Quasi-Frobenius Splittings and Calabi-Yau Varieties in Positive Characteristic

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Introduction

Let k be an algebraically closed field and X a smooth, proper variety over k of dimension $n > 0$.

Definition

X is a *Calabi-Yau* variety if $\omega_X \cong \mathscr{O}_X$ and for all $1 < i < n$ we have $h^i(X, \mathscr{O}_X) = 0$.

1. If $\dim(X) = 1$ then $X = E$ is an elliptic curve.
2. If $\dim(X) = 2$ then X is a K3 surface.

Witt Vectors

Let k be a perfect field of characteristic $p > 0$. Informally, the ring of *Witt vectors* with coefficients in k: $W(k)$ is the *minimal* complete discrete valuation ring of *characteristic 0* with residue field k such that the extension $W(k) \supseteq \mathbb{Z}_p$ is unramified and so p is a uniformiser. We write $W_n(k) = W(k)/p^{n+1}W(k)$ for the n-truncated Witt vectors.

Example

$W(\mathbb{F}_p) = \mathbb{Z}_p$

As a set, $W_2(k) = k \oplus k$ with addition and multiplication given as follows:

1. $(a_0, a_1) +_W (b_0, b_1) := (a_0 + b_0, a_1 + b_1 - \frac{1}{p}((a_0 + b_0)^p - a_0^p - b_0^p));$
2. $(a_0, a_1) \cdot_W (b_0, b_1) := (a_0 b_0, a_0^p b_1 + a_1 b_0^p + p a_1 b_1)$

The unit is given by $(1, 0)$ and we write $p = (0, 1)$ in $W_2(k)$.

Deformation Theory

Suppose that X is a smooth, proper variety over an algebraically closed field k. Let Art_k the category of artinian local rings $(A, \mathfrak{m}_A)$ with residue field $A/\mathfrak{m}_A = k$. A *(smooth) deformation* of X to A is a smooth A-scheme $\mathscr{X}$ such that $\mathscr{X}_k \cong X$. A *small extension* $A' \twoheadrightarrow A$ is a surjection in Art_k with kernel I satisfying $\mathfrak{m}_{A'} \cdot I = 0$.

Given a small extension $A' \to A$ and a deformation $\mathscr{X} \to \mathrm{Spec}(A)$ there is an associated *obstruction* given by an element of $H^2(X, \mathcal{T}_X) \cong \mathrm{Ext}^2(\Omega_X, \mathscr{O}_X)$. If this obstruction is zero then there exists a deformation $\mathscr{X}' \to \mathrm{Spec}(A')$ such that $\mathscr{X}'_A \cong \mathscr{X}$. Moreover, the set of such deformations up to isomorphism is given by $H^1(X, \mathcal{T}_X) \cong \mathrm{Ext}^1(\Omega_X, \mathscr{O}_X)$.

Definition

We say that deformations of X/k are *unobstructed* if the obstructions for all small extensions are zero.

Deformations in Mixed Characteristic

Classically, given a smooth proper variety $X/\mathbb{C}$ we are interested in smooth schemes $\mathscr{X}_1/D$, where $D = \mathbb{C}[t]/t^2$, that extend X. Consider the split exact sequence:

$$0 \longrightarrow \mathbb{C} \cong t \cdot D \longrightarrow D \longrightarrow \mathbb{C} \longrightarrow 0$$

where the section $\mathbb{C} \to D$ is given by setting $t = 0$. The base change of X with respect to the zero section gives a *trivial lifting* $\mathscr{X}_1^{\mathrm{triv}}$. Indeed, the set of small deformations over D (modulo isomorphism) of $X/\mathbb{C}$ is really a group with identity $\mathscr{X}_1^{\mathrm{triv}}$. On the other hand, if X/k is a smooth proper variety ($\mathrm{char}(k) = p > 0$, $k = \bar{k}$), then there is no distinguished lifting to $W_2(k)$. This is because the short exact sequence:

$$0 \longrightarrow k \cong pW_2(k) \longrightarrow W_2(k) \longrightarrow k \longrightarrow 0$$

does not admit a section. The isomorphism classes of small deformations of X are in fact a *(pseudo-)torsor* under $H^1(X, \mathcal{T}_X)$.

Main Theorem

Theorem (Bogomolov-Tian-Todorov)

Let $X/\mathbb{C}$ be a Calabi-Yau variety. Then deformations are unobstructed.

On the other hand, there exist examples of Calabi-Yau threefolds over algebraically closed fields k of characteristic $p > 0$ which admit no lifting to characteristic 0 [Hir99], and thus deformations of CY's in mixed characteristic are generically obstructed. The main theorem of Yobuko is the following [Yob19]:

Theorem (Yobuko)

Let X be a Calabi-Yau variety over an algebraically closed field of characteristic $p > 0$ with $\mathrm{ht}(X) < \infty$. Then X admits a smooth lifting to $W_2(k)$.

where $W_2(k)$ denotes the length 2 Witt vectors of k. The *(Artin-Mazur) height* $\mathrm{ht}(X)$ can be defined for any variety X/k. We will give the definition later. Indeed, the example of Hirokado has infinite height.

