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An Ad hoc Approximation to the Gauss Error Function and a Correction Method

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Abstract

We propose an algebraic type ad hoc approximation method, taking a simple form with two parameters, for the Gauss error function which is valid on a whole interval $(-\infty, \infty)$. To determine the parameters in the presented approximation formula we employ some appropriate constraints. Furthermore, we provide a correction method to improve the accuracy by adding an auxiliary term. The plausibility of the presented method is demonstrated by the results of the numerical implementation.

Mathematics Subject Classification: 62E17, 65D10

Keywords: Error function, ad hoc approximation, cumulative distribution function, correction method

1 Introduction

The Gauss error function defined below is a special function appearing in statistics and partial differential equations.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\,, \qquad -\infty < x < \infty\,, \tag{1}$$

which is strictly increasing and maps the real line $\mathbb{R}$ onto an interval $(-1, 1)$. The integral can not be represented in a closed form and its Taylor series expansion is

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \tag{2}$$

which converges for all $-\infty < x < \infty$ [2]. For $-1 < y < 1$ the inverse error function is defined as

$$\operatorname{erf}^{-1}(y) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} \left(\frac{\sqrt{\pi}}{2} y \right)^{2n+1}, \tag{3}$$

where $c_0 = 1$ and

$$c_n = \sum_{k=0}^{n-1} \frac{c_k c_{n-1-k}}{(k+1)(2k+1)}, \qquad n \geq 1.$$

For evaluation of the approximation error later, we refer to the following numerical approximation to the error function $\operatorname{erf}(x)$ given in the literature [16].

$$p(x) \;=\; \operatorname{sgn}(x) \cdot \{1 - g(x)\}\;, \qquad -\infty < x < \infty\,, \tag{4}$$

where

$$\begin{aligned} g(x) = \; t \cdot \exp\big(&-x^2 - 1.26551223 + 1.00002368\, t + 0.37409196\, t^2 + 0.09678418\, t^3 \\ &- 0.18628806\, t^4 + 0.27886807\, t^5 - 1.13520398\, t^6 + 1.48851587\, t^7 \\ &- 0.82215223\, t^8 + 0.17087277\, t^9\big) \end{aligned} \tag{5}$$

for $t = t(x) = 1/(1 + 0.5|x|)$. It is known that a maximal error of $p(x)$ over $(-\infty, \infty)$ is 1.2×10^{-7}. Due to the high accuracy, $p(x)$ will be substituted for the exact error function $\operatorname{erf}(x)$ in numerical implementation.

The error function $\operatorname{erf}(x)$ is used to represent the cumulative distribution function(cdf) $\Phi(\xi)$ as follows.

$$\Phi(\xi) \;:=\; \int_{-\infty}^{\xi} \phi(z)\, dz \;=\; \frac{1}{2} \left\{ 1 + \operatorname{erf}\left(\frac{\xi}{\sqrt{2}} \right) \right\}, \qquad -\infty < \xi < \infty \tag{6}$$

for $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, the probability density function of the standard normal distribution. There are many approximation formulas to the cumulative distribution function $\Phi(\xi)$ or the error function $\operatorname{erf}(x)$. Most formulas having high accuracy are based on the form of series expansion which requires quite many numerical coefficients [8, 11, 13–15]. Other approximations are the so-called "ad hoc approximations" which often take simple forms with few numerical coefficients [1, 6, 7, 9, 10, 17, 18]. For general classification and further comments on these approximations, one can see the literature [3] and [15]. However, it should be noted that lots of the existing approximation formulas are valid on some limited region and not invertible.

