Completely Integrable Hamiltonian Systems: The Two-Centre Problem

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Abstract

In this report, we give a description of completely integrable Hamiltonian systems and present one of the most famous examples of such a system: the two-centre problem. At the start of the first chapter, we briefly introduce the Lagrangian and Hamiltonian formalism of classical mechanics. After, we describe the symplectic structure of a system's phase space and how one can express the equations describing the system's evolution geometrically. In the second chapter, we define what it means for a system to be completely integrable and state the main theorem of the report: the Liouville-Arnold theorem. In order to make sense of the theorem, the Hamilton-Jacobi method is presented and we show that it can be used to solve two classical completely integrable systems: the harmonic oscillator and the Kepler problem. In the third and final chapter, the two-centre problem is discussed as another example of completely integrable Hamiltonian systems. A method similar to the one used to solve the Kepler problem is employed to give an analytic solution of the two-centre problem involving elliptic functions and integrals; a special set of initial conditions are chosen to simplify the solution to the problem.

Declaration

We declare that this report was composed by us and that the work contained therein is our own, except where explicitly stated otherwise in the text.

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Chapter 1

Introduction

1.1 Lagrangian and Hamiltonian formalisms

The Lagrangian formulation of mechanics is based on the observation that there are variational principles behind the fundamental laws of mechanics given by Newton's law. [MR98, p. 2] When studying an n-particle system one chooses a configuration space M, a real $3n$-dimensional smooth manifold with local coordinates q^i corresponding to the position of each particle. Then one defines the Lagrangian $L\left(q(t), \dot{q}(t), t\right)$ as the kinetic energy minus the potential energy. Here dots denote differentiation with respect to time t, and $q(t)$ denotes the $3n$ coordinates $q^i(t)$ (similarly for $\dot{q}(t)$).

Hamilton's principle of least action states that the curve in configuration space $q : [t_a, t_b] \to M$ traced out by the system evolving over time extremizes the action functional, which is given by

$$S[q] = \int_{t_a}^{t_b} L(q(t), \dot{q}(t), t)\, \mathrm{d}t.$$

where $\dot{q} \equiv \frac{\mathrm{d}q}{\mathrm{d}t}$.

In other words, for any small variation $\varepsilon(t)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}s} S[q + s\varepsilon]\bigg|_{s=0} = 0. \tag{1.1}$$

In the simplest case where the variations are endpoint-fixed ($\varepsilon(t_a) = \varepsilon(t_b) = 0$), using the chain rule and integrating by parts gives

$$\int_{t_a}^{t_b} \sum_{i=1}^{3n} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) \varepsilon^i \, \mathrm{d}t = 0.$$

Since ε is arbitrary, the fundamental lemma of variational calculus [Har20, p. 8] yields the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, 3n. \tag{1.2}$$

The Lagrangian is a scalar field whose domain is the $6n$-dimensional tangent bundle of the configuration space, TM, and time.

$$L : TM \times \mathbb{R} \to \mathbb{R}$$

The tangent bundle is also called the velocity phase space, which in this case corresponds to the fact that one evaluates L at $(q, v) = (q, \dot{q}) \in TM$ in local coordinates.

In the Hamiltonian formalism one focuses on the cotangent bundle T^*M. To make the switch, one defines the conjugate momenta by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{1.3}$$

The Lagrangian L is regular when the matrix whose entries are given by $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is invertible. When this is the case, the implicit function theorem guarantees that (1.3) can be inverted to obtain $\dot{q}^i = \dot{q}^i(q, p, t)$. One then defines the Hamiltonian by taking the Legendre transform of L:

$$H(q, p, t) = \sum_{i=1}^{3n} p_i \dot{q}^i - L\left(q, \dot{q}, t\right),$$

where each $\dot{q}^i$ is regarded as a function of the positions and momenta. Then the Euler Lagrange equations (1.2) are equivalent to Hamilton's equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \tag{1.4}$$

and the Hamiltonian is a scalar field whose domain is the cotangent bundle of M and time:

$$H : T^*M \times \mathbb{R} \to \mathbb{R}.$$

In most cases, the Lagrangian is given by

$$L(q, \dot{q}) = \sum_{i=1}^{3n} \frac{1}{2} m \left(\dot{q}^i\right)^2 - V(q).$$

Computing the conjugate momenta and performing the Legendre transform gives the Hamiltonian

$$H(q, p) = \sum_{i=1}^{3n} \frac{p_i^2}{2m} + V(q).$$

In this case, the Hamiltonian function is equal to the total energy of the system and it remains constant over time. From now on, we only consider this type of Hamiltonian functions.

Note: from now on, the Einstein summation convention will be assumed unless otherwise stated.

1.2 The symplectic geometry of the cotangent bundle

The Hamiltonian formalism restricts our focus to the $6n$-dimensional cotangent bundle T^*M of the $3n$-dimensional configuration space M, known as the phase space. Here, we introduce the notion of symplectic geometry which is present on the phase space. To start with, we define the notion of a symplectic manifold [Chu18, p. 4] as in.

Definition 1.2.1 (Symplectic manifold)**.** *A symplectic manifold M is a manifold of even dimension equipped with a closed, non-degenerate 2-form, called the symplectic form.*

Following the notation in [Fig20], the cotangent bundle T^*M comes equipped with a tautological one-form $\theta \in \Omega^1(T^*M)$ defined by

$$\theta(\alpha) = \alpha \circ (\pi_*)_\alpha$$

for each $\alpha \in T^*M$. Here $\pi : T^*M \to M$ is the canonical projection map given by $(q^i, p_i) \mapsto q^i$ and $(\pi_*)_\alpha : T_\alpha(T^*M) \to T_{\pi(\alpha)}M$ is its derivative at α. We write $\theta = \lambda_i \mathrm{d}q^i + \mu^i \mathrm{d}p_i$ for some $\lambda_i, \mu^i \in C^\infty(T^*M)$, and solving for λ_i and μ^i gives $\theta = p_i \mathrm{d}q^i$.

We define a 2-form on T^*M by

$$\omega = -\mathrm{d}\theta \in \Omega^2(T^*M). \tag{1.5}$$

In local coordinates ω is given by

$$\begin{aligned}\omega &= -\mathrm{d}(p_i \mathrm{d}q^i) = -\mathrm{d}p_i \wedge \mathrm{d}q^i - p_i \wedge \mathrm{d}^2 q^i \\ &= \mathrm{d}q^i \wedge \mathrm{d}p_i,\end{aligned}$$

This allows us to conclude that ω is closed, since $\mathrm{d}\omega = \mathrm{d}^2 q^i \wedge \mathrm{d}p_i + \mathrm{d}q^i \wedge \mathrm{d}^2 p_i = 0$.

We denote the set of vector fields on T^*M as $\mathfrak{X}(T^*M)$ and define the associated map to ω as

$$\begin{array}{rccc}\omega^\flat : & \mathfrak{X}(T^*M) & \to & \Omega^1(T^*M), \\ & X & \mapsto & \iota_X \omega,\end{array}$$

where $\iota_X \omega$ denotes the interior product of X on ω. A more suggestive notation for $\iota_X\omega$ is $\omega(X,-)$, and we will use these interchangeably. The following lemma is exercise 1 in [Fig20].

Lemma 1.2.1. *The associated map $\omega^\flat$ is a $C^\infty(T^*M)$-module isomorphism.*

Proof. Consider a point $a \in T^*M$ and a tangent vector $X_a \in T_a(T^*M)$, which is generally given by $\lambda^i(a)\frac{\partial}{\partial q^i}\big|_a + \mu_i(a)\frac{\partial}{\partial p_i}\big|_a$ for $\lambda^i, \mu_i \in C^\infty(T^*M)$. Then

$$\begin{aligned}
\omega_a^\flat(X_a) &= \omega_a(X_a,-) \\
&= \big((\mathrm{d}q^i)_a \wedge (\mathrm{d}p_i)_a\big)\left(\lambda^j(a)\frac{\partial}{\partial q^j}\bigg|_a + \mu_j(a)\frac{\partial}{\partial p_j}\bigg|_a, -\right) \\
&= \lambda^i(a)(\mathrm{d}p_i)_a - \mu_i(a)(\mathrm{d}q^i)_a.
\end{aligned}$$

This shows that $\omega_a^\flat : T_a(T^*M) \to T_a^*(T^*M)$ is injective, and hence an isomorphism by the rank-nullity theorem. Therefore the vector space isomorphisms $\omega_a^\flat : T_a(T^*M) \to T_a^*(T^*M)$ for each $a \in T^*M$ arrange into a $C^\infty(T^*M)$-module isomorphism $\mathfrak{X}(T^*M) \to \Omega^1(T^*M)$. $\square$

A consequence of Lemma 1.2.1 is that ω is non-degenerate. Given $X \in \mathfrak{X}(T^*M)$, the associated map $\omega^\flat$ being an isomorphism implies that

$$\omega^\flat(X)(Y) = 0 \quad \forall\, Y \in \mathfrak{X}(T^*M) \iff X = 0.$$

This is the same statement as

$$\omega(X,Y) = 0 \quad \forall\, Y \in \mathfrak{X}(T^*M) \iff X = 0,$$

which is the condition for non-degeneracy of ω. Therefore, from Definition 1.2.1, $\omega \in \Omega^2(T^*M)$ is a symplectic form and T^*M is a symplectic manifold. Hence, from now on, by 'Hamiltonian system' we will mean the n-particle system described by the triple

$$(T^*M, \omega, H) \tag{1.6}$$

where T^*M is the $6n$-dimensional cotangent bundle of the configuration space M, $\omega \in \Omega^2(T^*M)$ is the symplectic form on T^*M defined above and $H \in C^\infty(T^*M)$ is the Hamiltonian function.

Another consequence of Lemma 1.2.1 is that given a smooth function $f \in C^\infty(T^*M)$, there is a unique vector field X_f such that $\iota_{X_f}\omega = df$. We can compare the coefficients of $\iota_{X_f}\omega$ and df to obtain a local expression for $X_f := \lambda^i\frac{\partial}{\partial q^i} + \mu_i\frac{\partial}{\partial p_i}$.

$$\begin{array}{ccccc}
\iota_{X_f}\omega & = & \lambda^i\mathrm{d}p_i & - & \mu_i\mathrm{d}q^i, \\
df & = & \frac{\partial f}{\partial p_i}\mathrm{d}p_i & + & \frac{\partial f}{\partial q^i}\mathrm{d}q^i.
\end{array}$$

Therefore

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}.$$

From this we give the following definition.

Definition 1.2.2 (Hamiltonian vector field)**.** *A vector field* $Y \in \mathfrak{X}(T^*M)$ *is* Hamiltonian *if* $Y = X_f$ *for some* $f \in C^\infty(T^*M)$*. Equivalently,* Y *satisfies* $\iota_Y\omega = df$ *for some* $f \in C^\infty(T^*M)$*.*

Finally, we wish to draw our attention to diffeomorphisms $\psi : T^*M \to T^*M$ that leave the symplectic form ω unchanged under a pull-back by ψ, which will become relevant in the next section. Explicitly, the pull-back $\psi^* : \Omega^2(T^*M) \to \Omega^2(T^*M)$ is a map defined by

$$(\psi^*\alpha)(X, Y)(a) = \alpha_{\psi(a)}\left((\psi_*)_a X_a, (\psi_*)_a Y_a\right),$$

where $\alpha \in \Omega^2(T^*M)$ is a 2-form, $X, Y \in \mathfrak{X}(T^*M)$ are vector fields, $a \in T^*M$ is a point in T^*M and $(\psi_*)_a : T_a(T^*M) \to T_{\psi(a)}(T^*M)$ is the derivative of ψ at a.

Definition 1.2.3 (Symplectomorphism)**.** *A diffeomorphism* $\psi : T^*M \to T^*M$ *is a* symplectomorphism *if*

$$\psi^*\omega = \omega.$$

1.3 A geometric treatment of classical systems

Now we reformulate the Hamiltonian equations in a more geometric way. Let $I \subset \mathbb{R}$ be an interval and consider a curve $c : I \to T^*M$, given by $c(t) = (q^1(t), \dots, q^{3n}(t), p_1(t), \dots, p_{3n}(t))$. If its components satisfy Hamilton's equations (1.4), the velocity of c is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = \left(\frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_{3n}}, -\frac{\partial H}{\partial q^1}, \dots, -\frac{\partial H}{\partial q^{3n}}\right). \tag{1.7}$$

The right-hand side of (1.7) can be written as

$$\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\right)c(t),$$

which from Definition 1.2.2 of a Hamiltonian vector field is equal to $X_H(c(t))$. Hence Hamilton's equations can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = X_H(c(t)). \tag{1.8}$$

If a curve $c : I \to T^*M$ satisfies (1.8), we say that it is an *integral curve* of the Hamiltonian vector field X_H. Additionally, by the definition of a Hamiltonian vector field after Lemma 1.2.1, we have

$$\iota_{X_H}\omega = \mathrm{d}H. \tag{1.9}$$

We use (1.8) and (1.9) to prove the conservation of energy [MR98, Proposition 2.4.5].

