SCHOOL OF COMPUTATION, INFORMATION AND TECHNOLOGY — INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Bachelor's Thesis in Informatics

Verification of Convex Hull Algorithms in Isabelle/HOL

Author

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Verification of Convex Hull Algorithms in Isabelle/HOL

Titel der Abschlussarbeit

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I confirm that this bachelor's thesis is my own work and I have documented all sources and material used.

Munich, Submission date Author

Acknowledgments

Abstract

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1 Introduction

1.1 Section

Citation test [Lam94].

Acronyms must be added in `main.tex` and are referenced using macros. The first occurrence is automatically replaced with the long version of the acronym, while all subsequent usages use the abbreviation.

E.g. `\ac{TUM}, \ac{TUM}` $\Rightarrow$ Technical University of Munich (TUM), TUM

For more details, see the documentation of the `acronym` package[1].

1.1.1 Subsection

See Table 3.1, Figure 3.1, Figure 3.2, **??**.

Table 1.1: An example for a simple table.

A	B	C	D
1	2	1	2
2	3	2	3

$$\begin{tikzcd}
R_1 \arrow[r, dash] \arrow[d, dash] & R_2 \arrow[r, dash] \arrow[d, dash] & R_5 \\
R_3 \arrow[r, dash] & R_4 &
\end{tikzcd}$$

Figure 1.1: An example for a simple drawing.

!TeX root = ../main.tex

[1] `https://ctan.org/pkg/acronym`

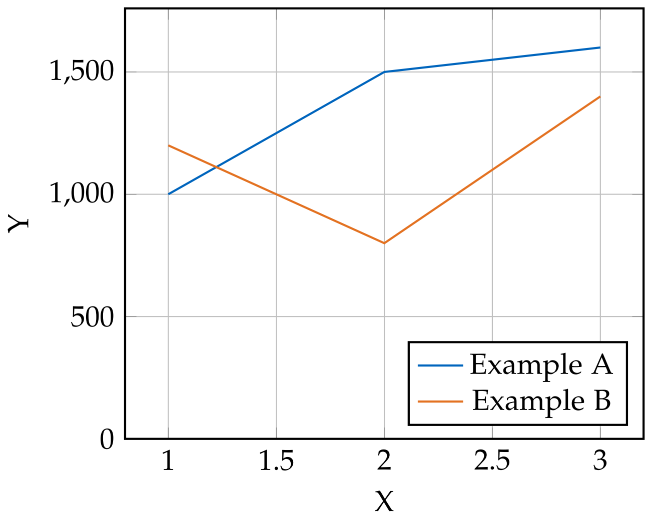

Figure 1.2: An example for a simple plot.

2 Definitions and Algorithms

2.1 Convex Hull

2.1.1 Basics

First the Convex Hull will be defined. A set $s \subseteq \mathbb{R}^2$ is convex if for every two points p and q in s it holds that all points on the line segment connecting p and q are in s again. This can be expressed, as the fact that the any convex combination of p and q has to be in s again. In Isabelle the convex predicate is defined exactly this way:

```
definition convex :: 'a real_vector set ⇒ bool where
convex s ⟷ (∀x∈s. ∀y∈s. ∀u≥0. ∀v≥s. u + v = 1 ⟶ u *R x + v *R y ∈ s)
```

The convex hull of a set s is the smallest convex set in which s is contained. There are several alternative ways in which the convex hull can be defined. One possible way is to define it as the intersection of all convex sets containing s, which is also the definition used in Isabelle/HOL. We have already seen the convex predicate, the hull predicate is defined as the intersection of all sets t that contain s and fulfill the predicate S.

```
definition hull :: (a' set ⇒ bool) ⇒ a' set ⇒ a' set where
S hull s = ⋂{t. S t ∧ s ⊆ t}
```

Consequently `convex hull s` refers to the intersection of all convex sets that contain s and therefore the convex hull of the set s. In the two dimensional case for a finite $s \subset \mathbb{R}^2$, the convex hull CH of s is a convex polygon and all the corners of this convex polygon are points from S (see figure 1). [De 00] As this thesis will focus on the two dimensional case and only give an outlook on the the three dimensional case, we will deal with computing the convex hull of $s \in \mathbb{R}^2$ in the following and therefore computing a convex polygon as representation of the convex hull of s. Assuming no three points in s are colinear, then the edges $E \subseteq s^2$ of the polygon can be described as exactly those $(p, q) \in s^2$ for which all points in s lie on the left of the vector $\vec{pq}$. Notice that the direction of the vector i.e. from p to q is relevant for expressing that a point lies on the left of the vector $\vec{pq}$. Of course the symmetric definition of E as those $(p, q) \in s^2$ for which all points in s lie on the right of the line $\vec{pq}$ works as well. The only difference is that in the set of directed edges we get, every edge now points into the opposite direction. Both definitions make sense, but because there is already infrastructure in place for first definition i.e. (p, q) is an edge if and only if all points in s are left of $\vec{pq}$, we will use this definition. But first we need to state the concept of a point q being left of the vector $\vec{pq}$ more precicsely, especially when there can be three colinear points in s.

