1) a) Bruk trapes-regelen på $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(x)\cos(2x)\,dx$

Trapesregelen: $\int_a^b f(x)\,dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right) = I_h$

Løsning:

- $a = \frac{\pi}{6}$
- $b = \frac{\pi}{3}$
- $f(a) = f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)\cdot\cos\left(\frac{\pi}{3}\right) = \frac{1}{4}$
- $f(b) = f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{4}$

$$\Longrightarrow \quad I \approx I_h = \left(\frac{\pi}{3} - \frac{\pi}{6}\right)\cdot\left(\frac{\frac{1}{4} - \frac{\sqrt{3}}{4}}{2}\right) = \frac{\pi}{6}\cdot\left(\frac{1}{8} - \frac{\sqrt{3}}{8}\right) = -0{,}0479126 1337\ldots$$

$$\approx -0{,}0479$$

b) Regn ut feilen: $E = |I - I_h|$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \underset{v'(x)}{\sin(x)}\,\underset{u(x)}{\cos(2x)}\,dx = -\cos(x)\cdot\cos(2x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} 2\cos(x)\sin(2x)\,dx$$

$$= -\cos(x)\cdot\cos(2x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \sin(x) + \sin(3x)\,dx$$

$$= -\cos(x)\cdot\cos(2x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \left[-\cos(x) - \frac{\cos(3x)}{3}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left[-\cos(x)\cdot\cos(2x) + \frac{4}{3}\cos^3(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{12}\left(5 - 3\cdot\sqrt{3}\right) \approx -0{,}016.$$

$$E = |I - I_h| = \left| \frac{1}{12}\left(5 - 3\sqrt{3}\right) - \frac{\pi}{6}\cdot\left(\frac{1-\sqrt{3}}{8}\right) \right|$$

$$= 0{,}031567\ldots$$

c) Finn øvre grense for feilen E.

→ Sjekk at dette faktisk er større enn feilen du fant (E).

Kan bruke Python for å finne $\max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|$

Feil estimat for trapesregelen er gitt ved:

$$|I - I_h| \leq \frac{(b-a)^3}{12} \max_{\xi \in [a,b]} |f''(\xi)| \quad \text{hvor } f(x) \in C^{n+1}[a,b]$$

$$f(x) = \sin(x)\cos(2x)$$

$$f''(x) = \frac{1}{2}\left(\sin(x) - 9\sin(3x)\right) \quad \text{(wolfram)}$$

$$\max_{\xi \in [a,b]} |f''(\xi)| = \frac{17}{4}$$

ved $\xi = \frac{\pi}{6}$ (Python)

$$\rightarrow |I - I_h| \leq \frac{(b-a)^3}{12} \max_{\xi \in [a,b]} |f''(\xi)|$$

$$\rightarrow |I - I_h| \leq \frac{\left(\frac{\pi}{6}\right)^3}{12} \cdot \frac{17}{4}$$

$$= 0{,}0508\ldots \text{ (øvre grense for } E)$$

Med vår $E = 0{,}0315\ldots < 0{,}0508$ ok!

2) Gauss Legendre

• 4 noder på intervallet $x \in [-1,1]$

$$x_0 = \frac{-1}{35}\sqrt{525+70\sqrt{30}}, \quad x_1 = \frac{-1}{35}\sqrt{525-70\sqrt{30}}$$

$$x_2 = \frac{1}{35}\sqrt{525-70\sqrt{30}}, \quad x_3 = \frac{1}{35}\sqrt{525+70\sqrt{30}}$$

og vektene:

$$w_0 = \frac{1}{36}\left(18-\sqrt{30}\right), \quad w_1 = \frac{1}{36}\left(18+\sqrt{30}\right)$$

$$w_2 = \frac{1}{36}\left(18+\sqrt{30}\right), \quad w_3 = \frac{1}{36}\left(18-\sqrt{30}\right)$$

Generelt:

$$\boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{2}\int_{-1}^{1} f(\tilde{x})\,d\tilde{x} = \frac{b-a}{2}\sum_{i=0}^{n} w_i \cdot f(\tilde{x}_i)}$$

Nodene transformert: $\tilde{x}_i = \left(\frac{b-a}{2}\right)\cdot x_i + \frac{b+a}{2}$

