Categorification of the Legendre-Fenchel transform

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1 Introduction

Category theory gives us a formal way of understanding mathematical structures. In this report, we show how the Legendre-Fenchel transform and Galois correspondences arise from enriched category theory, through a construction called the nucleus of a profunctor.

The contents of this report follow Simon Willerton's 2015 paper "The Legendre-Fenchel transform from a category theoretic perspective" [Wil15] closely. The very useful blog posts [Wil13], [Wil14a], [Wil14b] and [Wil14c] at the n-Category Café cover a lot of the content in the paper. However, this report has been written with an audience of undergraduate mathematics students in mind, so many of the concepts that are assumed as 'obvious' in Willerton's paper are made explicit here. Indeed, this should hopefully be accessible to a second year student with some experience in abstract algebra and analysis, and no knowledge of category theory.

Our aim is to outline and explain how a Galois correspondence

$$R^* : \mathcal{P}_{\mathrm{cl}}(X) \cong \mathcal{P}_{\mathrm{cl}}(Y)^{\mathrm{op}} : R_*$$

arising from a relation on sets and the Legendre-Fenchel duality

$$\mathbb{L}^* : \mathrm{Cvx}(V, \overline{\mathbb{R}}) \cong \mathrm{Cvx}(V^*, \overline{\mathbb{R}}) : \mathbb{L}_*$$

both arise from the nucleus of a profunctor construction. Apart from showing how these two results from separate parts of mathematics are closely related, viewing the latter from a category theoretic perspective will allow us to say more about the Legendre-Fenchel transform than we would have been able to from a classical perspective.

A useful set of reference points in this report are the following tables, that attempt to summarize and compare the most important ideas.

Table 1: Comparing the Legendre-Fenchel transform to sets and relations.
Table 2: Comparing quantales, semirings and monoidal categories.
Table 3: Truth and $\overline{\mathbb{R}}$ as monoidal categories.
Table 4: Translating from enriched categories to preorders and $\overline{\mathbb{R}}$-spaces.
Table 5: Categorifying linear algebra.

2 The Legendre-Fenchel transform

In this section we introduce the Legendre-Fenchel transfrom and its most salient features. In what follows, $\overline{\mathbb{R}}$ denotes the set of extended real numbers $\mathbb{R} \cup \{-\infty, +\infty\}$ and V is a locally convex real vector space.

2.1 Convex functions

Before talking about the Legendre-Fenchel transform, we must understand what a convex function is; we present this in a similar way to [Bel14].

Definition 2.1. Let V be a real vector space. A subset $W \subset V$ is *convex* if for every $w_1, w_2 \in W$, we have $(1-t)w_1 + tw_2 \in W$ for each $0 \leqslant t \leqslant 1$.

This means that the straight line segment connecting w_1 and w_2 lies in W. We call a set that is not convex a non-convex set.

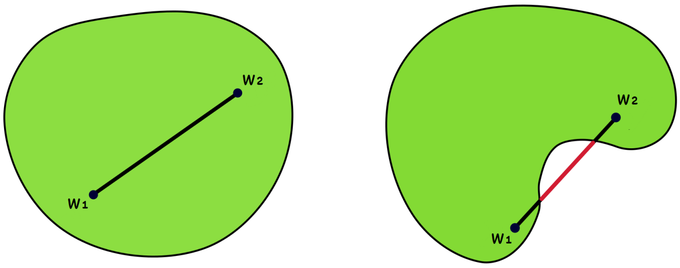

Figure 1: Convex and non-convex sets

Before stating what a convex function is, we first define its epigraph. Intuitively, this is the area 'above' the graph.

Definition 2.2. Let $f : V \to \overline{\mathbb{R}}$ be a function. The *epigraph* of f is the set

$$\operatorname{epi} f = \{(x, y) \in V \times \mathbb{R} : y \geqslant f(x)\}.$$

Definition 2.3. Let $f : V \to \overline{\mathbb{R}}$ be a function. We say that f is *convex* if $\operatorname{epi} f$ is a convex subset of $V \times \mathbb{R}$.

To gain a better understanding of Definitions 2.2 and 2.3, we can consider the case $V = \mathbb{R}$. In Figure 2 we see what it means for a function to be convex; the epigraph, which is the area above the graph, is a convex set. This results in the graph having a U-shape, which is a common feature of convex functions. This feature leads to the following fact about convex functions: if

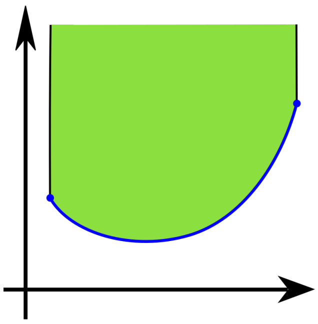

Figure 2: A convex function

we take a convex function and draw a line segment between two points on its graph, the segment will lie above the graph. Often, convex functions are defined in this way.

2.2 Legendre-Fenchel duality

The Legendre-Fenchel transform of a function is often referred to as its convex conjugate. We now define it and explore how it relates to convex functions. Here, V^* refers to the algebraic dual of V, which is the vector space consisting of all the linear functionals on V (linear maps $V \to \mathbb{R}$), together with the vector space structure of pointwise addition and scalar multiplication.

Definition 2.4. Let $f : V \to \overline{\mathbb{R}}$. The *Legendre-Fenchel transform* of f is $f^* : V^* \to \overline{\mathbb{R}}$, defined by

$$f^*(k) = \sup_{x \in V} \{\langle k, x\rangle - f(x)\}. \tag{1}$$

By the angled brackets we mean the natural pairing $\langle -, - \rangle : V^* \times V \to \mathbb{R}$, so $\langle k, x\rangle$ can be thought of as 'k evaluated at x'. Again, we consider the case where $V = \mathbb{R}$, and also where $f : \mathbb{R} \to \overline{\mathbb{R}}$ is continuously differentiable. Then the natural pairing $\langle k, x\rangle$ becomes the product kx. So $f^*(k)$ is the maximum value of $kx - f(x)$ as x varies, which we can find by differentiation. Therefore $f^*(k) = kx(k) - f(x(k))$, where $x(k)$ satisfies $f'(x(k)) = k$. If we also impose the conditions that f' be monotonically increasing and $f'(x) \to -\infty$ as $x \to -\infty$ and $f'(x) \to +\infty$ as $x \to +\infty$, we ensure that f^* is well-defined for all k. This simplified formulation of the Legendre-Fenchel transform is used in physics to switch between Hamiltonian and Lagrangian dynamics [Wil13]:

$$H = \sum_i p_i \dot{q}_i - L.$$

We now state and prove the first theorem that explains why the Legendre-Fenchel transform is often known as the convex conjugate.

Theorem 2.1. *Let $f : V \to \overline{\mathbb{R}}$ be a function. Then $f^* : V^* \to \overline{\mathbb{R}}$ is convex.*

Proof. We want to show that $\operatorname{epi} f^*$ is a convex set, which is the statement $(k_1, y_1), (k_2, y_2) \in \operatorname{epi} f^* \implies (1-t)(k_1, y_1) + t(k_2, y_2) \in \operatorname{epi} f^*$ for each $0 \leqslant t \leqslant 1$. If we notice that $(1-t)(k_1, y_1) + t(k_2, y_2) = ((1-t)k_1 + tk_2, (1-t)y_1 + ty_2)$, we see that we want to show that $y_1 \geqslant f^*(k_1)$ and $y_2 \geqslant f^*(k_2)$ imply $(1-t)y_1 + ty_2 \geqslant f^*((1-t)k_1 + tk_2)$ for every $0 \leqslant t \leqslant 1$.

$$\begin{aligned} f^*((1-t)k_1 + tk_2) &= \sup_{x \in V} \{\langle (1-t)k_1 + tk_2, x\rangle - f(x)\} \\ &= \sup_{x \in V} \{(1-t)\langle k_1, x\rangle + t\langle k_2, x\rangle - f(x)\} \\ &= \sup_{x \in V} \{(1-t)(\langle k_1, x\rangle - f(x)) + t(\langle k_2, x\rangle - f(x))\} \\ &\leqslant \sup_{x \in V} \{(1-t)(\langle k_1, x\rangle - f(x))\} + \sup_{x \in V} \{t(\langle k_2, x\rangle - f(x))\} \\ &= (1-t)\sup_{x \in V} \{\langle k_1, x\rangle - f(x)\} + t\sup_{x \in V} \{\langle k_2, x\rangle - f(x)\} \\ &= (1-t)f^*(k_1) + tf^*(k_2) \\ &\leqslant (1-t)y_1 + ty_2. \end{aligned}$$

So $(1-t)(k_1, y_1) + t(k_2, y_2) \in \operatorname{epi} f^*$ as needed. □

The double dual of V, denoted by V^{**}, is canonically isomorphic to V, so we can write $V = V^{**}$. Therefore, the Legendre-Fenchel transform of $g : V^* \to \overline{\mathbb{R}}$ is a function $V \to \overline{\mathbb{R}}$ defined by

$$g^*(x) = \sup_{k \in V^*} \{\langle k, x\rangle - g(k)\}. \tag{2}$$

Thus we get a pair of transforms between function spaces which we can represent as follows:

$$\mathbb{L}^* : \operatorname{Fun}\left(V, \overline{\mathbb{R}}\right) \rightleftarrows \operatorname{Fun}\left(V^*, \overline{\mathbb{R}}\right) : \mathbb{L}_*. \tag{3}$$

$\operatorname{Fun}\left(V, \overline{\mathbb{R}}\right)$ denotes the space of functions $V \to \overline{\mathbb{R}}$, and similarly for $\operatorname{Fun}\left(V^*, \overline{\mathbb{R}}\right)$. The following proposition shows how f and f^{**} are related.

Proposition 2.1. *Let $f : V \to \overline{\mathbb{R}}$ be a function. Then $f \geqslant f^{**}$, in the sense that $f(x) \geqslant f^{**}(x)$ for each $x \in V$.*

Proof. Let $k \in V^*$. Then

$$\begin{aligned} f^*(k) &= \sup_{x \in V} \{\langle k, x \rangle - f(x)\} \\ \Longrightarrow f^*(k) &\geqslant \langle k, x \rangle - f(x) \quad \forall\, x \in V \\ \Longrightarrow f(x) &\geqslant \langle k, x \rangle - f^*(k) \quad \forall\, x \in V. \end{aligned}$$

This holds for arbitrary $k \in V^*$, so

$$f(x) \geqslant \sup_{k \in V^*} \{\langle k, x \rangle - f^*(k)\} = f^{**}(x) \qquad \forall\, x \in V.$$

Hence $f \geqslant f^{**}$. □

We can actually say more than this. It turns out that f^{**} is the pointwise largest convex function such that $f \geqslant f^{**}$, so it is often referred to as the convex hull of f [Tou05, page 7]. When do we get equality between the two? Before answering this, we introduce two new concepts that take care of the cases where the function in question attains infinite values.

Definition 2.5. Let $f : V \to \overline{\mathbb{R}}$ be a convex function. We say that f is *proper* if $f(x) < +\infty$ for at least one $x \in V$ and f does not take the value $-\infty$.

Definition 2.6. Let $f : V \to \overline{\mathbb{R}}$ be a function. We say that f is *lower semi-continuous* if $\operatorname{epi} f$ is a closed subset of $V \times \mathbb{R}$.

Theorem 2.2 (Fenchel-Moreau)**.** *Let $f : V \to \overline{\mathbb{R}}$ be a function. Then $f^{**} = f \iff f$ is a proper convex and lower semi-continuous function.*

Proof. See [LL88, page 89] and [Bel14, page 3]. □

Another property of the Legendre-Fenchel transform that we outline is an example of what is known as a *Galois connection*, which we will explore further in Section 3.1.

Proposition 2.2. *Let $f : V \to \overline{\mathbb{R}}$ and $g : V^* \to \overline{\mathbb{R}}$ be functions. Then*

$$f \geqslant g^* \iff f^* \leqslant g. \tag{4}$$

Proof.

$$\begin{aligned}
f \geqslant g^* &\iff f(x) \geqslant g^*(x) && \forall\, x \in V \\
&\iff f(x) \geqslant \sup_{k \in V^*} \{\langle k, x\rangle - g(k)\} && \forall\, x \in V \\
&\iff f(x) \geqslant \langle k, x\rangle - g(k) && \forall\, x \in V, k \in V^* \\
&\iff \langle k, x\rangle - f(x) \leqslant g(k) && \forall\, x \in V, k \in V^* \\
&\iff \sup_{x \in V} \{\langle k, x\rangle - f(x)\} \leqslant g(k) && \forall\, k \in V^* \\
&\iff f^*(k) \leqslant g(k) && \forall\, k \in V^* \\
&\iff f^* \leqslant g,
\end{aligned}$$

which is the desired result. □

In (3) we displayed the general structure that arises from the Legendre-Fenchel transform $\mathbb{L}^*$ and its inverse $\mathbb{L}_*$. Now we show that the operations $\mathbb{L}_* \circ \mathbb{L}^* : \text{Fun}\left(V, \overline{\mathbb{R}}\right) \to \text{Fun}\left(V, \overline{\mathbb{R}}\right)$ and $\mathbb{L}^* \circ \mathbb{L}_* : \text{Fun}\left(V^*, \overline{\mathbb{R}}\right) \to \text{Fun}\left(V^*, \overline{\mathbb{R}}\right)$ have some interesting properties.

We use the symbol $\ltimes$ to represent a preorder, which is a reflexive ($a \ltimes a$ for all a) and transitive ($a \ltimes b$ and $b \ltimes c$ imply $a \ltimes c$) binary relation. A preordered set is a set equipped with a preorder; an example that will be relevant later is $\left(\overline{\mathbb{R}}, \geqslant\right)$.

Definition 2.7. Let $(X, \ltimes)$ be a preordered set. A map $C : X \to X$ is a *closure operator* if:

- It is *extensive*: $x \ltimes C(x) \quad \forall\, x \in X$.
- It is *monotone*: $x \ltimes y \implies C(x) \ltimes C(y) \quad \forall\, x, y \in X$.
- It is *idempotent*: $C\left(C(x)\right) = C(x) \quad \forall\, x \in X$.

If we equip $\text{Fun}\left(V, \overline{\mathbb{R}}\right)$ with the preorder $\succcurlyeq$, where $f_1 \succcurlyeq f_2 \iff f_1(x) \geqslant f_2(x) \quad \forall\, x \in V$, then $\mathbb{L}_* \circ \mathbb{L}^*$ is a closure operator. It is extensive as a consequence of Proposition 2.1; it is monotone because f^{**} is the pointwise largest convex function such that $f \succcurlyeq f^{**}$; it is idempotent because f^{**} convex $\implies f^{***} = f^* \implies f^{****} = f^{**}$ by Theorem 2.2. In an analogous manner, $\mathbb{L}^* \circ \mathbb{L}_*$ is also a closure operator when we equip $\text{Fun}\left(V^*, \overline{\mathbb{R}}\right)$ with the preorder $\preccurlyeq$; note how this is opposite to the preorder for $\text{Fun}\left(V, \overline{\mathbb{R}}\right)$.

Once a closure operator is defined, there is a notion of closed elements or closed subsets with respect to the operator.

Definition 2.8. Let $(X, \ltimes)$ be a preordered set and $C : X \to X$ a closure operator. We say that $x \in X$ is a *closed* element of X if it equals its closure: $C(x) = x$.

The closed elements of $\mathbb{L}_* \circ \mathbb{L}^*$ are the functions $f \in \mathrm{Fun}\left(V, \overline{\mathbb{R}}\right)$ that satisfy $\mathbb{L}_* \circ \mathbb{L}^*(f) = f$, or $f^{**} = f$. By the Fenchel-Moreau theorem, these are precisely the proper convex, lower semi-continuous functions $V \to \overline{\mathbb{R}}$. Again, the same holds for $\mathbb{L}^* \circ \mathbb{L}_*$. Therefore, $\mathbb{L}^*$ and $\mathbb{L}_*$ give mutually inverse bijections between the subsets of $\mathrm{Fun}\left(V, \overline{\mathbb{R}}\right)$ and $\mathrm{Fun}\left(V^*, \overline{\mathbb{R}}\right)$ consisting of the proper convex, lower semi-continuous functions.

$$\mathbb{L}^* : \mathrm{Cvx}\left(V, \overline{\mathbb{R}}\right) \cong \mathrm{Cvx}\left(V^*, \overline{\mathbb{R}}\right) : \mathbb{L}_*. \tag{5}$$

This is commonly referred to as Legendre-Fenchel duality, which is an example of a Galois correspondence; we will see an analogous example in Section 3.2.

3 Galois connections and correspondences

We dedicate this short section to showing how the type of structure that arises from the Legendre-Fenchel transform also appears when simply considering a relation between partially ordered sets, also known as *posets*. A partially ordered set $(X, \ltimes)$ is a preordered set as described at the start of Section 2.2, with the added condition that $\ltimes$ is also anti-symmetric ($a \ltimes b$ and $b \ltimes a$ imply $a = b$).

