Homework 7 stat phys

I Permutation cycles and determinants

A. Preparation

Notice that for $\sigma = (i_1 \dots i_k) \in S_{2n}$

$$\sigma = (i_1 i_2)(i_2 i_3) \dots (i_{k-1} i_k)$$

$$\Rightarrow \quad \operatorname{sign}\sigma = (-1)^{k-1} = (-1)^{1-k} = (-1)^{2n+1-k}$$

$$= (-1)^{\underbrace{(2n-k)}_{\text{cycles of length } 1} + 1} = (-1)^{\#\text{ of cycles}}$$

- As sign is multiplicative we conclude that for $\sigma = c_1 \dots c_k \in S_{2n}$ with c_i a cycle (including 1 cycles)

$$\operatorname{sign}\sigma = (-1)^{l(c_1)-1} \cdot \dots \cdot (-1)^{l(c_k)-1}$$

where $l(c_i)$ stands for the length of c_i $\qquad \sum_i l(c_i) = 2n$

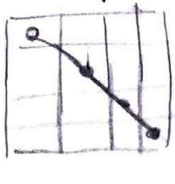
$$\Rightarrow \quad \operatorname{sign}\sigma = (-1)^{-k} = (-1)^k$$

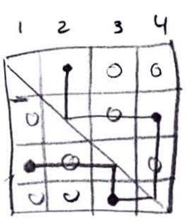
This indeed proves formula (3) where weight of i-th cycle stands for $u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_{k-1} i_k}$ for $c_i = i_1 \dots i_k$

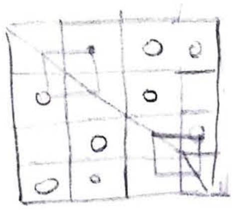
- $$\det \tilde{u}_{2\times 2} = \begin{pmatrix} 1 & \gamma \tanh\beta & \cdot & 0 \\ \cdot & 1 & \cdot & \gamma \tanh\beta \\ \gamma \tanh\beta & \cdot & 1 & \cdot \\ \cdot & 0 & \gamma \tanh\beta & \end{pmatrix} \qquad (1 2 4 3)$$

$$= \underbrace{1}_{(-1)^4} - \underbrace{\gamma^4 \tanh^4\beta}_{(1 2 4 3)} \quad \text{— only non vanishing}$$

trivial permutation







$= 0$

$(1 2)(3 4)$

$$\det \mathcal{U}_{2\times 2} = 1 - \gamma^{4} \tanh^4\beta = 1 + \tanh^4\beta$$