Degeneration of Hodge de-Rham Spectral Sequence

Theorem (Deligne-Illusie)

Let X/k be a smooth, proper scheme over an algebraically closed field of characteristic $p \geq \dim(X)$. Suppose that X admits a lifting to $W_2(k)$. Then the Hodge-de Rham spectral sequence:

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \implies H^{p+q}_{\mathrm{dR}}(X)$$

degenerates at the E_1-page.

Suppose that $X/\mathbb{C}$ is a smooth proper variety. Then, by degenerating to a prime $p \geq \dim(X)$ (of good reduction), Deligne-Illusie [DI87] give a purely algebraic proof of the degeneration of the Hodge-de Rham spectral sequence (well known for compact Kähler manifolds).

On the other hand, there is a counter-example to this phenomenon for a non-liftable variety due to Mumford [Mum61]. Morally, the behaviour of $W_2(k)$-liftable varieties is 'closer' varieties in characteristic 0.

Frobenius Splitting

Let X be a smooth, proper variety over k. We say that X is *Frobenius split* if the absolute Frobenius map:

$$F^{\#} : \mathscr{O}_X \longrightarrow F_* \mathscr{O}_X$$

admits a section as an $\mathscr{O}_X$-module homomorphism. Namely, there exists an $\mathscr{O}_X$-module homomorphism $\varphi : F_* \mathscr{O}_X \to \mathscr{O}_X$ such that $\varphi \circ F^{\#} = \mathrm{Id}$.

Example

$X = \mathbb{A}^n_k = \mathrm{Spec}(k[t_1, ..., t_n])$ is Frobenius split. If t^I is a monomial where $I \in \mathbb{N}^n$, then a splitting $\varphi : k[t_1, ..., t_n] \to F_*(k[t_1, ..., t_n])$ is induced by:

$$\varphi(t^I) = \begin{cases} t^J & \text{if } I = pJ,\ J \in \mathbb{N}^n \\ 0 & \text{otherwise} \end{cases}$$

In fact, if X/k is a smooth *affine* variety, then X is Frobenius split.

Frobenius Splitting

Given a smooth projective variety X/k, a Frobenius splitting always exists affine-locally, however, Frobenius split varieties are quite special. For example they satisfy strong vanishing properties [BK07]:

Proposition

Let X be a proper Frobenius split variety and $\mathscr{L}$ an ample line bundle. Then $H^i(X, \mathscr{L}) = 0$ for all $i > 0$.

If $X = C$ is a genus $g \geq 2$ curve, then ω_X is ample.

Example

Let C/k be a smooth proper curve of genus $g \geq 2$. Then C is not Frobenius split.

On the other hand, homogenising the splitting of affine space $\mathbb{A}^n_k$, one may show that projective space $\mathbb{P}^n_k$ is also Frobenius split.

Sketch of Proof

The key relationship between Frobenius splittings and liftings to $W_2(k)$ is given by the following theorem [Jos07]:

Theorem (Joshi)

Let X/k be a smooth, proper variety. If X is Frobenius split then there exists a flat lifting to $W_2(k)$.

Yobuko defines a new invariant called the *Frobenius split height* $\mathrm{ht}^s(X)$ of X. We say that X is *quasi-Frobenius split* if $\mathrm{ht}^s(X) < \infty$. Indeed, if $\mathrm{ht}^s(X) = 1$ then X is Frobenius split and we will show that, quite generally, any quasi-Frobenius split, smooth variety is $W_2(k)$-liftable. Finally, we will show that for a Calabi-Yau variety X, the notions of height and Frobenius split height coincide which will conclude the proof.

Elliptic Curves

We saw that curves of general type are never Frobenius split, whereas $\mathbb{P}^1_k$ is Frobenius split. What about elliptic curves E/k? Recall that the multiplication by p map $[p] : E \to E$ is of degree p^2. However, in characteristic $p > 0$ it factors through the relative Frobenius $F_{E/k} : E \to E^{(p)}$. In particular, the group of p-torsion points $E[p](k)$ is isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$ where $i = 0$ or 1. We call i the *p-rank* of E.

Definition

We say that an elliptic curve E/k is *ordinary* if it has p-rank 1 and *supersingular* if it has p-rank 0.

One can show that an ordinary elliptic curve is Frobenius split, however, a supersingular elliptic curve is not. On the other hand, $H^2(E, \mathcal{T}_E) = 0$ as $\dim(E) = 1$ so we may always lift an elliptic curve E/k to $W_2(k)$.

K3 Surfaces

Let X/k be a K3 surface. If k is of characteristic 0, it is well known that (when X is algebraic) the Picard group $\mathrm{Pic}(X)$ is free of rank ρ where $1 \leq \rho \leq 20$.

However, when k has characteristic $p > 0$, there exist K3 surfaces with $\rho = 22$. We refer to a K3 surface of rank 22 as a *Shioda supersingular K3* surface.

It is not hard to calculate that $H^2(X, \mathcal{T}_X) = 0$ for a K3 surface. In fact, Deligne was able to show that every K3 surface over an algebraically closed field of characteristic $p > 0$ admits a lift to characteristic 0 (he showed that the deformation functor is unobstructed) [DI81].

So we need to look in dimension ≥ 3 to find non-liftable Calabi-Yau varieties.

Formal Groups

Let k be an algebraically closed field of characteristic $p > 0$.

