1a $\quad f(x; r, R) = \frac{1}{2\pi}\left[1 + \frac{r}{R}\cos x\right]$, $x \in [0, 2\pi]$, $0 < r < R$

define $Y \sim U(0, 2\pi)$, so pdf of Y: $g(y) = \frac{1}{2\pi}$, $\quad 0 \le y \le 2\pi$

Note that g & f having the same support $[0, 2\pi]$

Also, $\left|\frac{f(x)}{g(x)}\right| = \left|1 + \frac{r}{R}\cos x\right| \le 2$, so f/g is bounded in $[0, 2\pi]$

Now, let $M = \sup_x \left\{\frac{f(x; r, R)}{g(x)}\right\} = \sup_x \left\{1 + \frac{r}{R}\cos x\right\}$

$= 1 + \frac{r}{R}$, occurs at $x = 0$ or 2π

Note that Y is our proposed uniform distribution

Algorithm: Step 1: simulate $Y \sim U(0, 2\pi)$

simulate $U \sim U(0, 1)$

Step 2: calculate the ratio $\frac{f(Y)}{Mg(Y)}$ and compare with U

If $U < \frac{f(Y)}{Mg(Y)}$

then set $X = Y$, the accepted sample of X

and return X

else, back to step 1 and repeat

2a $\quad b_1, b_2 \sim \exp(\lambda)$, independent , $\lambda = 0.01$

$$\pi(b_1, b_2) = \lambda \exp(-\lambda b_1) \cdot \lambda \exp(-\lambda b_2)$$
$$= \lambda^2 \exp(-\lambda(b_1+b_2))$$
$$\propto \exp(-\lambda(b_1+b_2))$$

then given $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma^2 = 0.01$, $i = 1, 2, \dots, 10$, take $n = 10$

so $y_i \sim N(b_1 \exp\{1 - \exp(-x_i/b_2)\}, \sigma^2)$

so, $\pi(y \mid x, b_1, b_2)$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2\sigma^2}\left(y_i - b_1 \exp\left\{1 - \exp\left(\frac{-x_i}{b_2}\right)\right\}\right)^2\right]$$

$$\propto \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}\left[y_i^2 - 2b_1 y_i \exp\left\{1-\exp\left(\frac{-x_i}{b_2}\right)\right\} + b_1^2 \exp\left\{2\left(1-\exp\left(\frac{-x_i}{b_2}\right)\right)\right\}\right]\right]$$

$$\propto \exp\left[\frac{-1}{2\sigma^2}\left(-2b_1 \sum_{i=1}^{n} y_i \exp\left\{1-\exp\left(\frac{-x_i}{b_2}\right)\right\} + b_1^2 \sum_{i=1}^{n} \exp\left\{2 - 2\exp\left(\frac{-x_i}{b_2}\right)\right\}\right)\right]$$

$$\propto \exp\left[\frac{b_1}{\sigma^2}\sum_{i=1}^{n} y_i \exp\left\{1-\exp\left(\frac{-x_i}{b_2}\right)\right\} - \frac{b_1^2}{2\sigma^2}\sum_{i=1}^{n} \exp\left\{2-2\exp\left(\frac{-x_i}{b_2}\right)\right\}\right]$$

Hence, the posterior distribution, and put back the value

$$\pi(b_1, b_2 \mid y, x) \propto \pi(y \mid x, b_1, b_2) \cdot \pi(b_1, b_2)$$
$$\propto \exp\left[100 b_1 \sum_{i=1}^{10} y_i \exp\left\{1 - \exp\left(\frac{-x_i}{b_2}\right)\right\} - 50 b_1^2 \sum_{i=1}^{10} \exp\left\{2 - 2\exp\left(\frac{-x_i}{b_2}\right)\right\} - 0.01(b_1 + b_2)\right]$$

This the detail of the algorithm, in R, ~~plug~~ plug back the value of $\sigma^2 = 0.01$, $\lambda = 0.01$

