Two-asset model with recursive preferences

David Kraus

1 Income and Returns

Income $\tilde{z}$ follows a Poisson process that takes values in $\tilde{z} \in \{z^L, z^H\}$ with transition intensities $\{\lambda^L, \lambda^H\}$.

Asset returns for the safe asset are deterministic and follow $dP_t = rP_t dt$.

Asset returns for the risky asset are stochastic and follow a diffusion process as per $dQ_t = \mu Q_t dt + \nu Q_t dW_t$.

2 Household Problem in Sequence Form

As in the Merton model, we define the parameters $a_t = b_t P_t + \pi_t Q_t$ and $\theta_t = \frac{\pi_t Q_t}{a_t}$, such that θ_t denotes the share of risky assets. The household problem in sequence form then becomes,

$$\max_{c_t,\theta_t} \mathbb{E}_0[\int_0^\infty f(c_t, V_t)dt]$$

such that,

$$da_t = (z_t w_t + r_t a_t + \theta_t(\mu - r_t)a_t - c_t)dt + \theta_t a_t \nu dW_t$$

$$a_t \geq \underline{a}$$

Here, $f(\cdot)$ denotes the normalized aggregator, and $V_t = \max_{c_t,\theta_t} \mathbb{E}_0[\int_t^\infty f(c_s, V_s)ds]$. In particular,

$$f(c, V) = \frac{\beta}{1-\sigma}(1-\gamma)V \left[\left(\frac{c}{((1-\gamma)V)^{\frac{1}{1-\gamma}}} \right)^{1-\sigma} - 1 \right]$$

3 Household Problem in Recursive Form

For now, let us assume that $r_t = r$ and $w_t = w$ in order to obtain a stationary HJB. Using Ito's Lemma and the fact that the flow payoff is not discounted this yields,

$$0 = \max_{c,\theta}\{f(c, V_j(a)) + V_j'(a)(z^j w + ra + \theta(\mu - r)a - c) + \frac{1}{2}V_j''(a)a^2\theta^2\nu^2 + \lambda^j(V_{-j}(a) + V_j(a))\}$$

We can resolve the max operator and obtain the implicit policy functions $c(a)$ and $\theta(a)$ by taking F.O.C.'s with respect to c and θ. Using the functional form of the normalized aggregator $f(\cdot)$, where $1/\sigma$ denotes the IES and γ denotes the coefficient of RRA, this yields,

$$c^{-\sigma}\beta((1-\gamma)V_j(a))^{\frac{\sigma-\gamma}{1-\gamma}} = V_j'(a) \implies c(a) = \left[\frac{1}{\beta}V_j'(a)((1-\gamma)V_j(a))^{\frac{\gamma-\sigma}{1-\gamma}}\right]^{-1/\sigma}$$

$$(\mu - r)aV_j'(a) + \theta a^2\nu^2 V_j''(a) = 0 \implies \theta(a) = -\frac{(\mu - r)V_j'(a)}{a\nu^2 V_j''(a)}$$

4 Boundary Conditions

The borrowing constraint allows us to impose the following boundary condition,

$$V_j'(\underline{a}) \geq f'(\underline{c}^*, V_j)$$

Here, $\underline{c}^*$ denotes the consumption level of staying put at the borrowing constraint. Note that we need to know $\theta(\underline{a})$ to impose this boundary condition, since $\theta(a)$ determines the consumption level of staying put. We can again circumvent this issue by imposing a no-borrowing constraint, i.e., $\underline{a} = 0$.

In order to impose the second boundary condition, we use the following proposition. The proposition and its proof are directly analogous to the second part of Proposition 2 in Achdou et al. (2021).

Proposition 1: *With recursive utility, asymptotic individual policy functions as $a \to \infty$ are given by,*

$$\theta_j(a) \sim \frac{\mu - r}{\gamma\nu^2}$$

$$c_j(a) \sim \left(\frac{\beta}{\sigma} - \frac{(1-\sigma)(\mu - r)^2}{2\gamma\sigma\nu^2} - \frac{r(1-\sigma)}{\sigma}\right)a$$

In order to prove Proposition 1 we first derive two auxiliary lemmas. Lemma 1 derives an analytical solution for the household problem without income and without the borrowing constraint. Lemma 2 shows a certain homogeneity property of the value function. We can then show that the solution of the household problem presented in Lemma 1 is equivalent to the solution of the household problem as wealth tends to infinity.