3.1 Galois connections

Definition 3.1. Let X and Y be sets. A *relation* R between X and Y is a subset $R \subseteq X \times Y$. We say

$$(x, y) \in R \iff x\,R\,y \iff x \text{ is related to } y.$$

Suppose we have two sets X and Y and a relation R between them. A classic example is from algebraic geometry, where we have $X = \mathbb{C}^n$ and $Y = \mathbb{C}\,[x_1, \ldots, x_n]$, with $x\,R\,p \iff p(x) = 0$. We denote by $\mathcal{P}(X)$ the power set of X, ordered by inclusion $\subseteq$. Similarly, $\mathcal{P}(Y)^{\mathrm{op}}$ is the power set of Y ordered by containment $\supseteq$. The relation R gives rise to two functions

$$R^* : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y)^{\mathrm{op}} : R_* \tag{6}$$

defined by

$$\begin{aligned} R^*(S) &= \{y \in Y : s\,R\,y \text{ for all } s \in S\}\,, \\ R_*(T) &= \{x \in X : x\,R\,t \text{ for all } t \in T\}\,. \end{aligned}$$

We now show that R^* and R_* form a Galois connection, as in Proposition 2.2. Notice how the proof follows a similar pattern as before.

Proposition 3.1. *Let $S \subseteq X$ and $T \subseteq Y$. Then*

$$S \subseteq R_*(T) \iff R^*(S) \supseteq T. \tag{7}$$

Proof.

$$\begin{aligned} S \subseteq R_*(T) &\iff s \in R_*(T) && \forall\, s \in S \\ &\iff s\,R\,t && \forall\, s \in S, t \in T \\ &\iff t \in R^*(S) && \forall\, t \in T \\ &\iff R^*(S) \supseteq T. \end{aligned}$$

∎

3.2 Galois correspondences

In a similar way to before, we may consider the compositions $R_* \circ R^*$ and $R^* \circ R_*$, which turn out to be closure operators just like $\mathbb{L}_* \circ \mathbb{L}^*$ and $\mathbb{L}^* \circ \mathbb{L}_*$.

Proposition 3.2. *The functions $R_* \circ R^* : \mathcal{P}(X) \to \mathcal{P}(X)$ and $R^* \circ R_* : \mathcal{P}(Y)^{\mathrm{op}} \to \mathcal{P}(Y)^{\mathrm{op}}$ are closure operators.*

Proof. We show it for $R_* \circ R^*$.

- Let $S \subseteq X$, then

$$\begin{aligned} R_* \circ R^*(S) &= \{x \in X : x\,R\,t \text{ for all } t \in R^*(S)\} \\ &= \{x \in X : x\,R\,t \text{ for all } t \text{ such that } s\,R\,t \ \ \forall\, s \in S\} \\ &= S \cup \{x \in X \backslash S : x\,R\,t \text{ for all } t \text{ such that } s\,R\,t \ \ \forall\, s \in S\}\,. \end{aligned}$$

  So $S \subseteq R_* \circ R^*(S)$. Hence $R_* \circ R^*$ is extensive.

- Let $S_1 \subseteq S_2$ be subsets of X. If the elements of a subset of Y are related to all elements of S_2, then they are certainly related to all elements of S_1, so $R^*(S_1) \supseteq R^*(S_2)$. Using the same logic again, we get $R_* \circ R^*(S_1) \subseteq R_* \circ R^*(S_2)$. Hence $R_* \circ R^*$ is monotone.

- For each $S \subseteq X$, the set $R_* \circ R^*(S)$ is the set S plus all the other elements of X that are related to all the elements of Y to which the elements of S are related to (this is a wordy way of describing the last equality in the first bullet point). So if we apply $R_* \circ R^*$ again, we don't gain any new elements as all the ones outside of S have already been accounted for. Hence $(R_* \circ R^*)^2 = R_* \circ R^*$ is idempotent.

The same will hold in a similar way for $R^* \circ R_*$ but with the opposite order, as $\mathcal{P}(Y)^{\mathrm{op}}$ is ordered by containment $\supseteq$ instead of inclusion $\subseteq$. $\square$

Just as in the previous section, if we restrict $\mathcal{P}(X)$ and $\mathcal{P}(Y)^{\mathrm{op}}$ to the subsets that are fixed by $R_* \circ R^*$ and $R^* \circ R_*$ (denoted by $\mathcal{P}_{\mathrm{cl}}(X)$ and $\mathcal{P}_{\mathrm{cl}}(Y)^{\mathrm{op}}$), we get a bijection

$$R^* : \mathcal{P}_{\mathrm{cl}}(X) \cong \mathcal{P}_{\mathrm{cl}}(Y)^{\mathrm{op}} : R_* \tag{8}$$

known as a *Galois correspondence*.

Returning to the example $X = \mathbb{C}^n$ and $Y = \mathbb{C}\,[x_1, \ldots, x_n]$, the closure of $S \subseteq X$ is its closure in the Zariski topology and the closure of $T \subseteq Y$ is the radical of the ideal generated by T.

Clearly, the constructions in Section 2 and Section 3 have a very similar structure. Later, we will see that these arise from enriching categories over $\overline{\mathbb{R}}$ and over Truth, which can themselves be viewed as special types of categories. For now, we compare the similarities between what we have just seen below.

Legendre-Fenchel transform	**Sets and relations**
$\mathrm{Fun}\left(V, \overline{\mathbb{R}}\right)$ ordered by $\geqslant$	$\mathcal{P}(X)$ ordered by $\subseteq$
$\mathrm{Fun}\left(V^*, \overline{\mathbb{R}}\right)$ ordered by $\leqslant$	$\mathcal{P}(Y)^{\mathrm{op}}$ ordered by $\supseteq$
$\mathbb{L}^* : \mathrm{Fun}\left(V, \overline{\mathbb{R}}\right) \to \mathrm{Fun}\left(V^*, \overline{\mathbb{R}}\right)$ $\mathbb{L}^*(f)(k) = \sup_{x \in V} \{\langle k, x\rangle - f(x)\}$	$R^* : \mathcal{P}(X) \to \mathcal{P}(Y)^{\mathrm{op}}$ $R^*(S) = \{y \in Y : s\, R\, y \text{ for all } s \in S\}$
$\mathbb{L}_* : \mathrm{Fun}\left(V^*, \overline{\mathbb{R}}\right) \to \mathrm{Fun}\left(V, \overline{\mathbb{R}}\right)$ $\mathbb{L}_*(g)(x) = \sup_{k \in V^*} \{\langle k, x\rangle - g(k)\}$	$R_* : \mathcal{P}(Y)^{\mathrm{op}} \to \mathcal{P}(X)$ $R_*(T) = \{x \in X : x\, R\, t \text{ for all } t \in T\}$
Galois connection $f \geqslant \mathbb{L}_*(g) \iff \mathbb{L}^*(f) \leqslant g$	Galois connection $S \subseteq R_*(T) \iff R^*(S) \supseteq T$
Closure operators $\mathbb{L}_* \circ \mathbb{L}^* : \mathrm{Fun}\left(V, \overline{\mathbb{R}}\right) \to \mathrm{Fun}\left(V, \overline{\mathbb{R}}\right)$ $\mathbb{L}^* \circ \mathbb{L}_* : \mathrm{Fun}\left(V^*, \overline{\mathbb{R}}\right) \to \mathrm{Fun}\left(V^*, \overline{\mathbb{R}}\right)$	Closure operators $R_* \circ R^* : \mathcal{P}(X) \to \mathcal{P}(X)$ $R^* \circ R_* : \mathcal{P}(Y)^{\mathrm{op}} \to \mathcal{P}(Y)^{\mathrm{op}}$
Legendre-Fenchel duality $\mathbb{L}^* : \mathrm{Cvx}\left(V, \overline{\mathbb{R}}\right) \cong \mathrm{Cvx}\left(V^*, \overline{\mathbb{R}}\right) : \mathbb{L}_*$	Galois correspondence $R^* : \mathcal{P}_{\mathrm{cl}}(X) \cong \mathcal{P}_{\mathrm{cl}}(Y)^{\mathrm{op}} : R_*$

Table 1: Comparing the Legendre-Fenchel transform to sets and relations.

4 Category theory

In this section we aim to give a brief overview of the concepts and tools in category theory that will be useful in understanding Galois correspondences and Legendre-Fenchel transforms. At the start, we will assume that the reader has no knowledge of category theory and will follow roughly the same structure as [Lei14], summarizing important definitions and results. We suggest that inexperienced readers refer to this source for a more thorough explanation of each concept, if needed. A good alternative for a very short introduction to category theory that goes into more detail than we do here is [Che19].

4.1 Categories

We start by defining what a category is.

Definition 4.1. A *category* $\mathscr{A}$ consists of:

- A collection of objects $\mathrm{ob}(\mathscr{A})$.
- For each $A_1, A_2 \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A_1, A_2)$ of maps/arrows/morphisms from A_1 to A_2.
- For each $A_1, A_2, A_3 \in \mathrm{ob}(\mathscr{A})$, a function called composition:

$$\begin{array}{ccc} \mathscr{A}(A_2, A_3) \times \mathscr{A}(A_1, A_2) & \to & \mathscr{A}(A_1, A_3) \\ (g, f) & \mapsto & g \circ f. \end{array}$$

- For each $A \in \mathrm{ob}(\mathscr{A})$, a map $1_A \in \mathscr{A}(A, A)$ called the identity on A.

We will write $A \in \mathscr{A}$ to mean $A \in \mathrm{ob}(\mathscr{A})$ and gf to mean $g \circ f$. Categories vary considerably between each other and can represent many different types of mathematical structures. Below we give some examples.

Example 4.1. **Set** is the category whose objects are sets and whose maps are functions. Similarly, **Grp** is the category whose objects are groups and whose maps are group homomorphisms.

Example 4.2. We can view a preordered set as a category $\mathscr{A}$ where there is at most one map between each pair of objects. This is known as a *thin* category. That is, if we view the unique map $A_1 \to A_2$ as the statement $A_1 \ltimes A_2$, then the reflexivity and transitivity of $\ltimes$ follow from the identity and composition conditions in Definition 4.1.

Definition 4.2. The *dual* of a category $\mathscr{A}$, written as $\mathscr{A}^{\mathrm{op}}$, is defined by reversing all the arrows: $\mathrm{ob}\left(\mathscr{A}^{\mathrm{op}}\right) = \mathrm{ob}\left(\mathscr{A}\right)$ and $\mathscr{A}^{\mathrm{op}}\left(A_1, A_2\right) = \mathscr{A}\left(A_2, A_1\right)$ for each $A_1, A_2 \in \mathscr{A}$.

We can think of the product of two categories in the following way.

Definition 4.3. Given two categories $\mathscr{A}$ and $\mathscr{B}$, the *product category* $\mathscr{A} \times \mathscr{B}$ is defined as:

$$\mathrm{ob}\left(\mathscr{A} \times \mathscr{B}\right) = \mathrm{ob}\left(\mathscr{A}\right) \times \mathrm{ob}\left(\mathscr{B}\right),$$
$$\left(\mathscr{A} \times \mathscr{B}\right)\left(\left(A_1, B_1\right), \left(A_2, B_2\right)\right) = \mathscr{A}\left(A_1, A_2\right) \times \mathscr{B}\left(B_1, B_2\right)$$

for each $A_1, A_2 \in \mathscr{A}$ and $B_1, B_2 \in \mathscr{B}$. The composition and identity of $\mathscr{A} \times \mathscr{B}$ is defined in the same way as Definition 4.1.

Just like sets have subsets and groups have subgroups, categories have subcategories.

Definition 4.4. Let $\mathscr{A}$ be a category. A *subcategory* $\mathscr{S}$ of $\mathscr{A}$ consists of a subset of objects $\mathrm{ob}\left(\mathscr{S}\right) \subseteq \mathrm{ob}\left(\mathscr{A}\right)$, and for each $A_1, A_2 \in \mathscr{S}$ a subset $\mathscr{S}\left(A_1, A_2\right) \subseteq \mathscr{A}\left(A_1, A_2\right)$. We say that $\mathscr{S}$ is a *full subcategory* if $\mathscr{S}\left(A_1, A_2\right) = \mathscr{A}\left(A_1, A_2\right)$ for each $A_1, A_2 \in \mathscr{S}$.

In many categories, some objects are essentially the same. For example, we know what it means for two objects $G_1, G_2 \in \mathbf{Grp}$ to be isomorphic outside the context of category theory. We can generalize this notion to arbitrary categories as follows.

Definition 4.5. A map $f : A_1 \to A_2$ in a category $\mathscr{A}$ is an *isomorphism* if there exists a map $g : A_2 \to A_1$ such that $gf = 1_{A_1}$ and $fg = 1_{A_2}$. We say that A_1 and A_2 are *isomorphic* and write $A_1 \cong A_2$.

4.2 Functors

Whenever we encounter a new type of mathematical object, it is sensible to consider whether there is an intuitive notion of a map between two of these objects. For categories, this map is called a functor.

Definition 4.6. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A *functor* $F : \mathscr{A} \to \mathscr{B}$ consists of:

- A function
$$\begin{array}{ccc} \mathrm{ob}\left(\mathscr{A}\right) & \to & \mathrm{ob}\left(\mathscr{B}\right) \\ A & \mapsto & F(A). \end{array}$$

- For each $A_1, A_2 \in \mathscr{A}$, a function

$$\begin{array}{ccc} \mathscr{A}(A_1, A_2) & \to & \mathscr{B}(F(A_1), F(A_2)) \\ f & \mapsto & F(f). \end{array}$$

It satisfies the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ for each $A_1 \xrightarrow{f} A_2 \xrightarrow{f'} A_3$. (9)
- $F(1_A) = 1_{F(A)}$ for each $A \in \mathscr{A}$. (10)

Example 4.3. We denote by $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ the identity functor that sends $A \mapsto A$ for each $A \in \mathscr{A}$ and $f \mapsto f$ for each $f \in \mathscr{A}(A_1, A_2)$.

Example 4.4. If we regard preordered sets $(S, \ltimes_S)$ and $(T, \ltimes_T)$ as categories as in Example 4.2, a functor $F : S \to T$ is an order preserving map. To see this, notice that if there is a map $f : s_1 \to s_2$ in S, then there is a map $F(f) : F(s_1) \to F(s_2)$ in T. This is the same as the statement $s_1 \ltimes_S s_2 \implies F(s_1) \ltimes_T F(s_2)$.

Definition 4.7. Let $\mathscr{A}$ be a category. A *presheaf* on $\mathscr{A}$ is a functor $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$.

4.3 Natural transformations

Now that we have defined functors, we may once again consider whether there is a suitable map between them. These maps are called natural transformations.

Definition 4.8. Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $\mathscr{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathscr{B}$ be functors. A *natural transformation* $\alpha : F \to G$ consists of a family $(F(A) \to G(A))_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that, for each $A_1, A_2 \in \mathscr{A}$ and each $A_1 \xrightarrow{f} A_2$, the square

$$\begin{tikzcd} F(A_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{A_1}"'] & F(A_2) \arrow[d, "\alpha_{A_2}"] \\ G(A_1) \arrow[r, "G(f)"'] & G(A_2) \end{tikzcd} \tag{11}$$

commutes; in other words $G(f) \circ \alpha_{A_1} = \alpha_{A_2} \circ F(f)$. This last condition is called a *naturality axiom*. We may draw the following diagram to represent a natural transformation:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$$

The definition of a natural transformation might seem rather confusing and arbitrary to some, and it is not clear why this is the right notion of a map between functors. Therefore we give an example that attempts to highlight how one can understand a mathematical construction with a natural transformation.

Example 4.5. [Lei14, page 29] Let **CRing** be the category whose objects are commutative rings and whose maps are ring homomorphisms, and let **Mon** be the category whose objects are monoids and whose maps are monoid homomorphisms.

Fix a natural number n. For any $R \in \mathbf{CRing}$, the set $M_n(R)$ consisting of $n \times n$ matrices with entries in R forms a monoid under multiplication. Also, any ring homomorphism $R \to S$ induces a monoid homomorphism $M_n(R) \to M_n(S)$. Therefore we have a functor $M_n : \mathbf{CRing} \to \mathbf{Mon}$.

We have another functor $U : \mathbf{CRing} \to \mathbf{Mon}$, where $U(R)$ is simply the underlying monoid R and $U(f)$ is the underlying monoid homomorphism f. This is commonly known as a 'forgetful' functor.

We can understand the 'determinant of an $n \times n$ matrix' as a natural transformation

$$\begin{tikzcd} \mathbf{CRing} \arrow[r, bend left, "M_n"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow \mathrm{det}"] & \mathbf{Mon} \end{tikzcd}$$

One can use the familiar properties of determinants to check that det is a well-defined natural transformation, i.e. it satisfies the naturality axiom (11).

