Homework 1 stat mech

1 Some necesary conditions for the formula to make sense are for $\langle x \rangle$, $Var(x)$ to exists

$$P(|x - \langle x \rangle| \geq \epsilon) \leq \frac{Var(x)}{\epsilon^2}$$

It turns out that the proof just require this two conditions for both the continous and discrete case.

2 a) Let X such that equality holds

$$P(|X - \langle x \rangle| \geq \epsilon) = \frac{Var(X)}{\epsilon^2}$$

Then let $Y := X - \langle X \rangle$ and $\sigma^2 = Var(Y) = Var X$

$P(|Y| \geq \epsilon) = \frac{\sigma^2}{\epsilon^2}$, we can compute the RHS and LHS as follows

$$P(|Y| \geq \epsilon) = \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) dy\, \pi(y), \quad \frac{\sigma^2}{\epsilon^2} = \frac{1}{\epsilon^2}\int_{-\infty}^{\infty} dy\, y^2 \pi(y)$$

$$\Rightarrow \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) dy \left(\frac{y^2}{\epsilon^2} - 1\right)\pi(y) + \int_{-\epsilon}^{\epsilon} \frac{y^2}{\epsilon^2}\pi(y) = 0$$

As both terms are positive, they have both to be 0.
Then $\pi(y) = 0$ for all y st $|y| < \epsilon$ and $y \neq 0$
And $\pi(y) = 0$ for all y st $|y| > \epsilon$.

Hence Y is a discrete random variable whose values are $\{\epsilon, -\epsilon, 0\}$. To have 0 average ($\langle y \rangle = 0$) we must have

$P(Y=\epsilon) = P(Y=-\epsilon)$ Then we set $P(Y=\epsilon) = P(Y=-\epsilon) = \frac{1}{k}$
Then $P(Y=0) = 1 - \frac{1}{2k}$

We can then compute $\langle Y \rangle = \frac{1}{k}(\epsilon - \epsilon) = 0$ $\sigma^2 = \langle Y^2 \rangle = \frac{1}{k}(\epsilon^2 + \epsilon^2) = \frac{\epsilon^2}{2k}$

Then $P(|Y| \geq \epsilon) = \frac{1}{2K} = \frac{\epsilon^2}{2K} \frac{1}{\epsilon^2}$. Saturating Chebyshev's in

Then we have proven that every RV X that saturates Chebyshev's inequality has a distribution

$$\pi_X(x) = \frac{1}{K}\left(\delta(x-\epsilon-\mu) + \delta(x+\epsilon-\mu) + \delta(\mu)\right)$$

for some constants $\boxed{x, K, \mu}$

6 It has already be shown it is not possible.

Rényi's formula.

1 $\langle X \rangle = 0$, $\sigma^2 = \langle X^2 \rangle = \int_{-1}^{1} dx\, x^2 = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} \Rightarrow \boxed{\sigma^2 = \frac{2}{3}}$

2. The variance of the sum of independent random variables is the sum of the variances. Then $\boxed{\sigma_n^2 = \frac{2}{3}n}$

3.

4 Chebychev's inequality is way to big for this distribution. Is allways way to far to the right.

5 For small x The Cantelli inequality is more restrictive.

6 No, The graph shows that for some values chebyshev's inequality is actually better. The Hoeffding bound is

$$e^{-\epsilon^2/2n}$$

It can have this behaviour because it is particular to this distribution. Not a general statement as Chebyshev's inequality.

Also we recall chebyshev's inequality can be saturated at a single point. Not everywhere

Lévy distributions. two simple demonstration

1. $\pi_\xi(x) = \frac{\alpha}{x^{1+\alpha}}$, $x \geq 1$

$$\int_1^\infty dx \frac{\alpha}{x^{1+\alpha}} = - \frac{\alpha}{x^\alpha} \frac{1}{\alpha} \Big|_1^\infty = \begin{cases} 1 & \alpha > 0 \\ -\infty & \text{otherwise} \end{cases}$$

Hence the function is normalized in particular for $\alpha = 1.25$, $\alpha = 5$

2. $$\int_1^\infty dx \frac{x\, \alpha}{x^{1+\alpha}} = \int_1^\infty \frac{dx\, \alpha}{x^\alpha} = \frac{\alpha}{(1-\alpha)} \frac{1}{x^{\alpha-1}} \Big|_1^\infty = \begin{cases} \frac{\alpha}{\alpha - 1} & \alpha > 1 \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha = 1.25$ $\quad \langle x \rangle = \frac{1.25}{0.25} = 5$

for $\alpha = 5$ $\quad \langle x \rangle = \frac{5}{4} = 1.25$

3 For $\gamma = -0.2$ the histogram of $\bar{x}_n - \langle x \rangle$ is pretty sharp and symmetric around 0

4 But for $\gamma = -0.8$ the situation is pretty differen there are very few points above zero but very far and a lot of points below zero but very close