Lemma 1: *Let $f(\cdot)$ denote the normalized aggregator. Consider the problem,*

$$0 = \max_{c,\theta}\{f(c, V(a)) + V'(a)(ra + \theta(\mu - r)a - c) + \frac{1}{2}V''(a)\theta^2\nu^2a^2\} \tag{1}$$

The optimal policy functions that solve this problem are given by,

$$\theta(a) = \frac{\mu - r}{\gamma\nu^2}$$

$$c(a) = \left(\frac{\beta}{\sigma} - \frac{(1-\sigma)(\mu - r)^2}{2\gamma\sigma\nu^2} - \frac{r(1-\sigma)}{\sigma}\right)a$$

Proof Lemma 1: Let us first define,

$$H(m, p) := \max_{c}\{f(c, m) - cp\}$$

$$G(p, q) := \max_{\theta}\{p\theta(\mu - r)a + \frac{1}{2}q\theta^2\nu^2a^2\}$$

Resolving the max operators gives us,

$$H(m, p) = \frac{\sigma}{1-\sigma}\beta^{1/\sigma}((1-\gamma)m)^{\frac{\sigma-\gamma}{\sigma(1-\gamma)}}p^{\frac{\sigma-1}{\sigma}} - \frac{\beta(1-\gamma)}{1-\sigma}m$$

$$G(p, q) = -\frac{p^2(\mu - r)^2}{2q\nu^2}$$

Now, we can rewrite the problem as,

$$0 = H(V(a), V'(a)) + G(V'(a), V''(a)) + raV'(a)$$

Let us guess the solution $V(a) = Aa^{1-\gamma}$. Then, $V'(a) = (1-\gamma)Aa^{-\gamma}$ and $V''(a) = -\gamma(1-\gamma)Aa^{-\gamma-1}$. It is straightforward to verify that this solution satisfies the HJB for some $A \in \mathbb{R}$, where A depends on the parameters of the problem.

Furthermore, the F.O.C. with respect to consumption yields,

$$V'(a) = \frac{\partial f}{\partial c}(a) \Leftrightarrow c(a) = \beta^{1/\sigma}(1-\gamma)^{\frac{\sigma-1}{\sigma(1-\gamma)}} A^{\frac{\sigma-1}{\sigma(1-\gamma)}} a$$

Putting everything together, we can now write,

$$\begin{aligned}
&0 = H(V(a), V'(a)) + G(V'(a), V''(a)) + raV'(a) \\
\Leftrightarrow\ &0 = Aa^{1-\gamma}\left(\frac{\sigma}{1-\sigma}\beta^{1/\sigma}(1-\gamma)^{\frac{2\sigma-\sigma\gamma-1}{\sigma(1-\gamma)}} A^{\frac{\sigma-1}{\sigma(1-\gamma)}} - \frac{\beta(1-\gamma)}{1-\sigma} + \frac{(1-\gamma)(\mu-r)^2}{2\gamma\nu^2} + r(1-\gamma)\right) \\
\Leftrightarrow\ &0 = \frac{\sigma}{1-\sigma}\beta^{1/\sigma}(1-\gamma)^{\frac{2\sigma-\sigma\gamma-1}{\sigma(1-\gamma)}} A^{\frac{\sigma-1}{\sigma(1-\gamma)}} - \frac{\beta(1-\gamma)}{1-\sigma} + \frac{(1-\gamma)(\mu-r)^2}{2\gamma\nu^2} + r(1-\gamma) \\
\Leftrightarrow\ &0 = \frac{\sigma c(a)}{(1-\sigma)a} - \frac{\beta}{1-\sigma} + \frac{(\mu-r)^2}{2\gamma\nu^2} + r
\end{aligned}$$

From which we can derive,

$$c(a) = \left(\frac{\beta}{\sigma} - \frac{(1-\sigma)(\mu-r)^2}{2\gamma\sigma\nu^2} - \frac{r(1-\sigma)}{\sigma}\right) a$$

Now, the F.O.C. with respect to the risky asset share yields,

$$V'(a)(\mu-r)a + V''(a)\theta\nu^2 a^2 = 0$$

Which finally gives us,

$$\theta(a) = -\frac{V'(a)(\mu-r)}{V''(a)\nu^2 a} = \frac{\mu-r}{\gamma\nu^2}$$

$\square$

Lemma 2: *Consider the problem,*

$$0 = \max_{c,\theta}\{f(c, V_j(a)) + V_j'(a)(z^j w + ra + \theta(\mu-r)a - c) + \frac{1}{2}V_j''(a)a^2\theta^2\nu^2 + \lambda^j(V_{-j}(a) + V_j(a))\} \tag{2}$$

Then, for any $\xi > 0$,

$$V_j(\xi a) = \xi^{1-\gamma} V_{\xi,j}(a)$$

where $V_{\xi,j}$ solves the problem,

$$0 = \max_{c,\theta}\{f(c, V_{\xi,j}(a)) + V_{\xi,j}'(a)(\frac{z^j w}{\xi} + ra + \theta(\mu-r)a - c) + \frac{1}{2}V_{\xi,j}''(a)a^2\theta^2\nu^2 + \lambda^j(V_{\xi,-j}(a) + V_{\xi,j}(a))\} \tag{3}$$

