Final Project: Measurement of the Gravitational Constant G

Abstract:

The goal of this experiment is to measure the gravitational constant, $G = 6.7430 \times 10^{-11}\,\frac{Nm^2}{kg^2}$ [1]. This was done in two ways. The first method employed involved equating Hooke's Law to Newton's Gravitational Law. The gravitational constant was found to be $G = (6.72 \pm 0.68) \times 10^{-11}\frac{Nm^2}{kg^2}$, using this method. The second method used, known as the Cavendish experiment, involves measuring the gravitational attraction between objects. The gravitational constant for the first iteration of the Cavendish method, which involved oscillatory motion between the large masses, was determined to be $G = (4.8 \pm 0.2) \times 10^{-7}\frac{Nm^2}{kg^2}$. The second iteration of the Cavendish method, which used the equalization of torques between the torsion in the torsional band and the torques acting on the system by the masses, determined the gravitational constant to be $G = (6.4 \pm 1.6) \times 10^{-11}\frac{Nm^2}{kg^2}$.

Motivations:

The gravitational constant has been measured over a dozen times in the last 40 years, but the results have varied by a lot more than would be expected from random and systematic. It has been found that measured G values oscillate over time like a sine wave with period of 5.9 years. It is believed that something else, something unknown is affecting this measurement [2]. Researchers have proposed that the variations in G are caused by changing motions in Earth's core, or perhaps some other geophysical process. This means that the measurement of G is still relevant in modern day physics despite having been first measured by Henry Cavendish in his 1798 experiment [1]. The gravitational constant is used commonly in astrophysics calculations, classical mechanics, and even civil engineering to design infrastructure to withstand gravitational loads.

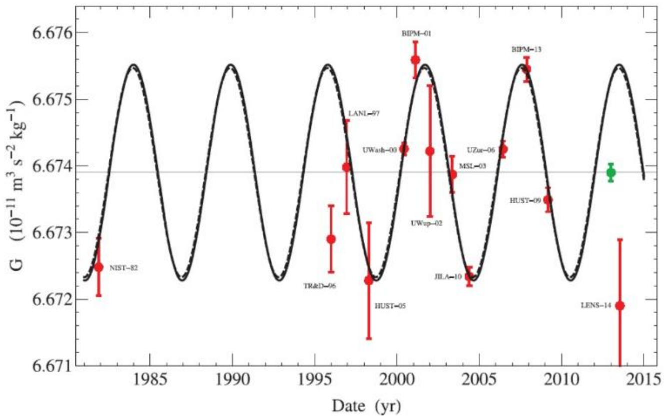


Figure 1: Sine wave which illustrates how G changes over time. [2]

Method 1: Mass on a Spring

Theory:

A mass on a spring is one of the simplest yet effective methods of measuring the gravitational constant, G. A spring, pointed directly downwards with a mass attached to the end, will stretch in accordance with Hooke's Law, where $F = -k\Delta x$, where F is the force applied, k is the spring constant of the specific spring, and Δx is the displacement of the end of the spring. The minus sign just indicates the direction in which the force acts (downwards). From this, we can derive that the force applied, F, is equivalent to the gravitational force on the mass, and from there the gravitational constant can be calculated.

Experimental Setup:

1. **Torsion Pendulum:** The heart of the experiment was a spring with known spring constant hanging downwards from some fixed point, where the mass was attached to the hanging end. The extension of the spring is then proportional to the gravitational force on the mass acting on the spring.
2. **Masses:** One or more masses was attached to the end of the spring which then feels a gravitational force directed towards the Earth. This force then acted upon the spring, extending it.
3. **Displacement Measurement:** The displacement of the mass when it extends due to the force of gravity was the main driver in the determination of the force acting upon the spring, which will then be translated into gravitational force.

