Writing Portfolio 2

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Proof Count: 18

Theorem 10.1. *Consider lists made from the letters* $\{T, H, E, O, R, Y\}$.
(a) The number of lists of length 4 with repetition is 6^4.
(b) The number of lists of length 4 without repetition is $(6)_4$.
(c) The number of lists of length 4 that begin with a T *with repetition is* 6^3.
(d) The number of lists of length 4 that begin with a T *without repetition is* $(5)_3$.
(e) The number of lists of length 4 which do not begin with T *with repetition is* $5(6^3)$.
(f) The number of lists of length 4 which do not begin with T *without repetition is* $5(5)_3$.

Proof. (a) Since there are 6 letters and 4 to choose from a list where repetition is allowed. Thus, by the multiplicative counting principle, the total number of lists is $6 \times 6 \times 6 \times 6 = 6^4$ or 1290 lists.

(b) Since repetition is not allowed in the first position, you have 6 choices in the second position, you have 5 choices, and so on. Thus by the multiplicative counting principle, the number of lists of length 4 is $6 \times 5 \times 4 \times 3 = (6)_4$ or 360 lists.

(c) Since T is required we have 3 choices left for the list of length 4. Thus by the multiplicative counting principle, the number of lists is $6 \times 6 \times 6 = 6^3$ or 216.

(d) Since T is required we have 3 choices left for the list of length 4 since repetition is not allowed we have 5 to choose from because we can't begin with T. Thus by the multiplicative counting principle, the number of lists is $5 \times 4 \times 3 = (5)_3$ or 60 lists.

(e) Since 6^4 is the number of lists with repetition and 6^3 is the number of lists with repetition that begin with T. Then $6^4 - 6^3$ is the number of lists of length 4 that do not begin with T with repetition or $5 \times 6 \times 6 = 5(6)^3$. Thus 1080 lists by the subtraction principle.

(f) Since $(6)_4$ is the number of lists without repetition and $(5)_3$ is the number of lists without repetition and begins with T. Then $(6)_4 - (5)_3$ is the number of lists which do not begin with T without repetition or $5 \times 5 \times 4 \times 3$. Thus the number of lists is 300 by the subtraction principle.

□

Theorem 11.1. *Five cards are dealt off of a standard 52-card deck and lined up in a row.*
(a) The number of lineups with at least one red card is $(52)_5 - (26)_5$.
(b) The number of lineups in which not all cards are red is $(52)_5 - (26)_5$.
(c) The number of lineups in which no card is a club is $(52 - 13)_5$.
(d) The number of lineups with either all black or all hearts is $(26)_5 + (13)_5$.
(e) The number of lineups with all 5 cards of the same suit is $4(13)_5$.
(f) The number of lineups with all 5 cards of the same color is $2(26)_5$.
(g) The number of lineups with exactly one of the 5 cards being a queen is $5(4)(48)_4$.

Proof. Note that a standard deck has 52 cards, with 26 red cards (hearts and diamonds) and 26 black cards (spades and clubs). There are 13 hearts, 13 diamonds, 13 spades, and 13 clubs. Note repetition is not allowed since you can not have the same card twice.

(a) Since the number of lineups for any choice of 5 cards, arranging those 5 cards from 52 is $52 \times 51 \times 50 \times 49 \times 48$ or $(52)_5$. Since the number of lineups where all 5 cards are black is $26 \times 25 \times 24 \times 23 \times 22$ or $(26)_5$ by the multiplicative counting principle. Thus the number of lineups that have at least one red card is $(52)_5 - (26)_5$ by the subtraction principle.

(b) This is the same as part (a). The number of lineups with at least one red card is the same as The number of lineups in which not all cards are red but with black cards instead. "The number of lineups with at least one black card".

(c) Since the number of lineups for any choice of 5 cards is $(52)_5$. Since there are 13 clubs then there are 52 - 13 = 39 non-club cards in the deck thus the number of lineups in which no card is a club is $(52-13)_5$ by the multiplicative counting principle.

(d) Since the number of lineups where all 5 cards are black is $26 \times 25 \times 24 \times 23 \times 22$ or $(26)_5$ and since the number of lineups where all 5 cards are hearts is $13 \times 12 \times 11 \times 10 \times 9$ or $(13)_5$. Thus the total number of lineups is $(26)_5 + (13)_5$ by the addition principle.

(e) The number of lineups where all 5 cards are the same suit is $13 \times 12 \times 11 \times 10 \times 9 = (13)_5$. Thus the total number of lineups is $4(13)_5$ by the multiplicative counting principle.

(f) Since the number of lineups where all 5 cards are all black is $26 \times 25 \times 24 \times 23 \times 22 = (26)_5$. Thus the number of lineups with all 5 cards of the same color is $2(26)_5$ by the multiplicative counting principle.