Now in the 2×2 Ising model

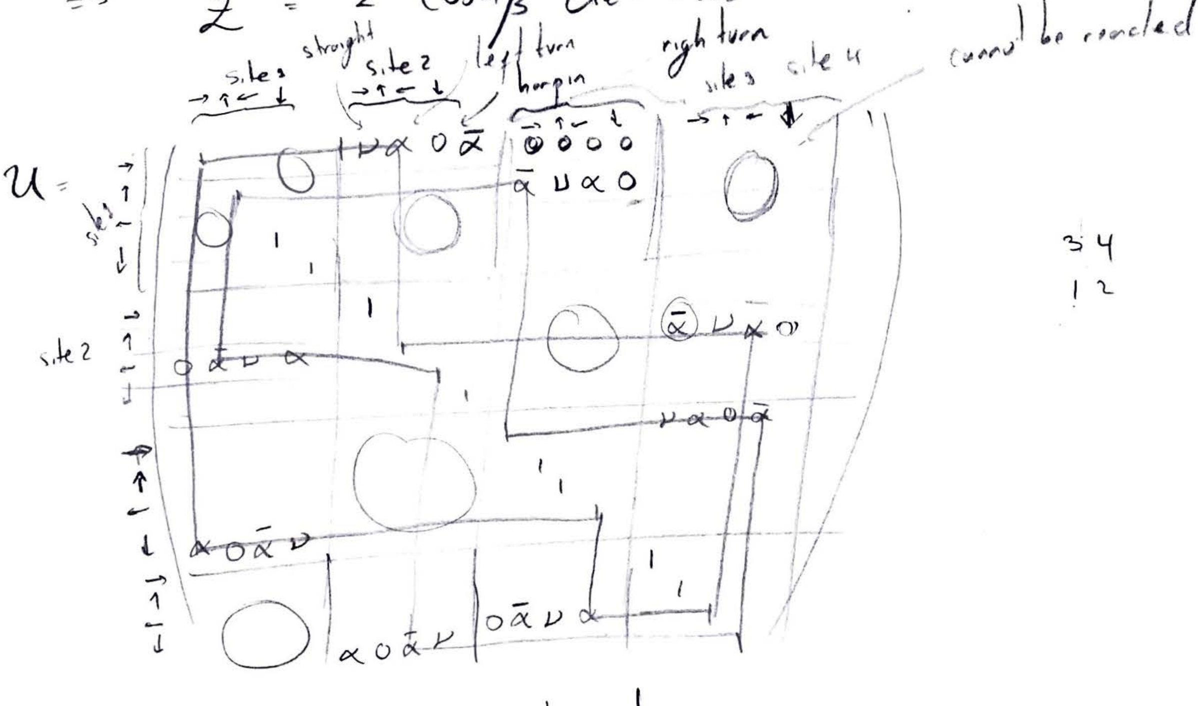
pairs $\qquad\qquad v = \tanh\beta$

$$Z = \sum_{\{\sigma\}} \prod_{\langle ij\rangle} \underbrace{e^{\beta\sigma_i\sigma_j}}_{\cosh\beta + \sigma_i\sigma_j \sinh\beta J} = \sum_{\{\sigma\}} (\cosh\beta J)^4 \prod_{\langle ij\rangle} (1 + \sigma_i\sigma_j v)$$

$\square$ only closed path

$$Z = (\cosh\beta J)^4\, 2^4 (1 + v^4)$$

$$\Rightarrow \quad Z = 2^4 \cosh^4\beta \,\det \mathcal{U}_{2\times 2}$$



- All the matrix is explained
- The non trivial permutation are given by the two paths

$$\det \mathcal{U} = \underset{\text{Trivial path}}{1} - \underset{1^{st}\text{ path}}{\alpha^4} - \underset{2^{nd}\text{ path}}{\bar\alpha^4} + \underbrace{\alpha^4\bar\alpha^4}_{\text{both}} = 1 + v^4 + v^4 + v^8$$

$$\boxed{\det \mathcal{U} = (1 + v^4)^2} \quad \Rightarrow \quad (\det \hat{\mathcal{U}})^2 = \det \mathcal{U}$$

The matrix can be written as

$$\boxed{\mathcal{U}_{2\times 2} = \begin{pmatrix} 1 & \mathcal{U}_{\rightarrow} & \mathcal{U}_{\uparrow} & \cdot \\ \mathcal{U}_{\leftarrow} & 1 & \cdot & \mathcal{U}_{\uparrow} \\ \mathcal{U}_{\downarrow} & \cdot & 1 & \mathcal{U}_{\rightarrow} \\ \cdot & \mathcal{U}_{\downarrow} & \mathcal{U}_{\leftarrow} & 1 \end{pmatrix}}$$

Kac-Ward Matrix 4x4 Ising

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

all empty slots are 0, and to simplify notation will write → ↑ ↓ ← instead of $\mathcal{U}_{\to}, \mathcal{U}_{\uparrow}, \mathcal{U}_{\downarrow}, \mathcal{U}_{\leftarrow}$

$\mathcal{U}_{4\times4} =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	→			↑											
2	←	1	→			↑										
3		←	1	→			↑									
4			←	1				↑								
5	↓				1	→			↑							
6		↓			←	1	→			↑						
7			↓			←	1	→			↑					
8				↓			←	1				↑				
9					↓				1	→			↑			
10						↓			←	1	→			↑		
11							↓			←	1	→			↑	
12								↓			←	1				↑
13									↓				1	→		
14										↓			←	1	→	
15											↓			←	1	→
16												↓			←	1

- The program just compute the determinant of $\mathcal{U}_{2\times2}$
- As explained before $\det \mathcal{U} = (\det \hat{\mathcal{U}})^2$

Then we need a square root.

- $$\det \mathcal{U}_{4\times4} = \Big(1 + \underbrace{9}_{\text{4-edges loops}}\nu^4 + \underbrace{12}_{\text{6-edges loops}}\nu^6 + \underbrace{50}_{\text{8 edges loops}}\nu^8 + \overbrace{92}^{\text{10-edges loops}}\nu^{10} + \underbrace{158}_{\text{12-edges loops}}\nu^{12} + \underbrace{116}_{\text{14-edges loops}}\nu^{14} + \overbrace{69}^{\text{16 edges loops}}\nu^{16} + \underbrace{4}_{\text{18-edges loops}}\nu^{18} + \overbrace{\nu^{20}}^{\text{20 edges loop}}\Big)^2$$

Then indeed

$$Z = 2^{16} \cosh\beta^{16} \sqrt{\det \mathcal{U}_{4\times4}}$$