Exploring Financial Dynamics: A Comprehensive Study of the Black-Scholes Partial Differential Equation Model

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Abstract:

This project undertakes an exhaustive exploration of the Black-Scholes Partial Differential Equation (PDE) model, a pioneering framework in quantitative finance for pricing European-style options. Originating in the early 1970s, the Black-Scholes model has played a pivotal role in shaping modern financial markets. Our investigation encompasses the foundational principles, assumptions, and real-world applications of the model, aiming to provide a thorough understanding of its strengths, limitations, and adaptability.

Key Terms:

1. Option: An option is a contract giving the buyer the right, but not the obligation, to buy or sell an underlying asset (a stock or index) at a specific price on or before a certain date. An option is a security, just like a stock or bond, and constitutes a binding contract with strictly defined terms and properties. [1]
2. Security: The term "security" is defined broadly to include a wide array of investments, such as stocks, bonds, notes, debentures, limited partnership interests, oil and gas interests, and investment contracts. [2]
3. Derivative: A derivative is a financial instrument whose value is derived from an underlying asset, commodity, or index. A derivative comprises a contract between two parties who agree to take action in the future if certain conditions are met, most commonly to exchange an item of value. [3]
4. Volatility: Volatility measures how much the price of a security, derivative, or index fluctuates. [4]
5. Brownian Motion: The unpredictable, random movement of small particles in a fluid caused by the constant barrage of surrounding medium molecules.
6. Itô Process: An Itô process is a type of stochastic process described by Japanese mathematician Kiyoshi Itô, which can be written as the sum of the integral of a process over time and of another process over a Brownian motion. [12]
7. Put Option: A put option is a contract that gives its holder the right to sell a number of equity shares at the strike price, before the option's expiry. If an investor owns shares of a stock and owns a put option, the option is exercised when the stock price falls below the strike price. [13]
8. Call Option: A call option is a contract between a buyer and a seller to purchase a certain stock at a certain price up until a defined expiration date. The buyer of a call has the right, not the obligation, to exercise the call and purchase the stocks. [14]

Introduction:

In the dynamic realm of financial markets, the Black-Scholes Partial Differential Equation (PDE) model stands as a cornerstone in understanding and predicting the behaviour of derivative securities. The Black-Scholes model, which was created in the early 1970s by Fischer Black, Myron Scholes, and Robert Merton, transformed the area of quantitative finance by offering a mathematical framework for valuing options in the European manner.

This project delves into the intricate landscape of the Black-Scholes PDE model, aiming to unravel its underlying principles, assumptions, and implications. The Black-Scholes equation has become a fundamental tool for risk management, options pricing, and investment strategy formulation, making it an essential subject of study for financial analysts, mathematicians, and economists alike.

Our exploration will commence by explaining the foundation upon which the Black-Scholes model rests, including the assumptions that govern its applicability and the derivation of the PDE itself. We will dissect the components of the equation, investigating the roles played by volatility, interest rates, and underlying asset prices in determining option prices. Moreover, we will scrutinize the limitations and challenges associated with the Black-Scholes model, offering insights into its real-world applications and areas where it may fall short.

Historical Background:

The Black-Scholes model was developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes who were all finance professors at MIT at the time. [5] Before the Black-Scholes model, there was a lack of a unified and rigorous mathematical framework for valuing options. Traditional methods used by financial practitioners were often ad-hoc, relying on rules of thumb and intuition. The need for a more systematic approach to options pricing became increasingly apparent as financial markets evolved and the demand for derivatives grew. In 1970, Fischer Black, an economist and consultant, developed the groundwork for the Black-Scholes model. His initial work laid the foundation for understanding the relationship between risk and return in financial markets. Myron Scholes, a Canadian economist, and Robert Merton, an American economist, collaborated with Fischer Black to refine and extend the initial ideas. Their collaboration led to the publication of the groundbreaking paper titled "The Pricing of Options and Corporate Liabilities" in the Journal of Political Economy in 1973. The Black-Scholes model quickly gained recognition for its elegance and practicality. It provided a closed-form solution for calculating the theoretical price of European options, which was a significant advancement. The model's impact was felt not only in academia but also in financial markets, where it became a standard tool for options pricing and risk management. In 1997, Myron Scholes and Robert Merton were awarded the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (commonly referred to as the Nobel Prize in Economic Sciences) for their contributions to the development of the Black-Scholes model. Unfortunately, Fischer Black had passed away in 1995 and was not eligible for the Nobel Prize. [6]

Assumptions of the Black-Scholes Model:

1. <u>Geometric Brownian Motion:</u> The underlying asset's price movement is assumed by the model to follow a geometric Brownian motion. This implies that the logarithm of the asset's price evolves as a random walk with constant drift and volatility. The geometric Brownian motion assumption allows for the modelling of continuous, random price movements.
2. <u>Constant Volatility</u>: The model assumes that the volatility of the underlying asset's returns is constant over the life of the option. This assumption is made to simplify the mathematical

calculations and is one of the model's limitations, as volatility in real markets may vary over time.