Given categories $\mathscr{A}$ and $\mathscr{B}$, we can now construct the category of functors from $\mathscr{A}$ to $\mathscr{B}$: $[\mathscr{A}, \mathscr{B}]$.

$$\mathrm{ob}\left([\mathscr{A}, \mathscr{B}]\right) = \{\text{functors } \mathscr{A} \to \mathscr{B}\},$$

$$[\mathscr{A}, \mathscr{B}]\,(F, G) = \{\text{natural transformations } F \to G\}.$$

Note that this means that we can compose natural transformations α and β by defining $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ [Lei14, page 30]. By referring to $[\mathscr{A}, \mathscr{B}]$, we can say when a natural transformation is a natural isomorphism.

Definition 4.9. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A *natural isomorphism* between functors $\mathscr{A} \to \mathscr{B}$ is an isomorphism in the category of functors $[\mathscr{A}, \mathscr{B}]$.

A useful equivalent version of this definition is given by the following lemma which is stated in [Lei14]; we show the proof for it. It states that one can check the individual components of a natural transformation to determine whether it is a natural isomorphism.

Lemma 4.1. *The natural transformation* $\mathscr{A} \underset{G}{\overset{F}{\Downarrow\alpha}} \mathscr{B}$ *is a natural isomorphism* $\iff F(A) \xrightarrow{\alpha_A} G(A)$ *is an isomorphism in* $\mathscr{B}$ *for each* $A \in \mathscr{A}$.

Proof. The object $\alpha \in [\mathscr{A}, \mathscr{B}](F, G)$ is a natural isomorphism
$\iff$ there exists an $\alpha' \in [\mathscr{A}, \mathscr{B}](G, F)$ such that $\alpha' \circ \alpha = 1_F$ and $\alpha \circ \alpha' = 1_G$
$\iff$ $(\alpha' \circ \alpha)_A = 1_{F(A)}$ and $(\alpha \circ \alpha')_A = 1_{G(A)}$ for all $A \in \mathscr{A}$
$\iff$ $\alpha'_A \circ \alpha_A = 1_{F(A)}$ and $\alpha_A \circ \alpha'_A = 1_{G(A)}$ for all $A \in \mathscr{A}$
$\iff$ α_A is an isomorphism in $\mathscr{B}$ for all $A \in \mathscr{A}$. □

We motivated the definition of an isomorphism in a category by the idea that two objects that are isomorphic are essentially the same. Therefore, if two functors in $[\mathscr{A}, \mathscr{B}]$ are isomorphic, we can regard them as essentially the same object. With this in mind, we can talk about the correct notion of 'sameness' of two categories, which is looser than isomorphism; we write a slightly modified version of Definition 4.5.

Definition 4.10. An *equivalence* between two categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ such that $GF \cong 1_{\mathscr{A}}$ and $FG \cong 1_{\mathscr{B}}$, or equivalently, there exists a pair of natural isomorphisms

$$\eta : 1_{\mathscr{A}} \to GF, \quad \varepsilon : FG \to 1_{\mathscr{B}}.$$

We say that $\mathscr{A}$ and $\mathscr{B}$ are *equivalent* and write $\mathscr{A} \simeq \mathscr{B}$, and that F and G are *equivalences*.

4.4 Adjunctions

Adjunctions are arguably the most interesting and important aspects of category theory. Saunders Mac Lane, one of the founders of category theory, wrote "adjunctions arise everywhere" in his textbook *Categories for the Working Mathematician*. We present their definition and an example that will be relevant in the next sections.

Definition 4.11. Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be categories and functors. We say that F is *left adjoint* to G, and G is *right adjoint* to F if

$$\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B)) \tag{12}$$

for each $A \in \mathscr{A}, B \in \mathscr{B}$. We call a choice of isomorphism (12) an *adjunction* between F and G and write $F \dashv G$. We label the adjunction by a horizontal bar:

$$\begin{array}{ccc} \big(F(A) \xrightarrow{g} B\big) & \mapsto & \big(A \xrightarrow{\overline{g}} G(B)\big), \\ \big(F(A) \xrightarrow{\overline{f}} B\big) & \leftarrow\!\shortmid & \big(A \xrightarrow{f} G(B)\big), \end{array}$$

so that $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$. Furthermore, given maps g, q in $\mathscr{B}$ and maps f, p in $\mathscr{A}$, the adjunction has to satisfy the following naturality axioms:

$$\overline{\big(F(A) \xrightarrow{g} B_1 \xrightarrow{q} B_2\big)} = \big(A \xrightarrow{\overline{g}} G(B_1) \xrightarrow{G(q)} G(B_2)\big), \tag{13}$$

by which we mean $\overline{q \circ g} = G(q) \circ \overline{g}$, or equivalently $q \circ g = \overline{G(q) \circ \overline{g}}$.

$$\overline{\big(A_2 \xrightarrow{p} A_1 \xrightarrow{f} G(B)\big)} = \big(F(A_2) \xrightarrow{F(p)} F(A_1) \xrightarrow{\overline{f}} B\big), \tag{14}$$

by which we mean $\overline{f \circ p} = \overline{f} \circ F(p)$, or equivalently $f \circ p = \overline{\overline{f} \circ F(p)}$.

Example 4.6. Given two sets A and B, we can form two new sets from them: their cartesian product $A \times B$ and the set of functions between them, $\mathrm{Fun}(A, B)$. So if we fix a set B, we can define the functors

$$\begin{array}{rccc} - \times B : & \mathbf{Set} & \to & \mathbf{Set} \\ & A & \mapsto & A \times B \end{array}$$

and

$$\begin{array}{rccc} \mathrm{Fun}(B, -) : & \mathbf{Set} & \to & \mathbf{Set} \\ & C & \mapsto & \mathrm{Fun}(B, C). \end{array}$$

The first one is left adjoint to the latter, so that for every $A, C \in \mathbf{Set}$ we have an isomorphism

$$\mathbf{Set}\,(A \times B, C) \cong \mathbf{Set}\,(A, \mathrm{Fun}(B, C))\,. \tag{15}$$

To see this, given any map $g : A \times B \to C$, define $\overline{g} : A \to \mathrm{Fun}(B, C)$ by $\overline{g}(a)(b) = g(a, b)$. Similarly, given any map $f : A \to \mathrm{Fun}(B, C)$, define $\overline{f} : A \times B \to C$ by $\overline{f}(a, b) = f(a)(b)$. The naturality axioms (13) and (14) also hold.

We can restrict an adjunction to an equivalence between certain subcategories containing the elements that are fixed up to isomorphism by the adjunction. The following theorem is based on Exercise 2.2.11 in [Lei14], whose solution relies on the fact that an adjunction is equivalent to a pair $(1_{\mathscr{A}} \xrightarrow{\eta} GF, FG \xrightarrow{\varepsilon} 1_{\mathscr{B}})$ of natural transformations that satisfy certain 'triangle identities'. We instead give a proof of it based only on the category theoretic ideas that we have outlined so far.

Theorem 4.1. *Let* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ *be categories and functors and suppose* $F \dashv G$. *Define* $\mathrm{Fix}(GF)$ *to be the full subcategory of* $\mathscr{A}$ *such that* $\mathrm{ob}\,(\mathrm{Fix}(GF)) = \{A \in \mathscr{A} : GF(A) \cong A\}$. *Define* $\mathrm{Fix}(FG)$ *similarly. Then* F *and* G *restrict to equivalences* F' *and* G' *giving*

$$\mathrm{Fix}(GF) \simeq \mathrm{Fix}(FG).$$

Proof. Firstly, note that the restrictions of F and G, given by

$$\begin{array}{rccc} F': & \mathrm{Fix}(GF) & \to & \mathrm{Fix}(FG) \\ & A & \mapsto & F(A) \\ & f & \mapsto & F(f), \\ G': & \mathrm{Fix}(FG) & \to & \mathrm{Fix}(GF) \\ & B & \mapsto & G(B) \\ & g & \mapsto & G(g) \end{array}$$

are well-defined on both objects and maps. For objects, this is because $A \in \mathrm{Fix}(GF) \implies A \cong GF(A) \implies F(A) \cong F\,(GF(A)) = FG\,(F(A)) \implies F(A) \in \mathrm{Fix}(FG)$ and similarly for G'. For maps, this is because $\mathrm{Fix}(GF)$ and $\mathrm{Fix}(FG)$ are full subcategories of $\mathscr{A}$ and $\mathscr{B}$.
Following Definition 4.10, we wish to find natural isomorphisms $\eta : 1_{\mathscr{A}} \to G'F'$ and $\varepsilon : F'G' \to 1_{\mathscr{B}}$, from which the equivalence will follow. Consider $\eta : 1_{\mathscr{A}} \to G'F'$ defined by

$$A \xrightarrow{\eta_A} GF(A) = \overline{F(A) \xrightarrow{1_{F(A)}} F(A)}$$

for each $A \in \mathrm{Fix}(GF)$. To show that this is indeed a natural transformation, we use the naturality axioms of the adjunction, given by (13) and (14), to show that the naturality axiom of a natural transformation, given by (11), holds. In other words we want to show that this square commutes for each $A_1, A_2 \in \mathscr{A}$ and $f \in \mathscr{A}\,(A_1, A_2)$:

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "\eta_{A_1}"'] & A_2 \arrow[d, "\eta_{A_2}"] \\ GF(A_1) \arrow[r, "GF(f)"'] & GF(A_2) \end{tikzcd}$$

We evaluate $\eta_{A_2} \circ f$:

$$
\begin{aligned}
& \big(A_1 \xrightarrow{f} A_2 \xrightarrow{\eta_{A_2}} GF(A_2)\big) & \\
= \; & \big(A_1 \xrightarrow{f} A_2 \xrightarrow{\overline{1_{F(A_2)}}} GF(A_2)\big) & \text{(definition of } \eta\text{)} \\
= \; & \overline{\big(F(A_1) \xrightarrow{F(f)} F(A_2) \xrightarrow{1_{F(A_2)}} F(A_2)\big)} & \text{(from (14))} \\
= \; & \overline{\big(F(A_1) \xrightarrow{F(f)} F(A_2)\big)} & \\
= \; & \big(A_1 \xrightarrow{\overline{F(f)}} GF(A_2)\big). & \text{(adjunction (12))}
\end{aligned}
$$

Similarly for $GF(f) \circ \eta_{A_1}$:

$$
\begin{aligned}
& \big(A_1 \xrightarrow{\eta_{A_1}} GF(A_1) \xrightarrow{GF(f)} GF(A_2)\big) & \\
= \; & \big(A_1 \xrightarrow{\overline{1_{F(A_1)}}} GF(A_1) \xrightarrow{GF(f)} GF(A_2)\big) & \text{(definition of } \eta\text{)} \\
= \; & \overline{\big(F(A_1) \xrightarrow{1_{F(A_1)}} F(A_1) \xrightarrow{F(f)} F(A_2)\big)} & \text{(from (13))} \\
= \; & \overline{\big(F(A_1) \xrightarrow{F(f)} F(A_2)\big)} & \\
= \; & \big(A_1 \xrightarrow{\overline{F(f)}} GF(A_2)\big). & \text{(adjunction (12))}
\end{aligned}
$$

So $\eta_{A_2} \circ f = \overline{F(f)} = GF(f) \circ \eta_{A_1}$, hence η is a natural transformation. Furthermore, note that $A \xrightarrow{\eta_A} GF(A)$ is an isomorphism in $\mathscr{A}$ for each $A \in \mathrm{Fix}(GF)$, so η is a natural isomorphism by Lemma 4.1. We may define $\varepsilon : F'G' \to 1_{\mathscr{B}}$ in an analogous manner:

$$\big(FG(B) \xrightarrow{\varepsilon_B} B\big) = \overline{\big(G(B) \xrightarrow{1_{G(B)}} G(B)\big)},$$

and do the same steps as before to show that it is a natural isomorphism. $\square$

4.5 Products and coproducts

Denote a set containing a single element by 1. We may consider an element x of a set X as a function $1 \to X$ which takes in the single element of 1 and outputs x. We can generalize this idea to an arbitrary set A and consider the element $x \in X$ as a function $A \to X$.

Now let Y be another set, and consider the cartesian product $X \times Y$. Every element of this set consists of an element of X together with an element of Y. Using the above idea, we can say that every map $A \to X \times Y$ corresponds to a pair of maps $(A \to X, A \to Y)$, giving us a bijection

$$\mathrm{Fun}(A, X \times Y) \cong \mathrm{Fun}(A, X) \times \mathrm{Fun}(A, Y).$$

The bijection is defined by composing the map $f : A \to X \times Y$ with the projection maps

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times Y & \xrightarrow{p_2} & Y \\ x & \longmapsfrom & (x, y) & \longmapsto & y. \end{array}$$

This way of thinking about products leads us to the following definition for products of objects in a category.

Definition 4.12. Let $\mathscr{A}$ be a category and $X, Y \in \mathscr{A}$. A *product* of X and Y consists of an object $X \times Y$ and maps $p_1 : X \times Y \to X, p_2 : X \times Y \to Y$, such that for all objects $A \in \mathscr{A}$ and maps $f_1 : A \to X, f_2 : A \to Y$, there exists a unique $\tilde{f} : A \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, dashed, "\tilde{f}"] \arrow[dr, "f_2"] & \\ X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

Example 4.7. Consider the preordered set $\left(\overline{\mathbb{R}}, \geqslant\right)$ viewed as thin category, as in Example 4.2. The product of $x, y \in \overline{\mathbb{R}}$ satisfies $x \times y \geqslant x$ and $x \times y \geqslant y$, and also $a \geqslant x, a \geqslant y \implies a \geqslant x \times y$. Therefore $x \times y = \max\{x, y\}$.

The coproduct in a category is defined dually to the product; all we do is we switch the direction of the arrows.

Definition 4.13. Let $\mathscr{A}$ be a category and $X, Y \in \mathscr{A}$. A *coproduct* of X and Y consists of an object $X + Y$ and maps $p_1 : X \to X + Y, p_2 : Y \to X + Y$, such that for all objects $A \in \mathscr{A}$ and maps $f_1 : X \to A, f_2 : Y \to A$, there exists a unique $\tilde{f} : X + Y \to A$ such that the following diagram commutes:

$$\begin{tikzcd} & A & \\ X \arrow[ur, "f_1"] \arrow[r, "p_1"'] & X + Y \arrow[u, dashed, "\tilde{f}"] & Y \arrow[ul, "f_2"'] \arrow[l, "p_2"] \end{tikzcd}$$

Example 4.8. Dually to Example 4.7, the coproduct of x and y in $\left(\overline{\mathbb{R}}, \geqslant\right)$ is $\min\{x, y\}$.

We should not limit ourselves to considering products and coproducts of only two objects. The above definitions can easily be generalized to describe the products and coproducts of an arbitrary number of objects.

Definition 4.14. Let $\mathscr{A}$ be a category and $(X_i)_{i\in I}$ be a family of objects in $\mathscr{A}$.
A product of $(X_i)_{i\in I}$ consists of an object $\prod_{i\in I} X_i$ and a family of maps

$$\left(p_i : \prod_{i\in I} X_i \to X_i\right)_{i\in I}$$

such that for every $A \in \mathscr{A}$ and every family of maps

$$(f_i : A \to X_i)_{i\in I},$$

there exists a unique map $\tilde{f} : A \to \prod_{i\in I} X_i$ such that $p_i \circ \tilde{f} = f_i$ for all $i \in I$.
A coproduct of $(X_i)_{i\in I}$ consists of an object $\coprod_{i\in I} X_i$ and a family of maps

$$\left(p_i : X_i \to: \coprod_{i\in I} X_i\right)_{i\in I}$$

such that for every $A \in \mathscr{A}$ and every family of maps

$$(f_i : X_i \to A)_{i\in I},$$

there exists a unique map $\tilde{f} : \coprod_{i\in I} X_i \to A$ such that $\tilde{f} \circ p_i = f_i$ for all $i \in I$.

Example 4.9. The product and coproduct of $(x_i)_{i\in I}$ in $\left(\overline{\mathbb{R}}, \geqslant\right)$ are $\sup_{i\in I}\{x_i\}$ and $\inf_{i\in I}\{x_i\}$ respectively.

5 Quantales, semirings and monoidal categories

In this section we outline how special types of categories can be be viewed as more familiar structures. A one-sentence summary of what follows is: *Commutative quantales* are equivalent to *complete, commutative, idempotent semirings*, which are equivalent to *complete, cocomplete, thin, skeletal, closed, symmetric, monoidal categories.* This will be of interest to us later on, as we will be enriching over these categories and it is often helpful to view them from a different perspective.

