Exercise #2

28. August 2023

Exercises marked with a (J) should be handed together with a Jupyter notebook.

Problems marked with 4N or 4D must be submitted only by the respective course, while unmarked problems must be submitted by both courses.

Optional exercises will not be corrected.

Problem 1. (Taylor Polynomials)

a) Compute all Taylor polynomials of $f(x) = -2x^4 + 2x^2 - 3x + 2$ around $x_0 = -1$

b) Compute the Taylor series of $g(x) = e^{1-2x}$ around $x_0 = 0$

Problem 2. (Polynomial interpolation - **J**)

a) Compute by hand the Lagrangian cardinal functions for the points

$$x_0 = -1, \qquad x_1 = 0, \qquad x_2 = 1, \qquad x_3 = 2.$$

b) Consider the function

$$f(x) = 2^{x^2-4-x}$$

and interpolate it by a polynomial of minimal degree by hand using the results above.

c) Implement a Python algorithm that performs the same task above for arbitrary functions and arbitrary interpolation points. The algorithm only returns the value of the interpolated function in an array of points which is given as input. What is the maximal error $e(x) = |f(x) - p(x)|$ on the intervals $[-1, 2]$ and $[-5, 5]$? Plot $f(x)$ and $p(x)$ in the same plot, on the interval $[-1, 2]$.

d) The sequence of Chebyshev nodes on the reference interval $[-1, 1]$ is

$$z_k = \cos \frac{\pi}{2} \frac{2k+1}{n} \in [-1, 1], \; k = 0, \dots, n-1$$

and can be transported on a general interval $[a, b]$ by

$$x_k = \frac{a+b}{2} + \frac{b-a}{2} z_k.$$

Find the Chebyshev nodes on the interval $[-1, 2]$ for $n = 3$.

e) Interpolate $f(x)$ numerically in the Chebyshev nodes for $n = 3$. What is the maximal error on the intervals $[-1, 2]$ and $[-5, 5]$? Plot $f(x)$ and the interpolating polynomial in the same plot, on the interval $[-1, 2]$.

f) Plot the error as a function in x, for both interpolations (Lagrange and Chebyshev), on the interval $[-1, 2]$. Plot both errors in the same plot.

Problem 3. (J)

For this problem we really recommend Python and not to do this with pen & paper.
Consider a helicopter flying at a fixed altitute (z = constant), preparing to land. Due to strong winds, the pilot is having trouble keeping the helicopter steady for landing. The mission control is trying to help, and has access to the x and y coordinates of the helicopter's center of gravity for different times t_i:

i	0	1	2	3	4	5	6	7
t_i	0	0.8976	1.7952	2.6928	3.5904	4.4880	5.3856	6.2832
x_i	1	1.5984	−0.6564	−1.6828	−0.1191	0.2114	−0.3514	1
y_i	0	0.7818	0.9750	0.4339	−0.4339	−0.975	−0.7818	0

To understand what is happening, mission control needs to have an idea of the *trajectory* the helicopter is describing mid-air.

a) Use Lagrange polynomials to interpolate $x(t)$ and plot how x evolves in time.

b) Use the same cardinal functions to interpolate $y(t)$ and plot how y evolves in time.

c) Based on the interpolations, plot the helicopter's approximate trajectory in space.

d) The sample points were taken from a curve parametrized by $x(t) = \cos(t) + \sin(2t)$ and $y(t) = \sin(t)$. Plot the exact and predicted curves in the same graph and comment on the differences between them.

Problem 4. (Trigonometric interpolation)

In our first lectures, we saw how polynomials can be used to interpolate functions or discrete datasets. That is actually not the only possibility, and in fact we can also use trigonometric functions for interpolation. Namely, the set $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}$ is a common basis for that.

Let $y = f(x) = \cos^2 x$ be the function we wish to interpolate, for which we consider three data points:

i	0	1	2
x_i	0	$\pi/6$	$\pi/2$
y_i	1	$\frac{3}{4}$	0

Our task here is to construct a trigonometric function $p(x)$ that goes through these three points. Since $f(x)$ is an even function, it suffices to use only cosine terms for the interpolation. That is, we are looking for some

$$p(x) = \alpha_0 + \alpha_1 \cos x + \alpha_2 \cos 2x$$

such that $p(x_i) = y_i$ for $i = 0, 1, 2$, with α_0, α_1 and α_2 being three real coefficients to be determined.

a) Based on the data in table, compute the interpolation coefficients α_0, α_1 and α_2.

b) Show that, for this particular case, we have $p(x) = f(x)$ for all $x \in \mathbb{R}$, that is, the interpolant is identical to the function being interpolated.
Hint: remember the trigonometric identity $\cos 2x = \cos^2 x - \sin^2 x$.