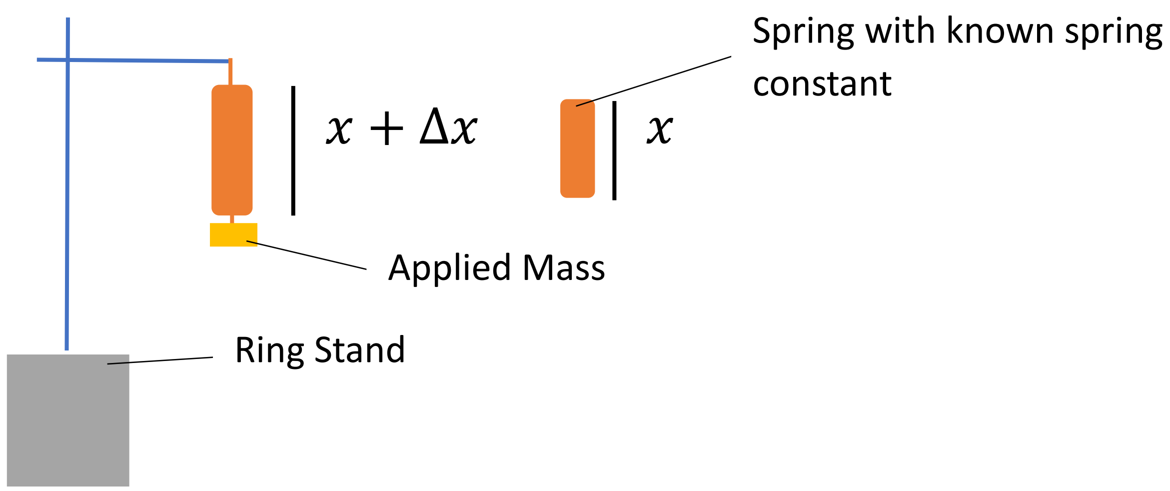


Figure 2: Model of the spring-mass setup

Procedure:

1. **Initial Equilibrium:** The spring system must first be oriented in a way where there is no gravitational force extending the spring, so that a measurement can be taken on the total length of the system when the spring is not extended, so that a change in position can then be measured.
2. **Release of the mass:** The system was then placed in a position to where it was hanging freely with the mass at the end of the spring, at which time the mass extended the spring and was left to rest until it came to a complete stop and was no longer oscillating (if it began oscillating when released).
3. **Data Collection:** The displacement of each mass can then be calculated using Hooke's Law to determine the force acting on the spring in each case, which was then used to calculate G, which was then averaged together from the multiple trials to provide a more accurate result.
4. **Measurement of k:** We will use springs of known spring constants. The springs which we will use have k values of 10 N/m, 20 N/m and 40 N/m respectively. The uncertainty as per the manufacturer on these values is ±10%.

Calculation of G:

The formula for calculating G using the spring is relatively straightforward, as all that is necessary is setting Hooke's law equal to the equation for gravitational force, from which algebra can be performed to solve for G in terms of known values:

$$F = \frac{GMm}{R^2} \tag{1}$$

$$F = -k\Delta x \tag{2}$$

$$-k\Delta x = \frac{GMm}{R^2} \tag{3}$$

$$G = \frac{-k\Delta x R^2}{Mm} \tag{4}$$

Where:

- G is the gravitational constant.
- ω is the period of oscillation for the spring.
- k is the spring constant of the given spring.
- Δx is the displacement of the mass on the spring.
- M is the mass of the Earth (known).
- m is the mass of the small mass on the spring.

The uncertainty on this G value can be calculated using the following equation:

$$dG = \sqrt{\left(dk\frac{xR^2}{mM}\right)^2 + \left(dx\frac{kR^2}{mM}\right)^2 + \left(-dm\frac{kxR^2}{mM^2}\right)^2} \tag{5}$$

Where:

- dk is the uncertainty on the spring constant.
- dm is the uncertainty on the mass measurement.
- dx is the uncertainty on the extension of the spring measurement.
- All other values are as previously stated.