(g) Since there are 5 possible positions for the queen and there are 4 queens. There are 48 cards to choose from with queens removed and 4 possible positions for them. Thus the number of lineups with exactly one of the 5 cards being a queen is $5(4)(48)_4$ by the multiplicative counting principle. □

Theorem 12.4. *The argument, Reductio ad Absurdum, is valid.*

$$\begin{array}{r} \neg p \\ q \\ \neg q \\ \hline \therefore p \end{array}$$

Proof.

p	$\neg p$	q	$\neg q$	$\neg p \land q$	$\neg p \land q \land \neg q$	$[\neg p \land q \land \neg q] \to p$
T	F	T	F	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	F	T
F	T	F	T	F	F	T

Since $[\neg p \land q \land \neg q] \to p$ is always true because $\neg p \land q \land \neg q$s always false since q and $\neg q$ cannot both be true. Therefore, the argument is valid, as shown by the truth table. □

Theorem 13.1. *The number of anagrams of the word "PEPPERMINT" is 302,400.*

Proof. The number of total combinations is 10! but P appears three times and E appears two times. Since the order in the P's and E's appear doesn't matter. There are 3! ways to arrange P and 2! ways to arrange E. Thus the number of anagrams is $\frac{10!}{3!2!} = \frac{10*9*8*7*6*5*(2*2)*3!}{3!2} = 10*9*8*7*6*5*2 = 302,400$ by the division counting principle. □

Theorem 14.6. *Consider the set of length-5 lists drawn from the set A, B, C, D, E, F, G, H, I. The number of lists with no repetition and not in alphabetical order is 14,994.*

Proof. Since we are forming lists of length 5 from a set of 9 elements without repetition, the total number of possible permutations is $\frac{9!}{4!} = 15120$. The number of lists that are in alphabetical order is $\frac{9!}{5!4!} = 120$. The permutations over count, thus by the subtractive principle of counting the total number of lists is $15120 - 120 = 14,994$. $\square$

Theorem 15.9. *The number of lists (x, y, z) of three integers such that $0 < x \leq y \leq z \leq 100$ is* $\binom{102}{3}$

Proof. Let any three numbers be selected and ordered to match the inequality $0 < x \leq y \leq z \leq 100$. This is the same as counting multichoose $\left(\binom{n}{k}\right)$. Where n is the number of choices ($n = 100$) and $k = 3$. Then number of multisets is $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$. Thus $\left(\binom{100}{3}\right) = \binom{100+2-1}{3} = \binom{102}{3}$ $\square$

Theorem 16.4. *Consider the lists drawn from the set T, H, E, O, R, Y with repetition allowed.*
(a) The number of 4-letter lists that do not begin with T, or do not end in Y is $10(6^3) - 5^2(6^2)$.
(b) The number of 4-letter lists in which the sequence of letters T, H, E appears consecutively in that order is $2(6)$.
(c) The number of 6-letter lists are there in which the sequence of letters T, H, E appears consecutively in that order is $4(6^3) - 1$.

Proof. (a) Let A be the set of lists that do not start with T then the number of lists is $5(6^3)$. Let B be the set of lists that do not end with Y then the number of lists is $(6^3)5$ then $A \cup B = A + B - A \cap B$ where $A \cap B$ is $5 * (6^2) * 5$. Thus the number of lists that do not begin with T, or do not end in Y is $5(6^3) + (6^3)5 - 5 * 5(6^2) = 2(5(6^3)) - 5^2(6^2) = 10(6^3) - 5^2(6^2)$.

(b) There are only two ways to have the sequence T, H, E in a 4-letter list that is $\{(x, T, H, E), (T, H, E, x)\}$ where x can be any letter from the set. Thus the number of lists is $2(6)$.

(c) This argument is the same as (b) but there are 4 positions for the sequence T, H, E, and 6^3 chooses for the remaining positions. Then the number of lists is $4(6^3)$ but there is one list (T, H, E, T, H, E) that has been counted twice so we must remove one of the relations. Thus the number of lists is $4(6^3) - 1$. $\square$

Theorem 17.1. *If six integers are chosen at random, then at least two of them will have the same remainder when divided by 5.*

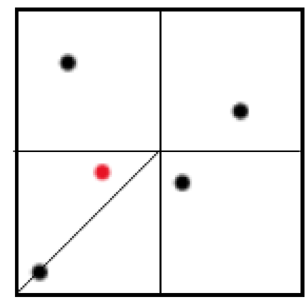
Proof. This is the same as working in mod 5 the only possible remainders when working mod 5 are $\{0, 1, 2, 3, 4\}$ let 6 random integers be chosen there are only 5 remainders then can have. Thus by the pigeonhole principle, at least two of them will have the same remainder. $\square$

