Homework 2

I. As $p(\bar{x}|\mu,\sigma^2)$ is positive for every $(\bar{x},\mu,\sigma^2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{>0}$ we can maximize $l(\mu,\sigma^2|\bar{x}) = \log p(\bar{x}|\mu,\sigma^2)$

$$l(\mu,\sigma|\bar{x}) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_i (x_i-\mu)$$

$$\left.\frac{\partial L}{\partial \mu}\right|_{\substack{\mu=\hat{\mu}\\ \sigma=\hat{\sigma}}} = 0 \Rightarrow -\frac{1}{\sigma^2}\sum_i (x_i-\hat{\mu}) = 0 \Rightarrow \hat{\mu} = \frac{1}{n}\sum_i x_i \Rightarrow \boxed{\hat{\mu} = \bar{x}_n}$$

$$\left.\frac{\partial L}{\partial \sigma}\right|_{\substack{\mu=\hat{\mu}\\ \sigma=\hat{\sigma}}} = 0 \Rightarrow -\frac{n}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3}\sum_i (x_i-\hat{\mu})^2 \Rightarrow \boxed{\hat{\sigma}^2 = \frac{1}{n}\sum_i (x_i - \bar{x}_n)^2}$$

II. • $X \sim \mathcal{U}([0,\theta]) \Rightarrow p(x|\theta) = \begin{cases} 1/\theta & x \in [0,\theta] \\ 0 & \text{otherwise} \end{cases}$

Now $\mathcal{L}(\theta|\bar{x}) = \prod_i p(x_i|\theta) = \frac{1}{\theta^n}$ if $\forall i\ x_i \in [0,\theta]$ otherwise $\mathcal{L}(\theta|\bar{x}) = 0$

i.e $\boxed{\mathcal{L}(\theta|\bar{x}) = \begin{cases} \frac{1}{\theta^n} & \theta \geq \max_i\{x_i\} \geq 0 \\ 0 & \text{otherwise} \end{cases}}$

• The function $\theta \longmapsto \frac{1}{\theta^n}$ is a strictly decreasing function (for $\theta > 0$). Then the maximum of $\mathcal{L}(\theta|\bar{x})$ is achieved at $\boxed{\hat{\theta} = \max_i\{x_i\}}$

• This is quite strange because $\hat{\theta} = x_i$ for some i. And we would expect that in order to get a measurement x_i we would need to have $\theta > x_i$.

Also, as $\hat{\theta} = x_i$ for some i, The MLE is the smallest posible value of θ compatible with the measurement.

III A For $m < k$ $P_N(m) = 0$

Now the number of ways to pick k balls whose maximum is m is equal to the number of ways to pick $(k-1)$ balls smaller than m, ie $\binom{m-1}{k-1}$. And the total way to pick k balls from the thank is $\binom{N}{k}$. Then

$$\boxed{P_N(m) = \frac{\binom{m-1}{k-1}}{\binom{N}{k}}}$$

B Now $k=4 \Rightarrow P_N(14) \propto \frac{(N-4)!}{N!}$ (For $N \geq 14$)

As it turns out this function is decreasing in N

Hence, the MLE is $\boxed{N=14}$ then there apply the same considerations as in $\boxed{\text{II A}}$

IV German Tank Problem.

There is no disernable difference between A and B.
Hence A is much better as it is computationally less expensive.