2b. ~~Consider $f(b_1, b_2) = \log(\pi(b_1, b_2 | y, x))$~~

by ignoring the ~~proportionality constant~~ normalizing constant,

let

$$f(b_1, b_2) = \log(\pi(b_1, b_2 | y, x))$$
$$= \frac{b_1}{\sigma^2}\sum_{i=1}^{n} y_i \exp\left\{1 - \exp\left(\frac{-x_i}{b_2}\right)\right\}$$
$$- \frac{b_1^2}{2\sigma^2}\sum_{i=1}^{n} \exp\left\{2\left(1 - \exp\left(\frac{-x_i}{b_2}\right)\right)\right\}$$
$$- \lambda(b_1 + b_2)$$

$$\frac{\partial f}{\partial b_1} = \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i \exp\left\{1 - \exp\left(-\frac{x_i}{b_2}\right)\right\} - \frac{2b_1}{2\sigma^2}\sum_{i=1}^{n} \exp\left\{2 - 2\exp\left(-\frac{x_i}{b_2}\right)\right\} - \lambda$$

$$\frac{\partial f}{\partial b_2} = \frac{b_1}{\sigma^2}\sum_{i=1}^{n} y_i \exp\left\{1 - \exp\left(\frac{-x_i}{b_2}\right)\right\}\cdot\left(-\exp\left(\frac{-x_i}{b_2}\right)\right)\cdot\left(\frac{x_i}{b_2^2}\right)$$
$$- \frac{b_1^2}{2\sigma^2}\sum_{i=1}^{n} \exp\left(2 - 2\exp\left(-\frac{x_i}{b_2}\right)\right)\cdot\left(-2\exp\left(-\frac{x_i}{b_2}\right)\right)\left(\frac{x_i}{b_2^2}\right)$$
$$- \lambda$$

Using Steepest Ascent Algorithm:

Step 1:
starting from an initial guess $\underline{x}^0$, and get the gradient vector $\nabla f(\underline{x}^0)$
where $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial b_1} \\ \frac{\partial f}{\partial b_2} \end{bmatrix}$

Step 2: Calculate $\underline{x}(\delta) = \underline{x}^0 + \delta \nabla f(\underline{x}^0)$

Step 3: If $f(\underline{x}(\delta)) \geq f(\underline{x}^0) \Rightarrow$ next iterate $\underline{x}' = \underline{x}^0 + \delta \nabla f(\underline{x}^0)$ then replace x^0 by our new $\underline{x}'$
else make smaller δ and back to step 2

Step 4: repeat step 2 & 3 until $|f(\underline{x}^n) - f(\underline{x}^{n-1})| <$ tolerance

2c we wish to find $\pi(b_2 \mid x, y) = \int_0^\infty \pi(b_1, b_2 \mid x, y)\, db_1$

let $g(b_1) = g_{b_2}(b_1) = -\log \pi(b_1, b_2 \mid x, y)$, for fixed b_2

let $\hat{b}_1$ be the value of b_1 that minimize $g_{b_2}(b_1)$

Note that we have, using my notations, $f = \log \pi(b_1, b_2 \mid x, y)$ up to proportionality
so, $f = \log(K\pi(b_1, b_2 \mid x, y)) = \log K - g$, where K is normalizing constant

$$\frac{\partial f}{\partial b_1} = \frac{1}{\sigma^2}\sum y_i \exp\left\{1-\exp\left(\frac{-x_i}{b_2}\right)\right\} - \frac{b_1}{\sigma^2}\sum \exp\left\{2-2\exp\left(\frac{-x_i}{b_2}\right)\right\} - \lambda$$

$$\frac{\partial f}{\partial b_2} = \frac{b_1}{\sigma^2}\sum y_i \exp\left\{1-\exp\left(\frac{-x_i}{b_2}\right)\right\}\left(-\exp\left(\frac{-x_i}{b_2}\right)\right)\left(\frac{x_i}{b_2^2}\right)$$
$$- \frac{b_1^2}{2\sigma^2}\sum \exp\left\{2-2\exp\left(\frac{-x_i}{b_2}\right)\right\}\left(-2\exp\left(\frac{-x_i}{b_2}\right)\right)\left(\frac{x_i}{b_2^2}\right)$$
$$-\lambda$$