2.1.2 Orientation

Figure x shows the convex hull of the points $s = \{p_0, p_1, p_2, p_3\}$ in the form of a convex polygon. When using the previous definition, (p_1, p_2), (p_2, p_3) and (p_1, p_3) would be edges of the convex polygon, because it holds that all points in s are left of $\vec{p_1 p_2}$, left of $\vec{p_2 p_3}$ and left of $\vec{p_1 p_3}$. This is an unintuitive definition which should be avoided. Therefore we define the condition for (p, q) to be an edge of the convex hull polygon more precicsely. $(p, q) \in s^2$ is an edge of the convex hull polygon if and only if all points $r \in s$ are either strictly left of the vector $\vec{pq}$ (p, q and r are not colinear) or r is contained in the closed segment between p and q. The second part can be written as `r ∈ closed_segment p q` in Isabelle where `closed_segment` is defined as:

```
definition closed_segment :: 'a::real_vector ⇒ 'a ⇒ 'a set
where closed_segment a b = {(1 - u) *R a + u *R b | u::real. 0 ≤ u ∧ u ≤ 1 }
```

The fact that r lies strictly left of $\vec{pq}$ can be expressed differently by stating that (p, q, r) are making a strictly counterclockwise turn. The three points are written as a tuple as it is again necessary to state the order of p, q and r when talking about a counterclockwise turn. In the following a counterclockwise turn will always refer to a strict counterclockwise turn. Checking if a point r lies strictly left of a vector is an operation that is essential for almost all convex hull algorithms. To check if the points $((x_1, y_1), (x_2, y_2), (x_3, y_3))$ make a counterclockwise turn, we can look at the sign of the determinant of the following matrix.

$$\det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

If the determinant is positive, we know that the sequence $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ makes a counterclockwise turn, if the determinant is zero we know that the three points are colinear and if the determinant is negative, we know the the sequence $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ makes a clockwise turn. In Isabelle the function that calculates the above determinant for three points is called `det3`.

```
fun det3:: point ⇒ point ⇒ point ⇒ real where
det3 (x1, y1) (x2, y2) (x3, y3) =
x1 * y2 + y1 * x3 + x2 * y3 - y2 * x3 - y1 * x2 - x1 * y3"
```

Based on `det3` the `ccw'` predicate is defined, which expresses that three points (p,q,r) make a counterclockwise turn.

```
definition ccw' p q r ⟷ 0 < det3 p q r
```

Lastly the predicate `ccw'_seg p q r` holds if and only if r either lies counterclockwise of $\vec{pq}$ or r is contained in the closed segment between p and q.

```
definition ccw'_seg p q r = ccw' p q r ∨ r ∈ closed_segment p q
```

2.1.3 Order

In both algorithms we need to do the following operation. Given an corner p of the convex polygon, find another corner by searching for a point q such that for all other points $r \in s$ either `ccw' p q r` or `r ∈ closed_segment p q` holds. In short, we search for a q that fulfills `∀ r ∈ s. ccw'_seg p q r`. Intuitively it makes sense that given a finite $s \subseteq \mathbb{R}^2$ and a corner of the convex hull polygon, we can find a unique next corner. Figuratively speaking, we rotate a line that starts in p counterclockwise until we hit a point q, which is going to be the next corner. If we hit several points at the same time, we are just going to take the point further away from p. Now to translate this into a formal framework, we start with the previous definition of finding a q that fulfills $\forall r \in s$. `(ccw'_seg p)` q r. If `(ccw'_seg p)` is a total order on s, we know that such a q exists. That's because q is the minimum respect to the ordering `(ccw'_seg p)`. For `(ccw'_seg p)` to be a total order and for later proofs it is necessary that we derive some form of transitivity for the counterclockwise orientation. For example it should hold that if `(ccw'_seg p a b)` and `(ccw'_seg p b c)` holds, then `(ccw'_seg p a c)` should hold as well. The same implication should hold when using the `(ccw'_seg p)` ordering instead of `(ccw'_seg p)`. Altough straightforward, this kind of transitivity does not always hold as the following examples shows. Clearly `(ccw'` p_1 p_2 p_3`)` holds and also `(ccw'` p_0 p_3 p_4`)`, but `(ccw'` p_1 p_2 p_4`)` does not hold, instead `(ccw'` p_1 p_4 p_2`)` holds. So in order for transitivity to hold, we need to restrict the set on which transitivity is supposed to hold. It can be shown that if there exists a p_0 such that for all $r \in s$ it holds that `ccw'_seg` p_0 p_1 `r` holds, then `(ccw'_seg` p_1`)` is transitive on s. This restriction avoids the counterexample for general transitivity from above. Transitivity also holds if there exists a point p_0 such that all $r \in s$ are lexicographically bigger than p_0, meaning $\forall r \in s$. `lex` p_0 `r` holds, where `lex` is defined as.

```
definition lex:: point ⇒ point ⇒ bool where
"lex p q ⟷ (fst p < fst q ∨ fst p = fst q ∧ snd p < snd q ∨ p = q)"
```