5.1 Quantales

A commutative quantale is a poset $(Q, \ltimes)$ such that every subset $W \subseteq Q$ has the following two objects associated to it: a *meet*, denoted by $\bigwedge_{x\in W} x$, and a *join*, denoted by $\bigvee_{x\in W} x$. These satisfy

$$\bigwedge_{x\in W} x \ltimes w \quad \text{and} \quad w \ltimes \bigvee_{x\in W} x \tag{16}$$

for all $w \in W$. Moreover, Q is equipped with an associative and commutative operation $\otimes : Q \times Q \to Q$ with unit $\mathbb{I} \in Q$, which distributes over arbitrary joins:

$$q \otimes \Big(\bigvee_{x\in W} x\Big) = \bigvee_{x\in W} (q \otimes x)$$

for each $q \in Q$. The *residuation* is defined as $[-,-] : Q \times Q \to Q$ by

$$[b,c] = \bigvee_{a\otimes b \ltimes c} a. \tag{17}$$

Proposition 5.1. *The residuation in a quantale Q satisfies the* adjunction property *for each $a, b, c \in Q$:*

$$a \otimes b \ltimes c \iff a \ltimes [b,c]. \tag{18}$$

Proof. Suppose $a \otimes b \ltimes c$. Then by definition of joins we have $a \ltimes \bigvee_{d\otimes b\ltimes c} d$. But $\bigvee_{d\otimes b\ltimes c} d = [b,c]$, so the $\implies$ implication follows.
Conversely, suppose $a \ltimes [b,c]$. Taking the product with b on both sides gives $a \otimes b \ltimes \big(\bigvee_{d\otimes b\ltimes c} d\big) \otimes b = \bigvee_{d\otimes b\ltimes c} (d \otimes b)$. But note that $\bigvee_{d\otimes b\ltimes c} (d \otimes b) \ltimes \bigvee_{d\otimes b\ltimes c} c = c$, so the $\impliedby$ implication follows from transitivity of $\ltimes$. $\square$

5.2 Semirings

The usual definition of a ring is a set R equipped with binary operations $\oplus$ and $\otimes$ such that $(R, \oplus)$ is an abelian group and $(R, \otimes)$ is a monoid, with $\otimes$ distributing over $\oplus$. A semiring S is the same as a ring, except that $(S, \oplus)$ is a monoid and not an abelian group. One can think of this as a 'ring without negatives'. If $\oplus$ and $\otimes$ are commutative then S is a *commutative* semiring, and S is *idempotent* if $a \oplus a = a$ for every $a \in S$.

Proposition 5.2. *In a commutative, idempotent semiring S, the relation*

$$a \ltimes b \iff a \oplus b = b$$

is a partial order.

Proof. Consider the following points:

- Idempotency of S is equivalent to $a \ltimes a$ for every $a \in S$.
- Suppose $a \ltimes b$ and $b \ltimes a$. Then $a \oplus b = b$ and $b \oplus a = a$. But $a \oplus b = b \oplus a$ since S is commutative, so $a = b$.
- Suppose $a \ltimes b$ and $b \ltimes c$. Then $a \oplus b = b$ and $b \oplus c = c$. Therefore $a \oplus c = a \oplus (b \oplus c) = (a \oplus b) \oplus c = b \oplus c = c$, where we have used the associativity of $\oplus$. Hence $a \ltimes c$.

We have shown that $\ltimes$ is reflexive, anti-symmetric and transitive, so it is a partial order. $\square$

As a result of this, we can think of $\oplus$ as the join in a commutative, idempotent semiring S, since given any subset $W \subseteq S$ and any $w \in W$, the sum $\bigoplus_{x \in W} x$ satisfies

$$w \oplus \left(\bigoplus_{x \in W} x \right) = (w \oplus w) \oplus \left(\bigoplus_{x \in W'} x \right) = w \oplus \left(\bigoplus_{x \in W'} x \right) = \bigoplus_{x \in W} x,$$

where we have written $W' := W \backslash \{w\}$. This is exactly the same statement as $w \ltimes \bigoplus_{x \in W} x$, which is analogous to the definition of a join in a commutative quantale given by (16). Therefore, just as in (17), we may define the residuation

$$[b, c] = \bigoplus_{a \otimes b \ltimes c} a$$

so that the adjuction property $a \otimes b \ltimes c \iff a \ltimes [b, c]$ holds. Finally, we say that S is *complete* if $\otimes$ distributes over arbitrary joins:

$$a \otimes \left(\bigoplus_{x \in W} x \right) = \bigoplus_{x \in W} (a \otimes x)$$

for each $a \in S$ and each $W \subseteq S$.

5.3 Monoidal categories

In Example 4.2 we saw how we can think of a preordered set as a *thin* category, where the preorder is defined by $a \ltimes b \iff \exists$ a map $a \to b$. If every isomorphism in this category is an equality then $\ltimes$ becomes a partial order, as it is also anti-symmetric. A category in which every isomorphism is an equality is known as a *skeletal* category. Therefore we can think of a poset as a thin, skeletal category $\mathcal{V}$. The products and coproducts in $\mathcal{V}$ as described in Section 4.5 correspond to meets and joins. This is because, given a family of objects $(X_i)_{i \in I}$, these satisfy

$$\exists \text{ a map } \prod_{i \in I} X_i \to X_i \quad \text{and} \quad \exists \text{ a map } X_i \to \coprod_{i \in I} X_i \quad \forall\, i \in I.$$

Compare this to (16). We say that $\mathcal{V}$ is *complete* if arbitrary families of objects have products, and that it is *cocomplete* if arbitrary families of objects have coproducts.

The category $\mathcal{V}$ is *monoidal* if it is equipped with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ that is associative on the set of objects and has a unit object $\mathbb{I} \in \mathcal{V}$. Moreover, $\mathcal{V}$ is *symmetric* if $\otimes$ is commutative. Further, $\mathcal{V}$ is *closed* if there is a functor $[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ with the property that $(- \otimes b)$ is adjoint to $[b, -]$ for fixed $b \in \mathcal{V}$, in a similar way to Example 4.6. Unpacking this statement we get that for each $a, c \in \mathcal{V}$

$$\mathcal{V}(a \otimes b, c) \cong \mathcal{V}(a, [b, c]). \tag{19}$$

When $\mathcal{V}$ is thin, both sides contain at most one map, so this simplifies to

$$\exists \text{ a map } a \otimes b \to c \iff \exists \text{ a map } a \to [b, c]$$

which is exactly the adjunction property (18).

The one-sentence summary given at the start of this section was: *Commutative quantales* are equivalent to *complete, commutative, idempotent semirings*, which are equivalent to *complete, cocomplete, thin, skeletal, closed, symmetric, monoidal categories.* We show this explicitly in the table on the next page.

Commutative quantale Q	**Complete, commutative, idempotent semiring S**	**Complete, cocomplete, thin, skeletal, closed, symmetric, monoidal category $\mathcal{V}$**
Poset $(Q, \ltimes)$.	$a \ltimes b \iff a \oplus b = b$ is a partial order because $(S, \oplus)$ is an idempotent, commutative monoid.	$\mathcal{V}$ is thin and skeletal, so $a \ltimes b \iff \exists$ a map $a \to b$ is a partial order.
Join $\bigvee$ of $W \subseteq Q$ satisfies $w \ltimes \bigvee_{x \in W}$ for each $w \in W$.	Join $\bigoplus$ of $W \subseteq S$ satisfies $w \ltimes \bigoplus_{x \in W}$ for each $w \in W$.	The coproduct of $(X_i)_{i \in I}$ has a map $X_i \to \coprod_{i \in I} X_i$ for each $i \in I$.
Meet $\bigwedge$ of $W \subseteq Q$ satisfies $\bigwedge_{x \in W} \ltimes w$ for each $w \in W$.	Meet is defined in terms of join.	The product of $(X_i)_{i \in I}$ has a map $\prod_{i \in I} X_i \to X_i$ for each $i \in I$.
Join exists for each $W \subseteq Q$.	S is complete.	$\mathcal{V}$ is cocomplete (has coproducts)
Meet exists for each $W \subseteq Q$.	S is complete.	$\mathcal{V}$ is complete (has products).
$\otimes : Q \times Q \to Q$.	$\otimes : S \times S \to S$.	$\mathcal{V}$ is monoidal: it is equipped with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.
$\otimes$ is associative.	$(S, \otimes)$ is a monoid.	$\otimes$ is associative on objects.
$\otimes$ is unital.	$(S, \otimes)$ is a monoid.	$\otimes$ is unital on objects.
$\otimes$ distributes over arbitrary joins.	S is complete.	$\otimes$ distributes over products.
Residuation: $$[b, c] = \bigvee_{a \otimes b \ltimes c} a.$$	Residuation: $$[b, c] = \bigoplus_{a \otimes b \ltimes c} a.$$	$\mathcal{V}$ is closed: there exists a functor $[-, -] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ such that $(- \otimes b)$ is left adjoint to $[b, -]$ for fixed $b \in \mathcal{V}$.
Adjunction property: $a \otimes b \ltimes c \iff a \ltimes [b, c]$.	Adjunction property: $a \otimes b \ltimes c \iff a \ltimes [b, c]$.	$\mathcal{V}$ is closed and thin: the adjuction $(- \otimes b) \dashv [b, -]$ says $a \otimes b \to c \iff a \to [b, c]$.

Table 2: Comparing quantales, semirings and monoidal categories.

5.4 Truth and $\overline{\mathbb{R}}$

We now turn our attention to the two categories we will be enriching over: Truth and $\overline{\mathbb{R}}$.

Truth consists of two elements: false and true, with the only non-identity map being $\mathsf{false} \to \mathsf{true}$; this map defines a partial order, which we denote by $\vdash$. The product (which we think of as the meet) is the logical 'and' operator and the coproduct (which we think of as the join) is the logical 'or' operator. The functor axioms (9) and (10) give us that the monoidal product is exactly the same as the meet, while the residuation (17) is implication.

Logical and $\bigwedge$	**Logical or** $\bigvee$	**Implication** $\Rightarrow$
$\mathsf{true} \bigwedge \mathsf{true} = \mathsf{true}$	$\mathsf{true} \bigvee \mathsf{true} = \mathsf{true}$	$\mathsf{true} \Rightarrow \mathsf{true} = \mathsf{true}$
$\mathsf{true} \bigwedge \mathsf{false} = \mathsf{false}$	$\mathsf{true} \bigvee \mathsf{false} = \mathsf{true}$	$\mathsf{true} \Rightarrow \mathsf{false} = \mathsf{false}$
$\mathsf{false} \bigwedge \mathsf{true} = \mathsf{false}$	$\mathsf{false} \bigvee \mathsf{true} = \mathsf{true}$	$\mathsf{false} \Rightarrow \mathsf{true} = \mathsf{true}$
$\mathsf{false} \bigwedge \mathsf{false} = \mathsf{false}$	$\mathsf{false} \bigvee \mathsf{false} = \mathsf{false}$	$\mathsf{false} \Rightarrow \mathsf{false} = \mathsf{true}$

Truth tables for $\bigwedge$, $\bigvee$ and $\Rightarrow$.

The extended real numbers are the set $\overline{\mathbb{R}}$ equipped with the partial order $\geqslant$. The meet and join are the supremum and infimum respectively, from Example 4.9. The monoidal product is addition, so the residuation (17) is subtraction because

$$[b, c] = \bigvee_{a \otimes b \ltimes c} a = \inf_{a+b \geqslant c} \{a\} = c - b.$$

We summarize everything below.

	Truth	$\overline{\mathbb{R}}$
Partial order $\ltimes$	$\vdash$	$\geqslant$
Meet $\bigwedge$	Logical and	sup
Join $\bigvee$	Logical or	inf
Monoidal product $\otimes$	Logical and	Addition $+$
Monoidal unit $\mathbb{I}$	true	0
Residuation $[b, c]$	Implication $b \Rightarrow c$	Subtraction $c - b$

Table 3: Truth and $\overline{\mathbb{R}}$ as monoidal categories.

6 Enriched category theory

We now begin translating concepts from ordinary category theory to enriched category theory. In what follows, $\mathcal{V}$ will be assumed to be a complete, cocomplete, thin, skeletal, closed, symmetric monoidal category, as described in the previous section. These restrictions on $\mathcal{V}$ will allow us to ignore the full generality of some of the concepts in this section, so that we may focus only on what will be useful in understanding the Legendre-Fenchel transform. At the end of each subsection, we discuss the relevant concepts in the context of enriching over $\mathcal{V} = \mathsf{Truth}$ and $\mathcal{V} = \overline{\mathbb{R}}$.

6.1 $\mathcal{V}$-categories

Definition 4.1 says that an *ordinary* category $\mathscr{A}$ consists of a set of objects $\mathrm{ob}(\mathscr{A})$ together with:

- for every $A_1, A_2 \in \mathscr{A}$, a set $\mathscr{A}(A_1, A_2) \in \mathrm{ob}(\mathbf{Set})$.
- for every A_1, A_2, A_3, a function $\mathscr{A}(A_2, A_3) \times \mathscr{A}(A_1, A_2) \to \mathscr{A}(A_1, A_3)$ in **Set** called composition.
- for every $A \in \mathscr{A}$, a map $1_A \in \mathscr{A}(A, A)$ in **Set** called the identity, which we may think of as a map $1 \to \mathscr{A}(A, A)$ as in Section 4.5.

We make slight modifications to the three bullet points above to define *enriched* categories.

Definition 6.1. A *category enriched over* $\mathcal{V}$ or a $\mathcal{V}$*-category*, $\mathscr{A}$, consists of a set of objects $\mathrm{ob}(\mathscr{A})$ together with:

- For every $A_1, A_2 \in \mathscr{A}$, a set $\mathscr{A}(A_1, A_2) \in \mathrm{ob}(\mathcal{V})$.
- For every A_1, A_2, A_3, a function $\mathscr{A}(A_2, A_3) \otimes \mathscr{A}(A_1, A_2) \to \mathscr{A}(A_1, A_3)$ in $\mathcal{V}$ called composition.
- For every $A \in \mathscr{A}$, a map $\mathbb{I} \to \mathscr{A}(A, A)$ in $\mathcal{V}$ called the identity.

It is worth highlighting how isomorphisms are understood in enriched categories, as this is slightly different to Definition 4.5.

Definition 6.2. Let $\mathscr{A}$ be a $\mathcal{V}$-category. We say that $A_1, A_2 \in \mathscr{A}$ are isomorphic, and write $A_1 \cong A_2$, if there exist maps $f : \mathbb{I} \to \mathscr{A}(A_1, A_2)$ and $g : \mathbb{I} \to \mathscr{A}(A_2, A_1)$ in $\mathcal{V}$ such that the composition $\mathbb{I} \otimes \mathbb{I} \to \mathscr{A}(A_2, A_1) \otimes \mathscr{A}(A_1, A_2) \to \mathscr{A}(A_1, A_1)$, where the first map is $g \otimes f$ and the second is the composition in $\mathscr{A}$, is the identity (similarly for $f \otimes g$). In our case that $\mathcal{V}$ is thin, the last condition is trivially satisfied.

This gives a hint to the underlying partial order that exists in a $\mathcal{V}$-category $\mathscr{A}$; it is given by

$$A_1 \ltimes_{\mathscr{A}} A_2 \iff \exists \text{ a map } \mathbb{I} \to \mathscr{A}(A_1, A_2) \text{ in } \mathcal{V}. \tag{20}$$

The reflexivity and transitivity of $\ltimes_{\mathscr{A}}$ are given by the identity and composition conditions in Definition 6.1, while its anti-symmetry is given by $\mathcal{V}$ being skeletal. We now consider the cases where one enriches over Truth and $\overline{\mathbb{R}}$.

Truth. A Truth-category R is a set where each pair of elements $r_1, r_2 \in R$ has a truth value $R(r_1, r_2)$ associated to them. In the case where $R(r_1, r_2) = \mathsf{true}$, we write $r_1 \leqslant_R r_2$. As of now, we will write $\lceil a \rceil$ to denote the truth value of the statement a, so $R(r_1, r_2) = \lceil r_1 \leqslant_R r_2 \rceil$. The composition condition in Definition 6.1 corresponds to

$$\lceil r_1 \leqslant_R r_2 \rceil \text{ AND } \lceil r_2 \leqslant_R r_3 \rceil \vdash \lceil r_1 \leqslant_R r_3 \rceil.$$

This means that when $\lceil r_1 \leqslant_R r_2 \rceil = \mathsf{true}$ and $\lceil r_2 \leqslant_R r_2 \rceil = \mathsf{true}$ we have $\lceil r_1 \leqslant_R r_3 \rceil = \mathsf{true}$, since true only maps to true. This is the same as the statement $r_1 \leqslant_R r_2, r_2 \leqslant_R r_3 \implies r_1 \leqslant_R r_3$, so $\leqslant_R$ is transitive. The identity condition, on the other hand, is

$$\mathsf{true} \vdash \lceil r \leqslant_R r \rceil,$$

which means that true maps to $\lceil r \leqslant_R r \rceil$. But again, true only maps to true so this is the same as saying $r \leqslant_R r$, so $\leqslant_R$ is reflexive. Therefore $(R, \leqslant_R)$ is a *preordered set.* If $r_1 \leqslant_R r_2$ and $r_2 \leqslant_R r_1$, then $r_1 \cong r_2$ by Definition 6.2. So if R is a skeletal Truth-category, then $\leqslant_R$ is a partial order. The underlying preorder (20) of R is exactly the same as the preorder just described.