Proposition 1.3.1. *The function* $H \in C^\infty(T^*M)$ *is conserved along its integral curves.*

Proof. We compute $\frac{\mathrm{d}H}{\mathrm{d}t}$ along an integral curve $c(t)$.

$$\begin{aligned}
\frac{\mathrm{d}H(c(t))}{\mathrm{d}t} &= \mathrm{d}H_{c(t)}\left(\frac{\mathrm{d}c(t)}{\mathrm{d}t}\right) \\
&= (\iota_{X_H}\omega)_{c(t)}\left(\frac{\mathrm{d}c(t)}{\mathrm{d}t}\right) && \text{(by (1.8))} \\
&= (\iota_{X_H}\omega)_{c(t)}\big(X_H(c(t))\big) && \text{(by (1.9))} \\
&= \omega\big(X_H(c(t)), X_H(c(t))\big) && \text{(by definition of } \iota_{X_H}\omega) \\
&= 0. && \text{(by skew-symmetry of } \omega)
\end{aligned}$$

Hence H is conserved along $c(t)$. □

The proof above hints at how one can determine whether a smooth function $f \in C^\infty(T^*M)$ is a conserved quantity of a Hamiltonian system by computing Poisson brackets, which we define here.

Definition 1.3.1 (Poisson bracket)**.** *The Poisson bracket on the cotangent bundle is a skew-symmetric, $\mathbb{R}$-linear map*

$$\{-,-\} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$$

given by

$$\{f, g\} = \omega\left(X_f, X_g\right).$$

A function $f \in C^\infty(T^*M)$ is said to be a *conserved quantity* or an *integral* of the Hamiltonian system (1.6) if it is integral along integral curves of X_H. Below we give a characterisation of this in terms of Poisson brackets.

Proposition 1.3.2. *A smooth function f is a first integral of the Hamiltonian system if and only if the Poisson bracket $\{f, H\}$ is identically zero along integral curves of H.*

Proof. The proof is analogous to the one for Proposition 1.3.1. We compute $\frac{\mathrm{d}f}{\mathrm{d}t}$ along the integral curve $c(t)$.

$$\begin{aligned}
\frac{\mathrm{d}f(c(t))}{\mathrm{d}t} &= \mathrm{d}f_{c(t)}\left(\frac{\mathrm{d}c(t)}{\mathrm{d}t}\right) \\
&= \left(\iota_{X_f}\omega\right)_{c(t)}\big(X_H(c(t)\big) \\
&= \omega\big(X_f(c(t)), X_H(c(t))\big) \\
&= \{f, H\}(c(t))
\end{aligned}$$

Hence $\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \iff \{f, H\} = 0$ along $c(t)$. □

When the Poisson bracket of f and g is identically zero, we say that f and g are in *involution* or that they *Poisson commute.* One can check that for $f, g, h \in C^\infty(T^*M)$, the Poisson bracket satisfies *Jacobi's identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \tag{1.10}$$

This is shown in [MR98, Proposition 2.7.3]. This makes the vector space $C^\infty(T^*M)$ into a Lie algebra. Moreover, it implies that given integrals $f, g \in C^\infty(T^*M)$ of the Hamiltonian system (1.6), the Poisson bracket $\{f, g\}$ is also an integral by Proposition 1.3.2, since

$$\{\{f,g\},H\} = \{f,\{g,H\}\} + \{g,\{f,H\}\} = 0.$$

This means that it is possible to generate an infinite amount of integrals for a Hamiltonian system by repeatedly taking Poisson brackets. In the next chapter we will explain why this stops being useful at a certain point.

For computational purposes, it is useful to have a local coordinate expression for $\{f, g\}$. Firstly, notice that

$$\mathrm{d}q^i(X_f) = \mathrm{d}q^i\left(\frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j}\right) = -\frac{\partial f}{\partial p_i}.$$

Similarly,

$$\mathrm{d}p_i(X_f) = \frac{\partial f}{\partial q^i}.$$

So for $f, g \in C^\infty(T^*M)$, we have

$$\begin{aligned}\{f,g\} &= \omega\left(X_f, X_g\right)\\ &= \mathrm{d}q^i(X_f)\mathrm{d}p_i(X_g) - \mathrm{d}q^i(X_g)\mathrm{d}p_i(X_f)\\ &= \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.\end{aligned}$$

We make two remarks that follow from the coordinate expression of $\{f, g\}$. Firstly, Hamilton's equations (1.4) can be written as

$$\dot{q}^i = \{q^i, H\}, \quad \dot{p}_i = \{p_i, H\}. \tag{1.11}$$

Secondly, the coordinates (q, p) on T^*M satisfy the following relations:

$$\{q^i, q^j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q^i, p_j\} = \delta^i_j. \tag{1.12}$$

Coordinates (q, p) that satisfy (1.12) are called *Darboux coordinates.* A diffeomorphism $\psi : T^*M \to T^*M$ induces a change of variables $(q, p) \mapsto (Q, P)$, where $Q^i = q^i \circ \psi$ and $p_i = p_i \circ \psi$. One can demand that the new coordinates also satisfy the above relations to get the following definition, which is similar to [GPS00, p. 388].

Definition 1.3.2 (Canonical transformation)**.** *Given a local coordinate chart (q, p) on T^*M, a diffeomorphism $\psi : T^*M \to T^*M$ is a* canonical transformation *if*

$$\{Q^i, Q^j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q^i, P_j\} = \delta^i_j,$$

where $Q^i = q^i \circ \psi$ and $P_i = p_i \circ \psi$.

Since all smooth functions can be written as linear combinations of the local coordinates, an equivalent statement of the above definition given in [Jov11, p. 7] is that $\psi : T^*M \to T^*M$ is a canonical transformation if and only if

$$\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$$

for all $f, g \in C^\infty(T^*M)$.

We have outlined two properties that a diffeomorphism on T^*M can have; it can be a symplectomorphism (Definition 1.2.3), or it can be a canonical transformation (Definition 1.3.2). It turns out that these are equivalent, so we can use the terms interchangeably. Before proving this claim, we give a useful lemma presented in [MR98, Proposition 5.4.4].

Lemma 1.3.1. *Let $\psi : T^*M \to T^*M$ be a diffeomorphism and $f \in C^\infty(T^*M)$. Then ψ is a symplectomorphism if and only if*

$$X_{f\circ\psi} = \psi_*^{-1} X_f. \tag{1.13}$$

Proof. Suppose ψ is a symplectomorphism, and let $Y \in \mathfrak{X}(T^*M)$. We evaluate $\mathrm{d}(f \circ \psi)(Y)$ in two different ways. Firstly, by Definition 1.2.2 of a Hamiltonian vector field,

$$\begin{aligned}
\mathrm{d}(f \circ \psi)(Y) &= \left(\iota_{X_{f\circ\psi}}\omega\right)(Y) \\
&= \omega\left(X_{f\circ\psi},\, Y\right) \\
&= \left(\psi^*\omega\right)\left(X_{f\circ\psi},\, Y\right) \qquad (\psi \text{ is a symplectomorphism}) \\
&= \omega\left(\psi_* X_{f\circ\psi},\, \psi_* Y\right).
\end{aligned}$$

On the other hand, using the chain rule we get

$$\begin{aligned}
\mathrm{d}(f \circ \psi)(Y) &= \mathrm{d}f\left(\psi_* Y\right) \\
&= \iota_{X_f}\left(\psi_* Y\right) \qquad \text{(Definition 1.2.2)} \\
&= \omega\left(X_f,\, \psi_* Y\right).
\end{aligned}$$

So $\omega\left(\psi_* X_{f\circ\psi},\, \psi_* Y\right) = \omega\left(X_f,\, \psi_* Y\right)$ for all $Y \in \mathfrak{X}(T^*M)$. Then by nondegeneracy of ω we obtain

$$\psi_* X_{f\circ\psi} = X_f,$$

from which (1.13) follows. The other direction is proved by working the above proof backwards. $\square$

Proposition 1.3.3. *A diffeomorphism $\psi : T^*M \to T^*M$ is a symplectomorphism if and only if it is a canonical transformation.*

Proof. We wish to show that $\psi^*\omega = \omega$ if and only if for every $f, g \in C^\infty(T^*M)$ we have $\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$. As before, we assume that ψ is a symplectomorphism and prove one direction, and the other direction will follow from working the proof backwards. Let $a \in T^*M$. Then

$$\begin{aligned}
\{f \circ \psi, g \circ \psi\}(a) &= \omega\left(X_{f\circ\psi},\, X_{g\circ\psi}\right)(a) \\
&= \left(\psi^*\omega\right)\left(X_{f\circ\psi},\, X_{g\circ\psi}\right)(a) \qquad (\psi \text{ is a symplectomorphism}) \\
&= \left(\psi^*\omega\right)\left(\psi_*^{-1}X_f,\, \psi_*^{-1}X_g\right)(a) \qquad\qquad\qquad \text{(Lemma 1.3.1)} \\
&= \omega\left(\psi_*\psi_*^{-1}X_f,\, \psi_*\psi_*^{-1}X_g\right)(\psi(a)) \\
&= \omega\left(X_f,\, X_g\right)(\psi(a)) \\
&= \left(\{f, g\} \circ \psi\right)(a),
\end{aligned}$$

so $\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$ as needed. Notice that, for reasons of clarity, we have abused notation in the equations above by not writing explicitly that we have to evaluate everything at a. For example, the expression $\psi_*\psi_*^{-1}X_f$ should be understood as $\left(\psi_*\right)_a \left(\psi_*^{-1}\right)_{\psi(a)} \left(X_f\right)_a$. $\square$

A key property of canonical transformations is that they preserve the form of Hamilton's equations. More precisely, given a Hamiltonian system (1.6) and a canonical transformation $\psi : T^*M \to T^*M$, we obtain a new Hamiltonian $K = H \circ \psi$, often referred to as the 'Kamiltonian'. Using the form of Hamilton's equations given in (1.11), evaluating one of the partial derivatives of K gives

$$\begin{aligned}
\frac{\partial K}{\partial P_i} &= \{Q^i, K\} \\
&= \{q^i \circ \psi, H \circ \psi\} \\
&= \{q^i, H\} \circ \psi \qquad (\psi \text{ is a canonical transformation}) \\
&= \dot{q}^i \circ H \qquad\qquad\qquad\qquad \text{(from (1.11))} \\
&= \dot{Q}^i.
\end{aligned}$$

We obtain the other equation in a similar fashion. In conclusion, from Proposition 1.3.3 we learn that a diffeomorphism $\psi : T^*M \to T^*M$ that preserves the symplectic form ω is the same as a diffeomorphism that preserves the Poission relations (1.12), which in turn preserves Hamilton's equations (1.4).

1.4 Some remarks on the geometry of Hamiltonian systems

The sections above discussed a particular type of Hamiltonian system, which is the n-particle classical system where the quantum effects are ignored. As we stated previously, to construct a Hamiltonian system in a symplectic geometric context, all we need are

- a manifold equipped with a symplectic 2-form;
- a smooth real-valued function on the manifold.

This is more general than the classical definition, which only includes the systems determined by the two differential equations (1.4). Another example of a

Hamiltonian system is given by [MR98, p. 68]:

- a Hilbert space $\mathcal{H}$ with Hermitian inner product $\langle , \rangle$, equipped with a symplectic form defined by $\omega(\psi_1, \psi_2) = -2\hbar \ \mathrm{Im}(\psi_1, \psi_2)$, where $\psi_i \in \mathcal{H}$ are vectors in the Hilbert space;
- a smooth function $H : \mathcal{H} \to \mathbb{R}$.

The Hamiltonian vector field X_H satisfies

$$\langle iX_H\psi_1, \psi_2\rangle = \langle \psi_1, iX_H\psi_2\rangle,$$

which is saying that iX_H is a Hermitian operator. Then, from (1.8),

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = X_H(\psi).$$

Recognising that the vector field here is a linear operator of the Hilbert space, define $\hat{H} = i\hbar X_H$. Using Dirac notation, we obtain the general time-dependent Schrödinger equation:

$$i\hbar|\psi\rangle = \hat{H}|\psi\rangle$$

As undergraduates students, we have often been told that the dynamics in the quantum world is very different from the classical world. However, what we have outlined above shows that they still share the same symplectic structure.

It is also possible to formulate general relativity in a symplectic geometric context, as in [FS92]. This is due to the fact that geodesics can be understood as flows of a certain Hamiltonian vector field defined on the cotangent space of a manifold. Hence it is possible to find a symplectic form from the metric on a Riemannian manifold.

Chapter 2

Integrable Systems

2.1 Complete integrability and the Liouville-Arnold theorem

In the previous chapter, we explained how Jacobi's identity (1.10) on $C^\infty(T^*M)$ gives us a way of generating infinitely many integrals of the Hamiltonian system (1.6). Clearly, since T^*M is $6n$-dimensional, there cannot be more than $6n$ independent functions on T^*M. Therefore, repeatedly taking Poisson brackets eventually stops yielding new independent integrals of the system. To define what it means for functions on T^*M to be independent, we first outline a definition from [Asv90, p. 869] which says what it means for 1-forms on a manifold to be independent.

Definition 2.1.1. *Let $\alpha_1, \ldots, \alpha_k$ be 1-forms on T^*M. We say that they are* independent *at the point $(q,p) \in T^*M$ if $\alpha_1(q,p) \ldots, \alpha_k(q,p)$ span a k-dimensional subspace in the cotangent space $T^*_{(q,p)}(T^*M)$. The 1-forms are independent on T^*M if the points of T^*M where $\alpha_1, \ldots, \alpha_k$ are not independent form a set of measure zero.*

The last sentence involving the set of measure zero can be intuitively understood as 'the 1-forms are independent almost everywhere on T^*M'. From this we can describe independence of a collection of smooth functions.

Definition 2.1.2. *Let $f_1, \ldots, f_k$ be smooth functions on T^*M. We say that they are* independent *on T^*M if the 1-forms $df_1, \ldots, df_k$ are independent on T^*M.*

A useful statement that is equivalent to the above is that the smooth functions $f_1, \ldots, f_k$ are independent on T^*M if the wedge product $df_1 \wedge \cdots \wedge df_k$ is nonzero. We use this fact in Appendices B.1 and C.1 when showing that the Kepler problem and the two-centre problem are completely integrable.