On the other hand, the error function also appears in solutions of the heat equation with piecewise constant initial condition as

$$\begin{aligned} u_t(x,t) - u_{xx}(x,t) = 0\,, \qquad & -\infty < x < \infty\,, \quad (t>0) \\ u(x,0) = 1\,, \qquad & -1 \le x \le 1 \end{aligned} \tag{7}$$

whose solution is

$$u(x,t) \;=\; \frac{1}{2}\left\{\mathrm{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x-1}{2\sqrt{t}}\right)\right\}. \tag{8}$$

The purpose of this paper is to propose a new ad hoc approximate error function and to develop an efficient correction method which can highly improve the accuracy. In Section 2 we introduce a simple algebraic type approximation to the error function which has its inverse of a closed form. The presented approximation formula consists of the terms $(x+\alpha)^m$ and $(x-\alpha)^m$ only, and the two parameters m and α are to be determined numerically by some appropriate constraints. Absolute difference error is less than 2.1×10^{-3} over the whole interval $(-\infty, \infty)$ for $m = 2.4850$, a specially chosen value.

In Section 3, exploring the global behavior of the difference error of the presented approximation, we propose a correction method by adding an auxiliary term. One can see that its maximal error is improved to 3.8×10^{-4}. Moreover, applying the proposed correction method to the recent approximation formula with a maximal error 6.1×10^{-5}, we can obtain an improved maximal error of 5.6×10^{-6}.

2 An algebraic type ad hoc approximation

In this work, focusing on the simplicity and usefulness over the whole region $(-\infty, \infty)$, we propose a new algebraic type approximation formula for the error function $\mathrm{erf}(x)$.

For some real numbers $m > 1$ and $\alpha > 0$, set a function

$$f_m(x) \;=\; \frac{(\alpha+x)^m - (\alpha-x)^m}{(\alpha+x)^m + (\alpha-x)^m}\,, \qquad -\alpha \le x \le \alpha \tag{9}$$

and then, using the signum function $\mathrm{sgn}(x)$, we define

$$\tau_m(x) \;=\; \begin{cases} f_m(x)\,, & -\alpha \le x \le \alpha \\ \mathrm{sgn}(x)\,, & |x| > \alpha \end{cases} \tag{10}$$

to approximate $\mathrm{erf}(x)$ for $-\infty < x < \infty$. Noting the first derivative

$$\tau_m'(x) \;=\; \begin{cases} \frac{4m\alpha(\alpha^2-x^2)^{m-1}}{\{(\alpha+x)^m+(\alpha-x)^m\}^2}\,, & -\alpha \le x \le \alpha \\ 0\,, & |x| > \alpha \end{cases}, \tag{11}$$

we can see that $\tau_m(x)$ is in $C^1(-\infty,\infty)$, at least. Furthermore, the approximate error function $\tau_m(x)$ has more properties summarized in the theorem below. The proof is omitted as it is clear from the definition of $\tau_m(x)$ in (9) and (10).

Theorem 2.1 *For a real number $m > 1$, $\tau_m(x)$ has the following properties.*

(*a*) *$\tau_m(x)$ is an odd function mapping the real line $\mathbb{R}$ onto an interval $[-1,1]$, and it is strictly increasing on a finite interval $[-\alpha,\alpha]$.*

(*b*) *$\tau_m(-\alpha) = -1$, $\tau_m(\alpha) = 1$ and*

$$\tau_m^{(j)}(\pm\alpha) = 0\,, \qquad j = 1, 2, \cdots, \lceil m \rceil - 1\,,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. In addition, $\tau_m(0) = 0$ with $\tau_m'(0) = \frac{m}{\alpha}$.

(*c*) *$\tau_m(x) \in C^\infty(-\alpha,\alpha) \cap C^{\lceil m \rceil - 1}(-\infty,\infty)$.*

(*d*) *The inverse function of τ_m is*

$$\tau_m^{-1}(y) \;=\; \alpha\,\frac{(1+y)^{1/m} - (1-y)^{1/m}}{(1+y)^{1/m} + (1-y)^{1/m}}, \qquad -1 \le y \le 1.$$

From this theorem we can see that the proposed function $f_m(x)$ has similar properties of the sigmoidal transformation introduced by Prössdorf and Rathsfeld [12] as

$$\gamma_r(x) = \frac{x^r}{x^r + (1-x)^r}$$

which grows from 0 to 1 strictly on a unit interval $[0,1]$ and it will rapidly blow up or sink outside $[0,1]$. In fact, the sigmoidal transformation $\gamma_r(x)$ is usually used in evaluation of weak singular integrals over a finite interval [4, 5, 19].

