Consumption and portfolio choice with recursive preferences and uninsurable labour income risk

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Outline

Motivation

- Inequality matters for macroeconomics
- Incomplete markets can explain why
- Decoupling RRA and IES is arguably also important
  - A priori no reason for reciprocal relationship as per power utility
  - Coupled parameters confound respective effects
  - Improve ability to quantitatively match empirical moments
  - Long-run risk + EZ preferences shows promise in resolving asset pricing puzzles (Bansal & Yaron, 2004)

Model overview

- Unit continuum of households
- Households face idiosyncratic labour income shocks and an exogenous borrowing constraint (uninsurable labour income risk)
- Households can invest in a risk-free and a risky asset
- Partial equilibrium approach, i.e., prices are given
- Time is continuous, $t \in [0, +\infty)$
  - FOCs hold with equality on the interior of the state space
  - Adjoint relationship between HJB generator and KF

Household problem

Preferences: The individual lifetime utility of a household is given by,

$$V_0 = \max_{\{c_t,\theta_t\}_{t\geq 0}} \mathbb{E}_0[\int_0^\infty f(c_t, V_t)dt]$$

Where $f(\cdot)$ denotes the normalized aggregator,

$$f(c, V) = \frac{\beta}{1-\sigma}(1-\gamma)V\left[\left(\frac{c}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-\sigma} - 1\right]$$

Furthermore, $V_t = \max_{\{c_t,\theta_t\}_{t\geq 0}} \mathbb{E}_0[\int_t^\infty f(c_s, V_s)ds]$

Idiosyncratic labour income risk: Labour income follows a two-state Poisson process, where $\tilde{z} \in \{z^L, z^H\}$ with transition intensities $\{\lambda^L, \lambda^H\}$

Asset returns: Safe asset returns are deterministic and follow,

$$dP_t = rP_t dt$$

Risky asset returns are stochastic and follow a diffusion process,

$$dQ_t = \mu Q_t dt + \nu Q_t dW_t$$

Budget, borrowing and short-sale constraint:

$$da_t = (z_t + r_t a_t + \theta_t(\mu - r)a_t - c_t)dt + \theta_t a_t \nu dW_t$$

$$a_t \geq -\phi$$

$$\frac{a_t + \phi}{a_t} \geq \theta_t \geq 0$$

Definition: The household problem in sequence form becomes,

$$\max_{\{c_t,\theta_t\}_{t\geq 0}} \mathbb{E}_0[\int_0^{\infty} f(c_t, V_t)dt], \text{ such that}$$

$$\tilde{z} \in \{z^L, z^H\} \text{ Poisson with transition intensities } \{\lambda^L, \lambda^H\}$$

$$da_t = (z_t + r_t a_t + \theta_t(\mu - r)a_t - c_t)dt + \theta_t a_t \nu dW_t$$

$$a_t \geq -\phi$$

$$\frac{a_t + \phi}{a_t} \geq \theta_t \geq 0$$

Given $(z_0, a_0) \in \{z^L, z^H\} \times [-\phi, +\infty)$ initial income and wealth

Solution: A solution to the household problem is a stochastic process $\{c_t, \theta_t\}_{t\geq 0}$

Recursive representation

- Intuition from dynamic programming: a recursive representation significantly simplifies the problem
- In continuous time, solving the problem in sequence form is equivalent to solving a partial differential equation, namely,

$$0 = \max_{c,\theta}\{f(c, V(a, z^j)) + \mathbb{E}\frac{dV}{dt}\}$$

- Applying Ito's Lemma this yields,

$$\begin{aligned} 0 = \max_{c,\theta}\{f(c, V(a, z^j)) + V_a(a, z^j)(z^j w + ra + \theta(\mu - r)a - c) \\ + \frac{1}{2}V_{aa}(a, z^j)a^2\theta^2\nu^2 + \lambda^j(V(a, z^{-j})(a) - V(a, z^j))\} \end{aligned}$$

- The max operator can be resolved by taking FOCs w.r.t. c and θ and substituting

Boundary conditions

- The first boundary condition is provided by (or rather enforces) the borrowing constraint, namely,

$$V_a(-\phi, z^j) \geq f_c(\underline{c}^*, V(-\phi, z^j))$$

- The second boundary condition requires some more work. In particular, we use the following proposition,