~~so,~~ $\cancel{\frac{\partial^2 f}{\partial b_1 \partial b_2} = \frac{-1}{\sigma^2}\sum y_i \exp\left\{1-\exp\left(\frac{-x_i}{b_2}\right)\right\}\left(\exp\left(\frac{-x_i}{b_2}\right)\right)\left(\frac{x_i}{b_2^2}\right)}$
$\cancel{+ \frac{2b_1}{\sigma^2}\sum \exp\left\{2-2\exp\left(\frac{-x_i}{b_2}\right)\right\}\left(\exp\left(\frac{-x_i}{b_2}\right)\right)\left(\frac{x_i}{b_2^2}\right)}$

and since $g = \log K - f \Rightarrow \frac{\partial g}{\partial b_1} = -\frac{\partial f}{\partial b_1}$; $\frac{\partial g}{\partial b_2} = -\frac{\partial f}{\partial b_2}$

$$\frac{\partial^2 g}{\partial b_1^2} = -\frac{\partial^2 f}{\partial b_1^2} = \frac{1}{\sigma^2}\sum_{i=1}^n \exp\left\{2-2\exp\left(\frac{-x_i}{b_2}\right)\right\}$$

Now, by fixing b_2, $\frac{dg(b_1)}{db_1} = -\frac{\partial f}{\partial b_1}$, $\frac{d^2 g(b_1)}{db_1^2} = -\frac{\partial^2 f}{\partial b_1^2}$
and for $\frac{dg(b_1)}{db_1} = 0 \Rightarrow \frac{\partial f}{\partial b_1} = 0$, and solve b_1:

$$\Rightarrow b_1 = \frac{\sum y_i \exp\left\{1-\exp\left(\frac{-x_i}{b_2}\right)\right\} - \lambda\sigma^2}{\sum \exp\left\{2-2\exp\left(\frac{-x_i}{b_2}\right)\right\}}$$

~~let that value be $\tilde{b}_1$, now, consider first derivative test (quite trivial by factorizing it)~~

b_1	$b_1 < \tilde{b}_1$	$b_1 = \tilde{b}_1$	$b_1 > \tilde{b}_1$
$\frac{\partial f}{\partial b_1}$	+	0	−
$\frac{dg}{db_1}$	−	0	+

~~$\Rightarrow$ g is minimized at $\tilde{b}_1$ so, $\hat{b}_1 = \tilde{b}_1$~~

2c Continue :

To test if $\tilde{b}_1$ minimize g, consider

$$\left.\frac{d^2 g(b_1)}{db_1^2}\right|_{b_1=\tilde{b}_1} = \frac{1}{\sigma^2}\sum \exp\left\{2 - 2\exp\left(\frac{-x_i}{b_2}\right)\right\}$$

$$> 0$$

So by 2^{nd} derivative test, we know that $\tilde{b}_1$ minimize g

take $\hat{b}_1 = \tilde{b}_1$

Now, since we have the Hessian of g at $b_1 = \hat{b}_1$, name it H

so, we can apply the formula

$$\pi(b_2 \mid x, y) = \int_0^\infty \exp(-g(b_1))\, db_1$$

$$= e^{-g(\hat{b}_1)} \int_0^\infty \exp\left\{-\tfrac{1}{2}(b_1-\hat{b}_1)^T H (b_1-\hat{b}_1)\right\} db_1$$

$$= \pi(\hat{b}_1, b_2 \mid x, y)\, \frac{\sqrt{2\pi}}{\sqrt{H}} \left[1 - \Phi\left(\frac{0-\hat{b}_1}{1/\sqrt{H}}\right)\right]$$

$$= \pi(\hat{b}_1, b_2 \mid x, y)\, \frac{\sqrt{2\pi}}{\sqrt{H}} \left[\Phi\left(\hat{b}_1 \sqrt{H}\right)\right]$$

(since integral is $N(\hat{b}, \frac{1}{H})$ pdf)

then plug back the value $\sigma^2 = 0.01$, $\lambda = 0.01$

$\Phi(\cdot)$ is the ~~cdf of standard normal~~ cumulative probability of standard normal

$\pi(\hat{b}_1, b_2 \mid x, y)$ - the FULL posterior distribution, where the value can be easily compute in R