To check if p is lexicographically smaller than q, we check if p_x is smaller than q_x. If they are equal we check if $p_y \leq q_y$ holds. Now given for our reference set $ps \subseteq \mathbb{R}^2$ if ($\forall q \in ps.$ `ccw'_seg p_stl p_last q`) $\vee$ ($\forall q \in ps.$ `lex p_last q`) holds, then the following lemmas can be proven.

```
lemma ccw'_seg_trans:
assumes "p ∈ ps" "q ∈ ps" "k ∈ ps"
assumes "ccw'_seg p_last p q" "ccw'_seg p_last k p"
shows "ccw'_seg p_last k q"

lemma ccw'_seg_total:
assumes "p ∈ ps" "q ∈ ps"
shows "ccw'_seg p_last p q ∨ ccw'_seg p_last q p"

lemma ccw'_seg_antisymmetric:
assumes "ccw'_seg p_last p q ∧ ccw'_seg p_last q p"
shows "p = q"
```

Reflexivity directly follows from the definition of `ccw'_seg`. Therefore we know that there exists a unique q such that $\forall r \in ps.$ `(ccw'_seg p_last) q r`. Notice how ps was defined using p_last.

2.1.4 Convex Polygon

Both algorithms calculate the convex polygon that corresponds to the convex hull of the input set $s \subseteq \mathbb{R}^2$. This convex polygon is described by a list of points from s that are the corners of this convex polygon. So far, we just always just stated that the convex polygon corresponds to the convex hull, yet it is not obvious that this is the case. Therefore we require a description of a convex polygon in Isabelle/HOL and we need to know that this description is indeed equivalent to `convex hull`, which is defined as Intersection of all convex sets that contain s. To be more precicse, we require a proof that the convex hull of the corners of such a convex polygon corresponds to the set of all points that lie within the polygon. This fact was proven for a list of corners `p0 # ps` that should represent a convex polygon by Simon Hanssen.

```
lemma polygon_eq_convex_hull:
assumes turns_only_left (p0 # ps)
    and sorted_wrt (ccw' p0) ps
    and 2 ≤ length ps
  shows list_all (encompasses p) (polychain_of (p0 # ps @ [p0]))
    ⟷ p ∈ convex hull (set (p0 # ps))"
```

To understand this proof, we need to first look at the definitions of all the predicates used. First `turns_only_left l` for a list l expresses that every three consecutive points in the list are turning counterclockwise. This ensures that every interior angle of the polygon is less than or equal 180°, which is one of the typical definitions of a convex polygon.

```
fun turns_only_left :: "point list ⇒ bool" where
"turns_only_left (p#q#r#ps) ⟷ ccw' p q r ∧ turns_only_left (q#r#ps)"|
"turns_only_left _ = True"
```

Next `sorted_wrt (ccw' p0)`, where p0 is the start or our list of corners, states that for every corner p in the list, all corners that are behind it in the list, lie counterclockwise of $\overrightarrow{p0p}$.

```
fun sorted_wrt :: "('a ⇒ 'a ⇒ bool) ⇒ 'a list ⇒ bool" where
orted_wrt P [] = True" |
orted_wrt P (x # ys) = ((∀ y ∈ set ys. P x y) ∧ sorted_wrt P ys)"
```

This property avoids degenerations as shown in Figure x. Lastly $2 \leq$ `length ps` is needed, as the definition does not work in the case of two corners, where the polygon is just a closed segment between two points. Now given a list `p0 # ps` fulfills these properties, then we know that this list describes a list of corners of a convex polygon and the following statement holds.

```
list_all (encompasses p) (polychain_of (p0 # ps @ [p0]))
    ⟷ p ∈ convex hull (set (p0 # ps))
```

Where `polychain_of (p0 # ps @ [p0])` is just the list of all tuples of two consecutive points in the list and `encompasses p seg = det3 (fst seg) (snd seg) p` ≥ 0 states that p lies counterclockwise (or colinear) of the vector $\overrightarrow{(fst\ seg)(snd\ seg)}$. With $(fst\ seg)$ being the first point in the tuple seg and $(snd\ seg)$ being the second point in the tuple `seg`.

```
fun polychain_of where
"polychain_of [] = []"
"polychain_of [p2] = []"
"polychain_of (p1#p2#ps) = (p1, p2) # polychain_of (p2 # ps)"
```

The `list_all P l` predicate states that the condition `P` has to hold for every element in the list `l`. Consequently `list_all (encompasses p) (polychain_of (p0 # ps @ [p0]))` states that p lies inside the polygon defined by `p0 # ps` as it requires that p lies counterclockwise (or colinear) of every edge of the polygon. Therefore the lemma `polygon_eq_convex_hull` states that a point p lies inside the convex polygon defined by `p0 # ps` if and only if p is in the convex hull of `set (p0 # ps)`. Now we have the definition of a convex polygon and the proof that the convex hull of the corners of such a polygon corresponds to the set of all points that lie within the polygon. Based on this we can show that the inspected algorithms, for an input set s, compute a convex polygon according to the definition and that the convex hull that corresponds to this convex polygon is indeed the convex hull of s.