Results:

We noticed that there was a 5cm offset for all three springs in use, i.e., the springs were extending 5cm less than was expected for the given spring constants. Our hypothesis is that the true equilibrium point of the spring is 5cm shorter than the length of the spring, and they can't compress that far based on the physical limitations of the coils. This is factored in by adding 5cm to all Δx measurements. Eight separate trials were carried out using springs with spring constants of 10 N/m, 20 N/m, and 40 N/m respectively and various weights. Two variables were measured; the mass attached to the spring (m), and the extension of the spring when the mass was attached (Δx). The mass attached was measured using an electronic balance and the spring extension was measured using a meter stick. The control variable was the mass attached to the spring and the response variable was the extension of the stretched spring. The uncertainty on the spring constants were taken to be $\pm 10\%$ as given in the manufacturer's specifications. The uncertainty on m was taken to be ± 0.1 gram and the uncertainty on Δx was taken to be ± 0.2 cm. Using equations (4) and (5), the Gravitational constant and the corresponding uncertainty can be calculated. G was determined to be $G = (6.72 \pm 0.68) \times 10^{-11}\frac{Nm^2}{kg^2}$.

Finding the Torsional Constant of the Wire

Theory:

An important aspect in the Cavendish experiment is knowing the torsional constant of the wire that the system of smaller masses is suspended from, as it dictates the amount of torque acting on the system at given angle θ by the equation:

$$\tau = -\kappa\theta \tag{6}$$

To find the torsional constant of the wire, a new setup is needed, where the wire suspends a cylindrical mass of known moment of inertia *I,* as shown in figure 3:

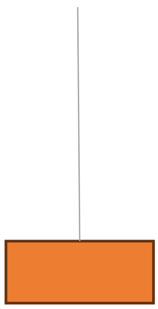


Figure 3: Cylindrical mass suspended from wire to find κ

Once the system is set up, the mass is set into oscillatory motion about the axis of the wire with small angles of motion so that the small angle approximation can be used, resulting in the approximate period of motion being described as:

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{7}$$

In this way, when the values of I and T can be measured, the equation can be rearranged to solve for the torsional coefficient *κ.* The mass used had a moment of inertia about its central axis of $I = (2.0 \pm 0.2) \times 10^{-6} kg * m^2$. The first torsional band that was used was then put into this oscillatory motion, yielding an oscillatory period of $t = (42.9 \pm 0.1)\ s$, which yielded a torsional constant of $\kappa = (1.1 \pm .1) \times 10^{-9} \frac{N*m}{rad}$. However, in the process of experimentation, this torsional band snapped under the weight of the system, and so a sturdier wire was needed instead. This wire, when put under the same experimentation, had an average oscillatory period of $t = (9.6 \pm 0.1)\ s$, resulting in a torsional constant of $\kappa = (2.1 \pm .2) \times 10^{-8} \frac{N*m}{rad}$, which is the used value going forward. This experiment relies on very small torsional constants to get accurate data, so having to increase the tortional constant we were using by a factor of approximately 20 was unfortunate, but ultimately unavoidable.

Method 2: Cavendish Experiment

<u>Theory:</u>

The main way in which we are going to measure G is using the Cavendish method. To carry out the experiment, a light, rigid rod about 11cm long was used. Two small lead spheres of mass 22g were attached to the ends of the rod and the rod suspended by a thin wire. When the rod becomes twisted, the torsion of the wire begins to exert a torsional force that is proportional to the angle of rotation of the rod. The more twist of the wire, the more the system pushes *backwards* to restore itself towards the original position. Two large lead spheres of mass 1600g, are brought near the smaller spheres attached to the rods. Since all masses attract, the large spheres exert a gravitational force upon the smaller spheres and twist a measurable amount. The system then will eventually come to rest. From equation (6), the torque on the torsion wire is proportional to the deflection angle θ of the balance.

<u>Experimental Set-Up:</u>

We have a Helium Neon Laser which is reflected off two mirrors before striking the system's mirror. This mirror is situated at the centre of the rod connecting the small masses. In set-up 1, the rod rotates since the large and small masses attract to one another the reflected laser will sweep a certain angle for its period of oscillation (θ). The reflected laser is incident on three photodetectors placed 1.0cm apart. From the incident light on the photodiodes, we determine the period of oscillation, T. Since the laser will precess past all of the photodiodes, a meter stick is placed behind them so that, during the oscillations, the position of the laser can be measured at each apex, from which the angle θ can be found. For this set-up the large masses are placed at stationary positions and the small masses are free to rotate. In set-up 2, both the large and small masses are rotatable. Instead of a photodetector the detection method is just a ruler.