Definition

A *formal group* of *dimension* n over k is a homomorphism $k[\![X_1, ..., X_n]\!] \to k[\![Y_1, ..., Y_n]\!] \hat{\otimes}_k k[\![Z_1, ..., Z_n]\!]$ given by a tuple $f(Y, Z) = (f_i(Y, Z))$ of n power series in $2n$ variables $Y = (Y_1, ..., Y_n)$ and $Z = (Z_1, ..., Z_n)$ satisfying:

1. $f(X, 0) = f(0, X) = X$;
2. $f(X, f(Y, Z)) = f(f(X, Y), Z)$;
3. $f(X, Y) = X + Y +$ higher order terms

We say that $f(X, Y)$ is *commutative* if $f(X, Y) = f(Y, X)$. This defines a group law on the formal scheme $\mathrm{Spf}(k[\![X_1, ..., X_n]\!])$ with multiplication induced by f. Given $m > 0$, we inductively write $[m](X) = f(X, [m-1](X)) \in k[\![X_1, ..., X_n]\!]$ for multiplication by m.

Examples of Formal Groups

In characteristic $p > 0$, we have that:

$$[p](X) = X^{p^h} + \text{higher order terms}$$

for an integer $h > 0$ called the *height* of the formal group. Note that, it is possible for $[p](X) = 0$, in which case we write $h = \infty$.

Example

1. Let $\mathbb{G}_{m,k} = \operatorname{Spec}(k[T, T^{-1}])$ be the algebraic torus. We have an associated formal group law:

$$f(X, Y) = (X + 1)(Y + 1) - 1 = X + Y + XY$$

   The multiplicative formal group has height $h = 1$.

2. Associated to $\mathbb{G}_{a,k} = \operatorname{Spec}(k[T])$ we have the additive formal group law:

$$f(X, Y) = X + Y$$

   The additive formal group has height $h = \infty$.

Artin-Mazur Formal Groups

Let X be a smooth, proper variety over an algebraically closed field of characteristic $p > 0$. Let A be an artinian k-algebra and Pic_X/k the Picard variety associated to k. Let $\widehat{\mathrm{Pic}}_X$ denote the formal completion of Pic_X with respect to the identity. Then there is a natural isomorphism $\widehat{\mathrm{Pic}}_X(A) \cong \ker(H^1_{\text{ét}}(X_A, \mathbb{G}_m) \longrightarrow H^1_{\text{ét}}(X, \mathbb{G}_m))$.

Theorem (Artin-Mazur)

Suppose that X/k is a Calabi-Yau variety of dimension n and let Art_k denote the category of artinian k-algebras. Then the functor:

$$\begin{aligned} \Phi_X : \mathrm{Art}_k &\longrightarrow \mathrm{Ab} \\ A &\longmapsto \ker(H^n_{\text{ét}}(X_A, \mathbb{G}_m) \longrightarrow H^n_{\text{ét}}(X, \mathbb{G}_m)) \end{aligned}$$

is pro-representable by a formal group (also denoted by Φ_X) of height 1.

[AM77]. We define the *height* of X to be $\mathrm{ht}(X) = \mathrm{ht}(\Phi_X)$. For an elliptic curve E, the formal group Φ_E coincides with $\hat{E}$.

Artin-Mazur Formal Groups of K3 Surfaces

Recall that the *Brauer group* of a K3 surface X/k is given by:

$$\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$$

Unlike the Picard group $\mathrm{Pic}(X)$, there is not usually a k-scheme that parametrises the Brauer group. However, Φ_X can be thought of as a *formal Brauer group* $\widehat{\mathrm{Br}}_X$ as for an artinian k-algebra A we have:

$$\Phi_X(A) = \ker(H^2_{\text{ét}}(X_A, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(X, \mathbb{G}_m))$$

Theorem (Artin-Mazur)

Let X/k be a K3 surface, $k = \bar{k}$ of characteristic $p > 0$. Then:

$$\mathrm{ht}(X) \in \{1, ..., 10\} \cup \{\infty\}$$

When $h = \mathrm{ht}(X) < \infty$, we have that $\rho = \mathrm{rk}(\mathrm{Pic}(X)) = 22 - h$. We say that X is *Artin supersingular* if $h = \infty$. Thanks to the proof of the Tate conjecture for K3 surfaces, we have that $h = \infty \iff \rho = 22$ and so we can say X is *supersingular* without ambiguity.

Witt Vectors (again)

Consider the polynomials

$$W_0 = X_0,\, W_1 = X_0^p + pX_1,\, ...,\, W_n = X_0^{p^n} + pX_1^{p^{n-1}} + ... + p^n X_n$$

in $\mathbb{Z}[X_0, X_1, ..., X_n]$. Then, one can show that there exists polynomials $S_i(X,Y), P_i(X,Y) \in \mathbb{Z}[X_0, ..., X_n, Y_0, ..., Y_n]$ for $i = 0, 1, ..., n$ such that:

$$\begin{aligned} W_n(S_0, ..., S_n) &= W_n(X_0, ..., X_n) + W_n(Y_0, ..., Y_n) \\ W_n(P_0, ..., P_n) &= W_n(X_0, ..., X_n) \cdot W_n(Y_0, ..., Y_n) \end{aligned}$$

