# Writing Portfolio 3

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Proof Count: 17

**Theorem 19.6.** *If $n \in N$, then*

$$1 \cdot 3 + 2 \cdot 4 + \ldots + n(n+2) = \tfrac{n(n+1)(2n+7)}{6}$$

*Proof.* Consider $n = 1$ then for the LHS $1(1+2) = 3$, for the RHS $\frac{1(1+1)(2(1)+7)}{6} = \frac{18}{6} = 3$. So it holds for the base case.
Assume it hold for some $n = k$, that is $1 \cdot 3 + 2 \cdot 4 + \ldots + k(k+2) = \frac{k(k+1)(2k+7)}{6}$.
Then for $n = k+1$ we have $1 \cdot 3 + 2 \cdot 4 + \ldots + k(k+2) + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$ for the LHS. We also have $\frac{(k+)(k+3)(2(k+1)+7)}{6}$, for the RHS. Factor out $(k+1)$ from both terms on the left-hand side:

$$\begin{aligned}
\text{LHS} &= (k+1) \cdot \frac{k(2k+7) + 6(k+3)}{6} \\
&= (k+1) \cdot \frac{2k^2 + 13k + 18}{6} \\
&= \frac{(k+1)(2k^2 + 13k + 18)}{6}
\end{aligned}$$

Now for the RHS:

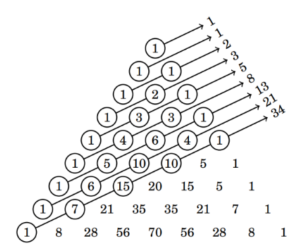
$$\begin{aligned}
\text{RHS} &= \frac{(k+1)(k+2)(2k+9)}{6} \\
&= \frac{(k+1)(2k^2 + 13k + 18)}{6}
\end{aligned}$$

Thus, the formula holds for $n = k+1$ so By the principle of mathematical induction, the formula is true for all $n \in N$ □

**Theorem 20.4.** *If $n \in N$ and $f_n$ is the nth of the Fibonacci Sequence, then*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = f_{n+1}$$

*The above formula shows that the indicated diagonals of Pascal's triangle sum to Fibonacci numbers.*




*Proof.* When $n$ is even, then $n = 2m$ for $m \in \mathbb{Z}$. For this case: $\lfloor \frac{n}{2} \rfloor = m$ and $\lfloor \frac{n+1}{2} \rfloor = m$ Then the formula becomes

$$\sum_{k=0}^{m} = \binom{2m-k}{k}$$

Then for $m = 0$ and $m = 1$

$$\sum_{k=0}^{0} = \binom{2(0)-k}{k} = \binom{0}{0} = 1$$

$$\sum_{k=0}^{1} = \binom{2(1)-k}{k} = \binom{2}{0} + \binom{1}{1} = 1 + 1 = 2$$

The Fibonacci number $f_{2(0)+1} = f_1 = 1$ and $f_{2(1)+1} = f_2 = 2$ so the base case's hold. Assume that it holds for some $m = i$, That is

$$\sum_{k=0}^{i} = \binom{2(i)-k}{k} = f_{2(i)+1} = f_{2i+1}$$

Then consider $m = i+1$

$$\sum_{k=0}^{i+1} = \binom{2(i+1)-k}{k} = f_{2(i+1)+1} = f_{2i+3}$$

Rewriting the sum as

$$\sum_{k=0}^{i+1} = \binom{2(i+1)-k}{k} = \sum_{k=0}^{i} = \binom{2(i+1)-k}{k} + \binom{2(i+1)-i+1}{i+1} = \sum_{k=0}^{i} = \binom{2(i+1)-k}{k} + \binom{i+1}{i+1}$$

$$= \binom{2(i+1)-k}{k} + 1 = f_{2(i)+1} + f_{2(i+1)+1} = f_{2i+1} + f_{2i+3} = f_{2i+4}$$

Thus by the principle of mathematical induction and our assumption, it is shown that the formula holds for all $n \in \mathbb{N}$

