Exercise 5

Problem 1

Laplace transform:

a) $f(t) = (t-2)^4 = \left((t-2)(t-2)\right)^2 = (t^2-4t+4)(t^2-4t+4)$

$= t^4 - 4t^3 + 4t^2 - 4t^3 + 16t^2 - \cancel{16t}\ 16t + 4t^2 - 16t + 16$

$= t^4 - 8t^3 + 24t^2 - 32t + 16$

$\mathcal{L}[f(t)] = \frac{4!}{s^5} - \frac{8\cdot 3!}{s^4} + \frac{24\cdot 2!}{s^3} - \frac{32}{s^2} + \frac{16}{s}$

$= \frac{24}{s^5} - \frac{48}{s^4} + \frac{48}{s^3} - \frac{32}{s^2} + \frac{16}{s}$

$= \frac{8}{s^5}\left(3s^4 - 4s^3 + 6s^2 - 6s + 3\right)$

b) $f(t) = te^{-t}$ shift teorem: $e^{at}f(t) = F(s-a)$

$= \frac{1!}{(s-a)^2} = \frac{1}{(s+1)^2}$

c) $f(t) = e^{-5t}\sin(t)$ same: $e^{at}f(t) = F(s-a)$

$\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1^2}$

$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{(s+5)^2+1} = \frac{1}{s^2+10s+26}$

d) $f(t) = e^{-2t}\cos^2(3t) - 3t^2 e^{3t}$

nytter den trigonometriske identiteten:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2(3t) = \frac{1+\cos 6t}{2} = \frac{1}{2}(1+\cos 6t)$$

$$\mathcal{L}(f(t)) = \mathcal{L}\left[e^{-2t}\cdot\frac{1}{2}(1+\cos 6t) - 3t^2 e^{3t}\right]$$

bruker skiftteoremet på begge ledd

$$= \frac{1}{2}\mathcal{L}\left[e^{-2t}(1+\cos 6t)\right] - 3\mathcal{L}\left[e^{3t}t^2\right]$$

$$= \frac{1}{2}\left(\frac{1}{s+2} + \frac{s+2}{(s+2)^2+36}\right) - 3\left(\frac{2!}{(s+3)^3}\right)$$

$$= \underline{\underline{\frac{1}{2s+4} + \frac{s+2}{(s+2)^2+6^2} - \frac{6}{(s-3)^3}}}$$

Problem 2 Inverse Laplace transform

a) $F(s) = \frac{2s}{s^2 - 3}$

fra tabell 9

$f(t) = 2\cosh\sqrt{3}t$

b) $F(s) = \frac{s^2+s+1}{s^3+s} = \frac{s^2+s}{s^3+s} + \frac{1}{s^3+s}$

$= \frac{s+1}{s^2+1} + \frac{1}{s^3+s}$

$= \frac{s}{s^2+1^2} + \frac{1}{s^2+1^2} + \frac{1}{s^3+s}$

$f(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+1^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2+1^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^3+s}\right]$

$= \cos t + \sin t$

På nytt!

$\frac{s^2+s+1}{s^3+s} = \frac{s^2+s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1)+s(Bs+C)}{s(s^2+1)}$

$= \frac{As^2+A+Bs^2+Cs}{s(s^2+1)}$

$\Rightarrow s^2+s+1 = As^2+A+Bs^2+Cs$

$A = 1 \quad B = 0 \quad C = 1$

$$F(s) = \frac{1}{s} + \frac{1}{s^2+1}$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \underline{\underline{1 + \sin(t)}}$$

c) $$F(s) = \frac{1}{(s-1)^2(s+1)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+1}$$

„cover-up" method gir: $\underline{A \underset{x=1}{=} \frac{1}{2}}$

$$\underline{C \underset{x=-1}{=} \frac{1}{4}}$$

$$\frac{1}{(s-1)^2(s+1)} = \frac{1}{2(s-1)^2} + \frac{B}{s-1} + \frac{1}{4(s+1)}$$

for $s=0$ gir B: $1 = \frac{1}{2} + \frac{B}{-1} + \frac{1}{4}$

$$\Rightarrow \underline{B = -\frac{1}{4}}$$

Får da:

$$F(s) = \frac{1}{2(s-1)^2} - \frac{1}{4(s-1)} + \frac{1}{4(s+1)}$$

$$f(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$= \underline{\underline{\frac{1}{2}te^{t} - \frac{1}{4}e^{t} + \frac{1}{4}e^{-t}}}$$

Problem 3

a) True. Laplace is a linear transformation.

b) False. Since $\mathcal{F}(f \cdot g) = \int_0^\infty e^{-st} f(t) g(t)\, dt$

and $\mathcal{F}(f) \cdot \mathcal{F}(g) = \int_0^\infty e^{-st} f(t)\, dt \int_0^\infty e^{-st} g(t)\, dt$

which is not the same generally.

c) True. $\mathcal{F}(f) = \int_0^\infty e^{-st} f(t)\, dt$ is positive/0

for all $f(t) \geq 0$ & $t \geq 0$.

d) True. if $|f(t)| \leq a e^{bt}$ for all $t \geq 0$

& f continous, then a $F(s)$ exists.