3. <u>Risk-Free Interest Rate:</u> The model assumes the existence of a risk-free interest rate, denoted by 'r.' This rate is the return that can be earned with certainty from investing in a risk-free asset, such as a government bond. The assumption of a constant risk-free rate simplifies the discounting of future cash flows.
4. <u>Continuous Trading:</u> The model assumes that trading is continuous, meaning that investors can buy and sell the underlying asset and options at any time. This assumption allows for the creation of a continuously hedged portfolio, a key concept in the derivation of the Black-Scholes equation.
5. <u>No Dividends:</u> The original Black-Scholes model assumes that the underlying asset does not pay any dividends during the life of the option. This assumption facilitates the modelling of the asset's price dynamics and the calculation of the option's value.
6. <u>European-Style Options</u>: The concept is intended exclusively for European-style options, which have an expiration date and can only be exercised at that time. This simplifies the option pricing calculation compared to American-style options, which can be exercised at any time before expiration.
7. <u>Market Efficiency:</u> The model is specifically designed for European-style options, which can only be exercised at the expiration date. This simplifies the option pricing calculation compared to American-style options, which can be exercised at any time before expiration.

Derivation of the Black-Scholes Partial Differential Equation:

To derive the Black-Scholes Equation, we need to cover the basic ideas behind a stochastic process, enough to understand an Itô process and Itô's Lemma. We only cover Standard Brownian Motion to understand the theoretical processes behind stochastic calculus. This Standard Brownian motion is sometimes referred to as a Wiener Process.

<u>Basic Definitions of Standard Brownian Motion:</u>
We must cover a few basic definitions before we start:

1. Brownian Motion at time t is defined as B_t.
2. Standard Brownian Motion begins at the origin, $B_0 = 0$.
3. The difference in Brownian Motion from time t_1 to time t_2 is written as $B_2 - B_1$, where B_1 is the Brownian Motion at time t_1 and B_2 is the Brownian Motion at time t_2.
4. $B_2 - B_1$ is distributed normally with a mean, $\mu = 0$ and a standard deviation, $\sigma = t_2 - t_1$.
5. dB_t is the derivative of B_t and the integral (without limits of integration) of dB_t can be written as B_t.

<u>Intuition of Brownian Motion:</u>
In figure 1, we see an example of $f'(x)$. This is a real valued continuous function. We can do many things with this function. We can integrate $f'(x)$ from point a to point b, we can take the

derivative of $f'(x)$ at any point and it can be shown that $f'(x)$ is continuous at any point.

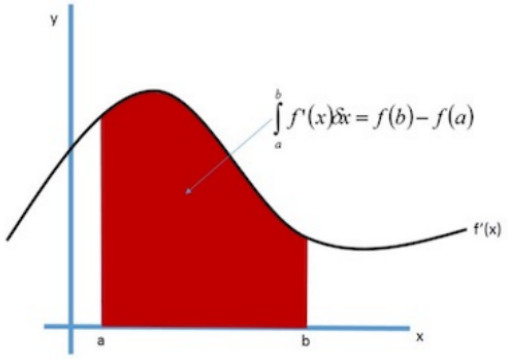

Figure 1: A real valued function $f'(x)$ with a smooth curve. [7]

Now, let's look at Brownian Motion. Using computer software such as Python or Mathematica, it is possible to simulate a 2-D Brownian Motion from one time period to another. In figure 2, we see a single simulated Brownian Motion walk with standard parameters from time 0 to time 1. This motion doesn't follow a normal or predictable paths compared to the path of a function such as $f'(x)$. Here, the sample oscillates in an unpredictable and random way, based on the normal distribution. Also, this motion would be different every time it's simulated. Theoretically, it's impossible to replicate the same motion more than once. [8] This leads to several problems. We are now unable to quantify the motion using a function, we cannot prove that the motion is continuous from point a to point b, we can't integrate using Newtonian methods with respect to time from point a to point b and we can't take the derivative with respect to time at any point. A point to note is that while individual simulations of Brownian Motion don't yield any kind of pattern, multiple simulations do. As the number of simulations increases, we begin to see the emergence of the normal distribution curve horizontally. You can see this normal distribution shape in figure 3.

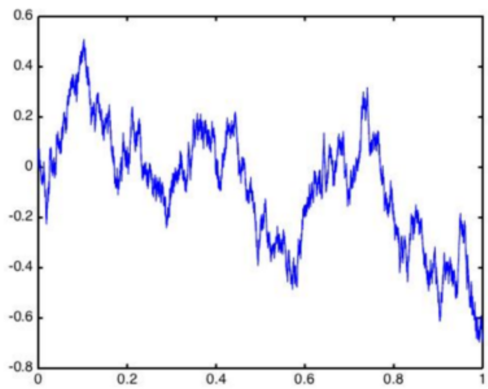

Figure 2: Graph of Standard Brownian Motion from time t=0 to time t=15. [9]

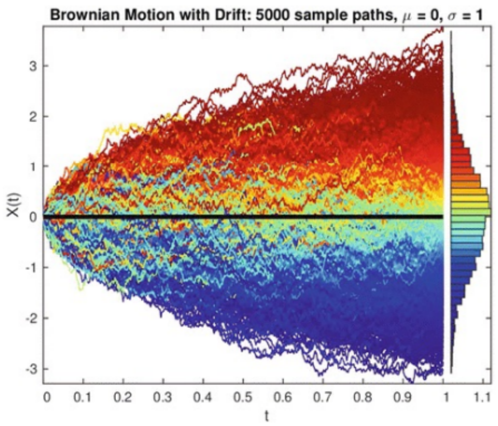

Figure 3: 5000 sample paths of Standard Brownian Motion. [8]

Derivatives with Respect to Brownian Motion:

B_t is not differentiable intuitively, at least not in a conventional sense. However, if we investigate B_t using a methodology akin to that of Newtonian calculus, we get some intriguing conclusions. We can calculate a function's derivative in Newtonian calculus,

$$g(t) = g(0)e^{rt}$$

very easily. The derivative is,

$$\frac{dg}{dt} = rg(0)e^{rt}$$

Now if we add Brownian Motion with a scalar alpha term to $g(t)$ to make a new equation $s(t)$,

$$s(t) = s(0)e^{rt+\alpha B_t}$$

When trying to differentiate $s(t)$ we find,

$$\frac{ds}{dt} = \left[r + \alpha\frac{dB_t}{dt}\right]s(0)e^{rt+\alpha B_t} = \left[r + \alpha\frac{dB_t}{dt}\right]s(t) = rs(t) + \alpha s(t)\frac{dB_t}{dt}$$

However, at this time, we have no way of quantifying,

$$\frac{dB_t}{dt}$$

We can attempt a separation of variables,

$$\frac{ds}{dt} = rs(t) + \alpha s(t)\frac{dB_t}{dt}$$

Becomes,

$$ds = rs(t)dt + \alpha s(t)\frac{dB_t}{dt}dt$$

The dt's cancel in the second term to give,

$$ds = rs(t)dt + \alpha s(t)dB_t$$

Again, we have a term we are not familiar with,

$$dB_t.$$

Therefore, it does not seem that we can currently take the derivative of a process that involves a term of Brownian Motion. We will return to this discussion later.

Integration with Respect to Brownian Motion:

Due to the fact that we cannot take the derivative of terms with Brownian Motion, or a stochastic process we will look at how integration with respect to Brownian Motion should behave in theory.

Constant Function: Let $x(t) = k$, a constant. Taking the integral of this from 0 to t_1 with respect to Brownian Motion,

$$\int_0^{t_1} x(t)dB_t$$

Calculating this using Newtonian methods,

$$\int_0^{t_1} x(t)dB_t = \int_0^{t_1} kdB_t = k\int_0^{t_1} dB_t = k\left[B_{t_1} - B_0\right] = kB_{t_1}$$

From this, it is evident that integrating a constant with respect to Brownian Motion gives us the constant k, multiplied by the Brownian Motion at the upper limit of integral.

Step Function: A piecewise continuous function with a finite number of parts is called a step function. [10] $\Phi(t)$ is a step function with n steps. The value of this step function at $i = 1, 2, 3, \ldots, n$ is denoted as k_i. When this is integrated it is the same as taking the sum of the integral of each constant with respect to Brownian Motion. So,

$$\int_a^b \Phi(t)dB_t = \sum_{i=1}^{n} \int_a^b k_i dB_t$$

Calculating this as a sum of all the integrals of each constant k_i,

$$\sum_{i=1}^{n} \int_a^b k_i dB_t = \sum_{i=1}^{n} k_i\left[B_b - B_a\right]$$

From the basic definitions we know that it is normally distributed so we can say,

$$\left[B_b - B_a\right] \sim N(0, (b - a))$$

From this we can conclude that the integral of a function with a term involving Brownian Motion will have a normal distribution.

Well Behaved Function: Integration usually begins with a Riemann sum in Newtonian calculus. The Riemann sum for a real and continuous function, $f(t)$, to find the area underneath the curve from point a to point b is given as,

$$Area = \sum_a^b f(t)\Delta t$$

Sending the length of Δt to zero,

$$\sum_{a}^{b} f(t)\Delta t = \int_{a}^{b} f(t)dt$$

This result is the standard Newtonian integral. This can be applied to Brownian Motion in a similar way. Our integral with a step function can be defined as shown previously as,

$$\sum_{i=1}^{n} \int_{a}^{b} k_i dB_t = \sum_{i=1}^{n} k_i \left[B_b - B_a\right]$$

Where,

$$\left[B_b - B_a\right] \sim N\left(0, K^2(b - a)\right)$$

If we want to evaluate an integral $f(t)$ with respect to Brownian motion, we can partition the function $f(t)$ into n number of partitions of widths Δt. Evaluating a well-behaved function by means of a Riemann sum, we can compare this to the way in which the step function was integrated. By this same process as before, should we want to calculate, $\int_{a}^{b} f(t)dB_t$, we can approximate it to a step function, and divide $f(t)$ into n partitions with length Δt. We get a closer and closer estimate of the function as smaller and smaller partitions. So,

$$\int_{a}^{b} f(t)dB_t \approx \int_{a}^{b} f(t)dt \sim N(0, \int_{a}^{b} [f(t)]^2\, dt)$$

As it is impossible to completely calculate a function to Brownian Motion, it only gives us an estimate of the integral.

<u>Random Variable:</u> Y_t is a random variable defined as,

$$Y_t = y_0 \quad \left(0 < t < t_1\right)$$

$$Y_t = y_1 \quad \left(t_1 < t < 1\right)$$

If we integrate said random variable with respect to Brownian Motion and $t < t_1$,

$$\int_{0}^{t} Y_t dB_t = y_0 B_t$$

Should $t > t_1$, the integral is,

$$\int_{0}^{t} Y_t dB_t = y_0 B_{t_1} + y_1\left(B_t + B_{t_1}\right)$$

If we put these together, the entire integral can be written as,

$$\int_{0}^{t} Y_t dB_t = y_0 B_t \quad \left(0 < t < t_1\right)$$

$$\int_{0}^{t} Y_t dB_t = y_0 B_t + y_1(B_1 + B_t) \quad \left(t_1 < t < 1\right)$$

Therefore, as a result, we can calculate the integral with respect to Brownian Motion of a random variable.

