COURSE SUMMARY NOTES

METHODS UNIT 3

Contents

1 Differentiation

Notation

The derivative of a function $f(x)$ or y can be written as $f'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ respectively.

First Principles

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The Power Rule

Given $y = a \cdot x^n$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a \cdot n \cdot x^{n-1}$$

Note:

1. Use negative indices rather than having the x in the denominator.
2. Use fractional indices rather than using the root sign.

The Product Rule

Given $y = u(x) \cdot v(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v \cdot u' + u \cdot v'$$

Note:

1. Identify a product of two functions of x.

The Quotient Rule

Given $y = \frac{u(x)}{v(x)}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Note:

1. Identify a function of x dividing another function of x.
2. Often the question will ask you not to simplify.

The Chain Rule

Given $y = u(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Note:

1. Identitfy a function of a function of x.

2 Applications of Differentiation

Sketching Gradient Functions

$f(x)$	$\Leftrightarrow$	$f'(x)$
Stationary Points	$\Leftrightarrow$	$x-$intercepts
Point of Inflection	$\Rightarrow$	Turning Point
Gradient $+ve$	$\Leftrightarrow$	Function above the $x-$ axis
Gradient $-ve$	$\Leftrightarrow$	Function below the $x-$ axis

Sketching Functions

Intercepts

$$y - \text{intercept } (x = 0)$$
$$x - \text{intercept } (y = 0)$$

Turning Points

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

If Turning Point is a local maximum (concave down) then:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -ve$$

If Turning Point is a local minimum (concave up) then:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = +ve$$

Note:

1. To avoid possible error of the second derivative test in identifying the nature of a turning point (i.e. $f(x) = x^4$) check $\frac{\mathrm{d}y}{\mathrm{d}x}$ "suffciently close" to either side of $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Point of Inflection

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

If horizontal point of inflection then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Note:

1. Check $\frac{\mathrm{d}y}{\mathrm{d}x}$ either side of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ to determine nature of point of inflection.

Sketching Polynomial Quotients

$$y - \text{intercept } (x = 0)$$
$$x - \text{intercept } (y = 0)$$
$$x \to \pm\infty$$
$$y \to \pm\infty$$

Optimisation

At local maximum or minimum

$$\frac{dy}{dx} = 0$$

Note:

1. If given a function to optimise in 2 variables, you will be given a relationship between those variables so you can use substitution to obtain a function to optimise in 1 variable.

Related Rates

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note:

1. Identitfy what you are being asked to find i.e. $\frac{dy}{dx}$, the rate of change of y with respect to x
2. Identify the relationships between the two required variables through the "linking" variable u i.e. $\frac{dy}{du}$ and $\frac{du}{dx}$

Small Change

$$\frac{\delta y}{\delta x} \approx \frac{\mathrm{d}y}{\mathrm{d}x}$$

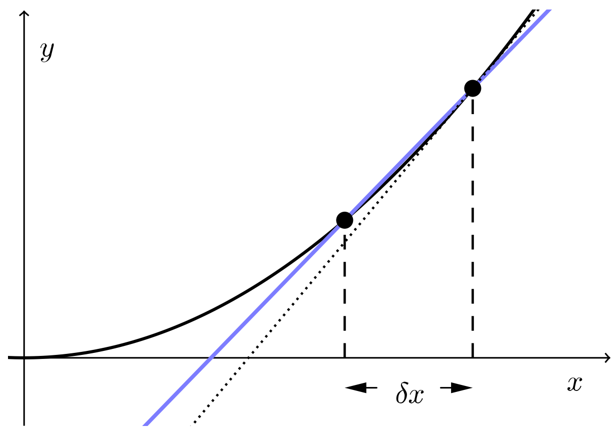

Figure 2.1: The derivative (the dotted line) can be a useful estimation for the average rate of change (the blue line).

