Mathe 4 Exercise 1

Problem 1

$u = t^5 + \sin(xy)$

$u_y = x\cos(xy)$

$u_t = 5t^4$

$u_{xx} = \frac{\partial}{\partial x}\left(y\cos(xy)\right) = -y^2\sin(xy)$

$u_{xy} = \frac{\partial}{\partial x}\left(x\cos(xy)\right) = \cos(xy) - xy\sin(xy)$

$u_{yx} = \frac{\partial}{\partial y}\left(y\cos(xy)\right) = \cos(xy) - xy\sin(xy)$

$u = \cos(txy)$

$u_y = -tx\sin(txy)$

$u_t = -xy\sin(txy)$

$u_{xx} = \frac{\partial}{\partial x}\left(-ty\sin(txy)\right) = -t^2y^2\cos(txy)$

$u_{xy} = \frac{\partial}{\partial x}\left(-tx\sin(txy)\right) = -t\sin(txy) - t^2xy\cos(txy)$

$u_{yx} = \frac{\partial}{\partial y}\left(-ty\sin(txy)\right) = -t\sin(txy) - t^2xy\cos(txy)$

$$u = e^{-t}\sin(x)\ln(y)$$

$$u_y = \frac{e^{-t}\sin(x)}{y}$$

$$u_t = \sin(x)\ln(y)\cdot\frac{\partial}{\partial t}\left(e^{-t}\right) = -\sin(x)\ln(y)\,e^{-t}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(e^{-t}\cos(x)\ln(y)\right) = -e^{-t}\sin(x)\ln(y)$$

$$u_{xy} = \frac{\partial}{\partial x}\left(\frac{e^{-t}\sin(x)}{y}\right) = \frac{e^{-t}\cos(x)}{y}$$

$$u_{yx} = \frac{\partial}{\partial y}\left(e^{-t}\cos(x)\ln(y)\right) = \frac{e^{-t}\cos(x)}{y}$$

$$u = e^{-x}\sqrt{x^2+y}$$

$$u_y = \frac{e^{-x}}{2\sqrt{x^2+y}}$$

$$u_t = 0$$

$$u_{xx} = \frac{\partial}{\partial x}\left(-e^{-x}\sqrt{x^2+y} + \frac{\cancel{2}xe^{-x}}{\cancel{2}\sqrt{x^2+y}}\right)$$

$$= e^{-x}\sqrt{x^2+y} - \frac{\cancel{2}xe^{-x}}{\cancel{2}\sqrt{x^2+y}} + \left(\frac{(e^{-x}-xe^{-x})\sqrt{x^2+y} - \frac{\cancel{2}x^2e^{-x}}{\cancel{2}\sqrt{x^2+y}}}{x^2+y}\right)$$

$$= \frac{e^{-x}(x^2+y)^2}{(x^2+y)\sqrt{x^2+y}} - \frac{xe^{-x}(x^2+y)}{(x^2+y)\sqrt{x^2+y}} + \frac{(e^{-x}-xe^{-x})(x^2+y) - x^2e^{-x}}{(x^2+y)\sqrt{x^2+y}}$$

$$= \frac{e^{-x}(x^2+y^2) - xe^{-x}(x^2+y) + (e^{-x}-xe^{-x})(x^2+y) - x^2e^{-x}}{(x^2+y)\sqrt{x^2+y}}$$

$$u_{xy} = \frac{\partial}{\partial x}\left(\frac{e^{-x}}{2\sqrt{x^2+y}}\right)$$

$$= \frac{1}{2}\left(\frac{-e^{-x}\sqrt{x^2+y} - \frac{e^{-x}\cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+y}}}{x^2+y}\right) = \frac{-e^{-x}}{2\sqrt{x^2+y}} - \frac{xe^{-x}}{(x^2+y)^{3/2}}$$

$$u_{yx} = \frac{\partial}{\partial y}\left(-e^{-x}\sqrt{x^2+y} + \frac{e^{-x}\cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+y}}\right)$$

$$= -e^{-x}\frac{\partial}{\partial y}\sqrt{x^2+y} + xe^{-x}\frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y}}$$

$$= \frac{-e^{-x}}{2\sqrt{x^2+y}} + xe^{-x}\left(-\frac{1}{x^2+y}\right)\cdot\frac{1}{2\sqrt{x^2+y}}$$

$$= -\frac{e^{-x}}{2\sqrt{x^2+y}} - \frac{xe^{-x}}{2(x^2+y)^{3/2}}$$

$u = (t^2 e^t) \cos(x)$

$u_y = 0$

$u_t = \cos(x)\left(2te^t + t^2 e^t\right)$

$= te^t \cos(x)\,(t+2)$

$u_{xx} = \frac{\partial}{\partial x}\left(-t^2 e^t \sin x\right) = -t^2 e^t \cos x$