2.2 Jarvis-March Algorithm

2.2.1 Definition of the Algorithm

The Jarvis March or Gift-Wrapping Algorithm is a simple output-sensitive way of calculating the convex hull of a given finite set $S \subseteq \mathbb{R}^2$ of points. It calculates the convex hull by calculating the corresponding convex polygon and returning an ordered list of the corners of the polygon. The algorithm has runtime O(n * h), where n is the number of points in S and h is the number of points that lie on the convex hull or the number of corners on the calculated polygon to be more precicse. The algorithm starts by choosing a point that is guaranteed to lie on the convex hull, for example $p_0 = min_y min_x S$ (the lexicographical minimum). Then the next corner of the convex polygon is found by searching a p_1 such that every point $r \in s$ lies counterclockwise of $\vec{p_0 p_1}$ or is contained in the closed segment between p_0 and p_1, meaning $\forall r \in ps.$ `(ccw'_seg` p_0`)` p_1 `r` should hold. As explained in 2.1.3 we know that such a p_1 exists, because $\forall r \in ps.$ `lex` p_0 `r` holds. In Isabelle the definition for finding the minimum with respect to the total order `(ccw'_seg` p_0`)` is.

```
definition ccw'_seg_min :: " point set ⇒ point" where
"ccw'_seg_min ps = (THE p. p ∈ ps ∧ (∀ q ∈ ps. ccw'_seg p0 p q))"
```

Now from 2.1.1, we know that (p_0, p_1) is an edge of the wanted convex polygon and we know that p_1 is once again a point on the convex hull and a corner of the polygon as (p_0, p_1) fulfills $\forall r \in ps.$ `(ccw'_seg` p_0`)` p_1 `r`. Therefore we can repeat the previous step and search for a p_2 that fulfills $\forall r \in ps.$ `(ccw'_seg` p_1`)` p_2 `r`. Once again according to 2.1.3, we know that $\forall r \in ps.$ `(ccw'_seg` p_0`)` p_1 `r` holds and therefore `(ccw'_seg` p_1`)` is a total order and a unique p_2 exists. Again p_2 has to be a corner of the convex polygon and (p_1, p_2) an edge on of the polygon. The algorithm continues until a $p_h = p_0$ is found to be the next point and stops, because the first corner of the polygon is encountered again. The ordered sequence of points $p_0, p_q, ..., p_{h-1}$ are the corners of the convex polygon and $(p_0, p_1), (p_1, p_2)..., (p_{h-2}, p_{h-1}), (p_{h-1}, p_0)$ are the edges of the polygon. This repeated finding of the next corner is defined as the function `wrap`, where q is the last minimum that was found and ps is the current set of points we want to find the convex hull of.

```
function wrap :: "point ⇒ point set ⇒ point list" where
"wrap q ps =
(if q = p0 then [] else q#(wrap (ccw'_seg_min q ps) (ps - {q}) ) )"
```

The last minimum q is prepended to the list of corners we will return, if we not yet arrived at the first corner $p0$ again. The next corners are found by recursively calling `wrap` with the next corner or minimum `ccw'_seg_min q ps` and the set $ps - \{q\}$. q can be removed from the set of points we search for the next corner, as q can not be a corner of the polygon again. Lastly the algorithm Javis March is defined by an inital call to wrap, but $p0$ is this time not removed from the set ps we search for the next corner, because $p0$ is the only corner we can and must encounter twice.

```
definition "jarvis_march = to_set (wrap (ccw'_seg_min p0 ps) ps)"
```

The `to_set` function just turns the list or corners into the appropriate definition of the set of points that lie inside the polygon (see 2.1.4).

```
fun to_set :: "point list ⇒ point set" where
o_set [] = {p0}" |
o_set [p] = closed_segment p0 p" |
o_set qs = {p. list_all (encompasses p) (polychain_of (p0#qs@[p0]))}"
```

The special cases of `wrap` returning an empty list or an list with only one element need more explaination. If `(wrap (ccw'_seg_min p0 ps) ps) = []`, then we know `ccw'_seg_min p0 ps = p0` has to hold and therefore $\forall r \in ps.$ `ccw'_seg` p_0 p_0 `r`. Intuitively it should be clear, that the only point r that fulfills `ccw'_seg` p_0 p_0 `r` is $p0$ itself and therefore ps has to only contain $p0$ and the convex hull of a single point is a set containing this very point. If `(wrap (ccw'_seg_min p0 ps) ps) = [p]`, then we know $\forall r \in ps.$ `ccw'_seg` p_0 `p r` and $\forall r \in ps.$ `ccw'_seg p` p_0 `r`. Again from geometric intuition it should be clear that $\forall r \in ps.$ `r ∈ closed_segment p0 p` should hold, as `ccw'` p_0 `p r` or `ccw' p` p_0 `r` instantly leads to a contradiction. The last case of the `to_set` function just applies the definition for the set of points inside inside a convex polygon, as introduced in 2.1.4.

