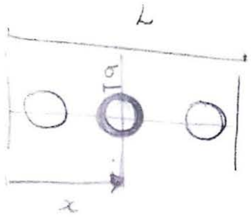
Homework 5

- 

$Z = 1$ if the configuration is possible, 0 otherwise.
If we let the middle particle be at position x
Then the first particle behaves as if it where in a box of size $x-\sigma$, and the third as if where in box of size $L-x-\sigma$.

Then
$$Z^{\text{indist}}_{3,L} = \int_{3\sigma}^{L-3\sigma} dx \; Z_{1,x-\sigma} \, Z_{1,L-x-\sigma}$$

- $$Z = \int_{3\sigma}^{L-3\sigma} dx \, (x-\sigma-2\sigma)(L-x-\sigma-2\sigma)$$

$$= -\int_{3\sigma}^{L-3\sigma} dx \, (x-3\sigma)(x-L-3\sigma)$$

the change of variable $y = x-3\sigma$ yields

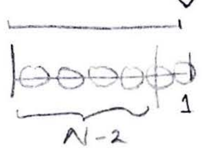
$$= -\int_0^{L-6\sigma} dy \, y(y-L) = -\frac{y^3}{3}\Big|_0^{(L-6\sigma)} + L\frac{y^2}{2}\Big|_0^{(L-6\sigma)}$$

$$= (L-6\sigma)^3\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{6}(L-6\sigma)^3$$

$$\boxed{Z^{\text{ind}}_{3,L} = \frac{1}{6}(L-6\sigma)^3}$$

(if the particles where distinguishable we would have to multiply by $3! = 6$)

- For arbitrarly N



$$Z_{N,L} = \int_{2(N-2)\sigma+\sigma}^{L-3\sigma} dx \; Z_{N-2,x-\sigma} \, Z_{1,L-x-\sigma}$$

$$\boxed{Z_{N,L} = \int_{2N\sigma-3\sigma}^{L-3\sigma} dx \; Z_{N-2,x-\sigma} \, Z_{1,L-x-\sigma}}$$

Now we can prove $Z^{\text{ind}}_N = \frac{1}{N!}(L-2N\sigma)^N$ by induction on N

$$Z_{N,L} = \int_{2N\sigma-3\sigma}^{L-3\sigma} \frac{1}{(N-2)!}\left(x-\sigma-2(N-2)\sigma\right)^{N-2}\left(L-x-\sigma-2\sigma\right)$$

$$= -\int_{2N\sigma-3\sigma}^{L-3\sigma} \frac{1}{(N-2)!}\left(x-(2N\sigma-3\sigma)\right)^{N-2}\left(x-(L-3\sigma)\right)$$

The change of variables $y = x-(2N\sigma-3\sigma)$ yields

$$Z_{N,L} = -\int_0^{L-2N\sigma} dy\, \frac{1}{(N-2)!}\, y^{N-2}\left(y+2N\sigma-L\right)$$

$$= \overset{0}{\overbrace{-\frac{1}{(N-2)!}\,\frac{y^{N-1}}{(N-1)}\left(y+2N\sigma-L\right)\Big|_0^{L-2N\sigma}}} + \int_0^{L-2N\sigma} dy\, \frac{1}{(N-1)!}\, y^{N-1}$$

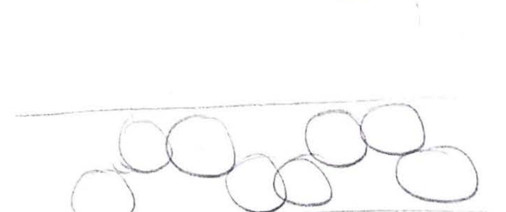
$$\boxed{Z^{ind}_{N,L} = \frac{1}{N!}\left(L-2N\sigma\right)^{N}}$$

II Density profile of the one dimensional Hard-Sphere model.

