COURSE SUMMARY NOTES

METHODS UNIT 1

Contents

1 Trigonometry

Unit Circle

The Unit Circle is a circle with radius 1 unit on the Cartesian plane centred at the origin.

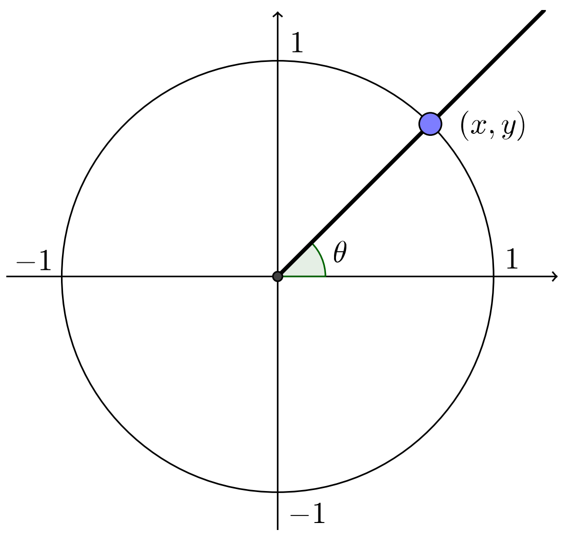


Figure 1.1: A ray is constructed from the origin with an angle measured from the positive x-axis anticlockwise. The point of intersection of this ray and the unit circle has coordinates (x, y).

Trigonometric Functions

We then define functions that relate the angle to the rectangular coordinate system as follows.

$$\sin\theta = y \tag{1.1}$$

$$\cos\theta = x \tag{1.2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{1.3}$$

Non-Right Angle Triangle Trigonometry

Note:

1. Be aware of the ambiguous case when solving for an angle in a triangle using sine.

$$\text{Area}_{\triangle} = \frac{1}{2}ab\sin C \tag{1.4}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{1.5}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{1.6}$$

Exact Values

Extracurricular:

1. Proving the following exact values can be done from geometric constructions on the Unit Circle.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Angles of Inclination of a Line

$$\tan\theta = m \tag{1.7}$$

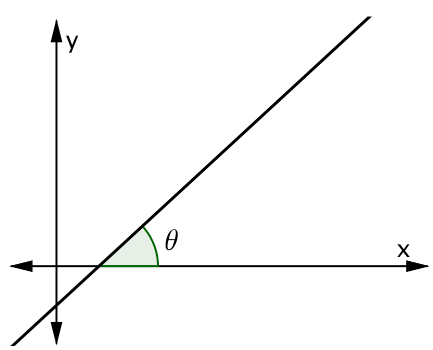


Figure 1.2: The angle of inclination of a line is the angle from the positive x-axis anti-clockwise to the line.

2 Radians

Definition

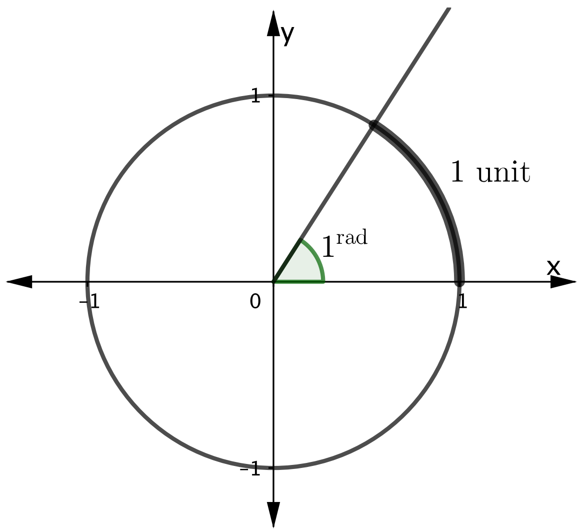


Figure 2.1: 1 radian is the angle subtended by an arc length of 1 unit on the unit circle.