Case 2: When n is odd, then $n = 2m+1$ for $m \in \mathbb{Z}$, For this case: $\lfloor \frac{n}{2} \rfloor = m$ and $\lfloor \frac{n+1}{2} \rfloor = m+1$ Then the formula becomes

$$\sum_{k=0}^{m} = \binom{(2m+1)-k}{k} = f_{2m+2}$$

When $m = 0$, we have $n = 2 \cdot 0 + 1 = 1$ $\sum_{k=0}^{0} \binom{(1-k)}{k} = 1$. So it holds for the base case.

Assume it hold for some $m = j$, then consider $m = j+1$

$$\sum_{k=0}^{j+1} \binom{(2(j+1)+1)-k}{k} = f_{2(j+1)+2},$$

Rewrite this sum by separating the last term

$$\sum_{k=0}^{j+1} \binom{(2j+3)-k}{k} = \sum_{k=0}^{j} \binom{(2j+3)-k}{k} + \binom{(2j+3)-(j+1)}{j+1} = f_{2j+3} + f_{2j+2} = f_{2j+4}.\text{(From the inductive step)}$$

Thus, the result holds for when $n$ is odd by mathematical induction.

□

**Theorem 21.8.** *Suppose $a, b \in R$. If $a$ is rational and $ab$ is irrational, then $b$ is irrational.*

*Proof.* Suppose that $a$ is rational, $ab$ is irrational, and $b$ is rational. If $a$ is rational it can be expressed as $\frac{c}{d}$ where $d \neq 0$ and b can also be expressed as $\frac{g}{h}$ where $h \neq 0$. Then $ab = \frac{cg}{dh}$, since $c$ is rational and $g$ then $cg$ is rational. Since $d$ is rational and $h$ is rational then $dh$ is rational. This implies that $\frac{cg}{dh}$ is rational, this is a contradiction from the assumption that $ab$ is irrational. Therefore, our assumption that $b$ is rational must be false. Hence, $b$ must be irrational. □

**Theorem 20.6.** *For all integers $m, n \geq 1$, it holds $f_n \mid f_{nm}$.*

*Proof.* Assume there exists a smallest pair of integers $n$ and $m$ which $m, n \geq 1$ for which $f_n \nmid f_{nm}$.
Recall that Fibonacci numbers satisfy the recurrence relation: $f_{n+2} = f_{n+1} + f_n$.
Consider the first three Fibonacci numbers $\{1, 1, 2\}$ then $1 \cdot 1 \mid 1$ and $2 \cdot 1 \mid 1$ so it works for the base case.
Let $f_{nm}$ be a Fibonacci number and assume $f_n \nmid f_{nm}$ but $f_{nm} = gcd(f_n, f_m) = f_{gcd(nm)}$ by the GCD identity for Fibonacci numbers. Then the $gcd(f_n)$ is also the $gcd(f_{nm})$ which implies that $f_n \mid f_{nm}$.
Thus there is no counterexample to $f_n \mid f_{nm}$. □

**Theorem 23.2.** *Let $A$ be a set and let $R$ be an irreflexive relation on $A$. Then any subset $R$ is also irreflexive.*

*Proof.* Let $S \subseteq R$ then $\forall a \in A, (a, a) \notin R$. Let $(r, r) \notin S$ then $(r, r) \notin R$ since $\forall x \in S$, $x$ must also be in $R$. Therefore, the subset $S$ inherits the irreflexive property from $R$ □

**Theorem 24.14.** *Let $A$ be a set and let $R$ be a relation on $A$. Then $R$ is reflexive if and only if $\not R$ is irreflexive.*

*Proof.* This is true by the definition of reflexivity, $\forall a \in A, (a, a) \in R$ then for irreflexivity $\forall a \in A, (a, a) \notin R$. This implies that $\not R$ is irreflexivity because all elements that are in $R$ are reflexive, so elements that are in $\not R$ must be irreflexive. □

