HW 13 - stat-phys

I Density of states

$$N(E) = \sum_{E_x=0}^{E}\sum_{E_y=0}^{E-E_x} 1 = \sum_{E_x=0}^{E} E - E_x + 1 = (E+1)^2 - \sum_{E_x=0}^{E} E_x$$

$$= (E+1)^2 - \frac{1}{2}E(E+1) = \boxed{\frac{(E+1)(E+2)}{2} = N(E)}$$

Using the integral

$$N(E) = \sum_{E_x, E_y, E_z=0}^{E} \delta_{E_x+E_y+E_z-E,0} = \sum_{E_x,E_y,E_z=0} \int \frac{d\lambda}{2\pi} e^{-iE\lambda} e^{i(E_x+E_y+E_z)}$$

$$= \sum_{E_x,E_y,E_z=0}^{E} \int \frac{dz}{2\pi i} \frac{1}{z^{E+1}} z^{E_x} z^{E_y} z^{E_z} = \int \frac{dz}{2\pi i}\frac{1}{z^{E+1}}\left(\frac{1-z^{E+1}}{1-z}\right)^3$$

$z = e^{i\lambda}$

$$= \operatorname{Res}_{z=0}\left(\frac{1}{z^{E+1}}\left(\frac{1-z^{E+1}}{1-z}\right)^3\right)$$

and noting $\left(\frac{1}{1-z}\right)^3 = \frac{1}{2}\left(1\times 2 + 2\times 3 z + 3\times 4 z^2 + 4\times 5 z^3 + \dots\right)$

we conclude

$$\boxed{N(E) = \frac{1}{2}(E+1)(E+2)}$$

III Integration:

$$Z_N = \sum_{\{o\}} e^{-\beta E(\{o\})} = \sum_{E} e^{-\beta E} N(E)$$

$$\int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{-iN\lambda} \prod_i e^{in_i\lambda}$$

$$= \sum_{\{n_i\}} e^{-\beta(n_0E_0 + n_1E_1 + \dots + n_iE_i + \dots)} \overbrace{\delta_{n_0+n_1+\dots+n_i+\dots, N}}$$

$$= \sum_{\{n_i\}} \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{-iN\lambda} e^{n_0(-\beta E_0 + i\lambda)} e^{n_1(-\beta E_1 + i\lambda)} \dots$$

$$Z = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{-i\lambda N} \prod_i \frac{1}{1 - e^{-\beta E_i + i\lambda}} = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{-i\lambda N} \prod_E \underbrace{\left(\frac{1}{1 - e^{-\beta E + i\lambda}} \right)}_{f_E(\beta,\lambda)}$$

$$\boxed{Z = \int_{-\pi+i\epsilon}^{\pi+i\epsilon} \frac{d\lambda}{2\pi} e^{-i\lambda N} \prod_E f_E(\beta,\lambda)^{N(E)}}$$

For $E \neq 0$ there is no pole and moving the integration up a little changes nothing. For $E = 0$ we realize we should have written

$$Z = \sum_{n_0} \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{-i\lambda N} e^{i\lambda n_0} \prod_{E>0} f_E(\beta,\lambda)^{N(E)}$$

for higher n_0 gives 0

$$= \sum_{n_0=0}^{N} \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} \frac{1}{e^{i\lambda(N-n_0)}} \prod_{E>0} f_E(\beta,\lambda)^{N(E)}$$

$$= \int_{S^1} \frac{dz}{2\pi i z} \frac{1}{z^N} \left(\frac{1 - z^{N+1}}{1 - z} \right) \prod_{E>0} \left(\frac{1}{1 - e^{-\beta E} z} \right)^{N(E)} \qquad z = e^{i\lambda}$$

$$= \operatorname*{Res}_{z=0} \left(\frac{1}{z^{N+1}} \left(\frac{1 - z^{N+1}}{1 - z} \right) \prod_{E>0} \left(\frac{1}{1 - e^{-\beta E} z} \right)^{N(E)} \right)$$

$$= \operatorname*{Res}_{z=0} \left(\frac{1}{z^{N+1}} \frac{1}{1 - z} \prod_{E>0} \left(\frac{1}{1 - e^{-\beta E} z} \right)^{N(E)} \right) + \cancel{\operatorname*{Res}_{z=0} \left(- \frac{1}{1 - z} \prod_{E>0} \left(\frac{1}{1 - e^{-\beta E} z} \right)^{N(E)} \right)}$$

$$= \int \frac{dz}{2\pi i} \frac{1}{z^{N+1}} \frac{1}{1 - z} \prod_{E>0} \left(\frac{1}{1 - e^{-\beta E} z} \right)^{N(E)}$$

circle of radius $e^{-\epsilon}$

any number smaller of than ϵ

$$z = e^{-\epsilon} e^{i\theta} \qquad \lambda = \theta + i\epsilon$$

$$dz = e^{-\epsilon} i e^{i\theta} d\theta$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-iN\lambda} \frac{1}{1 - e^{i\lambda}} \prod_{E>0} f(\epsilon,\beta)^{N(E)}$$

$$\boxed{Z = \int_{-\pi+i\epsilon}^{\pi+i\epsilon} \frac{d\lambda}{2\pi} e^{-iN\lambda} \frac{1}{1 - e^{i\lambda}} \prod_{E>0} f_E(\lambda,\beta)^{N(E)}}$$

Surprisingly enough the formula for f_0 is $\frac{1}{1 - e^{i\lambda}}$

Then

$$\boxed{Z_N(\beta) = \int_{-\pi+i\epsilon}^{\pi+i\epsilon} \frac{d\lambda}{2\pi} e^{-iN\lambda} \prod_{E\geq 0} f_E(\lambda,\beta)^{N(E)}}$$

A Explicit integration

B Experiments with the partition function integration

The value of μ is given by

- $$\frac{1}{i}\frac{\partial}{\partial\lambda}\left\{-iN\lambda - \overset{\text{missing }(}{\sum_E} N(E)\log\left(1-e^{-\beta E+i\lambda}\right)\right\} = 0$$

$$-N + \sum_E N(E)\frac{e^{-\beta E+i\lambda}}{1-e^{-\beta E+i\lambda}} = 0$$

$$\boxed{N = \sum_E N(E)\frac{1}{e^{-\beta E+i\lambda}-1}}$$ writing $\lambda = -i\beta\mu$ $(\mu<0)$

- $$\boxed{N = \sum_E N(E)\frac{1}{e^{-\beta(E-\mu)}-1}}$$

A plot for the variation of the integration function when changing $\text{Im}\,\lambda$ is given