Common Angles

$$\pi = 180^\circ$$

$$\frac{\pi}{6} = 30^\circ$$

$$\frac{\pi}{4} = 45^\circ$$

$$\frac{\pi}{3} = 60^\circ$$

$$2\pi = 360^\circ$$

Radian Conversion

$$\theta^{\mathrm{r}} = \frac{\pi \cdot \theta^{\circ}}{180} \tag{2.1}$$

$$\theta^{\circ} = \frac{180 \cdot \theta^{\mathrm{r}}}{\pi} \tag{2.2}$$

Note:

1. Most conversions are done considering the factions of 180° or π radians.

Arc Length, Area of Sector and Area of Segment

$$\text{Arc Length} = r\theta \tag{2.3}$$

$$\text{Area of Sector} = \frac{1}{2}r^2\theta \tag{2.4}$$

$$\text{Area of Segment} = \frac{1}{2}r^2\left(\theta - \sin\theta\right) \tag{2.5}$$

3 Function

Definition

A function is a special relationship where each input has a single output. It is often written as "$f(x)$" where x is the input value and f is the name of the function.

Note:

1. A graphical check that a relation is a function (i.e. not a many-to-many or one-to-many) is the vertical line test.

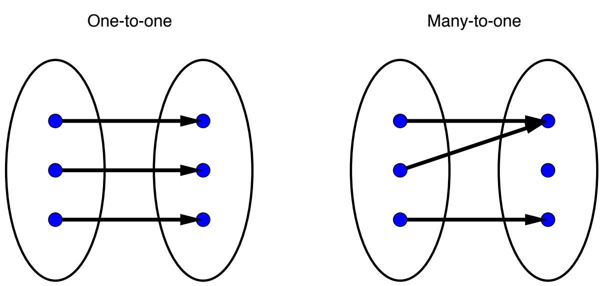


Figure 3.1: A function is a one-to-one or a many-to-one mapping of elements from the domain to the range.

Notation

$\in$ is an element of
$\mathbb{Z}$ is the set of Integers.
$\mathbb{Q}$ is the set of Rational Numbers.
$\mathbb{R}$ is the set of Real Numbers.
Domain$\{x \in \mathbb{R}\}$ e.g. with restriction Domain$\{x \in \mathbb{R} | x > 3\}$
Range$\{y \in \mathbb{R}\}$ e.g. with restriction Range$\{y \in \mathbb{R} | y \neq 0\}$

Natural Domain

The natural domain of a function is the maximum set of x values for which the function is defined.

Determining the domain and range of a function.

1. Graphical approach.

2. Algebraic approach.

4 Linear Functions

Forms of Linear Functions

1. General Form: $y = mx + c$
2. Gradient/Point Form: $y - y_1 = m(x - x_1)$
3. Intercept Form: $ax + by = c$

Gradient

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \tag{4.1}$$

Parallel and Perpendicular Lines

Parallel lines have equal gradients.

$$m_1 = m_2$$

Perpendicular lines have negative reciprocal gradients.

$$m_1 = -\frac{1}{m_2}$$

Midpoint

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{4.2}$$

Distance Between Points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{4.3}$$

Direct Proportion

$$y \propto x$$

$$y = kx$$

5 Quadratic Functions

Forms of Quadratic Equations

1. General Form: $y = ax^2 + bx + c$
   - y-intercept: $(0, c)$
   - Line of symmetry: $\frac{-b}{2a}$
2. x-intercept Form: $y = a(x-m)(x-n)$
   - x-intercepts: $(m, 0)$ and $(n, 0)$
3. Turning Point Form: $y = a(x-p)^2 + q$
   - Turning Point: (p, q)
   - Line of Symmetry $x = p$

Note:

1. Not all functions have x-intercepts and so not all quadratics can be written in factorised form.
2. Note the connection between turning point form and linear transformations of $f(x) = x^2 \longrightarrow af(bx+c)+d = a(bx+c)^2+d$.

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.1}$$

Note:

1. Prove the quadratic formula using completing the square.

Discriminant

$$\Delta = b^2 - 4ac \tag{5.2}$$

- if D a square number then quadratic is factorisable.
- if $D = 0$ then quadratic has 1 root.
- if D is a positive number then quadratic has 2 roots.
- if D is a negative number then quadratic has 0 roots.