2.2.2 Jarvis March is Convex Hull

In the following let $ps \subseteq \mathbb{R}^2$ be the finite set of points of which we want to calculate the convex hull and let $p0 = min_y min_x ps$ be the lexicographical minimum with which we start Jarvis March, i.e. our first corner of the convex polygon. In Isabelle terms, we assume `∀p ∈ ps. lex p0 p , p0 ∈ ps and finite ps`. First we need to show that the recursive wrap function terminates.

```
lemma wrap_dom:
assumes q ∈ qs ∧ p0 ∈ qs
assumes "qs ⊆ ps"
assumes "q = p0 ∨ (∀q' ∈ qs. ccw'_seg p_stl q q')"
shows "wrap_dom (q,qs)"
```

This lemma follows from the step by step description of 2.2.1. In every step our last minimum q was either equal to $p0$ (in the beginning) which fulfills $\forall r \in ps.$ `lex` $\mathtt{p_0}$ `r` or our last minimum fulfilled $\forall r \in ps.$ `ccw'_seg p q r` (found with `wrap`) for some p. In both cases `(ccw'_seg q)` is a total order and a new minimum q_{next} such that $\forall r \in ps.$ `ccw'_seg q q_{next} r` holds, exists (see 2.1.3). So `ccw'_seg_min q qs` and therefore every recursive call to `wrap` is well-defined. Additionally the size of the set with which `wrap` is recursively called decreases in every iteration. Hence the call `(wrap (ccw'_seg_min p0 ps) ps)` will terminate. Now we need to show that the list that `(wrap (ccw'_seg_min p0 ps) ps)` returns represents a correct convex polygon.

```
lemma wrap_sorted:
  shows "sorted_wrt (ccw' p0) (wrap (ccw'_seg_min p0 ps) ps)"

lemma wrap_turns_left:
  shows "turns_only_left (wrap (ccw'_seg_min p0 ps) ps)"
```

We will start with the proof of `sorted_wrt (ccw' p0) (wrap (ccw'_seg_min p0 ps) ps)`. To do this, we first show that the inner call `wrap (ccw'_seg_min q qs) (qs - {q})`, where we assume that $\forall r \in qs.$ `ccw'_seg p q r` holds for the last minimum q and some p, produces a list that is `sorted_wrt (ccw' p0)`.

```
lemma wrap_sorted_ind:
assumes wrap (ccw'_seg_min q qs) (qs - {q}) = ls
assumes q ∈ qs ∧ p0 ∈ qs
assumes qs ⊆ ps
assumes (∀r ∈ qs. ccw'_seg p q r) ∧ (p0 ≠ q)
shows sorted_wrt (ccw' p0) ls
```

The proof works by induction over the list ls. In the inductive case, we assume `ls = a # b # rs` and that the induction hypothesis holds for `b # rs`. Meaning we want to show `sorted_wrt (ccw' p0) a # b # rs` and assume that `sorted_wrt (ccw' p0) b # rs` already holds additional to the other assumptions like $q \in qs$ and `∀r ∈ qs. ccw'_seg p q r .` As the `sorted_wrt (ccw' p0)` predicate only makes sense for list of at least length two, we can assume `ls = a # b # rs` in the inductive case. Due to the properties of `wrap`, `a` and `b` are minima defined by the `ccw'_seg_min` function, for example `a = (ccw'_seg_min q qs)` has to hold. From this, one can show that $\forall r \in (qs - \{a, b\})$.

`ccw'_seg a b r` and $a \neq p0 \wedge b \neq p0$ holds. From $a \neq p0 \wedge b \neq p0$ and $p0 \in qs$, we know that $p0 \in (qs - \{a, b\})$ holds and with that `ccw'_seg a b p0` has to hold. But because we assumed $p0$ to be the lexicographical minimum of ps and $a \in ps \wedge b \in ps$, we know `lex p0 a ∧ lex p0 b` has to hold. From geometric intuition it should be clear, that if `lex p0 a ∧ lex p0 b` and $a \neq p0 \wedge b \neq p0$, it follows that `p0 ∈ closed_segment a b` can not hold. Hence with `ccw'_seg a b p0` we know, that `ccw' a b p0` and therefore `ccw' p0 a b` has to hold. Using the induction hypothesis `sorted_wrt (ccw' p0) b # rs`, we know that $\forall r \in$ `set rs. ccw' p0 b r` holds. From $\forall r \in ps$. `lex` p_0 `r`, we know that `(ccw' p0)` is a total order on ps and therefore also transitive on ps. Each element in the list rs is indirectly picked from ps, which implies `(set rs) ⊆ ps`. So in the end, from $\forall r \in$ `set rs. ccw' p0 b r` and `ccw' p0 a b` follows with transitivity $\forall r \in$ `set (b#rs). ccw' p0 a r`. Again together with the induction hypothesis `sorted_wrt (ccw' p0) a # b # rs` follows, which is what we wanted to show. The final lemma for `sorted_wrt (ccw' p0) (wrap (ccw'_seg_min p0 ps) ps)` works very similar, but a slightly different approach is needed, because in the first step the old minimum $p0$ is not removed from ps for the call to `wrap`.