Finding a Small Change

When asked to find δy (a small change in y).

$$\delta y \approx \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \delta x$$

Note:

1. Language in question will be "Use calculus to find the approximate change ..."

Finding a Percentage Change

When asked to find $\frac{\delta y}{y}$ (a percentage change in y).

$$\frac{\delta y}{y} \approx \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \delta x \cdot \frac{1}{y}$$

Note:

1. If $f(x) = ax^n$ with a $k\%$ change in x then percentage change in y is $(n \cdot k)\%$

Finding a Marginal Change

When asked to find δy when $\delta x = 1$ (a small change in y given the change in x is 1).

$$\delta y \approx \frac{\mathrm{d}y}{\mathrm{d}x} \cdot 1$$

Note:

1. If $R(x)$ is the revenue function for producing x units and $C(x)$ is the cost function for producing x units then the profit function is $P(x) = R(x) - C(x)$.

3 Integration

Power Rule

$$\int ax^n \, \mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$$

Extracurricular:

1. Integration from first principles is not covered in this course. See `http://www.mathshelper.co.uk/Integration%20FFP.pdf`

Chain Rule

$$\int f'(x) \, [f(x)]^n \, \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + c$$

Note:

1. Identify the derivative function.
2. Factor any remaining scalar outside on the integral.
3. Don't forget to divide by $n+1$.

Rectilinear Motion

$$x(t) \to \text{Displacement of a body as a function of time, } t$$

$$\frac{\mathrm{d}\, x(t)}{\mathrm{d}\, t} = v(t) \to \text{Velocity of a body as a function of time, } t$$

$$\frac{\mathrm{d}^2\, x(t)}{\mathrm{d}\, t^2} = \frac{\mathrm{d}\, v(t)}{\mathrm{d}\, t} = a(t) \to \text{Acceleration of a body as function of time, } t$$

Note:

1. $x(0)$ is initial displacement.
2. $v(0)$ is initial velocity
3. $a(0)$ is initial acceleration.
4. If the displacement is on the positive side of the origin then the displacement is positive.
5. If the direction of travel is in the positive direction then the velocity is positive.
6. Visualise acceleration as a force. If the force is acting in the positive direction then the acceleration is positive.

4 Area Under a Curve

Definition of Integral

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x) \cdot \delta x = \int_a^b f(x)\,\mathrm{d}x$$

Definite Integral

$$\int\limits_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Note:

1. There is no need to consider the constant of integration, c, for a definite integral.

Area Under a Curve

$$\text{Area} = \int\limits_a^b |\, f(x)\,|\,\mathrm{d}x$$

Note:

1. Care must be taken to understand the difference between a definite integral and the area under a curve.
2. To find the area under a curve that crosses the $x-$axis (by hand) we must find the $x-$intercepts and separate the integral.

Area Between Two Curves

$$\text{Area} = \int\limits_a^b |\, f(x) - g(x)\,|\,\mathrm{d}x$$

Note:

1. To find the area between two curves (by hand) we must find the intercepts of the curves and separate the integral.
2. Upper curve $-$ lower curve will give a positive result.

Total Change

Total change in A when t changes from a to b is given by:

$$\text{Total Change in } A = \int\limits_a^b \frac{\mathrm{d}A}{\mathrm{d}t}\,\mathrm{d}t$$

5 The Fundemental Theorem of Calculus

Part 1 of The Fundamental Theorem of Calculus

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \tag{5.1}$$

Part 2 of The Fundamental Theorem of Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x) \tag{5.2}$$

Note:

1. This result is used to demonstrate your understanding of the relationship between integration and differentiation.
2. Take care to use the chain rule if there is a function of x in the upper limit of the integral.
3. If the variable x is in the lower limit of the integral the resulting function will be the negative of the original.

6 The Exponential Function

Definition

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{6.1}$$

$$e^a = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n \tag{6.2}$$

Note:

1. The limit $\lim_{x\to 0}\frac{e^x-1}{x} = 1$ is used to differentiate e^x by first principles.

Differentiation of Exponential Functions

$$\frac{d\,e^x}{dx} = e^x \tag{6.3}$$

$$\frac{d\,e^{f(x)}}{dx} = f'(x)\cdot e^{f(x)} \tag{6.4}$$

Integral of Exponential Functions

$$\int e^x\,dx = e^x + c \tag{6.5}$$

$$\int f'(x)\cdot e^{f(x)}\,dx = e^{f(x)} + c \tag{6.6}$$

Growth and Decay

Given

$$\frac{d\,P(t)}{dt} = kP(t) \tag{6.7}$$

then

$$P(t) = P_0\cdot e^{kt} \tag{6.8}$$

Note:

1. k must be the **continuous** rate of growth or decay.

