Their work does not take into account the occurrence of a different rule for the asset amortization. In many practical applications (both for lease and mortgage contracts) the customers receiving the assets choose to pay back their debt by constant periodic principal installments (the rule is known as *Italian amortization*). Up to now this common rule has been totally ignored in models formalization.

The objective of this paper is twofold. First of all we innovate with respect to previous modeling approaches by introducing a general model to select financial assets *at multiple dates*. The motivation derives from the practical need of finding alternative and possibly more effective formulations for the problem of asset selection in ABS to achieve a better utilization of the long term loan. Secondly, we analyze the frequently encountered practical case in which the assets (lease or mortgage contracts) are paid back by constant periodic principal installments (*Italian amortization* rule). In this way the paper aims to provide the analysis of an alternative amortization rule available in practice as well as the development of better tools for the institutions responsible for the planning and management of ABSs.

Before defining the new model we should give a more detailed sketch of the ABS process. To help the reader in visualizing and better understanding the structure of an ABS process the following description is summarized in figure 1. The SPV issues notes on the financial market receiving funds from institutional investors who purchase the notes and hold them until maturity subject to the availability of acceptable short-term financing. The proceeds obtained by the notes' issuance are used by the SPV to make revolving purchases of the unrated assets from the originator. The latter receives a long term loan which is payable solely by assets. In particular, the originator has to select the assets to be handed over for the loan reimbursement. These assets are "converted into" the notes issued by the SPV.

The assets which are included in an ABS process have to be selected in a way such that the sum of their outstanding principals never exceeds the outstanding principal of the received loan (from now on simply *the main outstanding principal*) at any point in time. Now in order to maximize the financial gain of the operation the critical problem for the originator consists of minimizing the gap between the main outstanding principal and the

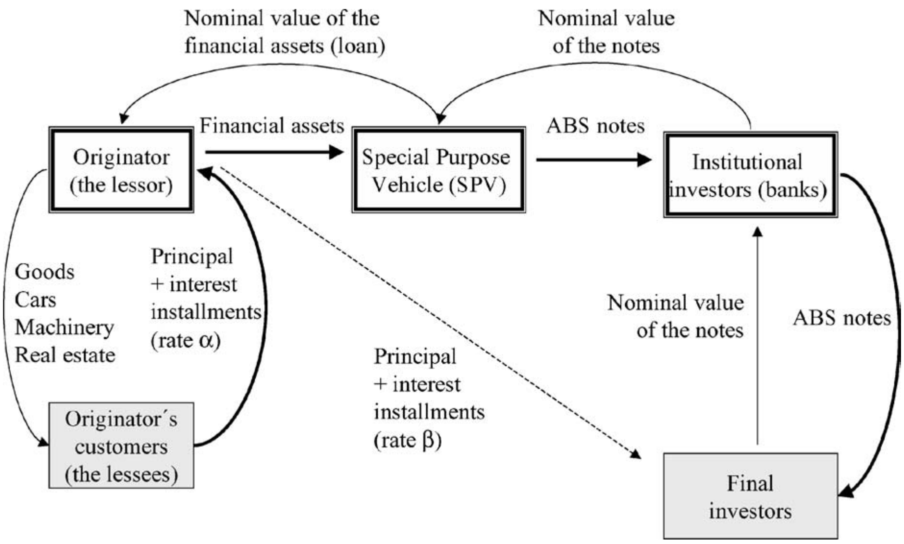

Figure 1. Asset-backed securitization process.

outstanding principal of the selected assets over all points in time. This gap constitutes a loss of profit due to missing more profitable investments with higher yields.

Actually, the area of the main outstanding principal covered by the sum of outstanding principals of the handed-over assets yields a return for the originator (e.g. the lessor) depending on the difference between the percent interest rate per year (say α) that the originator got from its customers (e.g. the lessees) and the lower percent interest rate (say β) paid to the note holders. If the sum of the outstanding principals of the selected assets has a global reimbursement profile which decreases more rapidly than that of the main outstanding principal, then the originator gets funds from its customers in advance with respect to the deadline at which it should pay the capital installment to the SPV. Such funds have to be reinvested in some predefined type of investments indicated in the ABS agreement. These investments last for a brief period (from the date in which they are available to the following date of reimbursement for the main loan) and usually yield a very low interest rate (say $\gamma > 0$). Given the rate β payed for the notes it frequently happens that $\gamma - \beta$ is close to zero and may also be negative involving a loss for the originator. This justifies the interest in minimizing the gap between the two profiles and stresses the importance of studying alternative shapes for the outstanding principals which allow the sum of the outstanding principals of the assets to better fit the main outstanding principal.

Another important aspect in an ABS process is the risk of assets prepayment (cf. Schwartz and Torous [18]). A decline in interest rates may cause an earlier repayment of the outstanding principal to the originator by some of the lessees. Clearly, such a prepayment decreases the sum of the outstanding principals of the assets and hence has a negative effect on the value of the objective function over time since the gap towards the main outstanding principal increases.

For some types of assets such as auto loans or credit card receivables this prepayment is unusual. However, leasing-like assets do face the risk of interest-rate based prepayment. Since prepayment events are non-predictable they cannot be taken explicitly into account in a deterministic off-line optimization model. Implicitly, it is assumed that all assets have the same probability of prepayment. In all cases where the risk of early paybacks is particularly high, a re-optimization of the whole ABS process at a later point in time is strongly recommended.

Concerning the time line, in our case the assets are handed over by the originator and purchased by the SPV starting at a *closing date* (initial date for the loan) and on a fixed basis thereafter during the so called *revolving period*. Each date at which a purchase takes place is called *settlement date*. The assets handed over by the originator at the closing date and thereafter at the settlement dates are collectively referred to as the *initial* and *subsequent portfolios*, respectively. Issued notes yield an interest payable on periodic bases (usually quarterly) and are redeemed at different final maturity dates. For this reason, notes are divided into tranches characterized by different deadlines.

The reimbursement to the holders of the principals of a tranche of notes corresponds to a reimbursement installment of the main outstanding principal. Hence, the outline of the outstanding principal of the loan has as many installments (steps) as the number of tranches of notes with different maturity issued on the market.

The main source of payment of interest and principal on notes are recoveries arising out of the assets. In particular, the cash-flow deriving from assets is used by the SPV to satisfy its obligations to the holders of notes.

Naturally, the outstanding principal of an asset depends on the rule used for amortization. As mentioned above, two different rules mainly appear in practice. In the first rule, usually known as *French amortization*, the general periodic installment (sum of periodic interests and principal installment) is constant over time. In this case the customers who hold assets (mortgage or lease contracts) have to pay the same constant amount at each deadline. Since the principal installments increase geometrically over time (see figure 2(b)), the outstanding principal can be approximated by a concave piece-wise linear function.

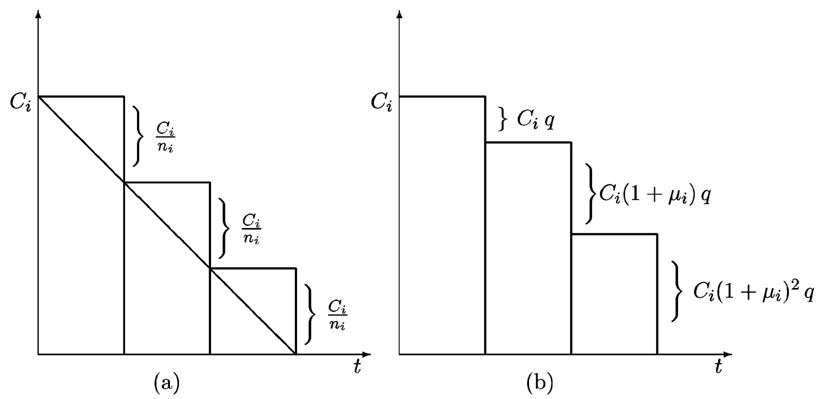

Figure 2. Italian versus French amortization: Outstanding principal of an asset with $n_i = 3$. Define $q := \frac{\mu_i}{(1+\mu_i)^{n_i}-1}$.