→ Dette gir oss for $\boxed{\int_{-3}^{3} e^x\,dx}$, $a=-3$, $b=3$

$$\tilde{x}_0 = \left(\frac{3+3}{2}\right)\cdot x_0 + \frac{0}{2} = \underline{\underline{3\cdot x_0}}$$

$$\tilde{x}_1 = \underline{\underline{3\cdot x_1}}$$

$$\tilde{x}_2 = \underline{\underline{3\cdot x_2}}$$

$$\tilde{x}_3 = \underline{\underline{3\cdot x_3}}$$

Python for utregninga:

$$\int_{-3}^{3} e^x\,dx \approx 3\cdot\sum_{i=0}^{n} w_i\cdot f(\tilde{x}_i) = \underline{\underline{20{,}028688\ldots}}$$

Nesten likt

$$\text{Eksakt: } \int_{-3}^{3} e^x\,dx = \underline{\underline{20{,}036\ldots}}$$

b) $E = \frac{(b-a)^{2n+1} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi)$ for en $\xi \in [a,b]$ om f er $2n$ ganger kontinuerlig deriverbar.

→ feil i oppg. tekst, skal være $(2n)!$, ikke $(2n!)$

Basert på utrykket over, hvilken feilestimater forventer du for den sammensatte Gauss-Legendre kvadraturen, G_m?

Gi en kort forklaring ved at intervallet (a,b) er delt inn i m delintervall, hver med lengde $h = \frac{b-a}{m}$

Løsning:

$$E = \frac{(b-a)^{2n+1} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi) \implies$$ Oppgitt, ikke sammensatt av flere delintervall.

Bruker $h = \frac{b-a}{m} \leftrightarrow m \cdot h = b - a$ og skriver om:

$$E = \frac{(m \cdot h)^{2n+1} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi) \overset{m=1}{=} \frac{h^{2n+1} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi)$$

→ ikke sammensatt av flere intervaller

Sammensatt:

$$E_m = \sum_{i=0}^{m-1} \frac{(b-a)^{2n+1} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi)$$

$$= \sum_{i=0}^{m-1} \frac{h^{2n+1} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi)$$

$$= m \cdot \frac{(h)^{2n+1} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi), \quad m \cdot h = (b-a)$$

$$= \frac{(m \cdot h) \cdot h^{2n} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi)$$

$$= \frac{(b-a) \cdot h^{2n} \cdot (n!)^4}{(2n+1) \cdot ((2n)!)^3} \cdot f^{(2n)}(\xi)$$

Vi har gått fra:

$$E = \frac{h^{2n+1} \cdot (n!)^4}{(2n+1)\cdot((2n)!)^3} \cdot f^{2n}(\xi) \quad \text{til} \quad E_m = \frac{(b-a)\cdot h^{2n} \cdot (n!)^4}{(2n+1)\cdot((2n)!)^3} \cdot f^{2n}(\xi)$$

der $h = \frac{b-a}{m}$; $\begin{cases} E: h = \frac{b-a}{1} = b-a \\ E_m: h = \frac{b-a}{m} \end{cases}$ $\longrightarrow$ Feilen E_m blir mindre om antall intervaller m øker

c) $f(x) = \frac{x^8}{8!}$

i) Bruk E til å regne ut feilen E_1 til $G_1 = \int_{-3}^{3} f(x)\,dx$

Løsning:

Litt usikker nå om det kun er dette...

$$E_1(-3,3) = \int_{-3}^{3} \frac{x^8}{8!}\,dx - G_1(-3,3) = \frac{(6)^{2n+1} \cdot (n!)^4}{(2n+1)\cdot((2n)!)^3} \cdot f^{2n}(\xi)$$

ii) splitter $[-3,3]$ til $(-3,0)$ og $(0,3)$

$\rightarrow$ Regn ut E_2 for den nå sammensatte kvadraturen:

$$E_2 = \frac{(b-a)\cdot h^{2n} \cdot (n!)^4}{(2n+1)\cdot((2n)!)^3} \cdot f^{2n}(\xi) \quad , \quad m = 2$$

$$= \frac{(b-a) \cdot \left(\frac{b-a}{m}\right)^{2n} \cdot (n!)^4}{(2n+1)\cdot((2n)!)^3} \cdot f^{2n}(\xi)$$

$$= \frac{6 \cdot (3)^{2n} \cdot (n!)^4}{(2n+1)\cdot((2n)!)^3} \cdot f^{2n}(\xi)$$

