Exercise 4

a) Plottet i jupyter

Problem 1. b) $f(x) = (1-3^x)x^2 + 4(x-1)\cdot 3^x + 4(1-x)$

$I = [-2, 3]$

$f(-2) = (1-3^{-2})(-2)^2 + 4(-2-1)\cdot 3^{-2} + 4(1+2)$

$= \left(1 - \frac{1}{9}\right)4 - \frac{12}{9} + 12 = 4 - \frac{4}{9} - \frac{12}{9} + 12$

$= 16 - \frac{16}{9} > 0$

$f(3) = (1-3^3)\cdot 9 + 4(2)\cdot 3^3 + 4(-2)$

$= -26\cdot 9 + 8\cdot 27 - 8 = -9\cdot 26 + 8\cdot 26 = \underline{-26}$

$f(-2)\cdot f(3) < 0 \implies$ Finnes minimum 1 rot til f på $I = [-2, 3]$

$C_0 = \frac{-2+3}{2} = \frac{1}{2}$

$f(\frac{1}{2}) = (1-\sqrt{3})\frac{1}{4} + 2\cdot\sqrt{3} + 2 = \frac{1}{4} - \frac{\sqrt{3}}{4} - 2\sqrt{3} + 2$

$\approx -1{,}647$

$f(-2)\cdot f(\frac{1}{2}) < 0$

$I_1 = [-2, \frac{1}{2}]$

$C_1 = \frac{-2+\frac{1}{2}}{2} = \frac{-\frac{4}{2}+\frac{1}{2}}{2} = -\frac{3}{4}$ $\quad f(-\frac{3}{4}) \approx 4{,}245$

$I_2 = [-\frac{3}{4}, \frac{1}{2}]$ $\quad C_2 = \frac{-\frac{3}{4}+\frac{1}{2}}{2} = -\frac{1}{8}$ $\quad f(-\frac{1}{8}) \approx 0{,}579$

$I_3 = [-\frac{1}{8}, \frac{1}{2}]$ $\quad C_3 = \frac{-\frac{1}{8}+\frac{1}{2}}{2} = \frac{\frac{3}{8}}{2} = \frac{3}{16}$ $\quad f(\frac{3}{16}) \approx -0{,}751$ *

* $I_4 = [-\frac{1}{8}, \frac{3}{16}]$

$C_4 = \frac{-\frac{2}{16}+\frac{3}{16}}{2}$

$= \underline{\underline{\frac{1}{32}}}$

$$e_4 \leq \frac{3+2}{2^5} = \underline{\underline{\frac{5}{32}}}$$

Ser fra plot at eksakt løsning er $\underline{0}$.

$$\left|\frac{1}{32} - 0\right| \leq \frac{5}{32} \quad \underline{\text{OK!}}$$

c) $k = \log_2\left(\frac{b-a}{2 \cdot 10^{-3}}\right) = \log_2\left(\frac{5}{\frac{2}{1000}}\right) = \log_2(2500) \approx \underline{11{,}28}$

Vi trenger altså $\underline{\underline{k = 12}}$ iterasjoner for å garantere feil på under 10^{-3}.

Problem 2: Fixed point method

$$\cos(e^{-x}) = 2\sqrt{x}$$

a) Show that the fixed-point method

$$x = g(x) \quad , \quad g(x) = \frac{\cos^2(e^{-x})}{4} \quad \text{has}$$

a unique solution $\hat{x} \geq 0$

Let initial guess $x_0 = 0$:

$$x_1 = g(0) = \frac{\cos^2(1)}{4} = \frac{1}{4}$$

$$x_2 = g\left(\tfrac{1}{4}\right) = \frac{\cos^2\left(\frac{1}{4\sqrt{e}}\right)}{4} = \frac{\cos^2(4\sqrt{e})}{4} = 0{,}0200$$

$x_3 = g(0{,}0200) = 0{,}077523$

$x_4 = g(0{,}077523) = 0{,}0904537$

$x_5 = g(0{,}0904537) = 0{,}0933197$

$x_6 = g(x_5) = 0{,}09395241$

$x_7 = g(x_6) = 0{,}0940919$

$x_8 = g(x_7) = 0{,}0941227$

$x_9 = 0{,}0941295$

$x_{10} = 0{,}0941310$

$x_{11} = 0{,}0941313$

$x_{12} = 0{,}0941514$

$x_{13} = 0{,}0941314$ Convergence

Anvender fix-punkt teoremet:

1) $|g'(x)| \leq L < 1$ for alle $x \in [a,b]$?

$$g'(x) = \frac{e^{-x}\cos(e^{-x})\sin(e^{-x})}{2}$$

$\underset{\max x\in[0,\infty)}{|g'(x)|}$ kan aldri være ≥ 1, fordi telleren aldri kan bli større enn 1 for alle $x \geq 0$.
Så $g'(x)$ vil være < 1 for alle $x \in [0, \infty)$.

$\Rightarrow$ $|g'(x)| \leq L < 1$ for alle $x \in [a,b]$ ✓ ok!

2) $g(x) \in [a,b)$ for alle $x \in [a,b]$?

Siden $b = \infty$ trenger vi kun å se på a.

$g(x)$ - min $a = 0,227$ for $x = \log\left(\frac{2}{7\pi}\right)$

$\Rightarrow$ $g(x) \in (a,b)$ for alle $x [a,b]$ ✓ ok!

Altså: Fra teoremet, må $x = g(x)$ ha en unik løsning $\hat{x} \geq 0$ □

b) Øvre grense for $e = |\hat{x} - x|$:

Har gjort 9 iterasjoner:

$$e_9 \leq \frac{L}{1-L} |x_9 - x_8| = \frac{L}{1-L} |0{,}094131\ldots - 0{,}0941309\ldots|$$

$$= \frac{L}{1-L} |4{,}2065 \cdot 10^{-7}|$$

$$\max_{x \in [0, \infty)} |g'(x)| = 0{,}227 \Rightarrow L = 0{,}227$$

$\Rightarrow$ $\underline{e_9 \leq 1{,}235 \cdot 10^{-7}}$ er en øvre grense

Problem 3:

$$f(x) = \cos(x) - \sqrt{x}$$

a) Newtons metode, 2 steg, $x_0 = 1$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$f'(x) = -\sin(x) - \frac{1}{2\sqrt{x}}$$

$$x_1 = 1 - \frac{f(1)}{f'(1)} \approx 0{,}6573$$

$$x_2 = 0{,}6573 - \frac{f(0{,}6573)}{f'(0{,}6573)} \approx \underline{\underline{0{,}6417}}$$

Problem 4:

a) Newtons metode

b) $x_0 = 0{,}5$

$$f(x) = \cos(x) + \log(x)$$

$$f'(x) = -\sin(x) + \frac{1}{x}$$

while error > tol:

$$dx = -\frac{f(x)}{f'(x)}$$

$$x \mathrel{+}= dx \Rightarrow x_{new} = x - \frac{f(x)}{f'(x)}$$

update(error)

c) theres a minus too much in front of dx in: $x \mathrel{+}= -dx$

Remove

Since it's already present in the dx variable.