$\overline{\mathbb{R}}$. An $\overline{\mathbb{R}}$-enriched category X is a set where each pair of elements $x_1, x_2 \in X$ has a number $X(x_1, x_2) \in \overline{\mathbb{R}}$ associated to it, which we relabel to $\mathrm{d}(x_1, x_2)$ to emphasize that we think of this as a distance. The composition and identity conditions are

- $\mathrm{d}(x_2, x_3) + \mathrm{d}(x_1, x_2) \geqslant \mathrm{d}(x_1, x_3)$,
- $0 \geqslant \mathrm{d}(x, x)$.

The first one is the triangle identity, and one can prove that the second one simply says that either $\mathrm{d}(x, x) = 0$ or $\mathrm{d}(x, x) = -\infty$. Therefore we may view an $\overline{\mathbb{R}}$-category as a generalized metric space, where the metric d is not necessarily symmetric and negative distances are allowed. We call this an $\overline{\mathbb{R}}$-*space.* An example of an $\overline{\mathbb{R}}$-space is $\overline{\mathbb{R}}$ itself, where the metric is residuation: $\mathrm{d}(x_1, x_2) = x_2 - x_1$. The underlying partial order (20) of X is labeled by $\succcurlyeq$ and is given by

$$x_1 \succcurlyeq x_2 \iff 0 \geqslant \mathrm{d}(x_1, x_2). \tag{21}$$

6.2 $\mathcal{V}$-functors

Just as we have done for a categories, we can slightly modify Definition 4.6 to define a functor between enriched categories.

Definition 6.3. Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathcal{V}$-categories. Then a *$\mathcal{V}$-functor* $F : \mathscr{A} \to \mathscr{B}$ consists of:

- A function $F : \mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B})$.
- For each $A_1, A_2 \in \mathscr{A}$, a map $\mathscr{A}(A_1, A_2) \to \mathscr{B}(F(A_1), F(A_2))$ in $\mathcal{V}$.

Truth. If R, S are preordered sets (i.e.Truth-categories), a Truth-functor $F : R \to S$ satisfies

$$\lceil r_1 \leqslant_R r_2 \rceil \vdash \lceil F(r_1) \leqslant_S F(r_2) \rceil$$

from the second bullet point. Unpacking this we get $r_1 \leqslant_R r_2 \implies F(r_1) \leqslant_S F(r_2)$, so F is an *order preserving function* $R \to S$.

$\overline{\mathbb{R}}$. If X, Y are $\overline{\mathbb{R}}$-spaces (i.e. $\overline{\mathbb{R}}$-categories), an $\overline{\mathbb{R}}$-functor $F : X \to Y$ satisfies

$$\mathrm{d}(x_1, x_2) \geqslant \mathrm{d}\left(F(x_1), F(x_2)\right) \tag{22}$$

so F is a *distance non-increasing function.*

6.3 Natural transformation objects

In ordinary category theory, we label the category of functors $\mathscr{A} \to \mathscr{B}$ by $[\mathscr{A}, \mathscr{B}]$, and say that a natural transformation
$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$
is an element of the set $[\mathscr{A}, \mathscr{B}](F, G)$. In the enriched setting, the category of $\mathcal{V}$-functors $[\mathscr{A}, \mathscr{B}]$ is itself a $\mathcal{V}$-category given that $\mathcal{V}$ is complete, and $[\mathscr{A}, \mathscr{B}](F, G)$ is an element of $\mathcal{V}$. We define it as follows.

Definition 6.4. Let $\mathscr{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathscr{B}$ be $\mathcal{V}$-functors. The *$\mathcal{V}$-object of natural transformations* is

$$[\mathscr{A}, \mathscr{B}](F, G) = \bigwedge_{A \in A} \mathscr{B}\left(F(A), G(A)\right). \tag{23}$$

Truth. If we have preordered sets $(R, \leqslant_R)$ and $(S, \leqslant_S)$, we get a preordered set $\left([R, S], \leqslant_{[R,S]}\right)$ consisting of order-preserving functions (i.e.Truth-functors) $R \to S$. If F, G are such functions, then the truth value of $F \leqslant_{[R,S]} G$ is the truth value of $[R,S](F,G)$. The meet $\bigwedge$ in Truth is the logical and operator (see Table 3), so we have

$$\lceil F \leqslant_{[R,S]} G \rceil = \lceil \forall\, r \in R : F(r) \leqslant_S G(r) \rceil.$$

Therefore the preorder $\leqslant_{[R,S]}$ is a *domination relation*: $F \leqslant_{[R,S]} G \iff F(r) \leqslant_S G(r)$ for all $r \in R$.

$\overline{\mathbb{R}}$. Given $\overline{\mathbb{R}}$-spaces X and Y, we get another $\overline{\mathbb{R}}$-space $[X,Y]$ consisting of distance non-increasing functions (i.e. $\overline{\mathbb{R}}$-functors) $X \to Y$. The meet in $\overline{\mathbb{R}}$ is sup, so the metric in $[X,Y]$ is given by

$$\mathrm{d}(F,G) = \sup_{x \in X} \{\mathrm{d}\left(F(x), G(x)\right)\}. \tag{24}$$

This is the *asymmetric sup metric on functions.*

6.4 Enriched products and coproducts

In Section 4.5, the definition of a product in an ordinary category was motivated by observing that $\mathrm{Fun}(A, X \times Y) \cong \mathrm{Fun}(A,X) \times \mathrm{Fun}(A,Y)$ for sets A, X, Y. We label the product in a $\mathcal{V}$-category $\mathscr{A}$ by Π and define it in such a way so that it satisfies a similar property, keeping in mind that we think of the product in $\mathcal{V}$ as the meet:

$$\begin{array}{rcll} \mathrm{Fun}\,(A, X \times Y) & \cong & \mathrm{Fun}(A,X) \times \mathrm{Fun}(A,Y) & \quad\text{(sets)} \\ \mathscr{A}\,(A, X\,\Pi\,Y) & \cong & \mathscr{A}(A,X) \bigwedge \mathscr{A}(A,Y) & \quad\text{(enriched CT)} \end{array}$$

Again, we can generalize this idea to arbitrary families of objects to get the following definition.

Definition 6.5. Let $\mathscr{A}$ be a $\mathcal{V}$-category. A *product* of a family of objects $(X_i)_{i\in I}$ in $\mathscr{A}$ consists of an object $\prod_{i\in I} X_i$ such that

$$\mathscr{A}\Big(A, \prod_{i\in I} X_i\Big) = \bigwedge_{i \in I} \mathscr{A}\,(A, X_i) \quad \forall\, A \in \mathscr{A}.$$

Notice how the isomorphism has become an equality here, since $\mathcal{V}$ is skeletal. The same goes for coproducts in a $\mathcal{V}$-category.

Definition 6.6. Let $\mathscr{A}$ be a $\mathcal{V}$-category. A *coproduct* of a family of objects $(X_i)_{i\in I}$ in $\mathscr{A}$ consists of an object $\coprod_{i\in I} X_i$ such that

$$\mathscr{A}\Big(\coprod_{i\in I} X_i, A\Big) = \bigwedge_{i \in I} \mathscr{A}\,(X_i, A) \quad \forall\, A \in \mathscr{A}.$$

A $\mathcal{V}$-category is said to have products if all families of objects have a product, and similarly for coproducts.

Truth. We use the above definitions to identify what the product and coproduct are in a preordered set R. Similarly to the previous section, we have a meet which corresponds to the logical and operator, giving us a $\forall$ symbol. So given arbitrary $r \in R$ we have:

$$R\Big(r, \prod_{i\in I} r_i\Big) = \lceil \forall\, i \in I : r \leqslant_R r_i \rceil,$$

which is just like saying

$$r \leqslant_R \prod_{i\in I} r_i \iff r \leqslant_R r_i \quad \forall\, i \in I.$$

Comparing this to (16) we conclude that the product $\prod$ in R is the same as the meet $\bigwedge$. Similarly we get

$$\coprod_{i\in I} r_i \leqslant_R r \iff r_i \leqslant_R r \quad \forall\, i \in I,$$

so the coproduct $\coprod$ in R is the same as the join $\bigvee$.

$\overline{\mathbb{R}}$**.** Applying the definitions of products and coproducts in an $\overline{\mathbb{R}}$-space X gives that $\prod_{i\in I} x_i$ and $\coprod_{i\in I} x_i$ satisfy

$$\mathrm{d}\Big(y, \prod_{i\in I} x_i\Big) = \sup_{i\in I}\{\mathrm{d}(y, x_i)\} \quad \text{and} \quad \mathrm{d}\Big(\coprod_{i\in I} x_i, y\Big) = \sup_{i\in I}\{\mathrm{d}(x_i, y)\}$$

for every $y \in X$.

6.5 Tensors and cotensors

In Section 5.3 we stated that a category $\mathcal{V}$ is complete (cocomplete) if it has products (coproducts). In order to describe completeness and cocompleteness of $\mathcal{V}$-categories we require an understanding of additional structures called cotensors and tensors. It can be useful to think of these analogously to scalar multiplication in a vector space.

Definition 6.7. Let $\mathscr{A}$ be a $\mathcal{V}$-category. Then $\mathscr{A}$ is *cotensored* if there exists a function $\pitchfork : \mathrm{ob}(\mathcal{V}) \times \mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{A})$ such that

$$\mathscr{A}\,(A_1, v \pitchfork A_2) = [v, \mathscr{A}\,(A_1, A_2)] \quad \forall\, A_1, A_2 \in \mathscr{A}, v \in \mathcal{V}.$$

A $\mathcal{V}$-category is *complete* if it has products and is cotensored. Given the restrictions that we have placed on $\mathcal{V}$ (closed, symmetric monoidal), it is possible to view $\mathcal{V}$ as a $\mathcal{V}$-category. A useful fact is that $\mathcal{V}$ is itself a complete $\mathcal{V}$-category.

Lemma 6.1. *$\mathcal{V}$ is complete when viewed as a $\mathcal{V}$-category. Its product $\prod$ is given by the meet $\bigwedge$ and its cotensor $\pitchfork$ is residuation: $v_1 \pitchfork v_2 = [v_1, v_2]$.*

Proof. See [Wil15, page 16]. □

The tensor in a $\mathcal{V}$-category is defined dually to the cotensor.

Definition 6.8. Let $\mathscr{A}$ be a $\mathcal{V}$-category. Then $\mathscr{A}$ is *tensored* if there exists a function $\odot : \mathrm{ob}(\mathcal{V}) \times \mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{A})$ such that

$$\mathscr{A}\,(v \odot A_2, A_1) = [v, \mathscr{A}\,(A_2, A_1)] \quad \forall\, A_1, A_2 \in \mathscr{A}, v \in \mathcal{V}.$$

A $\mathcal{V}$-category is *cocomplete* if it has coproducts and is tensored. Dually to before, $\mathcal{V}$ is also cocomplete when viewed as a $\mathcal{V}$-category.

Lemma 6.2. *$\mathcal{V}$ is cocomplete when viewed as a $\mathcal{V}$-category. Its coproduct $\coprod$ is given by the join $\bigvee$ and its tensor $\odot$ is the monoidal product: $v_1 \odot v_2 = v_1 \otimes v_2$.*

Proof. Dual to the proof of Lemma 6.1. □

Truth. In a preordered set R, we use the fact that residuation corresponds to implication ($[b, c] = \lceil b \Rightarrow c \rceil$) in Truth to find out what $\pitchfork$ gives:

$$R(r_1, \mathsf{true} \pitchfork r_2) = \lceil \mathsf{true} \Rightarrow R(r_1, r_2) \rceil = R(r_1, r_2),$$

since $\lceil \mathsf{true} \Rightarrow \mathsf{true} \rceil = \mathsf{true}$ and $\lceil \mathsf{true} \Rightarrow \mathsf{false} \rceil = \mathsf{false}$. This can only be possible if $\mathsf{true} \pitchfork r_2 = r_2$ for every $r_2 \in R$. On the other hand,

$$R(r_1, \mathsf{false} \pitchfork r_2) = \lceil \mathsf{false} \Rightarrow R(r_1, r_2) \rceil = \mathsf{true},$$

since $\lceil \mathsf{false} \Rightarrow \mathsf{false} \rceil = \lceil \mathsf{false} \Rightarrow \mathsf{true} \rceil = \mathsf{true}$. Therefore we must have $r_1 \leqslant_R \mathsf{false} \pitchfork r_2$ for every $r_1, r_2 \in R$, so we denote it by $\mathsf{false} \pitchfork r_2 := \bigwedge_{\varnothing}$.

For the tensor $\odot$ we can do the same to get $\mathsf{true} \odot r_2 = r_2$ and $\mathsf{false} \odot r_2 \leqslant_R r_1$ for every $r_1, r_2 \in R$, so $\mathsf{false} \odot r_2 := \bigvee_{\varnothing}$.

$\overline{\mathbb{R}}$. In an $\overline{\mathbb{R}}$-space X the residuation is subtraction ($[b, c] = c - b$) so we get

$$\mathrm{d}\,(x_1, c \pitchfork x_2) = \mathrm{d}(x_1, x_2) - c \quad \text{and} \quad \mathrm{d}\,(c \odot x_2, x_1) = d(x_2, x_1) - c$$

for every $c \in \overline{\mathbb{R}}$.

6.6 Presheaves and copresheaves

Definition 4.7 says that a presheaf on an ordinary category $\mathscr{A}$ is a functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$. In enriched category theory we have, loosely speaking, substituted **Set** with $\mathcal{V}$. Therefore we define a *presheaf* on a $\mathcal{V}$-category $\mathscr{A}$ to be $\mathcal{V}$-functor $\mathscr{A}^{\mathrm{op}} \to \mathcal{V}$. Note that, once again, we have regarded $\mathcal{V}$ as a $\mathcal{V}$-category, which is possible thanks to its 'nice' properties. A *copresheaf*, on the other hand, is a $\mathcal{V}$-functor $\mathscr{A} \to \mathcal{V}$.

From these we can form the *presheaf category* $\widehat{\mathscr{A}} := [\mathscr{A}^{\mathrm{op}}, \mathcal{V}]$ and the *opcopresheaf category* $\widecheck{\mathscr{A}} := [\mathscr{A}, \mathcal{V}]^{\mathrm{op}}$, which are both $\mathcal{V}$-categories. It turns out that they are also complete and cocomplete, with (co)products and (co)tensors defined pointwise.

Proposition 6.1. *Let $\mathscr{A}$ be a $\mathcal{V}$-category. The presheaf category $\widehat{\mathscr{A}}$ is complete and cocomplete. For every family of presheaves $(P_i)_{i\in I} \in \widehat{\mathscr{A}}$ and every $A \in \mathscr{A}$, the (co)products and (co)tensors are defined as follows:*

$$\Big(\prod_{i\in I} P_i\Big)(A) = \bigwedge_{i\in I} P_i(A); \qquad \Big(\coprod_{i\in I} P_i\Big)(A) = \bigvee_{i\in I} P_i(A); \tag{25}$$

$$(v \pitchfork P)(A) = v \pitchfork P(A); \qquad (v \odot P)(A) = v \odot P(A). \tag{26}$$

Proof. See [Wil15, page 18]. □

The expressions for the cotensor and tensor simplify further thanks to Lemma 6.1 and Lemma 6.2: $v \pitchfork P(A) = [v, P(A)]$ and $v \odot P(A) = v \otimes P(A)$. The dual to the above proposition for the opcopresheaf category $\widecheck{\mathscr{A}}$ is similar, with a noteworthy difference. In $\widecheck{\mathscr{A}}$, all the maps are reversed with respect to $\widehat{\mathscr{A}}$ (see Definition 4.2), so the product in $\widecheck{\mathscr{A}}$ is the coproduct in $\widehat{\mathscr{A}}$ and vice versa. Also, the cotensor in $\widecheck{\mathscr{A}}$ is the tensor in $\widehat{\mathscr{A}}$ and vice versa.