The second rule, known in practice as *Italian amortization*, establishes that not the general but only the periodic principal installments are constant over time. Commercial customers who, as a practice, tend to divide the cost of a lease or of a mortgage over the periods constituting the reimbursement term, typically appreciate this rule as an equitable way of allocating the original cost among periods. In this case, the outstanding principal of each asset has a very simple step-wise decreasing shape which is independent from the internal rate of amortization. For each asset i let us define as C_i the initial value of the outstanding principal (i.e. the purchasing price of a property such as machinery, vehicle or real estate) and as n_i the number of installments. Figure 2 illustrates the outlines of the outstanding principals for the same asset i computed using Italian amortization (part (a)) and using French amortization (part (b)). The internal rate μ_i is used to compute the outstanding principal in the French amortization case. Note that, for the case of Italian amortization, all the steps have the same size so that the points $(k, (n_i - k)\frac{C_i}{n_i})$ for $k = 0, \ldots, n_i$ lie on the same line. Hence, the outstanding principal can be approximated by a linear function.

In this paper we will focus on *Italian* instead of *French amortization*. The choice is consistent with our aim to give relevance to those practical aspects which, up to now, have been ignored (i.e. the use of possible alternative rules of amortization in ABS processes). Moreover, this choice has been further confirmed by the recent interest shown by institutions in charge with ABSs in the evaluation of the possible economic advantages of applying a different amortization rule. In particular, we propose a general model for selecting financial assets all over the ABS duration and not only at a single date as in [12], provided that all assets are available at the initial date of the securitization. Although such model can be used for any type of amortization rule chosen by the originator, we will analyze its solution for the case of Italian amortization exploiting the nice features this amortization variant provides.

For this problem four heuristic algorithms will be proposed in Section 3, two based on the solution of the LP-relaxation of the model with greedy extension, one exploiting the multi-knapsack structure of the model and finally a heuristic combining these two aspects.

Two of the heuristics are shown to be $1/2$-approximation algorithms. Although, reaching at least one half of the optimal value will not strike practitioners as very spectacular, one should keep in mind that this is the result of a worst–case analysis and the first theoretical certificate of any algorithm in this application area.

Extensive computational results are reported in Section 4. Based on the cooperation with a commercial bank specialized in the leasing business a model for randomly generated instances was developed to simulate typical sets of assets. For these instances the performance of the four heuristic algorithms in comparison to the optimal solution value computed by a standard optimization software package was evaluated. It turned out that the deviation from optimality of all four heuristics as well as their running times are quite small.

2. Formal problem definition

Formalizing the selection problem of the originator at every settlement date the objective of the problem is to reduce the "distance between the outstanding principal of the loan and that of the portfolio of assets" as much as possible. Assets prepayment is not explicitly included in the model.

We assume that all assets are available at the beginning of the ABS. This assumption is consistent with the originator's decision to allocate assets among different securitizations before their starting date.

As time unit we take the periodicity of the reimbursement of assets which is assumed to be the same for all the assets. In practice this unit is equal to one month. The reimbursement dates for the main loan are assumed to happen at multiples r of this time unit (usually with a frequency not lower than every 12 or 18 months). Therefore, we introduce a discretized time horizon $0, 1, 2, \ldots$ such that the dates of the principal reimbursement for each asset and for the main loan as well as the settlement dates are points in this sequence. Let $D := \{t_0 = 0, t_1, \ldots, t_T\}$ be the set of dates for the assets selection (the set of settlement dates) and let its index set be referred to by $\mathcal{T} := \{0, 1, 2, \ldots, T\}$. We define as $\tilde{D}$ the set of the dates at which the principal installments of the main loan are reimbursed. We assume that $\tilde{D} = \{t_1, t_2, \ldots, t_T, t_{T+1} = n\}$, i.e. all the settlement dates but the first one are also reimbursement dates. Note that $n = r * (T + 1)$ is the date of the last reimbursement of the main outstanding principal and since $t_0 = 0$ it is also the main loan duration.

The set S with $|S| = m$ contains all assets available at the closing date, i.e. all assets that have been started before (tails of assets) or that begin at the closing date. Each asset i, $i \in S$, is characterized by the initial value of its outstanding principal $C_i = C_{i0}$ and by the number of principal installments n_i. Note that n_i also corresponds to the deadline of asset i.

The problem of selecting the assets at each time t_j, $j \in \mathcal{T}$, in such a way that the sum of their outstanding principals never exceeds the outstanding principal of the main loan can be formulated as a linear integer program with the following binary variables:

$$x_{ij} = \begin{cases} 1 \text{ if asset } i \text{ is included in the portfolio at time } t_j, \\ 0 \text{ otherwise.} \end{cases}$$

$$\text{(ABS)} \quad v := \max \sum_{j=0}^{T} \sum_{i=1}^{m} p_{ij} x_{ij} \tag{1}$$

$$\sum_{j=0}^{T} x_{ij} \leq 1 \quad \forall\, i \in S \tag{2}$$

$$\sum_{i=1}^{m} C_{ij} \sum_{k=0}^{j} x_{ik} \leq C_{0j} \quad \forall\, j \in \mathcal{T} \tag{3}$$

$$x_{ij} \in \{0, 1\} \quad i \in S, \quad j \in \mathcal{T}, \tag{4}$$

where C_{0j} is the outstanding principal of the main loan at time t_j, with $C_{00} = C$, n is the main loan duration and $T + 1$ is the number of main loan installments.

For $n_i > t_j$ the value C_{ij} represents the residual outstanding principal of asset i, $i \in S$, at time t_j while the value p_{ij} associated with each asset i in the objective function represents the sum of asset outstanding principals from time t_j until its deadline n_i. Both are equal to zero if $n_i \leq t_j$.

The set of constraints (2) implies that each asset can be selected at most once. Constraints (3) state that at each time t_j, $j \in \mathcal{T}$, the sum of the outstanding principal of the assets selected before and at time t_j must not exceed the outstanding principal of the main loan at the same date. Finally, constraints (4) are the usual binary conditions.

As originally stated the objective function establishes the minimization of the gap between the main outstanding principal and the sum of the outstanding principal of the assets over time. Thus, we can consider for each asset the area it covers in a (time, principal) domain. In this setup we want to minimize the gap between the area given by the main loan and the sum of areas given by the selected assets depending on their selection time

$$\min \quad \frac{C}{2}(n+r) - \sum_{j=0}^{T} \sum_{i=1}^{m} p_{ij} x_{ij} \tag{5}$$

which is equivalent to (1).

Note that problem (ABS) is a general model which is valid independently from the type of amortization both for the assets and the main outstanding principal. However, the amortization rule based on constant periodic principal installments, with respect to other amortization variants, allows for each asset a simpler determination of the residual outstanding principal at a given date as well as of the sum of outstanding principals from a date to the asset maturity. More precisely, the computation of the (ABS) coefficients, for assets with $n_i > t_j$, in this case are as follows:

$$\begin{aligned} p_{ij} &= \sum_{s=0}^{n_i - t_j} \frac{C_i}{n_i}(n_i - t_j - s) = \frac{C_i}{2n_i}(n_i - t_j)(n_i - t_j + 1) \\ C_{ij} &= \frac{C_i}{n_i}(n_i - t_j), \end{aligned}$$

otherwise they are equal to zero. In the remainder of the paper we will concentrate on this case of constant principal installments, from now on simply referred to as Italian amortization.

