# Exercise 6

Problem 1 4N only

Problem 2 a) $y'' + 4y = \delta(t-\pi)$, $y(0) = 8$ $y'(0) = 0$

$$s^2 Y(s) - s8 + 4Y(s) = e^{-\pi s}$$

$$Y(s)(s^2+4) - 8s = e^{-\pi s}$$

$$Y(s) = \frac{e^{-\pi s} + 8s}{s^2+2^2} = e^{-\pi s}\frac{1}{s^2+2^2} + \frac{8s}{s^2+2^2}$$

shift theorem

$$y(t) = u(t-a)f(t-a) + 8\cos 2t$$

$$= u(t-\pi)\mathcal{L}^{-1}\left[\frac{2}{2}\cdot\frac{1}{s^2+2^2}\right] + 8\cos 2t$$

$$\underline{y(t) = u(t-\pi)\cdot\frac{1}{2}\sin(2(t-\pi)) + 8\cos 2t}$$

hvor u er Heaviside-step funksjonen:

$$u \begin{cases} 0 & t < \pi \\ 1 & t > \pi \end{cases}$$

b) $y'' + 3y' + 2y = 10(\sin t + \delta(t-1))$

$y(0) = 1 \qquad y'(0) = -1$

$$s^2 Y(s) - s + 1 + 3(sY(s) - 1) + 2Y(s) = 10(\sin t + \delta(t-1))$$

$$s^2 Y(s) - s + 1 + 3sY(s) - 3 + 2Y(s) = 10\sin t + 10\delta(t-1)$$

$$Y(s)(s^2 + 3s + 2) - s - 2 = \frac{10}{s^2+1} + 10e^{-s}$$

$$Y(s) = \frac{\frac{10}{s^2+1} + 10e^{-s} - s - 2}{s^2 + 3s + 2}$$

$$= \frac{10}{(s^2+1)(s^2+3s+2)} + \frac{10e^{-s}}{s^2+3s+2} - \frac{s}{s^2+3s+2} - \frac{2}{s^2+3s+2}$$

$$= \frac{10}{(s^2+1)(s+1)(s+2)} + 10\frac{e^{-s}}{(s+1)(s+2)} - \frac{s}{(s+1)(s+2)} - \frac{2}{(s+1)(s+2)}$$

Delbrøksoppspaltning:

$$\frac{1}{(s+1)(s+2)} \stackrel{\wedge}{=} \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow 1 = A(s+2) + B(s+1)$$

$s = -2: \quad 1 = -B \Rightarrow \underline{B = -1}$

$s = -1: \quad 1 = A \Rightarrow \underline{A = 1}$

$$\Rightarrow -\frac{s}{(s+1)(s+2)} = -\left(\frac{s}{s+1} - \frac{s}{s+2}\right) = -\frac{s}{s+1} + \frac{s}{s+2}$$

$$\Rightarrow -\frac{2}{(s+1)(s+2)} = -\left(\frac{2}{s+1} - \frac{2}{s+2}\right) = -\frac{2}{s+1} + \frac{2}{s+2}$$

Delbrøksopspaltning:

$$\frac{1}{(s^2+1)(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+1}$$

$$1 = A(s^2+1)(s+2) + B(s^2+1)(s+1) + (Cs+D)(s+1)(s+2)$$

$$s=-1: \quad 1 = 2A \Rightarrow \underline{A = \frac{1}{2}}$$

$$s=-2: \quad 1 = \frac{1}{2}\cdot 5 \cdot 0 + B(5)\cdot(-1) + 0$$

$$1 = -5B \Rightarrow \underline{B = -\frac{1}{5}}$$

$$s=0: \quad 2\cdot\frac{1}{2} - \frac{1}{5} + Cs+D\,(1)(2) = 1$$

$$2C\overset{0}{s} + 2D = 1 - 1 + \frac{1}{5}$$

$$\underline{D = \frac{1}{10}}$$

$$s=1: \quad 1 = \frac{1}{2}(1+1)(3) + \left(-\frac{1}{5}\right)(2)(2) + \left(C + \frac{1}{10}\right)(2)(3)$$

$$6C = 1 - 3 + \frac{4}{5} - \frac{6}{10}$$

$$6C = -2 + \frac{8}{10} - \frac{6}{10} = -2 + \frac{2}{10} = -\frac{18}{10}$$

$$C = -\frac{18}{60} = \underline{-\frac{3}{10}}$$

$$Y(s) = \frac{1 \cdot 10}{2(s+1)} - \frac{10}{5(s+2)} + \frac{\left(-\frac{3}{10} \cdot s + \frac{1}{10}\right) \cdot 10}{s^2+1}$$