Procedure:

1. We first placed the large masses in fixed positions, and the smaller masses are allowed to come to equilibrium with the new torques acting on them by the larger masses.
2. The small masses are then slightly bumped to knock them out of the equilibrium position. The equilibrium position is the point at which the small masses come to rest when the large masses are not present in the system.
3. We then leave the system to run for a couple of minutes. The laser will oscillate backwards and forwards hitting the photodetectors as it does so. The small masses are attracted to the large masses due to the gravitational force between them. During this time, the distance that the laser travels along the meter stick is recorded as well, so as to find the oscillatory angle θ.
4. Next, we look at set up 2. We must first determine the equilibrium position. We leave the small masses oscillating until they come to rest and record where this position is on the ruler.
5. We then place the large masses on a rotatable mechanism. We first place them in one position. We then measure the separation between the small and large spheres. After some time, the system comes to rest. We record this position on the metre stick.
6. We then place the large spheres in the opposite position. Again, the separation between the small and large spheres is measured. The position at which the system comes to rest is recorded. Using trigonometry, we can determine the necessary angle θ.

Set-Up 1:

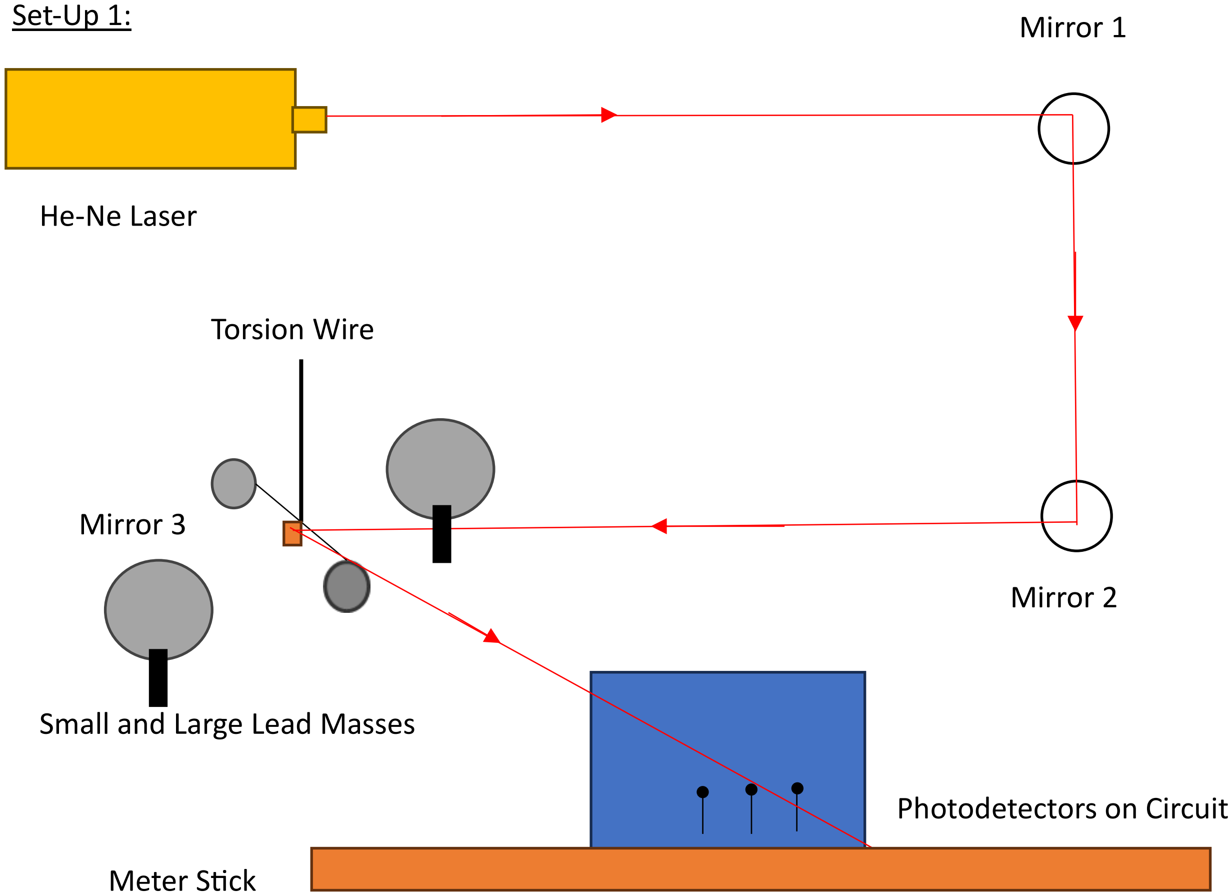
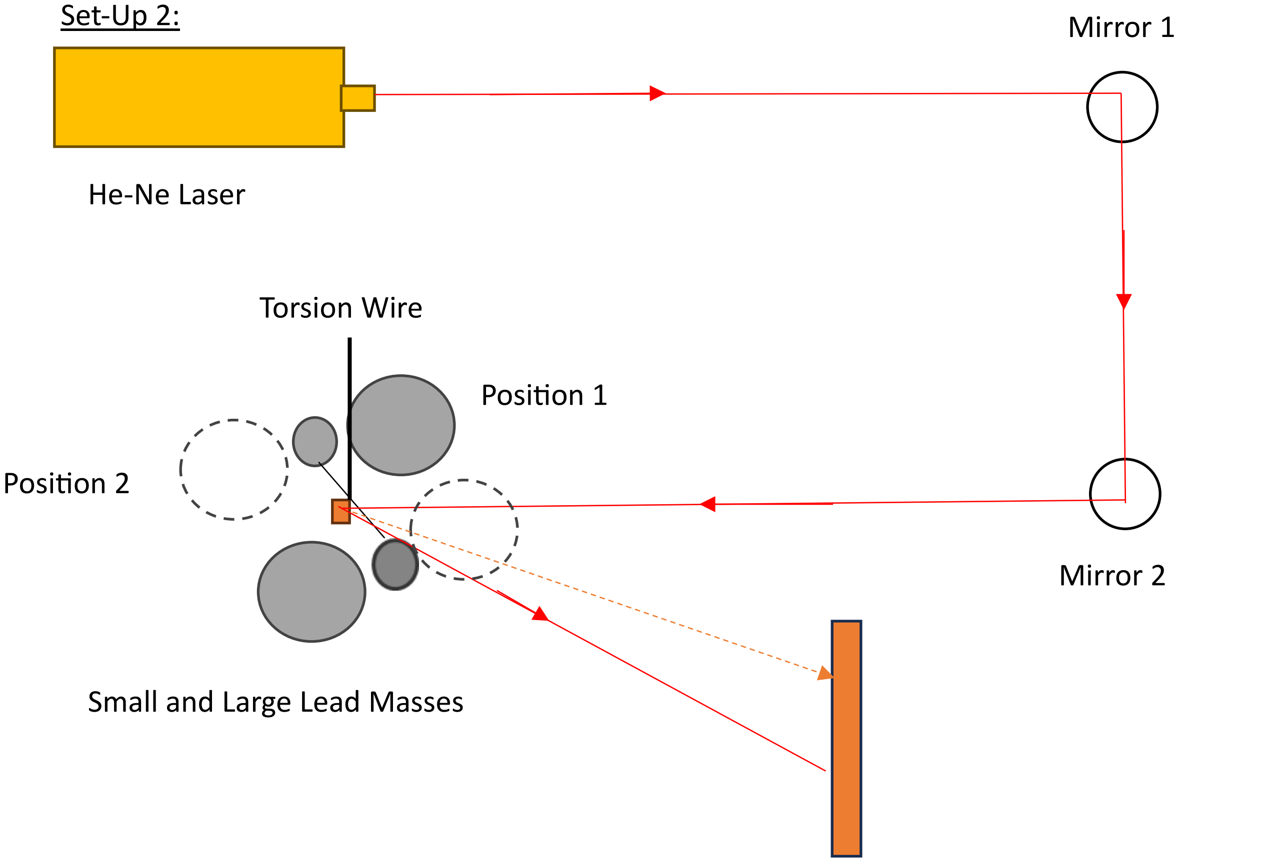


Figure 4: Setup of Cavendish method 1, where the laser oscillates across multiple photodiodes.