The Liouville-Arnold theorem reveals that we do not need $6n$ independent functions on T^*M to 'solve' the system, but only $3n$. By 'solve', we mean to integrate in quadratures, which according to [Car+15, p. 3] means that you can determine the solutions by means of a finite number of algebraic operations and integrations of known functions. Before stating the theorem, we describe the complete integrability condition.

Definition 2.1.3 (Complete integrability). *Let $f_1, \dots, f_{3n} \in C^\infty(T^*M)$ be $3n$ independent smooth functions. Suppose that the functions are in involution:*

$$\{f_i, f_j\} = 0, \quad i, j = 1, \dots, 3n.$$

Then the Hamiltonian system (1.6) with Hamiltonian function $H = f_1$ is completely integrable.

We remark that in general, it is hard to find completely integrable Hamiltonian systems. Usually, a generic system will not satisfy the above conditions, but it will split into the sum of a completely integrable part and a perturbation. Moreover, the definition does not hint at the existence of any systematic way of finding each integral. Nevertheless, when one actually finds a completely integrable system, the following theorem becomes very useful. We state in a similar way to [Arn78, p. 272] and [Jov11, p. 8].

Theorem 2.1.1 (Liouville-Arnold). *Suppose we have a completely integrable system (1.6) with smooth Poisson-commuting integrals $f_1, \dots, f_{3n} \in C^\infty(T^*M)$. Given real numbers $c_1, \dots, c_{3n}$, consider the level set given by*

$$(T^*M)_c = \{(q, p) \in T^*M : f_i(q, p) = c_i, \ i = 1, \dots, 3n\}.$$

Then

1. *$(T^*M)_c$ is a smooth submanifold of T^*M.*
2. *If $(T^*M)_c$ is compact and connected, then it is diffeomorphic to the $3n$-dimensional torus.*
3. *In a neighbourhood of $(T^*M)_c$ there exists a canonical transformation to coordinates (φ, I), called* action-angle variables*, such that the level sets of the actions $I_1, \dots, I_{3n}$ are invariant tori and the Kamiltonian $K = K(I_1, \dots, I_{3n})$ only depends on the actions. Thus, Hamilton's equations are linearized:*
$$\dot{I}_i = 0, \quad \dot{\varphi}_i = \frac{\partial K}{\partial I_i} = \omega_i(I).$$
4. *Hamilton's equations can be integrated by quadratures.*

Proof. See [Arn78, chapter 10]. □

2.2 The Hamilton-Jacobi method

The third point in the Liouville-Arnold theorem guarantees that for a Hamiltonian system (1.6) with enough integrals in involution, one can find a canonical transformation to action-angle variables (φ, I) so that the Kamiltonian $K = K(I_1, \dots, I_{3n})$ only depends on the actions. In this section we discuss one of the methods of finding such a transformation, known as the Hamilton-Jacobi method. For the moment we consider cases where H can depend on t. Even though we will not need such a high level of generality in the next chapter, doing this will help to highlight some of the features of the Hamilton-Jacobi method.

2.2.1 Generating functions

Firstly, a common way of writing Hamilton's principle (1.1) is presented in [Ste16, p. 61]:

$$\delta \int_{t_a}^{t_b} \big(p_i \dot{q}^i - H(q,p,t)\big)\,\mathrm{d}t = 0. \tag{2.1}$$

In order for the diffeomorphism $(q,p) \mapsto (Q,P)$ to be a canonical transformation, the new coordinates must also obey Hamilton's equations, so they also obey Hamilton's principle (2.1):

$$\delta \int_{t_a}^{t_b} \Big(P_i \dot{Q}^i - K(Q,P,t)\Big)\,\mathrm{d}t = 0.$$

For this to be true, we require

$$p_i \dot{q}^i - H = P_i \dot{Q}^i - K + \dot{F}, \tag{2.2}$$

for sume function $F \in C^\infty(T^*M)$, which is called the *generating function* of the canonical transformation. The generating function will depend on some mixture of the old and new variables (q,p) and (Q,P) and will have to satisfy (2.2). There are four cases of this, and we highlight one of them.

When $F = F_2(q,P,t)$, its total time derivative is

$$\dot{F} = \frac{\partial F_2}{\partial q^i}\dot{q}^i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t}.$$

Therefore

$$p_i \dot{q}^i - H = P_i \dot{Q}^i - K + \frac{\partial F_2}{\partial q^i}\dot{q}^i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t}.$$

The old and new variables are assumed to be independent, so for both sides to equal we require

$$p_i = \frac{\partial F_2}{\partial q^i}.$$

However, to cancel the remaining terms we need to subtract $Q^i P_i$ from F, so that

$$\dot{F} = \frac{\partial F_2}{\partial q^i}\dot{q}^i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t} - \dot{Q}^i P_i - Q^i \dot{P}_i,$$

and so

$$Q^i = \frac{\partial F_2}{\partial P_i}.$$

Moreover, we obtain an expression for K in terms of H:

$$K = H + \frac{\partial F_2}{\partial t}.$$

The function F_2 is referred to as a *type-2* generating function. We include 3 other types in the table below.

Function	Transformations
$F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q^i},\ P_i = -\frac{\partial F_1}{\partial Q^i}$
$F_2(q, P, t) - Q^i P_i$	$p_i = \frac{\partial F_2}{\partial q^i},\ Q^i = \frac{\partial F_2}{\partial P_i}$
$F_3(p, Q, t) + q^i p_i$	$q^i = -\frac{\partial F_3}{\partial p_i},\ P_i = -\frac{\partial F_3}{\partial Q^i}$
$F_4(p, P, t) + q^i p_i - Q^i P_i$	$q^i = -\frac{\partial F_4}{\partial p_i},\ Q^i = \frac{\partial F_4}{\partial P_i}$

Table 2.1: Type-n generating functions. Adapted from [Ste16, p. 64].

For each generating function F_i, the relationship between the old and new Hamiltonian is always the same:

$$K = H + \frac{\partial F_i}{\partial t}. \tag{2.3}$$

2.2.2 Hamilton's principal function

We can ensure that the transformed coordinates (Q, P) are all constant by demanding that K be a constant, which we may set to zero. If F is a type-2 generating function, then (2.3) becomes

$$H\left(q^1, \ldots, q^{3n}, \frac{\partial F_2}{\partial q^1}, \ldots, \frac{\partial F_2}{\partial q^{3n}}, t\right) + \frac{\partial F_2}{\partial t} = 0, \tag{2.4}$$

where we used $p_i = \frac{\partial F_2}{\partial q^i}$. This is known as the *Hamilton-Jacobi equation*. It is a partial differential equation in $3n + 1$ variables $q^1, \ldots, q^{3n}, t$ for the generating function, whose solution (if it exists) is denoted by S and is called *Hamilton's principal function*. Following the discussion in [GPS00, p. 432], a complete solution to (2.4) can be written in the form

$$S = S(q, \alpha, t),$$

where $\alpha_1, \ldots, \alpha_{3n}$ are independent constants of integration that are not solely additive (physically, these correspond to the integrals of the system). At this point, we may take the new momenta P_i to be these constants of integration. Then, given initial conditions for (q, p) at time t_0, we can find the value of α_i by evaluating each side of

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q^i} \tag{2.5}$$

at t_0. The other transformation given in Table 2.1 is

$$Q^i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}, \tag{2.6}$$

which evaluated at t_0 gives us the values of the new constant coordinates $Q^i = \beta^i$. Inverting (2.6) and finally evaluating (2.5) gives us the original phase space coordinates $q^i = q^i(\beta, \alpha, t)$ and $p_i = p_i(\beta, \alpha, t)$ as a function of t, solving Hamilton's

equations.
From above, we see that solving the Hamilton-Jacobi equation corresponds to solving Hamilton's equations. Hence the Hamilton-Jacobi method establishes an equivalence between the $6n$ first-order differential equations (1.4) and the first-order partial differential equation in $3n + 1$ variables (2.4). We now look at the case where the Hamiltonian does not depend explicitly on time.

2.2.3 Hamilton's characteristic function

When H does not depend explicitly on time, we seek a solution to the Hamilton-Jacobi equation of the form

$$S(q, \alpha, t) = W(q, \alpha) - g(\alpha, t). \tag{2.7}$$

The function $W(q, \alpha)$ is referred to as *Hamilton's characteristic function*, while $g(\alpha, t)$ is a function to be determined. Putting this ansatz in (2.4) gives

$$H\left(q^1, \ldots, q^{3n}, \frac{\partial W}{\partial q^1}, \ldots, \frac{\partial W}{\partial q^{3n}}\right) = \frac{\partial g}{\partial t}.$$

The left side is independent of t while the right side is not, so they must equal a separation constant α_1 which corresponds to the constant energy H. Hence $g(\alpha, t) = \alpha_1 t$. The new equation

$$H\left(q^1, \ldots, q^{3n}, \frac{\partial W}{\partial q^1}, \ldots, \frac{\partial W}{\partial q^{3n}}\right) = \alpha_1 \tag{2.8}$$

is referred to as the *time-independent Hamilton-Jacobi equation.* Similarly to before, we may set the new momenta P_i equal to α_i. Then, since the Kamiltonian satisfies $K = \alpha_1$, the transformed Hamilton equations for the coordinates Q^i become

$$\dot{Q}^1 = \frac{\partial K}{\partial \alpha_1} = 1$$

and, for $i = 2, \ldots, 3n$ we have

$$\dot{Q}^i = \frac{\partial K}{\partial \alpha_i} = 0.$$

So $Q^1 = t + \beta^1$ and $Q^i = \beta^i$ for $i = 2, \ldots, 3n$. Then one can solve for the original coordinates (q, p) by writing $S = W - \alpha_1 t$ and proceeding in a similar way to the previous section.

Once again, the problem of solving $6n$ first-order differential equations for $(q(t), p(t))$ is translated into the problem of solving a first-order partial differential equation in $3n$ variables for the characteristic function W. We have shown that once one does this, the Hamiltonian obtained from the transformation generated by W depends only on the new momenta $P_i = \alpha_i$, and so the equations of motion are linearized. The Liouville-Arnold Theorem guarantees that such an transformation does indeed exist for systems with sufficient independent integrals of motion, of which we give an example now.

2.2.4 A simple example: the harmonic oscillator

As in [Ste16, p. 75], we consider the one-dimensional harmonic oscillator as an example. The Hamiltonian is

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

The time-independet Hamilton-Jacobi equation (2.8) becomes

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = \alpha,$$

which upon rearrangement and integration yields

$$W = \pm \int \sqrt{2m\alpha - m^2\omega^2 q^2}\,\mathrm{d}q.$$

Then Hamilton's principal function is

$$S = -\alpha t \pm \int \sqrt{2m\alpha - m^2\omega^2 q^2}\,\mathrm{d}q.$$

We may now solve for the transformed position $Q = \beta$ by using (2.6) and differentiating inside the integral sign:

$$\begin{aligned}\beta = \frac{\partial S}{\partial \alpha} &= -t \pm \int \frac{\partial}{\partial \alpha}\sqrt{2m\alpha - m^2\omega^2 q^2}\,\mathrm{d}q \\ &= -t \pm m \int \frac{\mathrm{d}q}{\sqrt{2m\alpha - m^2\omega^2 q^2}} \\ &= -t \pm \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m\omega^2}{2\alpha}}q\right),\end{aligned}$$

where we have absorbed the constant of integration into β. We can invert this to solve for the original position q:

$$q = \pm\sqrt{\frac{2\alpha}{m\omega^2}}\sin\left(\omega(t+\beta)\right). \tag{2.9}$$

Finally, we use (2.5) and (2.9) to obtain p:

$$\begin{aligned}p = \frac{\partial S}{\partial q} &= \pm\sqrt{2m\alpha - m^2\omega^2 q^2} \\ &= \pm\sqrt{2m\alpha}\cos\left(\omega(t+\beta)\right).\end{aligned}$$

The constants α and β relate to the initial conditions $(q(t_0), p(t_0))$. We may shift β appropriately to take the positive square roots, giving us the the expected solution of the simple harmonic oscillator with energy $\alpha = E$.

2.3 The Kepler problem

Before presenting the two-centre problem in the next chapter, we give a simpler version of it that serves as good practice for solving completely integrable systems; the Kepler problem.

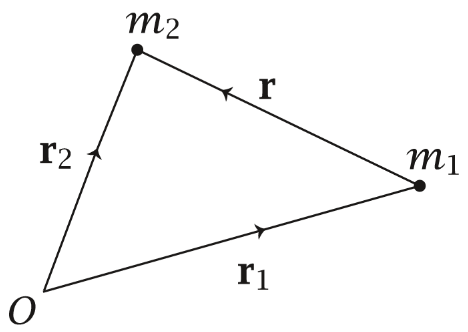

Figure 2.1: The Kepler problem. Adapted from [Mat08, p. 33].