Now we want to show the second part `turns_only_left (wrap (ccw'_seg_min p0 ps) ps)`. Again we start with the inner call `wrap (ccw'_seg_min q qs) (qs - {q})`.

```
lemma wrap_turns_left_ind:
assumes "wrap (ccw'_seg_min q qs) (qs - {q}) = ls"
assumes q ∈ qs ∧ p0 ∈ qs
assumes qs ⊆ ps
assumes (∀r ∈ qs. ccw'_seg p q r) ∧ (p0 ≠ q)
shows " turns_only_left ls"
```

Similar to before in the inductive case, we assume `ls = k # q # p # rs` and can show using the other assumptions and the induction hypothesis that `turns_only_left q # p # rs` already holds. Now we want to show `turns_only_left k # q # p # rs`, which in this case only requires to show `ccw' k q p`, because of the induction hypothesis. Again the points k q and p are defined by `ccw'_seg_min` and we can show that $\forall r \in (qs - \{k, q\})$. `ccw'_seg k q r` and `ccw'_seg k q p` holds. Now if `p ∈ closed_segment k q` would hold, then k, q and p would be colinear. Also from the previously shown wrap_sorted_ind we know, that `ccw' p0 k q` and `ccw' p0 q p` hold. From this and the fact that k q and p are colinear it would follow that `q ∈ closed_segment k p`, which should be apparent from an geometric viewpoint. But if `p ∈ closed_segment k q` and `q ∈ closed_segment k p` holds, then it can be shown that $p = q$ has to hold, which is a contradiction to for example `ccw' p0 q p`. Therefore `p ∈ closed_segment k q` can not hold and with `ccw'_seg k q p`, it follows that `ccw' k q p` has to hold, which shows the lemma. So `(wrap (ccw'_seg_min p0 ps) ps)` does indeed produce a convex polygon. Finally we can show the lemma `jarvis_eq_convex_hull`.

```
lemma jarvis_eq_convex_hull:
"jarvis_march p0 ps = convex hull ps"
```

Applying the definition of the jarvis_march function, we have to show `to_set (wrap (ccw'_seg_min p0 ps) ps) = convex hull ps`. Why this equality holds if `wrap (ccw'_seg_min p0 ps) ps` returns an empty list or a list with one element was already explained when the `to_set` function was defined (see 2.2.1). If the list `ls = wrap (ccw'_seg_min p0 ps) ps` contains at least two points, we know from `wrap_sorted` and `wrap_turns_left`, that *ls* correctly represents a convex polygon. As *ls* represents a correct convex polygon and contains at least two points, we know `jarvis_march p0 ps = to_set ls` is the set of points inside the corresponding polygon.

```
to_set ls = {p. list_all (encompasses p) polychain_of (p0#ls@[p0])}
```

With `wrap_sorted`, `wrap_turns_left` and `polygon_eq_convex_hull`, we then know that the set of points inside the polygon is equal to the convex hull of the corners.

```
{p. list_all (encompasses p) polychain_of (p0#ls@[p0])} = convex hull (set p0#ls)
```

This is almost what we wanted to show. All points in *ls* are points from *ps*, they are precicsely the points that were identified as corners of the convex polygon that is the convex hull of *ps*. When we start with a call `(wrap (ccw'_seg_min p0 ps) ps)` to the wrap function, we know that in every recursive call to `wrap`, one point will be removed from *ps* until $p0$ is encountered and ls ist returned. The points that are removed from *ps* throughout the recursion are exactly the points in *ls*. With that we also know, that the points `ps - (set ls)` are in the set that is considered for the next corner in every recursive step of `(wrap (ccw'_seg_min p0 ps) ps)`. This implies that for every r in `ps - (set ls)` and every two consecutive corners p, q in *ls* it holds that `ccw'_seg p q r`. This is because q is found as minimum with respect to `(ccw'_seg p)` and r has to be in the set in which we search the minimum. Therefore this r lies inside the polygon defined by *ls*, as `ccw'_seg p q r` $\Longrightarrow$ `det3 p q r` $\geq$ `0` and therefore we know that r lies counterclockwise (or colinear) of every edge of the polygon. The edges of the polygon are exactly the elements of the list `polychain_of (p0#ls@[p0])`. Meaning we know `list_all (encompasses r) polychain_of (p0#ls@[p0])` holds and the inside of the polygon is equal to the convex hull of the corners, hence we also know `r` $\in$ `convex hull (set p0#ls)`. Finally, we know `ps - set (ls)` $\subseteq$ `convex hull (set p0#ls)` and `set (ls)` $\subseteq$ `convex hull (set p0#ls)`, which implies `convex hull (set p0#ls) = convex hull ps`. Using this equality and the previously obtained `jarvis_march p0 ps = convex hull (set p0#ls)`, we get the final assertion.