More generally, if A is an $\mathbb{F}_p$-algebra we can define a ring $W_n(A)$ given by $A^{\oplus n}$ as a set with addition and multiplication:

$$\begin{aligned} a +_{W_n} b &= (S_0(a,b), ..., S_n(a,b)) \\ a \cdot_{W_n} b &= (P_0(a,b), ..., P_n(a,b)) \end{aligned}$$

For example, $S_0 = X_0 + Y_0$ and $S_1 = X_0 + Y_0 - \frac{1}{p}((X_1 + Y_1)^p - X_1^p - Y_1^p)$

Witt Vectors (again)

We have the following additive group homomorphisms associated to the Witt vectors of an $\mathbb{F}_p$-algebra A:

1. The *Frobenius*:
$$F : W_m(A) \longrightarrow W_m(A), \quad (a_0, ..., a_{m-1}) \longmapsto (a_0^p, ..., a_{m-1}^p);$$
2. The *Verschiebung*:
$$V : W_m(A) \longrightarrow W_{m+1}(A), \quad (a_0, ..., a_{m-1}) \longmapsto (0, a_0, ..., a_{m-1});$$
3. The *restriction*:
$$R : W_{m+1}(A) \longrightarrow W_m(A), \quad (a_0, ..., a_m) \longmapsto (a_0, ..., a_{m-1}).$$

Taking the limit, we define the Witt vectors associated to A to be:
$$W(A) := \varprojlim W_m(A)$$
under the restrictions R. The ring of Witt vectors has associated Frobenius and Verschiebung maps defined similarly to above.

Dieudonné Modules

Let k be an algebraically closed field of characteristic $p > 0$. As k is perfect, the lift of Frobenius $\sigma : W(k) \to W(k)$ given by $(a_0, a_1, ...) \mapsto (a_0^p, a_1^p, ...)$ is a ring automorphism. We define the *Dieudonné ring* $D(k)$ to be the non-commutative ring obtained by adjoining elements F, V to $W(k)$ with the relations:

$$F \cdot \lambda = \lambda^\sigma, \quad V \cdot \lambda = \lambda^{\sigma^{-1}}, \quad F \cdot V = V \cdot F = p$$

for any element $\lambda \in W(k)$. We use the notation $D(k) = \frac{W(k)\{F,V\}}{(FV-p)}$. There is a fully faithful functor between the category of formal groups over k and the category of *Dieudonné modules* [Dem06]:

$$\mathbb{D} : \mathrm{FGrp}_k \longrightarrow D(k)\text{–Mod}$$

where for a formal group Γ we define $\mathbb{D}(\Gamma) = \varprojlim \mathrm{Hom}(\Gamma, \mathbb{W}_n)$ where $\mathbb{W}_n$ is the *Witt scheme* of length n. Informally, the Dieudonné module allows us to classify formal groups in terms of linear algebra via the actions Frobenius and Verschiebung.

Properties of Dieudonné Modules

Let M be the Dieudonné module of a formal group Γ (e.g. $M = \mathbb{D}(\Phi_X)$ for a Calabi-Yau X). Then we have the following:

1. M is free as a $W(k)$-module;
2. $\mathrm{rk}_{W(k)} M = \dim(\Gamma)$.

Example

1. Let $M = \mathbb{D}(\hat{\mathbb{G}}_a)$. Then M is the free $W(k)$-module generated by e say with $F(e) = 0$ and $V(e) = pe$.
2. Let $M = \mathbb{D}(\hat{\mathbb{G}}_m)$. Then $M = W(k) \cdot e$ with $F(e) = pe$ and $V(e) = e$.

Many properties of a formal group Γ can be read off from the Dieudonné module. For example, letting $K = \mathrm{Frac}(W(k))$:

$$\mathrm{ht}(\Gamma) = \begin{cases} \dim_K \mathbb{D}(\Gamma) \otimes_{W(k)} K & \text{if} \quad \mathbb{D}(\Gamma) \otimes_{W(k)} K \neq 0 \\ \infty & \text{otherwise} \end{cases}$$

Differential Calculus in Positive Characteristic

Let X/k be a smooth proper variety of dimension n over an algebraically closed field of characteristic $p > 0$. Let Ω^1_X denote the sheaf of Kähler differentials. We write $B^1\Omega_X = \operatorname{Im}(\mathscr{O}_X \xrightarrow{d} \Omega^1_X)$ for the subsheaf of *exact* 1-forms and $Z^1\Omega_X = \ker(\Omega^1_X \to \Omega^2_X)$ for the sheaf of *closed* 1-forms. Recall that the differential $d : \mathscr{O}_X \to \Omega^1_X$ satisfies the Leibniz rule $d(fg) = f \cdot dg + g \cdot df$ for local sections $f, g \in \mathscr{O}_X$. Of course this is not a morphism of $\mathscr{O}_X$-modules, however, it is $\mathscr{O}_X^p$-linear:

$$d(f^p g) = pf^{p-1}g \cdot df + f^p \cdot dg = f^p \cdot dg$$

In this way, the derivation becomes $\mathscr{O}_X$-linear on the complex $0 \to F_*\mathscr{O}_X \to F_*\Omega^1_X \to \ldots$ and we may think of:

$$B^1\Omega_X \subseteq Z^1\Omega_X \subseteq F_*\Omega^1_X$$

as $\mathscr{O}_X$-*submodules.*

Let $\gamma : \Omega^1_X \to Z^1\Omega_X/B^1\Omega_X$ be defined by $\gamma(f \cdot dg) = f^p g^{p-1} \cdot dg$.