Due to the particular form of the outstanding principal computed with Italian amortization, the problem of selecting the assets only at the closing date of the ABS can be reduced to the solution of a 0–1 knapsack problem (KP), i.e. an integer linear program with a single constraint, binary variables and positive coefficients. This will be shown in the following.

Let X be the set of assets handed over at the closing date $t_0 = 0$. For convenience let $X(t) := \{i \in X \mid n_i > t\}$. Let $\bar{D} = \{t_0 = 0, t_1, \ldots, t_T, t_{T+1} = n\}$ be the set of dates of reimbursement of the main loan plus the closing date. Note that $\bar{D} = \tilde{D} \cup \{t_0 = 0\} = D \cup \{t_{T+1} = n\}$ and that the time values in $\bar{D}$ are multiples of r and hence $t_{j+1} = t_j + r$ holds for any $t_j \in \bar{D}$. Then we define the sum of outstanding principals of these assets at any date $t \in \bar{D}$ as $w(t) : \bar{D} \to \mathbb{R}$ with

$$w(t) := \sum_{i \in X(t)} \frac{C_i}{n_i}(n_i - t).$$

Clearly, $w(t)$ is a decreasing function. Moreover, $w(t)$ is linear as long as the set $X(t)$ does not change. Taking into account that an increase of t may eliminate assets from $X(t)$ with small n_i, we get the following property.

Lemma 1. *The function $w(t)$ is convex.*

Proof: Consider the differences between two consecutive function values $\delta(t) := w(t) - w(t+r)$. Using the fact that $i \in X(t) \setminus X(t+r)$ implies $n_i \leq t + r$ and since every term of $w(t)$ is increasing with n_i we can evaluate and then bound $\delta(t)$ for any $t \in \bar{D}$ from both sides by

$$r \sum_{i \in X(t+r)} \frac{C_i}{n_i} \leq \delta(t) = r \sum_{i \in X(t+r)} \frac{C_i}{n_i} + \sum_{i \in X(t) \setminus X(t+r)} \frac{C_i}{n_i}(n_i - t) \leq r \sum_{i \in X(t)} \frac{C_i}{n_i}.$$

Since $X(t) \supseteq X(t+r)$ for every $t \in \bar{D}$ there is always $\delta(t) \geq \delta(t+r)$. But this suffices to show the convexity of $w(t)$. □

Lemma 2. *If $\sum_{i \in X} C_i \leq C$ then*

$$w(t) = \sum_{i \in X(t)} \frac{C_i}{n_i}(n_i - t) \leq \frac{C}{n}(n - t) \quad \forall\, t \in \bar{D}.$$

Proof: For $t = 0$ the statement $w(0) \leq C$ follows trivially from the condition. Since $n_i \leq n$ for all $i \in X$, we get $w(n) = 0$ and the statement holds for $t = n$. This means that $\frac{C}{n}(n-t)$ is a linear function in t which is greater or equal than $w(t)$ at two points $t = 0$ and $t = n$. Now the statement follows immediately from Lemma 1. □

Lemma 2 means that if assets are selected only at the closing date, any set of assets which does not exceed the outstanding principal C at this point will also fulfill conditions (3) on the outstanding principals at all later points in time. Therefore, we can conclude for the case where both assets and main loan outstanding principals are computed with Italian amortization:

Theorem 3. *If assets are selected only at time t_0 then the optimal solution of (ABS) is given by the solution of the following instance of (KP):*

$$\begin{aligned} \max \quad & \sum_{i \in S} p_i x_i \\ & \sum_{i \in S} C_i x_i \leq C \\ & \quad x_i \in \{0, 1\}, \quad i \in S, \end{aligned}$$

with $p_i = \frac{C_i}{2}(n_i + 1)$.

Let us assume to compare two main loans with the same initial value C, the same duration n and the same number of installments $T+1$, the first computed using Italian amortization and the second one using French amortization.

Then, it is easy to show that, for any $t_j \in \bar{D}$ the outstanding principal in the former case will always be smaller than or equal to that in the latter case. More precisely, the outstanding principal at time $t_j < n$ with French amortization is given by:

$$C_{t_j} = C(1+\mu)^j \frac{(1+\mu)^{T+1-j}-1}{(1+\mu)^{T+1}-1} = C\frac{(1+\mu)^{T+1}-(1+\mu)^j}{(1+\mu)^{T+1}-1},$$

where $\mu > 0$ is the internal rate of amortization.

One can check by elementary calculation that the outstanding principal computed according to French amortization is a piece-wise concave function and the following simple result holds:

Observation 4.

$$\frac{C}{n}(n-t_j) \leq C_{t_j}, \quad \forall\, t_j \in \bar{D}.$$

As a direct consequence of Observation 4 we obtain the following result:

Corollary 5. *If the main outstanding principal follows French amortization, then the problem of a unique date selection of assets computed with Italian amortization can be solved by the* (*KP*) *instance given in Theorem* 3.

Since, in practice, the main outstanding principal always has identical or increasing principal installments, it follows from Corollary 5 that, using Italian amortization as financial rule for the assets, we do not need to care about the shape of the main outstanding principal.

The same general result does not hold for the case in which financial assets are computed with French amortization (see [12]). In this case the function representing the sum of assets outstanding principals over time is concave as long as the set $X(t)$ does not change. Thus, an additional constraint is needed for every reimbursement date of the main outstanding principal even if assets are selected only at the closing date. A result similar to that stated in Theorem 3 applies only under special additional conditions.

3. Approximation algorithms for (ABS)

Solving model (ABS) to optimality requires, in general, the intractable task of solving a $(T+1)$-dimensional knapsack problem. Hence, we will proceed by developing approximation algorithms and heuristics for (ABS). Given $\varepsilon \in (0, 1]$, an algorithm is an ε-approximation algorithm for (ABS) if it computes for every instance of (ABS) a feasible solution with objective value larger than or equal to $\varepsilon\, v$.

Algorithm ABS-LP:

for $j = 0$ to T
  for $i = 1$ to m
    choose $x_{ij}^{LP} \in [0, 1]$ as large as possible
    such that (2) and (3) are fulfilled.

Figure 3. Greedy algorithm for the LP-relaxation of (ABS).

x_{i0}^{LP}	1	...	1	λ_0	0	...	0	0	0	...	0	0	...
x_{i1}^{LP}	0	...	0	$1-\lambda_0$	1	...	1	λ_1	0	...	0	0	...
x_{i2}^{LP}	0	...	0	0	0	...	0	$1-\lambda_1$	1	...	1	λ_2	...

Figure 4. Typical structure of the solution x_{ij}^{LP} for t_0, t_1, t_2 generated by ABS-LP.

In analogy to standard knapsack terminology we will use the notion of “efficiency” throughout this section. The efficiency of an asset i at time t_j is given by $\frac{p_{ij}}{C_{ij}} = \frac{1}{2}(n_i - t_j + 1)$, which is increasing in n_i for all j. Hence, it will be convenient to assume throughout the remainder of this section that the assets are sorted in decreasing order of duration n_i, i.e. $n_1 \geq n_2 \geq \cdots \geq n_m$.