**Theorem 25.1.** *Let $(X, \leq)$ be a poset such that for every $x, y \in X$ either $x \leq y$ or $y \leq x$. Then is $X$ said to be a totally ordered set. Likewise, we say $\leq$ is total order (or linear order).*
*(a) The poset $(N, \leq)$ is totally ordered.*
*(b) The poset $(N, |)$ is not totally ordered.*

*Proof.* $(a)$ For any two natural numbers $a$ and $b$, either $a \leq b$ or $b \leq a$. Thus the relation is total order.
$(b)$ Consider this counter-example, if you take the numbers 2 and 3 from $\mathbb{N}$, neither 2 divides 3 nor 3 divides 2. Since there exist elements in which neither holds true, the relation is not a total order. □

**Theorem 26.5.** *Suppose $R$ is a reflexive and symmetric relation on a finite set $A$. Define a relation $S$ on $A$ by declaring $xSy$ if and only if for some $n \in N$ there are elements $x_1, x_2, \ldots, x_n \in A$ satisfying $xRx_1, x_1Rx_2, x_2Rx_3, x_3Rx_4, \ldots, x_{n-1}Rx_n$, and $x_nRy$. Then $S$ is an equivalence relation and $R \subseteq S$. In fact, $S$ is the unique smallest equivalence relation on $A$ containing $R$.*

*Proof.* Since $R$ is reflexive, $\forall x, (x, x) \in R$ so $xRx$. Hence, $xSx$ is true, Thus $S$ is reflexive.
Since $R$ is symmetric, if $xSy$ then there exists a list $x = x_1, x_2, \ldots, x_n = y$. By symmetry of $R$, we can reverse the order of this list to obtain $ySx$, proving that $S$ is symmetric.
If $xSy$ and $ySz$, there exist list $x = x_1, x_2, \ldots, x_n = y$ and $y = y_1, y_2, \ldots, y_m = z$ Combining these list results in a list that will connect $x$ and $z$, showing $xSz$, Thus $S$ is transitivity.
Suppose there is another equivalence relation $T$ containing $R$. Since $S$ is the set of all pairs related by lists in $R$, and any equivalence relation containing $R$ must also relate all elements connected by these lists, it follows that $S \subseteq T$.
Thus $S$ is the unique smallest equivalence relation containing $R$.

□

**Theorem 27.6.** *The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + sin(x)$ has a unique x-intercept.*

*Proof.* For this $f$ to have one unique x-intercept $f$ must be continuous on $\mathbb{R}$. $f$ must also have a constant derivative where is it positive or negative everywhere, that is the derivative is never zero.

Let $\epsilon > 0$ be given and choose $\delta = \max\{1, \frac{\epsilon-2}{2}\}$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$

$$\begin{aligned}
|2x + \sin(x) - 2y - \sin(y)| &= |2(x-y) + (\sin(x) - \sin(y))| \\
|2(x-y)| + |\sin(x) - \sin(y) &\leq |2(x-y)| + |\sin(x) - \sin(y)| \quad \text{(by the triangle inequality)} \\
&= |\sin(x) - \sin(y)| + 2|x-y| \\
&\leq |\sin(x) - \sin(y)| + 2\delta \\
\leq 2 + 2 \cdot \tfrac{\epsilon-2}{2} &\quad \text{(Since max of } |\sin(x) - \sin(y)| = 2 \text{ )} \\
&= \epsilon
\end{aligned}$$

Thus $f(x)$ is continuous on $\mathbb{R}$
lemma

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

note that $\sin x \leq x \leq \tan x$

$$\begin{aligned}
\sin x \leq x \leq \tan x \; &= \tfrac{\sin x}{\sin x} \leq \tfrac{x}{\sin x} \leq \tfrac{\sin x}{\cos x} \cdot \tfrac{1}{\sin x} \\
1 &\leq \tfrac{x}{\sin x} \leq \tfrac{1}{\cos x} \\
1 &\leq \tfrac{\sin x}{x} \leq \cos x \\
\lim_{x\to 0} 1 &\leq \lim_{x\to 0} \tfrac{\sin x}{x} \leq \lim_{x \to 0} \cos x \\
1 &\leq \lim_{x\to 0} \tfrac{\sin x}{x} \leq 1 \quad \text{(by squeeze theorem)}
\end{aligned}$$

Thus $\lim_{x\to 0} \frac{\sin x}{x} = 1$
Lemma Show that $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$ by using trigonometric identities.