2.2.3 Computability

In 2.2.2, the lemma `wrap_dom` used, that if we call `wrap` with a valid last minimum like $p0 = min_y\, min_x\, ps$ in `(wrap (ccw'_seg_min p0 ps) ps)`, then in every recursive step with the last minimum being $p \in qs$, `(ccw'_seg p)` is a total order and transitive on the current set $qs \subseteq ps$. From the fact that `(ccw'_seg p)` is transitive it should be clear, that the minimum can be found by looking at every element in qs once, which takes $O(|qs|) \leq O(|ps|) = O(n)$, if n is the number of points in the input set for Jarvis March. Comparing two points according to the `(ccw'_seg p)` predicate can be achieved with the following computable function, which can be shown to be equivalent to the `ccw'_seg` predicate.

```
definition ccw'_seg_fun :: "point ⇒ point ⇒ point ⇒ bool" where
"ccw'_seg_fun p q r =
(det3 p q r > 0 ∨ (det3 p q r = 0 ∧ dist p r ≤ dist p q) )"

lemma ccw'_seg_fun_iff_ccw'_seg:
  assumes p ∈ ps ∧ q ∈ ps
  shows ccw'_seg_fun p_last p q ⟷ ccw'_seg p_last p q
```

p and q have to be in some ps, which fulfills $\forall r \in ps.$ `ccw'_seg p_stl p_last r` $\vee$ $\forall q \in ps.$ `lex p_last r` for everything to be well-defined. Computing the determinant and distance between two points (`dist`) only takes a few arithmetic operations, therefore comparing two points can be seen as an operation that takes $O(1)$. To sum up, given there are h corners in the convex polygon and n points in the input set ps, we have to find h times the minimum with respect to a total order which takes $O(n)$ steps and therefore we get a runtime of $O(h \cdot n)$. The algorithm is simpler than the Graham Scan or the Chan's algorithm and has a worse runtime than both unless h is small. Graham Scan achieves a $O(n \cdot log(n))$ runtime and Chan's algorithm a $O(n \cdot log(h))$ runtime. If h is small Jarvis March can be faster than Graham Scan.

2.3 Graham Scan

2.4 Chans Algorithm

Citation test [Lam94].

Acronyms must be added in `main.tex` and are referenced using macros. The first occurrence is automatically replaced with the long version of the acronym, while all subsequent usages use the abbreviation.

E.g. `\ac{TUM}, \ac{TUM}` ⇒ TUM, TUM

For more details, see the documentation of the `acronym` package[1].

2.4.1 Subsection

See Table 3.1, Figure 3.1, Figure 3.2, **??**.

Table 2.1: An example for a simple table.

A	B	C	D
1	2	1	2
2	3	2	3

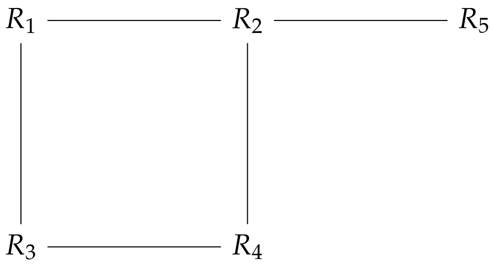

Figure 2.1: An example for a simple drawing.

!TeX root = ../main.tex

[1] `https://ctan.org/pkg/acronym`

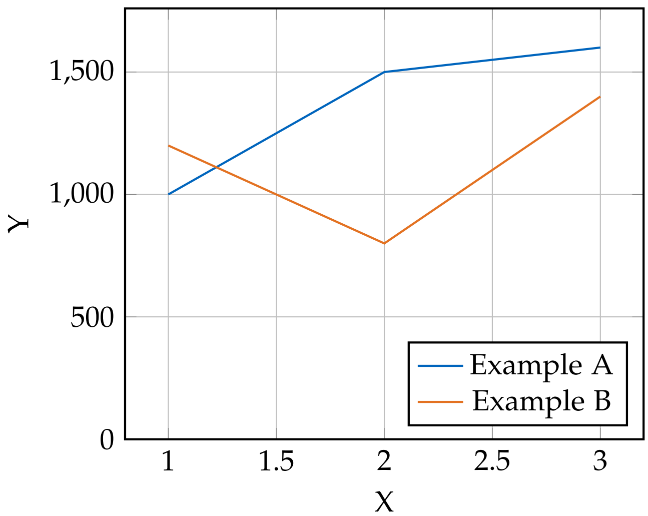

Figure 2.2: An example for a simple plot.

3 Definitions and Algorithms

3.1 Convex Hull

First the Convex Hull will be defined. A set $S \subseteq \mathbb{R}^2$ is convex if for every two points p and q in S it holds that all points on the line segment connecting p and q are in S again. This can be expressed, as the fact that the any convex combination of p and q has to be in S again, i.e. $\{x | \exists u, v \geq 0. p * u + q * v = x\} \subseteq S$ has to hold. The convex hull of a set S is the smallest convex set in which S is contained. There are several ways in which the convex hull can be defined. The convex hull CH of S is the intersection of all convex sets containing S, which is also the definition Isabelle/HOL is going to use. But the convex hull can also be defined as the set of all convex combinations of points in S, which can be proven equivalent to the previous definition. In the two dimensional case for a finite $S \subset \mathbb{R}^2$, the convex hull CH of S is a convex polygon, where the corners of this convex polygon are points from S (see figure 1). [De 00] An edge connecting two points $(p, q) \in S^2$ is an edge of the convex polygon iff. all points lie to the left of the line $\overline{pq}$ connecting p and q. Similarly the set of all edges of the convex polygon can be defined as all $(p, q) \in S^2$ for which all points in S lie to the right of $\overline{pq}$ As this thesis will focus on the two dimensional case and only give an outlook on the the three dimensional case, the examined algorithms compute a convex polygon for a given $S \subset \mathbb{R}^2$.