The setup is as above. We consider the motion of two bodies of mass m_1 and m_2 moving under the influence of their mutual gravitational attraction. We denote their respective positions from the origin O at time t by the vectors $\boldsymbol{r}_1(t) = (x_1(t), y_1(t), z_1(t))$ and $\boldsymbol{r}_2(t) = (x_2(t), y_2(t), z_2(t))$. We assume that the bodies cannot occupy the same point at the same time, so that the configuration space of the system is $\mathbb{R}^3 \times \mathbb{R}^3$ minus the diagonal. The potential is

$$V = -\frac{\mu}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|},$$

where $\mu = Gm_1m_2$ and $G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is the gravitational constant. Hence the Lagrangian of the system is

$$L = \frac{1}{2}m_1\|\dot{\boldsymbol{r}}_1\|^2 + \frac{1}{2}m_2\|\dot{\boldsymbol{r}}_2\|^2 + \frac{\mu}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|}.$$

The potential depends on the positions of both bodies, so we introduce a transformation on configuration space $(\boldsymbol{r}_1, \boldsymbol{r}_2) \mapsto (\boldsymbol{R}, \boldsymbol{r})$ so that V only depends on $\boldsymbol{r}$. This is given by

$$\begin{aligned} \boldsymbol{R} &= \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2, \\ \boldsymbol{r} &= \boldsymbol{r}_1 - \boldsymbol{r}_2, \end{aligned}$$

where $M = m_1 + m_2$ is the total mass of the system. The Lagrangian can then be written as

$$L = \frac{1}{2}M\|\dot{\boldsymbol{R}}\|^2 + \frac{1}{2}m\|\dot{\boldsymbol{r}}\|^2 + \frac{\mu}{\|\boldsymbol{r}\|},$$

where $\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$ is the reduced mass of the system. The Euler-Lagrange equation (1.2) for the centre of mass $\boldsymbol{R}$ is simply $\ddot{\boldsymbol{R}} = 0$, which is solved by

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + t\dot{\boldsymbol{R}}(0).$$

Hence from now on, we focus on the nontrivial relative motion in 6-dimensional velocity phase space with coordinates $(\boldsymbol{r}, \dot{\boldsymbol{r}}) = (x, y, z, \dot{x}, \dot{y}, \dot{z})$ described by the Lagrangian

$$L = \frac{1}{2}m\|\dot{\boldsymbol{r}}\|^2 + \frac{\mu}{\|\boldsymbol{r}\|}.$$

This is spherically symmetric, so we introduce the transformation to spherical coordinates $(x, y, z) \mapsto (r, \theta, \varphi)$. Then

$$\begin{aligned}
\boldsymbol{r} &= (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) := r\boldsymbol{e}_r,\\
\dot{\boldsymbol{r}} &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\sin\theta\dot{\varphi}\boldsymbol{e}_\varphi,
\end{aligned}$$

where $\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{r}}{\partial \theta} / \left\|\frac{\partial \boldsymbol{r}}{\partial \theta}\right\|$ and $\boldsymbol{e}_\varphi = \frac{\partial \boldsymbol{r}}{\partial \varphi} / \left\|\frac{\partial \boldsymbol{r}}{\partial \varphi}\right\|$. Then the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) + \frac{\mu}{r}.$$

The conjugate momenta are

$$\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r},\\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},\\
p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi},
\end{aligned}$$

and so the Hamiltonian is given by

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\varphi^2\right) - \frac{\mu}{r} := E.$$

The time-independent Hamilton-Jacobi equation (2.8) is

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial W}{\partial \varphi}\right)^2\right] - \frac{\mu}{r} = \alpha_r. \tag{2.10}$$

We seek a separable solution as in [GPS00, p. 450]:

$$W = W_r(r, \alpha) + W_\theta(\theta, \alpha) + W_\varphi(\varphi, \alpha)$$

with $\alpha = (\alpha_r, \alpha_\theta, \alpha_\varphi)$. The first of the new momenta α_r is the constant energy E.

Also, p_φ is a constant because $\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$. Then we choose $\alpha_\varphi = p_\varphi$, which gives

$$\frac{\partial W_\varphi}{\partial \varphi} = p_\varphi \implies W_\varphi = p_\varphi \varphi.$$

Substituting this into (2.10) and rearranging, we obtain

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta} = r^2\left(2m\left(E + \frac{\mu}{r}\right) - \left(\frac{\partial W_r}{\partial r}\right)^2\right). \tag{2.11}$$

The left-hand side of (2.11) depends only on θ and the right-hand side on r, so each side must be constant. Therefore we may define α_θ as

$$\alpha_\theta^2 = \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta}, \tag{2.12}$$

which reduces (2.10) into

$$\left(\frac{\partial W_r}{\partial r}\right)^2 = 2m\left(E + \frac{\mu}{r}\right) - \frac{\alpha_\theta^2}{r^2}. \tag{2.13}$$

Equations (2.12) and (2.13) give W_θ and W_r as indefinite integrals:

$$\begin{aligned} W_\theta &= \int \sqrt{\alpha_\theta^2 - \frac{p_\varphi^2}{\sin^2\theta}}\,\mathrm{d}\theta, \\ W_r &= \int \sqrt{2m\left(E + \frac{\mu}{r}\right) - \frac{\alpha_\theta^2}{r^2}}\,\mathrm{d}r. \end{aligned}$$

Thus Hamilton's principal function (2.7) is

$$S = W - Et = p_\varphi\varphi + \int \sqrt{\alpha_\theta^2 - \frac{p_\varphi^2}{\sin^2\theta}}\,\mathrm{d}\theta + \int \sqrt{2m\left(E + \frac{\mu}{r}\right) - \frac{\alpha_\theta^2}{r^2}}\,\mathrm{d}r - Et.$$

From (2.6) we find expressions for the new coordinates $(\beta^r, \beta^\theta, \beta^\varphi)$, which can later be calculated given a set of initial conditions $(r_0, \theta_0, \varphi_0) = (r(t_0), \theta(t_0), \varphi(t_0))$.

$$\begin{aligned} \beta^r &= \frac{\partial S}{\partial \alpha_r} = \frac{\partial S}{\partial E} \\ &= -t + \frac{\partial}{\partial E}\sqrt{2m\left(E + \mu/r\right) - \alpha_\theta^2/r^2}\,\mathrm{d}r \\ &= -t + m\int \frac{\mathrm{d}r}{\sqrt{2m(E + \mu/r) - \alpha_\theta^2/r^2}} \\ &= -t - m\int \frac{\mathrm{d}u}{u^2\sqrt{2mE + 2m\mu u - \alpha_\theta^2 u^2}}. \qquad (u = 1/r) \end{aligned}$$

The integral can be solved by elementary methods [GR14, section 2.269] to give

$t = t(r)$. Next,

$$
\begin{aligned}
\beta^\varphi &= \frac{\partial S}{\partial \alpha_\varphi} = \frac{\partial S}{\partial p_\varphi} \\
&= \varphi + \frac{\partial}{\partial p_\varphi} \int \sqrt{\alpha_\theta^2 - p_\varphi^2 / \sin^2\theta}\, \mathrm{d}\theta \\
&= \varphi - p_\varphi \int \frac{\mathrm{d}\theta}{\sin^2\theta \sqrt{\alpha_\theta^2 - p_\varphi^2/\sin^2\theta}} \\
&= \varphi + \frac{1}{2} \int \frac{\mathrm{d}u}{u\sqrt{(\alpha_\theta/p_\varphi)^2 - 1 - u^2}}, \qquad\qquad (u = \cot^2\theta)
\end{aligned}
$$

which can be solved [GR14, section 2.275] for $\varphi = \varphi(\theta)$ and can be inverted to find $\theta = \theta(\varphi)$. Finally,

$$
\begin{aligned}
\beta^\theta &= \frac{\partial S}{\partial \alpha_\theta} \\
&= \frac{\partial}{\partial \alpha_\theta} \int \sqrt{\alpha_\theta^2 - p_\varphi^2/\sin^2\theta}\, \mathrm{d}\theta + \frac{\partial}{\partial \alpha_\theta} \int \sqrt{2m\,(E + \mu/r) - \alpha_\theta^2/r^2}\, \mathrm{d}r \\
&= \alpha_\theta \int \frac{\mathrm{d}\theta}{\sqrt{\alpha_\theta^2 - p_\varphi^2/\sin^2\theta}} - \alpha_\theta \int \frac{\mathrm{d}r}{r^2\sqrt{2m\,(E+\mu/r) - \alpha_\theta^2/r^2}} \\
&= -\frac{\alpha_\theta}{2p_\varphi} \int \frac{\mathrm{d}u}{u(u+1)\sqrt{\left(\alpha_\theta/p_\varphi^2\right)^2 - 1 - u^2}} + \alpha_\theta \int \frac{\mathrm{d}u}{\sqrt{2mE + 2m\mu u - \alpha_\theta^2 u^2}}\, \mathrm{d}u,
\end{aligned}
$$

where we used the same substitutions as above. Once again, these integrals can be solved [GR14, sections 2.282 and 2.261] to give $r = r(\theta) = r(\theta(\varphi))$. We can see that all the integrals involved in the solution of the Kepler problem are of the following form:

$$
\int \frac{\mathrm{d}u}{u^n(u+1)^m\sqrt{au^2 + bu + c}}, \tag{2.14}
$$

where $n, m = 0, 1, 2$. These can all be solved by elementary methods as shown in section 2.2 of [GR14]. In the next chapter, we will see that the two-centre problem gives rise to integrals with quartic polynomials in the denominator, whose solutions are given in section 5.1 of [GR14].

The new momenta α_θ and $\alpha_\varphi = p_\varphi$ are related to the angular momentum of the system, which is given by

$$
\begin{aligned}
\boldsymbol{L} &= \boldsymbol{r} \times (m\dot{\boldsymbol{r}}) \\
&= r\boldsymbol{e}_r \times (m\dot{r}\boldsymbol{e}_r + mr\dot{\theta}\boldsymbol{e}_\theta + mr\sin\theta\dot{\varphi}\boldsymbol{e}_\varphi) \\
&= r\boldsymbol{e}_{\boldsymbol{r}} \times (p_r\boldsymbol{e}_r + \frac{p_\theta}{r}\boldsymbol{e}_\theta + \frac{p_\varphi}{r\sin\theta}\boldsymbol{e}_\varphi) \\
&= p_\theta\boldsymbol{e}_\varphi - \frac{p_\varphi}{\sin\theta}\boldsymbol{e}_\theta.
\end{aligned}
$$

Thus the z component is $L_z = p_\varphi = \alpha_\varphi$ and the magnitude is

$$\ell^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} = \left(\frac{\partial W_\theta}{\partial\theta}\right)^2 + \frac{\alpha_\varphi^2}{\sin^2\theta} = \alpha_\theta^2.$$

From this we see that ℓ^2 and p_φ are conserved quantities of the system. In Appendix B.1 we show that the integrals H, ℓ^2 and p_φ are independent and in involution, which by the Definition 2.1.3 makes the Kepler problem completely integrable. Then the Liouville-Arnold theorem guarantees that we can transform to action-angle variables; for a bounded system ($E < 0$), the action variables can be calculated in [Hor20, p. 80]:

$$\begin{aligned}
J_r &= \frac{1}{2\pi}\oint \frac{\partial W_r}{\partial r}\mathrm{d}r = \frac{1}{2\pi}\oint \sqrt{2m(E+\mu/r) - \alpha_\theta^2/r^2}\,\mathrm{d}r = -\alpha_\theta + \frac{\mu}{2}\sqrt{\frac{2m}{|E|}},\\
J_\theta &= \frac{1}{2\pi}\oint \frac{\partial W_\theta}{\partial\theta}\mathrm{d}\theta = \frac{1}{2\pi}\oint \sqrt{\alpha_\theta^2 - \alpha_\varphi^2/\sin^2\theta}\,\mathrm{d}\theta = \alpha_\theta - \alpha_\varphi,\\
J_\varphi &= \alpha_\varphi.
\end{aligned}$$

Summing these up gives

$$J_r + J_\theta + J_\varphi = \frac{\mu}{2}\sqrt{\frac{2m}{|E|}},$$

so the energy is

$$H = -|E| = -\frac{m\mu^2}{2(J_r + J_\theta + J_\varphi)^2}.$$

By the Hamilton's equations, the velocities of the angle variables $\omega_i = \dot{\varphi}_i = \frac{\partial H}{\partial J_i}$ are all the same:

$$\omega_i = \frac{\partial H}{\partial J_r} = \frac{\partial H}{\partial J_\theta} = \frac{\partial H}{\partial J_\varphi} = \frac{2}{\mu}\sqrt{\frac{2|E|^3}{m}}.$$

It can be shown that the semi-major axis a of the elliptic orbit is $a = \mu/(2|E|)$, from which we obtain *Kepler's third law*:

$$T^2 = \left(\frac{2\pi}{\omega}\right)^2 = \frac{\pi^2 m}{2\mu}\frac{\mu^3}{|E|^3} = \frac{4\pi^2 m}{\mu}a^3.$$

There is actually another conserved quantity of the system that is of interest, which is the Laplace-Runge-Lenz vector

$$\boldsymbol{A} = m\dot{\boldsymbol{r}} \times \boldsymbol{L} - m\mu\boldsymbol{e_r}. \tag{2.15}$$

In Appendix B.1 we show that the three phase space functions $H, p_\varphi, \|\boldsymbol{A}\|^2$ are in involution, where $\|\boldsymbol{A}\|^2$ is given by

$$\|\boldsymbol{A}\|^2 = m^2\mu^2 + 2mE\ell^2.$$

Therefore we may use these three as an alternative to H, ℓ^2, p_φ to show that the Kepler problem is completely integrable. Moreover,

$$\boldsymbol{A} \cdot \boldsymbol{L} = 0.$$

Hence the Laplace-Runge-Lenz vector contributes only one additional independent conserved quantity. So the Kepler system has five independent first integrals: energy, three components of angular momentum and one component of the Laplace-Runge-Lenz vector. In three degrees of freedom, the existence of five globally defined and functionally independent integrals of motion (called *maximal superintegrability*) immediately implies that all finite trajectories are closed, as stated in [Eva90, p. 5674].