iii) Regn ut $\frac{E_2}{E_1}$

$$\frac{E_2}{E_1} = \frac{\frac{6\cdot(3)^{2n}\cdot\cancel{(n!)^4}}{\cancel{(2n+1)}\cdot\cancel{((2n)!)^3}}\cdot\cancel{f^{(2n)}(\xi)}}{\frac{(6)^{2n+1}\cdot\cancel{(n!)^4}}{\cancel{(2n+1)}\cdot\cancel{((2n)!)^3}}\cdot\cancel{f^{(2n)}(\xi)}}$$

$$= \frac{6\cdot 3^{2n}}{6^{2n+1}}$$

$$= \frac{\cancel{2}\cdot \cancel{3^{2n+1}}}{2^{2n+\cancel{1}}\cdot \cancel{3^{2n+1}}} = \underline{\underline{\frac{1}{2^n}}} \quad \longleftrightarrow \quad$$

Generelt Svar: $\frac{1}{m^n}$?

3a) Vis at at den sammensatte Simpsons regelen er gitt ved:

$$S_n = \frac{1}{3}\cdot h\cdot\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots\dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Skrevet om, fra boka

$$L_? = \frac{h}{3}\left[f(x_0) + 4\cdot\sum_{i=1} f(x_{2i-1}) + 2\cdot\sum_{i=1}^{(n/2)-1} f(x_{2i}) + f(x_n)\right]$$

Ikke sammensatt: $n = 2$ (Antall intervaller (=? hs)) $(=m\cdot 2)$, $h = \frac{b-a}{n} \iff \frac{b-a}{2}$

$$S = \frac{b-a}{6}\left(f(a) + 4\cdot f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Tredelt: $a \rightarrow a+h \rightarrow b$
→ Likt avstand

Skriver om: $n=2$

$$= \frac{b-a}{2}\cdot\frac{1}{3}\left(f(a) + 4\sum_{i=1}^{n/2} f(\underbrace{a+h}_{=x_{2i-1}}) + 2\cancel{\sum_{i=1}^{(n/2)-1} f(x_{2i})} + f(b)\right]$$

$= 0$ ledd, $\sum_{i=1}^{0}$

Sum med ett ledd

$h = \frac{b-a}{2}$, $a+h = \frac{a+b}{2}$

$n = 2\cdot m, m>1$

$$= \underline{\underline{\frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(b)\right] = S_n}}$$

(for $n = 2\cdot m$, $m>1$)

(Beklager rot)

3c) Finn n slik at $E \leq 10^{-3}$ for $\int_0^1 e^{-x} dx$

$$E = \frac{h^4}{180}(b-a) \cdot \max_{x \in (a,b)} \{f^{(4)}(x)\} \quad , h = \frac{b-a}{n}$$

$\rightarrow f^{(4)} = e^{-x}$, max på $(0,1) = f^{(4)}(0) = e^0 = \underline{\underline{1}}$

$$\rightarrow E = \frac{h^4}{180}(b-a) \quad , h = \frac{b-a}{n} \quad \rightarrow E = \frac{(b-a)^5}{180 \cdot n^4}$$

$$= \frac{1}{180 \cdot n^4} \leq 10^{-3}$$

$$n = \sqrt[4]{\frac{10^3}{180}} = 1{,}53\ldots \longrightarrow \lceil 1{,}53 \rceil = \underline{\underline{2}}$$

$\rightarrow$ må minst ha 2 intervaller

Fra python får jeg $E = 0{,}17\ldots$ ved $n = 1$, $> 10^{-3}$

$E = 0{,}0002$ ved $n = 2$, $< 10^{-3}$

Øving 3 Eirik Tveiten

In [2]:
```
# Importing the necessary libraries
import numpy as np
import matplotlib.pyplot as plt
```

Task 1

In [133]:
```
# Finner maks verdi for f''(x) på [pi/6 , pi/3]

# f''(x):
t = lambda x: 0.5 * (np.sin(x) - 9 * np.sin(3*x))

# Intervallet med 1000 steg:
I = np.linspace(np.pi/6, np.pi/3, 1000)
t_x = t(I)

# Finner maks verdi:
tmax = np.max(abs(t_x))

print("Maks verdi er :", tmax)
```

```
Maks verdi er : 4.25
```

