Exercise #2

Problem 1

a) Taylor polynomials of $f(x) = -2x^4 + 2x^2 - 3x + 2$ around $x_0 = -1$

$$f(x) \approx f(x_0) + \sum_{n=1}^{4} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

Computing derivatives:	with $x_0 = -1$:
$f'(x) = -8x^3 + 4x - 3$	$8 - 4 - 3 = 1$
$f''(x) = -24x^2 + 4$	$-24 + 4 = -20$
$f^{(3)}(x) = -48x$	48
$f^{(4)}(x) = -48$	-48

$f(x_0) = \cancel{-2} \cancel{+2} + 3 + 2 = 5$

$$T_4(x) = 5 + \frac{1 \cdot (x+1)^1}{1!} - \frac{20(x+1)^2}{2!} + \frac{48(x+1)^3}{3!} - \frac{48(x+1)^4}{4!}$$

$$= 5 + x + 1 - \left(10\left(x^2 + 2x + 1\right)\right) + \left(8\left(x^2 + 2x + 1\right)(x+1)\right)$$

$$- \left(2\left(x^2 + 2x + 1\right)\left(x^2 + 2x + 1\right)\right)$$

$$= x + 6 - 10x^2 - 20x - 10 + 8\left(x^3 + 2x^2 + x + x^2 + 2x + 1\right)$$

$$- 2\left(x^4 + 2x^3 + x^2 + 2x^3 + 4x^2 + 2x + x^2 + 2x + 1\right)$$

$$= -10x^2 - 19x - 4 + \cancel{8x^3} + 24x^2 + 24x + 8$$

$$- 2x^4 - \cancel{8x^3} - 12x^2 - 8x - 2$$

$$\underline{\underline{T_4(x) = -2x^4 + 2x^2 - 3x + 2}}$$

b) Taylor series of $g(x) = e^{1-2x}$ around $x_0 = 0$

Formula: $g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(x_0)}{k!} (x - x_0)^k \overset{x_0 = 0}{\Longrightarrow} \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k$

$g(x) = e^{1-2x}$ $\quad x_0 => 0 \Rightarrow g(0) = e$

$g'(x) = -2e^{1-2x}$ $\quad x_0 = 0 \Rightarrow g'(0) = -2e$

$g''(x) = 4e^{1-2x}$ $\quad x_0 = 0 \Rightarrow g''(0) = 4e$

$g^{(3)}(x) = -8e^{1-2x}$ $\quad x_0 = 0 \Rightarrow g^{(3)}(0) = -8e$

$\vdots$

$g^{(n)}(x) = (-1)^n 2^n e^{1-2x} \Rightarrow g^{(n)}(0) = (-2)^n \cdot e$

Thus, the taylor series of $g(x)$ around $x_0 = 0$ is:

$$\underline{e^{1-2x} = \sum_{k=0}^{\infty} \frac{(-2)^k \cdot e}{k!} x^k}$$

Problem 2 a) Lagrangian cardinal functions of the points $x_0 = -1$ $x_1 = 0$ $x_2 = 1$ $x_3 = 2$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \left(\frac{x - x_j}{x_i - x_j} \right)$$

$i=0$

$$l_0(x) = \underset{j=1}{\left(\frac{x-0}{-1-0} \right)} \cdot \underset{j=2}{\left(\frac{x-1}{-1-1} \right)} \cdot \underset{j=3}{\left(\frac{x-2}{-1-2} \right)} = -x \cdot \left(\frac{x-1}{-2} \right) \cdot \left(\frac{x-2}{-3} \right)$$

$$= -\frac{x(x-1)(x-2)}{6} = \frac{(-x^2+x)(x-2)}{6} = \frac{-x^3+2x^2+x^2-2x}{6}$$

$$\boxed{= -\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x}$$

$i=1$

$$l_1(x) = \underset{j=0}{\left(\frac{x+1}{0+1} \right)} \cdot \underset{j=2}{\left(\frac{x-1}{0-1} \right)} \cdot \underset{j=3}{\left(\frac{x-2}{-2} \right)} = (x+1)(-x+1)\left(\frac{x-2}{-2} \right)$$

$$= (-x^2+1)\left(-\frac{x}{2}+1\right) = \frac{1}{2}x^3 - x^2 + \frac{1}{2}x + 1$$