3.1. Linear relaxation and greedy heuristics

The first algorithm we present will be based on the LP-relaxation of (ABS) where (4) is replaced by $x_{ij} \in [0, 1]$, $i \in S$, $j \in \mathcal{T}$. The corresponding optimal solution vector x_{ij}^{LP} has a surprisingly simple structure and can be computed by a straightforward greedy procedure. To avoid a lot of technical notations we will not explicitly describe this solution structure but rather state in figure 3 the greedy algorithm ABS-LP generating x_{ij}^{LP}. The resulting block structure is illustrated by figure 4, where $\lambda_0, \lambda_1, \ldots$ denote the fractional part of an asset chosen such that (3) is fulfilled with equality.

Lemma 6. *The solution x_{ij}^{LP} generated by algorithm* ABS-LP *is an optimal solution for the LP-relaxation of* (*ABS*).

Proof: Let x_{ij} be an arbitrary feasible solution of the LP-relaxation of (ABS) where no local improvement, i.e. an increase of a single variable without violating feasibility, is possible. If there exists $r < s$ with $\sum_{\ell=0}^{T} x_{r\ell} < 1$ and $x_{sk} > 0$ for some $k \in \mathcal{T}$ we can improve the solution by increasing x_{rk} and decreasing x_{sk} (by an amount which preserves feasibility) since asset r has a higher efficiency than asset s for all starting times t_k.

Hence, we can assume that there exists some asset b such that $\sum_{\ell=0}^{T} x_{i\ell} = 1$ for $i = 1, \ldots, b-1$, $\sum_{\ell=0}^{T} x_{b\ell} < 1$ and $\sum_{\ell=0}^{T} x_{i\ell} = 0$ for all $i > b$.

Now we can perform iteratively the following cyclic exchange step: Let $j \in \mathcal{T}$ be the smallest index of a point in time where the structure implied by algorithm ABS-LP is violated and let $r \in \{1, \dots, b-1\}$ be the smallest index of an asset which causes such a violation, i.e. $x_{rj} < 1$ and $x_{sj} > 0$ for some $s > r$.

From the above assumption we have $\sum_{\ell=0}^{T} x_{r\ell} = 1$. Hence, there exists some point in time $k \in \mathcal{T}$, $k > j$, with $x_{rk} > 0$.

We will now show that the following exchange yields a new feasible solution x'_{ij} while not decreasing the objective function value. For some suitably small $\varepsilon > 0$ set $x'_{rj} := x_{rj} + \varepsilon$ and $x'_{sj} := x_{sj} - \frac{\varepsilon C_{rj}}{C_{sj}}$ which clearly preserves (3). To guarantee (2) we have to choose $x'_{rk} := x_{rk} - \varepsilon$ which opens the possibility to increase x_{sk}. The amount of such an increase is bounded by (2) and (3). The first resp. second constraint is clearly preserved by an increase by $\frac{\varepsilon C_{rj}}{C_{sj}}$ resp. $\frac{\varepsilon C_{rk}}{C_{sk}}$. A following straightforward calculation shows that the second expression is always at least as large as the first one.

$$
\begin{aligned}
\frac{C_{rj}}{C_{sj}} - \frac{C_{rk}}{C_{sk}} &= \frac{C_r n_s}{C_s n_r}\left(\frac{n_r - t_j}{n_s - t_j} - \frac{n_r - t_k}{n_s - t_k}\right) \\
&= \frac{C_r n_s}{C_s n_r}\left(\frac{-n_r t_k - n_s t_j + n_s t_k + n_r t_j}{(n_s - t_j)(n_s - t_k)}\right) \\
&= \frac{C_r n_s}{C_s n_r}\frac{(n_r - n_s)(t_j - t_k)}{(n_s - t_j)(n_s - t_k)} < 0
\end{aligned}
$$

Hence, we set $x'_{sk} := x_{sk} + \frac{\varepsilon C_{rj}}{C_{sj}}$.

Determining by a basic calculation the change Δ of the objective function value caused by this exchange yields

$$
\begin{aligned}
\Delta &= \varepsilon\left(p_{rj} - p_{rk} - \frac{C_{rj}}{C_{sj}}(p_{sj} - p_{sk})\right) \\
&= \frac{\varepsilon\, C_r}{2n_r}\Bigg((n_r - t_j)(n_r - t_j + 1) - (n_r - t_k)(n_r - t_k + 1) \\
&\quad - (n_r - t_j)(n_s - t_j + 1) + \frac{n_r - t_j}{n_s - t_j}(n_s - t_k)(n_s - t_k + 1)\Bigg) \\
&= \frac{\varepsilon\, C_r}{2n_r(n_s - t_j)}\big(n_r t_k^2 - n_r t_k + n_r t_j^2 - n_s t_j^2 + 2n_s t_j t_k - n_s t_j \\
&\quad -2n_r t_j t_k + n_r t_j - n_s t_k^2 + n_s t_k\big) \\
&= \frac{\varepsilon\, C_r}{2n_r(n_s - t_j)}(n_r - n_s)(t_k - t_j)(t_k - t_j - 1) \ \geq\ 0\,.
\end{aligned}
$$

Repeating this exchange operation for all $r < s$ with $x_{rj} < 1$ and $x_{sj} > 0$ we can transform the given solution into the solution generated by algorithm ABS-LP for time t_j. Performing the same procedure for all subsequent points in time we can transform any solution into x_{ij}^{LP} while not decreasing the objective function value. $\square$

```
Algorithm ABS-Greedy:
    for j = 0 to T
        for i = 1 to m
            set x_ij = 1, if sum_{k=0}^{j-1} x_ik = 0 and (3) is not violated.
```

Algorithm ABS-Greedy:

for $j = 0$ to T

  for $i = 1$ to m

    set $x_{ij} = 1$, if $\sum_{k=0}^{j-1} x_{ik} = 0$ and (3) is not violated.

Figure 5. Greedy algorithm for (ABS).

From the structure of the optimal solution vector x^{LP} we can derive an approximation algorithm ABS-Greedy in the following way. For every point in time t_j we consider all available assets in the given order of decreasing duration and add them to the portfolio if this is possible. A formal description of this approach is given in figure 5. The running time of ABS-Greedy is trivially $O(mT)$.

Although this very simple algorithm works very well in practice (see Section 4) instances can be constructed along the lines of worst case examples for the greedy algorithm for (KP) (cf. [10]) where ABS-Greedy performs arbitrarily bad compared to the optimal solution. Hence we will introduce below in figure 6 a version ABS-ext-Greedy of the greedy approach avoiding this drawback.

This variant is slightly more complicated than the classical greedy algorithm for (KP) because the first fractional item (asset) has to be taken special care of. In particular, we want to avoid to leave out the "break item" s, i.e. the first asset causing a violation of constraint (3) at time t_0, if it contributes a very large profit. Therefore, we compute the time t_r as the earliest time where the break item s can be included in the portfolio together with the assets of longer duration 1 to $s-1$. Then it will be determined by a comparison of profits whether this "break item" should be selected already at time t_0 and all assets 1 to $s-1$ later at time t_r or the other way round, namely selecting asset s at time t_r and the assets 1 to $s-1$ at the closing time t_0. In the special case, where no t_r exists, the algorithm reduces to the decision between selecting at time t_0 only asset s or assets 1 to $s-1$.

Algorithm ABS-ext-Greedy :

compute asset $s := \min\{j \mid \sum_{i=1}^{j} C_{i0} > C_{00}\}$.

compute time $t_r := \min\{t \mid \sum_{i=1}^{s} C_{ir} \leq C_{0r}\}$.

if $\sum_{i=1}^{s-1}(p_{i0} - p_{ir}) \geq p_{s0} - p_{sr}$ then

  select assets 1 up to $s-1$ at time t_0 and asset s at time t_r.

else

  select asset s at time t_0 and assets 1 up to $s-1$ at time t_r.