Proposition 1: *With recursive utility, asymptotic individual policy functions as $a \to \infty$ are given by,*

$$\theta(a, z^j) \sim \frac{\mu - r}{\gamma \nu^2}$$

$$c(a, z^j) \sim \left(\frac{\beta}{\sigma} - \frac{(1-\sigma)(\mu - r)^2}{2\gamma\sigma\nu^2} - \frac{r(1-\sigma)}{\sigma} \right) a$$

- By the FOC w.r.t. θ this allows us to impose,

$$V_{aa}(a, z^j) \sim -\gamma \frac{V_a(a, z^j)}{a}, \text{ as } a \to \infty$$

The stationary wealth distribution

- By the adjoint relationship between the HJB generator and the KF,

$$\frac{dg(a,z^j)}{dt} = -\frac{d}{da}[(z^j w + ra + \theta(a,z^j)(\mu - r)a - c(a,z^j))g(a,z^j)] \\ + \frac{1}{2}\frac{d^2}{da^2}[a^2\theta(a,z^j)^2\nu^2 g(a,z^j)] - \lambda^j g(z^j,a) + \lambda^{-j} g(z^{-j},a)$$

- Finding the stationary wealth distribution simply means solving,

$$\frac{dg(a,z^j)}{dt} = 0$$

Numerical method

- Discretize the state space into a matrix of size $I \times 2$
- Approximate the first derivative with an upwind scheme and the second derivative with a central difference
- Update the value function using a semi-implicit one-step scheme
- Consistency + stability + monotonicity $\implies$ convergence to unique viscosity solution (Barles & Souganidis, 1991)
- Solve KF equation "for free" by adjoint relationship, i.e., if $\mathbf{A}$ discretizes $\mathcal{A}V = \mathbb{E}\frac{dV}{dt}$ then $\mathbf{A}^{\intercal}$ discretizes $\mathcal{A}^{*}g$
- Obtain stationary wealth distribution by solving the eigenvalue problem,

$$\mathbf{A}^{\intercal}\mathbf{g} = 0$$

Calibration

Parameter	Value
Preferences	
β (discount rate)	0.05
γ (coefficient of relative risk aversion)	4
ψ (intertemporal elasticity of substitution)	0.5
Labour income	
$\{z^L, z^H\}$ (income levels)	$\{0.03, 0.09\}$
$\{\lambda^L, \lambda^H\}$ (transition intensities)	$\{0.5, 0.5\}$
Asset returns	
r (risk-free rate)	0.03
μ (expected equity returns)	0.07
ν (volatility of equity returns)	0.2
ϕ (borrowing constraint)	0

Results

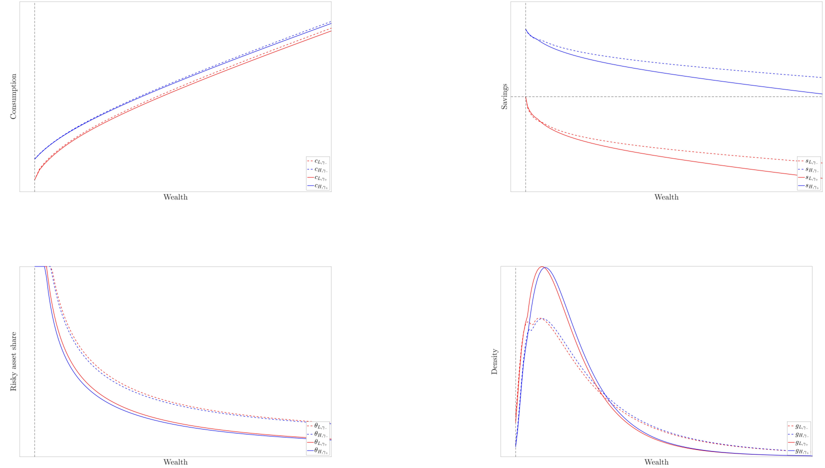

Figure: Individual policy functions and the stationary wealth distribution for an increase in risk aversion from $\gamma = 4$ to $\gamma = 5$

