

Homework 6, Statistical Mechanics: Concepts and applications

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In lectures 05 and 06 we treated the question of spontaneous symmetry breaking in the 2d Ising models, following Peierls (1936), but also in the one-dimensional Ising model with $1/r^2$ interactions, following D. J. Thouless' (1969) in the paper that was the beginning of the Kosterlitz-Thouless story leading up to the 2016 Nobel prize (see Kosterlitz 2016 for a partly historical yet mostly scientific account). We then moved on to discuss the transfer-matrix solution of the two-dimensional Ising model, following Onsager (1944) and Schultz et al (1964), approximately halfway through the papers.

I. AN INTEGRAL IN THE ONE-DIMENSIONAL ISING MODEL WITH LONG-RANGE INTERACTIONS

The Ising model with long-range interaction is defined by the energy

$$E(i, j) = -J \frac{\sigma_i \sigma_j}{|r_i - r_j|^2}, \quad (1)$$

where the $\sigma = \pm 1$ are Ising spins. Suppose that the spins are on a line of length L , and that they are separated by a lattice spacing $a \ll L$. Suppose that there is a domain wall at position $L/2$ (with nearest lattice sites at $L/2 - a/2$ and at $L/2 + a/2$). All the spins left of the interface are equal to -1 and all the spins to the right are equal to $+1$. The excitation energy of the interface is given by:

$$E = J' \int_0^{L/2-a/2} \int_{L/2+a/2}^L \frac{dxdy}{(x-y)^2} \quad (2)$$

- Justify eq. (2).... Why is this a good formula, and why do we install a microscopic lengthscale a
- Actually compute this integral.
- Explain, by taking into account the entropy of the domain walls, why the one-dimensional Ising model with an energy function as in eq. (2) can be expected to have a phase transition at a finite temperature.

- What would you expect to be the phase behavior of the Ising model with interaction

$$E(i, j) = -J' \frac{\sigma_i \sigma_j}{|r_i - r_j|^{2+\epsilon}} \quad (3)$$

with $\epsilon \pm 0$?

NB: The Thouless paper (Phys Rev 187, 732 (1969)) is only two pages long, but it contains results that are stronger than those by illustrious authors Dyson, Anderson and Ruelle, from the same period.

II. PARTITION FUNCTION OF THE 2×2 ISING MODEL WITH PERIODIC BOUNDARY CONDITIONS

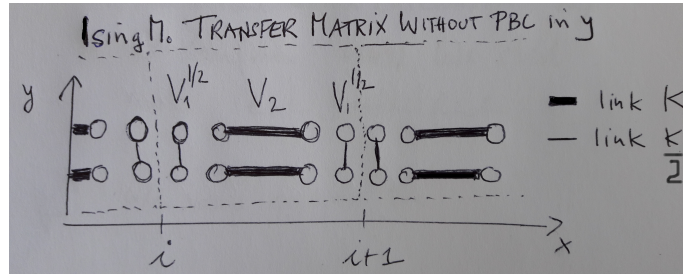


FIG. 1: Sketch of the transfer matrix for the $2 \times M$ Ising model without periodic boundary conditions in y . The matrices V_1 and V_2 can be found in the mathematica notebook file.

In the second part of lecture 5, we discussed the Ising model on a stripe of width 2, without periodic boundary conditions in y . It is given by

$$T = V_1^{1/2} V_2 V_1^{1/2} \quad (4)$$

(see Fig. 1). The precise values of V_1 and of V_2 were discussed in the lecture, but they can also be found in the mathematica notebook file on the webpage (where $V_1^{1/2}$ is called “V1sq”).

- In which way do we have to modify the transfer matrix if we introduce periodic boundary conditions in y ? (Note that periodic boundary conditions with two spins are somewhat artificial, as we then have two spins interact in two ways).
- Use this new transfer matrix to compute explicitly the partition function of the 2×2 Ising model with (artificial) periodic boundary conditions both in x and in y . Check your calculation with an (explicit) enumeration on paper of the 16 configurations of the 2×2 Ising model.

- Explain how you would obtain the partition function of the 4×4 Ising model with periodic boundary conditions, but just as a proof-of-concept without doing any actual computation. What is the dimension of the transfer matrix?
- Explain how you would obtain the free energy per particle of the $4 \times M$ Ising model with periodic boundary conditions, for any finite M and also in the limit $M \rightarrow \infty$, again without doing any detailed calculations.