- $$\pi(x) = \frac{1}{Z_{N,L}} \sum_k \overbrace{\binom{N-1}{k}}^{\text{combinatory factor accounting to picking the sphere at } x} \underbrace{Z_{k,x-\sigma}}_{\text{partition function of the balls before the one at } x} \underbrace{Z_{N-1-k,L-x-\sigma}}_{\text{partition function of the balls after the one at } x}$$

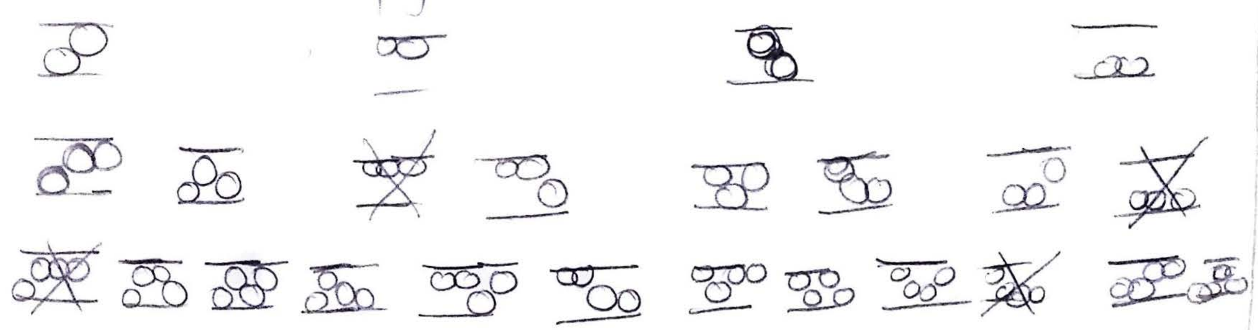
III Quasi one-dimensional Jammed disks and their transfer matrix

The longest is



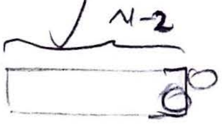
The shortest is

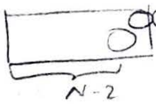
Let J_N the number of configurations of jammed disks
Then for the first couple of n's



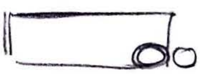
n	Configurations	J_n
2		4
3		6
4		10
5	...	16

- The general idea is to consider the last two balls

 $\rightsquigarrow$ All are valid $\frac{1}{2} J_{N-1}$ (all J_{N-1} that finish down)

 $\rightarrow$ Just valid if previous is down  $\Rightarrow \frac{1}{2} J_{N-2}$

 $\longrightarrow \frac{1}{2} J_{N-1}$

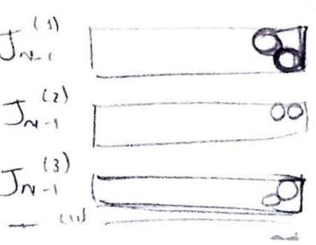
 $\longrightarrow \frac{1}{2} J_{N-2}$

- Then $J_N = \frac{1}{2} J_{N-1} + \frac{1}{2} J_{N-2} + \frac{1}{2} J_{N-1} + \frac{1}{2} J_{N-2}$

$$\boxed{J_N = J_{N-1} + J_{N-2}}$$

And the first ones are given by $\boxed{J_1 = 2 , J_2 = 4}$

- For the transfer matrix we let $J_N^{(i)}$ $\rightarrow i = 1,2,3,4$ the # of Jammed configuration of N disk which end up in the configuration of two balls 1,2,3,4



N ball

$\uparrow \longrightarrow J_{N-1}^{(1)}$ added to $J_N^{(3)}$
$\downarrow \longrightarrow J_{N-1}^{(1)}$ added to $J_N^{(4)}$

$\times$
$\downarrow \longrightarrow J_{N-1}^{(2)}$ added to $J_N^{(1)}$

$\uparrow \longrightarrow J_{N-1}^{(3)}$ added to $J_N^{(2)}$
$\downarrow \longrightarrow J_{N-1}^{(3)}$ added to $J_N^{(1)}$
$\uparrow \longrightarrow J_{N-1}^{(4)}$ added to $J_N^{(3)}$

$$J_N^{(1)} = J_{N-1}^{(2)} + J_{N-1}^{(3)}$$
$$J_N^{(2)} = J_{N-1}^{(3)}$$
$$J_N^{(3)} = J_{N-1}^{(4)} + J_{N-1}^{(1)}$$
$$J_N^{(4)} = J_{N-1}^{(1)}$$