Theorem (Cartier)

Suppose that X/k is a smooth scheme. Then γ admits an inverse:

$$C : Z^1\Omega_X/B^1\Omega_X \longrightarrow \Omega^1_X$$

called the Cartier operator.

Let $X = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$. Then we have that $F_*k[t]$ is a free $k[t]$-module of rank p with basis $\{1, t, ..., t^{p-1}\}$ i.e. we can uniquely express every polyonimal $f(t) \in k[t]$ as:

$$f(t) = f_0(t)^p + f_1(t)^p t + ... + f_{p-1}(t)^p t^{p-1}$$

We then have that:

$$\begin{aligned} C : Z^1\Omega_{k[t]} &\longrightarrow \Omega^1_{k[t]} \\ f(t)dt &\longmapsto f_{p-1}(t)dt \end{aligned}$$

Cartier Operator

Alternatively phrased, we have a short exact sequence of $\mathscr{O}_X$-modules:

$$0 \to B^1\Omega_X \to Z^1\Omega_X \xrightarrow{C} \Omega^1_X \to 0$$

Inductively, we define the $\mathscr{O}_X$-submodules:

$$B^m\Omega_X \subseteq Z^m\Omega_X \subseteq (F^m)_*\Omega^1_X$$

by setting:

$$B^{m+1}\Omega_X = \gamma(B^m\Omega_X) \quad \text{and} \quad Z^{m+1}\Omega_X = \gamma(Z^m\Omega_X)$$

We have the *Illusie sheaves* $B^m\Omega_X$, for each $m > 0$ [Ill79]. They will play an important role throughout. Note that, by construction, a local section $f \in \mathscr{O}_X$ acts on a local section $\omega \in Z^m\Omega_X$ by $f^{p^m}\omega$.

Witt Sheaves

Let X be a smooth k-scheme. Given a commutative k-algebra A, recall that we defined the rings $W_m(A)$ for $m \geq 0$ and $W(A) = \varprojlim W_m(A)$. Similarly, we can define the *Serre Witt vector sheaf* $W_m\mathscr{O}_X$ by setting:

$$(W_m\mathscr{O}_X)(U) = W_m(\mathscr{O}_X(U))$$

for all opens $U \subseteq X$. Again, we have the associated Frobenius, Verschiebung and restriction morphisms (of sheaves of abelian groups):

$$\begin{aligned}
F &: W_m\mathscr{O}_X \longrightarrow W_m\mathscr{O}_X, \quad (f_0, ..., f_{m-1}) \longmapsto (f_0^p, ..., f_{m-1}^p); \\
V &: W_m\mathscr{O}_X \longrightarrow W_{m+1}\mathscr{O}_X, \quad (f_0, ..., f_{m-1}) \longmapsto (0, f_0, ..., f_{m-1}); \\
R &: W_m\mathscr{O}_X \longrightarrow W_m\mathscr{O}_X, \quad (f_0, ..., f_m) \longmapsto (f_0, ..., f_{m-1}).
\end{aligned}$$

The Illusie and Serre Witt vector sheaves are related by [Ser58, Prop. 8]:

$$0 \to W_m\mathscr{O}_X \xrightarrow{F} F_*W_m\mathscr{O}_X \xrightarrow{D_m} B^m\Omega_X \to 0$$

where $D_m(f_0, ..., f_{m-1}) = df_{m-1} + f_{m-2}^{p-1}df_{m-2} + ... + f_0^{p^{m-1}-1}df_0$.

Frobenius Liftability

Note that for $m = 1$ we have the short exact sequence $\mathcal{E}_1$:

$$0 \to \mathscr{O}_X \xrightarrow{F} F_*\mathscr{O}_X \xrightarrow{d} B^1\Omega_X \to 0$$

Recall that, given a smooth scheme X/k, we obtain an obstruction $\mathrm{ob}_X \in H^2(X, \mathcal{T}_X) \cong \mathrm{Ext}^2(\Omega_X, \mathscr{O}_X)$ to lifting X to $\mathscr{X}_1/W_2(k)$.

Definition

We say that a smooth scheme X/k is Frobenius liftable if it admits a smooth lifting $\mathscr{X}_1/W_2(k)$ and a morphism $F_1 : \mathscr{X}_1 \to \mathscr{X}_1$ that extends the absolute Frobenius $F : X \to X$.

Lemma (Srinivas-Mehta)

1. *The obstruction to lifting the pair* (X, F) *is given by* $\mathrm{ob}_{(X,F)} \in H^1(X, \mathscr{H}om(\Omega^1_X, B^1\Omega_X)) \cong \mathrm{Ext}^1(\Omega^1_X, B^1\Omega_X)$
2. *The set of possible liftings* $\mathrm{Rel}(X, F)$ *is a pseudo-torsor under* $\mathrm{Hom}(\Omega^1_X, B^1\Omega_X)$