$i=2$

$$l_2(x) = \underset{j=0}{\left(\frac{x+1}{1+1} \right)} \cdot \underset{j=1}{\left(\frac{x}{1} \right)} \cdot \underset{j=3}{\left(\frac{x-2}{1-2} \right)} = \frac{x(x+1)(x-2)}{-2}$$

$$= \frac{(x^2+x)(x-2)}{-2} = \frac{x^3-2x^2+x^2-2x}{-2}$$

$$= -\frac{1}{2}x^3 + \frac{1}{2}x^2 + x$$

$i=3$

$$l_3(x) = \underset{j=0}{\left(\frac{x+1}{2+1} \right)} \cdot \underset{j=1}{\left(\frac{x}{2} \right)} \cdot \underset{j=2}{\left(\frac{x-1}{2-1} \right)} = \frac{x(x+1)(x-1)}{6}$$

$$= \frac{x^3-x}{6}$$

b) $f(x) = 2^{x^2-x-4}$ interpolate f

$$p(x) = \sum_{i=0}^{3} y_i l_i(x)$$

$$y_0 = 2^{1+1-4} = 2^{-2} = \frac{1}{4}$$

$$y_1 = 2^{-4} = \frac{1}{16}$$

$$y_2 = 2^{1-1-4} = \frac{1}{16}$$

$$y_3 = 2^{4-2-4} = \frac{1}{4}$$

$$p(x) = \frac{1}{4}\left(-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x\right) + \frac{1}{16}\left(\frac{1}{2}x^3 - x^2 + \frac{1}{2}x + 1\right)$$

$$+ \frac{1}{16}\left(-\frac{1}{2}x^3 + \frac{1}{2}x^2 + x\right) + \frac{1}{4}\left(\frac{1}{6}x^3 - \frac{1}{6}x\right)$$

$$= \cancel{-\frac{1}{24}x^3} + \underline{\frac{1}{8}x^2} - \underline{\frac{1}{12}x} + \cancel{\frac{1}{32}x^3} - \underline{\frac{1}{16}x^2} + \underline{\frac{1}{32}x} + \frac{1}{16}$$

$$\cancel{-\frac{1}{32}x^3} + \underline{\frac{1}{32}x^2} + \underline{\frac{1}{16}x} + \cancel{\frac{1}{24}x^3} - \underline{\frac{1}{24}x}$$

$$= \frac{4}{32}x^2 - \frac{2}{32}x^2 + \frac{1}{32}x^2 - \frac{3}{24}x + \frac{3}{32}x + \frac{1}{16}$$

$$= \frac{3}{32}x^2 - \frac{3}{32}x + \frac{1}{16}$$

Problem 4

a)

i	0	1	2
x_i	0	$\pi/6$	$\pi/2$
y_i	1	$3/4$	0

$f(x) = \cos^2 x$

$p(x_i) = y_i$

$\Rightarrow p(x_0) = \alpha_0 + \alpha_1 \cos(0) + \alpha_2 \cos(2\cdot 0) = 1$

$\Rightarrow \alpha_0 + \alpha_1 + \alpha_2 = 1$

$p(x_1) = \alpha_0 + \alpha_1 \cos(\pi/6) + \alpha_2 \cos(\pi/3) = 3/4$

$\Rightarrow \alpha_0 + \alpha_1 \frac{\sqrt{3}}{2} + \frac{1}{2}\alpha_2 = 3/4$

$p(x_2) = \alpha_0 + \alpha_1 \cos(\pi/2) + \alpha_2 \cos(\pi) = 0$

$\Rightarrow \alpha_0 + \alpha_1 \cdot 0 - \alpha_2 = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{3}{4} \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$\downarrow$ RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

$\alpha_0 = 1/2$

$\alpha_1 = 0$

$\alpha_2 = 1/2$

$p(x) = 1/2 + 1/2 \cos(2x)$

Hint: $\cos 2x = \cos^2 x - \sin^2 x$

b) Show that $\underbrace{P(x)}_{\frac{1}{2}+\frac{1}{2}\cos(2x)} = \underbrace{f(x)}_{\cos^2(x)}$

$$\cos^2 x = 1 - \sin^2 x \quad (\text{since } \cos^2 x + \sin^2 x = 1)$$

$$\cos^2 x = 1 - \cos^2(x) + \cos(2x) \quad (\text{hint})$$

$$2\cos^2 x = 1 + \cos(2x)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) \quad \square$$