Task 2

In [132]:
```
# a)

# Definerer xi og wi
x0 = (-1/35) * np.sqrt(525 + 70*np.sqrt(30))
x1 = (-1/35) * np.sqrt(525 - 70*np.sqrt(30))
x2 = (1/35) * np.sqrt(525 - 70*np.sqrt(30))
x3 = (1/35) * np.sqrt(525 + 70*np.sqrt(30))

x_i = np.array([x0, x1, x2, x3])

w0 = (1/36)*(18-np.sqrt(30))
w1 = (1/36)*(18+np.sqrt(30))
w2 = (1/36)*(18+np.sqrt(30))
w3 = (1/36)*(18-np.sqrt(30))

w_i = np.array([w0, w1, w2, w3])

# Definerer funksjonen f(x):

f2 = lambda x: np.exp(x)

# Her er x_i * 3 = x~, se skriftlig 2a.

Gh = np.sum(w_i * f2(x_i * 3)) * 3

print(Gh)
```

```
20.028688395290693
```

Task 3

In [116]:
```
# Defining the interval boundaries and the funciton to integrate
a=0
b=1
def f(x):
    return np.exp(-x)
```

In [48]:
```
#Defining an array with the number of intervals
ms = np.arange(2,10,2)
```

In [136]:
```
# Definition of the Simpson integrating function. Inputs:
#  f: funciton to integrate
#  a: interval start
#  b: interval end
#  m: number of subintervals
# Outputs:
#  int: value of the integral

def compositeSimpson(f, a, b, m):
    # INSERT CODE HERE
    h = (b-a) /m
    x = np.linspace(a, b, m+1)
    return (h / 3) * (f(x[0]) + 4 * np.sum(f(x[1:m:2])) + 2 * np.sum(f(x[2:m-1:2])) + f(x[m]))

# Skrev "direkte av" det jeg hadde i 3a
```

In [144]:
```
# Exact integral value
I_exact = (np.e - 1) / np.e # Wolfram

print(I_exact)
print(compositeSimpson(f, a, b, 10))

print()
# Ser at det må minst være 2 intervaller for en feil mindre enn 0.001
print(I_exact - compositeSimpson(f, a, b, 1))
print(I_exact - compositeSimpson(f, a, b, 2))
```

```
0.6321205588285577
0.6321209095890152

0.17616074510474355
-0.00021312117510496886
```

In [139]:
```
# Array of errors
errs = [np.abs(I_exact - compositeSimpson(f,a,b,m)) for m in ms]
print('Subintervals: ', ms)
print('Errors: ', errs)
```

```
Subintervals:  [2 4 6 8]
Errors:  [0.00021312117510496886, 1.361649197462178e-05, 2.700772859354217e-06, 8.557761845828793e-07]
```

Her (over) er allerede første ledd (n=2) mindre enn kravet. Tilsvarer det jeg fant skriftlig.

In [7]:
```
#Estimated convergence order (should be 4 for Simpson's rule)
approxp = [ np.log(errs[i+1]/errs[i]) / (np.log(ms[i]/ms[i+1]))  for i in range(ms.size-1) ]
```

Out[7]:
```
[3.9682469683548565, 3.989846955083258, 3.9949808838244567]
```

Oppgave 4

In [153]:
```
def g(x) :
    return x**2

def Method(f, a, b, m):
    xs = np.linspace (a , b, m+1)
    ys = [f(x) for x in xs]
    s = ys [0] + ys [ -1] + 2* sum( ys [1: -1])
    return s

m = 10

print(Method(g, 0 ,1 ,3)) # Skal printe m/3

# Løsning
#print(m/3), neida :)
```

```
2.111111111111111
```

Min løsning:

"If the method had been implemented correctly, running Method(f,0,1,m) should return an

output equal to 1/3 regardless of the input m"

Betyr dette 1/3 av m, 1/3 av det numeriske integralet på [a,b] eller alltid bare selve tallet 1/3 ("equal to 1/3 regardless of m")??

In [156]:
```
# min "Method"

def metode(f, a, b , m):
    xs = np.linspace (a , b, m+1)
    ys = [f(x) for x in xs]
    s = (1/2) * (ys[0] + ys[-1]) + sum(ys[1:-1]) # Trapesregelen
    return s

print(metode(g, 0 ,1 ,m))
print(metode(g, 0 ,1 ,30)) # Svarene blir m/3
```

```
3.3500000000000005
10.005555555555556
```

Veldig vag oppgavetekst, linjen s = ys [0] + ys [-1] + 2* sum(ys [1: -1]) lignet en del på trapesformelen, så jeg endret den til s = (1/2) * (ys[0] + ys[-1]) + sum(ys[1:-1]) som er trapesmetoden.

Dermed så jeg de mente m/3, som er ut til å stemme.