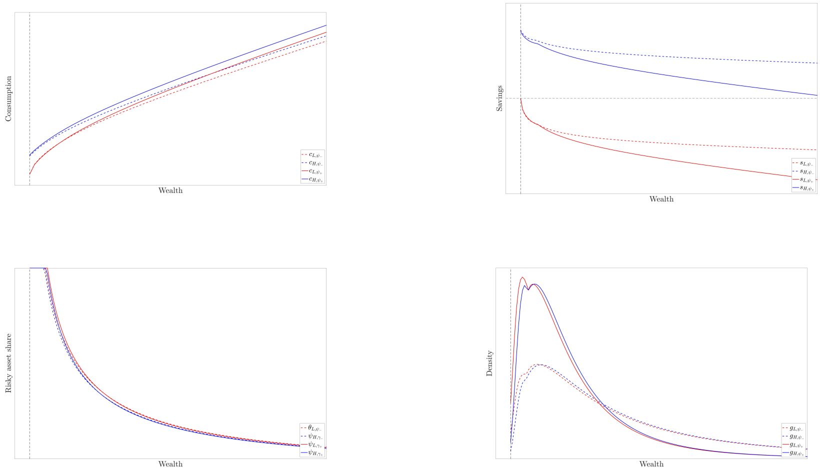

Figure: Individual policy functions and the stationary wealth distribution for an increase in the intertemporal elasticity of substitution from $\psi = 0.4$ to $\psi = 0.6$

Extensions

- Cointegrated labour and equity markets to explain increasing risky asset share (Benzoni et al., 2007)
- Stock market entry cost to explain limited participation (Gomes & Michaelides, 2007)
- General equilibrium with aggregate risk. This complicates the problem (a lot) and leads to a "monster equation". Moving away from rational expectations could resolve this issue (Moll, 2024)

Thank you!

Some more results

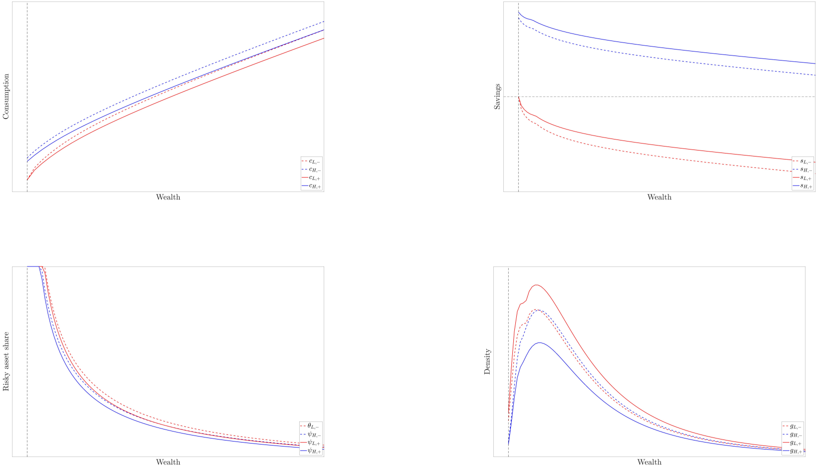

Figure: Individual policy functions and the stationary wealth distribution for an increase in labour income risk

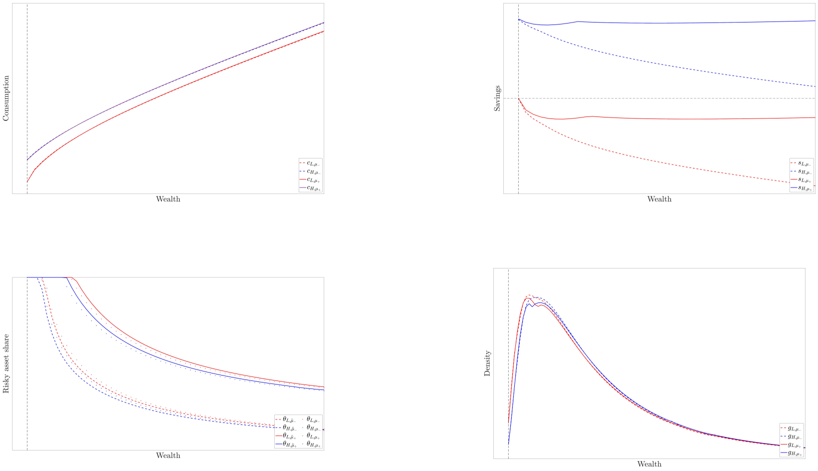

Figure: Individual policy functions and the stationary wealth distribution for a time-varying equity premium