Theorem 18.11. *For all $n, k, m \in \mathbb{N}$,* $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$

Proof. Consider the right-hand side $\binom{n}{k}$ implies that we are counting subsets, and $\binom{k}{m}$ implies that we are counting a subset within k or distinguished members of m.
Consider the left-hand side $\binom{n}{m}$ implies that we are counting a distinguished member m because we have already chosen m we are counting subsets of $\binom{n-m}{k-m}$.
Now consider the right-hand side's binomial coefficient $\frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!} = \frac{n!}{m!(n-k)!(k-m)!}$
Now consider the left-hand side's binomial coefficient $\frac{n!}{m!(n-m)!}\frac{(n-m)!}{(k-m)!((n-m)-(k-m))!} = \frac{n!}{m!(n-k)!(k-m)!}$
Thus it is shown that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$. $\square$

Theorem 17.4. *For any five points on a square whose side-length is one unit, at least two of these points are within $\frac{\sqrt{2}}{2}$ units of each other.*

Proof. Consider the unit square with side-lenght 1. Then partition the unit square into four quadrants by dividing it into four equal squares with side lengths equal to $\frac{1}{2}$. The furthest any two points can be in one of the quadrants is $\frac{\sqrt{2}}{2}$ because we have to place five Points at least two of them will be in the same quadrant by the pigeonhole principle. Thus at least two points have to be within $\frac{\sqrt{2}}{2}$.



□

Theorem 12.5. *The argument, Constructive Dilemma, is valid.*

$$\begin{array}{r} p \vee q \\ p \to r \\ q \to s \\ \hline \therefore r \vee s \end{array}$$

Proof.

p	q	r	s	$p \vee q$	$p \to r$	$q \to s$	$r \vee s$	$(p \vee q) \wedge (p \to r)$	$(p \vee q) \wedge (p \to r) \wedge (q \to s)$	$([p \vee q) \wedge (p \to r) \wedge (q \to s)] \to [r \vee s]$
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	F	T	T	F	T
T	T	F	T	T	F	T	T	F	F	T
T	T	F	F	T	F	F	F	F	F	T
T	F	T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T	F	F	T
T	F	F	F	T	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T	T	T
F	T	T	F	T	T	F	T	T	F	T
F	T	F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	F	F	T	F	T
F	F	T	T	F	T	T	T	F	F	T
F	F	T	F	F	T	T	T	F	F	T
F	F	F	T	F	T	T	T	F	F	T
F	F	F	F	F	T	T	F	F	F	T

As we can see from the last column of the truth table the argument always ends in a truth. Thus Constructive Dilemma is valid. □

Theorem 15.11. *A bag contains 50 pennies, 50 nickels, 50 dimes, and 50 quarters. You reach in and grab 30 coins. The number of ways this is possible is* $\binom{33}{4}$

Proof. Let n be the number of coins grabbed ($n = 30$) and let k be the different types of coins $k = 4$. There are $\left(\binom{n}{k}\right)$ multichoice coin combinations. Then number of multisets is $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$. Thus $\left(\binom{30}{4}\right) = \binom{30+4-1}{4} = \binom{33}{4}$ □

Theorem 15.3. *You have a dollar in pennies, a dollar in nickels, a dollar in dimes, and a dollar in quarters. You give a friend four coins. The number of ways this can be done is 35.*

Proof. Case 1 brute force lists all the possible combinations let the list be {p, n, d, q} where, pennies = p, nickels = n, dimes = d, quarters = q. The set of lists is
$\{(4, 0, 0, 0), (3, 1, 0, 0), (3, 0, 1, 0), (2, 2, 0, 0), (2, 1, 1, 0), (2, 0, 2, 0), (1, 3, 0, 0), (1, 2, 1, 0), (1, 1, 2, 0), (1, 0, 3, 0), (0, 4, 0, 0), (0, 3, 1, 0), (0, 2, 2, 0), (0, 1, 3, 0), (0, 0, 4, 0), (3, 0, 0, 1), (2, 1, 0, 1), (2, 0, 1, 1), (1, 2, 0, 1), (1, 1, 1, 1), (1, 0, 2, 1), (0, 3, 0, 1), (0, 2, 1, 1), (0, 1, 2, 1), (0, 0, 3, 1), (2, 0, 0, 2), (1, 1, 0, 2), (1, 0, 1, 2), (0, 2, 0, 2), (0, 1, 1, 2), (0, 0, 2, 2), (1, 0, 0, 3), (0, 1, 0, 3), (0, 0, 1, 3), (0, 0, 0, 4)\}$.
Case 2 Multisets, Let n be the number of coins given to a friend ($n = 4$), and let k be the different types of coins $k = 4$. There are $\left(\binom{n}{k}\right)$ multichoice coin combinations. Then number of multisets is $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$. Thus $\left(\binom{4}{4}\right) = \binom{4+4-1}{4} = \binom{7}{4} = 35$

□

Theorem 10.5. *Consider 8-bit binary strings such as 10011011 or 00001010.*
(a) The total number of 8-bit strings is 2^8.
(b) The total number of 8-bit strings that end in 0 is 2^7.
(c) The total number of 8-bit strings that have 1's for their second and fourth digits is 2^6.
(d) The total number of 8-bit strings that have 1's for their second or fourth digits is $2^8 - 2^6$.