Figure 1 illustrates behavior of the proposed function $\tau_m(x)$, for $m=2$ and 10, with α based on the relation (13), which implies that $\tau_m(x)$ will approximate the error function $\mathrm{erf}(x)$ accurately for some m between 2 and 10.

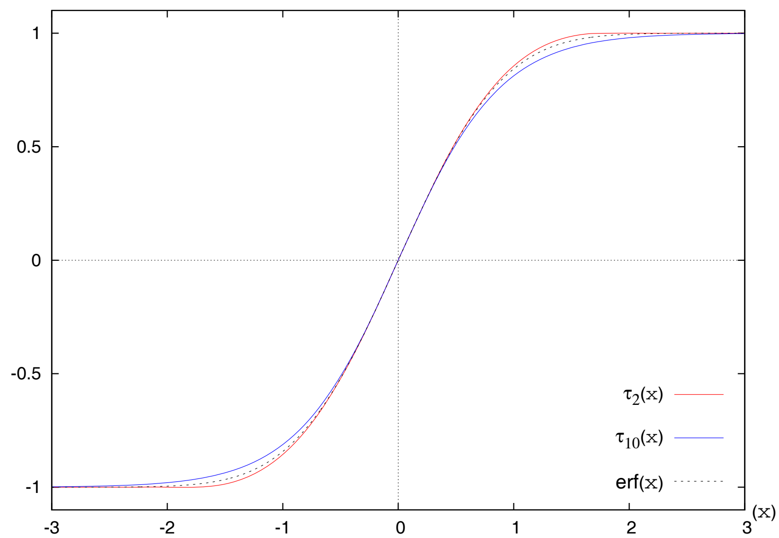

Figure 1. Graphs of $\tau_m(x)$ with $m=2$ and 10 compared with the error function $\mathrm{erf}(x)$.

To determine the unknown parameters α and m, we will employ the following constraints for the error function related with the standard normal distribution $\phi(\xi)=\Phi'(\xi)$ in (6):

(C1) $\phi(0)=\frac{1}{2\sqrt{2}}\mathrm{erf}'(0)=\frac{1}{\sqrt{2\pi}}$, that is, $\mathrm{erf}'(0)=\frac{2}{\sqrt{\pi}}$.

(C2) For a random variable X related to the standard normal distribution,

$$\mathrm{Var}(X) \;=\; \frac{1}{2\sqrt{2}}\int_{-\infty}^{\infty} x^2 \mathrm{erf}'\left(\frac{x}{\sqrt{2}}\right)\, dx \;=\; 1\,,$$

that is,

$$2\int_0^{\infty} t^2 \mathrm{erf}'\left(t\right)\, dt \;=\; 1\,. \tag{12}$$

Replacing $\mathrm{erf}'(0)$ by $\tau_m'(0) = \frac{m}{\alpha}$ in the condition (C1), we have a relation between m and α as

$$\alpha \,=\, \frac{\sqrt{\pi}}{2}\, m\,. \tag{13}$$

In the condition (C2) or the equation (12), replacing $\mathrm{erf}'(t)$ by $\tau_m'(t)$ given in (11), we have

$$2\int_0^{\alpha} t^2 \tau_m'(t)\; dt \,=\, 2\left\{\alpha^2 - 2\int_0^{\alpha} t\tau_m(t)\; dt\right\} \,=\, 1\,, \tag{14}$$

where α is substituted by the relation (13). A numerical solution m^* of the above equation for m can be found as follows by using a computing software such as *Mathematica*, for example,

$$m^* \,=\, 2.5673 \tag{15}$$

and $\alpha = 2.2752$ from (13).