Chapter 3

The Two-Centre Problem

A concise description of the main problem that we will be considering from now on is given in [WDR04, p. 266]. The two-centre problem was first considered by Euler in 1760; it describes the motion of a test particle in the field of two space fixed (unmoving) Newtonian centres of attraction.

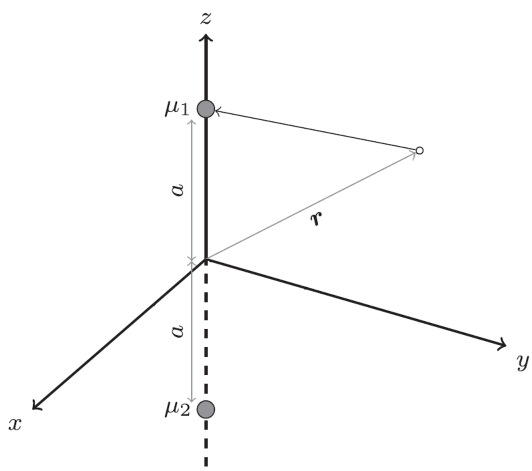

Figure 3.1: Setup of the two-centre problem. Adapted from [BI16, p. 3481].

To begin with, we orient the coordinate axes so that the two fixed centres are found at

$$\boldsymbol{f}_1 = (0, 0, -a), \quad \boldsymbol{f}_2 = (0, 0, +a).$$

The i^{th} fixed centre gives rise to a potential of the form

$$-\frac{\mu_i}{r_i},$$

where $r_i = \|\boldsymbol{r} - \boldsymbol{f}_i\|$ and $\mu_i = GM_i m > 0$, where M_i is the mass of the i^{th} centre and m is the mass of the test particle. The Lagrangian of the system is given by

$$L = T - V = \frac{1}{2}m\|\dot{\boldsymbol{r}}\|^2 + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2},$$

and since the potential is velocity independent, the Hamiltonian is

$$H = T + V = \frac{\|\boldsymbol{p}\|^2}{2m} - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}.$$

For convenience, we rescale the variables (r, p) and the Hamiltonian H as in [WDR04, p. 268].

$$\begin{aligned} r &\longmapsto \frac{r}{a}, \\ p &\longmapsto \frac{p}{\sqrt{m(\mu_1 + \mu_2)/a}}, \\ H &\longmapsto \frac{H}{(\mu_1 + \mu_2)/a}. \end{aligned}$$

After this change, the Lagrangian and Hamiltonian are

$$\begin{aligned} L &= \frac{\|\dot{\boldsymbol{r}}\|^2}{2} + \frac{\mu}{r_1} - \frac{1-\mu}{r_2}, \\ H &= \frac{\|\boldsymbol{p}\|^2}{2} - \frac{\mu}{r_1} - \frac{1-\mu}{r_2}, \end{aligned}$$

where we defined $\mu = \frac{\mu_1}{\mu_1 + \mu_2} \in (0, 1)$. Without loss of generality we can assume that $\mu_1 \leq \mu_2$, so that $\mu \leq \frac{1}{2}$.

3.1 Separation of variables

3.1.1 Cylindrical symmetry

From the setup of the system, it is clear that it is invariant under rotation about z-axis. This motivates us to introduce a transformation to cylindrical coordinates $(x, y, z) \mapsto (\rho, \varphi, z) \in \mathbb{R}^+ \times [0, 2\pi) \times \mathbb{R}$, given by

$$(x, y, z) = (\rho \cos\varphi, \rho \sin\varphi, z).$$

Then we have

$$\begin{aligned} \dot{\boldsymbol{r}} &= (\dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi, \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi, \dot{z}), \\ \boldsymbol{r}_1 &= (\rho\cos\varphi, \rho\sin\varphi, z + 1), \\ \boldsymbol{r}_2 &= (\rho\cos\varphi, \rho\sin\varphi, z - 1). \end{aligned}$$

So the terms in L are transformed to

$$\begin{aligned} \|\dot{\boldsymbol{r}}\|^2 &= \dot{\rho}^2 + \rho^2\dot{\varphi} + \dot{z}^2, \\ r_1 &= (\rho^2 + (z+1)^2)^{\frac{1}{2}}, \\ r_2 &= (\rho^2 + (z-1)^2)^{\frac{1}{2}}. \end{aligned}$$

These expressions enable us to write down the Lagrangian in cylindrical form as

$$L = \frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) + \frac{\mu}{(\rho^2 + (z+1)^2)^{\frac{1}{2}}} + \frac{1-\mu}{(\rho^2 + (z-1)^2)^{\frac{1}{2}}}.$$

The conjugate momenta are

$$\begin{aligned} p_\rho &= \frac{\partial L}{\partial \dot{\rho}} = \dot{\rho}, \\ p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = \rho^2\dot{\varphi}, \\ p_z &= \frac{\partial L}{\partial \dot{z}} = \dot{z}. \end{aligned}$$

As in the Kepler problem, p_φ is conserved and we denote it by l. Thus L can be written in a form that is independent of $\dot{\varphi}$:

$$L = \frac{1}{2}\left(\dot{\rho}^2 + \frac{l^2}{\rho^2} + \dot{z}^2\right) + \frac{\mu}{(\rho^2 + (z+1)^2)^{\frac{1}{2}}} + \frac{1-\mu}{(\rho^2 + (z-1)^2)^{\frac{1}{2}}}. \tag{3.1}$$

As in [WDR04, p. 269], we consider the Jacobian of the transformation to cylindrical coordinates:

$$J_{(\rho,\varphi,z)\mapsto(x,y,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\sin\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

This is singular on the z-axis, when $\rho = 0$. The trajectory of the test particle might cross the z-axis, so we need another transformation.

3.1.2 Elliptic coordinates

We now introduce two variables $(\xi, \eta) \in [1, \infty) \times [-1, 1]$ such that $r_1 = \xi + \eta$ and $r_2 = \xi - \eta$. Then

$$\begin{aligned} \xi &= \frac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\rho^2 + (z+1)^2\right)^{\frac{1}{2}} + \frac{1}{2}\left(\rho^2 + (z-1)^2\right)^{\frac{1}{2}}, \\ \eta &= \frac{1}{2}(r_1 - r_2) = \frac{1}{2}\left(\rho^2 + (z+1)^2\right)^{\frac{1}{2}} - \frac{1}{2}\left(\rho^2 + (z-1)^2\right)^{\frac{1}{2}}. \end{aligned}$$

Inverting gives

$$\rho = \left((\xi^2 - 1)(1 - \eta^2)\right)^{\frac{1}{2}}$$

according to [WDR04, p. 268], and

$$z = \xi\eta.$$

The Jacobian of this transformation is given by

$$J_{(\xi,\varphi,\eta)\mapsto(\rho,\varphi,z)} = \begin{vmatrix} \frac{\partial\rho}{\partial\xi} & \frac{\partial\rho}{\partial\varphi} & \frac{\partial\rho}{\partial\eta} \\ \frac{\partial\varphi}{\partial\xi} & \frac{\partial\varphi}{\partial\varphi} & \frac{\partial\varphi}{\partial\eta} \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\eta} \end{vmatrix} = \begin{vmatrix} \frac{\partial\rho}{\partial\xi} & 0 & \frac{\partial\rho}{\partial\eta} \\ 0 & 1 & 0 \\ \frac{\partial z}{\partial\xi} & 0 & \frac{\partial z}{\partial\eta} \end{vmatrix} = \frac{\partial\rho}{\partial\xi}\frac{\partial z}{\partial\eta} - \frac{\partial\rho}{\partial\eta}\frac{\partial z}{\partial\xi}$$
$$= \frac{\xi^2(1-\eta^2)}{\rho} + \frac{\eta^2(\xi^2-1)}{\rho} = \frac{\xi^2-\eta^2}{\rho}.$$

Then the Jacobian of the combined transformation is

$$J_{(\xi,\varphi,\eta)\mapsto(x,y,z)} = J_{(\rho,\varphi,z)\mapsto(x,y,z)}J_{(\xi,\varphi,\eta)\mapsto(\rho,\varphi,z)} = \xi^2-\eta^2 = (\xi+\eta)(\xi-\eta) = r_1 r_2.$$

This has a clear geometric meaning; the transformation is singular only at the two fixed centres ($r_1 = 0$ or $r_2 = 0$), which happens when the test particle collides with one of the two bodies.

Given the Lagrangian (3.1), we now wish to write the Hamiltonian H in elliptical coordinates. We start by transforming the kinetic part. The terms that are transformed are $\dot{\rho}$ and $\dot{z}$:

$$\begin{aligned}\dot{\rho} &= \frac{\partial\rho}{\partial\xi}\dot{\xi} + \frac{\partial\rho}{\partial\eta}\dot{\eta} \\ &= \left((\xi^2-1)(1-\eta^2)\right)^{-\frac{1}{2}}\left((1-\eta^2)\xi\dot{\xi} - (\xi^2-1)\eta\dot{\eta}\right) \\ &= \sqrt{\frac{1-\eta^2}{\xi^2-1}}\xi\dot{\xi} - \sqrt{\frac{\xi^2-1}{1-\eta^2}}\eta\dot{\eta},\end{aligned}$$

and

$$\begin{aligned}\dot{z} &= \frac{\partial z}{\partial\xi}\dot{\xi} + \frac{\partial z}{\partial\eta}\dot{\eta} \\ &= \eta\dot{\xi} + \xi\dot{\eta}.\end{aligned}$$

Squaring each of them and summing

$$\begin{aligned}\dot{\rho}^2 + \dot{z}^2 &= \frac{1-\eta^2}{\xi^2-1}\xi^2\dot{\xi}^2 + \frac{\xi^2-1}{1-\eta^2}\eta^2\dot{\eta}^2 + \eta^2\dot{\xi}^2 + \xi^2\dot{\eta}^2 \\ &= \left(\frac{1-\eta^2}{\xi^2-1}\xi^2 + \eta^2\right)\dot{\xi}^2 + \left(\frac{\xi^2-1}{1-\eta^2}\eta^2 + \xi^2\right)\dot{\eta}^2 \\ &= \frac{\xi^2-\eta^2}{\xi^2-1}\dot{\xi}^2 + \frac{\xi^2-\eta^2}{1-\eta^2}\dot{\eta}^2.\end{aligned}$$

Hence the kinetic part of H is

$$
\begin{aligned}
T &= \frac{1}{2}\left(\dot{\rho}^2 + \dot{z}^2 + \frac{l^2}{\rho^2}\right) \\
&= \frac{1}{2}\left(\frac{\xi^2-\eta^2}{\xi^2-1}\dot{\xi}^2 + \frac{\xi^2-\eta^2}{1-\eta^2}\dot{\eta}^2 + \frac{l^2}{(\xi^2-1)(1-\eta^2)}\right),
\end{aligned}
$$

where we used $\rho = \left((\xi^2-1)(1-\eta^2)\right)^{\frac{1}{2}}$. The conjugate momenta are

$$
p_\xi = \frac{\partial L}{\partial \dot{\xi}} = \frac{\partial T}{\partial \dot{\xi}} = \frac{\xi^2-\eta^2}{\xi^2-1}\dot{\xi}, \tag{3.2}
$$

$$
p_\eta = \frac{\partial L}{\partial \dot{\eta}} = \frac{\partial T}{\partial \dot{\eta}} = \frac{\xi^2-\eta^2}{1-\eta^2}\dot{\eta}. \tag{3.3}
$$

Inverting (3.2) and (3.3) to obtain $\dot{\xi}$ and $\dot{\eta}$ in terms of ξ, η, p_ξ, p_η turns T into

$$
T = \frac{1}{2}\left(\frac{\xi^2-1}{\xi^2-\eta^2}p_\xi + \frac{1-\eta^2}{\xi^2-\eta^2}p_\eta + \frac{l^2}{(\xi^2-1)(1-\eta^2)}\right).
$$

We factor out $\frac{1}{\xi^2-\eta^2}$ and split the last fraction into a ξ-dependent part and a η-dependent part to write T in separated form as

$$
\begin{aligned}
T &= \frac{1}{2(\xi^2-\eta^2)}\left(p_\xi^2(\xi^2-1) + \frac{l^2}{\xi^2-1} + p_\eta^2(1-\eta^2) + \frac{l^2}{1-\eta^2}\right) \\
&:= \frac{1}{\xi^2-\eta^2}(T_\xi + T_\eta),
\end{aligned}
$$

where we defined $T_\xi = \frac{1}{2}\left(p_\xi^2(\xi^2-1) + \frac{l^2}{\xi^2-1}\right)$ and $T_\eta = \frac{1}{2}\left(p_\eta^2(1-\eta^2) + \frac{l^2}{1-\eta^2}\right)$ as functions of (ξ, p_ξ) and (η, p_η) respectively.