Set-Up 2:



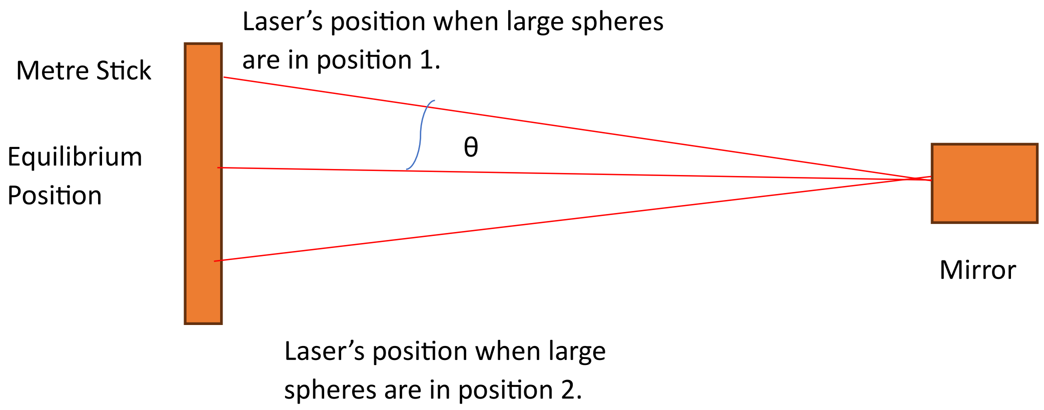


Figure 4: Setup of Cavendish method 2, where the beam is deflected based on equilibrium of torques between gravity and torsion.

<u>Calculation of G:</u>

To begin with the first Cavendish set-up, the large masses were placed very near to the smaller masses so that the separation between the centre of mass between each small and large mass was approximately 85mm, with 1mm of separation between the edges of the masses. The large masses averaged at 1.64kg, and the smaller masses average at 22g. From there, the system was given a very slight perturbation to go into oscillatory motion. Using the photodiodes to determine the position of the beam as a function of time, a sinusoidal fit was able to be produced, as shown in figure 5:

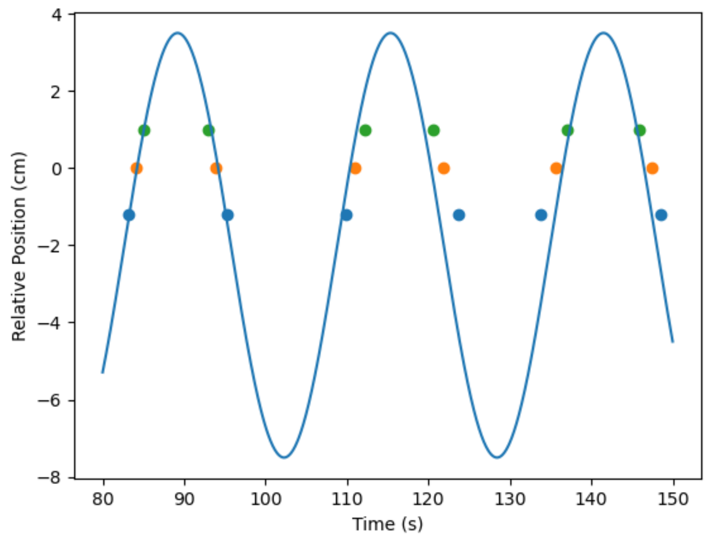


Figure 5: Model fit of laser position, centred on middle photodiode, as a function of time

This model fit provided an oscillatory period of T=21.6s, and an angle θ of 0.15 radians. From there, we were able to use the following equation:

$$G = \frac{2\pi^2 R r^2 \theta}{M T^2} \tag{8}$$

Where R is the radius from the wire to the small masses, measured to be 99.5mm, r is the separation between the centres of masses between the small and large masses, θ is the angle of rotation of the system, M is the mass of the larger masses, and T is the period of motion.