Solving Quadratic Equations

1. Solving Quadratic Equations of the form $ax^2 + bx$

$$\begin{aligned} 3x^2 &= -x \\ 3x^2 + x &= 0 \\ x(3x+1) &= 0 \\ x &= 0 \text{ and } x = -\frac{1}{3} \end{aligned}$$

2. Solving Quadratic Equations of the form $ax^2 + c$

$$\begin{aligned} 4x^2 - 8 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

3. Solving Quadratic Equations of the form $x^2 + bx + c$ (factorisable)

$$\begin{aligned} x^2 &= 7x - 12 \\ x^2 - 7x + 12 &= 0 \\ (x-4)(x-3) &= 0 \\ x = 3 \text{ and } x &= 4 \end{aligned}$$

4. Solving Quadratic Equations of the form $ax^2 + bx + c$ (factorisable and a is a common factor)

$$\begin{aligned} 2x^2 + 4x - 6 &= 0 \\ 2(x^2 + 2x - 3) &= 0 \\ 2(x+3)(x-1) &= 0 \\ x = -3 \text{ and } x &= 1 \end{aligned}$$

5. Solving Quadratic Equations of the form $ax^2 + bx + c$ (factorisable and a is NOT a common factor)

$$\begin{aligned} 2x^2 - x - 3 &= 0 \\ \frac{(2x+2)(2x-3)}{2} &= 0 \\ (x+1)(2x-3) &= 0 \\ x = -1 \text{ and } x &= \frac{3}{2} \end{aligned}$$

6. Solving Quadratic Equations (NOT factorisable)

$$\begin{aligned} 2x^2 - x - 13 &= 0 \\ 2\left(x^2 - \frac{1}{2}x - \frac{13}{2}\right) &= 0 \\ x^2 - \frac{1}{2}x - \frac{13}{2} &= 0 \\ \left(x - \frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{13}{2} &= 0 \\ \left(x - \frac{1}{4}\right)^2 &= \frac{105}{16} \\ x &= \frac{1 \pm \sqrt{105}}{4} \end{aligned}$$

6 Polynomials

Definition

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots + a_2 x^2 + a_1 x + a_0$$

Cubic Functions

$$y = ax^3 + bx^2 + cx + d$$
$$y = a(x-m)(x-p)(x-q)$$

Base Cubic Function $f(x) = x^3$

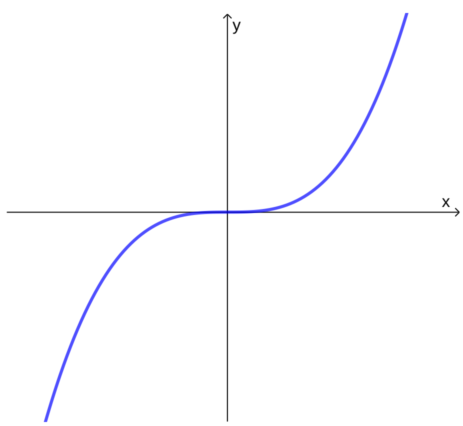


Figure 6.1: The base cubic function $f(x) = x^3$.

Linear Transformations of Functions

$$f(x) \rightarrow \underset{3}{a} f(\underset{2}{b} x + \underset{1}{c}) + \underset{4}{d}$$

Note:

1. When determining the equation of a function from a graph use $af\left(b(x+k)\right) + d$

Transformation Types

c: Horizontal Translation by $-c$ units.
b: Horizontal Dilation by scale factor $\frac{1}{b}$.
a: Vertical Dilation by scale factor a.
d: Vertical Translation by d units.

Note:

1. When a dilation is negative it is a reflection. If b is negative there is a reflection about the y-axis. If a is negative there is a reflection about the x- axis.

Transformation Description

c: Each point on the function is translated c units left.
b: The distance each point is from the y axis is multiplied by $\frac{1}{b}$.
a: The distance each point is from the x axis is multiplied by a.
d: Each point on the function is translated d units up.