Itô Calculus:

Each of these examples, concepts and procedures emphasises a recurring theme. The idea is that every process should be divided into two parts: one that can be assessed using Newtonian methods and the other that must be evaluated using stochastic methods. This appears from using a discrete situation in which we can distinguish between the certainty of our t terms and the unpredictability of our Brownian Motion terms. The Itô Process's approach and mentality are based on this concept of separation.

Itô Process:

A sophisticated method known as an Itô Process divides a stochastic process, x_t, into the sum of two integrals: one related to time and the other to Brownian Motion.

$$x_t = x_0 + \int_0^t \sigma_t dB_t t + \int_0^t \mu_t dt$$

In this case, σ and μ are the processes that are reliant on time and can be integrated with regard to Brownian Motion and time, respectively, whereas x_0 is a constant factor.

Itô Differential Equation:

If we take the derivative of the Itô Process we obtain the subsequent differential equation,

$$dx_t = \mu_t dt + \sigma_t dB_t$$

But since we are unable to take the derivative with regard to Brownian Motion using standard methods, this equation remains unevaluable. This will be used as our stochastic process derivative so that we may perform the computations we want to.

Itô's Lemma Pre-Remarks:

One equation that is comparable to the "chain rule" in Newtonian calculus is Itô's Lemma. I have provided an informal proof of Itô's Lemma using a Taylor power series approximation, which offers the contextual rationale underlying the formal proof, even though the formal proof is outside the purview of this report.

Itô's Lemma: If $f\left(x, x_t\right)$ is a twice differentiable scalar function, where x_t is an Itô Process. Then,

$$df = (\mu_t \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2)dt + \frac{\partial f}{\partial x}\sigma_t dB_t$$

A prerequisite for deriving the Black-Scholes Equation is Itô's Lemma.

Taylor Series Approximation of Itô's Lemma: Given our conditions for an Itô Differential Equation, let x_t be an Itô Process that meets them.

$$dx_t = \mu_t dt + \sigma_t dB_t$$

Generally, we may express the Taylor expansion of a two-variable, twice-differentiable scalar function $f(x, t)$ as,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2$$

We may rewrite df as follows if we equate x_t and $\mu_t + \sigma_t dB_t$ for dx,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\left(\mu_t dt + \sigma_t dB_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\mu_t^2 dt + 2\mu_t \sigma_t dt dB_t + \sigma_t^2 dB_t^2\right)$$

dt^2 and $dtdB_t$ will go to zero faster than dt and dB_t^2 when we take the limit of df as $dt \to 0$. This will lead us to the conclusion,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\mu_t dt + \frac{\partial f}{\partial x}\sigma_t dB_t + 0 + 0 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dB_t^2$$

Replace dB_t^2 with dt because of the quadratic variance of a Wiener Process, [11],

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\mu_t dt + \frac{\partial f}{\partial x}\sigma_t dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dt$$

dt and dB_t terms are finally factored out and separated,

$$df = \left(\mu_t \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2\right)dt + \frac{\partial f}{\partial x}\sigma_t dB_t$$

Which is Ito’s Lemma.

Black-Scholes Equation:

The following formula can be used to determine the value of an option, V, of a stock price S if it follows an Itô Process, with r being the interest rate,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Derivation:

Let S be an Itô Process-following stock price. Consequently, S’s Itô Differential Equation is,

$$dS = \mu S dt + \sigma S dz \quad (1)$$

Assume that the price of a call option or other derivative that is reliant on S has a twice differentiable function, denoted by V. Itô's Lemma allows us to write dV as,

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma S dz \quad (2)$$

Take note of how σ and μ are now dependent on S rather than t. It is also possible to write (1) and (2) separately over a time interval Δt as,

$$\Delta S = \mu S \Delta t + \sigma S \Delta z \quad (3)$$

And,

$$\Delta V = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2\right)\Delta t + \frac{\partial V}{\partial S}\sigma S \Delta z \quad (4)$$

Currently, S and V adhere to the identical Itô Process. As a result, we can price an option and do away with the Itô Process if we choose a portfolio. We will choose a portfolio of short 1 derivative and long $\frac{\partial V}{\partial S}$ stock shares. We'll soon make evident why we chose this portfolio. The value of our portfolio is denoted by Π. By definition,

$$\Pi = -V + \frac{\partial V}{\partial S} S \quad (5)$$

The discrete form of this equation over the time interval t can instead be expressed as,

$$\Delta \Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S \quad (6)$$

We shall now replace our (3) and (4) in (6),

$$\Delta \Pi = -\left[\left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2\right)\Delta t + \frac{\partial V}{\partial S}\sigma S \Delta z\right] + \frac{\partial V}{\partial S}[\mu S \Delta t + \sigma S \Delta z] \quad (7)$$

$$\Delta \Pi = -\frac{\partial V}{\partial S}\mu S \Delta t - \frac{\partial V}{\partial t}\Delta t - \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2 \Delta t - \frac{\partial V}{\partial S}\sigma S \Delta z + \frac{\partial V}{\partial S}\mu S \Delta t + \frac{\partial V}{\partial S}\sigma S \Delta z \quad (8)$$

$$\Delta \Pi = -\frac{\partial V}{\partial t}\Delta t - \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2 \Delta t \quad (9)$$

$$\Delta \Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2\right)\Delta t \quad (10)$$