Proof Lemma 2: We consider,

$$V_j(\xi a) = \xi^{1-\gamma} V_{\xi,j}(a)$$

Hence,

$$V_j(a) = \xi^{1-\gamma} V_{\xi,j}(a/\xi)$$

$$V_j'(a) = \xi^{-\gamma} V_{\xi,j}'(a/\xi)$$

$$V_j''(a) = \xi^{-\gamma-1} V_{\xi,j}''(a/\xi)$$

We can plug these expressions into equation (2). We use the same definitions of H and G as in the proof of Lemma 1. This then gives us,

$$\begin{aligned} 0 = H(\xi^{1-\gamma} V_{\xi,j}(a/\xi), \xi^{-\gamma} V_{\xi,j}'(a/\xi)) + G(\xi^{-\gamma} V_{\xi,j}'(a/\xi), \xi^{-\gamma-1} V_{\xi,j}''(a/\xi)) \\ + \xi^{-\gamma} V_{\xi,j}'(a/\xi)(z^j w + ra) + \lambda^j (\xi^{1-\gamma} V_{\xi,-j}(a/\xi) - \xi^{1-\gamma} V_{\xi,j}(a/\xi)) \end{aligned}$$

Some computation yields,

$$H(\xi^{1-\gamma} V_{\xi,j}(a/\xi), \xi^{-\gamma} V_{\xi,j}'(a/\xi)) = \xi^{1-\gamma} H(V_{\xi,j}(a/\xi), V_{\xi,j}'(a/\xi))$$

Similarly,

$$G(\xi^{-\gamma} V_{\xi,j}'(a/\xi), \xi^{-\gamma-1} V_{\xi,j}''(a/\xi)) = \xi^{1-\gamma} G(V_{\xi,j}'(a/\xi), V_{\xi,j}''(a/\xi))$$

Plugging in these expressions, dividing both sides by $\xi^{1-\gamma}$, and writing by slight abuse of notation $a = a/\xi$, we then finally obtain,

$$0 = \max_{c,\theta} \{ f(c, V_{\xi,j}(a)) + V_{\xi,j}'(a)(\frac{z^j w}{\xi} + ra + \theta(\mu - r)a - c) + \frac{1}{2} V_{\xi,j}''(a) a^2 \theta^2 \nu^2 + \lambda(V_{\xi,-j}(a) + V_{\xi,j}(a)) \}$$

$\square$

Proof Proposition 1: We first derive the asymptotic consumption policy function. From the F.O.C. with respect to consumption and our expression for $V_{\xi,j}$ we obtain,

$$c_j(a) = \left[\frac{1}{\beta} V_j'(a)((1-\gamma)V_j(a))^{\frac{\gamma-\sigma}{1-\gamma}} \right]^{-1/\sigma} = \left[\frac{1}{\beta} \xi^{-\gamma} V_{\xi,j}'(a/\xi)((1-\gamma)\xi^{1-\gamma} V_{\xi,j}(a/\xi))^{\frac{\gamma-\sigma}{1-\gamma}} \right]^{-1/\sigma} = \xi c_{\xi,j}(a/\xi)$$

In particular, for $\xi = a$ this gives us,

$$c_j(a) = c_{\xi,j}(1) \cdot a$$

We let $\xi = a \to \infty$ and obtain,

$$\lim_{a\to\infty} \frac{c_j(a)}{a} = \lim_{\xi\to\infty} c_{\xi,j}(1) = \frac{\beta}{\sigma} - \frac{(1-\sigma)(\mu-r)^2}{2\gamma\sigma\nu^2} - \frac{r(1-\sigma)}{\sigma}$$

We first note that by Lemma 2, $c_{\xi,j}$ is the consumption policy function solving equation (3). We then note that equation (3) converges to equation (1) as $\xi \to \infty$. Hence, by Lemma 1, we can write the second equality.

Hence, we finally have that,

$$c_j(a) \sim \left(\frac{\beta}{\sigma} - \frac{(1-\sigma)(\mu-r)^2}{2\gamma\sigma\nu^2} - \frac{r(1-\sigma)}{\sigma}\right) a$$

Similarly, from the F.O.C. with respect to risky asset share and our expression for $V_{\xi,j}$ we obtain,

$$\theta_j(a) = -\frac{(\mu-r)V_j'(a)}{a\nu^2 V_j''(a)} = -\frac{(\mu-r)V_{\xi,j}'(a/\xi)}{\frac{a}{\xi}\nu^2 V_{\xi,j}''(a/\xi)} = \theta_{\xi,j}(a/\xi)$$

In particular, for $\xi = a$ this gives us,

$$\theta_j(a) = \theta_{\xi,j}(1)$$

We let $\xi = a \to \infty$ and obtain,

$$\lim_{a\to\infty} \theta_j(a) = \lim_{\xi\to\infty} \theta_{\xi,j}(1) = \frac{\mu-r}{\gamma\nu^2}$$

The second equality follows from Lemma 1 and Lemma 2 by the same argument as before. Hence, we have,

$$\theta_j(a) \sim \frac{\mu-r}{\gamma\nu^2}$$

□

We will use the following property to impose our second boundary condition. By Proposition 2 and the F.O.C. with respect to the risky asset share we have that as $a \to \infty$,

$$\theta(a) = -\frac{(\mu-r)V_j'(a)}{a\nu^2 V_j''(a)} \sim \frac{\mu-r}{\gamma\nu^2} \Leftrightarrow V_j''(a) \sim -\gamma\frac{V_j'(a)}{a}$$

We can solve for the stationary distribution by solving the KF equation with reflecting barriers at both boundaries of the state space. We can solve for the general equilibrium by imposing a market clearing condition for the asset market.