7 Calculus of Trigonometric Functions

Identities

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{7.1}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0 \tag{7.2}$$

Extracurricular:

1. Proof of the identity 7.1 can be shown geometrically on the unit circle using the Sandwich Theorem.
2. Proof of the identity 7.2 can be shown using the first identity and the Product Law of Limits.

Derivative of $\sin x$

$$\frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x \tag{7.3}$$

Derivative of $\cos x$

$$\frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x \tag{7.4}$$

Derivative of $\tan x$

$$\frac{\mathrm{d}\tan x}{\mathrm{d}x} = \frac{1}{\cos^2 x} \tag{7.5}$$

Integral of $\sin x$

$$\int \sin x\,\mathrm{d}x = -\cos x + c \tag{7.6}$$

Integral of $\cos x$

$$\int \cos x\,\mathrm{d}x = \sin x + c \tag{7.7}$$

8 Discrete Random Variables

A discrete random variable X is a distinct value, the value of which depends on a random process $\mathrm{P}(\mathrm{X} = x)$

Discrete Probability Function

$$\mathrm{P}(\mathrm{X} = x) = \begin{cases} \mathrm{p}_i & \text{for } x = x_1, x_2, \ldots, x_n \\ 0 & \text{for all other values of } x \end{cases}$$

Note:

1. $0 \leq \mathrm{p}_i \leq 1$
2. $\sum_{i=1}^{n} \mathrm{p}_i = 1$

x	x_1	x_2	$\ldots$	x_n
$P(X = x)$	p_1	p_2	$\ldots$	p_n

Expected Value

$$\mathrm{E}(X) = \sum_{i=1}^{n} x_i \cdot \mathrm{p}_i \tag{8.1}$$

Note:

1. $\mathrm{E}(X)$, the expected value, is the mean μ

Variance

$$\mathrm{Var}(\mathrm{X}) = \sum_{i=1}^{n} \left(\mathrm{p}_i \cdot (x_i - \mu)^2 \right) \tag{8.2}$$

$$= \mathrm{E}(\mathrm{X}^2) - (\mathrm{E}(\mathrm{X}))^2 \tag{8.3}$$

Note:

1. Standard deviation $\sigma = \sqrt{\mathrm{Var}(X)}$

Linear Transformations of Data

$$\mathrm{E}(aX + b) = \mathrm{aE}(X) + b \tag{8.4}$$

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X) \tag{8.5}$$

$$\sigma(aX + b) = |a|\, \sigma(X) \tag{8.6}$$

Extracurricular:

1. Prove 8.4 and 8.5

9 Binomial Distribution

Bernoulli Trial

x	0	1
$\mathrm{P}(x)$	$(1-p)$	p

Note:

1. μ, the mean, is the expected value $\mathrm{E}(x)$

$$\mu = p \tag{9.1}$$

$$\sigma = \sqrt{p(1-p)} \tag{9.2}$$

Binomial Distribution

A Binomial Distribution occurs from repeating **independent** Bernoulli Trials. Parameter n is the number of trials. Parameter p is the probability of success.

$$X \sim \mathrm{Bin}(n, p) \tag{9.3}$$

$$P(x) = {}^{n}\mathrm{C}_{x} \cdot p^{x} \cdot (1-p)^{n-x} \tag{9.4}$$

$$\mu = n \cdot p \tag{9.5}$$

$$\sigma = \sqrt{n \cdot p \cdot (1-p)} \tag{9.6}$$

Testing for Improvement

To test if a program has been effective in causing e improvements from n trials. Calculate the probability, t, of the number of improvements occurred by chance.

$$\text{Assume } p = 0.5 \tag{9.7}$$

$$\text{then } X \sim Bin(n, 0.5) \tag{9.8}$$

$$P(X \geq e) = t \tag{9.9}$$