$i = s + 1$

for $j = r + 1$ to T

  while (3) is fulfilled

    select asset i at time t_j .

  increment i *the break item is omitted in every period*

Figure 6. Modified greedy algorithm for (ABS).

After this decision a simplified greedy algorithm deals with the remaining assets. In every remaining time period assets are selected one after the other until constraint (3) is violated for the first time. The asset causing this violation is omitted and the next time period considered.

The running time of the main computation of ABS-ext-Greedy is $O(m+T)$, since both assets and time periods are considered in increasing order without going back. However the initialization of the output requires $O(mT)$ time. This can be easily reduced by reporting the selected assets explicitly. Naturally, in practice the second part of ABS-ext-Greedy can be modified by plugging in elements from ABS-Greedy (i.e. trying to select every asset at every time period).

Theorem 7. *Algorithm* ABS-ext-Greedy *is a* $\frac{1}{2}$*-approximation algorithm for* (*ABS*).

Proof: We will deal separately with the part of the objective function generated between t_0 and t_r (the earliest point in time where $\sum_{i=1}^{s} C_{ir} \leq C_{0r}$) and between t_r and t_T. Denoting the solution vector computed by ABS-ext-Greedy as x^H we have for the first part

$$
\begin{aligned}
2\sum_{i=1}^{s}(p_{i0}-p_{ir})x_{i0}^{H} &= 2\max\left\{\sum_{i=1}^{s-1}(p_{i0}-p_{ir}),\ p_{s0}-p_{sr}\right\} \geq \sum_{i=1}^{s}(p_{i0}-p_{ir})\\
&\geq \sum_{k=0}^{r-1}\sum_{i=1}^{s}(p_{ik}-p_{ir})x_{ik}^{LP},
\end{aligned}
$$

which proves the $\frac{1}{2}$ approximation for the first part.

Concerning the contribution to the objective function generated after time t_r we will perform the proof for some interval between t_k and t_{k+1}, $r \leq k < T$, thus proving the claim for the remaining part of the objective.

The only difference between the profit of the heuristic solution and the solution of the LP-relaxation for the current interval is given by a single fractional variable x_{bk} for some asset b, because both solutions are generated by the same greedy approach. However, we know that on one hand $\sum_{i=1}^{b-1} C_i > C \geq C_b$. On the other hand $n_i \geq n_b$, $\forall\, i < b$. But now we can prove

$$
\begin{aligned}
\sum_{i=1}^{b-1}\left(p_{ik}-p_{i(k+1)}\right) &= \sum_{i=1}^{b-1}\frac{C_i}{2n_i}((n_i-t_k)(n_i-t_k+1)\\
&\quad -(n_i-t_{k+1})(n_i-t_{k+1}+1))\\
&= \sum_{i=1}^{b-1}\frac{C_i}{2n_i}(t_{k+1}-t_k)(2n_i-t_k-t_{k+1}+1)\\
&\geq \sum_{i=1}^{b-1}\frac{C_i}{2n_b}(t_{k+1}-t_k)(2n_b-t_k-t_{k+1}+1)\\
&\geq \frac{C_b}{2n_b}(t_{k+1}-t_k)(2n_b-t_k-t_{k+1}+1) = p_{bk}-p_{b(k+1)}\,,
\end{aligned}
$$

where the first inequality is valid because $(p_{ik} - p_{i(k+1)})$ is increasing with n_i for $n_i > t_k$.

This shows that the heuristic reaches at least half of the LP-solution value for the interval between t_k and t_{k+1}.

An example showing that ABS-ext-Greedy can actually reach this performance ratio of $\frac{1}{2}$ asymptotically is given below. □

Example. Consider a long term loan with normalized initial outstanding principal $C = 1$ and duration $n = \ell(\ell + 1)$ depending on a parameter $\ell \geq 1$. There is $t_1 = t_T = 1$ and three assets are given with the following data:

i	1	2	3
C_i	$\frac{\ell}{\ell(\ell+1)-1}$	1	1
n_i	$\ell(\ell+1)$	$\ell+1$	ℓ
$2p_{i0}$	$\ell + \frac{2\ell}{\ell(\ell+1)-1}$	$\ell+2$	$\ell+1$
$2p_{i1}$	ℓ	ℓ	$\ell-1$
C_{i1}	$\frac{1}{\ell+1}$	$\frac{\ell}{\ell+1}$	$\frac{\ell-1}{\ell}$

Obviously, exactly one of the three assets can be selected at t_0. Since $C_{01} = \frac{\ell^2+\ell-1}{\ell(\ell+1)}$, assets 1 and 3 are the only pair of assets which may be selected at the same time at time t_1.

Performing ABS-ext-Greedy for this example we have $j = T + 1$ and $p_{20} > p_{10}$ which yields as solution that only asset 2 is selected at time t_0 with an objective value $v^H = \frac{\ell}{2} + 1$ whereas the optimal solution consists of selecting asset 3 at t_0 and asset 1 at t_1 with $v = \ell + \frac{1}{2}$. Hence, we have for this example

$$\lim_{\ell \to \infty} \frac{v^H}{v} = \frac{1}{2}.$$

3.2. Knapsack based heuristics

Another obvious approach to (ABS), which may perform better than the adaptation of the greedy approach and the LP-relaxation, is the successive solution of $T + 1$ different knapsack problems. This means that we first consider only the assets at time t_0 and solve the resulting subproblem, which is the instance of (KP) given in Theorem 3. Then we optimize separately the portfolio at time t_1 and so on.

Although it may seem strange to propose the solution of an $\mathcal{NP}$-hard problem as a heuristic, it is well known that the knapsack problem is by no means a hopeless problem but can be tackled by several promising strategies. We will comment on the applied solution method in Section 4.

The resulting heuristic ABS-Knap is performed in $T+1$ steps by solving iteratively for $j = 0, \dots, T$ the following instance KP_j of (KP) with optimal variable values x_{ij}^*.

$$\text{KP}_j \quad v_j^K = \max \quad \sum_{i=1}^{m} p_{ij} x_{ij}$$

$$\sum_{k=0}^{j-1} x_{ik}^* + x_{ij} \le 1 \quad \forall\, i \in S \tag{6}$$

$$\sum_{i=1}^{m} C_{ij} \left(\sum_{k=0}^{j-1} x_{ik}^* + x_{ij} \right) \le C_{0j} \tag{7}$$

$$x_{ij} \in \{0,1\}, \quad i = 1, \dots, m$$

Note that KP_j is indeed a standard knapsack problem since constraint (6) can be handled by eliminating from consideration any asset i where $\sum_{k=0}^{j-1} x_{ik}^* = 1$. The overall solution value of this heuristic is given by $v^K = \sum_{j=0}^{T} v_j^K$.

It can be shown that the worst–case performance of ABS-Knap is the same as for ABS-ext-Greedy.

Theorem 8. *Algorithm* ABS-Knap *is a* $\frac{1}{2}$*-approximation algorithm for* (*ABS*).

Proof: We will compare the $T+1$ parts of v^K with the corresponding parts of the solution of the LP-relaxation using the structure and notation given in Lemma 6. Therefore, we define $v_j^{LP} := \sum_{i=1}^{m} p_{ij} x_{ij}^{LP}$. It is well known (see e.g. [10, ch. 2.2]) that $v_0^K \ge \frac{1}{2} v_0^{LP}$.