In the absence of a stochastic variable, Δz, this portfolio is essentially risk-free over the duration Δt. We have developed a portfolio that will yield instantaneous rates of return over the time period t, because there are no arbitrage opportunities, asset trading is continuous, and all securities have the same short-term constant interest rate. Thus, $\Delta\Pi$ can be written as,

$$\Delta \Pi = r \Pi \Delta t \quad (11)$$

We can now enter our (5) and (10) into (11), and the following equation is what we get,

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2\right)\Delta t = r\left(-V + \frac{\partial V}{\partial S} S\right)\Delta t \quad (12)$$

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2 = r\left(-V + \frac{\partial V}{\partial S} S\right) \quad (13)$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{dS^2}\sigma^2 S^2 = r\left(V - \frac{\partial V}{\partial S} S\right) \quad (14)$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{dS^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad (15)$$

Which is the Black-Scholes PDE, meaning our derivation is finished.

Solving the Black-Scholes Partial Differential Equation to obtain Black Scholes Formula:

The Black-Scholes PDE can be numerically solved using common numerical analysis techniques, such as a kind of finite difference method, once the PDE, along with boundary and terminal conditions, has been derived for a derivative. It is feasible to find an accurate formula in some circumstances. Black and Scholes achieved this in the instance of a European call. The answer is straightforward conceptually. Given that the underlying stock price S_t in the Black-Scholes model moves in a geometric Brownian motion, the distribution of S_T, conditional on its price S_t at time t, is a log-normal distribution. Next, the derivative's price is just the discounted expected payoff

$E\left[e^{-r(T-t)}K(S_T)|S_t\right]$. If the reward function K is analytically tractable, $E\left[e^{-r(T-t)}K(S_T)|S_t\right]$ may be computed analytically; if not, it can be computed numerically.

Remember that the PDE above has boundary conditions in order to perform this for a call option,

$$C(0,t) = 0 \text{ for all } t$$

$$C(S,t) \sim S - Ke^{-r(T-t)} \text{ as } S \to \infty$$

$$C(S,T) = max\{S - K, 0\}$$

The option's value at the moment of maturity is provided by the final condition. As S approaches 0 or infinity, more circumstances could arise. In general, different boundary conditions will yield different solutions, so some financial insight should be used to choose appropriate conditions for the given situation. For instance, common conditions used in other situations are delta to vanish as S goes to 0 and gamma to vanish as S goes to infinity; these will yield the same formula as the conditions above. The PDE's solution provides the option's value at any previous time, $E[max\{S - K\},\ 0\}]$. We identify the PDE as a Cauchy–Euler equation, which may be solved by applying the change-of-variable transformation to convert it into a diffusion equation,

$$\tau = T - t$$

$$u = Ce^{r\tau}$$

$$x = ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau$$

The Black-Scholes PDE then transforms into a diffusion equation,

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$$

$C(S,T) = max\{S - K, 0\}$ was the terminal condition but it now becomes an initial condition,

$$u(x,0) = u_0(x) := K\left(e^{max\{x,0\}} - 1\right) = K\left(e^x - 1\right)H(x)$$

Where $H(x)$ is the Heaviside step function. The Heaviside step function, or the unit step function, usually denoted by H or θ (but sometimes u, 1 or 𝟙), is a step function named after Oliver Heaviside, the value of which is zero for negative arguments and one for positive arguments. [16] In the $S,\ t$ coordinate system, the Heaviside function translates to the enforcement of the boundary data, which demands that when $t = T$,

$$C(S,T) = 0 \ \forall\, S < K$$

With the assumption that both $S, K > 0$. Under this supposition, it is equal to the maximum function across all x in the real numbers, excluding $x = 0$. Because it is not true for $x = 0$, the equality between the max function and the Heaviside function above is only valid in the context of distributions. This is significant even if it is subtle because it eliminates the necessity for the Heaviside function to be defined or even finite at $x = 0$. Given an initial value function, $u(x, 0)$, and the diffusion equation to solve, we have used the traditional convolution method.

$$u(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} u_0(y)exp\left[-\frac{(x-y)^2}{2\sigma^2\tau}\right]dy$$

Which, following some manipulation, produces,

$$u(x, \tau) = Ke^{x+\frac{1}{2}\sigma^2\tau}N(d_+) - KN(d_-),$$

Where $N(....)$ is the standard normal cumulative distribution function and,

$$d_+ = \frac{1}{\sigma\sqrt{\tau}}[(x + \frac{1}{2}\sigma^2\tau) + \frac{1}{2}\sigma^2\tau]$$

$$d_- = \frac{1}{\sigma\sqrt{\tau}}[(x + \frac{1}{2}\sigma^2\tau) - \frac{1}{2}\sigma^2\tau]$$

Up to time translation, these are the same answers that Fischer Black found in 1976. The Black-Scholes equation's solution can be obtained by returning μ, x, τ, and v to their initial set of variables. This final Black-Scholes model is given as,

$$C = N(d_1)S_t - N(d_2)Ke^{-rt}$$

$$\text{where, } d_1 = \frac{ln(\frac{S_t}{K})+(r+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{t}$$

This is for European call options. For European put options, the Black-Scholes Model formula similar but the formula involves $N(-d_1)$ and $N(-d_2)$.