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x} &= \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \\
&= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)} \quad \text{(by Pythagorean identity)} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 0. \\
&\text{(Since the limit of } \frac{\sin x}{x} \text{ is equal to 0 from previous Limma so the whole limit is 0)}
\end{aligned}$$

Let $f'(x) = \lim_{h\to 0} \frac{2(x+h)+(\sin(x+h))-2(x)-(\sin(x))}{h}$

$$\lim_{h\to 0} \frac{2(x+h)+(\sin(x+h))-2(x)-(\sin(x))}{h}$$
$$= \lim_{h\to 0} \frac{2h+(sin(x+h)-sin(x))}{h}$$
$$= \lim_{h\to 0} \frac{2h}{h} + \frac{(sin(x+h)-sin(x))}{h}$$
$$= \lim_{h\to 0} 2 + \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \quad \text{(by the sum trig identity)}$$
$$= \lim_{h\to 0} 2 + \frac{\sin x(\cos h - 1)\cos x \cdot \sin h}{h}$$
$$= \lim_{h\to 0} 2 + \lim_{h\to 0} -\sin x\frac{1-\cos h}{h} + \lim_{h\to 0} \cos x\frac{\sin h}{h}$$
$$= \quad \text{From our previous Lemma we know that } \frac{1-\cos h}{h} = 0 \text{ so the whole part is 0.}$$
$$\text{We also now that } \lim_{h\to 0} \frac{\sin h}{h}) = 1 \text{ from a previous Lemma so } \lim_{h\to 0} \cos x \cdot 1 = \cos x$$

Thus $f'(x) = 2 + \cos x$. Note that $-1 \leq \cos x \leq 1$, since $-1$ is the minimum of $\cos x$ then $f'(x) = 2-1 = 1 > 0$. Since $f'(x)$ is also greater then the 0 everywhere. This implies that there is only one real solution by the mean value theorem. Therefore since $f(x)$ is continuous on $\mathbb{R}$ and $f'(x) > 0$ everywhere. The function $f(x)$ has one unique x-intercept. □

**Theorem 27.5.** *Every real solution of $x^3 + x + 3 = 0$ is irrational.*

*Proof.* Suppose that there exists a rational root, $r = \frac{p}{q}$

$$x^3 + x + 3 = 0$$
$$x^3 + x = -3$$
$$x(x^2 + 1) = -3$$

This implies that $\{0, i, -i\}$ are solutions but zero is the only real solution, check for extraneous solutions

$$0^3 + 0 = -3$$
$$0 = -3$$

This is a contradiction zero does not equal negative three. Thus every real solution of $x^3 + x + 3 = 0$ is irrational. □

**Theorem 21.1.** *Suppose $n \in Z$. If $n$ is odd, then $n^2$ is odd.*

*Proof.* Let $n = 2k+1, k \in \mathbb{Z}$ then $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ where $(2k^2+2k) \in \mathbb{Z}$. Thus $n^2$ is odd.
Suppose that $n$ is odd and $n^2$ is even, then $n = 2l+1, l \in \mathbb{Z}$ then $n^2 = (2l+1)^2 = 4l^2+4l+1 = 2(2l^2+2l)+1$ this implies that $n^2$ is odd a contradiction of the assumption, thus $n^2$ is odd. □