Considering an arbitrary part $j \in \{1, \dots, T\}$ of the objective function value we will prove that the solution of the LP-relaxation of KP_j denoted by x_{ij}^{*R} with value v_j^R is always at least as large as the corresponding part of the solution for the LP-relaxation of (ABS).

$$v_j^R \ge v_j^{LP}. \tag{8}$$

By the standard knapsack argument we get with (8) $v_j^K \ge \frac{1}{2} v_j^R \ge \frac{1}{2} v_j^{LP}$ and are done.

To show (8), note that x_{ij}^{LP} is also a relaxed optimal solution of the maximization of $\sum_{i=1}^{m} C_{ij} \sum_{k=0}^{j-1} x_{ik}$, i.e. the left hand side of (7) contributed by x_{ik}^*, as can be seen by an analogous application of Lemma 6 (without going into the details of the computation). Note that the efficiency of an item i in this artificial problem with respect to constraint k, $k = 0, 1, \dots, j-1$, would be $\frac{C_{ij}}{C_{ik}} = 1 - \frac{t_j - t_k}{n_i - t_k}$ which is increasing in n_i thus leading to the same solution structure as described in figure 4. Hence we have

$$\sum_{i=1}^{m} C_{ij} \sum_{k=0}^{j-1} x_{ik}^{LP} \ge \sum_{i=1}^{m} C_{ij} \sum_{k=0}^{j-1} x_{ik}^*.$$

However, in every LP-relaxation of KP_j constraint (7) will be satisfied with equality. Therefore, $\sum_{i=1}^{m} C_{ij} \sum_{k=0}^{j} x_{ik}^{LP} = \sum_{i=1}^{m} C_{ij} (\sum_{k=0}^{j-1} x_{ik}^* + x_{ij}^{*R})$ implies $\sum_{i=1}^{m} C_{ij} x_{ij}^{LP} \le \sum_{i=1}^{m} C_{ij} x_{ij}^{*R}$.

From the characterization of x_{ij}^{LP} in Lemma 6 it follows that there exists some $b \in \{1, \ldots, m\}$ such that $\sum_{k=0}^{j} x_{ik}^{LP} = 1,\ \forall\, i < b, 0 \le x_{bj}^{LP} < 1$ and $\sum_{k=0}^{j} x_{ik}^{LP} = 0,\ \forall\, i > b$. This means also that $\sum_{i=1}^{b-1} C_{ij} \le C_{0j}$ and $\sum_{i=1}^{b} C_{ij} > C_{0j}$.

Since the efficiency of an item in KP_j is given by $\frac{1}{2}(n_i - t_j + 1)$ and hence again increasing in n_i, x_{ij}^{*R} is generated by selecting all items which were not selected by x_{ik}^{*}, $k = 0, 1, \ldots, j-1$ in decreasing order of n_i until (7), or equivalently (3), is satisfied to equality. But since $\sum_{i=1}^{b} C_{ij} > C_{0j}$ we must have also $x_{ij}^{*R} = 0,\ \forall\, i > b$ no matter how x_{ik}^{*R} was chosen for $k < j$.

Comparing v_j^R and v_j^{LP} we note that the only differences may result from items I_+ which are selected by x_{ij}^{*R} but not (or not completely) by x_{ij}^{LP} (since they were already selected at some earlier point in time) and from items I_- which are (at least partially) selected by x_{ij}^{LP} but not by x_{ij}^{*R} because they were already chosen by some x_{ik}^{*}, $k < j$. It follows from above that both I_+ and I_- are subsets of $\{1, \ldots, b\}$.

Putting together the capacities at time t_j we get

$$\sum_{i=1}^{b} C_{ij} x_{ij}^{*R} = \sum_{i=1}^{m} C_{ij} x_{ij}^{*R} \ge \sum_{i=1}^{m} C_{ij} x_{ij}^{LP} = \sum_{i=1}^{b} C_{ij} x_{ij}^{*R} - \sum_{i \in I_+} C_{ij} + \sum_{i \in I_-} C_{ij}$$

implying $\sum_{i \in I_+} C_{ij} \ge \sum_{i \in I_-} C_{ij}$. Using this inequality and summarizing the above we get

$$\begin{aligned} 2v_j^R &= \sum_{i=1}^{m} C_{ij}(n_i - t_j + 1) x_{ij}^{*R} \\ &= \sum_{i=1}^{m} C_{ij}(n_i - t_j + 1) x_{ij}^{LP} + \sum_{i \in I_+} C_{ij}(n_i - t_j + 1) - \sum_{i \in I_-} C_{ij}(n_i - t_j + 1) \\ &\ge \sum_{i=1}^{m} C_{ij}(n_i - t_j + 1) x_{ij}^{LP} = 2v_j^{LP}, \end{aligned}$$

since $n_r \ge n_s$ for $r \in I_+$, $s \in I_-$.

The existence of an instance where this performance ratio of $\frac{1}{2}$ is (asymptotically) attained, is established by the previous example where ABS-Knap yields the same solution as ABS-ext-Greedy. □

In the experiments of Section 4 it has turned out that the assets selected at time t_0 contribute by far the largest part of the objective function. For the scenario on which the experiments were based this part delivered on average more than 90% of the optimal solution value.

To exploit this behavior algorithmically, it should be noted that on one hand the selection at time t_0 is crucial and should be done as good as possible, whereas on the other hand the remaining points in time will not induce extreme deviations from the optimum and it is not worth to spend large amounts of computation time on them.

Realizing these properties, a mixture of ABS-Knap and ABS-Greedy comes to mind. The resulting algorithm ABS-Comb solves a knapsack problem for t_0, i.e. the above instance

KP_0, and applies ABS-Greedy for the remaining points in time t_1 until t_T. The details of this method are obvious.

4. Computational experiments for a real-world scenario

In this section we will discuss the practical performance of the four approximation algorithms presented in the previous section and compare their output with the optimal solution computed by the standard software package CPLEX©.

A practical situation of asset-backed securitization occurs in the Italian bank *Banca per il Leasing—Italease S.p.A.* (Milano). As suggested by this bank we have used Italian amortization to evaluate the performance of our easily implementable algorithms. In the typical business cases of this bank the general model of (ABS) appears with the following data. The time unit of asset reimbursement is one month for all assets. The reimbursement of the main loan usually takes place with a frequency not lower than every 12 months but also variants with 18 or 24 months were considered. The financial assets are leasing contracts which may be roughly divided into five classes depending on the underlying objects.

1. Class A_1: motor vehicles, automobiles, commercial vans; small investments lower than 75.000 Euro.
2. Class A_2: plant and machinery; small investments lower than 75.000 Euro.
3. Class A_3: motor vehicles, commercial vans, trucks; large investments greater than 75.000 Euro.
4. Class A_4: plant and machinery; large investments greater than 75.000 Euro.
5. Class A_5: real estate; large investments up to 2 million Euro.

The distribution of the assets among these classes, i.e. the relative proportion h_s, $s = 1, \ldots, 5$, of the total assets belonging to class A_s, may vary from time to time. To generate realistic data for the experiments two scenarios were developed according to the suggestions of Italease.

1. Model A: $h_s := 1/5, \forall s$.
2. Model B:
   h_3 uniformly distributed in $[0, 0.3]$
   h_4 uniformly distributed in $[0, 0.2]$
   h_5 uniformly distributed in $[0, 0.15]$
   $h_1 = h_2 := \frac{1}{2}(1 - h_3 - h_4 - h_5)$.