Proposition 6.2. *Let $\mathscr{A}$ be a $\mathcal{V}$-category. The opcopresheaf category $\widecheck{\mathscr{A}}$ is complete and cocomplete. For every family of copresheaves $(Q_i)_{i\in I} \in \widecheck{\mathscr{B}}$ and every $B \in \mathscr{B}$, the (co)products and (co)tensors are defined as follows:*

$$\Big(\prod_{i\in I} Q_i\Big)(A) = \bigvee_{i\in I} Q_i(A); \qquad \Big(\coprod_{i\in I} Q_i\Big)(A) = \bigwedge_{i\in I} Q_i(A); \tag{27}$$

$$(v \pitchfork Q)(A) = v \odot Q(A); \qquad (v \odot Q)(A) = v \pitchfork Q(A). \tag{28}$$

Proof. Follows from Proposition 6.1. □

Truth. Given a preordered set R, a presheaf $P : R^{\mathrm{op}} \to \mathsf{Truth}$ sends every element of R to either false or true, so it is completely determined by the subset $\tilde{P} := P^{-1}(\mathsf{true})$. This subset is *downward closed*: If $r_1 \in \tilde{P}$ and $r_2 \leqslant_R r_1$, then $r_2 \in \tilde{P}$. To see this, note that by functoriality of P, there exists a map $R^{\mathrm{op}}(r_1, r_2) \to \mathsf{Truth}(P(r_1), P(r_2))$, where we view Truth as a Truth-category. The statement $r_1 \in \tilde{P}$ is equivalent to $P(r_1) = \mathsf{true}$, and the statement $r_2 \leqslant_R r_1$ is equivalent to $R^{\mathrm{op}}(r_1, r_2) = \mathsf{true}$, so we have a map $\mathsf{true} \to \mathsf{Truth}(\mathsf{true}, P(r_2))$. This in turn is the same as saying $\mathsf{Truth}(\mathsf{true}, P(r_2)) = \mathsf{true}$, so $P(r_2) = \mathsf{true}$ which gives $r_2 \in \tilde{P}$.

Therefore the presheaf category $\widehat{R} := [R^{\mathrm{op}}, \mathsf{Truth}]$ corresponds to the *set of downward closed subsets* of R. The domination relation $\leqslant_{\widehat{R}}$ (i.e. the natural transformation object) on this set becomes inclusion $\subseteq$:

$$\begin{aligned} P_1 \leqslant_{\widehat{R}} P_2 &\iff \forall\, r \in R : P_1(r) \vdash P_2(r) \\ &\iff \forall\, r \in R : P_1(r) = \mathsf{true} \Rightarrow P_2(r) = \mathsf{true} \\ &\iff \forall\, r \in R : r \in \tilde{P}_1 \Rightarrow r \in \tilde{P}_2 \\ &\iff \tilde{P}_1 \subseteq \tilde{P}_2. \end{aligned}$$

Dually to all of the above, the copresheaf $Q : R \to \mathsf{Truth}$ corresponds to an *upward closed* set: if $r_1 \in \tilde{Q}$ and $r_1 \leqslant_R r_2$, then $r_2 \in \tilde{Q}$. The opcopresheaf category $\check{R} := [R, \mathsf{Truth}]^{\mathrm{op}}$ is the *set of upward closed subsets* of R ordered by containment $\supseteq$.

$\overline{\mathbb{R}}$**.** A presheaf $P : X^{\mathrm{op}} \to \overline{\mathbb{R}}$ on an $\overline{\mathbb{R}}$-space X is an $\overline{\mathbb{R}}$-valued function satisfying

$$\mathrm{d}(x_1, x_2) \geqslant P(x_1) - P(x_2),$$

since all $\overline{\mathbb{R}}$-functors satisfy (22) and the metric in $\overline{\mathbb{R}}$ is residuation. Furthermore, the asymmetric sup metric (24) in the $\overline{\mathbb{R}}$-space $\widehat{X} := [X^{\mathrm{op}}, \overline{\mathbb{R}}]$ is

$$\mathrm{d}\,(P_1, P_2) = \sup_{x \in X}\{P_2(x) - P_1(x)\}. \tag{29}$$

The ordering on $\widehat{X}$ is given by the underlying partial order (21), which in this case is a domination relation:

$$\begin{aligned} P_1 \geqslant P_2 &\iff 0 \geqslant \mathrm{d}(P_1, P_2) \\ &\iff 0 \geqslant \sup_{x \in X}\{P_2(x) - P_1(x)\} \\ &\iff P_1(x) \geqslant P_2(x) \quad \forall\, x \in X. \end{aligned}$$

On the other hand, a copresheaf $Q : X \to \overline{\mathbb{R}}$ satisfies

$$\mathrm{d}(x_1, x_2) \geqslant Q(x_2) - Q(x_1),$$

and the metric in $\check{X} := [X, \overline{\mathbb{R}}]^{\mathrm{op}}$ is

$$\mathrm{d}(Q_1, Q_2) = \sup_{x \in X}\{Q_1(x) - Q_2(x)\}. \tag{30}$$

Note how the signs switch because we are dealing with the dual category. The ordering on $\check{X}$ is once again a domination relation:

$$Q_1 \leqslant Q_2 \iff Q_1(x) \leqslant Q_2(x) \quad \forall\, x \in X.$$

6.7 Enriched adjunctions

From now on, all $\mathcal{V}$-categories should be assumed to be skeletal. We define an adjunction in enriched category theory in the same way as we have done before. Two $\mathcal{V}$-functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ are adjoint if

$$\mathscr{B}(F(A), B) = \mathscr{A}(A, G(B))$$

in $\mathcal{V}$ for each $A \in \mathscr{A}, B \in \mathscr{B}$. Just as in Section 4.4, we define the full subcategories containing the objects that are fixed by the adjunction:
$\mathrm{ob}\,(\mathrm{Fix}(GF)) = \{A \in \mathscr{A} : GF(A) = A\}$ and $\mathrm{ob}\,(\mathrm{Fix}(FG)) = \{B \in \mathscr{B} : FG(B) = B\}$.
From Theorem 4.1 we get

$$\mathrm{Fix}(GF) \cong \mathrm{Fix}(FG).$$

Notice how all isomorphisms have turned into equalities and equivalences have turned into isomorphisms, since $\mathscr{A}$ and $\mathscr{B}$ are skeletal. The fixed categories are isomorphic to another full subcategory $\mathrm{Inv}(F, G)$ defined by

$$\mathrm{ob}\,(\mathrm{Inv}(F,G)) = \{(A, B) \in \mathscr{A} \times \mathscr{B} : F(A) = B, A = G(B)\}\,.$$

The isomorphisms are given by

$$\begin{array}{rcc} \alpha : \mathrm{Inv}(F,G) & \rightleftarrows & \mathrm{Fix}(GF) : \beta \\ \alpha(A,B) & = & A \\ \beta(A) & = & (A, F(A)), \end{array} \qquad \begin{array}{rcc} \gamma : \mathrm{Inv}(F,G) & \rightleftarrows & \mathrm{Fix}(FG) : \delta \\ \gamma(A,B) & = & B \\ \delta(B) & = & (G(B), B). \end{array}$$

Any of the three isomorphic categories $\mathrm{Fix}(GF)$, $\mathrm{Fix}(FG)$, $\mathrm{Inv}(F, G)$ is known as the *invariant part* of the adjunction.

Truth. Two order preserving functions between preordered sets $R \underset{G}{\overset{F}{\rightleftarrows}} S$ are adjoint if

$$F(r) \leqslant_S s \iff r \leqslant_R G(s)$$

for every $r \in R, s \in S$. This is exactly a Galois connection (see Proposition 2.2 and Proposition 3.1). The functions GF and FG are the closure operators arising from the Galois connection and the invariant part of the adjunction is the Galois correspondence:

$$F : \mathrm{Fix}(GF) \cong \mathrm{Fix}(FG) : G,$$

just like (5) and (8). $\mathrm{Fix}(GF)$ corresponds to the closed elements of R and $\mathrm{Fix}(FG)$ to the closed elements of S.

$\overline{\mathbb{R}}$. Two distance non-increasing functions between $\overline{\mathbb{R}}$-spaces $X \underset{G}{\overset{F}{\rightleftarrows}} Y$ are adjoint if

$$\mathrm{d}(F(x), y) = \mathrm{d}(x, G(y))$$

for every $x \in X, y \in Y$. The invariant part of the adjunction $\mathrm{Inv}(F, G)$ identifies $\overline{\mathbb{R}}$ with the diagonal in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

6.8 Profunctors

The last concept from enriched category theory that we will need is a profunctor. Before introducing it, we define the tensor product of two $\mathcal{V}$-categories, which is similar to Definition 4.3.

Definition 6.9. Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathcal{V}$-categories. The *tensor product* $\mathscr{A} \otimes \mathscr{B}$ is a $\mathcal{V}$-category whose objects are given by $\mathrm{ob}(\mathscr{A} \otimes \mathscr{B}) = \mathrm{ob}(\mathscr{A}) \times \mathrm{ob}(\mathscr{A})$ and whose hom-objects are given by

$$(\mathscr{A} \otimes \mathscr{B})\left((A_1, B_1), (A_2, B_2)\right) = \mathscr{A}(A_1, A_2) \otimes \mathscr{B}(B_1, B_2),$$

where $\otimes$ is the monoidal product of $\mathcal{V}$.

Definition 6.10. Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathcal{V}$-categories. A *profunctor* $\mathbb{M} : \mathscr{A} \rightsquigarrow \mathscr{B}$ is a $\mathcal{V}$-functor $\mathbb{M} : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathcal{V}$.

We denote by $\boldsymbol{\mathcal{V}}\mathbf{Cat}$ the ordinary category whose objects are $\mathcal{V}$-categories and whose maps are $\mathcal{V}$-functors. If we equip $\boldsymbol{\mathcal{V}}\mathbf{Cat}$ with the tensor product $\otimes$ defined above, we get a monoidal category. It turns out that $\boldsymbol{\mathcal{V}}\mathbf{Cat}$ is closed, in the sense that

$$\boldsymbol{\mathcal{V}}\mathbf{Cat}\left(\mathscr{A} \otimes \mathscr{B}, \mathscr{C}\right) \cong \boldsymbol{\mathcal{V}}\mathbf{Cat}\left(\mathscr{A}, [\mathscr{B}, \mathscr{C}]\right) \tag{31}$$

for all $\mathcal{V}$-categories $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$. This should be compared to the adjunction (15) in Example 4.6 and the adjunction property (18). This is why we use the notation $\otimes$ to mean the monoidal product in $\mathcal{V}$ and the tensor product $\boldsymbol{\mathcal{V}}\mathbf{Cat}$, and $[-,-]$ to mean residuation in $\mathcal{V}$ and the $\mathcal{V}$-category of $\mathcal{V}$-functors in $\boldsymbol{\mathcal{V}}\mathbf{Cat}$.

In particular, this means that every profunctor $\mathbb{M} : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathcal{V}$ corresponds to a $\mathcal{V}$-functor $\overline{\mathbb{M}} : \mathscr{A} \to \check{\mathscr{B}}$, or equivalently to a $\mathcal{V}$-functor $\overline{\mathbb{M}} : \mathscr{B} \to \widehat{\mathscr{A}}$ (since $\mathcal{V}$ is symmetric). In the next section we will see how this corresponds to the idea that a matrix gives rise to linear maps.

Truth. A profunctor $\mathbb{M} : R \rightsquigarrow S$ between preordered sets can be viewed as a relation:

$$\mathbb{M}(r,s) = \mathsf{true} \iff r\,\mathbb{M}\,s \iff r \text{ is related to } s.$$

$\overline{\mathbb{R}}$. There is some degree of choice involved in understanding a profunctor in the context of $\overline{\mathbb{R}}$-spaces. What will be of interest to us is the natural pairing between an $\overline{\mathbb{R}}$-space V and its algebraic dual V^*. This is given by

$$\begin{array}{rccc} \mathbb{L}: & V^{\mathrm{op}} \times V^* & \to & \overline{\mathbb{R}} \\ & (x,k) & \mapsto & \langle k, x\rangle. \end{array} \tag{32}$$

	$\mathcal{V} = \textsf{Truth}$	$\mathcal{V} = \overline{\mathbb{R}}$
$\mathcal{V}$-category $\mathscr{A}$	Preordered set $(R, \leqslant_R)$	$\overline{\mathbb{R}}$-space X
$\mathcal{V}$-functor $F : \mathscr{A} \to \mathscr{B}$	Order-preserving function $F : R \to S$	Distance non-increasing function $F : X \to Y$
Natural transformation $[\mathscr{A}, \mathscr{B}](F, G)$	Domination relation $F \leqslant_{[R,S]} G \iff F(r) \leqslant_S G(r) \quad \forall r \in R$	Asymmetric sup metric on functions $\mathrm{d}(F, G) = \sup_{x \in X}\{\mathrm{d}(F(x), G(x))\}$
Product $\prod$	Meet $\bigwedge$	$\mathrm{d}\Big(y, \prod_{i \in I} x_i\Big) = \sup_{i \in I}\{\mathrm{d}(y, x_i)\}$
Coproduct $\coprod$	Join $\bigvee$	$\mathrm{d}(\coprod_{i \in I} x_i, y\Big) = \sup_{i \in I}\{\mathrm{d}(x_i, y)\}$
Cotensor $\pitchfork$	$\textsf{true} \pitchfork r = r$, $\textsf{false} \pitchfork r = \bigwedge_{\varnothing}$	$\mathrm{d}(x_1, c \pitchfork x_2) = \mathrm{d}(x_1, x_2) - c$
Tensor $\odot$	$\textsf{true} \pitchfork r = r$, $\textsf{false} \odot r = \bigvee_{\varnothing}$	$\mathrm{d}(c \odot x_2, x_1) = \mathrm{d}(x_2, x_1) - c$
Presheaf $P : \mathscr{A}^{\mathrm{op}} \to \mathcal{V}$	Downward closed subset $\tilde{P} = P^{-1}(\textsf{true}) \subseteq R$	$\overline{\mathbb{R}}$-valued function satisfying $\mathrm{d}(x_1, x_2) \geqslant P(x_1) - P(x_2)$
Copresheaf $Q : \mathscr{A} \to \mathcal{V}$	Upward closed subset $\tilde{Q} = Q^{-1}(\textsf{true}) \subseteq R$	$\overline{\mathbb{R}}$-valued function satisfying $\mathrm{d}(x_1, x_2) \geqslant Q(x_2) - Q(x_1)$
Presheaf category $\widehat{\mathscr{A}} = [\mathscr{A}^{\mathrm{op}}, \mathcal{V}]$	Set $\widehat{R} = [R^{\mathrm{op}}, \textsf{Truth}]$ of downward closed subsets ordered by inclusion $\subseteq$	$\overline{\mathbb{R}}$-space $\widehat{X} = [X^{\mathrm{op}}, \overline{\mathbb{R}}]$ ordered by $\geqslant$, $\mathrm{d}(P_1, P_2) = \sup_{x \in X}\{P_2(x) - P_1(x)\}$
Opcopresheaf category $\widecheck{\mathscr{A}} = [\mathscr{A}, \mathcal{V}]^{\mathrm{op}}$	Set $\check{R} = [R, \textsf{Truth}]^{\mathrm{op}}$ of upward closed subsets ordered by containment $\supseteq$	$\overline{\mathbb{R}}$-space $\check{X} = [X, \overline{\mathbb{R}}]^{\mathrm{op}}$ ordered by $\leqslant$, $\mathrm{d}(Q_1, Q_2) = \sup_{x \in X}\{Q_1(x) - Q_2(x)\}$
Adjunction $\mathscr{B}(F(A), B) = \mathscr{A}(A, G(B))$	Galois connection $F(r) \leqslant_S s \iff r \leqslant_R G(s)$	$\overline{\mathbb{R}}$-adjunction $\mathrm{d}(F(x), y) = \mathrm{d}(x, G(y))$
Profunctor $\mathbb{M} : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathcal{V}$	Relation $\mathbb{M}(r, s) = \textsf{true} \iff r \, \mathbb{M} \, s$	Natural pairing $\langle -, - \rangle : V^* \times V \to \overline{\mathbb{R}}$

Table 4: Translating from enriched categories to preorders and $\overline{\mathbb{R}}$-spaces, adapted from [Wil15, page 13].

7 Categorified linear algebra

In this section we use the concepts discussed in the previous section to draw a parallel between linear algebra and enriched category theory. This will be the last step needed to understand the Legendre-Fenchel transform.

We show how we can think of the presheaf and opcopresheaf $\mathcal{V}$-categories as 'vector spaces' over the semiring $\mathcal{V}$. After, we explore what the equivalent of a matrix is in enriched category theory and draw some further parallels. Along the way we will use a useful analogy regarding sets and functions. Here $\underline{m}$ and $\underline{n}$ denote the sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$.

Linear algebra	**Sets and functions**	**Enriched category theory**
Vector spaces $\mathbb{R}^m$ and $\mathbb{R}^n$	Sets of functions $\mathbf{Set}(\underline{m}, \mathbb{R})$ and $\mathbf{Set}(\underline{n}, \mathbb{R})$	Functor categories $\widehat{\mathscr{A}}$ and $\check{\mathscr{B}}$
Real $m \times n$ matrix M	Function $M : \underline{m} \times \underline{n} \to \mathbb{R}$	Profunctor $\mathbb{M} : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathcal{V}$
Linear maps $M^* : \mathbb{R}^m \to \mathbb{R}^n$ $M_* : \mathbb{R}^n \to \mathbb{R}^m$	Functions $M^* : \mathbf{Set}(\underline{m}, \mathbb{R}) \to \mathbf{Set}(\underline{n}, \mathbb{R})$ $M_* : \mathbf{Set}(\underline{n}, \mathbb{R}) \to \mathbf{Set}(\underline{m}, \mathbb{R})$	Functors $\mathbb{M}^* : \widehat{\mathscr{A}} \to \check{\mathscr{B}}$ $\mathbb{M}_* : \check{\mathscr{B}} \to \widehat{\mathscr{A}}$
$\langle M^* v, w \rangle = \langle v, M_* w \rangle$	-	$\mathbb{M}^* \dashv \mathbb{M}_*$

Table 5: Categorifying linear algebra.