$u_{xy} = \frac{\partial}{\partial x} 0 = 0$

$u_{yx} = \frac{\partial}{\partial y}\left(-t^2 e^t \sin(x)\right) = 0$

$u = \sin(t) e^{-y} + \cos(t) e^{-x}$

$u_y = -\sin(t) e^{-y}$

$u_t = e^{-y}\cos(t) - e^{-x}\sin(t)$

$u_{xx} = \frac{\partial}{\partial x}\left(-\cos(t) e^{-x}\right) = \cos(t) e^{-x}$

$u_{xy} = \frac{\partial}{\partial x}\left(-\sin(t) e^{-y}\right) = 0$

$u_{yx} = \frac{\partial}{\partial y}\left(-\cos(t) e^{-x}\right) = 0$

Problem 2

$$u = -x^2 + y^2$$

$$u_{xx} = \frac{\partial}{\partial x}(-2x) = -2$$

$$u_{yy} = \frac{\partial}{\partial y}(2y) = 2$$

$$-2 + 2 = 0$$

$$u = \sin x \cosh y$$

$$u_{xx} = \frac{\partial}{\partial x}(\cosh y \cos x) = -\cosh y \sin x$$

$$u_{yy} = \frac{\partial}{\partial y}(\sin x \sinh y) = \sin x \cosh y$$

$$-\cosh y \sin x + \sin x \cosh y = 0$$

$$u = \arctan \frac{y}{x}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{1}{\left(\frac{y}{x}\right)^2 + 1} y \frac{\partial}{\partial x}\left(\frac{1}{x}\right)\right) = \frac{\partial}{\partial x}\left(\frac{y}{\left(\frac{y}{x}\right)^2 + 1} \cdot -\frac{1}{x^2}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{y^2 + x^2}\right)$$

$$= -y \frac{\partial}{\partial x}\left(\frac{1}{y^2 + x^2}\right) = -y \cdot -\frac{1}{(y^2+x^2)^2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{\partial}{\partial y}\left(\frac{1}{\left(\frac{y}{x}\right)^2+1} \frac{\partial}{\partial y}\left(\frac{y}{x}\right) \right)$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{\frac{y^2}{x}+x} \right) = \frac{\partial}{\partial y}\left(\frac{x}{y^2+x^2} \right)$$

$$= x \cdot -\frac{1}{(y^2+x^2)^2} \cdot 2y = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

$$u = \frac{y}{x^2+y^2}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(-\frac{2xy}{(y^2+x^2)^2} \right)$$

$$= \frac{-2y}{(y^2+x^2)^2} + -\frac{(-2xy)}{(y^2+x^2)^4} \frac{\partial}{\partial x}\left((y^2+x^2)^2\right)$$

$$= \frac{-2y}{(y^2+x^2)^2} + \frac{2xy \cdot 2\cancel{(y^2+x^2)} \cdot 2x}{(y^2+x^2)^{\cancel{4}}}$$

$$= \frac{8x^2y}{(x^2+y^2)^3} - \frac{2y}{(y^2+x^2)^2}$$

$$u_{yy} = \frac{\partial}{\partial y}\left(\frac{1}{y^2+x^2} + -\frac{y}{(y^2+x^2)^2}\cdot 2y\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{y^2+x^2} - \frac{2y^2}{(y^2+x^2)^2}\right)$$

$$= -\frac{2y}{(y^2+x^2)^2} - \left(\frac{4y}{(y^2+x^2)^2} + 2y^2 \cdot -\frac{1}{(y^2+x^2)^4}\cdot 2(y^2+x^2)\cdot 2y\right)$$

$$= \frac{-2y}{(y^2+x^2)^2} - \frac{4y}{(y^2+x^2)^2} + \frac{8y^3}{(y^2+x^2)^3}$$

$$= \frac{-6y}{(y^2+x^2)^2} + \frac{8y^3}{(y^2+x^2)^3}$$

$$\underline{y_{xx} + y_{yy} \neq 0}$$