**Theorem 19.4.** *If $n \in \mathbb{N}$, then $2^0 + 2^1 + 2^2 + ... + 2^n = 2^{n+1} - 1$.*

*Proof.* Proof by Mathematical Induction Base Case: For $n = 0$,

$$2^0 = 1 \quad \text{and} \quad 2^{0+1} - 1 = 2 - 1 = 1,$$

so the statement holds for $n = 0$.
Inductive Step: Assume the statement holds for some $n = k$; that is,

$$2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

Then consider $n = k + 1$, that is

$$2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1.$$

from the inductive hypothesis,

$$2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

Adding $2^{k+1}$ to both sides gives:

$$2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2^{k+2} - 1.$$

Thus, the statement holds for $n = k + 1$.

Combinatorial Proof: Note that the sum $2^0 + 2^1 + 2^2 + \cdots + 2^n$ represents the total number of subsets of a set with $n+1$ elements. For a set with $n+1$ elements, there are $2^{n+1}$ subsets in total, including the empty set. Excluding the empty set leaves $2^{n+1} - 1$, which is all the non-empty subsets, which matches the premise. Thus $2^0 + 2^1 + 2^2 + ... + 2^n = 2^{n+1} - 1$. □

**Theorem 18.4.** *If $p$ is prime and $k$ is an integer for which $0 < k < p$, then $p$ divides $\binom{p}{k}$*

*Proof.* Assume for the sake of contradiction that a counterexample exists where $p$ is prime and $k$ is an integer for which $0 < k < p$, and $p$ does not divide $\binom{p}{k}$. Then consider the binomial coefficient.

$$\begin{aligned}
\binom{p}{k} &= \frac{p!}{k!(p-k)!} \\
&= \frac{p \cdot (p-1)!}{k!(p-k)!} \\
\text{(divide by } p) &= \frac{p \cdot (p-1)!}{\frac{k!(p-k)!}{p}} \\
&= \frac{p \cdot (p-1)!}{p \cdot k!(p-k)!} \\
&= \frac{(p-1)!}{k!(p-k)!}
\end{aligned}$$

Therefore, $\binom{p}{k}$ contains a factor of p which implies that $p$ divides $\binom{p}{k}$ this is a contradiction of the assume there exists a counterexample. Thus there is no counterexample so we conclude that $p$ divides $\binom{p}{k}$ for all $k$ such that $0 < k < p$. □

**Theorem 3.14159.** *Consider the function $f(x) = x + \sin(x)$, obtain an approximation of $\pi$ with high accuracy using the fact that that $\pi$ is a fixed point of $f(x)$ and that $f$ maps the interval $[\frac{3\pi}{4}, \frac{5\pi}{4}]$ to itself. We have also shown that $f'(x) = 1 + \cos x$ from a previous theorem.*

*Proof.*

$$
\begin{aligned}
|x + \cos(x) - y - \cos y| &= L|x - y| \\
|1 + \cos(x) - 1 - \cos y| &\leq 1|x - y| + |\cos x - \cos y| \\
1|x - y| + |\cos x - \cos y| &\leq 1|x - y| + (1 - \frac{\sqrt{2}}{2}) \quad \text{(since the maximum occurs at } \cos\frac{3\pi}{4} - \cos\pi) \\
0 \leq |\cos x - \cos y| &\leq (1 - \frac{\sqrt{2}}{2}) \leq 1
\end{aligned}
$$

This implies $f$ has a Lipschitz constant less then 1, then by the Banach fixed-point theorem, it is varied that $\pi$ is a fixed point on the interval so we can use a fixpoint iteration. Choose 3 for the iteration since it is in the interval.

$$
\begin{aligned}
\pi &= f(f(f(\dots(3)\dots))) \\
\pi &= \sin 3 + 3 = 3.14112 \\
\pi &= \sin 3.14112 + 3.14112 = 3.14159264 \\
\pi &= \sin 3.14159264 + 3.14159264 = 3.141593654\dots
\end{aligned}
$$

Thus we have an approximation of $\pi$ with high accuracy (higher then the float points allowed on my calculator). $\square$