With the secondary set-up, the large masses were once again placed very near the smaller masses, with 1mm of edge separation in the first position and 2.5mm of edge separation in the second position, for a total separation of centres of mass being 85mm and 86.5mm, respectively. The laser was then pointed to a screen that was 3.50$\pm$ 0.02m away. Between the two equilibrium points, the laser moved 6$\pm$2mm along the screen, providing an angle θ of 0.0017$\pm$0.0005 radians. This could then be used in the following equation to solve for G:

$$G\left(\frac{2MmR}{r^2}\right) = -\kappa\theta \tag{9}$$

Results:

From these two different methods of measurement, two vastly different values of G were measured. For the primary set-up, when all values were plugged into (8) to solve for G, a final value of $G = (4.8 \pm 0.2) \times 10^{-7} \frac{Nm^2}{kg^2}$ was found, which is approximately 4 orders of magnitude higher than the accepted value of G. However, when all of the appropriate values were plugged into (9) and the equation was rearranged to solve for G, a final value of $G = (6.4 \pm 1.6) \times 10^{-11} \frac{Nm^2}{kg^2}$ was measured. This value is much closer to the accepted value, however there is a relatively high uncertainty associated with this value.

Conclusion:

From this experiment, we attempted to measure the value of the gravitational constant G, using 3 different methods. The first was using a series of hanging springs of known spring constant k and masses of known mass m, equating the universal law of gravitation (1) and Hooke's law (2) to then solve for the gravitational constant G, ultimately resulting in a value of $G = (6.72 \pm 0.68) \times 10^{-11} \frac{Nm^2}{kg^2}$, which is within 1 bound of uncertainty to the accepted value of $G = 6.7430 \times 10^{-11} \frac{Nm^2}{kg^2}$. The next method, involving the Cavendish method and oscillatory motion, provided a much less accurate result of $G = (4.8 \pm 0.2) \times 10^{-7} \frac{Nm^2}{kg^2}$. It is suspected that the unfortunate change in torsional wire may be the cause of this great deviation from the expected value, since it would have acted a greater torque on the system and caused the torques resulting from the gravitation to be largely irrelevant. The third and final method, however, was much more precise, which ultimately yielded a value of $G = (6.4 \pm 1.6) \times 10^{-11} \frac{Nm^2}{kg^2}$, which is once again within one bound of uncertainty to the accepted value, however the uncertainty associated with this measurement is extremely high, likely due to the high uncertainty on the positional measurement of the beam on the screen that was 3.5m away. Reducing this uncertainty would require a refocusing of the laser at that distance, minimizing the uncertainty associated with that measurement.

References:
1. Wikipedia Contributors. (2019, November 15). *Gravitational constant*. Wikipedia; Wikimedia Foundation. https://en.wikipedia.org/wiki/Gravitational_constant.
2. Zyga, L., & Phys.org. (n.d.). *Why do measurements of the gravitational constant vary so much?* Phys.org. Retrieved November 1, 2023, from https://phys.org/news/2015-04-gravitational-constant-vary.html#:~:text=The%20variations%20in%20G%20are.
3. Wikipedia Contributors. (2019, January 15). *Cavendish experiment*. Wikipedia; Wikimedia Foundation. https://en.wikipedia.org/wiki/Cavendish_experiment.
4. Xue, C., Liu, J.-P., Li, Q., Wu, J.-F., Yang, S.-Q., Liu, Q., Shao, C.-G., Tu, L.-C., Hu, Z.-K., & Luo, J. (2020). Precision measurement of the Newtonian gravitational constant. *National Science Review*, *7*(12), 1803–1817. https://doi.org/10.1093/nsr/nwaa165.