3.2 Jarvis-March Algorithm

The Jarvis March or Gift-Wrapping Algorithm is a simple output-sensitive way of calculating the convex hull of a given finite set $S \subseteq \mathbb{R}^2$ of points. It calculates the convex hull by calculating the corresponding convex polygon and returning an ordered list of the corners of the polygon. The algorithm has runtime O(n * h), where n is the number of points in S and h is the number of points that lie on the convex hull or the number of corners on the calculated polygon to be more precicse. First we will assume that no three points in S are colinear. The algorithm starts by choosing a point that is guaranteed to lie on the convex hull, for example a $p_0 = min_y min_x S$. Then the next corner of the convex polygon is found by searching a p_1 such that all points in S lie to the left of the line $\overline{p_0 p_1}$. As explained in 3.1 we know that (p_0, p_1) is an edge of the wanted convex polygon and we know that q is once again a point on the conex hull, i.e. a corner of the polygon. Therefore we can repeat the previous step and search for a p_2 such that all points in S lie left to the line $\overline{p_1 p_2}$. Again p_2 has to be a corner of the convex polygon and (p_1, p_2) an edge on of the polygon. The algorithm continues until a $p_h = p_0$ is found to be the next point and stops, because the first corner of the polygon is encountered again. The ordered sequence of points $p_0, p_q, ..., p_{h-1}$ are the corners of the convex polygon and $(p_0, p_1), (p_1, p_2)..., (p_{h-2}, p_{h-1}), (p_{h-1}, p_0)$ are the edges of the polygon. Now without the assumption that no three points are colinear, we require more rigorous definitions. Given a p_i that is a corner of the convex polygon the next corner p_{i+1} has to fulfill the following condition for all $q \in S$. Either q lies strictly left of $\overline{p_i p_{i+1}}$ (p_i , p_{i+1} and q are not colinear) or q is contained in the closed segment between p_i and p_{i+1}. In the following a point q lying strictly left of a line $\overline{p_i p_{i+1}}$ will be expressed as q lying counterclockwise of the line $\overline{p_i p_{i+1}}$. This clarification avoids, that points which are not a corner but still lie on the convex hull are ignored (see figure 2). The algorithm is simpler than the Graham Scan or the Chan's algorithm and has a worse runtime than both unless h is small. Graham Scan achieves a $O(nlog(n))$ runtime and Chan's algorithm a $O(nlog(h))$ runtime. If h is small Jarvis March can be faster than Graham Scan.

```
lemma turns_only_right st ⟹
turns_only_right (grahamsmarch qs st)
```

3.3 Graham Scan

3.4 Chans Algorithm

Citation test [Lam94].

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[1] `https://ctan.org/pkg/acronym`

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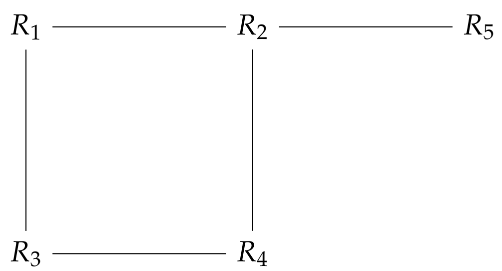

Figure 3.1: An example for a simple drawing.

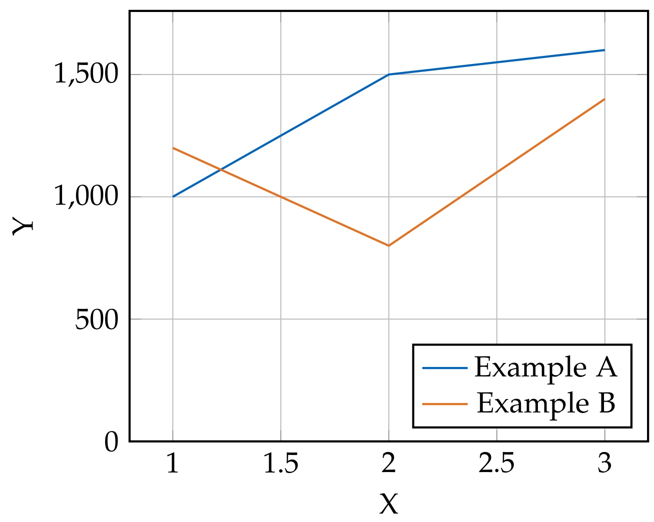

Figure 3.2: An example for a simple plot.

Abbreviations

TUM Technical University of Munich

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Bibliography

[De 00] M. De Berg. *Computational geometry: algorithms and applications*. Springer Science & Business Media, 2000.

[Lam94] L. Lamport. *LaTeX : A Documentation Preparation System User's Guide and Reference Manual*. Addison-Wesley Professional, 1994.