For an application we consider the following approximation formula for the cumulative distribution function $\Phi(\xi)$ in (6).

$$\begin{aligned} T_m(\xi) &= \frac{1}{2}\left\{1 + \tau_m\left(\frac{\xi}{\sqrt{2}}\right)\right\} \\ &= \begin{cases} \dfrac{\left(\sqrt{2}\alpha+\xi\right)^m}{\left(\sqrt{2}\alpha+\xi\right)^m + \left(\sqrt{2}\alpha-\xi\right)^m}\,, & -\sqrt{2}\alpha \le \xi \le \sqrt{2}\alpha \\ \left(1 + \mathrm{sgn}(\xi)\right)/2\,, & |\xi| > \sqrt{2}\alpha\,. \end{cases} \end{aligned} \tag{16}$$

Referring to Theorem 1, one can see that $T_m(\xi)$ is increasing from 0 to 1 and it is in $C^{\lceil m\rceil -1}(-\infty,\infty)$ for any $m > 1$. Table 1 includes numerical errors of the presented approximation $T_m(\xi)$ to $\Phi(\xi)$ for the values of $m = m^*$ given in (15) and a value $m = m_s = 2.4850$ specially chosen. Therein, we employed the approximation formula (4) for the error function, $\mathrm{erf}(x)$ in evaluation of the exact value of $\Phi(\xi)$ in (6). One can see that $T_m(\xi)$ with $m = m_s$ results in a little better approximation than that with $m = m^*$.

Table 1. Numerical results of the errors of the presented approximation $T_m(\xi)$ for $m = m^*$ and $m = m_s$.

m	α	$\Vert T_m(\xi) - \Phi(\xi)\Vert_\infty$	$\Vert T_m(\xi) - \Phi(\xi)\Vert_2$
$m^* = 2.5673$	2.2752	3.0×10^{-3}	4.1×10^{-3}
$m_s = 2.4850$	2.2023	2.1×10^{-3}	3.6×10^{-3}

Additionally, graphs of the differences $E(T_m)(\xi) := T_m(\xi) - \Phi(\xi)$ with $m = m^*$ and m_s are given in Figure 2. Therein, for a correction method introduced in the next section, we denote by a and b the locations of the minimum and the maximum, respectively, of $E(T_{m_s})(\xi)$ and it is observed that $|E(T_{m_s})(a)| \approx |E(T_{m_s})(b)|$. Indeed, we can find the numerical values of a, b as

$$a = 1.22529\,, \qquad b = 2.56483 \tag{17}$$

and

$$E(T_{m_s})(a) = -0.00214243\,, \qquad E(T_{m_s})(b) = 0.00215253\,. \tag{18}$$

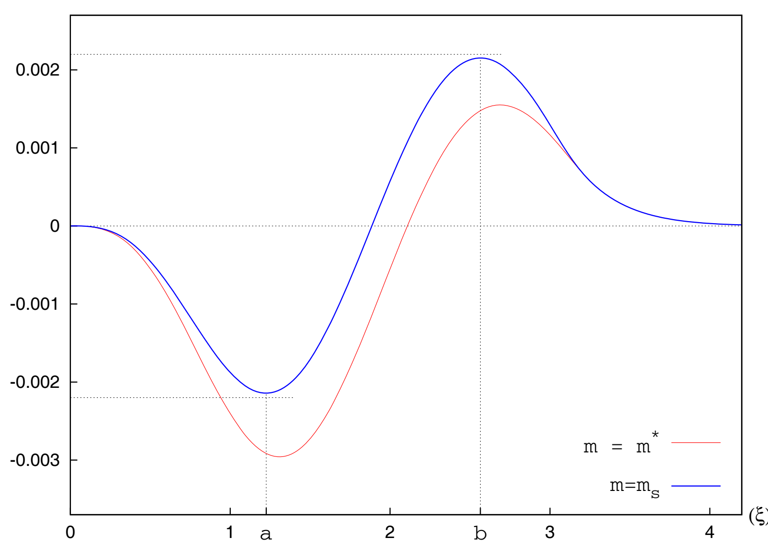

Figure 2. Differences $E(T_m)(\xi) = T_m(\xi) - \Phi(\xi)$ for $m = m^*$ and m_s.