Applications of Black Scholes and Basic Python Model:

```
import math
from scipy.stats import norm
```

```
S = 45   # Underlying Price
K = 40   # Strike Price
T = 2    # Time to Expiration
r = 0.1 # Risk-Free Rate
vol = 0.1 # Volatility (σ)
```

```
d1 = (math.log(S/K) + (r + 0.5 * vol**2)*T ) / (vol * math.sqrt(T))
d2 = d1 - (vol * math.sqrt(T))
C = S * norm.cdf(d1) - K * math.exp(-r * T) * norm.cdf(d2)
P = K * math.exp(-r * T) * norm.cdf(-d2) - S * norm.cdf(-d1)
print('The value of d1 is: ', round(d1, 4))
print('The value of d2 is: ', round(d2, 4))
print('The price of the call option is: $', round(C, 2))
print('The price of the put option is: $', round(P, 2))
```

```
The value of d1 is:  2.3178
The value of d2 is:  2.1764
The price of the call option is: $ 12.27
The price of the put option is: $ 0.02
```

r - This is the risk-free interest rate over the life of the option. The risk-free interest rate is an important factor in option pricing because it reflects the time value of money. Investors can earn a risk-free return by investing in risk-free assets such as government bonds. The inclusion of the risk-free interest rate in the Black-Scholes Model helps account for the opportunity cost of tying up capital in the option rather than investing it in a risk-free asset.
S – If I was to go into the stock market now and buy one share of the example stock used in this model it would cost me S.
K – If an option can be exercised only at the expiration date, it is called European. In this model we will deal with European call and put options.

European Call Option:

European call option (on a stock): Gives the holder the right (but no obligation) to buy one share of the stock (from the writer) at time T for the price K.

The value of the option at expiration is $(S_T - K)^+ = max(S_T - K, 0)$. For example, if at time 0 the stock price is $45 and the strike price is $42, then at T=1 (expiration is at time 1) the person who owns the option has the “option” to buy the stock for $42 when it is actually valued at $48. Therefore the value of the option at time 1 is $(S_T - K)^+ = (48 - 42)^+ = 6$

European Put Option:

European put option (on a stock) : Gives the holder the right (but no obligation) to sell one share of the stock (to the writer) at time T for the price K. The value of the option at expiration is $(K - S_T)^+ = max(K - S_T, 0)$. For example, if at time 0 the stock price is $45 and the strike price is $42, then at T=1 (expiration is at time 1) the person who owns the option has the “option” to buy the stock for $42 when it is actually valued at $48. Therefore the value of the option at time 1 is $(K - S_T)^+$=$(42 - 48)^+ = 0$

The Volatility (σ): Is the standard deviation of the stock’s prices. This tells us how much the stock moves.

$N(d_2)$: Finds the probability that the call option exercises in the money. That is it is finding the probability that is greater than the strike price at the time of expiration.

$N(d_1)$: This function gives the probability that a standard normal random variable will be less than or equal to. It's used in the Black-Scholes formula to determine the probability of the option expiring in-the-money.

Filling in values to our Black Scholes model in Python and varying the stock price, the volatility and the interest rate we see how these variables vary the price of our Option.

Stock Price	Strike Price	Time to Expriation	Volatility	Risk free interest rate	Theoretical Put Option Price	Theoretical Call Option Price
100	105	30 days	0.9	0.02	56.59	98.96
100	105	30 days	0.8	0.02	55.48	97.85
100	105	30 days	0.6	0.02	50.08	92.46
100	105	30 days	0.5	0.02	44.82	87.19
100	105	30 days	0.3	0.02	27.2	69.58
100	105	30 days	0.2	0.02	15.17	57.55
100	105	30 days	0.1	0.02	3.34	45.72
100	105	30 days	0.05	0.02	0.17	42.54
100	105	30 days	0.001	0.02	0	42.37
25	80	10 days	0.25	0.6	0	24.8
25	80	10 days	0.25	0.5	0	24.46
25	80	10 days	0.25	0.4	0	23.53
25	80	10 days	0.25	0.3	0.03	21.04
25	80	10 days	0.25	0.25	0.18	18.61
25	80	10 days	0.25	0.2	0.92	15.09
25	80	10 days	0.25	0.175	1.87	12.97
25	80	10 days	0.25	0.15	3.56	10.71
25	80	10 days	0.25	0.125	6.37	8.45
25	80	10 days	0.25	0.1	10.75	6.32
25	80	10 days	0.25	0.075	17.25	4.46
25	80	10 days	0.25	0.05	26.47	2.95
25	80	10 days	0.25	0.025	39.13	1.82
25	80	10 days	0.25	0.0125	46.99	1.39

Figure 4: Table showing the call and put options for various stocks calculated using the Black-Scholes formula.

From inspection the volatility and interest variables affect the black scholes model massively below we have graphed the volatility against price of the put and call options and interest against the price of the call and put options.