Appendix:

The raw data from mass on spring method is given in the table below:

$k\ \left(\frac{N}{m}\right)$	$m\ (g)$	$\Delta x\ (cm)$
10 ± 1	100 ± 0.1	$(5 \pm 0.2) + 5$
10 ± 1	150 ± 0.1	$(9.8 \pm 0.2) + 5$
20 ± 2	150 ± 0.1	$(1.7 \pm 0.2) + 5$
20 ± 2	199.9 ± 0.1	$(4.1 \pm 0.2) + 5$
20 ± 2	299.9 ± 0.1	$(8.8 \pm 0.2) + 5$
40 ± 4	299.9 ± 0.1	$(3.2 \pm 0.2) + 5$
40 ± 4	499.3 ± 0.1	$(8.1 \pm 0.2) + 5$
40 ± 4	599.3 ± 0.1	$(10.4 \pm 0.2) + 5$

The code used to calculate G and its uncertainty, dG, using the mass on the spring method is given below:

```
In [1]: import numpy as np

In [2]: def G(k,x,m):
            return (k)*(x)*((6.378e6+1650)**2)/((5.97219e24)*m)

In [3]: a=G(10,0.05+0.05,0.1)
        a

Out[3]: 6.814909459092895e-11

In [4]: b=G(10,0.098+0.05,0.15)
        b

Out[4]: 6.724043999638324e-11

In [5]: c=G(20,0.041+0.05,0.1999)
        c

Out[5]: 6.204669942745908e-11

In [6]: d=G(20,0.088+0.05,0.2999)
        d

Out[6]: 6.271807304800398e-11

In [7]: e=G(20,0.017+0.05,0.15)
        e

Out[7]: 6.08798578345632e-11

In [8]: f=G(40,0.081+0.05,0.4993)
        f

Out[8]: 7.152037966282149e-11
```

```
In [9]: g=G(40,0.032+0.05,0.299)
        g
Out[9]: 7.475887299606924e-11

In [10]: h=G(40,0.104+0.05,0.5993)
         h
Out[10]: 7.004812659438049e-11

In [11]: Gavg=(a+b+c+d+e+f+g+h)/8
         print('Average value of G =', Gavg)

         Average value of G = 6.71701930188262e-11

In [12]: def dG(k,x,m,dk,dx,dm):
             return np.sqrt(((dk*x*(6.378e6+1650)**2)/(m*5.97219e24))**2+((dx*k*(6.378e6+1650)**2)/(m*5.97219e24))**2+
                            ((dm*k*x*(6.378e6+1650)**2)/(m**2*5.97219e24))**2)

In [13]: a=dG(10,0.05+0.05,0.1,1,0.002,0.0001)
         a
Out[13]: 6.9502053834511055e-12

In [14]: b=dG(10,0.098+0.05,0.15,1,0.002,0.0001)
         b
Out[14]: 6.7853098925254565e-12

In [15]: c=dG(20,0.041+0.05,0.1999,2,0.002,0.0001)
         c
Out[15]: 6.352831751360528e-12

In [16]: d=dG(20,0.088+0.05,0.2999,2,0.002,0.0001)
         d
Out[16]: 6.337366022639003e-12

In [17]: e=dG(20,0.0170+0.05,0.15,2,0.002,0.0001)
         e
Out[17]: 6.353568356297128e-12

In [18]: f=dG(40,0.081+0.05,0.4993,4,0.002,0.0001)
         f
Out[18]: 7.234924246833486e-12

In [19]: g=dG(40,0.032+0.05,0.299,4,0.002,0.0001)
         g
Out[19]: 7.69508003531506e-12

In [20]: h=dG(40,0.104+0.05,0.5993,4,0.002,0.0001)
         h
Out[20]: 7.06364778876123 84e-12

In [21]: dGavg=((a+b+c+d+e+f+g+h)/8)
         print('Average uncertainty on G is',dGavg)

         Average uncertainty on G is 6.846616684647876e-12

In [22]: print('G=',Gavg,'+/-',dGavg)

         G= 6.71701930188262e-11 +/- 6.846616684647876e-12
```

Who Wrote What:

David wrote the abstract, motivations, mass on spring method, theory, experimental set-up, and procedure for Cavendish Method. All experimental figures were done by David also. Jacob provided the second half of the abstract, as well as the torsional constant section, and the calculations, results, and conclusion sections for the Cavendish setups.