For comparison of the presented approximation to the cumulative distribution function we recall the formula introduced by Lin [9], containing one numerical coefficient, as

$$Q^{\mathrm{L}}(\xi) \;=\; 1 - \left\{ 1 + \exp\left(\frac{4.2\pi\xi}{9 - \xi} \right) \right\}^{-1}. \tag{19}$$

Its absolute error is less than 6.8×10^{-3} over $0 \le \xi < 9$ and blows up for $\xi \ge 9$. In addition, Bryc [3] proposed two numerical coefficients formula

$$Q^{\mathrm{B}}(\xi) \,=\, 1 - \frac{\xi + 3.333}{\sqrt{2\pi}\xi^2 + 7.32\xi + 2 \times 3.333} \exp\left(-\frac{\xi^2}{2}\right) \tag{20}$$

whose absolute error is less than 7.1×10^{-4} over $0 \le \xi < \infty$. The accuracy of the presented approximation $T_m(\xi)$ with $m = m_s = 2.4850$ is between those of $Q^{\mathrm{L}}(\xi)$ and $Q^{\mathrm{B}}(\xi)$. Moreover, it should be noted that $Q^{\mathrm{L}}(\xi)$ and $Q^{\mathrm{B}}(\xi)$ are available for limited intervals while $T_m(\xi)$ is available for the whole interval $(-\infty, \infty)$ as $\tau_m(x)$ is an odd function approximating the error function $\mathrm{erf}(x)$ over the whole interval.

3 A correction method

In this section, taking $m = m_s$, we explore a method to improve the accuracy of the presented approximation $T_m(\xi)$. Exploring the graph of the difference error $E\,(T_m)\,(\xi) = T_m(\xi) - \Phi(\xi)$ in Figure 2 and referring to the correction method introduced in [20], we suggest an approximation to the error $E\,(T_m)\,(\xi)$ by an odd function in the form of

$$h(\xi) \,=\, A\left\{(\xi - c)e^{-r(\xi-c)^2} + (\xi + c)e^{-r(\xi+c)^2}\right\}, \qquad -\infty < \xi < \infty\,, \tag{21}$$

where A and $r > 0$ are additional parameters and $c = (a + b)/2$ for a and b in (17).

We note that

$$h'(\xi) \,=\, A\left\{e^{-r(\xi-c)^2}\left[1 - 2r(\xi - c)^2\right] + e^{-r(\xi+c)^2}\left[1 - 2r(\xi + c)^2\right]\right\}. \tag{22}$$

Employing a condition $h'(b) = 0$ (or $h'(a) = 0$, alternatively), we have

$$h'(b) \approx Ae^{-r(b-c)^2}\left[1 - 2r(b - c)^2\right] \,=\, 0\,. \tag{23}$$

This implies that

$$r \,=\, \frac{1}{2(b - c)^2} \,=\, \frac{2}{(b - a)^2}\,. \tag{24}$$

To determine A we take a condition

$$h(b) \,=\, E\,(T_m)\,(b)\,. \tag{25}$$

From (18) we may set

$$M \;:=\; E\left(T_m\right)(b) \;=\; 0.00215253\,. \tag{26}$$

Since $h(b) \approx A(b-c)e^{-r(b-c)^2}$ in (21), from (25) we have an approximate value of A as

$$A \;=\; \frac{M}{b-c}e^{r(b-c)^2} \;=\; \frac{2M}{b-a}\,e^{1/2}, \tag{27}$$

where the last equality holds because $r(b-c)^2 = 1/2$ from (24). As a result, using the values of r and A given by (24) and (27), we can evaluate the function $h(\xi)$ in (21).