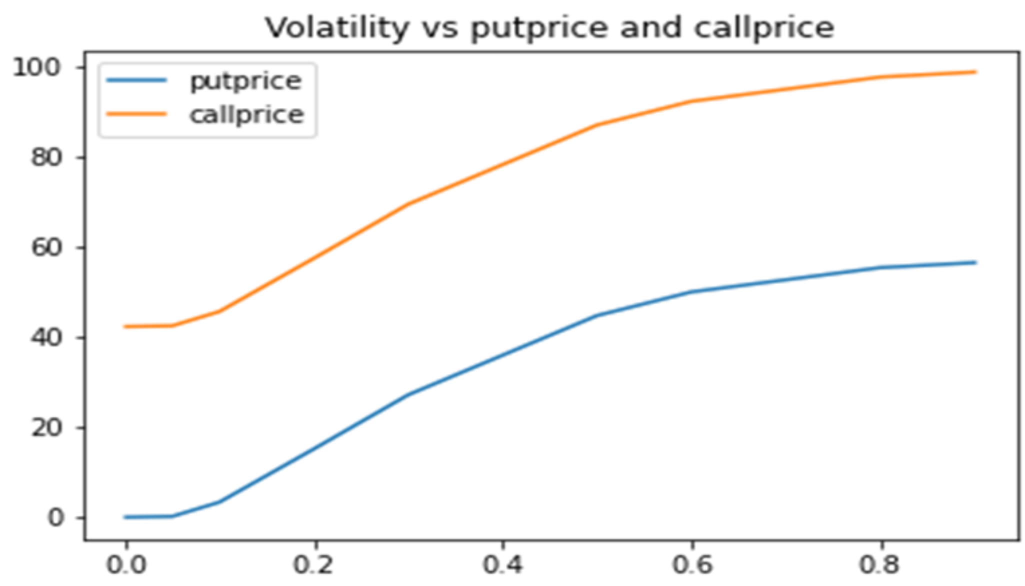

Figure 5: Graph showing how the volatility varies the prices of call and put options using Black-Scholes formula. Volatility is given on the x-axis and the call/put price at time t is on the y-axis.

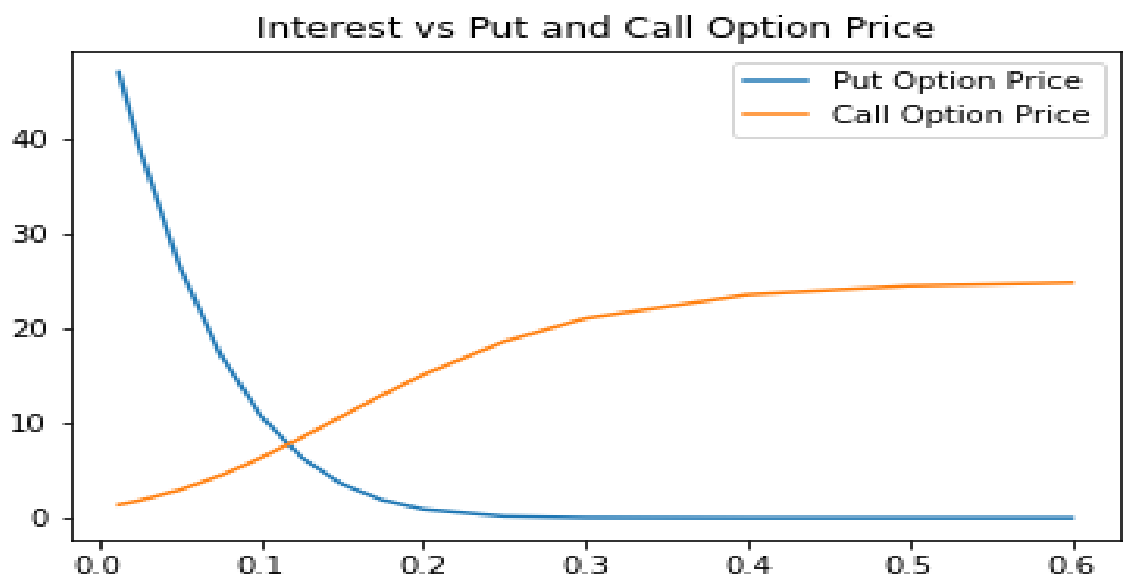

Figure 6: Graph showing how the interest varies the prices of call and put options using Black-Scholes formula. Interest is given on the x-axis and the call/put price at time t is on the y-axis.

Limitations of the Black-Scholes Model:

For the purpose of pricing European-style options, it is often useful. It does, however, have a number of drawbacks, and its presumptions might not always hold true in actual financial markets. The following are some of the Black-Scholes model's main drawbacks:

1. Constant Volatility Assumption: The model makes the assumption that volatility will be constant for the duration of the option, though this is rarely if ever the case in practice. The assumption that volatility would remain constant over time may result in incorrect option pricing.
2. Constant Interest Rate Assumption: The model makes the assumption that interest rates are always risk-free. In actuality, option prices are often impacted by changes in interest rates.
3. No Dividends: The original Black-Scholes model does not take into account dividend payments, which can be significant for certain stocks. This limitation is addressed by extensions of the model, such as the Black-Scholes-Merton model.
4. European-Style Options Only: The model is designed for European-style options, which can only be exercised at expiration. This affects the usefulness of the model as it cannot be used on alternate options such as, American-style which can be exercised at any time before expiration. While there are ways to adjust the model for American options, it adds complexity.
5. Market Frictions: The model assumes frictionless markets, with no transaction costs or taxes. In reality, these factors can impact trading and the profitability of options strategies.
6. Skewed and Fat Tailed Distributions: The Black-Scholes model assumes that stock prices follow a log-normal distribution. However, this often overlooks sudden market spikes or drops which can often occur especially in market crashes. These events often have effectively zero possibility in the model, which is of course inaccurate.
7. Limited time frame: The model is only effective over a short time-frame over a longer time frame the models assumptions will begin to break down. This can lead to issues when people incorrectly try to use Black-Scholes to try and predict future events.
8. Behavioural factors: The model does not consider behavioural factors, such as investor sentiment or market irrationality, which can influence option prices.
9. Limited to Single Underlying Asset: Only a single underlying asset is intended for use in the original model. For more complicated financial instruments or portfolios with numerous assets, it might not be as useful.