Obstructions to lifting Frobenius

Let $X = \bigcup U_\alpha$ be an open affine cover and suppose we pick a flat lifting $\mathscr{X}_1 = \bigcup \mathscr{U}_\alpha$. We can always lift the Frobenius affine-locally on smooth varieties so suppose we pick local liftings $F_\alpha : \mathscr{U}_\alpha \to \mathscr{U}_\alpha$ of $F|_{U_\alpha}$. Then, given a local section $x \in \mathscr{O}_{\mathscr{U}_\alpha}$, we have $F_\alpha(x) = x^p + p \cdot y$ for another local section y. So on overlaps $\mathcal{U}_{\alpha\beta}$ we have:

$$(F_\alpha - F_\beta)|_{\mathcal{U}_{\alpha\beta}} =: p \cdot h_{\alpha\beta}$$

where $h_{\alpha\beta}$ satisfies:

1. $h_{\alpha\beta}(x+y) = h_{\alpha\beta}(x) + h_{\alpha\beta}(y) + \frac{1}{p}(x^p + y^p - (x+y)^p)$;
2. $h_{\alpha\beta}(xy) = h_{\alpha\beta}(x) \cdot y^p + x^p \cdot h_{\alpha\beta}(y)$

Untangling this, we see that $(h_{\alpha\beta}, U_{\alpha\beta})$ defines a 1-cocycle for the sheaf of derivations on X valued in $F_*\mathscr{O}_X/\mathscr{O}_X \cong B^1\Omega_X$ i.e. an element:

$$\mathrm{ob}_{(X,F)} \in H^1(X, \mathscr{H}om(\Omega^1_X, B^1\Omega_X)) \cong \mathrm{Ext}^1(\Omega^1_X, B^1\Omega_X)$$

Similarly, if we have a lifting $(\mathscr{U}_\alpha, F_\alpha)$, we may modify it by a p-derivation $(\mathscr{U}_\alpha, \delta_\alpha)$ satisfying (1) and (2) to obtain another such lift.

Lifts of the Frobenius and the Cartier Operator

Recall that the Cartier operator induces a short exact sequence:

$$0 \to B^1\Omega_X \to Z^1\Omega_X \xrightarrow{C} \Omega^1_X \to 0$$

and thus defines an element in $\mathrm{Ext}^1(\Omega^1_X, B^1\Omega_X)$. In fact, this is precisely the obstruction $\mathrm{ob}_{(X,F)}$. Indeed, if $(\mathscr{X}_1, F_1)$ is a lifting of (X, F) then for some local section $x \in \mathscr{O}_{\mathscr{X}_1}$ we have again that $F_1(x) = x^p + p \cdot y$ for another local section y. On differential forms this gives the morphism of sheaves:

$$\begin{aligned} (F_1)_* : \Omega^1_{\mathscr{X}_1} &\longrightarrow (F_1)_*\Omega^1_{\mathscr{X}_1} \\ f \cdot dx &\longmapsto F_1(f) \cdot (px^{p-1}dx + pdy) \end{aligned}$$

Thus, 'dividing by p' we obtain a morphism of sheaves $\gamma :' \Omega^1_X \longrightarrow F_*\Omega^1_X$ sending $f \cdot dx \mapsto f^p x^{p-1} dx$ modulo exact forms, i.e. a section of the above exact sequence (cf. the inverse $\gamma = C^{-1}$).

Frobenius liftability vs. $W_2(k)$-liftability

The results of the previous slide illustrate the following:

Proposition

The obstruction $\mathrm{ob}_{(X,F)} \in \mathrm{Ext}^1(\Omega^1_X, B^1\Omega_X)$ *to lifting the pair* (X, F) *is given by the short exact sequence:*

$$0 \to B^1\Omega_X \to Z^1\Omega_X \xrightarrow{C} \Omega^1_X \to 0$$

Finally, we state the following without proof:

Proposition

Let $\delta : \mathrm{Ext}^1(\Omega^1_X, B^1\Omega_X) \to \mathrm{Ext}^2(\Omega^1_X, \mathscr{O}_X)$ *be the connecting map induced by the extension* $\mathcal{E}_1$*:*

$$0 \to \mathscr{O}_X \xrightarrow{F} F_*\mathscr{O}_X \xrightarrow{d} B^1\Omega_X \to 0$$

Then:

$$\delta(\mathrm{ob}_{(X,F)}) = \mathrm{ob}_X$$

Quasi-Frobenius Split Varieties

Let X be a k-scheme. For each $m > 0$, consider the diagram:

$$\begin{tikzcd} W_m\mathscr{O}_X \arrow[r, "F"] \arrow[d, "R^{m-1}"] & F_*W_m\mathscr{O}_X \\ \mathscr{O}_X & \end{tikzcd}$$

Definition

The *Frobenius-split height* $\mathrm{ht}^s(X)$ of X is the minimal $m > 0$ such that, there exists a $W_m\mathscr{O}_X$-linear homomorphism:

$$\varphi : F_*W_m\mathscr{O}_X \longrightarrow \mathscr{O}_X$$

such that $\varphi \circ F = R^{m-1}$. If no such $m > 0$ exists then $\mathrm{ht}^s(X) := \infty$. Otherwise, we say that X is *quasi-Frobenius split.*

Clearly, X is Frobenius split if and only if it has $\mathrm{ht}^s(X) = 1$.

Proof of Main Theorem 1

Theorem

Suppose that X/k is smooth and quasi-Frobenius split. Then X is $W_2(k)$-liftable.