Transformation of a point

$$(x, y) \to \left(\frac{x-c}{b}, ay + d\right)$$

7 Trigonometric Functions

Trigonometric Graphs

$y = \sin(x)$

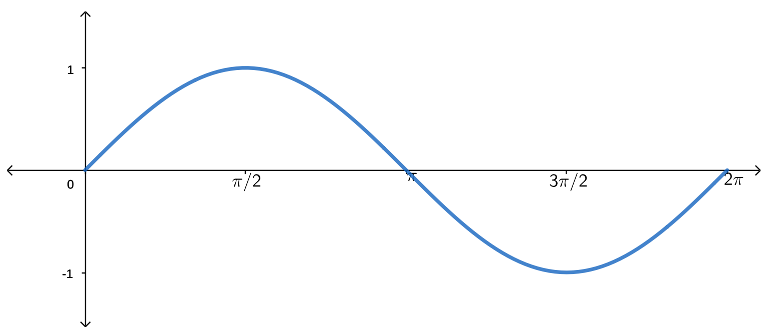


Figure 7.1: The sine function $f(x) = \sin(x)$.

$y = \cos(x)$

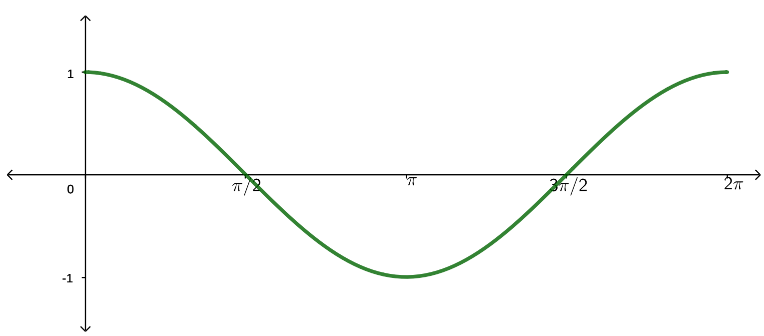


Figure 7.2: The sine function $f(x) = \cos(x)$.

$y = \tan(x)$

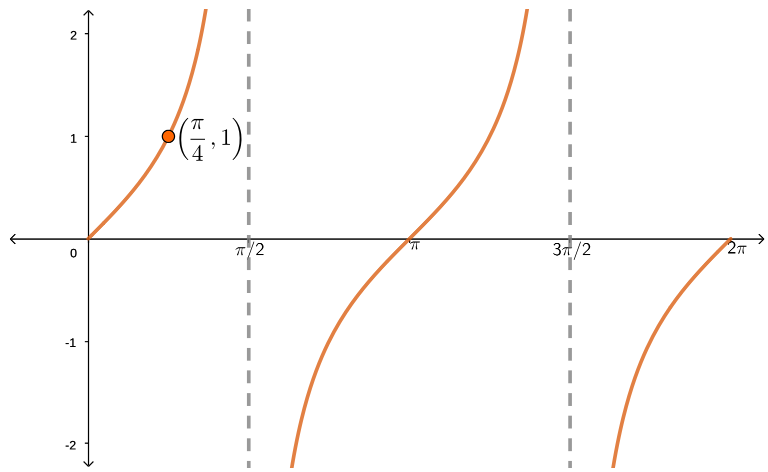


Figure 7.3: The sine function $f(x) = \tan(x)$.

Transformation c: Phase Shift

Translation parallel to the x-axis by $-c$ units.

The term **phase** refers to the change in the natural starting position for an oscillation.

$$y = \sin(x + c)$$
$$y = \cos(x + c)$$
$$y = \tan(x + c)$$

Transformation b: Period

Dilation parallel to the x-axis by scale factor $\frac{1}{|b|}$.

The term **period** is used to refer to the **length** of an oscillation. The **period** of the function $y = \sin(x)$ and $y = \cos(x)$ is 2π. The **period** of the function $y = \tan(x)$ is π.

$$y = \sin(bx)$$
$$y = \cos(bx)$$

Period is $\frac{2\pi}{|b|}$.

$$y = \tan(bx)$$

Period is $\frac{\pi}{|b|}$.

Transformation a: Amplitude

Dilation parallel to the y-axis by scale factor a.