On the other hand, the potential is

$$
\begin{aligned}
V &= -\frac{\mu}{r_1} - \frac{1-\mu}{r_2} \\
&= -\frac{\mu}{\xi+\eta} - \frac{1-\mu}{\xi-\eta} \\
&= \frac{1}{\xi^2-\eta^2}(-\xi - (1-2\mu)\eta) \\
&= \frac{1}{\xi^2-\eta^2}(V_\xi + V_\eta),
\end{aligned}
$$

where $V_\xi = -\xi$ and $V_\eta = -(1-2\mu)\eta$ as before. We combine the two by defining

$$H_\xi = T_\xi + V_\xi = \frac{1}{2}\left(p_\xi^2(\xi^2-1) + \frac{l^2}{\xi^2-1}\right) - \xi, \tag{3.4}$$

$$H_\eta = T_\eta + V_\eta = \frac{1}{2}\left(p_\eta^2(1-\eta^2) + \frac{l^2}{1-\eta^2}\right) - (1-2\mu)\eta. \tag{3.5}$$

Then the Hamiltonian in elliptic coordinates is given by

$$H = T + V = \frac{1}{\xi^2-\eta^2}(H_\xi + H_\eta). \tag{3.6}$$

By the conservation of energy (Proposition 1.3.1), H is an integral of the system. Following the method in [WDR04, p. 270], rearranging (3.6) so that one side only depends on ξ and the other only on η, we get the separation constant

$$G = \xi^2 H - H_\xi = \eta^2 H + H_\eta \tag{3.7}$$

that is also an integral of the system. A physical interpretation of the phase space function G is given in [WDR04, p. 270]. It turns out that $G = H + \Omega$, where

$$\Omega = \frac{1}{2}\left(\|\boldsymbol{L}\|^2 - p_x^2 - p_y^2\right) + (z+1)\frac{\mu}{r_1} - (z-1)\frac{1-\mu}{r_2}.$$

If we undo the rescaling of r, p and H at the start of this chapter, we see that $\boldsymbol{L}$ is the angular momentum. It also can be shown that as $a \to 0$ the quantity Ω becomes $\|\boldsymbol{L}\|^2$, while as $a \to \infty$ it is related to the Laplace-Runge-Lenz vector (2.15) in the Kepler problem.

We have now obtained three first integrals of motion for the two-centre problem: H, l and G. In Appendix C.1 we show that these three phase space functions are independent and in involution. Therefore the system is completely integrable and the Liouville-Arnold theorem tells us that we can solve it by quadratures.

3.2 Phase diagram in elliptic coordinates

In this section, we outline the procedure used to plot phase diagrams for the two-centre problem. The second point in the Theorem 2.1.1 states that when the phase space is compact and connected, it is diffeomorphic to a torus; we wish to verify this.

Let h and g denote the constant values of the phase space functions H and

G. Then from equations (3.4), (3.5) and (3.7) we have

$$\xi^2 h - g = H_\xi = \left(\frac{1}{2}(\xi^2+1)p_\xi^2 + \frac{l^2}{\xi^2-1}\right) - \xi,$$
$$g - \eta^2 h = H_\eta = \frac{1}{2}\left((1-\eta^2)p_\eta^2 + \frac{l^2}{1-\eta^2}\right) - (1-2\mu)\eta.$$

Rearranging for p_ξ and p_η we obtain

$$p_\xi^2 = \frac{2}{\xi^2-1}\left(h\xi^2 + \xi - g - \frac{l^2}{2(\xi^2-1)}\right), \tag{3.8}$$
$$p_\eta^2 = \frac{2}{1-\eta^2}\left(-h\eta^2 + (1-2\mu)\eta + g - \frac{l^2}{2(1-\eta^2)}\right). \tag{3.9}$$

Hence the relations between the elliptic coordinates and the corresponding momenta are fixed by the integrals of motion h, l and g. Then we can use $SageMath$ to plot the phase diagrams.

For convenience in this section, we set $l = 0$. We rewrite (3.8) and (3.9) to give

$$g = h\xi^2 + \xi - \frac{\xi^2-1}{2}p_\xi^2,$$
$$g = h\eta^2 - (1-2\mu)\eta + \frac{1-\eta^2}{2}p_\eta^2.$$

For practical reasons, we would like to expand the range of the coordinates, so we use an alternative definition of elliptic coordinates introduced by [WDR04, p. 269]:

$$\xi = \cosh\lambda,\ \eta = \sin\nu.$$

so that we can extend the range of phase diagram coordinates from $[1,\infty)\times[-1,1]$ to $\mathbb{R}\times[-\pi,\pi]$. Notice that

$$\begin{aligned} p_\lambda &= \frac{\partial L}{\partial \dot\xi}\frac{\partial \dot\xi}{\partial \dot\lambda} \\ &= \frac{\partial L}{\partial \dot\xi}\frac{\partial \xi}{\partial \lambda} \\ &= p_\xi \sinh\lambda. \end{aligned}$$

Similarly

$$p_\nu = p_\eta \frac{\partial \eta}{\partial \nu} = p_\eta \cos\nu.$$

Hence

$$g = h\cosh^2\lambda + \cosh\lambda - p_\lambda^2/2,$$
$$g = h\sin^2\nu - (1-2\mu)\sin\nu + p_\nu^2/2.$$

In Appendix C.2 we show an example of how these formulas are used to plot phase diagrams as the ones below, which correspond to the ones given in [WDR04, p. 277]. From the figures, we can see that when the level set of the phase space is connected and compact, it does indeed resemble a torus.

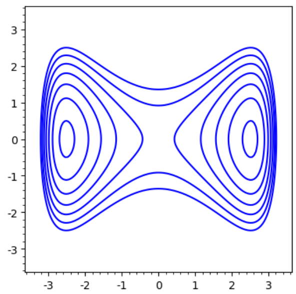

Figure 3.2: λ - p_λ phase diagram for $h = -0.08$, $g \in \{0, 0.5, ..., 3\}$, $l = 0$.

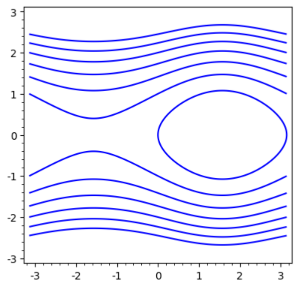

Figure 3.3: ν - p_ν phase diagram for $h = -0.08$, $g \in \{0, 0.5, ..., 3\}$, $l = 0$ and $\mu = 0.25$.

3.3 Equations of motion

We now square both sides of (3.2) and combine it with (3.8) to find the equation of motion for ξ.

$$\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^2 \dot{\xi}^2 = p_\xi^2 = \frac{2}{\xi^2 - 1}\left(\xi^2 h - g - \frac{l^2}{2(\xi^2 - 1)} + \xi\right),$$

which can be rearranged to obtain

$$\left(\xi^2 - \eta^2\right)^2 \dot{\xi}^2 = Q(\xi), \tag{3.10}$$

where $Q(\xi) := 2(\xi^2 - 1)\left(h\xi^2 + \xi - g\right) - l^2$ is a quartic polynomial in ξ depending on the integrals of the system h, g and l. Similarly, squaring both sides of (3.3) and combining it with (3.9) gives

$$\left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^2 \dot{\eta}^2 = p_\eta^2 = \frac{2}{1 - \eta^2}\left(g - \eta^2 h - \frac{l^2}{2(1 - \eta^2)} + (1 - 2\mu)\eta\right),$$

which simplifies to

$$\left(\xi^2 - \eta^2\right)^2 \dot{\eta}^2 = R(\eta), \tag{3.11}$$

where $R(\eta) = 2(1-\eta^2)\left(-h\eta^2 + (1-2\mu)\eta + g\right) - l^2$. The equations of motion above can be rewritten as

$$\left(\xi^2 - \eta^2\right) \frac{\mathrm{d}\xi}{\mathrm{d}t} = \sqrt{Q(\xi)},$$
$$\left(\xi^2 - \eta^2\right) \frac{\mathrm{d}\eta}{\mathrm{d}t} = \sqrt{R(\eta)},$$

from which we see that

$$\frac{\mathrm{d}\xi}{\sqrt{Q(\xi)}} = \frac{\mathrm{d}t}{\xi^2 - \eta^2} = \frac{\mathrm{d}\eta}{\sqrt{R(\eta)}}. \tag{3.12}$$

From (3.12) we conclude that

$$\frac{\mathrm{d}\xi}{\sqrt{Q(\xi)}} - \frac{\mathrm{d}\eta}{\sqrt{R(\eta)}} = 0 \tag{3.13}$$

and

$$\mathrm{d}t = \frac{\xi^2}{\sqrt{Q(\xi)}}\,\mathrm{d}\xi - \frac{\eta^2}{\sqrt{R(\eta)}}\,\mathrm{d}\eta. \tag{3.14}$$

We can integrate (3.13) and (3.14) to give the solution of the system in quadratures.

$$t + t_0 = \int \frac{\xi^2}{\sqrt{Q(\xi)}}\mathrm{d}\xi - \int \frac{\eta^2}{\sqrt{R(\eta)}}\mathrm{d}\eta, \tag{3.15}$$

$$0 = \int \frac{\mathrm{d}\xi}{\sqrt{Q(\xi)}} - \int \frac{\mathrm{d}\eta}{\sqrt{R(\eta)}}. \tag{3.16}$$

The integrals that are present here are elliptic integrals, which are more challenging than the ones of the form (2.14) found in the solution to the Kepler problem. We give more details on them in the next section.

3.4 Elliptic functions and integrals

A meromorphic function $f : \mathbb{C} \to \mathbb{C}$ is an *elliptic function* if it is doubly-periodic, which means that there exist $\omega_1, \omega_2 \in \mathbb{C}\backslash\{0\}$ such that $\omega_2/\omega_1 \notin \mathbb{R}$ and for all $z \in \mathbb{C}$:

$$f(z + \omega_1) = f(z + \omega_2) = f(z).$$

In this section, we introduce some standard elliptic functions and their relationship to integrals of the form (3.15) and (3.16).

3.4.1 Weierstrass functions

For ω_1, ω_2 as above, the *Weierstrass elliptic function* [GR14, section 8.160] is defined by the equation

$$\wp(z) = z^{-2} + \sum_{m,n=-\infty}^{\infty}{}' \left((z - \Omega_{mn})^{-2} - \Omega_{mn}^{-2} \right),$$

where $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$ and the prime of the sum means that terms with zero denominator are omitted, together with the condition

$$\frac{\mathrm{d}\wp(z)}{\mathrm{d}z} = -2 \sum_{m,n=-\infty}^{\infty} \frac{1}{(z - \Omega_{mn})^3}.$$

The function $\wp(z) - z^{-2}$ is Taylor expanded around $z = 0$ in [WW90, p.436] to get

$$\wp(z; g_2, g_3) = z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6).$$

The numbers g_2 and g_3 are called the *elliptic invariants*, which after comparing coefficients one can see that

$$g_2 = 60 \sum_{m,n=-\infty}^{\infty}{}' \Omega_{mn}^{-4}, \quad g_3 = 140 \sum_{m,n=-\infty}^{\infty}{}' \Omega_{mn}^{-6}.$$

Squaring and cubing $\wp(z; g_2, g_3)$ and $\wp'(z; g_2, g_3)$ respectively, we get

$$\begin{aligned} \wp(z)^3 &= z^{-6} + \frac{3}{20} g_2 z^{-2} + \frac{3}{28} g_3 + O(z^2) \\ \wp'(z)^2 &= 4z^{-6} - \frac{2}{5} g_2 z^{-2} - \frac{4}{7} g_3 + O(z^2), \end{aligned}$$

from which we obtain

$$\begin{aligned} \wp'(z)^2 - 4\wp^3(z) &= -g_2 z^{-2} - g_3 + O(z^2) \\ &= -g_2 \wp(z) - g_3 + O(z^2), \end{aligned}$$

where we used $\wp(z) = z^{-2} + O(z^2)$. Therefore, the function $\wp'(z)^2 - 4\wp^3(z) + g_2\wp(z) + g_3$ has terms of order z^2 and higher, so it is everywhere analytic. Following the reasoning in [WW90, p. 436], this makes it an elliptic function with no singularities, so it is a constant. On taking $z \to 0$ we see that this constant is zero. Hence the elliptic function $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{3.17}$$

The *Weierstrass zeta function* $\zeta(z)$ is defined by the equation

$$\frac{\mathrm{d}\zeta(z)}{\mathrm{d}z} = -\wp(z),$$

with the condition

$$\lim_{z\to 0}\left(\zeta(z) - z^{-1}\right) = 0.$$

By the uniform convergence of the series for $\wp - z^{-2}$, the function $\zeta(z)$ is written in [WW90, p. 445] as

$$\zeta(z) = z^{-1} + \sum_{m,n=-\infty}^{\infty}{}' \left((z - \Omega_{mn})^{-1} + \Omega_{mn}^{-1} + \Omega_{mn}^{-2} z\right).$$

The *Weierstrass sigma function* is defined by

$$\frac{\mathrm{d}}{\mathrm{d}z} \log \sigma(z) = \zeta(z),$$

with the condition

$$\lim_{z\to 0} \frac{\sigma(z)}{z} = 0.$$

Similarly to before, the function $\sigma(z)$ is wrriten in [WW90, p.447] as

$$\sigma(z) = z \prod_{m,n=-\infty}^{\infty}{}' \left(1 - \Omega_{mn}^{-1} z\right) \exp\left(\Omega_{mn}^{-1} z + \frac{1}{2}\Omega_{mn}^{-2} z^2\right).$$

3.4.2 Elliptic integrals

An *elliptic integral* in standard form is given by

$$z = \int_{x_0}^{x} \frac{\mathrm{d}t}{\sqrt{f(t)}}, \tag{3.18}$$

where $f(t) = a_4 t^4 + 4a_3 t^3 + 6a_2 t^2 + 4a_1 t + a_0$ is a quartic polynomial with no repeated factors. If the lower bound of the integral x_0 is a root of f, then we can find the solution as outlined in [WW90, p. 453]. Taylor expanding $f(t)$ around x_0 and using the substitution $s = (t - x_0)^{-1}$, we have

$$z = \int_{u}^{\infty} \frac{\mathrm{d}s}{\sqrt{4A_3 s^3 + 6A_2 s^2 + 4A_1 s + A_0}},$$

where $u = (x - x_0)^{-1}$ and

$$\begin{aligned} A_0 &= a_0, \quad A_1 = a_0 x_0 + a_1, \\ A_2 &= a_0 {x_0}^2 + 2a_1 x_0 + a_2, \\ A_3 &= a_0 {x_0}^3 + 3a_1 {x_0}^2 + 3a_2 x_0 + a_3. \end{aligned}$$