Despite these limitations, the Black-Scholes model has been influential and foundational in the field of financial derivatives. Traders and investors often use it as a starting point, but adjustments and alternative models may be necessary to better capture the complexities of real-world markets.

Conclusion:

To sum up, our investigation into financial dynamics, especially as seen via the Black-Scholes Partial Differential Equation (PDE) model, has been illuminating and thought-provoking. We explored the complexities of option pricing and risk management through an extensive investigation, obtaining insightful knowledge about the intricate realm of financial markets. The Black-Scholes PDE, which offers a mathematical framework for pricing options and risk management, has shown to be a potent tool for comprehending the dynamics of financial derivatives. Our research has improved our comprehension of the model's theoretical underpinnings and clarified its practical uses in actual financial situations. Throughout our exploration of the Black-Scholes model, we saw firsthand the fine equilibrium between risk and reward that characterises the financial world, from the model's assumptions to the PDE's derivation to its implications for option pricing. The model's capacity to reflect the subtleties of market dynamics was demonstrated by the sensitivity of option pricing to variables like volatility, time to expiration, and underlying asset values. But it's important to recognise the constraints and presumptions that the Black-Scholes model has. Financial models must always be improved and adjusted due to the dynamic nature of financial markets, which is marked by shifting volatility regimes and variable risk variables. For both researchers and practitioners, finding more reliable and accurate models is an ongoing problem. To sum up, our investigation into the Black-Scholes PDE model has improved our comprehension of financial dynamics and sparked an interest in going deeper and beyond the parameters of this ground-breaking model. Future research and innovation are encouraged by the dynamic interaction between theory and practice in the field of finance, which guarantees that our comprehension of financial dynamics will advance in tandem with the complexity of the markets we aim to understand.

References:

1. Stock Market Loss. (n.d.). What is an Option? Put and Call Option Explained. [online] Available at: https://www.stockmarketloss.com/practice/options/#:~:text=An%20option%20is%20a%20contract.
2. www.tn.gov. (n.d.). *What is a Security?* [online] Available at: https://www.tn.gov/commerce/securities/investors/what-is-a-security.html#:~:text=The%20term%20%22security%22%20is%20defined.
3. www.bankrate.com. (n.d.). What is a derivative? [online] Available at: https://www.fidelity.com/insights/investing-ideas/glossary-derivative#:~:text=A%20derivative%20is%20a%20financial.
4. Kuepper, J. (2019). Volatility Definition. [online] Investopedia. Available at: https://www.investopedia.com/terms/v/volatility.asp.
5. Investopedia. (n.d.). *What Is the Black-Scholes Model?* [online] Available at: https://www.investopedia.com/terms/b/blackscholes.asp#:~:text=Developed%20in%201973%20by%20Fischer.
6. Wikipedia Contributors (2019). *Black–Scholes model*. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_model.
7. T. Habb, What in the world will I ever use this for? Integration, Environmental Economics (2014).
8. A. Petters, and X. Dong, Stochastic calculus and geometric brownian motion mode, An Intro. Math. Finance. App. Springer (2016) 253-327.
9. D. Khoshnevisan, and Y. Xiao, Stochastic analysis and related topics, Springer (2017) 179- 206.
10. BYJUS. (n.d.). *Step Function - Definition, Domain and Range, Graph and Solved Examples*. [online] Available at: https://byjus.com/maths/step-function/#:~:text=In%20Mathematics%2C%20a%20step%20function.
11. H. Stark, and J.W. Woods, Probability, random processes, and estimation theory for engineers, Prentice-Hall, Inc. (1986).
12. www.tau.ac.il. (n.d.). *Ito process - Knowino*. [online] Available at: https://www.tau.ac.il/~tsirel/dump/Static/knowino.org/wiki/Ito_process.html#:~:text=An%20Ito%20process%20is%20a.
13. Investopedia. (n.d.). *How Is a Put Option Exercised?* [online] Available at: https://www.investopedia.com/ask/answers/06/putoptionexcercise.asp#:~:text=Key%20Takeaways.
14. www.fidelity.com. (n.d.). *Learn the basics about call options - Fidelity*. [online] Available at: https://www.fidelity.com/learning-center/investment-products/options/call-options-basics#:~:text=A%20call%20option%20is%20a.
15. WASHBURN, B. and DİK, M. (2021). Derivation of Black-Scholes Equation Using Itô's Lemma. *Proceedings of International Mathematical Sciences*. [online] Available at: doi:https://doi.org/10.47086/pims.956201.

16. Wikipedia Contributors (2019). *Heaviside step function*. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Heaviside_step_function.
17. Wikipedia Contributors (2020). Black–Scholes equation. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_equation.
18. Wikipedia Contributors (2019). *Black–Scholes model*. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_model.
19. "Black-Scholes in Python: Option Pricing Made Easy." *Www.youtube.com*, www.youtube.com/watch?v=Svmu_O6MH-4&t=657s.