The term **amplitude** is used to refer to the **magnitude** of an oscillation. The height from the **mean** value of the function.
The amplitude of $y = \sin(x)$ and $y = \cos(x)$ is 1. The amplitude of $y = \tan(x)$ is **undefined**.

$$y = a\sin(x)$$
$$y = a\cos(x)$$

Amplitude is $|a|$, where $|a|$ is the absolute value of a.

$$y = a\tan(x)$$

For a transformations of $\tan(x)$ identify where the point $\left(\frac{\pi}{4}, 1\right)$ has moved to.

Transformation d

Translation parallel to the y-axis by d units.

$$y = \sin(x) + d$$
$$y = \cos(x) + d$$
$$y = \tan(x) + d$$

Trigonometric Identities

Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1 \tag{7.1}$$

Addition Theorems

Note:

1. A geometric proof using the unit circle can be used to prove 7.2 and 7.3.
2. All of the following trigonometric Identities can be proven using 7.2 and 7.3.

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \tag{7.2}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \tag{7.3}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \tag{7.4}$$

Double Angle Theorems

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{7.5}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{7.6}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{7.7}$$

Complementary Angle Theorems

$$\sin(90-\theta) = \cos\theta \tag{7.8}$$

$$\cos(90-\theta) = \sin\theta \tag{7.9}$$

$$\tan(90-\theta) = \cot\theta \tag{7.10}$$

Supplementary Angle Theorems

$$\sin(180-\theta) = \sin\theta \tag{7.11}$$

$$\cos(180-\theta) = -\cos\theta \tag{7.12}$$

$$\tan(180-\theta) = -\tan\theta \tag{7.13}$$

Negative Angle Theorems

$$\sin(-\theta) = -\sin\theta \tag{7.14}$$

$$\cos(-\theta) = \cos\theta \tag{7.15}$$

$$\tan(-\theta) = -\tan\theta \tag{7.16}$$

Solutions to Trigonometric Equations

Solving equations involving trigonometric equation involves taking particular attention to the given domain.

Note:

1. Remember to consider the domain when solving trigonometric equations in calculator.

1. Isolate the trigonometric expression. Trigonometric identities may be required.

2. If needed substitute a new variable for the subject of the

3. Adjust the domain for the new variable.

4. Solve the new variable over the adjusted domain. **Use a diagram** of the unit circle to help.

5. Convert the solution for the original variable.

Example 1

Solve $\dfrac{\sin x}{2} = \dfrac{1}{4}$ for $-\pi \leq x \leq \pi$.

$\sin x = \dfrac{1}{2}$

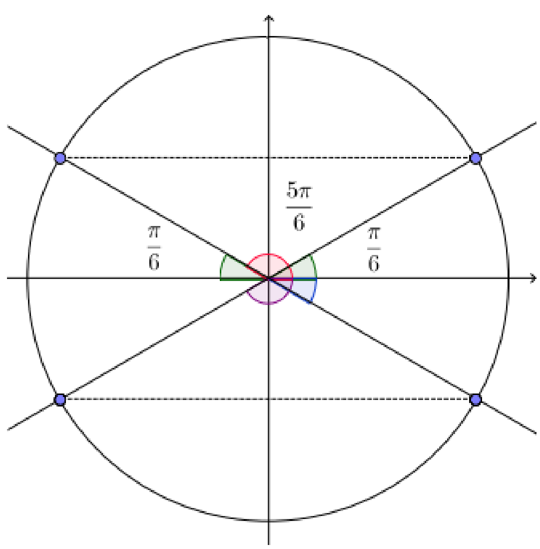


Figure 7.4: Reference angle: $\dfrac{\pi}{6}$

$\therefore x = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

Example 2

Solve $8\sin^2 x + 4\cos^2 x = 7$ for $0 \le x \le 2\pi$.

$$
\begin{aligned}
8\sin^2 x + 4(1 - \sin^2 x) &= 7 \\
8\sin^2 x + 4 - 4\sin^2 x &= 7 \\
4\sin^2 x &= 3 \\
\sin x &= \pm\frac{\sqrt{3}}{2}
\end{aligned}
$$