$$+ \frac{10 e^{-s}}{s+1} - \frac{10 e^{-s}}{s+2} - \frac{s}{s+1} + \frac{s}{s+2} - \frac{2}{s+1} + \frac{2}{s+2}$$

$$= \frac{5}{s+1} - \frac{\cancel{2}}{\cancel{s}+2} - \frac{3s-1}{s^2+1} + 10e^{-s} \cdot \frac{1}{s+1}$$

$$- 10e^{-s} \cdot \frac{1}{s+2} - \frac{s}{s+1} + \frac{s}{s+2} - \frac{2}{s+1} + \cancel{\frac{2}{s+2}}$$

$$y(t) = 5e^{-t} - 2e^{-t} - \left(3\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+1^2}\right]\right)$$

$$+ 10\left(u(t-1)\, e^{-(t-1)}\right)$$

$$- 10\left(u(t-1)\, e^{-2(t-1)}\right)$$

$$\mathcal{L}^{-1}\left[-\frac{s-1+1}{s+1} + \frac{s-2+2}{s+2}\right] \leftarrow \mathcal{L}^{-1}\left[-\frac{s+1}{s+1} - \frac{-1}{s+1} + \frac{s+2}{s+2} - \frac{2}{s+2}\right]$$

$$= 3e^{-t} - 3\cos t + \sin t + 10u(t-1)e^{1-t} - 10u(t-1)e^{2-2t}$$

$$- \cancel{\delta(t)} + e^{-t} + \cancel{\delta(t)} - 2e^{-2t}$$

$$y(t) = 4e^{-t} - 2e^{-2t} - 3\cos t + \sin t + 10u(t-1)e^{1-t} - 10u(t-1)e^{2-2t}$$

c) $y'' + 2y' + 5y = 25t - 100\delta(t-\pi)$, $y(0) = -2$

$y'(0) = 5$

LaPlace transformerer:

$$s^2 Y(s) + 2s - 5 + 2(sY(s) + 2) + 5Y(s) = \frac{25}{s^2} - 100e^{-\pi s}$$

$$Y(s)(s^2 + 2s + 5) + 2s - 1 = \frac{25}{s^2} - 100e^{-\pi s}$$

$$Y(s) = \frac{\frac{25}{s^2} - 100e^{-\pi s} - 2s + 1 - 4}{\underbrace{s^2 + 2s + 5 - 4}_{(s+1)^2}}$$

$$= \frac{25}{s^2(s+1)^2} - \frac{100e^{-\pi s}}{(s+1)^2} - \frac{2s}{(s+1)^2} - \frac{3}{(s+1)^2}$$

Huff

Problem 3 $\quad \dfrac{-3u(x)+4u(x+h)-u(x+2h)}{2h} \approx u'(x)$

Taylor expansion:

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2 u''(x) + \frac{1}{6}h^3 u^{(3)}(x) + O(h^4)$$

$$u(x+2h) = u(x) + 2hu'(x) + h^2 u''(x) + \frac{1}{3}h^3 u^{(3)}(x) + O(h^4)$$

Setter inn:

$$\frac{\cancel{-3u(x)} + \cancel{4u(x)} + 4hu'(x) + 2h^2 u''(x) + \frac{2}{3}h^3 u^{(3)}(x) - \cancel{u(x)} - 2hu'(x) - h^2 u''(x) - \frac{1}{3}h^3 u^{(3)}(x)}{2h}$$

$$= \frac{2hu'(x) + h^2 u''(x) + \frac{1}{3}h^3 u^{(3)}(x)}{2h} \approx u'(x)$$

$$= u'(x) + \frac{1}{2}hu''(x) + \frac{1}{6}h^2 u^{(3)}(x) \approx u'(x)$$

$$O(h^p) = \left|u'(x) - u'(x) + \frac{1}{2}hu''(x) + \frac{1}{6}h^2 u^{(3)}(x)\right|$$

$$= \left|\frac{1}{2}hu''(x) + \frac{1}{6}h^2 u^{(3)}(x)\right| = \underline{O(h^2)}$$

Convergence order is $\underline{\underline{p = 2}}$

b) $h = O(\varepsilon^k), \quad k > 0$

$h = \varepsilon^k$

siden vi har $O(h^2) \Rightarrow O(h^2) = O(\varepsilon^k)^2 = O(\varepsilon^{2k})$

Så $2k = 2 \Rightarrow \underline{\underline{k = 1}}$

Problem 4 4N only