Any vector $\vec{v} \in \mathbb{R}^m$ gives rise to a function $\underline{m} \to \mathbb{R}$, namely $i \mapsto v_i$. Therefore we may identify the space of vectors $\mathbb{R}^m$ with the set of functions $\mathbf{Set}(\underline{m}, \mathbb{R})$, which we then identify with the presheaf category $\widehat{\mathscr{A}} = [\mathscr{A}^{\mathrm{op}}, \mathcal{V}]$. We can look at the vector space structure on $\mathbf{Set}(\underline{m}, \mathbb{R})$ and follow it closely to describe the vector space structure on $\widehat{\mathscr{A}}$.

Addition in $\mathbf{Set}(\underline{m}, \mathbb{R})$ is defined pointwise; given $v_1, v_2 \in \mathbf{Set}(\underline{m}, \mathbb{R})$, the function $v_1 + v_2$ is given by

$$(v_1 + v_2)(i) = v_1(i) + v_2(i), \quad i = 1, \ldots, m.$$

Addition in the semiring $\mathcal{V}$ is given by the coproduct, which is the join. So given $P_1, P_2 \in \widehat{\mathscr{A}}$, the presheaf $P_1 \amalg P_2$ is given by

$$(P_1 \amalg P_2)(A) = P_1(A) \bigvee P_2(A), \quad A \in \mathscr{A},$$

thanks to Proposition 6.1(25).

Scalar multiplication in $\mathbf{Set}(\underline{m}, \mathbb{R})$ is also defined pointwise; given $c \in \mathbb{R}$ and $v \in \mathbf{Set}(\underline{m}, \mathbb{R})$, the function $c \cdot v$ is given by

$$(c \cdot v)(i) = c \cdot v(i), \quad i = 1, \ldots, m.$$

The scalar multiplication on the right is in $\mathbb{R}$. The 'multiplication' in $\mathcal{V}$, on the other hand, is the monoidal product $\otimes$. Proposition 6.1(26) and Lemma 6.2 indicate that we should take the tensor $\odot$ in $\widehat{\mathscr{A}}$ as the scalar multiplication; given $v \in \mathcal{V}$ and $P \in \widehat{\mathscr{A}}$, the presheaf $v \odot P$ is defined by

$$(v \odot P)(A) = v \otimes P(A), \quad A \in \mathscr{A}.$$

The distributive properties of $\bigvee$ and $\otimes$ in $\mathcal{V}$ ensure that analogue versions of vector space axioms are satisfied in $\widehat{\mathscr{A}}$.

We can do the same for $\check{\mathscr{B}}$ using Proposition 6.2 and Lemma 6.1; for each $Q_1, Q_2 \in \check{\mathscr{B}}$, the copresheaf $Q_1 \Pi Q_2$ is given by

$$(Q_1 \Pi Q_2)(B) = Q_1(B) \bigvee Q_2(B), \quad B \in \mathscr{B}.$$

Furthermore, given $v \in \mathcal{V}$ and $Q \in \check{\mathscr{B}}$, the copresheaf $v \pitchfork Q$ is given by

$$(v \pitchfork Q)(B) = v \otimes Q(B), \quad B \in \mathscr{B}.$$

One of the fundamental facts about vector spaces and linear maps is that, given two vector spaces V and W, any linear map $V \to W$ is completely determined by its action on basis vectors. To express this categorically, suppose V is an m-dimensional vector space, and $b : \underline{m} \to V$ picks out a basis of V. Then the pair (V, b) has the following universal property: given any vector space W and any function $f : \underline{m} \to W$, there exists a unique linear map $\tilde{f} : V \to W$ such that the following diagram commutes.

$$\begin{tikzcd} \underline{m} \arrow[r, "b"] \arrow[dr, "f"'] & V \arrow[d, dashed, "\tilde{f}"] \\ & W \end{tikzcd} \tag{33}$$

If $\vec{v} \in V$ is expressed in the basis b as $\vec{v} = \sum_{i=1}^{m} v_i \cdot b(i)$, we have

$$\tilde{f}(\vec{v}) = \sum_{i=1}^{m} v_i \cdot f(i). \tag{34}$$

We now think of this categorically. Firstly, for any $A \in \mathscr{A}$ let $H_A : \mathscr{A}^{\mathrm{op}} \to \mathcal{V}$ be the presheaf in $\widehat{\mathscr{A}}$ defined as follows:

- For objects $A_1 \in \mathscr{A}$, put $H_A(A_1) = \mathscr{A}(A_1, A)$.
- For maps $f : A_2 \to A_1$, define $H_A(f) : \mathscr{A}(A_1, A) \to \mathscr{A}(A_2, A)$ by $p \mapsto p \circ f$.

The $\mathcal{V}$-functor H_A is often denoted by $\mathscr{A}(-, A)$ for clarity. Every map $f : A_1 \to A_2$ induces a natural transformation $H_f : H_{A_1} \to H_{A_2}$, whose component at $A \in \mathscr{A}$ is

$$\begin{array}{ccc} H_{A_1}(A) = \mathscr{A}(A, A_1) & \to & H_{A_2}(A) = \mathscr{A}(A, A_2) \\ p & \mapsto & f \circ p. \end{array}$$

For any $\mathcal{V}$-category $\mathscr{A}$ we have a $\mathcal{V}$-functor $H_\bullet : \mathscr{A} \to \widehat{\mathscr{A}}$ defined by

$$\begin{array}{ccc} A & \mapsto & H_A, \\ f & \mapsto & H_f. \end{array}$$

This is called the *Yoneda embedding*. As a consequence of the Yoneda lemma [Lei14, page 94], the pair $(\mathscr{A}, H_\bullet)$ has the same type of universal property as before: given any cocomplete $\mathcal{V}$-category $\mathscr{C}$ and any $\mathcal{V}$-functor $F : \mathscr{A} \to \mathscr{C}$, there exists a unique colimit-preserving $\mathcal{V}$-functor $\tilde{F} : \widehat{\mathscr{A}} \to \mathscr{C}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathscr{A} \arrow[r, "H_\bullet"] \arrow[dr, "F"'] & \widehat{\mathscr{A}} \arrow[d, dashed, "\tilde{F}"] \\ & \mathscr{C} \end{tikzcd} \tag{35}$$

Compare this to linear algebra equivalent (33). For each $P \in \widehat{\mathscr{A}}$, we have

$$\tilde{F}(P) = \coprod_{A \in \mathscr{A}} P(A) \odot F(A). \tag{36}$$

Again, compare this to (34).

Everything described above dualizes, so that we get a similar result for $\check{\mathscr{B}}$. For every object $B \in \mathscr{B}$ and map $g : B_2 \to B_1$ we have a copresheaf $H^B : \mathscr{B} \to \mathcal{V}$ and a natural transformation $H^g : H^{B_1} \to H^{B_2}$. From these we obtain the functor $H^\bullet : \mathscr{B} \to \check{\mathscr{B}}$ defined by

$$\begin{array}{ccc} B & \mapsto & H^B, \\ g & \mapsto & H^g. \end{array}$$

The pair $(\mathscr{B}, H^\bullet)$ has the universal property: given any complete $\mathcal{V}$-category $\mathscr{D}$ and any $\mathcal{V}$-functor $G : \mathscr{B} \to \mathscr{D}$, there exists a unique limit-preserving $\mathcal{V}$-functor $\tilde{G} : \breve{\mathscr{B}} \to \mathscr{D}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "H^\bullet"] \arrow[dr, "G"'] & \breve{\mathscr{B}} \arrow[d, dashed, "\tilde{G}"] \\
& \mathscr{D}
\end{tikzcd} \tag{37}$$

For each $Q \in \breve{\mathscr{B}}$, we have

$$\tilde{G}(Q) = \prod_{B \in B} Q(B) \pitchfork G(B). \tag{38}$$

We now look at understanding matrices using sets and functions. We may think of a real $m \times n$ matrix M as a function $M : \underline{m} \times \underline{n} \to \mathbb{R}$ sending (i, j) to M_{ij}. The category of sets is closed (as shown in Example 4.6(15)), so we have a bijection

$$\mathbf{Set}(\underline{m} \times \underline{n}, \mathbb{R}) \cong \mathbf{Set}\left(\underline{m}, \mathrm{Fun}(\underline{n}, \mathbb{R})\right).$$

The set $\mathrm{Fun}(\underline{n}, \mathbb{R})$ is exactly the same as $\mathbf{Set}(\underline{n}, \mathbb{R})$. So the function M corresponds to the function

$$\begin{array}{rccc} \overline{M} : & \underline{m} & \to & \mathbf{Set}(\underline{n}, \mathbb{R}), \\ & i & \mapsto & \left(j \mapsto M_{ij}\right). \end{array} \tag{39}$$

We have just seen how we can give $\mathbf{Set}(\underline{m}, \mathbb{R})$ and $\mathbf{Set}(\underline{n}, \mathbb{R})$ a vector space structure to make them isomorphic to $\mathbb{R}^m$ and $\mathbb{R}^n$. Therefore, by the universal property (33), there exists a unique linear map $M^* : \mathbf{Set}(\underline{m}, \mathbb{R}) \to \mathbf{Set}(\underline{n}, \mathbb{R})$ making the following diagram commute.

$$\begin{tikzcd}
\underline{m} \arrow[r, "b"] \arrow[dr, "\overline{M}"'] & \mathbf{Set}(\underline{m}, \mathbb{R}) \arrow[d, dashed, "M^*"] \\
& \mathbf{Set}(\underline{n}, \mathbb{R})
\end{tikzcd} \tag{40}$$

We identify this with the linear map $M^* : \mathbb{R}^m \to \mathbb{R}^n$ arising from the matrix M.

By using the symmetry of $\times$, we can also think of M as the function $M : \underline{n} \times \underline{m} \to \mathbb{R}$. Repeating the same process again, we obtain the linear map $M_* : \mathbb{R}^n \to \mathbb{R}^m$. Clearly M^* and M_* are each other's transpose, so when we equip $\mathbb{R}^m$ and $\mathbb{R}^n$ with the standard inner products $\langle -, - \rangle_{\mathbb{R}^m}$ and $\langle -, - \rangle_{\mathbb{R}^n}$, we get:

$$\langle M^*(\vec{v}), \vec{w} \rangle_{\mathbb{R}^n} = \langle \vec{v}, M_*(\vec{w}) \rangle_{\mathbb{R}^m}$$

for every $\vec{v} \in \mathbb{R}^m, \vec{w} \in \mathbb{R}^n$. In the next section we explain the categorification of this process.

7.1 Nucleus of a profunctor

Firstly, we identify the matrix/function $M : \underline{m} \times \underline{n} \to \mathbb{R}$ with the profunctor $\mathbb{M} : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathcal{V}$. We mentioned briefly in Section 6.8 that $\boldsymbol{\mathcal{V}}\mathbf{Cat}$ is closed (31), so we have a bijection

$$\boldsymbol{\mathcal{V}}\mathbf{Cat}\left(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathcal{V}\right) \cong \boldsymbol{\mathcal{V}}\mathbf{Cat}\left(\mathscr{A}^{\mathrm{op}}, [\mathscr{B}, \mathcal{V}]\right).$$

Hence the profunctor $\mathbb{M}$ corresponds to the $\mathcal{V}$-functor $\overline{\mathbb{M}} : \mathscr{A} \to \check{\mathscr{B}}$. Now we use the universal property (35) with $F = \overline{\mathbb{M}}$ and $\mathscr{C} = \check{\mathscr{B}}$ to obtain the $\mathcal{V}$-functor $\mathbb{M}^* : \widehat{\mathscr{A}} \to \check{\mathscr{B}}$ that makes the diagram commute.

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "H_\bullet"] \arrow[dr, "\overline{\mathbb{M}}"'] & \widehat{\mathscr{A}} \arrow[d, dashed, "\mathbb{M}^*"] \\
& \check{\mathscr{B}}
\end{tikzcd}$$

For each $P \in \widehat{\mathscr{A}}$, we have (from (36))

$$\mathbb{M}^* P = \coprod_{A \in \mathscr{A}} P(A) \odot \overline{\mathbb{M}}(A).$$

When we evaluate $\mathbb{M}^* P$ at $B \in \mathscr{B}$ we get

$$\begin{aligned}
\mathbb{M}^* P(B) &= \Big(\coprod_{A \in \mathscr{A}} P(A) \odot \overline{\mathbb{M}}(A) \Big)(B) \\
&= \bigwedge_{A \in \mathscr{A}} \big(P(A) \odot \overline{\mathbb{M}}(A) \big)(B) && \text{(Definition 6.6 - coproduct)} \\
&= \bigwedge_{A \in \mathscr{A}} P(A) \pitchfork \mathbb{M}(A, B) && \text{(Proposition 6.2(28))} \\
&= \bigwedge_{A \in \mathscr{A}} \big[P(A), \mathbb{M}(A, B) \big]. && \text{(Lemma 6.1)}
\end{aligned}$$

Similarly to before, we use the symmetry of $\otimes$ to think about the functor $\mathbb{M} : \mathscr{B} \otimes \mathscr{A}^{\mathrm{op}} \to \mathcal{V}$. Using the fact that $\boldsymbol{\mathcal{V}}\mathbf{Cat}$ is closed and the universal property (37), we obtain the $\mathcal{V}$-functor $\mathbb{M}_* : \check{\mathscr{B}} \to \widehat{\mathscr{A}}$ that makes the diagram commute.

$$\begin{tikzcd} \mathscr{B} \arrow[r, "H^\bullet"] \arrow[dr, "\overline{\mathbb{M}}"'] & \check{\mathscr{B}} \arrow[d, dashed, "\mathbb{M}_*"] \\ & \widehat{\mathscr{A}} \end{tikzcd}$$

For each $Q \in \check{\mathscr{B}}$, we have (from (38))

$$\mathbb{M}_* Q = \prod_{B \in \mathscr{B}} Q(B) \pitchfork \overline{\mathbb{M}}(B).$$

When we evalaute $\mathbb{M}_* Q$ at $A \in \mathscr{A}$ we get

$$\mathbb{M}_* Q(A) = \bigwedge_{B \in \mathscr{B}} \big[Q(B), \mathbb{M}(A, B)\big]. \tag{41}$$

Just like M^* and M_* satisfy $\langle M^*(\vec{v}), \vec{w}\rangle_{\mathbb{R}^n} = \langle \vec{v}, M_*(\vec{w})\rangle_{\mathbb{R}^m}$ because they are mutually transpose, $\mathbb{M}^*$ and $\mathbb{M}_*$ satisfy the following.

Proposition 7.1. *The $\mathcal{V}$-functors $\mathbb{M}^* : \widehat{\mathscr{A}} \to \check{\mathscr{B}}$ and $\mathbb{M}_* : \check{\mathscr{B}} \to \widehat{\mathscr{A}}$ are adjoint:*

$$\check{\mathscr{B}}(\mathbb{M}^* P, Q) = \widehat{\mathscr{A}}(P, \mathbb{M}_* Q)$$

for every $P \in \widehat{\mathscr{A}}$ and $Q \in \check{\mathscr{B}}$.