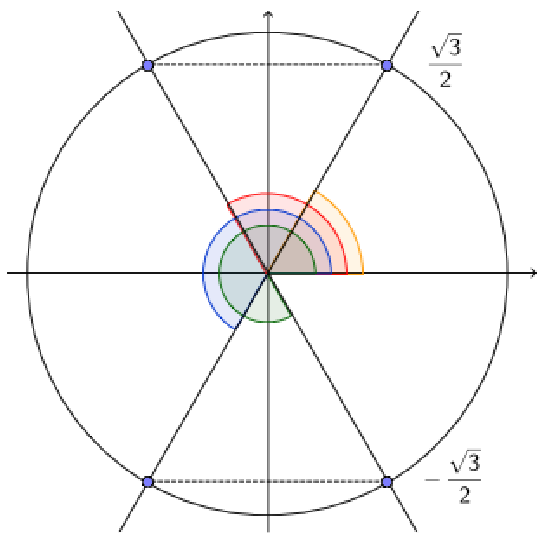


Figure 7.5: Reference angel: $\frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

Example 3

Solve $\sin(x + 30°) = \cos x$, for $0 \leq x \leq 360°$

$$\begin{aligned}
\sin(x + 30°) &= \cos x \\
\sin(x)\cos(30°) + \cos(x)\sin(30°) &= \cos x \\
\sin(x) \cdot \frac{\sqrt{3}}{2} + \cos(x) \cdot \frac{1}{2} &= \cos x \\
\frac{\sqrt{3}}{2} \cdot \sin(x) &= \frac{1}{2} \cdot \cos(x) \\
\tan(x) &= \frac{1}{\sqrt{3}}, \text{ for } 0 \leq x \leq 360°
\end{aligned}$$

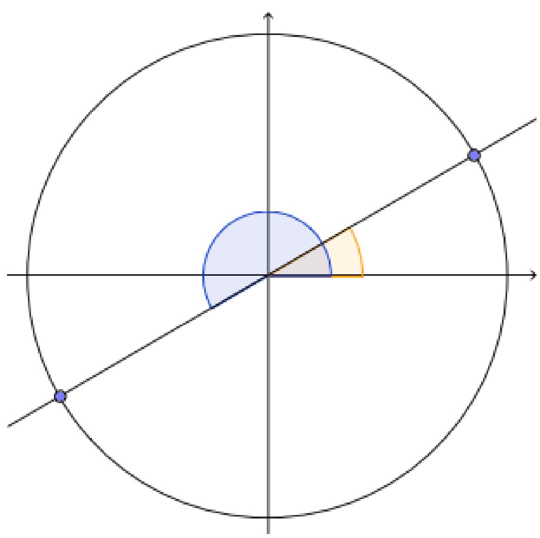


Figure 7.6: Reference angle: $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$

8 Probability

Notation

$$U = \{e_1, e_2, e_3\} : \text{The universal set containing the elements } e_1, e_2, e_2$$

$$e_1 \in U : e_1 \text{ is an element of } U$$

$$n(U) = 3 : \text{There are 3 elements in } U$$

$$\text{If } A = \{e_1, e_3\} \text{ then } A \subset U : A \text{ is a subset of } U$$

$$\overline{A} \text{ or } A' = \{e_2\} : \text{The compliment of } A$$

$$\cup : \text{Union “OR”}$$

$$\cap : \text{Intersection “AND”}$$

$$P(A|B) : \text{Probability of } A \text{ given } B$$

Calculating Probability

$$\mathrm{P}(E) = \frac{\mathrm{n}(E)}{\mathrm{n}(U)}$$

To find the probability of an event E we need to determine the number of ways outcome E can occur and the total number of outcomes in the universal set, U.
We can use the following techniques to help us solve probability problems:

1. List
2. 2 Way Table
3. Tree Diagram (weighted or non-weighted)
4. Venn Diagram
5. Counting Techniques

Probability Laws

Complimentary Events

$$P\left(\overline{A}\right) = 1 - P(A) \tag{8.1}$$

Conditional Probability

$$P\left(A|B\right) = \frac{P\left(A \cap B\right)}{P(B)} \tag{8.2}$$

$$\therefore P(A \cap B) = P(A|B) \cdot P(B) \tag{8.3}$$

Independent Events

$$P\left(A|B\right) = P(A) \tag{8.4}$$

$$\therefore P(A \cap B) = P(A) \cdot P(B) \tag{8.5}$$

Note:

1. 8.4 is the formal statement of independent events but in most applications 8.5 is used to solve problems.