We now set a corrected approximation to the cumulative distribution function $\Phi(\xi)$ as

$$\widetilde{T}_m(\xi) \;=\; T_m(\xi) \;-\; h(\xi)\,, \qquad -\infty < \xi < \infty\,. \tag{28}$$

One can see that $\widetilde{T}_m(\xi)$ will approximate $\Phi(\xi)$ accurately as much as $h(\xi)$ approximates $E\left(T_m\right)(\xi) = T_m(\xi) - \Phi(\xi)$ since

$$\widetilde{T}_m(\xi) - \Phi(\xi) \;=\; E\left(T_m\right)(\xi) \;-\; h(\xi)\,. \tag{29}$$

Table 2 includes numerical results of the errors of the corrected approximation $\widetilde{T}(\xi)$, with $m = m_s$ and $(a, b, M) = (1.22529,\, 2.56483,\, 0.00215253)$ as given in (17) and (26), for each $r = 1.1146$ based on (24) and $r = 1.3360$ particularly selected. The value of A is evaluated by (27). Figure 3 illustrates the enhanced accuracy of the corrected approximation $\widetilde{T}_m(\xi)$.

Table 2. Numerical results of the errors of the corrected approximation $\widetilde{T}(\xi)$ for $r = 1.1146$ and $r = 1.3360$.

$(r,\, A)$	$\lVert\widetilde{T}(\xi) - \Phi(\xi)\rVert_\infty$	$\lVert\widetilde{T}(\xi) - \Phi(\xi)\rVert_2$
(1.1146, 0.00529874)	3.8×10^{-4}	5.2×10^{-4}
(1.3360, 0.00585201)	1.8×10^{-4}	2.7×10^{-4}

Additionally, we consider more accurate approximation formula to the distribution function $\Phi(\xi)$,

$$W(\xi) \;=\; \frac{1}{2}\left\{1 + \omega\left(\frac{\xi}{\sqrt{2}}\right)\right\}\,, \qquad -\infty < \xi < \infty \tag{30}$$

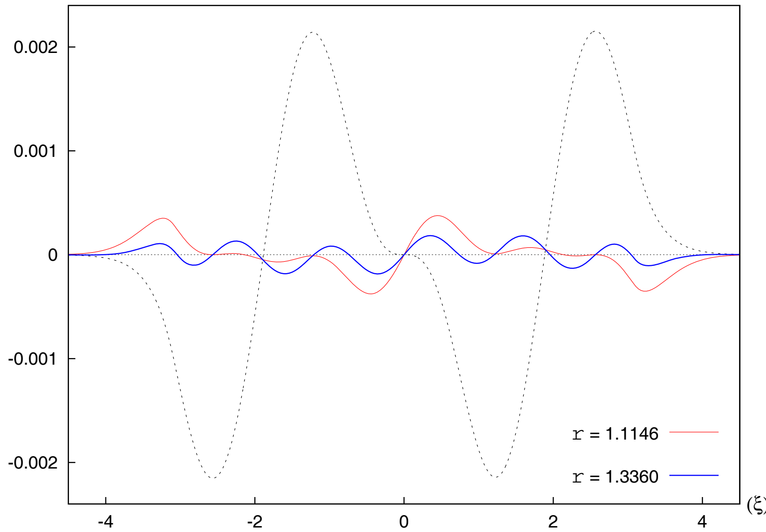

Figure 3. Differences $E(\widetilde{T}_m)(\xi) = \widetilde{T}_m(\xi) - \Phi(\xi)$, with $m = m_s$, for $r = 1.1146$ and $r = 1.3360$. The dotted line indicates $E(T_m)(\xi) = T_m(\xi) - \Phi(\xi)$.

based on the approximate error function introduced by Winitzki [17] as

$$\omega(x) \;=\; \mathrm{sgn}(x)\sqrt{1 - \exp\left(-\beta\, x^2\, \frac{4/\pi + \alpha x^2}{1 + \alpha x^2}\right)}\,, \qquad -\infty < x < \infty \tag{31}$$

with $\alpha = 0.147$ and $\beta = 1$. We can see that the absolute error of this formula is less than 6.1×10^{-5} for all $x \in (-\infty, \infty)$.