Note that by doing this we have turned the denominator in (3.18) from a quartic to a cubic. To remove the s^2 term in the cubic, we use the substitution $s =$

${A_3}^{-1}\left(\tau - \frac{1}{2}A_2\right)$ and we obtain

$$z = \int_v^\infty \frac{\mathrm{d}\tau}{\sqrt{4\tau^3 - g_2\tau - g_3}},$$

where v satisfies $u = {A_3}^{-1}\left(v - \frac{1}{2}A_2\right)$ and

$$g_2 = 3A_2^2 - 4A_1A_3 = a_0a_4 - 4a_1a_3 + 3a_2^2, \tag{3.19}$$
$$g_3 = 2A_1A_2A_3 - A_2^3 - A_0A_3^2 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4. \tag{3.20}$$

Thus, since $\wp(z; g_2, g_3)$ satisfies the differential equation (3.17), we obtain

$$v = \wp(z; g_2, g_3).$$

Undoing the substitutions we obtain x as a fraction of $\wp(z; g_2, g_3)$:

$$x = x_0 + \frac{\frac{1}{4}f'(x_0)}{\wp(z; g_2, g_3) - \frac{1}{24}f''(x_0)}. \tag{3.21}$$

3.5 Solving the equations of motion

We now use the content of the previous section to solve (3.15) and (3.16) for ξ and η. Similarly to [BI16, p. 3483], we consider whether $Q(\xi)$ and $R(\eta)$ have a root in $(\xi, \eta) \in [1, \infty) \times [-1, 1]$ in order to use (3.21). Recall that the equations of motion (3.10) and (3.11) are

$$(\xi^2 - \eta^2)^2\dot{\xi}^2 = Q(\xi),$$
$$(\xi^2 - \eta^2)^2\dot{\eta}^2 = R(\eta).$$

Notice that $Q(1) = -l^2 \leq 0$ and $R(\pm 1) = -l^2 \leq 0$. Since the left-hand side of the equations above must be real, we must have $Q(\xi) \geq 0$ for some $\xi \in (1, \infty)$ and $R(\eta) \geq 0$ for some $\eta \in (-1, 1)$. Therefore $P(\xi)$ and $Q(\eta)$ must have real roots in the domain, denoted by ξ_0 and η_0.

From (3.12), we introduce the *ficticious time* τ:

$$\tau = \int_0^t \frac{\mathrm{d}t'}{\xi(t')^2 - \eta(t')^2} = \int_{\xi_0}^{\xi} \frac{\mathrm{d}\xi'}{\sqrt{Q(\xi')}} = \int_{\eta_0}^{\eta} \frac{\mathrm{d}\eta'}{\sqrt{R(\eta')}}.$$

Using the solution of the standard elliptic integral (3.21) we have

$$\xi(\tau) = \xi_0 + \frac{\frac{1}{4}Q'(\xi_0)}{\wp(\tau; g_2(Q), g_3(Q)) - \frac{1}{24}Q''(\xi_0)}, \tag{3.22}$$

$$\eta(\tau) = \eta_0 + \frac{\frac{1}{4}R'(\eta_0)}{\wp(\tau; g_2(R), g_3(R)) - \frac{1}{24}R''(\eta_0)}, \tag{3.23}$$

where $g_2(Q)$ and $g_3(Q)$ are the elliptic invariants of Q, and similarly for $g_2(R)$ and $g_3(R)$; these can be computed from (3.19) and (3.20). Notice that these simple expressions are only possible when the initial conditions are

$$t = 0, \quad \xi(0) = \xi_0, \quad \eta(0) = \eta_0.$$

For simplicity, we assume this from now on.

Although we can write out the solution for the equations of motion elegantly using the fictitious time τ, it introduces some problems. Firstly, the angular momentum $\dot{\varphi}$ is not the same as $\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}$, and since $\mathrm{d}\tau$ depends on the position of the particle, we lose the conservation of momentum in terms of the fictitious time. Furthermore, τ is defined by an integral that depends on t. To be able to transform the values of the elliptic functions into information about the real motion, we still need an explicit expression of $t(\tau)$. Therefore, we now give give solutions for $t(\tau)$ and $\varphi(\tau)$.

The real time derivative of the fictitious time is

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\xi(\tau)^2 - \eta(\tau)^2},$$

which can be inverted to give

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \xi(\tau)^2 - \eta(\tau)^2. \tag{3.24}$$

The fictitious time derivative of φ is

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{l}{\rho(\tau)^2}\big(\xi(\tau)^2 - \eta(\tau)^2\big) = l\left(\frac{1}{\xi(\tau)^2 - 1} + \frac{1}{1 - \eta(\tau)^2}\right), \tag{3.25}$$

where we have used the fact that $l = \rho^2\dot{\varphi}$ and $\rho = \big((\xi^2 - 1)(1 - \eta^2)\big)^{\frac{1}{2}}$. Similarly to [BI16, p. 3485], we introduce the following notation for simplicity:

$$\begin{aligned} A_\xi &= \tfrac{1}{4}Q'(\xi_0), & A_\eta &= \tfrac{1}{4}R'(\eta_0), \\ B_\xi &= \tfrac{1}{24}Q''(\xi_0), & B_\eta &= \tfrac{1}{24}R''(\eta_0), \end{aligned}$$

and

$$\begin{aligned} \wp_\xi(\tau) &= \wp(\tau; g_2(Q), g_3(Q)), \\ \wp_\eta(\tau) &= \wp(\tau; g_2(R), g_3(R)). \end{aligned}$$

Then the expressions (3.22) and (3.23) for $\xi(\tau)$ and $\eta(\tau)$ can be written as

$$\xi(\tau) = \xi_0 + \frac{A_\xi}{\wp_\xi(\tau) - B_\xi}, \tag{3.26}$$

$$\eta(\tau) = \eta_0 + \frac{A_\eta}{\wp_\eta(\tau) - B_\eta}. \tag{3.27}$$

The derivative of t given by (3.24) is

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}\tau} = &\ \xi_0^2 + 2\xi_0 A_\xi \frac{1}{\wp_\xi(\tau) - B_\xi} + A_\xi^2 \frac{1}{\big(\wp_\xi(\tau) - B_\xi\big)^2} \\ &- \eta_0^2 - 2\eta_0 A_\eta \frac{1}{\wp_\eta(\tau) - B_\eta} - A_\eta^2 \frac{1}{\big(\wp_\eta(\tau) - B_\eta\big)^2}.\end{aligned}$$

By performing partial fraction decomposition with respect to $\wp_\xi(\tau)$ and $\wp_\eta(\tau)$ as mentioned in [BI16, p. 3485], we obtain the derivative of φ given by (3.25):

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = &\ \frac{lA_\xi}{2(\xi_0+1)^2} \frac{1}{\wp_\xi(\tau) + \frac{A_\xi - B_\xi(\xi_0+1)}{\xi_0+1}} \\ &- \frac{lA_\xi}{2(\xi_0-1)^2} \frac{1}{\wp_\xi(\tau) + \frac{A_\xi - B_\xi(\xi_0+1)}{\xi_0-1}} \\ &- \frac{lA_\eta}{2(\eta_0+1)^2} \frac{1}{\wp_\eta(\tau) + \frac{A_\eta - B_\eta(\eta_0+1)}{\eta_0+1}} \\ &+ \frac{lA_\eta}{2(\eta_0-1)^2} \frac{1}{\wp_\eta(\tau) + \frac{A_\eta - B_\eta(\eta_0+1)}{\eta_0-1}} \\ &+ \frac{l}{\xi_0^2 - 1} - \frac{l}{\eta_0^2 - 1}.\end{aligned}$$

In order to integrate the above expressions, we introduce the following integrals presented in [GR14, section 5.141]:

$$\begin{aligned}\mathcal{J}_1(u,v) &= \int \frac{\mathrm{d}u}{\wp(u) - \wp(v)} = \frac{1}{\wp'(v)} \left(\ln \frac{\sigma(u-v)}{\sigma(u+v)} + 2u\zeta(v) \right), \\ \mathcal{J}_2(u,v) &= \int \frac{\mathrm{d}u}{\big(\wp(u) - \wp(v)\big)^2} \\ &= \frac{1}{\wp'(v)^2} \big(-\zeta(u-v) - \zeta(u+v) - 2u\wp(v) - \wp''(v)\mathcal{J}_1(u,v) \big).\end{aligned}$$

Here $\zeta(u)$ and $\sigma(u)$ refer to the Weierstrass zeta and sigma functions introduced in Section 3.4.1, and we add a ξ or η subscript to $\mathcal{J}_1$ and $\mathcal{J}_2$ depending on whether $\wp_\xi$ or $\wp_\eta$ appears. With this convention, we may integrate $\frac{\mathrm{d}t}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}$:

$$\begin{aligned}t(\tau) = &\ \xi_0^2 \tau + 2\xi_0 A_\xi \mathcal{J}_{\xi,1}(\tau, b_\xi) + A_\xi^2 \mathcal{J}_{\xi,2}(\tau, b_\xi) \\ &- \eta_0^2 \tau - 2\eta_0 A_\eta \mathcal{J}_{\eta,1}(\tau, b_\eta) - A_\eta^2 \mathcal{J}_{\eta,2}(\tau, b_\eta),\end{aligned}$$

where just like in [BI16, p. 3486] we have defined

$$b_\xi = \wp_\xi^{-1}(B_\xi) = \int_{B_\xi}^{\infty} \frac{\mathrm{d}\tau}{\sqrt{4\tau^3 - g_2(Q)\tau - g_3(Q)}},$$
$$b_\eta = \wp_\eta^{-1}(B_\eta) = \int_{B_\eta}^{\infty} \frac{\mathrm{d}\tau}{\sqrt{4\tau^3 - g_2(R)\tau - g_3(R)}}.$$

Integrating $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$ gives

$$\begin{aligned}\varphi(\tau) = \quad & \varphi_0 \\ & + \frac{lA_\xi}{2(\xi_0+1)^2}\,\mathcal{J}_{\xi,1}(\tau,\nu_{\xi,1}) - \frac{lA_\xi}{2(\xi_0-1)^2}\,\mathcal{J}_{\xi,1}(\tau,\nu_{\xi,2}) \\ & - \frac{lA_\eta}{2(\eta_0+1)^2}\,\mathcal{J}_{\eta,1}(\tau,\nu_{\eta,1}) + \frac{lA_\eta}{2(\eta_0-1)^2}\,\mathcal{J}_{\eta,1}(\tau,\nu_{\eta,2}) \\ & + \frac{l\tau}{\xi_0^2-1} - \frac{l\tau}{\eta_0^2-1},\end{aligned}$$

where again the ν's are defined by

$$\nu_{\xi,1} = \wp_\xi^{-1}\left(-\frac{A_\xi - B_\xi(\xi_0+1)}{\xi_0+1}\right),$$
$$\nu_{\xi,2} = \wp_\xi^{-1}\left(-\frac{A_\xi - B_\xi(\xi_0+1)}{\xi_0-1}\right),$$
$$\nu_{\eta,1} = \wp_\eta^{-1}\left(-\frac{A_\eta - B_\eta(\eta_0+1)}{\eta_0+1}\right),$$
$$\nu_{\eta,2} = \wp_\eta^{-1}\left(-\frac{A_\eta - B_\eta(\eta_0+1)}{\eta_0-1}\right).$$

As for the conjugate momenta $p_\xi(\tau)$ and $p_\eta(\tau)$, we simply use (3.2) to obtain

$$p_\xi(\tau) = \frac{\xi(\tau)^2 - \eta(\tau)^2}{\xi(\tau)^2 - 1}\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{1}{\xi(\tau)^2 - 1}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{1}{\xi(\tau)^2 - 1}\frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau}.$$

The expressions for $\xi(\tau)$ and $\frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau}$ are readily obtained from (3.26) and can be inserted in above to obtain $p_\xi(\tau)$. Similarly, we may use (3.27) to give an explicit expression for

$$p_\eta(\tau) = \frac{1}{1 - \eta(\tau)^2}\frac{\mathrm{d}\eta(\tau)}{\mathrm{d}\tau}.$$

We now take a step back and look at what we have obtained. Initially, we transformed our phase space from (x, y, z, p_x, p_y, p_z) to $(\xi, \varphi, \eta, p_\xi, p_\varphi, p_\eta)$, and obtained the integrals of motion h, l and g. We introduced the fictitious time τ and gave expressions for the latter phase space coordinates in terms of it:

$$
\begin{aligned}
\xi(\tau) &= \xi_0 + \frac{A_\xi}{\wp_\xi(\tau) - B_\xi}, \\
\varphi(\tau) &= \varphi_0 + \frac{lA_\xi}{2(\xi_0+1)^2}\,\mathcal{J}_{\xi,1}(\tau,\nu_{\xi,1}) - \frac{lA_\xi}{2(\xi_0-1)^2}\,\mathcal{J}_{\xi,1}(\tau,\nu_{\xi,2}) - \cdots, \\
\eta(\tau) &= \eta_0 + \frac{A_\eta}{\wp_\eta(\tau) - B_\eta}, \\
p_\xi(\tau) &= \frac{1}{\xi(\tau)^2-1}\frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau}, \\
p_\varphi(\tau) &= l, \\
p_\eta(\tau) &= \frac{1}{1-\eta(\tau)^2}\frac{\mathrm{d}\eta(\tau)}{\mathrm{d}\tau}.
\end{aligned}
$$

In addition, we related the real time t to the fictitious time:

$$
t(\tau) = \xi_0^2\,\tau + 2\xi_0 A_\xi\,\mathcal{J}_{\xi,1}(\tau,b_\xi) + A_\xi^2\,\mathcal{J}_{\xi,2}(\tau,b_\xi) - \eta_0^2\,\tau - 2\eta_0 A_\eta\,\mathcal{J}_{\eta,1}(\tau,b_\eta) - A_\eta^2\,\mathcal{J}_{\eta,2}(\tau,b_\eta),
$$

so that for every range of τ there is a corresponding range of t. We can undo the transformations in Section 3.1 to obtain expressions for the cartesian coordinates:

$$
\begin{aligned}
x(\tau) &= \left((\xi(\tau)^2-1)(1-\eta(\tau)^2)\right)^{\frac{1}{2}}\cos\varphi(\tau), \\
y(\tau) &= \left((\xi(\tau)^2-1)(1-\eta(\tau)^2)\right)^{\frac{1}{2}}\sin\varphi(\tau), \\
z(\tau) &= \xi(\tau)\eta(\tau).
\end{aligned}
$$

A similar procedure yields the cartesian momenta, giving us the solution to the two-centre problem in terms of elliptic functions.