Inclusive Events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Mutually Exclusive Events

$$P(A \cap B) = 0 \tag{8.6}$$

$$\therefore P(A \cup B) = P(A) + P(B) \tag{8.7}$$

9 Counting Techniques

Multiplication Principle

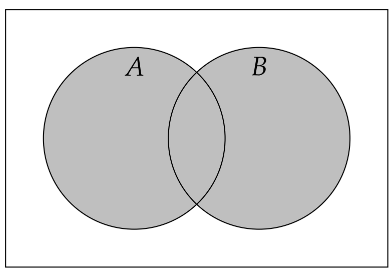
In combinatorics, the multiplication principle (the fundamental principle of counting) is the idea that if there are a ways of doing something and b ways of doing another thing, then there are $a \cdot b$ ways of performing both actions.

Hint:

1. We sometimes refer to this method of reasoning as the "Box Method".

Addition Principle

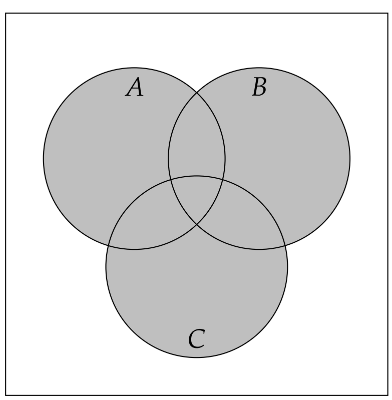
$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \tag{9.1}$$



Hint:

1. When breaking a problem down into countable parts we need to be careful we are not counting a set of outcomes twice.

$$\begin{aligned} n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) \\ - n(B \cap C) + n(A \cap B \cap C) \end{aligned} \tag{9.2}$$



Permutations

The number of ways of arranging r objects from n objects.

$$^{n}\mathrm{P}_r = \frac{n!}{(n-r)!} \tag{9.3}$$

Note:

1. Order is Important

Combinations

The number of ways of selecting r objects from n objects.

$$^{n}\mathrm{C}_r = \frac{n!}{(n-r)! \cdot r!} \tag{9.4}$$

Note:

1. Order is not Important

4 Steps to Solving Complicated Problems

1. Multiplication Principle: "AND" means multiply.
2. Addition Principle: "OR" means add.
3. The Restriction Principle: Deal with restrictions first.
4. The Selection Principle: Select objects before arranging them.

Pascal's Triangle

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & n=0 \\
 & & & & 1 & & 1 & & & & & & n=1 \\
 & & & 1 & & 2 & & 1 & & & & & n=2 \\
 & & 1 & & 3 & & 3 & & 1 & & & & n=3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & n=4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & n=5
\end{array}$$

Note:

1. The sum of all elements in each row of Pascal's triangle $\sum_{k=0}^{n} {}^{n}C_k = 2^n$.

$$\begin{array}{ccccccccccccc}
 & & & & & {}^0C_0 & & & & & & & n=0 \\
 & & & & {}^1C_0 & & {}^1C_1 & & & & & & n=1 \\
 & & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & & & & & n=2 \\
 & & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 & & & & n=3 \\
 & {}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4 & & & n=4 \\
{}^5C_0 & & {}^5C_1 & & {}^5C_2 & & {}^5C_3 & & {}^5C_4 & & {}^5C_5 & & n=5
\end{array}$$

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} {}^{n}C_k \cdot a^{n-k} \cdot b^k \tag{9.5}$$

$$= {}^{n}C_0 a^n b^0 + {}^{n}C_1 a^{n-1} b^1 + \cdots + {}^{n}C_{n-1} a^1 b^{n-1} + {}^{n}C_n a^0 b^n \tag{9.6}$$