In Model A we assign the same relevance to all classes. In the second case, we take into account the fact that the classes A_3, A_4, and A_5 correspond to more risky assets. Thus, the total number of assets available at the closing date belonging to these classes should not exceed e.g. two thirds of the total number of assets available for the selection. In Model B the expected number of these larger assets is only about one third of the total number of assets. Naturally, their outstanding principal covers a much higher proportion of the total outstanding principal.

Table 1. Generation of assets for the classes A_1 to A_5.

Class	$[C^s_{\min}, C^s_{\max}]$	average term (in months)	$[n^s_{\min}, n^s_{\max}]$
A_1	[5, 75]	35	[25, 45]
A_2	[5, 75]	48	[38, 58]
A_3	[75, 1500]	47	[37, 57]
A_4	[75, 1500]	54	[44, 64]
A_5	[75, 2000]	95	[85, 105]

For each asset i of class A_s, $s = 1, \ldots, 5$, the outstanding principal at the closing date C^s_i and the duration (i.e. number of installments) n^s_i are chosen independently from a uniform distribution with ranges $[C^s_{\min}, C^s_{\max}]$ resp. $[n^s_{\min}, n^s_{\max}]$ as given in Table 1. Based on empirical data from the bank the duration is assumed to lie within an interval of 20 months around the average value.

The total number of assets available at the closing date varies widely since there are many different securitizations being carried out by the bank. We investigated both small problems with 50 up to 200 assets, where the optimal solution of (ABS) could be determined with CPLEX in reasonable time, and larger (more realistic) instances with several thousand assets where only the approximation algorithms of Section 3 could be performed.

The duration of the main loan n is assumed to be in the range of [72, 120] (i.e. 6 to 10 years) with an expected value of 96 (8 years) but always longer than the longest available asset. The number of installments, i.e. the number of time periods $T + 1$, follows from the reimbursement period as given above. Following the real-world scenarios the outstanding principal C was chosen uniformly between $\frac{1}{2}$ and $\frac{2}{3}$ of the total sum of assets $\sum_{i \in S} C_i$. Also smaller selections of C were investigated.

4.1. Test environment

The tests were run on a standard PC with 400 MHz processor speed and 128 MB of RAM. The code was written in C++ under Windows NT©. Optimal solutions of (ABS) were computed with CPLEX© Version 6.6.1.

For all problem sizes we generated 10 instances of Model A and 10 of Model B. All values in the tables below are taken on average (resp. the maximum or minimum) over these 20 instances because the differences between Model A and Model B turned out to be not very significant for most performance parameters (see below).

The first question which is usually addressed in the evaluation of computational experiments is the running time of the algorithms. In our case we can provide a clear answer.

The running time of CPLEX to compute the optimal solution of (ABS) was quite high even for relatively small problems. The deviation between different problems of the same size was extremely large. Some instances could be solved in a couple of seconds whereas others were running for several minutes. Up to $m = 120$ all instances could be solved to

Table 2. Running time of 20 test instances. The numbers in brackets give the number of instances solved to optimality within the time limit of 15 minutes.

	Running times in seconds					
	CPLEX		ABS-Knap		ABS-Comb	
m	Av.	Max.	Av.	Max.	Av.	Max.
50	45	530	0.23	0.28	0.05	0.07
80	87	518	0.22	0.27	0.05	0.11
100	81	482	0.24	0.39	0.05	0.06
120	106	375	0.22	0.27	0.06	0.11
150	305	(14)	0.23	0.28	0.06	0.11
200	230	(17)	0.23	0.31	0.06	0.12

Table 3. Running time of 20 large test instances.

	Running times in seconds			
	ABS-Knap		ABS-Comb	
m	Av.	Max.	Av.	Max.
500	0.26	0.39	0.07	0.12
1000	0.30	0.41	0.09	0.14
1500	0.36	0.49	0.09	0.14
2000	0.39	0.63	0.12	0.16
2500	0.43	0.59	0.12	0.22
3000	0.45	0.61	0.14	0.21

optimality within the imposed time limit of 15 minutes. For problems of larger size there were still many instances solvable within only 20 seconds but others exceeded the time limit.

For ABS-Knap and ABS-Comb most of the running time is naturally consumed by the solution of knapsack problems. We used the code minknap by Pisinger [17] available from www.diku.dk/~pisinger/codes.html to solve the arising instances of (KP). As can be seen in Tables 2 and 3 the running time increases only very slightly with the number of assets even for large problems with up to 3000 assets. This can be explained by the strategy applied in minknap which follows a "core-concept" and is not directly dependent on the number of items. Moreover, the running time of these two knapsack based algorithms was extremely stable with very small deviations for different instances. A small number of instances could be solved much more quickly then the average but the maximum running time was not too far away from the average running time.

The remaining algorithms ABS-Greedy and ABS-ext-Greedy require only extremely simple operations after the sorting of the assets. Their running times were only fractions of a second and did not provide any meaningful data. The practical situation of the bank sometimes requires the solution of even larger instances (on average m is larger than 1000 and in some cases it may raise up to more than 10000). We omit the corresponding results since the behavior of all four approximation algorithms remains almost unchanged with a fairly moderate increase in running time. For these instances the corresponding optimal solutions from CPLEX are not available due to the excessive time and memory requirements. For the solution of these large-sized instances the use of heuristic algorithms clearly remains the preferable approach.

The input values of the outstanding principal generated according to Table 1 were taken as multiples of 100 Euro. Other variants of scaling the data had no clear consequences. The running time of minknap was slightly decreasing for smaller input coefficients (as might be expected) but this tendency was very weak with a considerable number of instances having even an increased running time for smaller coefficients. Rather strange was the behavior of CPLEX which reached a slight speedup for some instances but showed a considerable increase in running time for others.

To measure the quality of the approximate solutions the relative deviation from the optimal solution value was computed for all four algorithms. It can be seen from Table 4 that the brute-force strategy of ABS-ext-Greedy leaves a relatively large gap to the optimal solution value whereas the other three algorithms all perform extremely well leaving only a minimal gap. This gap decreases with the number of assets which is due to the fact that a larger number of randomly chosen assets will increase the chance to find a good solution even by a simple greedy strategy.

Between these three successful algorithms an ordering can be found with ABS-Knap yielding the best approximation quality followed by ABS-Comb and ABS-Greedy. This sequence gives also a decreasing ordering of their running times which means that in a time/quality space the three algorithms can be seen as lying on the efficient frontier. The ordering by quality, which may seem to be only weakly supported by Table 4, is further strengthened by Table 5 which gives the number of instances where each algorithm found the optimal solution value resp. the best known solution value.

Table 4. Deviation from optimality of 20 test instances.

	Percentage of deviation from the optimal solution value							
	ABS-ext-Greedy		ABS-Greedy		ABS-Knap		ABS-Comb	
m	Av.	Max.	Av.	Max.	Av.	Max.	Av.	Max.
50	3.789	15.128	0.331	1.579	0.124	0.376	0.150	0.386
80	2.623	5.957	0.143	0.377	0.100	0.335	0.128	0.345
100	1.636	4.593	0.053	0.161	0.029	0.160	0.049	0.168
120	1.697	6.689	0.049	0.150	0.027	0.134	0.041	0.151
150	1.549	5.096	0.033	0.113	0.016	0.088	0.025	0.093
200	1.272	4.284	0.008	0.024	0.001	0.014	0.006	0.023

Table 5. Number of instances out of 20 which were solved to optimality or as good as CPLEX after 15 minutes.

