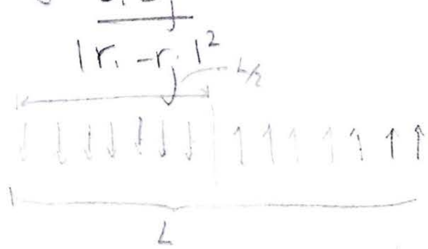
1/6 stat-phys

I An integral in the 1D ising model with long range interactions.

- $E(i,j) = -J \dfrac{\sigma_i \sigma_j}{|r_i - r_j|^2}$



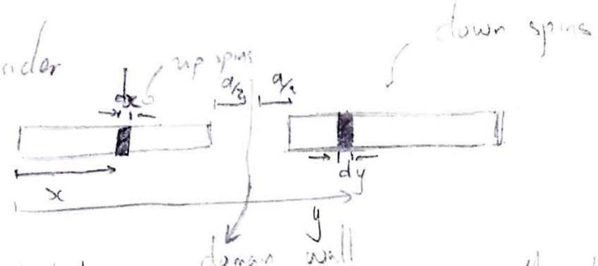
As flipping all the spins in one side (to get to the ground state) just changes the energy between pairs which one spin at each side

$$\Delta E = 2J \sum_{i=1}^{N/2} \sum_{j=N/2+1}^{N} \frac{1}{(r_i - r_j)^2}$$

where $r_j = (j-1)a$

We want to change this into an integral, but we notice that if we let $a \to 0$ $r_{N/2} - r_{N/2+1} \to 0$ Hence the formula diverges. Then we need to retain the microscopic parameter a.

Then we consider



The contribution to the sum of the two dark regions is

$$\frac{\frac{dx}{a}\frac{dy}{a}}{(x-y)^2}$$

~ One can think of $\frac{dx}{a}$ (or $\frac{dy}{a}$) as the ammount $d\sigma$ of spin in the length dx (if $dx = a$ $\frac{dx}{a} = 1$ hence there is just one spin)

$$\Rightarrow \boxed{\Delta E = \frac{2J}{a^2} \int_0^{L/2 - a/2} dx \int_{L/2 + a/2}^{L} dy \, \frac{1}{(x-y)^2}}$$

- $$\int_{L/2 + a/2}^{L} dy \int_0^{L/2 - a/2} dx \, \frac{1}{(x-y)^2} = \Big|_{L/2 + a/2}^{L} -\frac{1}{y} - \frac{1}{L/2 - a/2 - y}$$

$$= \log\left(\frac{L/2 + a/2}{L}\right) + \log\left(\frac{L - L/2 + a/2}{L/2 + a/2 - L/2 + a/2}\right)$$

$$= \log\left(\frac{1}{2}+\frac{a}{2L}\right) + \log\left(\frac{L}{2a}+\frac{1}{2}\right)$$

$$= \log\left(\left(\frac{1}{2}+\frac{a}{2L}\right)\left(\frac{1}{2}+\frac{L}{2a}\right)\right) = \log\left(\frac{1}{4}+\frac{L}{4a}+\frac{a}{4L}+\frac{1}{4}\right)$$

$$= \log\left(\frac{L}{4a}+\frac{a}{4L}+\frac{1}{2}\right) = \log\left(\frac{L}{4a}\left(1+2\frac{a}{L}+\left(\frac{a}{L}\right)^2\right)\right)$$

$$= \log\left(\frac{L}{4a}\left(1+a/L\right)^2\right)$$

Then

$$\Delta E = \frac{2J}{a^2}\log\left(\frac{L}{4a}(1+a/L)^2\right) = J'\log\left(\frac{L}{4a}(1+a/L)^2\right)$$

Recalling $\frac{L}{a} = N$ we get

$$\Delta E = J'\log\left(\frac{N}{4}(1+1/N)\right) \xrightarrow[N\to\infty]{} J'\log N$$