Replacing $\beta = 0.99985$ instead of 1 in (31), we can obtain an appropriate difference $E(W)(\xi) = W(\xi) - \Phi(\xi)$ to apply the presented correction method like (28). Indeed, it is found that $|E\,(W)\,(a_\omega)| \approx |E\,(W)\,(b_\omega)|$ for the extremal points a_ω and b_ω as

$$a_\omega = 1.00713\,, \qquad b_\omega = 2.24572 \tag{32}$$

with $c_\omega = (a_\omega + b_\omega)/2$, and we may set

$$M_\omega := E\,(W)\,(b_\omega) = 0.00005617\,. \tag{33}$$

Then we evaluate the parameters r and A based on the formulas in (24) and in (27), respectively, with $(a, b) = (a_\omega, b_\omega)$. Determining the function $h(x) \approx W(\xi) - \Phi(\xi)$ of the form given in (21), we have a corrected approximation,

$$\widetilde{W}(\xi) \;=\; W(\xi) \;-\; h(\xi)\,, \qquad -\infty < \xi < \infty\,. \tag{34}$$

The corrected approximation $\widetilde{W}(\xi)$ results in a maximal error of 5.6×10^{-6}. Figure 4 shows the global behavior of the difference error $E(\widetilde{W})(\xi) = \widetilde{W}(\xi) - \Phi(\xi)$ with $\beta = 0.99985$ and the parameters $(r, A) = (1.303696, 0.000149537)$, evaluated by the equations (24) and (27). It illustrates the improvement of the corrected approximation $\widetilde{W}(\xi)$, like $\widetilde{T}_m(\xi)$ as shown in Figure 3.

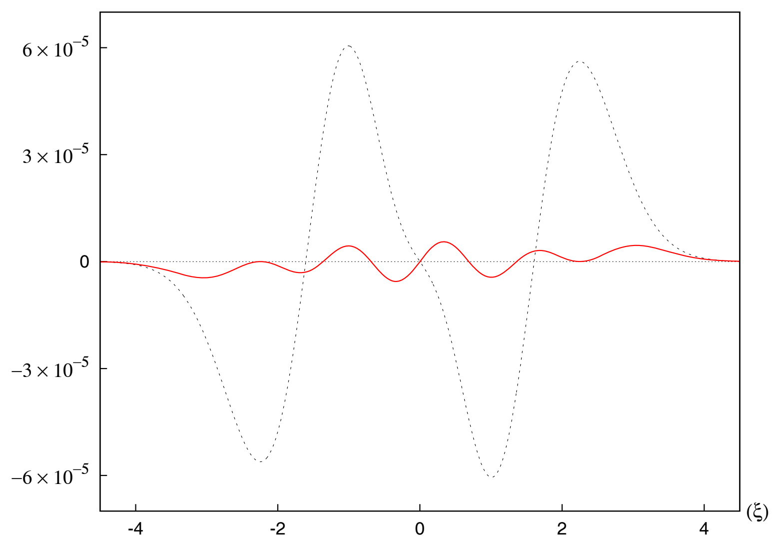

Figure 4. Differences $E(\widetilde{W})(\xi) = \widetilde{W}(\xi) - \Phi(\xi)$ with $\beta = 0.99985$. The dotted line indicates $E(W)(\xi) = W(\xi) - \Phi(\xi)$.

The corrected approximate formula for the error function $\mathrm{erf}(x)$ can be written by

$$\begin{aligned} \widetilde{\tau}_m(x) &:= 2\widetilde{T}_m(\sqrt{2}x) - 1 \\ &= \tau_m(x) - 2h(\sqrt{2}x) \end{aligned} \tag{35}$$

according to the equations (28) and (16). Similarly, from the equations (34) and (30) it follows that

$$\begin{aligned} \widetilde{w}(x) &:= 2\widetilde{W}(\sqrt{2}x) - 1 \\ &= w(x) - 2h(\sqrt{2}x). \end{aligned} \tag{36}$$

4 Conclusions

In this paper we proposed a new ad hoc approximate error function $\tau_m(x)$ which is useful over the whole interval $(-\infty, \infty)$ and has its inverse formula of a closed form. To determine the parameters included in the proposed approximation formula, some appropriate conditions are suggested.

Moreover, in approximation to the cumulative normal distribution function, we developed a correction method which highly improves the accuracy of the presented approximation. The correction method was also applied to an existing approximation of higher accuracy. Results of the numerical implementation demonstrate the plausibility of the presented method.

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