Consider the exact sequence:

$$0 \to W_m\mathscr{O}_X \xrightarrow{F} F_*W_m\mathscr{O}_X \xrightarrow{D_m} B^m\Omega_X \to 0$$

and pushout along the restriction $R^{m-1} : W_m\mathscr{O}_X \to \mathscr{O}_X$ to obtain the short exact sequence:

$$0 \to \mathscr{O}_X \to \mathcal{E}_m \to B^m\Omega_X \to 0$$

Then, this sequence splits if and only if there exists a quasi-Frobenius splitting $\varphi : F_*W_m\mathscr{O}_X \longrightarrow \mathscr{O}_X$. Therefore, we have that:

$$\mathrm{ht}^s(X) = \inf\{m > 0 \mid \mathcal{E}_m \text{ splits}\}$$

(if such an $m > 0$ exists, otherwise it is infinite).

Moreover, as the diagram:

$$\begin{tikzcd} F_*W_{m+1}\mathscr{O}_X \arrow[r, "D_{m+1}"] \arrow[d, "R"'] & B^{m+1}\Omega_X \arrow[d, "C"] \\ F_*W_m\mathscr{O}_X \arrow[r, "D_m"] & B^m\Omega_X \end{tikzcd}$$

commutes, we obtain the extension $\mathcal{E}_m$ by pulling back the exact sequence $\mathcal{E}_1$ along $C^{m-1} : B^m\Omega_X \to B^1\Omega_X$. In other words, we have a morphism of extensions:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_X \arrow[r] \arrow[d, "\sim"] & \mathcal{E}_m \arrow[r] \arrow[d] & B^m\Omega_X \arrow[r] \arrow[d, "C^{m-1}"] & 0 \\ 0 \arrow[r] & \mathscr{O}_X \arrow[r] & F_*\mathscr{O}_X \arrow[r] & B^1\Omega_X \arrow[r] & 0 \end{tikzcd}$$

Proof of Main Theorem 1

Suppose X/k is smooth with $\mathrm{ht}^s(X) < \infty$. Then the extension $\mathcal{E}_m$ is zero in $\mathrm{Ext}^1(B^m\Omega_X, \mathscr{O}_X)$. Let $x \in \mathrm{Ext}^1(\Omega_X, B^m\Omega_X)$ denote the extension in the top row of the below diagram:

$$\begin{tikzcd}
0 \arrow[r] & B^m\Omega_X \arrow[r] \arrow[d, "C^{m-1}"] & Z^m\Omega_X \arrow[r, "C^m"] \arrow[d, "C^{m-1}"] & \Omega_X \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & B^1\Omega_X \arrow[r] & Z^1\Omega_X \arrow[r] & \Omega_X \arrow[r, "C"] & 0
\end{tikzcd}$$

Consider the commutative diagram below:

$$\begin{tikzcd}
\mathrm{Ext}^1(\Omega^1_X, B^m\Omega_X) \otimes \mathrm{Ext}^1(B^m\Omega^1_X, \mathscr{O}_X) \arrow[r, "{(-,-)}"] \arrow[d, "C^{m-1}_*"] & \mathrm{Ext}^2(\Omega^1_X, \mathscr{O}_X) \arrow[d, "\sim"] \\
\mathrm{Ext}^1(\Omega_X, B^1\Omega_X) \otimes \mathrm{Ext}^1(B^1\Omega_X, \mathscr{O}_X) \arrow[r, "{(-,-)}"] \arrow[u, shift left=20, "(C^{m-1})^*"'] & \mathrm{Ext}^2(\Omega^1_X, \mathscr{O}_X)
\end{tikzcd}$$

where the vertical arrows represent pushout and pullback by C^{m-1} respectively, and the horizontal arrows are the Yoneda product.

Proof of Main Theorem 1

Then overall we have:

$$\begin{aligned}
\mathrm{ob}_X &= (\mathrm{ob}_{(X,F)}, \mathcal{E}_1) \\
&= (C_*^{m-1} x, \mathcal{E}_1) \\
&= (x, (C^{m-1})^* \mathcal{E}_1) \\
&= (x, \mathcal{E}_m) \\
&= 0
\end{aligned}$$

where by assumption $\mathcal{E}_m$ is the trivial extension.

Proof of Main Theorem 2

Theorem

Let X/k be a Calabi-Yau variety. Then $\mathrm{ht}^s(X) = \mathrm{ht}(X)$.

Equivalently, we can show that $\mathrm{ht}(X) \leq m$ if and only if $\mathcal{E}_m$ is zero in $\mathrm{Ext}^1(B^m\Omega_X, \mathscr{O}_X)$. Let X be a Calabi-Yau variety of dimension n and Φ_X the Artin-Mazur formal group of X. Recall that the height of X is given by the dimension of the Dieudonné module as a $K = \mathrm{Frac}(W(k))$ vector space. In fact, there is a canonical isomorphism:

$$\mathbb{D}(\Phi_X) \cong \varprojlim_m H^n(X, W_m\mathscr{O}_X)$$

The following characterisation of $\mathrm{ht}(X)$ is due to van der Geer and Katsura [GK03]:

Theorem (van der Geer-Katsuya)

The height of X is equal to the minimum number $m > 0$ such that F acts trivially on $H^n(X, W_m\mathscr{O}_X)$.

