Tree-based Multiple Hypothesis Testing with General FWER Control

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1. Mathematical Formulation

Here I consider a testing procedure without data splitting.

In the tree-structured multiple hypotheses setting, we say $\boldsymbol{X}_k$ is a data set to construct each kth p-value. Assume that, whenever j is a child node of i in the tree structure, $\boldsymbol{X}_j \subseteq \boldsymbol{X}_i$.

Let p_k be the p-value of the parent node and p_{k+1} be the p-value of the subsequent child of that parent node. When we fix some $\alpha \in (0, 1]$, our basic observation is

$$P(p_{k+1} > \alpha | p_k \leq \alpha_1) \leq P(p_{k+1} > \alpha | p_k \leq \alpha_2),$$

whenever $\alpha_1 \leq \alpha_2$. This is because, we can understand our procedure is actually using

$$p'_{k+1} = \max(p_1, ...p_k, p_{k+1}),$$

as a p-value implemented in the $k+1$th node, where p_i is all other ancestors of the p_{k+1}. This can be another (maybe equivalent) understanding of the **monotonicity condition** in Goeman and Solari (2010).

From this we can recall the **positive regression dependency on each one from a subset I_0(or PRDS on I_0)** from Benjamini and Yekutieli (2001), which states when D is a non-decreasing set(so whenever $x \in D$ and $x \leq y$, then $y \in D$),

$$P(\boldsymbol{X} \in D | X_i = X)$$

is non-decreasing in x.

We also provide some notation for the tree-structured hypotheses. Let us have a tree with depth of d where for some fixed integer k, k is a number of children for every parent node (so we consider *multiplicative trees*). Then we say N is the number of total nodes and n as a number of total clusters, where

$$N = \sum_{r=1}^{d} k^{r-1}$$
$$n = 1 + \sum_{r=1}^{d-1} k^{r-1}.$$

Then we can summarize our approach by

1. First, generate N many p-values from each node.
2. Then, after doing some *local, cluster-wise adjustments*, get n many p-values.

My idea is that,

- **Since $p_1, ..., p_n$ (so the p-values after local adjustments) are still has a PRDS property, we can try to directly apply Benjamini and Yekutieli (2001)'s approach to get general FDR or FWER control.**
- **We can also consider which kind of local adjustments give valid and more powerful tests.**

I will further check about,

- **Idea of data splitting used in the recent simulation.**
- **Prove explicitly the validness of locally adjusted p-values from different methods.**

REFERENCES

Yoav Benjamini and Daniel Yekutieli. The control of the false discovery rate in multiple testing under dependency. *The Annals of Statistics*, 38:6:3782–3810, 2001.

Jell J. Goeman and Aldo Solari. The sequential rejection principle of familywise error control. *The Annals of Statistics*, 38:6:3782–3810, 2010.