- If we put a domain wall at an arbitrary position νL over the chain $\left(\frac{a}{2} \le \nu L \le L - a/2\right)$ the energy will be

$$\Delta E = \frac{2J}{a^2}\int_0^{\nu L - a/2} dx \int_{\nu L + a/2}^{L} dy \frac{1}{(x-y)^2}$$ by the same argument

as before

$$\Delta E = J'\int_0^{\nu - 1/2N} d\tilde{x} \int_{\nu + 1/2N}^{1} d\tilde{y} \frac{1}{(\tilde{x}-\tilde{y})^2}$$ where $\tilde{x} = \frac{x}{L}$, $\tilde{y} = \frac{y}{L}$

$$= J'\int_{\nu+1/2N}^{1} -\frac{1}{\tilde{y}} + \frac{1}{\tilde{y}-\nu+1/2N} = J'\left(\ln\left(\nu+\frac{1}{2N}\right) + \ln\left(\frac{1-\nu-1/2N}{1/N}\right)\right)$$

$$\Delta E = J'\left(\ln\left(\nu+\frac{1}{2N}\right) + \ln N + \ln\left(1-\nu-\frac{1}{2N}\right)\right)$$

$$\boxed{\Delta E \longrightarrow J'\ln N}$$

And as there are N possible sites to put the domain wall

$$S = k\ln N$$

And the free energy is $J'\ln N - TK\ln N$

low temperatures then adding a domain wall increases the free energy so the system remains with no domain wall. At higher temperatures $(TK > J')$ the increase in energy is compensated by the increase in entropy hence there appear domain walls. We have then a critical temperature $T_c = \frac{J'}{K}$ where a phase transition occurs.

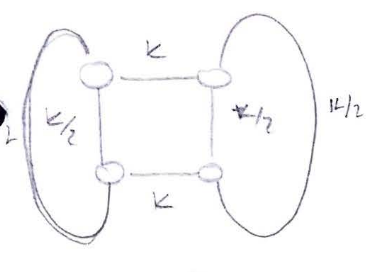
- $$E(i,j) = -\frac{J\sigma_i\sigma_j}{|r_i - r_j|^{2+\epsilon}}$$

If $\epsilon > 0$ the energy won't grow as fast. Hence there are domain walls at every positive temperature (No phase transition except possibly at $T=0$.

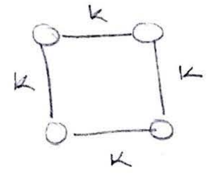
If $\epsilon < 0$ The energy will grow faster than the entropy and we will have always ordered state as the gain in energy will be greater than the change in entropy.

II Partition function 2x2 ising model periodic boundary conditions.

- We have to add an interaction between the up and low rows

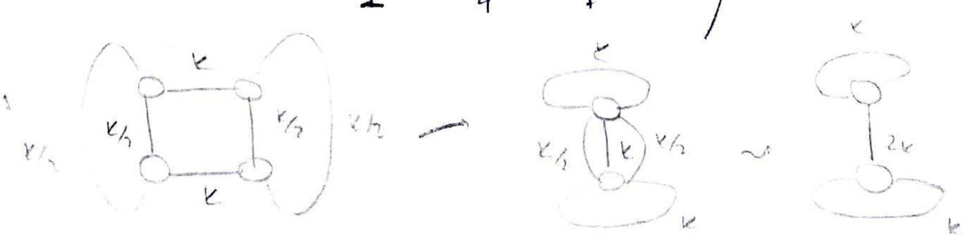


which is equivalent to setting the interaction between both of them to K



$$T = \begin{pmatrix} e^{4K} & 1 & 1 & 1 \\ 1 & 1 & e^{-4K} & 1 \\ 1 & e^{-4K} & 1 & 1 \\ 1 & 1 & 1 & e^{4K} \end{pmatrix}$$