Number of optimal solutions				
m	ABS-ext-Greedy	ABS-Greedy	ABS-Knap	ABS-Comb
50	0	0	1	0
80	0	0	1	0
100	0	1	5	1
120	0	0	3	0
150	0	0	8	1
200	0	4	14	5

Table 6. Deviation from the best solution computed by ABS-Knap of 20 test instances.

Percentage of deviation from the ABS-Knap value						
	ABS-ext-Greedy		ABS-Greedy		ABS-Comb	
m	Av.	Max.	Av.	Max.	Av.	Max.
500	1.075	3.003	0.005	0.031	0.004	0.016
1000	0.929	2.903	0.006	0.032	0.004	0.015
1500	0.934	3.014	0.006	0.031	0.003	0.017
2000	0.908	2.938	0.006	0.033	0.004	0.013
2500	0.895	2.975	0.007	0.035	0.004	0.011
3000	0.906	2.989	0.007	0.037	0.004	0.015

Also for larger problem instances algorithm ABS-Knap always produced the best solution among the four approximation methods. In Table 6 the deviation of the other three algorithms from these values are given. The ordering of ABS-Comb slightly dominating ABS-Greedy is preserved although these three algorithms produce solutions of almost the same quality. As can be expected ABS-ext-Greedy remains clearly behind.

As indicated in the motivation for introducing ABS-Comb most of the optimal solution value is determined by the assets selected at time t_0. Although this tendency deviates considerably among the instances (basically between 75% and 99%) there is a clear tendency that around 95% of the optimal solution value depends on the assets selected at the closing date as illustrated in Table 7.

This trend is slightly stronger for the approximation algorithms, which solve the instances starting "greedily" at t_0 and fill up later what is left, whereas the optimal solution sometimes leaves slightly more room at the beginning to exploit this freedom later on. This property looses significance if the reimbursement period of the main loan is increased to 18 or 24 months because in these cases the selection of assets at the later points in time clearly contributes even less profit to the objective function.

Table 7. Percentage of the solution value contributed by the assets selected at time t_0.

	Percentage of solution value selected at time t_0									
	CPLEX(Opt)		ABS-ext-Greedy		ABS-Greedy		ABS-Knap		ABS-Comb	
m	Av.	Min.	Av.	Min.	Av.	Min.	Av.	Min.	Av.	Min.
50	93.1	76.1	94.4	85.9	95.4	88.8	95.3	88.1	95.4	88.1
80	94.4	87.5	95.0	90.4	95.1	87.6	95.1	87.5	95.1	87.6
100	94.9	86.6	95.2	85.4	95.4	87.7	95.4	87.8	95.4	87.8
120	94.7	87.4	95.1	83.4	95.1	87.6	95.1	87.6	95.1	87.6
150	94.9	87.5	94.8	86.1	95.1	87.6	95.1	87.6	95.1	87.6
200	95.1	88.7	95.3	89.0	95.1	88.7	95.1	88.7	95.1	88.7

Table 8. Percentage of the solution value contributed by the assets selected at time t_0.

	Percentage of solution value selected at time t_0							
	ABS-ext-Greedy		ABS-Greedy		ABS-Knap		ABS-Comb	
m	Av.	Min.	Av.	Min.	Av.	Min.	Av.	Min.
500	95.5	90.9	94.9	88.2	95.0	88.3	95.0	88.3
1000	95.7	90.9	95.0	88.3	95.1	88.5	95.1	88.5
1500	95.7	90.9	94.9	88.2	94.9	88.2	94.9	88.3
2000	95.7	90.9	95.0	88.3	95.0	88.4	95.0	88.4
2500	95.8	90.9	95.0	88.2	95.0	88.3	95.0	88.4
3000	95.8	91.0	95.0	88.3	95.0	88.4	95.0	88.4

As can be seen in Table 8 there is almost no difference in this aspect for larger problems both for the different algorithms and for different problem sizes. The stochastic model of the data, although allowing for a flexible mixture of assets, seems to reach this value of around 95% for almost every instance through an “averaging effect” which always leads to the availability of an initial portfolio filling most of the available capacity of the main loan.

It might be interesting to compare the optimal solution value of (ABS) with a theoretical upper bound which would be given by filling the main loan completely with assets, e.g. by a single asset having exactly the same characteristics as the main loan. A straightforward computation for the test instances showed that on average the optimal solution reaches about 80% of this upper bound, however with considerable variations. For some instances less than 50% of this upper bound could be filled by the optimal solution. These numbers were also valid for instances with a larger number of assets. Such figures suggest that it would be a worthwhile approach to “reverse” the problem definition and try to find a main outstanding principal which fits the sum of assets outstanding principals as good as possible.

The differences between Models A and B were hardly significant which is the reason that we report the values together for all the 20 instances. The only noticeable effect was an increase of the variance for almost all reported values of Model B. It should be expected that the higher number of assets with small outstanding principal in Model B improves the performance of the greedy algorithms, but this tendency could be observed only to some extend. Clearly, ABS-ext-Greedy hardly gains from the increased number of small assets whereas we noticed a modest improvement of the solution quality for the other three algorithms.

Increasing the reimbursement period of the main loan from 12 to 18 resp. 24 months decreases the optimal solution value only by around 1% resp. 1.5% which is due to the fact stated above that the initial selection of assets at t_0 already defines 95% of the optimal solution value. The corresponding decrease of the number of constraints (3) leads to a significant acceleration of CPLEX but has little influence on the four approximation algorithms.

Choosing the outstanding principal of the main loan lower than above, e.g. less than $\frac{1}{5}$ of the total sum of assets instead of the random choice between $\frac{1}{2}$ and $\frac{2}{3}$, slows down both CPLEX and ABS-Knap, i.e. minknap, since the corresponding knapsack problems become more difficult to solve. In many cases ABS-Greedy reaches or slightly exceeds the quality of the solution computed by ABS-Knap for these problems.

5. Conclusion and future research

Summarizing the extended computational results reported in Section 4 it can be said that solving the proposed model to optimality even with the best available standard software tool can be done with full justification in reasonable time only for problems with up to 100 assets.

The four approximation algorithms developed in Section 3 may have a disappointing (though theoretically bounded) worst-case behavior, but perform very well for instances based on a real-world scenario, since their worst-case behavior occurs only for instances with very large assets which can be ruled out in practical applications.

Ranking the proposed algorithms in terms of solution values we have ABS-Knap followed closely by ABS-Comb and ABS-Greedy. Since the differences in the objective function values computed by these three algorithms are very small and even decreasing for larger problem instances whereas their running times are sorted in the opposite way, we would suggest to use ABS-Comb as the best algorithm for larger instances. It is considerably faster than ABS-Knap and produces very good results. If running time and simplicity of implementation is of high relevance ABS-Greedy may be used without inducing major drawbacks. For smaller instances ABS-Knap may be recommended since for these cases the running times hardly matter and the differences in the solution quality are more pronounced for some instances.

However, since the gap between the solution values computed by our heuristics and the optimal solution values, when available, was quite small, we are confident about the usability of our approach.

Moreover, according to the property emphasized in the computational results, namely that the selection of assets at the closing date is most important while the successive settlement dates have a much lower contribution to the objective function, the authors are investigating the scenario in which only two points are available for selection: the closing date t_0 and one more point t_1. For this special case more sophisticated procedures can be constructed. Furthermore, we can investigate an interesting bilevel optimization problem arising from the free choice for the originator of the second point t_1 for assets selection. This will be the subject of an upcoming paper.

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Note

1. Third parties, typically banks, may provide credit enhancement by issuing a Standby Letter of Credit or by providing reserve accounts.

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