Exercise 3

Problem 1

a) $$I = \int_{\pi/6}^{\pi/3} \sin(x)\cos(2x)\,dx$$

Trapezoidal rule: $$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} (x_{i+1}-x_i)\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)$$

$$I_h = (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

$$f(a) = \sin\left(\pi/6\right)\cdot\cos\left(\tfrac{1}{3}\pi\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$f(b) = \sin\left(\pi/3\right)\cdot\cos\left(\tfrac{2}{3}\pi\right) = \frac{\sqrt{3}}{2}\cdot -\frac{1}{2} = -\frac{\sqrt{3}}{4}$$

$$I_h = \left(\frac{\pi}{3}-\frac{\pi}{6}\right)\left(\frac{\frac{1}{4}-\frac{\sqrt{3}}{4}}{2}\right) = \frac{\pi}{6}\cdot\left(\frac{1}{8}-\frac{\sqrt{3}}{8}\right) \approx \underline{\underline{-0{,}048}}$$

b) $$E = |I - I_h|$$

$$\sin(x)\cos(2x) = \frac{1}{2}\left(\sin(x-2x)+\sin(x+2x)\right)$$

$$= \frac{1}{2}\left(\sin(3x)-\sin(x)\right)$$

$$I = \frac{1}{2}\left(\int_{\pi/6}^{\pi/3}\sin(3x)\,dx - \int_{\pi/6}^{\pi/3}\sin x\,dx\right) = \frac{1}{2}\left(\left[-\frac{1}{3}\cos(3x)\right]_{\pi/6}^{\pi/3} - \left[-\cos x\right]_{\pi/6}^{\pi/3}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{3}\cos(\pi)+\frac{1}{3}\cos\left(\frac{\pi}{2}\right)-\left(-\cos\left(\frac{\pi}{3}\right)+\cos\frac{\pi}{6}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}-\frac{\sqrt{3}}{2}\right) = \frac{5}{12}-\frac{\sqrt{3}}{4} \approx -0{,}016$$

$$E = |-0{,}0163 + 0{,}0479| = \underline{\underline{0{,}0316}}$$

c) $$|I - I_h| \leq \frac{(b-a)^3}{12} \max_{\xi \in [a,b]} |f''(\xi)|$$

$$f(x) = \sin(x)\cos(2x)$$

$$f'(x) = \cos(x)\cos(2x) - 2\sin(x)\sin(2x)$$

$$f''(x) = -\sin(x)\cos(2x) - 2\cos(x)\sin(2x)$$
$$-(2\cos(x)\sin(2x) + 4\sin(x)\cos(2x))$$

$$= \underline{-5\sin(x)\cos(2x) - 4\cos(x)\sin(2x)}$$

$$\max_{\xi \in [\frac{\pi}{6}, \frac{\pi}{3}]} |f''(\xi)| = \frac{17}{4}, \quad \xi = \frac{\pi}{6}$$

$$E \leq \frac{\left(\frac{\pi}{3} - \frac{\pi}{6}\right)^3}{12} \cdot \frac{17}{4} = \frac{\pi^3}{6^3 \cdot 12} \cdot \frac{17}{4} \approx 0{,}0508$$

$$\Rightarrow \underline{\underline{0{,}0316 \leq 0{,}0508}} \quad \text{OK!}$$

Problem 2

a) $$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\int_{-1}^{1} f\left(\frac{(b-a)}{2}\varphi + \frac{a+b}{2}\right)d\varphi = \frac{b-a}{2}\sum_{i=0}^{N} w_i f(\varphi_i)$$

$$\int_{-3}^{3} e^x\,dx = \frac{3+3}{2}\sum_{i=0}^{3} w_i f(\varphi_i)$$

$\varphi_0 = 3 \cdot x_0$

$\varphi_1 = 3x_1$

$\varphi_2 = 3x_2$

$\varphi_3 = 3x_3$

Python for utregning:

$$\int_{-3}^{3} e^x\,dx \approx 3\sum_{i=0}^{3} w_i f(\varphi_i) = \underline{\underline{20{,}0287}}$$

b) $$h = \frac{(b-a)}{m} \Rightarrow (b-a) = h \cdot m$$

$$E = \frac{(h \cdot m)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\varphi) \overset{m=1}{=} \frac{h^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\varphi)$$

$$E_m = \sum_{i=1}^{m} \frac{\left(\frac{b-a}{m}\right)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\varphi)$$

$$= m \cdot \frac{h^{2n+1}(n!)^4}{(2n+1)((2n)!)^3} f^{(2n)}(\varphi)$$

~~Siden h for $E = (b-a)$~~ siden $m = 1$, mens h for E_m vil gå mot 0 når $m \to \infty$, vil feilmarginen bli mindre når vi øker antall oppdelinger.

c) $$E_1 = \frac{6^{2n+1} (n!)^4}{(2n+1)((2n)!)^3} f^{2n}(\omega_e)$$

$$E_2 = 2 \cdot \frac{3^{2n+1} (n!)^4}{(2n+1)((2n)!)^3} f^{2n}(\omega_e)$$

$$\frac{E_2}{E_1} = \frac{2 \cdot 3^{2n+1}}{6^{2n+1}} = \frac{\cancel{6} \cdot 3^{2n}}{\cancel{6} \cdot 6^{2n}} = \left(\frac{3}{6}\right)^{2n} = \left(\frac{1}{2}\right)^{2n}$$

$$= \underline{\underline{\frac{1}{2^{2n}}}}$$

Problem 3 Simpson's for one segment:

$$\frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Simpson's for m segments of length $h = \frac{1}{m}$

$n = 2 \cdot m$ $$\frac{b-a}{6}\left(f(a) + \sum_{i=1}^{n/2} 4f(a+h) + \sum_{i=1}^{\frac{n}{2}-1} 2f(x_{2i}) + f(b)\right)$$

$$\underline{\underline{S_n = \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(b)\right]}}$$

c) $E = \frac{h^4}{180}(b-a) \max_{\xi \in [a,b]} |f^{(4)}(\xi)|$

$$|f^{(4)}(x)| = e^{-x}$$

$$\max_{[0,1]} e^{-x} = e^0 = 1$$

$$\frac{h^4}{180}(1-0) \leq 10^{-3}$$

$$\frac{h^4}{180} \leq \frac{1}{10^3} \qquad h = \frac{1}{m} \quad \Rightarrow \quad \frac{1}{m^4 \cdot 180} \leq \frac{1}{10^3}$$

$$\Rightarrow \quad m^4 \geq \frac{10^3}{180} \quad \Rightarrow \quad \sqrt[4]{m^4} \geq \sqrt[4]{\frac{10^3}{180}} \approx 1,535$$

$m = 2$

Når $\underline{m = 2}$ er $E \leq 10^{-3}$ for integralet $\int_0^1 e^{-x}\,dx$