Proof. (a) There are two choices 1 or 0 for a list of 8-bits is 2^8 by the multiplicative counting principle.

(b) The last bit is 0 and can't be chosen so there are 7 choices left for the lists. Thus the number of lists is 2^7 by the multiplicative counting principle.

(c) The second and fourth digits must be 1 and can't be chosen so there are 6 choices left for the lists. Thus the number of lists is 2^6 by the multiplicative counting principle.

(d) Let A be a set of 8-bit lists with 1 in the second space, and let B be a set of 8-bit lists with 1 in the fourth space. Then the union of A and B is $A \cup B$. Then the number of lists without overcounting the interaction is $A \cup B = A + B - (A \cap B)$. The interaction is the 8-bit lists that have 1's in the second and fourth space which is 2^6 from part c. Thus the number of lists is $2^7 + 2^7 - 2^6 = 128 + 128 - 64 = 256 - 64 = 2^8 + 2^6$. □

Theorem 11.2. *The number of integers between 0 and* $9999_(n - many)$ *with at least one repeated digit is*

$$(10^n - 1) - 9 \sum_{k=0}^{n-1} [(9)_k] \tag{1}$$

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Proof. Consider the first part of the equation $(10^n - 1) - 9$, 10^n gives the number with n digits, and minusing one gives the number $9999...9_n$. This counts all numbers from 0 to $9999...9_n$ Consider the second part of the equation the sum of the fall by k factorial gives all the combinations of the number with n many digits arranged in order. The 9 in front of the sum is the first digit and we can choose any digit from 1 to 9 for it and the factorial gives the remaining digits. This counts how many n-digit numbers have no repeated digits. Combining this gives all numbers from 0 to $9999...9_n$ minus the numbers have no repeated digits, which gives the number of integers with at least one repeated digit. Thus The number of integers between 0 and $9999_(n - many)$ with at least one repeated digit is

$$(10^n - 1) - 9\sum_{k=0}^{n-1}[(9)_k] \tag{2}$$

. By the subtractive principle of counting. □

Theorem 14.1. *Flip a coin ten times in a row. The number of outcomes with 3 heads and 7 tails is 120.*

Proof. Let n be the number of flips and let k be the number of heads. Then the number of outcomes is given by $\binom{n}{k} = \binom{10}{3} = \frac{10!}{3!(10-3)!} = 120$. Since counting heads is the same as counting the number of times we did not get tales. □

Theorem 10.2. *Airports are identified with 3-letter codes. For example, Richmond, Virginia has the code RIC, and Memphis, Tennessee has MEM. The total number of such airport codes is 263*

Proof. Since the number of letters that are in the alphabet is 26, we are choosing 3 with repetition and order matters. The number of total codes is 26^3 □

Theorem 1.15. *If $n \in \mathbb{Z}$, then n $n^2 + 3n + 4$ is even.*

Proof. Case 1: Let n be even that is $n = 2k$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 + 3n + 4 = (2k)^2 + 3(2k) + 4 &= 4k^2 + 6k + 4 \\ &= 2(2k^2 + 3k + 2) \end{aligned}$$

since $2k^2 + 3k + 2 \in \mathbb{Z}$ then $n^2 + 3n + 4$ is even.

Case 2: Let n is odd that is $n = 2l + 1$ where $l \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 + 3n + 4 = (2k+1)^2 + 3(2k+1) + 4 &= (4k^2 + 4k + 1) + (6k + 3) + 4 \\ &= 4k^2 + 10k + 8 \\ &= 2(2k^2 + 5k + 4) \end{aligned}$$

since $2k^2 + 5k + 4 \in \mathbb{Z}$ then $n^2 + 3n + 4$ is even. Thus, in both cases, $n^2 + 3n + 4$ is even. □

Theorem 1.36. *Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.*

Proof. Let a is even, then a^2 is divisible by 4 that is $a = 2k$ where $k \in \mathbb{Z}$. Then $a^2 = 4k^2$. Since a^2 is a multiple of 4, a^2 is divisible by 4. Thus, if a is even, then a^2 is divisible by 4. □