Proof. We evaluate both sides of the equation.

$$\begin{aligned}
\check{\mathscr{B}}(\mathbb{M}^* P, Q) &= \check{\mathscr{B}}\Big(\coprod_{A \in \mathscr{A}} P(A) \odot \overline{\mathbb{M}}(A), Q\Big) && \text{(from (36))} \\
&= \bigwedge_{A \in \mathscr{A}} \check{\mathscr{B}}\big(P(A) \odot \overline{\mathbb{M}}(A), Q\big) && \text{(Definition 6.6 - coproduct)} \\
&= \bigwedge_{A \in \mathscr{A}} \bigwedge_{B \in \mathscr{B}} \mathcal{V}\big(Q(B), \big(P(A) \odot \overline{\mathbb{M}}(A)\big)(B)\big) && \text{(natural transf. (23))} \\
&= \bigwedge_{A \in \mathscr{A}} \bigwedge_{B \in \mathscr{B}} \mathcal{V}\big(Q(B), P(A) \pitchfork \mathbb{M}(A, B)\big) && \text{(Proposition 6.2(28))} \\
&= \bigwedge_{A \in \mathscr{A}} \bigwedge_{B \in \mathscr{B}} \mathcal{V}\big(Q(B), [P(A), \mathbb{M}(A, B)]\big) && \text{(Lemma 6.1)} \\
&= \bigwedge_{A \in \mathscr{A}} \bigwedge_{B \in \mathscr{B}} \mathcal{V}\big(Q(B) \otimes P(A), \mathbb{M}(A, B)\big) && \text{($\mathcal{V}$ is closed (19))} \\
&= \bigwedge_{A \in \mathscr{A}} \bigwedge_{B \in \mathscr{B}} \mathcal{V}\big(P(A) \otimes Q(B), \mathbb{M}(A, B)\big). && \text{($\mathcal{V}$ is symmetric)}
\end{aligned}$$

Notice how in the third equality above, the quantities in the brackets switch places due to the 'op' in $\breve{\mathscr{B}} = [\mathscr{B}, \mathcal{V}]^{\mathrm{op}}$.

$$
\begin{aligned}
\widehat{\mathscr{A}}(P, \mathbb{M}_*Q) &= \widehat{\mathscr{A}}\Big(P, \prod_{B\in\mathscr{B}} Q(B) \pitchfork \overline{\mathbb{M}}(B)\Big) && \text{(from (38))}\\
&= \bigwedge_{B\in\mathscr{B}} \widehat{\mathscr{A}}\big(P, Q(B) \pitchfork \overline{\mathbb{M}}(B)\big) && \text{(Definition 6.5 - product)}\\
&= \bigwedge_{B\in\mathscr{B}} \bigwedge_{A\in\mathscr{A}} \mathcal{V}\left(P(A), \big(Q(B) \pitchfork \overline{\mathbb{M}}(B)\big)(A)\right) && \text{(natural transf. (23))}\\
&= \bigwedge_{B\in\mathscr{B}} \bigwedge_{A\in\mathscr{A}} \mathcal{V}\big(P(A), Q(B) \pitchfork \mathbb{M}(A, B)\big) && \text{(Proposition 6.1(26))}\\
&= \bigwedge_{B\in\mathscr{B}} \bigwedge_{A\in\mathscr{A}} \mathcal{V}\big(P(A), [Q(B), \mathbb{M}(A, B)]\big) && \text{(Lemma 6.1)}\\
&= \bigwedge_{B\in\mathscr{B}} \bigwedge_{A\in\mathscr{A}} \mathcal{V}\big(P(A) \otimes Q(B), \mathbb{M}(A, B)\big) && \text{($\mathcal{V}$ is closed (19))}\\
&= \bigwedge_{A\in\mathscr{A}} \bigwedge_{B\in\mathscr{B}} \mathcal{V}\big(P(A) \otimes Q(B), \mathbb{M}(A, B)\big). && \text{(meets commute)}
\end{aligned}
$$

The meets commute as a consequence of Proposition 6.2.8 in [Lei14, page 151]. Both sides are equal to $\bigwedge_{A\in\mathscr{A}} \bigwedge_{B\in\mathscr{B}} \mathcal{V}\big(P(A)\otimes Q(B), \mathbb{M}(A, B)\big)$. Therefore $\mathbb{M}^* \dashv \mathbb{M}_*$. □

All isomorphisms are equalities here since $\mathcal{V}$ is skeletal. The *nucleus of the profunctor* $\mathbb{M} : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathcal{V}$ is defined to be the invariant part of the adjunction $\mathbb{M}^* \dashv \mathbb{M}_*$, which is any of the three isomorphic subcategories $\mathrm{Fix}\,(\mathbb{M}_*\mathbb{M}^*)$, $\mathrm{Fix}\,(\mathbb{M}^*\mathbb{M}_*)$, $\mathrm{Inv}\,(\mathbb{M}^*, \mathbb{M}_*)$:

- $\mathrm{ob}\,(\mathrm{Fix}\,(\mathbb{M}_*\mathbb{M}^*)) = \{P \in \widehat{\mathscr{A}} : \mathbb{M}_*\mathbb{M}^*(P) = P\}$;
- $\mathrm{ob}\,(\mathrm{Fix}\,(\mathbb{M}^*\mathbb{M}_*)) = \{Q \in \breve{\mathscr{B}} : \mathbb{M}^*\mathbb{M}_*(Q) = Q\}$;
- $\mathrm{ob}\,(\mathrm{Inv}\,(\mathbb{M}^*, \mathbb{M}_*)) = \{(P, Q) \in \widehat{\mathscr{A}} \times \breve{\mathscr{B}} : \mathbb{M}^*(P) = Q, P = \mathbb{M}_*(Q)\}$.

8 Categorification

8.1 Galois connections and correspondences revisited

We now explain how we recover the content of Section 3 by enriching over $\mathcal{V} = \mathsf{Truth}$. Referring to Table 4 might be useful for what follows.

A set X can be thought of as a discrete preorder, where the relation is the identity; $x_1 \leqslant_X x_2 \iff x_1 = x_2$. This is the same thing as being a discrete Truth-category; $X(x_1, x_2) = \mathsf{false}$ if $x_1 \neq x_2$ and $X(x, x) = \mathsf{true}$. The same goes for the set Y.

With this preorder every set is both downward closed and upward closed, so a presheaf $S \in \widehat{X}$ and a copresheaf $T \in \check{Y}$ simply represent subsets $\tilde{S} \subseteq X$ and $\tilde{T} \subseteq Y$. Therefore the presheaf category $\widehat{X}$ is the set of subsets of X ordered by $\subseteq$, which is the same as $\mathcal{P}(X)$. Similarly, the opcopresheaf category $\check{Y}$ can be identified with $\mathcal{P}(Y)^{\mathrm{op}}$.

A profunctor $R : X^{\mathrm{op}} \otimes Y \to \mathsf{Truth}$ is a relation on the sets X and Y:

$$R(x, y) = \mathsf{true} \iff x \, R \, y \iff x \text{ is related to } y.$$

As explained in Section 7.1, from the profunctor R we obtain the two Truth-functors (6)

$$R^* : \widehat{X} \rightleftarrows \check{Y} : R_*, \text{ i.e.}$$
$$R^* : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y)^{\mathrm{op}} : R_*.$$

Recall from (41) that for every $T \in \check{Y}$, evaluating $R_*(T)$ at $x \in X$ gives

$$R_*(T)(x) = \bigwedge_{y \in Y} \big[T(y), R(x, y)\big].$$

In Truth, residuation is implication and the meet is logical 'and', so

$$\begin{aligned} R_*(T)(x) &= \lceil \forall \, y \in Y : T(y) \Rightarrow R(x, y) \rceil \\ &= \lceil \forall \, y \in Y : T(y) \Rightarrow x \, R \, y \rceil \end{aligned}$$

We identify the Truth-functor $T \in \check{Y}$ with the subset $\tilde{T} \in \mathcal{P}(Y)^{\mathrm{op}}$, where $y \in \tilde{T} \iff T(y) = \mathsf{true}$. Therefore $R_*(T)(x) = \mathsf{true} \iff x \, R \, t$ for all $t \in \tilde{T}$. So the associated set $\widetilde{R_*(T)} = R_*(T)^{-1}(\mathsf{true})$ is

$$\widetilde{R_*(T)} = \{x \in X : x \, R \, t \text{ for all } t \in \tilde{T}\}.$$

Similarly, for every $S \in \widehat{X}$ the set associated to $R^*(S)$ is

$$\widetilde{R^*(S)} = \{y \in Y : s \, R \, y \text{ for all } s \in \tilde{S}\}.$$

These are precisely the two functions arising from a relation R on X and Y described in Section 3.1

The Truth-functors R^* and R_* are adjoint by Proposition 7.1. Therefore

$$\check{Y}\left(R^*(S), T\right) = \widehat{X}\left(S, R_*(T)\right),$$

which is the Galois connection (7)

$$R^*(S) \supseteq T \iff S \subseteq R_*(T).$$

The nucleus of the profunctor R i.e. the invariant part of the adjunction $R^* \dashv R_*$ gives us a Galois correspondence (8) by Theorem 4.1:

$$R^* : \mathcal{P}_{\mathrm{cl}}(X) \cong \mathcal{P}_{\mathrm{cl}}(Y)^{\mathrm{op}} : R_*,$$

where we have identified $\mathrm{Fix}(R_* R^*) = \mathcal{P}_{\mathrm{cl}}(X)$ and $\mathrm{Fix}(R^* R_*) = \mathcal{P}_{\mathrm{cl}}(Y)^{\mathrm{op}}$.

8.2 The Legendre-Fenchel transform revisited

We now explain how we recover the content of Section 2 (and more) by enriching over $\mathcal{V} = \overline{\mathbb{R}}$. Again, referring to Table 4 might be useful for what follows.

A real vector space V can be thought of as a discrete $\overline{\mathbb{R}}$-space or $\overline{\mathbb{R}}$-category; $\mathrm{d}(x_1, x_2) = +\infty$ if $x_1 \neq x_2$ and $\mathrm{d}(x, x) = 0$. The same goes for the dual vector space V^*.

In this case every function is distance non-increasing, so a presheaf $f \in \widehat{V}$ and a copresheaf $g \in \widecheck{V^*}$ are simply $\overline{\mathbb{R}}$-valued functions. Therefore the presheaf category $\widehat{V}$ is the $\overline{\mathbb{R}}$-space of functions $V^{\mathrm{op}} \to \overline{\mathbb{R}}$ ordered pointwise by $\geqslant$; it is $\mathrm{Fun}(V, \overline{\mathbb{R}})$. Moreover, the metric on $\mathrm{Fun}(V, \overline{\mathbb{R}})$ is (29)

$$\mathrm{d}\left(f_1, f_2\right) = \sup_{x \in V}\{f_2(x) - f_1(x)\}.$$

Similarly, $\widecheck{V^*}$ can be identified with the $\overline{\mathbb{R}}$-space $\mathrm{Fun}(V^*, \overline{\mathbb{R}})$ and is ordered pointwise by $\leqslant$. The metric on $\mathrm{Fun}(V^*, \overline{\mathbb{R}})$ is (30)

$$\mathrm{d}\left(g_1, g_2\right) = \sup_{k \in V^*}\{g_1(k) - g_2(k)\}.$$

We identify the natural pairing between V and V^* as the profunctor (32)

$$\begin{aligned} \mathbb{L} : \quad V^{\mathrm{op}} \times V^* &\to \quad \overline{\mathbb{R}} \\ (x, k) \quad &\mapsto \quad \langle k, x \rangle. \end{aligned}$$

From Section 7.1 we obtain the two $\overline{\mathbb{R}}$-functors (3)

$$\mathbb{L}^* : \mathrm{Fun}(V, \overline{\mathbb{R}}) \rightleftarrows \mathrm{Fun}(V^*, \overline{\mathbb{R}}) : \mathbb{L}_*.$$

Recall that for every $f \in \mathrm{Fun}(V, \overline{\mathbb{R}})$, evaluating $\mathbb{L}^*(f)$ at $k \in V^*$ gives

$$\mathbb{L}^*(f)(k) = \bigwedge_{x \in V} \big[f(x), \mathbb{L}(x, k) \big].$$

In $\overline{\mathbb{R}}$, residuation is subtraction and the meet is sup, so

$$\begin{aligned} \mathbb{L}^*(f)(k) &= \sup_{x \in V} \{ \mathbb{L}(x, k) - f(x) \} \\ &= \sup_{x \in V} \{ \langle k, x \rangle - f(x) \}. \end{aligned}$$

Similarly for every $g \in \mathrm{Fun}(V^*, \overline{\mathbb{R}})$ we obtain

$$\mathbb{L}_*(g)(x) = \sup_{k \in V^*} \{ \langle k, x \rangle - g(k) \}.$$

These are precisely the Legendre-Fenchel transform and its inverse as defined in Section 2.2:

$$\begin{array}{cccc} \mathbb{L}^* : \ \mathrm{Fun}(V, \overline{\mathbb{R}}) & \rightleftarrows & \mathrm{Fun}(V^*, \overline{\mathbb{R}}) & : \mathbb{L}_* \\ f & \mapsto & f^*, & \\ g^* & \leftarrow\!\shortmid & g. & \end{array}$$

The $\overline{\mathbb{R}}$-functors $\mathbb{L}^*$ and $\mathbb{L}_*$ are adjoint by Proposition 7.1, meaning $\mathrm{d}\,(f^*, g) = \mathrm{d}\,(f, g^*)$, giving us the following theorem.

Theorem 8.1. *Let $f : V \to \overline{\mathbb{R}}$ and $g : V^* \to \overline{\mathbb{R}}$ be functions. Then*

$$\sup_{k \in V^*} \{ f^*(k) - g(k) \} = \sup_{x \in V} \{ g^*(x) - f(x) \}.$$

This is a stronger statement than the Galois connection (4).

Furthermore, $\mathbb{L}^* : \mathrm{Fun}(V, \overline{\mathbb{R}}) \to \mathrm{Fun}(V^*, \overline{\mathbb{R}})$ is a distance non-increasing function (22), so $\mathrm{d}(f_1, f_2) \geqslant \mathrm{d}(f_1^*, f_2^*)$.

Theorem 8.2. *Let $f_1, f_2 : V \to \overline{\mathbb{R}}$ be functions. Then*

$$\sup_{x \in V} \{ f_2(x) - f_1(x) \} \geqslant \sup_{k \in V^*} \{ f_1^*(k) - f_2^*(k) \}.$$

The nucleus of the profunctor $\mathbb{L}$, i.e. the invariant part of the adjunction $\mathbb{L}^* \dashv \mathbb{L}_*$ gives us the Legendre-Fenchel duality (5) by the Fenchel-Moreau theorem:

$$\mathbb{L}^* : \mathrm{Cvx}(V, \overline{\mathbb{R}}) \cong \mathrm{Cvx}(V^*, \overline{\mathbb{R}}) : \mathbb{L}_*,$$

where we have identified $\mathrm{Fix}(\mathbb{L}_*\mathbb{L}^*) = \mathrm{Cvx}(V, \overline{\mathbb{R}})$ and $\mathrm{Fix}(\mathbb{L}^*\mathbb{L}_*) = \mathrm{Cvx}(V^*, \overline{\mathbb{R}})$. This last theorem follows from this

Theorem 8.3 (Toland-Singer Duality)**.** *Let $f_1, f_2 : V \to \overline{\mathbb{R}}$ be functions and f_2 be proper convex and lower semi-continuous. Then*

$$\sup_{x \in V}\{f_2(x) - f_1(x)\} = \sup_{k \in V*}\{f_1^*(k) - f_2^*(k)\}.$$

Proof. We wish to show that $\mathrm{d}(f_1, f_2) = \mathrm{d}(f_1^*, f_2^*)$, from which the result follows. The function f_2 is proper convex and lower semi-continuous, so $f_2 = f_2^{**}$. Hence $\mathrm{d}(f_1, f_2) = \mathrm{d}(f_1, f_2^{**})$. Then, by Theorem 8.1, $\mathrm{d}(f_1, f_2^{**}) = \mathrm{d}(f_1^*, f_2^*)$. □

References

[Wil15] Simon Willerton. "The Legendre-Fenchel transform from a category theoretic perspective". In: (Jan. 2015). URL: https://arxiv.org/abs/1501.03791.

[Wil13] Simon Willerton. *The Nucleus of a Profunctor: Some Categorified Linear Algebra*. Aug. 2013. URL: https://golem.ph.utexas.edu/category/2013/08/the_nucleus_of_a_profunctor_so.html.

[Wil14a] Simon Willerton. *Galois Correspondences and Enriched Adjunctions*. Feb. 2014. URL: https://golem.ph.utexas.edu/category/2014/02/galois_correspondences_and_enr.html.

[Wil14b] Simon Willerton. *Enrichment and the Legendre–Fenchel Transform I*. Apr. 2014. URL: https://golem.ph.utexas.edu/category/2014/04/enrichment_and_the_legendrefen.html.

[Wil14c] Simon Willerton. *Enrichment and the Legendre–Fenchel Transform II*. May 2014. URL: https://golem.ph.utexas.edu/category/2014/05/enrichment_and_the_legendrefen_1.html.

[Bel14] Jordan Bell. *The Legendre transform*. Apr. 2014. URL: http://www.individual.utoronto.ca/jordanbell/notes/legendre.pdf.

[Tou05] Hugo Touchette. *Legendre-Fenchel transforms in a nutshell*. July 2005. URL: https://www.ise.ncsu.edu/fuzzy-neural/wp-content/uploads/sites/9/2019/01/or706-LF-transform-1.pdf.

[LL88] Hang-Chin Lai and Lai-Jui Lin. "The Fenchel-Moreau theorem for set functions". In: *Proceedings of the American Mathematical Society* (May 1988), pp. 85–90. URL: https://www.ams.org/journals/proc/1988-103-01/S0002-9939-1988-0938649-4/S0002-9939-1988-0938649-4.pdf.

[Lei14] Tom Leinster. *Basic Category Theory*. Cambridge University Press, Dec. 2014. URL: https://arxiv.org/abs/1612.09375.

[Che19] Evan Chen. *An Infinitely Large Napkin*. 2019, pp. 579–612. URL: https://web.evanchen.cc/napkin.html.