Chapter 4

Conclusion

We remind the reader that an n-particle Hamiltonian system is given by the triple

$$(T^*M, \omega, H)\,,$$

where $T^*M = \{q^i, p_i\}$ is the $6n$-dimensional phase space of the $3n$-dimensional configuration space $M = \{q^i\}$. The time evolution of the system is described by the $6n$ Hamilton equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

The main theorem in this report is the Liouville-Arnold theorem; it tells us is that it is sufficient to know $3n$ independent first integrals in involution to solve the system in quadratures. The Hamilton-Jacobi method gives us a way of finding the canonical transformation to action-angle variables that linearises Hamilton's equations and allows us to find such a solution.

In the case of the one-dimensional harmonic oscillator, the conservation of energy guarantees that the Hamiltonian H is itself a first integral, which allows us to readily solve the problem. Regarding the Kepler problem, we find that the first integrals are given by E, ℓ^2 and p_φ, where the last two are related to the angular momentum of the system. The presence of an additional conserved quantity given by the Laplace–Runge–Lenz vector makes the Kepler problem maximally superintegrable. After introducing appropriate substitutions, the integrals that appear in the solution are all of the form

$$\int \frac{\mathrm{d}u}{u^n(u+1)^m\sqrt{au^2+bu+c}},$$

where $n, m = 0, 1, 2$. These can be solved by elementary methods to arrive at a solution to the Kepler problem.

When considering the two-centre problem, we wish to solve for the motion of a test particle moving under the influence of the gravitational field generated by two fixed masses. The conservation of energy and the intrinsic cylindrical symmetry of the setup allow us to find two first integrals E and l. The last one is found by introducing a transformation to elliptic coordinates and separating variables to obtain the conserved quantity G. The integrals involved in the solution to the two-centre problem are of the form

$$\int \frac{\mathrm{d}t}{\sqrt{f(t)}},$$

where $f(t) = a_4 t^4 + 4a_3 t^3 + 6a_2 t^2 + 4a_1 t + a_0$ is a quartic polynomial. We introduce the fictitious time τ and give the solution in terms of the Weierstrass elliptic function $\wp(\tau)$, which belongs to a class of doubly-periodic meromorphic functions. In the special case where the initial conditions of the system correspond to a root of the polynomial $f(t)$, we obtain solutions for the positions (x, y, z) and time t in terms of τ.

The two-centre problem is but one of many completely integrable systems in classical mechanics. Other examples include the spherical pendulum and the Lagrangian top, and more illustrated in [Fas99, chapter 1]. There are others beyond classical mechanics, such as the Kortweg-de Vries equations. The more one looks into this topic, the more one can appreciate the power and scope of the Liouville-Arnold theorem.

Appendix A

Liouville's volume theorem

Given a Hamiltonian vector field $X_H \in \mathfrak{X}(T^*M)$, one can define the Lie derivative along the flow generated by X_H, denoted by $\mathcal{L}_{X_H}$. Cartan's formula gives us a simple way of writing this down:

$$\mathcal{L}_{X_H} = \mathrm{d}\iota_{X_H} + \iota_{X_H}\mathrm{d}.$$

The symplectic form ω is closed, so the Lie derivative of ω along X_H is

$$\mathcal{L}_{X_H}\omega = \mathrm{d}\iota_{X_H}\omega.$$

Then, from (1.9) we have $\iota_{X_H}\omega = \mathrm{d}H$. Therefore

$$\mathcal{L}_{X_H}\omega = \mathrm{d}^2 H = 0. \tag{A.1}$$

To explain the consequence of (A.1), we define the Liouville volume form on T^*M as ω^{3n}, where the power denotes taking the wedge product $3n$ times. Intuitively, this quantity represents the phase space volume. The following theorem states that it is constant along the flow generated by X_H.

Theorem A.0.1 (Liouville's volume theorem)**.** *Given a Hamiltonian system (1.6), the Lie derivative of the volume form ω^{3n} along the flow generated by X_H vanishes:*

$$\mathcal{L}_{X_H}\left(\omega^{3n}\right) = 0.$$

Proof. The Lie derivative obeys the Leibniz rule, so we have

$$\mathcal{L}_{X_H}\omega^{3n} = \mathcal{L}_{X_H}\left(\omega^{3n-1} \wedge \omega\right) = \left(\mathcal{L}_{X_H}\omega^{3n-1}\right) \wedge \omega + \omega^{3n-1} \wedge \left(\mathcal{L}_{X_H}\omega\right)$$

which from (A.1) equals

$$\left(\mathcal{L}_{X_H}\omega^{3n-1}\right) \wedge \omega.$$

Repeating the same procedure another $3n-2$ times, we obtain

$$\mathcal{L}_{X_H}\omega^{3n} = \left(\mathcal{L}_{X_H}\omega\right) \wedge \omega^{3n-1},$$

which vanishes once again from (A.1). $\square$

Appendix B

The Kepler problem

B.1 Integrability

First, we check that $\ell^2, L_z, \boldsymbol{A}$ and $\|\boldsymbol{A}\|^2$ are conserved quantities of the Kepler problem, and that the triples $\{H, \ell^2, L_z\}$ and $\{H, \|\boldsymbol{A}\|^2, L_z\}$ are in involution.

```
var('m mu x y z p_x p_y p_z')  #variables - m is reduced mass and mu=Gm1m2

R = vector([x,y,z])  #position vector
P = vector([p_x,p_y,p_z])  #momentum vector
L = R.cross_product(P)  #angular momentum vector
A = P.cross_product(L) - m*mu*R/R.norm()  #Laplace-Runge-Lenz vector

def poisson(f, g):  #poisson bracket of f and g
    b = 0
    for i in range(3):
        b += derivative(f,R[i])*derivative(g,P[i]) - derivative(f,P[i])*derivative(g,R[i])
    return b.simplify_full()

H = P*P/(2*m) - mu/R.norm()  #energy H
L2 = L*L  #square of L
Lx = L[0] #x component of L
Ly = L[1] #y component of L
Lz = L[2]  #z component of L
A2 = A*A  #square of A
Ax = A[0]  #x component of A
Ay = A[1]  #y component of A
Az = A[2]  #z component of A

conserved = [H, L2, Lz, A2, Ax, Ay, Az]  #integrals
c = ['H', 'L2', 'Lz', 'A2', 'Ax', 'Ay', 'Az']

for i in range(len(conserved)):
    print('{H,' + c[i] + '} = ' + str(poisson(H,conserved[i])))  #showing that all the integrals are conserved
print()
print('{L2,Lz} = ' + str(poisson(L2,Lz)))  #L2 and Lz are in involution
```

```
print()
print('{A2,Lz} = ' + str(poisson(A2,Lz)))  #A2 and Lz are in
    involution
```

Listing B.1: Checking that the first integrals of the Kepler problem are in involution.

```
{H,H} = 0
{H,L2} = 0
{H,Lz} = 0
{H,A2} = 0
{H,Ax} = 0
{H,Ay} = 0
{H,Az} = 0

{L2,Lz} = 0

{A2,Lz} = 0
```

Listing B.2: Output.

Next, we check that the phase space functions H, ℓ^2 and L_z are independent.

```
M = Manifold(6, 'M')  #cotangent bundle
U.<x,y,z,p_x,p_y,p_z> = M.chart()  #local coordinates
var('mu k')
R = vector([x,y,z])  #position vector
P = vector([p_x,p_y,p_z])  #momentum vector
L = R.cross_product(P)  #angular momentum vector
A = L.cross_product(P)/m - k*R/R.norm()  #Laplace-Runge-Lenz
    vector

H = M.scalar_field(P*P/(2*m) - k/R.norm())  #energy H
dH = H.differential()  #exterior derivative of H

L2 = M.scalar_field(L*L)  #square of L
dL2 = L2.differential()  #exterior derivative of L2

Lz = M.scalar_field(L[2])  #z component of L
dLz = Lz.differential()  #exterior derivative of Lz

A2 = M.scalar_field(A*A)  #square of A
dA2 = A2.differential()  #exterior derivative of A2

dH.wedge(dLz).wedge(L2) == 0 #if the wedge product does not
    vanish, H, L2, Lz are independent
```

Listing B.3: Checking that the first integrals are independent.

```
False
```

Listing B.4: Output.

Appendix C

The Two-Centre Problem

C.1 Integrability

Below, we check that H, l and G are Poisson-commuting, independent integrals of motion for the two-centre problem.

```
In [1]:
%display latex

M = Manifold(6, 'M')
U.<xi,eta,phi,p_xi,p_eta,p_phi> = M.chart()  #local coordinates
var('mu')

R = vector([xi,eta,phi])
P = vector([p_xi,p_eta,p_phi])

def poisson(f, g):
    b = 0
    for i in range(len(R)):
        b += derivative(f,R[i])*derivative(g,P[i]) - derivative(f
    ,P[i])*derivative(g,R[i])
    return b.simplify_full()

H_xi = (xi^2-1)*p_xi^2/2 + p_phi^2/(xi^2-1)/2 - xi
H_eta = (1-eta^2)*p_eta^2/2 + p_phi^2/(1-eta^2)/2 - (1-2*mu)*eta
H = (1/(xi^2-eta^2)) * (H_xi + H_eta)
G = xi^2*H - H_xi

scH = M.scalar_field(H)
dH = scH.differential()

scG = M.scalar_field(G)
dG = scG.differential()

scp = M.scalar_field(p_phi)
dp = scp.differential()

zero = M.scalar_field(0)
dzero = zero.differential()

print('{G,p_phi} = ' + str(poisson(G,p_phi)))
```

```
print('{H,p_phi} = ' + str(poisson(H,p_phi)))
print('{H,G} = ' + str(poisson(H,G)))
dH.wedge(dG).wedge(dp) == 0 #verify that H, G, and the angular momentum p
                                                                    #are independent by showing the volume form is not 0
```

Listing C.1: Verifying the Poisson brackets and the independence of the integrals of motion of the two-centre problem.

```
{G,p_phi} = 0
{H,p_phi} = 0
{H,G} = 0
Out[1]:
False
```

Listing C.2: Output.

C.2 Phase diagram

We display the code used to generate Figure 3.2 with *SageMath*.

```
In[1]:
var('p,q,h')   #define the coordinate as q and the conjugate momentum as p
               #here the coordinate is \lambda
h=-0.08                                    #energy
g(q,p) = h*(cosh(q))^2+cosh(q)-p^2*0.5  #the constant g as a function of \lambda and the corresponding conjugate momentum

P = Graphics()

P+=sum([implicit_plot(g(q,p)==x,(q,-3.5,3.5),(p,-3.5,3.5)) # setting the range of q, p of the contours
    for x in [0..3,step=0.5]])                              # setting the range of g of the contours
```

Listing C.3: Plotting λ - p_λ phase diagram

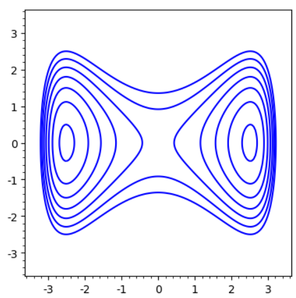

Figure C.1: λ - p_λ phase diagram for $h = -0.08$, $g \in \{0, 0.5, ..., 3\}$, $l = 0$.

Next, we do the same for Figure 3.3.

```
In[1]:
var('p,q,h')     #define the coordinate as q and the conjugate
    momentum as p
                 #here the coordinate is \nu
h=-0.08          #energy
s=0.25           #the relative strength \mu, 0.5 means the two
    centres have the same strength.
                 #0.25 means the attraction of top centre is 3
    times the bottom
g(q,p) = p^2*0.5+h*(sin(q))^2-(1-2*s)*sin(q) #the constant g as a
     function of lambda and the corresponding conjugate momentum
P = Graphics()

P+=sum([implicit_plot(g(q,p)==x,(q,-pi,+pi),(p,-3,3)) #setting
    the range of q, p of the contours
        for x in [0..3,step=0.5]])                     #setting
    the range of g of the contours
```

Listing C.4: Plotting ν - p_ν phase diagram.

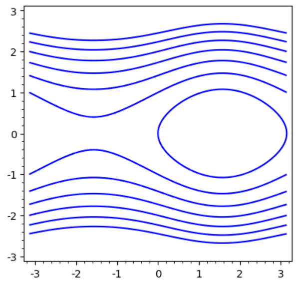

Figure C.2: ν - p_ν phase diagram for $h = -0.08$, $g \in \{0, 0.5, ..., 3\}$, $l = 0$ and $\mu = 0.25$.

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