- Now for $M=1$

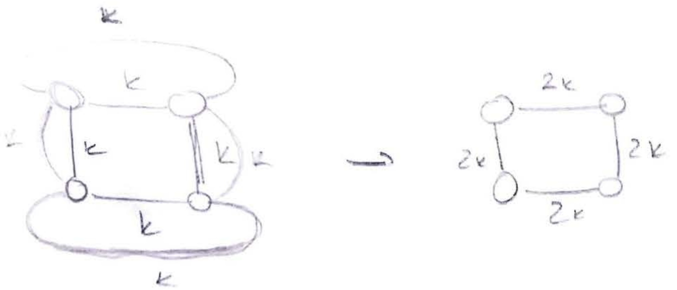


$\genfrac{}{}{0pt}{}{+}{+} \rightarrow e^{4K}$ $\quad \genfrac{}{}{0pt}{}{-}{-} \rightarrow e^{4K}$ $\Rightarrow$ $\boxed{Z = \mathrm{Tr}\, T = 2e^{4K} + 2}$

$\genfrac{}{}{0pt}{}{+}{-} \rightarrow 1$

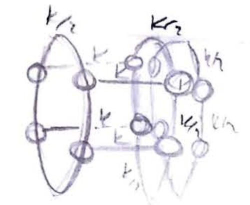
$\genfrac{}{}{0pt}{}{-}{+} \rightarrow 1$

$M = 2$

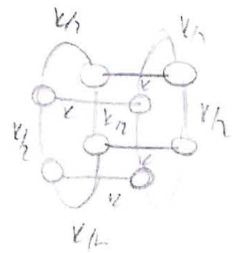


$\begin{smallmatrix}+&+\\+&+\end{smallmatrix}$	$\begin{smallmatrix}+&+\\+&-\end{smallmatrix}$	$\begin{smallmatrix}+&-\\+&+\end{smallmatrix}$	$\begin{smallmatrix}+&-\\+&-\end{smallmatrix}$	$\begin{smallmatrix}+&+\\-&+\end{smallmatrix}$	$\begin{smallmatrix}+&+\\-&-\end{smallmatrix}$	$\begin{smallmatrix}+&-\\-&+\end{smallmatrix}$	$\begin{smallmatrix}+&-\\-&-\end{smallmatrix}$
e^{8k}	1	1	1	1	1	e^{-8k}	1
$\begin{smallmatrix}-&+\\+&+\end{smallmatrix}$	$\begin{smallmatrix}-&+\\+&-\end{smallmatrix}$	$\begin{smallmatrix}-&-\\+&+\end{smallmatrix}$	$\begin{smallmatrix}-&-\\+&-\end{smallmatrix}$	$\begin{smallmatrix}-&+\\-&+\end{smallmatrix}$	$\begin{smallmatrix}-&+\\-&-\end{smallmatrix}$	$\begin{smallmatrix}-&-\\-&+\end{smallmatrix}$	$\begin{smallmatrix}-&-\\-&-\end{smallmatrix}$
1	e^{-8k}	1	1	1	1	1	e^{8k}

$$\boxed{Z = 12 + 2e^{8k} + 2e^{-8k} = \mathrm{Tr}(T^2)}$$



The fundamental unit is



The transfer matrix would then be $2^4 \times 2^4$ and for 4×4 we can just compute

$\mathrm{Tr}(T) \sim 4\times1$ $\quad \mathrm{Tr}(T^2) \sim 4\times2$ $\quad \mathrm{Tr}(T^3) \sim 4\times3$

$$\boxed{\mathrm{Tr}(T^4) \sim 4\times4}$$

We then have $Z = \mathrm{Tr}(T^m)$

We then can simply find $\lambda_1 \geq \dots \geq \lambda_{16}$ the eigenvalues of T and then

$$Z = \lambda_1^m \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^m + \dots + \left(\frac{\lambda_{16}}{\lambda_1}\right)^m\right) \xrightarrow{m \to \infty} \lambda_1^m$$

$$F = -\frac{1}{\beta} \ln Z = -\frac{m}{\beta} \ln \lambda_1 - \frac{1}{\beta} \ln\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^m + \dots + \left(\frac{\lambda_{16}}{\lambda_1}\right)^m\right)$$

$$\boxed{F \xrightarrow{m \to \infty} -\frac{m}{\beta} \ln \lambda_1}$$