Proof of Main Theorem 2

As a corollary, we have:

$$\dim_k H^i(X, B^m\Omega_X) = \begin{cases} \min\{m, \mathrm{ht}(X) - 1\} & \text{if } i = n-1, n \\ 0 & \text{otherwise} \end{cases}$$

Consider the exact sequence:

$$0 \to F_* B^{m-1}\Omega_X \xrightarrow{\iota} B^m\Omega_X \xrightarrow{C^{m-1}} B^1\Omega_X \to 0$$

Recall that $\mathcal{E}_1 \in \mathrm{Ext}^1(B^1\Omega_X, \mathscr{O}_X)$ and $(C^{m-1})^*\mathcal{E}_1 = \mathcal{E}_m \in \mathrm{Ext}^1(B^m\Omega_X, \mathscr{O}_X)$. As X is Calabi-Yau, it has trivial dualising sheaf $\omega_X \cong \mathscr{O}_X$ and so by Serre duality we have that:

$$\mathrm{Ext}^i(B^m\Omega_X, \mathscr{O}_X) \cong \mathrm{Ext}^i(B^m\Omega_X, \omega_X) \cong H^{n-i}(X, B^m\Omega_X)^\vee$$

for $i = 0, 1, ..., n$. In particular, $\mathrm{Ext}^2(B^1\Omega_X, \mathscr{O}_X) = 0$. Thus, we see that ι^* is surjective from the exact sequence:

Proof of Main Theorem 2

$$\begin{aligned} \mathrm{Ext}^1(B^1\Omega_X, \mathscr{O}_X) &\xrightarrow{(C^{m-1})^*} \mathrm{Ext}^1(B^m\Omega_X, \mathscr{O}_X) \xrightarrow{\iota^*} \mathrm{Ext}^1(F_*B^{m-1}\Omega_X, \mathscr{O}_X) \\ \mathcal{E}_1 &\longmapsto \mathcal{E}_m \qquad\qquad\qquad\qquad \xrightarrow{\delta} \mathrm{Ext}^2(B^1\Omega_X, \mathscr{O}_X) = 0 \end{aligned}$$

Moreover, $\dim_k \mathrm{Ext}^1(B^1\Omega_X, \mathscr{O}_X) = \min\{1, \mathrm{ht}(X) - 1\}$. Then, as $\mathcal{E}_1$ is zero if and only if $\mathrm{ht}(X) = 1$ we see that $\mathcal{E}_1$ generates $\mathrm{Ext}^1(B^1\Omega_X, \mathscr{O}_X)$. Finally, as F is finite, F_* is exact and so:

$$\mathrm{Ext}^i(F_*B^{m-1}\Omega_X, \mathscr{O}_X) \cong H^{n-i}(X, F_*B^{m-1}\Omega_X)^\vee \cong H^{n-i}(X, B^{m-1}\Omega_X)^\vee$$

Overall, $\mathcal{E}_m$ is the trivial extension if and only if ι^* is an isomorphism:

$$\iota^* : H^{n-1}(X, B^m\Omega_X)^\vee \to H^{n-1}(X, F_*B^{m-1}\Omega_X)^\vee \cong H^{n-1}(X, B^{m-1}\Omega_X)^\vee$$

which happens if and only if:

$$\min\{m, \mathrm{ht}(X) - 1\} = \min\{m - 1, \mathrm{ht}(X) - 1\}$$

i.e. $\mathrm{ht}(X) \leq m$.

Final Comments

Remarks

1. The discussion on liftings also holds for all small extensions $R : W_{n+1}(k) \to W_n(k)$. In particular, if X/k is a Calabi-Yau of finite height, then one can build a compatible system of lifts:

$$X \hookrightarrow \mathscr{X}_1 \hookrightarrow \dots \hookrightarrow \mathscr{X}_n \hookrightarrow \dots$$

   where $\mathscr{X}_n/W_n(k)$ to define a *formal lifting* $\widehat{\mathscr{X}} = \varinjlim \mathscr{X}_n$ of X. However, this does not imply that there is a $W(k)$-scheme $\mathscr{X}$ lifting X which algebraises $\widehat{\mathscr{X}}$.

2. By work of Mumford [Mum69], it is known that every polarised abelian variety A/k admits a lifting to an abelian scheme $\mathscr{A}/R$ without giving information on the characteristic 0 ring R. However, there are examples due to Norman [Nor81] of abelian varieties A/k that do not admit liftings to $W(k)$, but rather to *ramified extensions* $R \supseteq W(k)$.

Final Comments

Let A/k be an abelian variety of dimension g. Then $[p] : A \to A$ is finite of degree p^{2g}, however, in characteristic $p > 0$ we have that:

$$|A[p](k)| = p^i$$

where i is an integer $0 \leq i \leq g$ called the *p-rank* of A. Indeed, it is not hard to show that $\mathrm{ht}(A) = 2g - i$. When $i = g$ we say that A is *ordinary*. The theory of Serre-Tate moduli tells us that, for an ordinary abelian variety A, the deformation functor is unobstructed and pro-represented by a formal torus $\widehat{\mathbb{G}}_{m,W(k)}^{g^2}$ [LST64]. In particular, there exists a *canonical lifting* $\mathscr{A}^{\mathrm{can}}/W(k)$ corresponding to the identity of the formal torus which is the unique lifting of A (up to isomorphism) that lifts the absolute Frobenius F_A on A:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F_{\mathscr{A}}"] & \mathscr{A} \\
A \arrow[r, "F_A"] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

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